Appendix to "Steering the Climate System"

The first section provides additional background on solving control problems with pure state constraints. Section B contains proofs, derivations, and the analysis of the Hotelling-like policy in the main text. Section C describes the numerical example's calibration and solution. Section D adapts the carbon dioxide (CO_2) decay model of Golosov et al. (2014) to our setting and demonstrates that the main text's primary results still hold. Section E allows business-as-usual emissions to evolve over time. Section F numerically explores different degrees of inertia and the role of the discount rate in the base model. Section G provides a phase portrait analysis of the efficient policy. Section H derives the least-cost trajectory for a geoengineering control.

A Optimal control with pure state constraints

We solve our state-constrained control problem via a set of necessary conditions that will look unfamiliar to many economists. The standard approach to solving constrained control problems in economic applications is to embed the Hamiltonian inside of a Lagrangian and apply complementary slackness conditions. This approach requires that a "rank constraint qualification" hold: at any time t, the Jacobian of the binding constraints with respect to the controls must have full rank when evaluated at the optimal control vector $u(t)$ and optimal state vector $x(t)$.[1] Intuitively, the first-order conditions for maximizing a Lagrangian require that the regulator be able to choose its controls so as to have a first-order effect on each binding constraint.

We study a case in which the control $u(t)$ does not enter the constraint (i.e., we have a "pure" state constraint), so that the rank constraint qualification fails to hold. Our time t abatement control can affect a binding time t temperature constraint only by changing temperature at later times. Consider an interval over which a pure state constraint $h(t, x(t), u(t)) \geq 0$ binds. Assume one-dimensional controls and states, and note that being a pure state constraint means $\partial h(t, x(t), u(t))/\partial u(t) = 0$. To maintain the binding constraint $h(t, x(t), u(t)) = 0$, it must be true that $\mathrm{d}h(t, x(t), u(t))/\mathrm{d}t \triangleq h^1(t, x(t), u(t)) = 0$. Maintaining the pure state constraint requires steering the system so that its total derivative with respect to time is 0. If $\partial h^1(t, x(t), u(t))/\partial u(t) = 0$, then maintaining the constraint $h^1(t, x(t), u(t)) = 0$ requires that $\mathrm{d}h^1(t, x(t), u(t))/\mathrm{d}t \triangleq h^2(t, x(t), u(t)) = 0$. We continue this process until finding the first constraint $h^\rho(t, x(t), u(t))$ that includes the control variable $u(t)$. The pure state constraint is then said to be of order ρ. In our setting, the temperature constraint is of order 2 because its first time derivative depends on CO_2 but not abatement, and its second time derivative depends on abatement via the time derivative of CO_2. The policymaker's choice of time t abatement can immediately affect only the time t acceleration

[1] See, for instance, Caputo (2005, Chapter 6).

or deceleration of temperature, not the time t level or velocity of temperature.

Let the pure state constraint be of order ρ. The "indirect adjoining" approach used in our analysis builds the Lagrangian as if $h^\rho(t, x(t), u(t))$ were the relevant constraint. The rank constraint qualification would hold for a system constrained by $h^\rho(t, x(t), u(t)) \geq 0$, but we need to recognize that the true system must actually obey the constraint $h(t, x(t), u(t)) \geq 0$ and, over an interval over which the constraint binds, also $h^k(t, x(t), u(t)) = 0$ for $k < \rho$. Complementary slackness applies to the original constraint $h(t, x(t), u(t))$, not to $h^\rho(t, x(t), u(t))$. Critically, the costate variable on $x(t)$ can jump at the time that the constraint begins to bind.[2] The jump in the costate variable depends on both the partial derivatives of $h^k(t, x(t), u(t))$ with respect to $x(t)$ (for $k < \rho$) and on (the level and time derivatives of) the constraint multiplier.[3]

The survey by Hartl et al. (1995) is the best reference we have found for necessary conditions for control problems with pure state constraints. We adapt the necessary conditions from their Section 6, which presents the indirect adjoining approach to higher-order constraints.

B Formal analysis

This section contains proofs, an additional proposition, the derivation of equation (10), and the analysis of a CO_2 constraint in the main text's setting.

B.1 Proof of Proposition 1

We begin with a lemma that draws on the main text's analysis of the shadow cost of abatement along a least-cost path.

[2]Technically, the costate variable can jump at both the first time that the constraint binds (the "entry time") and the last time that the constraint binds (the "exit time"). However, the values of the costate variable and the constraint multiplier are not unique in that case, so we can normalize the costate variable to jump at only one of the two times. We here choose to allow a jump at the entry time and to impose continuity on the costate variable at the exit time.

[3]Imagine that $\partial h(t, x(t), u(t))/\partial x(t) \geq 0$. Then increasing the state variable helps satisfy the state constraint. Prior to the constraint binding, the costate variable for $x(t)$ includes the value induced by the effect of marginally increasing $x(t)$ on future times' constraints. Intuitively, the costate variable jumps at the time that the constraint begins to bind because the costate variable now includes only the marginal value of the state in meeting $h^\rho(t, x(t), u(t)) \geq 0$; the constraints $h^k(t, x(t), u(t)) = 0$ for $k < \rho$ now do not directly affect the level of the control. In our setting, increasing either CO_2 or (abstracting from a complication due to inertia) temperature makes it more difficult to satisfy the constraint in the future. The shadow costs of CO_2 and temperature initially include these dynamic costs induced by the constraint. Once the constraint begins to bind, the shadow costs of CO_2 and temperature jump down because the temperature constraint now enters the decision problem only as a constraint on the acceleration of temperature.

Lemma B-1. *Let τ indicate the first time $t > t_0$ at which $T(t) = \bar{T}$. Along any least-cost trajectory, τ is finite.*

Proof. The assumption that $E > \delta\left(F^{-1}(\bar{T}/s) - M_{pre}\right)$ implies that temperature along a least-cost path must either reach $\bar{T}$ in finite time or approach it asymptotically from below. Assume that there is no finite time τ at which the system attains $\bar{T}$. Either the system reaches $\bar{M}$ at some finite time and then remains there, or the system approaches $\bar{M}$ asymptotically. Therefore, either abatement reaches $E - \delta\left(F^{-1}(\bar{T}/s) - M_{pre}\right)$ at some finite time and then remains there, or abatement approaches $E - \delta\left(F^{-1}(\bar{T}/s) - M_{pre}\right)$ asymptotically. In either case,

$$\lim_{t\to\infty} \lambda_M(t) = C'\left(E - \delta\left(F^{-1}(\bar{T}/s) - M_{pre}\right)\right), \quad \lim_{t\to\infty} \dot{\lambda}_M(t) = 0.$$

Using equation (5), we have

$$\lim_{t\to\infty} \dot{\lambda}_M(t) = (r+\delta)\, C'\left(E - \delta\left(F^{-1}(\bar{T}/s) - M_{pre}\right)\right) - \phi\, s\, \lambda_T(t_0)\, e^{(r+\phi)(t-t_0)}\, F'(M(t)) = -\infty.$$

But $\dot{\lambda}_M(t)$ cannot approach both zero and negative infinity. We have a contradiction. The time τ must be finite. □

Now consider whether $M(t)$ is greater or less than $\bar{M}$ for $t \in (\tau - \epsilon, \tau)$, for some $\epsilon > 0$.[4] If $M(t) \leq \bar{M}$ for all $t \in (\tau - \epsilon, \tau)$, then equation (2) and $T(t) < \bar{T}$ imply that $T(\tau) = \bar{T}$ only as τ goes to infinity.[5] But Lemma B-1 showed that τ is finite. Therefore $M(t) > \bar{M}$ for some $t \in (\tau - \epsilon, \tau)$. Since this result holds for ϵ arbitrarily small, we then have that there exists some $\Delta > 0$ such that $M(t) > \bar{M}$ for all $t \in (\tau - \Delta, \tau)$.

Once temperature attains $\bar{T}$, CO_2 must remain no larger than $\bar{M}$ in order to prevent temperature from rising past the constraint. And the assumption that $E > \delta\left(F^{-1}(\bar{T}/s) - M_{pre}\right)$ implies that, along a least-cost trajectory, CO_2 must remain no less than $\bar{M}$ once temperature has attained $\bar{T}$. Therefore CO_2 must remain fixed at $\bar{M}$ once temperature attains $\bar{T}$. And because CO_2 must be strictly above $\bar{M}$ at some instant before temperature attains $\bar{T}$, there exists some time q such that $\dot{M}(t) \leq 0$ for all times $t \geq q$ and such that $\dot{M}(t) < 0$ for some time $t \geq q$. This establishes the first part of the proposition.

The second part of the proposition follows immediately from observing that a policymaker constrained to keep CO_2 no greater than the steady-state level $\bar{M}$ corresponding to $\bar{T}$ never lets temperature reach $\bar{T}$. Any path that satisfies the constrained CO_2 problem therefore also satisfies the corresponding constrained temperature problem. However, we have seen that the least-cost CO_2 trajectory must exceed $\bar{M}$ in the constrained temperature problem.

[4] We thank Larry Karp for catching an error in an earlier version of the following proof.

[5] Within the class of trajectories for which $M(t) \leq \bar{M}$ for $t \in (\tau - \epsilon, \tau)$, the trajectory that attains $\bar{T}$ at the earliest time fixes $M(t) = \bar{M}$ for all $t \in (\tau - \epsilon, \tau)$. In that case, equation (2) is an autonomous linear equation, which approaches its steady state $\bar{T}$ only asymptotically. Therefore, if $M(t) \leq \bar{M}$ for $t \in (\tau - \epsilon, \tau)$, then τ must be infinite.

The least-cost path that satisfies the temperature constraint therefore does not satisfy the corresponding CO_2 constraint. Constraining CO_2 introduces an additional binding constraint that strictly increases the cost of the least-cost policy pathway.

B.2 An additional proposition

Proposition B-2. *Let τ be the first time at which $T(t) = \bar{T}$, and let x be the last time prior to τ at which $M(t)$ is nondecreasing. If $\dot{M}(t_0) > 0$, then $x > t_0$, $\dot{\lambda}_M(x) > 0$, and there exists a unique time $y \in (x, \tau)$ at which $\lambda_M(t)$ reaches a maximum.*

Proof. First consider the CO_2 trajectory for times $t \in [t_0, \tau]$. We know by Proposition 1 that it is nonincreasing after some time prior to τ. Combined with the assumption that $\dot{M}(t_0) > 0$, we have that there exists a last time $x \in (t_0, \tau)$ at which $M(t)$ is nondecreasing. At this interior maximum, it must be the case that $\dot{M}(x) = 0$ and $\ddot{M}(x) < 0$. Differentiating equation (1), we have

$$\ddot{M}(t) = -\dot{A}(t) - \delta\, \dot{M}(t).$$

At a point where $\dot{M}(t) = 0$, $\ddot{M}(t) < 0$ if and only if $\dot{A}(t) > 0$. We know by equation (4) that marginal abatement cost equals the shadow cost of CO_2. This establishes that $\dot{\lambda}_M(x) > 0$.

At an interior maximum of $\lambda_M(t)$ in $[t_0, \tau]$, it must be the case that $\dot{\lambda}_M(t) = 0$ and $\ddot{\lambda}_M(t) \leq 0$. Differentiating equation (5), we have:

$$\ddot{\lambda}_M(t) = (r + \delta)\, \dot{\lambda}_M(t) + \Big[-F''(M(t))\, \dot{M}(t) - (r + \phi)F'(M(t))\Big]\, \phi\, s\, \lambda_T(t).$$

At a point where $\dot{\lambda}_M(t) = 0$, $\ddot{\lambda}_M(t) \leq 0$ if and only if $-\dot{M}(t)\, F''(M(t))/F'(M(t)) \leq r + \phi$. Recognizing that $F''(M(t)) < 0$, that $F'(M(t)) > 0$, and that $\dot{M}(t) < 0$ at all times $t \in (x, \tau)$, we have that $\ddot{\lambda}_M(t) < 0$ at any $t \in (x, \tau)$ for which $\dot{\lambda}_M(t) = 0$.

We have already seen that $\dot{\lambda}_M(x) > 0$. Now consider the first time τ when $T(t) = \bar{T}$. The proof of Proposition 1 shows that CO_2 must be strictly greater than $\bar{M}$ in the instants before τ: $M(\tau - \epsilon) = \bar{M} + \gamma$ for ϵ sufficiently small and $\epsilon, \gamma > 0$. In order to achieve the temperature limit at τ, abatement must be such that $[M(\tau) - M(\tau - \epsilon)]/\epsilon = -\gamma/\epsilon$. Letting ϵ and γ jointly go to 0, this relation implies that:

$$\dot{M}(\tau - \epsilon) = E - A(\tau - \epsilon) - \delta(\bar{M} - M_{pre} + \gamma) = -\frac{\gamma}{\epsilon},$$

which holds if and only if:

$$A(\tau - \epsilon) = E - \delta\left(\bar{M} - M_{pre}\right) + \left[\frac{\gamma}{\epsilon} - \delta\,\gamma\right].$$

As ϵ, γ jointly go to 0, the term in the brackets is strictly positive. To maintain temperature at $\bar{T}$ at time τ and beyond, abatement must satisfy $A(\tau) = E - \delta\left(\bar{M} - M_{pre}\right)$. Therefore abatement is greater in the instants before time τ. The main text shows that abatement is continuous at τ. Therefore, $\dot{\lambda}_M(\tau-\epsilon) < 0$. By the Intermediate Value Theorem, there exists some time $y \in (x, \tau-\epsilon)$ such that $\dot{\lambda}_M(y) = 0$. We have already established that $\ddot{\lambda}_M(y) < 0$ for all such y, so there is a unique maximum of $\lambda_M(t)$ between times x and τ.

□

B.3 Derivation of equation (10)

Substitute $\lambda_T(t)$ into equation (5):

$$(r+\delta)\lambda_M(t) - \dot{\lambda}_M(t) = \phi\, sF'(M(t))\, \lambda_T(t_0)\, e^{(r+\phi)(t-t_0)}.$$

Multiply by the integrating factor, integrate with respect to time, and rearrange:

$$\begin{aligned}
&(r+\delta)e^{-(r+\delta)(t-t_0)}\lambda_M(t) - e^{-(r+\delta)(t-t_0)}\dot{\lambda}_M(t) = e^{-(r+\delta)(t-t_0)}\phi\, sF'(M(t))\, \lambda_T(t_0)\, e^{(r+\phi)(t-t_0)} \\
\Leftrightarrow &\int_{t_0}^{t}\left[-(r+\delta)e^{-(r+\delta)(i-t_0)}\lambda_M(i) + e^{-(r+\delta)(i-t_0)}\dot{\lambda}_M(i)\right]\, di \\
&\qquad = \int_{t_0}^{t} -e^{-(r+\delta)(i-t_0)}\phi\, sF'(M(i))\, \lambda_T(t_0)\, e^{(r+\phi)(i-t_0)}\, di \\
\Leftrightarrow &e^{-(r+\delta)(t-t_0)}\lambda_M(t) - \lambda_M(t_0) = -\phi\, s\, \lambda_T(t_0)\int_{t_0}^{t} e^{(\phi-\delta)(i-t_0)} F'(M(i))\, di.
\end{aligned}$$

Substitute in $\lambda_T(t_0) = e^{-(r+\phi)(t-t_0)}\lambda_T(t)$ and rearrange:

$$\lambda_M(t_0) = e^{-[r+\delta](t-t_0)}\lambda_M(t) + e^{-[r+\delta](t-t_0)}\lambda_T(t)\int_{t_0}^{t} e^{-(\phi-\delta)(t-i)}\phi\, sF'(M(i))\, di.$$

B.4 Hotelling policy

Now consider the Hotelling-like policy in the main text's setting. Recall that this policy ignores the inertia in the climate system. It minimizes the cost of meeting the constraint $M(t) \leq \bar{M}$ (while ignoring temperature), where $\bar{M}$ is the unique steady-state CO_2 concentration implied by $\bar{T}$. The Hotelling trajectory solves:

$$\begin{aligned}
&\min_{A(\cdot)} \int_{t_0}^{\infty} e^{-r(t-t_0)}\, C(A(t))\, dt \\
\text{subject to } \dot{M}(t) =& E - A(t) - \delta(M(t) - M_{pre}), \\
M(t) \leq& \bar{M}, \\
M(t_0)& \text{ given.}
\end{aligned}$$

We follow the main text in ignoring the nonnegativity constraint on abatement. Define:

$$\begin{aligned} g^0(M(t), A(t)) =& \bar{M} - M(t) \geq 0, \\ g^1(M(t), A(t)) =& -\dot{M}(t). \end{aligned}$$

The state constraint is now of order one. As in other cases we have studied, there will be a first time τ at which the state constraint binds, and the state constraint will then bind forever after τ under a least-cost policy. The current-value Hamiltonian is

$$H(M(t),A(t), \lambda_M(t)) = C(A(t)) + \lambda_M(t)\,\left[E - A(t) - \delta(M(t) - M_{pre})\right].$$

The current-value Lagrangian is

$$H[t] + \nu(t)\left\{-E + A(t) + \delta(M(t) - M_{pre})\right\}.$$

The necessary conditions for a maximum are (Hartl et al., 1995):

$$C'(A(t)) =\lambda_M(t) - \nu(t), \tag{B-1}$$

$$\dot{\lambda}_M(t) =[r + \delta]\,\lambda_M(t) - \nu(t)\,\delta,$$

$$\nu(t)[\bar{M} - M(t)] =0, \quad \nu(t) \leq 0, \quad \dot{\nu}(t) \geq r\,\nu(t),$$

$$\lambda_M(\tau^-) =\lambda_M(\tau^+) - e^{r(\tau - t_0)}\,\eta_M, \tag{B-2}$$

$$H[\tau^-] =H[\tau^+],$$

$$\eta_M \leq 0,\ \eta_M \leq e^{-r(\tau - t_0)}\,\nu(\tau^+), \tag{B-3}$$

along with the transition equation, the initial condition on $M(t)$, and the state constraint.[6]

It is easy to see that we get the standard decay-adjusted Hotelling trajectory prior to time τ. After τ, abatement must be chosen so as to hold $\dot{M}(t) = 0$, as in the analysis of a temperature constraint. We need to consider whether abatement jumps at τ. Use equation (B-1) and substitute in from equation (B-2) to obtain:

$$C'(A(\tau^+)) =C'(A(\tau^-)) + e^{r(\tau - t_0)}\,\eta_M - \nu(\tau^+).$$

The conditions in (B-3) then imply that abatement either jumps down at τ (if $\eta_M < e^{-r(\tau - t_0)}\,\nu(\tau^+)$) or is continuous at τ (if $\eta_M = e^{-r(\tau - t_0)}\,\nu(\tau^+)$). Assume that abatement

[6] Formally, there are two more necessary conditions: that $\mathrm{d}H[t]/\mathrm{d}t = \mathrm{d}L[t]/\mathrm{d}t$, and that an omitted multiplier on the instantaneous payoff function be weakly positive. The first condition is always satisfied since at any time t either the constraints are binding or their Lagrange multipliers are zero. The second condition cannot be satisfied with a multiplier of zero because abatement would then always be either at its upper or lower bound (in order to maximize the Lagrangian), which cannot be optimal. Thus, as is typical in economic analysis, the omitted multiplier must be strictly positive and therefore ignorable. For ease of presentation, we ignore these two conditions here, in the main text, and in the remainder of the appendix.

jumps down at τ. We then have:

$$\begin{aligned}\dot{M}(\tau^-) =& E - A(\tau^-) - \delta(M(\tau^-) - M_{pre})\\ <& E - A(\tau^+) - \delta(M(\tau^-) - M_{pre})\\ =& E - A(\tau^+) - \delta(M(\tau^+) - M_{pre})\\ =& \dot{M}(\tau^+)\\ =& 0.\end{aligned}$$

Therefore, if abatement jumps down at τ, then $\dot{M}(\tau^-) < 0$, which would imply that CO_2 is declining towards $\bar{M}$ and thus that $M(t) > \bar{M}$ for some time $t < \tau$. But this would violate the state constraint. We have a contradiction. As a result, abatement must be continuous at τ and we must have $\eta_M = e^{-r(\tau - t_0)}\,\nu(\tau^+)$.

C Numerical calibration and solution

We calibrate the example to DICE-2007 (Nordhaus, 2008), as implemented with an annual timestep in Lemoine and Traeger (2014). All baseline runs use the 5.5% annual consumption discount rate ($r = 0.055$) generally consistent with this model.[7]

The full DICE model includes three carbon reservoirs. Lemoine and Traeger (2014) approximate DICE's full carbon dynamics by making the decay rate of CO_2 a function of the atmospheric CO_2 stock and time. Along the optimal path in DICE, the time-varying decay rate for CO_2 in excess of its pre-industrial level starts at 0.0141, declines to 0.0119 in 100 years, and declines to 0.0068 after 200 years. Using the average value over the first 100 years, we have $\delta = 0.0138$. We calibrate business-as-usual CO_2 emissions E to DICE's initial value. This yields $E = 9.97$ Gt C per year.

We follow much scientific literature in modeling forcing as $F(M(t)) = \alpha \ln(M(t)/M_{pre})$. We take $M_{pre} = 596.4$ Gt C, and we follow Ramaswamy et al. (2001, Table 6.2) in using $\alpha = 5.35$ W m^{-2}, which is approximately equivalent to the parameters used in DICE. The full DICE model includes two temperature reservoirs. Lemoine and Traeger (2014) simplify this setting by representing the deep ocean temperature as a function $\gamma_T(T, t)$ of surface temperature and time. In their discrete-time setting, the temperature transition becomes

$$T_{t+1} - T_t = C_T \left[F_{t+1} - \frac{\alpha \ln(2)}{cs} T_t - \left[1 - \gamma_T(T_t, t)\right] C_O\, T_t\right],$$

where we have used cs for climate sensitivity so as to avoid confusion with the present paper's notation. The present paper's parameter s gives equilibrium warming per unit of forcing,

[7]Technically, this setting with stationary output should use a discount rate no greater than 1.5% to be consistent with DICE-2007: consumption growth in the Ramsey equation is negative once we subtract the cost of abatement.

whereas DICE's $cs = 3$ gives equilibrium warming from doubled CO_2. Relating the two parameters, we have:

$$s = \frac{cs}{\alpha\ \ln(2)} = 0.809\ °\mathrm{C}\ \left[\mathrm{W\ m^{-2}}\right]^{-1}.$$

Using explicit Euler difference methods, we find:

$$\phi = \frac{C_T\left[F_{t+1} - \frac{\alpha\ \ln(2)}{cs} T_t - \left[1 - \gamma_T(T_t, t)\right] C_O\, T_t\right]}{s\, F_t - T_t}.$$

Along DICE's optimal trajectory, the inferred value of ϕ starts at 0.0129, falls to 0.0056 after 100 years, and falls to -0.0030 after 200 years (reflecting that the ocean begins transferring heat to the atmosphere as the CO_2 concentration declines). Using the average value over the first 100 years, we have $\phi = 0.0091$.

In DICE, the cost (as a fraction of time t output) of abating a fraction μ_t of business-as-usual emissions is $\Psi_t \mu_t^{a_2}$, where $a_2 = 2.8$ and

$$\Psi_t = \frac{a_0\, \sigma_t}{a_2}\left(1 - \frac{1 - e^{(t-t_0)\, g_\Psi}}{a_1}\right), \text{ with } \sigma_t = \sigma_0 \exp\left[\frac{g_{\sigma,0}}{\delta_\sigma}\left(1 - e^{-(t-t_0)\,\delta_\sigma}\right)\right].$$

The parameters are $a_0 = 1.17$, $a_1 = 2$, $g_\Psi = -0.005$, $\sigma_0 = 0.13$, $g_{\sigma,0} = -0.0073$, and $\delta_\sigma = 0.003$. Initial output Y (without adjusting for climate damages) in DICE is approximately 85 trillion dollars. We represent the cost of abatement $A(t)$ as

$$C\left(A(t)\right) = \Psi_{t_0}\left[\frac{A(t)}{E}\right]^{a_2} Y. \tag{C-4}$$

Finally, from DICE-2007, we have the initial CO_2 stock as $M_0 = 808.9$ Gt C, the initial global mean surface temperature as $T_0 = 0.7307$ °C (relative to 1900), and the initial time as $t_0 = 2005$.

To solve the four-dimensional system of differential equations defined in the main text, we begin with a triplet $(T(\tau), M(\tau), A(\tau))$ such that $T(\tau) = \bar{T}$, $M(\tau) = \bar{M}$, and $A(\tau)$ holds $\dot{M}(\tau) = 0$. From the Maximum Principle, we have $\lambda_M(\tau^-) = C'(A(\tau))$. We then seek the value of $\lambda_T(\tau^-)$ consistent with these conditions and with the initial conditions. For a given value of $\lambda_T(\tau^-)$, we use Matlab's ode23 solver with relative and absolute tolerances of 10^{-10} to solve the system of ordinary differential equations from τ but with time flowing in reverse.[8] In the resulting simulation, let x be the time t at which $M(t) = M_0$. At a solution to the system, it must also be the case that $T(x) = T_0$. An optimization routine searches for

[8] In general, we cannot solve the model forward by searching for the initial shadow costs $\lambda_M(t_0)$ and $\lambda_T(t_0)$ that lead the system to obey the terminal conditions because, as is typical of saddle-path stable systems, values slightly off the desired trajectories lead the system to a wildly different outcome. Our solution method is closely related to the "reverse shooting" technique described in Judd (1998).

the value of $\lambda_T(\tau^-)$ such that $T(x) = T_0$. At a solution, the values $\lambda_M(x)$ and $\lambda_T(x)$ are the efficient $\lambda_M(t_0)$ and $\lambda_T(t_0)$.[9] Using these initial values, we then simulate the model forward in actual time, setting $\lambda_M(t)$ to hold $M(t)$ constant at $M(\tau)$ for all times $t > \tau$. We use the trapezoidal method to approximate the integral of abatement cost over the mesh points.

D Extension to the decay model of Golosov et al. (2014)

Our main analysis assumes that the stock of CO_2 decays exponentially. In reality, the evolution of atmospheric CO_2 is more complex. We here extend our setting to the more realistic decay model of Golosov et al. (2014).

In Golosov et al. (2014), a fraction ψ_L of emissions remains forever, a fraction $(1-\psi_0)(1-\psi_L)$ decays immediately, and a fraction $\psi_0(1-\psi_L)$ decays geometrically at rate ψ. Their carbon decay model reduces to the main text's model when $\psi_L = 0$, $\psi_0 = 1$, and $\psi = \delta$. Let $M_1(t)$ be the stock of CO_2 that remains in the atmosphere forever and $M_2(t)$ be the stock of CO_2 that decays geometrically. We have the following equations of motion:

$$\begin{aligned}\dot{M}_1(t) =& \psi_L[E - A(t)],\\ \dot{M}_2(t) =& \psi_0(1-\psi_L)[E - A(t)] - \psi M_2(t).\end{aligned}$$

The total stock of CO_2 is the sum of the CO_2 in these two atmospheric reservoirs: $M(t) = M_1(t) + M_2(t)$. When we numerically implement this model, we follow their calibration in using $M_1(t_0) = 684$ Gt C, $M_2(t_0) = 118$ Gt C, $\psi_L = 0.2$, $\psi_0 = 0.393$, and $\psi = 0.0228/10$, where the latter adjusts for measuring time in years rather than in decades.[10]

The next subsection analyzes the least-cost policy trajectory with this new decay model, the subsequent subsection analyzes the least-cost Hotelling trajectory, and the third subsection reports numerical results.

D.1 Least-cost policy

We now consider least-cost policy. As in the main analysis, the CO_2 stock must equal $\bar{M}$ when the policymaker decides to finally let $T(t)$ reach $\bar{T}$, because otherwise the constraint $T(t) \leq \bar{T}$ either would be violated or would fail to bind in the following instant. As before, let τ be the first time at which $T(t) = \bar{T}$. The efficient policy trajectory must keep $M(t) = \bar{M}$ for all $t \geq \tau$.

[9] When solving for the Hotelling trajectory, we begin with $M(\tau)$ equal to $\bar{M}$ and $\lambda_M(\tau^-)$ equal to $C'(A(\tau))$. No search is necessary, as temperature can be effectively removed from the policymaker's problem.

[10] Golosov et al. (2014) abstract from inertia: they assume that temperature responds instantly to CO_2. They describe how to adjust their carbon decay model to mimic the combined effects of thermal inertia and carbon decay in the DICE model. We do not use this adjustment because we model inertia explicitly and we want to analyze robustness to their own carbon decay model.

Let $M_{min}(t)$ be the minimum CO_2 stock attainable at any time after t. If emissions (net of abatement) are strictly positive, then the infinite persistence of a fraction ψ_L of CO_2 means that $M_{min}(t)$ is increasing over time ($E(t) - A(t) > 0 \Leftrightarrow \dot{M}_{min}(t) > 0$). For any finite time t, the efficient policy path requires that $M_{min}(t) < \bar{M}$ so that the temperature constraint will not be violated at some later time. In order to prevent $M_{min}(t)$ from eventually becoming larger than $\bar{M}$, the efficient policy must have $A(t) \to E$ (at which point $\dot{M}_{min}(t) = 0$). We have thus seen that the policymaker must eventually eliminate all emissions. This result contrasts with the main text's setting with geometric decay, in which strictly positive emissions are consistent with holding the CO_2 stock fixed at $\bar{M}$ for all times $t \geq \tau$.[11]

We can also see that the policymaker eliminates all emissions only asymptotically. Imagine that the optimal path is such that $A(t) = E$ for finite t. At that time, the total stock of CO_2 would be declining because of the geometric decay represented by ψ. But this declining stock violates the condition that an efficient trajectory holds $M(t)$ fixed at $\bar{M}$ for all times $t \geq \tau$. For t sufficiently large, the efficient trajectory must have $A(t) \to E$ only asymptotically.

The least-cost abatement trajectory must solve:

$$
\begin{aligned}
&\min_{A(\cdot)} \int_{t_0}^{\infty} e^{-r(t-t_0)}\, C(A(t))\, dt \\
\text{subject to } \dot{M}_1(t) =& \psi_L[E - A(t)], \\
\dot{M}_2(t) =& \psi_0(1-\psi_L)[E - A(t)] - \psi M_2(t), \\
\dot{T}(t) =& \phi\left[s\, F(M_1(t) + M_2(t)) - T(t)\right], \\
A(t) \leq& E, \\
T(t) \leq& \bar{T}, \\
& M_1(t_0), M_2(t_0), T(t_0) \text{ given.}
\end{aligned}
$$

In contrast to the main text, we explicitly represent the nonnegativity constraint on net emissions $E - A(t)$. We will see in the numerical analysis that the new decay model makes this constraint relevant. Following the main text, define:

$$
\begin{aligned}
h^0(T(t), M_1(t), M_2(t), A(t)) =& \bar{T} - T(t) \geq 0, \\
h^1(T(t), M_1(t), M_2(t), A(t)) =& -\dot{T} = -\phi\left[s\, F(M_1(t) + M_2(t)) - T(t)\right], \\
h^2(T(t), M_1(t), M_2(t), A(t)) =& -\ddot{T} = -\phi\left[s\, F'\left(M_1(t) + M_2(t)\right)\left(\dot{M}_1(t) + \dot{M}_2(t)\right) - \dot{T}(t)\right] \\
=& -\phi s\, F'(M_1(t) + M_2(t))\left\{[\psi_0(1-\psi_L) + \psi_L][E - A(t)] - \psi M_2(t)\right\} \\
& + \phi^2\left[s\, F(M_1(t) + M_2(t)) - T(t)\right].
\end{aligned}
$$

[11] Further, because a fraction of emissions persists forever, the temperature limit here fixes cumulative emissions. In contrast, in the main text, we saw that recognizing inertia enabled the policymaker to increase cumulative emissions.

As in the main text, the state constraint is of order two. The current-value Hamiltonian is

$$
\begin{aligned}
H(M_1(t),&M_2(t), T(t), A(t), \lambda_{M1}(t), \lambda_{M2}(t), \lambda_T(t)) \\
&= C(A(t)) + \lambda_{M1}(t)\, \psi_L[E - A(t)] + \lambda_{M2}(t)\, \{\psi_0[1-\psi_L][E - A(t)] - \psi M_2(t)\} \\
&\quad + \lambda_T(t)\, \phi\, [s\, F(M_1(t) + M_2(t)) - T(t)].
\end{aligned}
$$

The current-value Lagrangian is

$$
\begin{aligned}
&H[t] + \mu(t)\, [A(t) - E] \\
&+ \nu(t)\Big\{ - \phi s\, F'(M_1(t) + M_2(t))\, \{[\psi_0(1-\psi_L) + \psi_L][E - A(t)] - \psi M_2(t)\} \\
&\qquad + \phi^2\, [s\, F(M_1(t) + M_2(t)) - T(t)]\Big\}.
\end{aligned}
$$

The necessary conditions for a maximum are (Hartl et al., 1995):

$$
\begin{aligned}
C'(A(t)) =& \lambda_{M1}(t)\psi_L + \lambda_{M2}(t)\psi_0[1-\psi_L] - \mu(t) \\
& - \nu(t)\phi\, s\, F'(M_1(t) + M_2(t))[\psi_0(1-\psi_L) + \psi_L], \qquad \text{(D-5)} \\
\dot{\lambda}_{M1}(t) =& r\lambda_{M1}(t) - \phi\, s\, F'(M_1(t) + M_2(t))\, \lambda_T(t) \\
& + \nu(t)\, \phi\, s\, F''(M_1(t) + M_2(t))\, \{[\psi_0\, (1-\psi_L) + \psi_L][E - A(t)] - \psi M_2(t)\} \\
& - \nu(t)\, \phi^2\, s\, F'(M_1(t) + M_2(t)), \\
\dot{\lambda}_{M2}(t) =& [r + \psi]\lambda_{M2}(t) - \phi\, s\, F'(M_1(t) + M_2(t))\, \lambda_T(t) \\
& + \nu(t)\, \phi\, s\, F''(M_1(t) + M_2(t))\, \{[\psi_0\, (1-\psi_L) + \psi_L][E - A(t)] - \psi M_2(t)\} \\
& - \nu(t)\, [\phi + \psi]\phi\, s\, F'(M_1(t) + M_2(t)), \\
\dot{\lambda}_T(t) =& [r + \phi]\lambda_T(t) + \nu(t)\phi^2, \\
\mu(t) \geq& 0,\ A(t) - E \leq 0,\ \mu(t)\, [A(t) - E] = 0, \\
\nu(t)[\bar{T} - T(t)] = 0,& \quad \nu(t) \leq 0, \quad \dot{\nu}(t) \geq r\, \nu(t), \quad \ddot{\nu}(t) \leq 2\, r\dot{\nu}(t) - r^2\, \nu(t), \\
\lambda_{M1}(\tau^-) =& \lambda_{M1}(\tau^+) - e^{r(\tau - t_0)}\, \eta^2_{M1}\phi\, s\, F'(M_1(t) + M_2(t)), \qquad \text{(D-6)} \\
\lambda_{M2}(\tau^-) =& \lambda_{M2}(\tau^+) - e^{r(\tau - t_0)}\, \eta^2_{M2}\phi\, s\, F'(M_1(t) + M_2(t)), \qquad \text{(D-7)} \\
\lambda_T(\tau^-) =& \lambda_T(\tau^+) - e^{r(\tau - t_0)}\, \eta^1_T + e^{r(\tau - t_0)}\, \eta^2_T\phi, \\
H[\tau^-] =& H[\tau^+], \\
\eta^1_x, \eta^2_x \leq 0,\ \eta^1_x \leq& - e^{-r(\tau - t_0)}\, \dot{\nu}(\tau^+) + r e^{-r(\tau - t_0)}\nu(\tau^+),\ \eta^2_x = e^{-r(\tau - t_0)}\, \nu(\tau^+) \ \text{ for } x \in \{T, M1, M2\}, \qquad \text{(D-8)}
\end{aligned}
$$

along with the transition equations, the initial conditions on $M_1(t)$, $M_2(t)$, and $T(t)$, and the state constraint.

For times $t \geq \tau$, abatement evolves so as to keep $h^1 = 0$, which requires $M_1(t) + M_2(t) = \bar{M}$. In order to maintain $h^2 = 0$ (i.e., in order to stay at $\bar{M}$), we need:

$$A(t) = E - \frac{\psi}{\psi_L + \psi_0(1-\psi_L)} \left[\bar{M} - M_1(t)\right], \tag{D-9}$$

for $t \geq \tau$. We therefore have:

$$A(\tau) = E - \frac{\psi}{\psi_L + \psi_0(1-\psi_L)} \left[\bar{M} - M_1(\tau)\right] \triangleq \bar{A}(M_1(\tau)).$$

Differentiate equation (D-9) with respect to time:

$$\dot{A}(t) = \frac{\psi\,\psi_L}{\psi_L + \psi_0(1-\psi_L)}[E - A(t)] \geq 0,$$

for $t \geq \tau$. Integrating from τ to $t \geq \tau$ yields:

$$A(t-\tau; M_1(\tau)) = E - e^{-\frac{\psi\,\psi_L}{\psi_L+\psi_0(1-\psi_L)}(t-\tau)}[E - \bar{A}(M_1(\tau))].$$

After temperature reaches $\bar{T}$, abatement rises from $A(\tau) = \bar{A}(M_1(\tau))$ towards E and attains E only asymptotically, as argued above.

From equation (D-5), we have

$$C'(A(\tau^-)) = \lambda_{M1}(\tau^-)\psi_L + \lambda_{M2}(\tau^-)\psi_0[1-\psi_L] - \mu(\tau^-),$$

and also that

$$C'(A(\tau^+)) = \lambda_{M1}(\tau^+)\psi_L + \lambda_{M2}(\tau^+)\psi_0[1-\psi_L] - \mu(\tau^+) - \nu(\tau^+)\phi\, s\, F'(M_1(\tau)+M_2(\tau))[\psi_0(1-\psi_L)+\psi_L].$$

Equations (D-6) and (D-7) and the conditions in (D-8) then imply that $A(\tau^-) = A(\tau^+)$. Thus, as in the main text, abatement is continuous at time τ.

In the remainder of this section, we study least-cost policy before the constraint binds. We have $\nu(t) = 0$ over these times $t \in [t_0, \tau)$. At these times, $C'(A(t)) = \lambda_{M1}(t)\psi_L + \lambda_{M2}(t)\psi_0[1-\psi_L] - \mu(t)$. If $\mu(t) > 0$, then $A(t) = E$ and $\mu(t)$ picks up the gap between the shadow cost of emissions and $C'(E)$ in equation (D-5).

Following the derivation for the main text, the costate equations on $M_1(t)$, $M_2(t)$, and $T(t)$ imply the following relationships:

$$\lambda_{M1}(t_0) = e^{-r(t-t_0)}\,\lambda_{M1}(t) + e^{-r(t-t_0)}\,\lambda_T(t) \int_{t_0}^{t} e^{-\phi(t-i)}\,\phi\, s\, F'(M(i))\, di,$$

$$\lambda_{M2}(t_0) = e^{-[r+\psi](t-t_0)}\,\lambda_{M2}(t) + e^{-[r+\psi](t-t_0)}\,\lambda_T(t) \int_{t_0}^{t} e^{-(\phi-\psi)(t-i)}\,\phi\, s\, F'(M(i))\, di.$$

These equations are exactly the same as in the main text, except with $M_1(t)$ lacking a geometric decay component. We therefore see that inertia enters $\lambda_{M1}(t)$ and $\lambda_{M2}(t)$ in the same way as it entered $\lambda_M(t)$ in the main text (modulo the geometric decay terms in the exponents). Further, equation (D-5) shows that marginal abatement cost (which defines the efficient emission tax) is linear in $\lambda_{M1}(t)$ and $\lambda_{M2}(t)$, just as it was linear in $\lambda_M(t)$ in the main text. The way that inertia affects the efficient tax on emissions is therefore qualitatively unchanged by the extension to the more realistic decay model of Golosov et al. (2014).

Write the cost of the remaining policy program at τ as a function of τ and $M_1(\tau)$:

$$W(\tau, M_1(\tau)) = \int_\tau^\infty e^{-r(i-\tau)} C(A(i-\tau; M_1(\tau)))\,\mathrm{d}i.$$

Along a least-cost path, the costate variables must be[12]

$$\lambda_{M1}(\tau^+) = \frac{\partial W(\tau, M_1(\tau))}{\partial M_1(\tau)} = \frac{\psi}{\psi_L + \psi_0(1-\psi_L)} \int_\tau^\infty e^{-\left(r + \frac{\psi\,\psi_L}{\psi_L+\psi_0(1-\psi_L)}\right)(i-\tau)} C'(A(i-\tau; M_1(\tau)))\,\mathrm{d}i$$

and

$$\lambda_{M2}(\tau^+) = \frac{\partial W(\tau, M_1(\tau))}{\partial M_2(\tau)} = 0.$$

From equations (D-6) and (D-7) and the conditions in (D-8), we then have:

$$\begin{aligned}
\lambda_{M1}(\tau^-) =& \frac{\psi}{\psi_L + \psi_0(1-\psi_L)} \int_\tau^\infty e^{-\left(r + \frac{\psi\,\psi_L}{\psi_L+\psi_0(1-\psi_L)}\right)(i-\tau)} C'(A(i-\tau; M_1(\tau)))\,\mathrm{d}i \\
& - \nu(\tau^+)\,\phi\, s\, F'(\bar{M}), && \text{(D-10)}\\
\lambda_{M2}(\tau^-) =& - \nu(\tau^+)\,\phi\, s\, F'(\bar{M}). && \text{(D-11)}
\end{aligned}$$

From equation (D-5), we then have:

$$\begin{aligned}
C'(\bar{A}(M_1(\tau))) =& \psi_L \frac{\psi}{\psi_L + \psi_0(1-\psi_L)} \int_\tau^\infty e^{-\left(r + \frac{\psi\,\psi_L}{\psi_L+\psi_0(1-\psi_L)}\right)(i-\tau)} C'(A(i-\tau; M_1(\tau)))\,\mathrm{d}i \\
& - [\psi_L + \psi_0[1-\psi_L]]\,\nu(\tau^+)\,\phi\, s\, F'(\bar{M}), && \text{(D-12)}
\end{aligned}$$

where we recognize that the abatement nonnegativity constraint does not bind at τ. (Suppose the constraint did bind at τ. We know that $M_2(\tau) > 0$, resulting in $M_1 + M_2 < \bar{M}$ in the

[12] If we had instead defined $\bar{A}$ as a function of $M_2(\tau)$ and left $M_1(\tau)$ as the residual, then we would obtain $\lambda_{M1}(\tau^+) = 0$ and $\lambda_{M2}(\tau^+) < 0$, with the difference between them being the exact same as in the given analysis. We will see that it is the difference that matters, as $\nu(\tau)$ will work to shift both multipliers' right-hand limits to match their left-hand limits. The results needed for the numerics will therefore be unchanged, as the inferred value of $\nu(\tau^+)$ will simply reflect whichever choice we make. However, the given presentation with both costate variables positive matches the reasonable assumption that the shadow costs should be positive.

next instant, but the efficient policy must maintain CO_2 at $\bar{M}$. The abatement nonnegativity constraint therefore cannot bind at τ.)

To numerically solve the model, we guess $\lambda_T(\tau^-)$ and $M_1(\tau)$. The guess for $M_1(\tau)$ implies $A(\tau)$, which in turn implies $\nu(\tau^+)$ from equation (D-12) and then $\lambda_{M1}(\tau^-)$ and $\lambda_{M2}(\tau^-)$ from equations (D-10) and (D-11).[13] We also know that $T(\tau) = \bar{T}$ and $M_2(\tau) = \bar{M} - M_1(\tau)$. We solve the system backwards in time, stopping when either $M_1(t)$ meets its initial condition or $A(t) = E$. In the latter case, we then simulate the system backwards from this new point, with $\mu(t)$ starting at zero (its value at the latest time that the constraint binds).[14] The differential equation for $\mu(t)$ comes from fixing $A(t) = E$ and then differentiating equation (D-5) with respect to time. We simulate the system backwards in time with $A(t) = E$ until we find a time at which $\mu(t)$ once again reaches zero, which is where the constraint that $E - A(t) \geq 0$ just started to bind. We then simulate the unconstrained system backwards in time from there, stopping when $M_1(t)$ meets its initial condition. Once we have found a time when $M_1(t)$ meets its initial condition, we check the initial conditions on $M_2(t)$ and $T(t)$. We iterate until we find values of $\lambda_T(\tau^-)$ and $M_1(\tau)$ that generate paths that satisfy these initial conditions.

D.2 Hotelling policy

Now consider the Hotelling-like policy under the decay model of Golosov et al. (2014). Recall that this policy ignores the inertia in the climate system. In the main analysis, it minimizes the cost of meeting the constraint $M(t) \leq \bar{M}$ (while ignoring temperature), where $\bar{M}$ is the unique steady-state CO_2 concentration implied by $\bar{T}$. Here, we study the problem of constraining $M_1(t) + M_2(t) \leq \bar{M}$, while ignoring temperature. The Hotelling trajectory solves:

$$\begin{aligned}
&\min_{A(\cdot)} \int_{t_0}^{\infty} e^{-r(t-t_0)}\, C(A(t))\, dt \\
\text{subject to } \dot{M}_1(t) =& \psi_L[E - A(t)], \\
\dot{M}_2(t) =& \psi_0(1-\psi_L)[E - A(t)] - \psi M_2(t), \\
A(t) \leq& E, \\
M_1(t) + M_2(t) \leq& \bar{M}, \\
& M_1(t_0), M_2(t_0) \text{ given.}
\end{aligned}$$

[13] We approximate the integral in equation (D-12) by starting from near $M_1(t) = \bar{M}$ and $M_2(t) = 0$, simulating backwards until reaching $M_1(\tau)$, and then using a Newton-Cotes formula to approximate the integral.

[14] In general, $\mu(t)$ need be only piecewise continuous, but continuity of $A(t)$ here ensures continuity of $\mu(t)$. See Caputo (2005, Chapter 6).

Define:

$$\begin{aligned} g^0(M_1(t), M_2(t), A(t)) =& \bar{M} - M_1(t) - M_2(t) \geq 0, \\ g^1(M_1(t), M_2(t), A(t)) =& - \dot{M}_1(t) - \dot{M}_2(t). \end{aligned}$$

The state constraint is now of order one. As in other cases we have studied, there will be a first time τ at which the state constraint binds, and it will bind forever after that time under a least-cost policy. The current-value Hamiltonian is

$$\begin{aligned} H(M_1(t), &M_2(t), A(t), \lambda_{M1}(t), \lambda_{M2}(t)) \\ &= C(A(t)) + \lambda_{M1}(t)\, \psi_L[E - A(t)] + \lambda_{M2}(t)\, \{\psi_0[1-\psi_L][E - A(t)] - \psi M_2(t)\}. \end{aligned}$$

The current-value Lagrangian is

$$H[t] + \mu(t)\, [A(t) - E] + \nu(t)\, \{-[\psi_0(1-\psi_L) + \psi_L][E - A(t)] + \psi M_2(t)\}.$$

The necessary conditions for a maximum are (Hartl et al., 1995):

$$\begin{aligned} C'(A(t)) =& \lambda_{M1}(t)\psi_L + \lambda_{M2}(t)\psi_0[1-\psi_L] - \mu(t) - \nu(t)\, [\psi_0(1-\psi_L) + \psi_L], && \text{(D-13)} \\ \dot{\lambda}_{M1}(t) =& r\lambda_{M1}(t), \\ \dot{\lambda}_{M2}(t) =& [r + \psi]\lambda_{M2}(t) - \nu(t)\, \psi, \\ \mu(t) \geq& 0,\ A(t) - E \leq 0,\ \mu(t)\, [A(t) - E] = 0, \\ \nu(t)[\bar{M} - M_1(t) - M_2(t)] = 0,& \quad \nu(t) \leq 0, \quad \dot{\nu}(t) \geq r\, \nu(t), \\ \lambda_{M1}(\tau^-) =& \lambda_{M1}(\tau^+) - e^{r(\tau - t_0)}\, \eta_{M1}, && \text{(D-14)} \\ \lambda_{M2}(\tau^-) =& \lambda_{M2}(\tau^+) - e^{r(\tau - t_0)}\, \eta_{M2}, && \text{(D-15)} \\ H[\tau^-] =& H[\tau^+], \\ \eta_x \leq& 0,\ \eta_x \leq e^{-r(\tau - t_0)}\, \nu(\tau^+) \ \text{ for } x \in \{M1, M2\}, && \text{(D-16)} \end{aligned}$$

along with the transition equations, the initial conditions on $M_1(t)$ and $M_2(t)$, and the state constraint.

For times $t \geq \tau$, abatement evolves so as to keep $g^1 = 0$. This requirement generates the same post-τ policy path as in the previous subsection. Now consider whether abatement is continuous at τ. Use equation (D-13) and substitute in from equations (D-14) and (D-15) to obtain:

$$C'(A(\tau^+)) = C'(A(\tau^-)) + e^{r(\tau - t_0)}\, [\psi_L\, \eta_{M1} + \psi_0\, (1-\psi_L)\, \eta_{M2}] - (\psi_0(1-\psi_L) + \psi_L)\, \nu(\tau^+).$$

The conditions in (D-16) then imply that abatement either jumps down at τ (if either $\eta_{M1} < e^{-r(\tau - t_0)}\, \nu(\tau^+)$ or $\eta_{M2} < e^{-r(\tau - t_0)}\, \nu(\tau^+)$) or is continuous at τ (if $\eta_{M1} = \eta_{M2} =$

$e^{-r(\tau-t_0)}\,\nu(\tau^+)$). Assume that abatement jumps down at τ. We then have:

$$\begin{aligned}\dot{M}_1(\tau^-)+\dot{M}_2(\tau^-)=&[\psi_L+\psi_0(1-\psi_L)][E-A(\tau^-)]-\psi M_2(\tau^-)\\ <&[\psi_L+\psi_0(1-\psi_L)][E-A(\tau^+)]-\psi M_2(\tau^-)\\ =&[\psi_L+\psi_0(1-\psi_L)][E-A(\tau^+)]-\psi M_2(\tau^+)\\ =&\dot{M}_1(\tau^+)+\dot{M}_2(\tau^+)\\ =&0.\end{aligned}$$

Therefore, if abatement jumps down at τ, then $\dot{M}_1(\tau^-)+\dot{M}_2(\tau^-)<0$, which would imply that total CO_2 is declining towards $\bar{M}$ and thus that $M_1(t)+M_2(t)>\bar{M}$ for some time $t<\tau$. But this would violate the state constraint. We have a contradiction. As a result, abatement must be continuous at τ and $\eta_{M1}=\eta_{M2}=e^{-r(\tau-t_0)}\,\nu(\tau^+)$.

The remaining analysis and the numerical methods are directly analogous to the previous subsection. Note that each shadow cost increases exponentially for $t\in(t_0,\tau)$. We therefore recover a Hotelling-like trajectory, modified for this decay model.

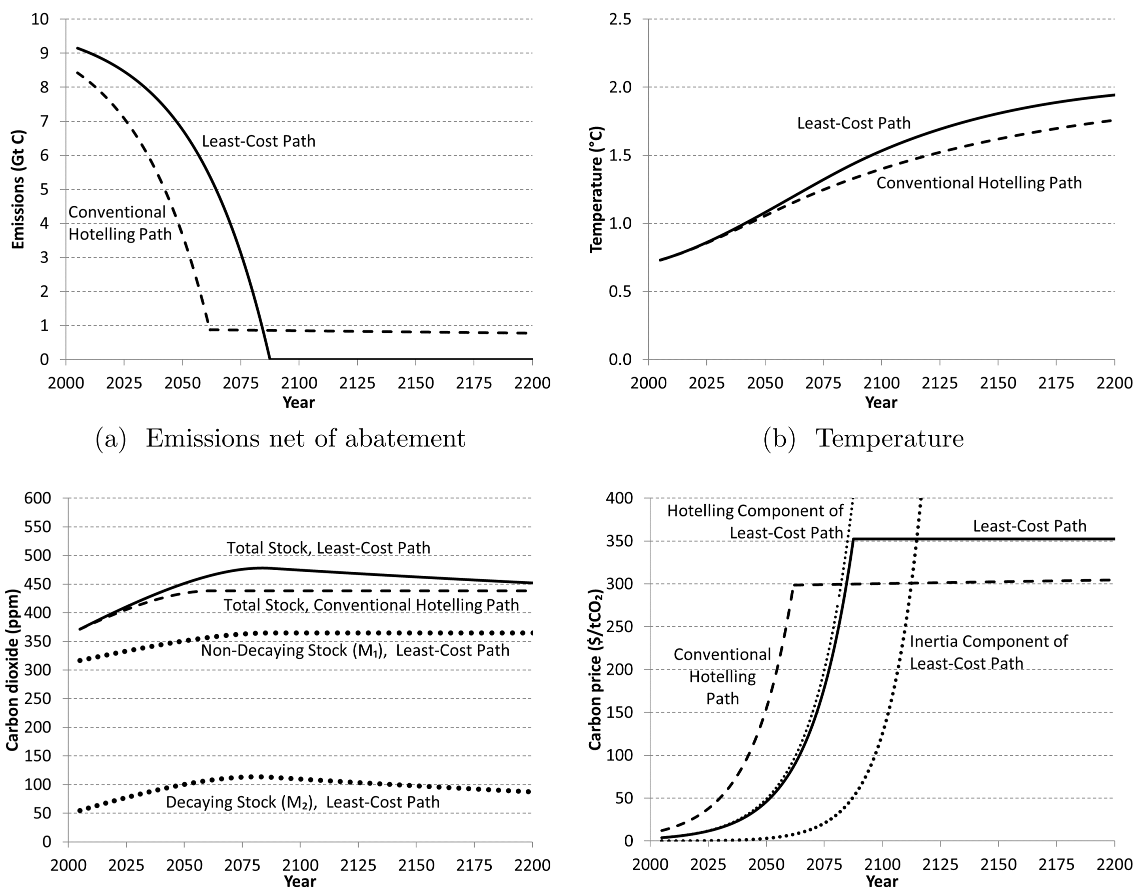
D.3 Numerical example

We now extend the numerical example from the main text to the decay model of Golosov et al. (2014). Figure D1 depicts the least-cost paths for emissions, temperature, each stock of carbon dioxide, and the emission tax implied by a 2 degree Celsius temperature constraint, along with the "Hotelling" paths generated by constraining $M(t)\leq\bar{M}$. This figure is the analogue of Figure 1 in the main text. As in the main text, we see that the Hotelling policy reduces emissions more aggressively than does the least-cost policy.[15] Temperature therefore increases more slowly under the Hotelling tax trajectory and only asymptotically approaches the constraint $\bar{T}$ (top right). As expected, the new decay model requires more substantial reductions in emissions than did the geometric decay model of the main text (top left). In particular, we now see that the nonnegativity constraint on net emissions binds throughout the twenty-second century. Around the year 2275 (past the end of the plot), the nonnegativity constraint stops binding. Abatement reaches $A(\tau)$ and temperature reaches $\bar{T}$ very shortly thereafter.

As in the main text, we see that the policymaker takes advantage of inertia to allow total CO_2 to overshoot $\bar{M}$, but now the magnitude of overshoot is reduced (bottom left). The dotted lines show that the overshoot is due entirely to the decaying stock $M_2(t)$. The non-decaying stock $M_1(t)$ cannot overshoot because it can never decline.

The bottom-right panel shows that the least-cost emission tax increases until abatement is equal to business-as-usual emissions. At this point, there are no more net emissions and abatement cannot rise further. The least-cost path increases slower than exponentially: the

[15]The kink in emissions under the Hotelling trajectory is due to reaching $\bar{M}$.



(a) Emissions net of abatement

(b) Temperature

(c) Carbon dioxide

(d) Carbon price

Figure D1: The least-cost trajectories (solid lines) for emissions, temperature, CO_2, and the carbon price for a temperature limit of $\bar{T} = 2°C$, using the carbon model of Golosov et al. (2014). Also, the conventional Hotelling-like paths (dashed lines), which are also the least-cost paths for the corresponding CO_2 constraint.

exponentially increasing Hotelling component (dotted) of the least-cost tax quickly exceeds the actual least-cost tax. Prior to reaching the maximum allowed value for abatement, the least-cost path is equal to its Hotelling component minus its inertia component. After reaching that maximum allowed value, the shadow value of the constraint accounts for the gap between the maximum allowed emission tax and the emission tax implied by the Hotelling and inertia components. We see that the inertia component becomes large around the same time that the constraint on net emissions begins to bind. This growing inertia component works to offset the exponentially increasing Hotelling component. At first, the gap between the maximum allowed emission tax and the shadow cost of emissions (which is the Hotelling component minus the inertia component) grows. This means that the shadow value of the constraint grows after it first binds. However, as the inertia component grows, that gap shrinks. The shadow value of the constraint then also begins declining, eventually falling back to zero. At that time (past the end of the plot), the constraint stops binding and temperature soon reaches $\bar{T}$.

Table D1 is the analogue of Table 1 in the main text. The new decay model restricts the policymaker much more severely than did the geometric decay model: some fraction of CO_2 now persists forever, so the policymaker must reduce emissions more aggressively to make up for the reduction in natural decay. Accordingly, all temperature limits imply a much more expensive policy than estimated in the main text. The conventional Hotelling trajectory is now around three times as expensive as the least-cost trajectory. In the main text's setting, recognizing the climate system's inertia saved a bit over $2 trillion in unnecessary costs when the temperature limit was 2 degrees Celsius. Here the savings are even greater: almost $13 trillion. The new decay setting increases the magnitude of spending and also the gains from getting policy right. As in the main text, recognizing inertia allows the policymaker to use a smaller emission tax in early years, reducing the initial emission tax to less than one-third of the Hotelling value when the temperature limit is 2 degrees Celsius. The emission tax still eventually reaches a higher level along the least-cost path than along the Hotelling path, but the percentage increase in the peak emission tax was greater in the main text's setting because there CO_2 was able to overshoot its steady-state level by a larger amount. In contrast, the presence of a permanent CO_2 stock here forces the policymaker to be less aggressive in overshooting the steady state level of CO_2.

Finally, while cumulative abatement over an infinite horizon is now fixed by the temperature limit, recognizing inertia does still allow the policymaker to reduce cumulative abatement over the next 200 years. The savings in the next 200 years' cumulative abatement are of roughly similar magnitudes as in the main text's setting, even though the required abatement is about twice as great. Therefore the savings as a percentage of cumulative abatement are much smaller here than in the main text's setting. Finally, note that because cumulative abatement over an infinite horizon is now fixed by $\bar{T}$, the monetary savings from using the least-cost path must ultimately be driven by discounting in the new decay setting.

Table D1: The present cost of each policy program, the initial carbon prices, the peak carbon prices, and cumulative abatement over the next 200 years, using the carbon model of Golosov et al. (2014).

	Temperature limit (°C)		
	2	**2.5**	**3**
Cost of efficient path from 2005–2205 ($billion)	6,750	1,046	130
Cost of Hotelling path from 2005–2205 ($billion)	19,397	3,994	686
CO_2 price along the efficient path in 2005 ($/t$CO_2$)	3.91	0.59	0.07
CO_2 price along the Hotelling path in 2005 ($/t$CO_2$)	12.24	2.35	0.39
Peak CO_2 price along the efficient path ($/t$CO_2$)	353	353	353
Peak CO_2 price along the Hotelling path ($/t$CO_2$)	305	289	271
Abatement from 2005–2205 along the efficient path (Gt C)	1,474	1,147	778
Abatement from 2005–2205 along the Hotelling path (Gt C)	1,551	1,281	986

E Nonstationary business-as-usual emissions

We now relax the assumption that business-as-usual emissions are constant. Let these emissions evolve exogenously, as $E(t)$. It is easy to see that the only necessary condition that changes in an interesting way is the condition that $h^2 = 0$, which gave us $A(\tau)$.[16] Our qualitative conclusions about the role of inertia in least-cost policy are therefore unchanged. The new condition that $h^2 = 0$ now pins down $A(t)$ for $t \geq \tau$ as

$$A(t) = E(t) - \delta(\bar{M} - M_{pre}).$$

We model the emission nonnegativity constraint as in Section D, which modifies the Maximum Principle's necessary condition to

$$C'(A(t)) = \lambda_M(t) - \mu(t) - \nu(t)\phi\, s\, F'(M(t)), \tag{E-17}$$

with $\mu(t) \geq 0$, $A(t) - E(t) \leq 0$, and $\mu(t)\,[A(t) - E(t)] = 0$.

To numerically solve this nonstationary setting, we guess τ and $\lambda_T(\tau^-)$. The guess for τ gives us $A(\tau)$ and thus $\lambda_M(\tau^-)$. We know $M(\tau) = \bar{M}$ and $T(\tau) = \bar{T}$. We solve the system

[16] Note in particular that h^0, h^1, and h^2 are unchanged except for the dependence of E on t in h^2. Even though we now have explicit time dependence in the problem, the other necessary conditions for a least-cost trajectory are unchanged because, from Hartl et al. (1995), the only partial derivatives with respect to time that would matter are those of h^0 and h^1, which are still zero.

backwards until reaching either time t_0 or a time when the nonnegativity constraint on net emissions binds. In the latter case, we then simulate the constrained system backwards until the shadow value of the constraint returns to zero (or to time t_0, whichever is later), and then simulate the unconstrained system backwards to time t_0.[17] During the interval for which the constraint binds, the shadow value of the constraint evolves according to the differential equation found by differentiating equation (E-17). Once we have reached t_0, we compare $T(t_0)$ and $M(t_0)$ to T_0 and M_0. We iterate until our guesses for τ and $\lambda_T(\tau^-)$ yield paths that satisfy the initial conditions.

We calibrate the evolution of business-as-usual emissions to total CO_2 emissions in the DICE model, with investment optimized and abatement fixed at zero. This calibration yields the following relationship for business-as-usual emissions, with emissions in Gt C and time in years:

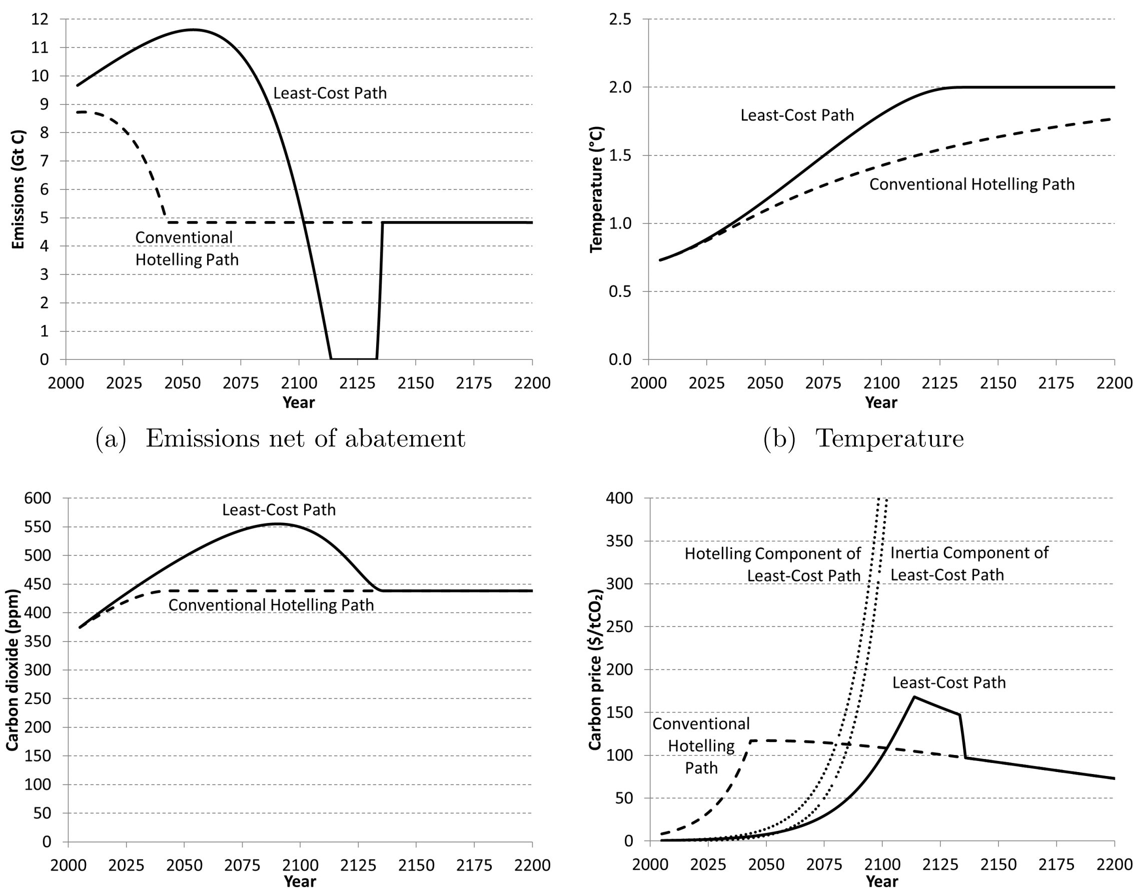
$$E(t) = 9.9662\, e^{0.0068(t-t_0)}.$$

This calibration has business-as-usual emissions increasing over time.

Figure E2 is the analogue of Figure 1 in the main text. We see that the least-cost path now has net emissions increase over the next 50 years, as business-as-usual emissions increase faster than does abatement (top left). However, abatement ramps up quickly near the end of the century, so that net emissions fall rapidly and the nonnegativity constraint begins to bind early in the next century. As in all other cases, the Hotelling policy abates emissions too aggressively over the next decades. As a result, temperature increases faster under the least-cost policy trajectory (top right). CO_2 overshoots its steady-state level $\bar{M}$ by a larger amount than in the setting with stationary emissions (bottom left). Finally, the bottom right panel shows the efficient emission tax and its components. As in Figure D1, the shadow cost of emissions along the least-cost trajectory is the difference between the Hotelling and inertia components. The efficient emission tax equals this shadow cost as long as the nonnegativity constraint on net emissions does not bind, and once that constraint binds, the shadow value of the constraint picks up the difference between the shadow cost of emissions and the maximal allowed emission tax. Once the constraint begins binding, its shadow value grows, but its shadow value eventually falls as the inertia component becomes larger relative to the Hotelling component (which makes the shadow cost of emissions fall back towards the maximal allowed emission tax). After the constraint ceases to bind, abatement quickly moves to $A(\tau)$ and temperature reaches $\bar{T}$.[18]

[17] See Section D for more on handling this constraint.

[18] Note that the efficient emission tax declines during the period in which the nonnegativity constraint binds and also in the period after τ, during which abatement holds CO_2 at $\bar{M}$. In these intervals, the change in abatement is exactly equal to the change in emissions ($\dot{A}(t) = \dot{E}(t)$). From equation (C-4), marginal abatement cost changes over these intervals in proportion to $(a_2-1)\,\dot{A}(t)/A(t) - a_2\,\dot{E}(t)/E(t)$. Thus, marginal abatement cost (but not total abatement cost) declines over these intervals if $a_2[E(t) - A(t)] - E(t) \leq 0$, which holds as long as $A(t)$ is not too much smaller than $E(t)$. Allowing Y to increase with business-as-usual emissions in equation (C-4) would introduce a force that would make marginal abatement cost more likely



(a) Emissions net of abatement (b) Temperature

(c) Carbon dioxide (d) Carbon price

Figure E2: The least-cost trajectories (solid lines) for emissions, temperature, CO_2, and the carbon price for a temperature limit of $\bar{T} = 2°C$, with business-as-usual emissions increasing over time. Also, the conventional Hotelling-like paths (dashed lines), which are also the least-cost paths for the corresponding CO_2 constraint.

Table E2: The present cost of each policy program, the initial carbon prices, the peak carbon prices, and cumulative abatement over the next 200 years, with business-as-usual emissions increasing over time.

	Temperature limit (°C)		
	2	**2.5**	**3**
Cost of efficient path from 2005–2205 ($billion)	604	122	26
Cost of Hotelling path from 2005–2205 ($billion)	5,489	1,475	379
CO_2 price along the efficient path in 2005 ($/t$CO_2$)	0.65	0.10	0.02
CO_2 price along the Hotelling path in 2005 ($/t$CO_2$)	8.38	1.57	0.28
Peak CO_2 price along the efficient path ($/t$CO_2$)	168	138	114
Peak CO_2 price along the Hotelling path ($/t$CO_2$)	117	88	69
Abatement from 2005–2205 along the efficient path (Gt C)	2,879	2,410	1,931
Abatement from 2005–2205 along the Hotelling path (Gt C)	3,167	2,783	2,384

Table E2 is the analogue of Table 1 in the main text. Unsurprisingly, having business-as-usual emissions increase exogenously raises the total cost of the policy program, though policy is still cheaper than in Section D where we used the carbon model of Golosov et al. (2014). The savings from using the least-cost policy are now greater than in the main text, so that ignoring inertia now costs almost $5 trillion under a 2 degree Celsius target (as opposed to just over $2 trillion in the main text). We again see that the least-cost policy uses a much lower initial carbon price and a much greater peak carbon price than does the Hotelling policy. Assuming that business-as-usual emissions increase exogenously leads to greater cumulative abatement under either policy. However, we see that using the least-cost policy instead of the Hotelling policy now reduces cumulative abatement by an even greater amount than in the main text's setting with stationary business-as-usual emissions.

F Alternate degrees of inertia and discounting

Figure F3 shows how the strength of inertia (left column) and the choice of discount rate (right column) affect the least-cost trajectories for achieving a 2°C temperature limit. Reducing inertia (i.e., increasing ϕ) means that the least-cost policy has to reduce emissions faster in order to avoid $\bar{T}$: temperature increases faster than in the baseline case even as

to be increasing over these two intervals.

CO_2 follows a lower trajectory (dashed lines). In contrast, increasing inertia (i.e., reducing ϕ) means that the effect of current CO_2 on temperature is delayed even further. The initial portion of the emission price trajectory is therefore lower and, in line with our analytic results, flatter. CO_2 now peaks over 100 ppm above $\bar{M}$ (dotted lines) even as temperature remains further from $\bar{T}$. However, even though increasing inertia lowers the initial carbon price, it does strongly raise the eventual peak carbon price (beyond the end of the plotted period) because the high degree of overshoot in CO_2 requires more aggressive abatement in order to return to $\bar{M}$.

The right column of Figure F3 shows the implications of reducing the annual consumption discount rate from the value of 5.5% used in DICE-2007 to the value of 1.4% used in Stern (2007). By raising the present cost of each unit of future abatement, the lower discount rate flattens the carbon price trajectory, which raises this century's carbon prices and lowers the next century's carbon prices. The initially higher carbon prices imply greater abatement early on, which lowers both the CO_2 and temperature trajectories. By increasing the present cost of future abatement, the lower discount rate reduces the economic importance of inertia. The more that CO_2 overshoots $\bar{M}$, the more abatement will eventually be needed to bring it back down to $\bar{M}$ before temperature reaches $\bar{T}$ (i.e., the higher the spike in the carbon price seen in the figures' bottom rows). As a result, the least-cost CO_2 trajectory overshoots $\bar{M}$ by only around 50 ppm under the lower discount rate, less than two-thirds of the overshoot under the higher discount rate, and the policy path is less peaked than with the higher discount rate.

G Phase portrait analysis

We now return to the setting and results of the main text. We construct conditional phase portraits for $t < \tau$ in order to better understand the evolution of abatement and CO_2 along a least-cost trajectory. Figure G4 depicts conditional phase portraits for a period with low temperature (top panel) and for a period with high temperature (bottom panel). These two snapshots correspond, respectively, to the early part of this century and to sometime late in this century or early in the next. The emission price (λ_M) is on the vertical axes, and CO_2 (M) is on the horizontal axes. Let $a(\cdot)$ denote the inverse of marginal abatement cost, so that $A(t) = a(\lambda_M(t))$. By the properties of $C(\cdot)$, we have that $a(0) = 0$ and $a'(\cdot) > 0$.

In each panel, the downward-sloping solid curve depicts, from equation (1), the M-nullcline:

$$M(t)|_{\dot{M}(t)=0} = \frac{1}{\delta}\left[E - a(\lambda_M(t))\right] + M_{pre}.$$

At these combinations of CO_2 and abatement, the CO_2 concentration is stationary. Decay increases in CO_2, so higher levels of CO_2 become stationary at lower levels of abatement. This curve is linear if abatement cost is quadratic. The downward-sloping dashed curve in

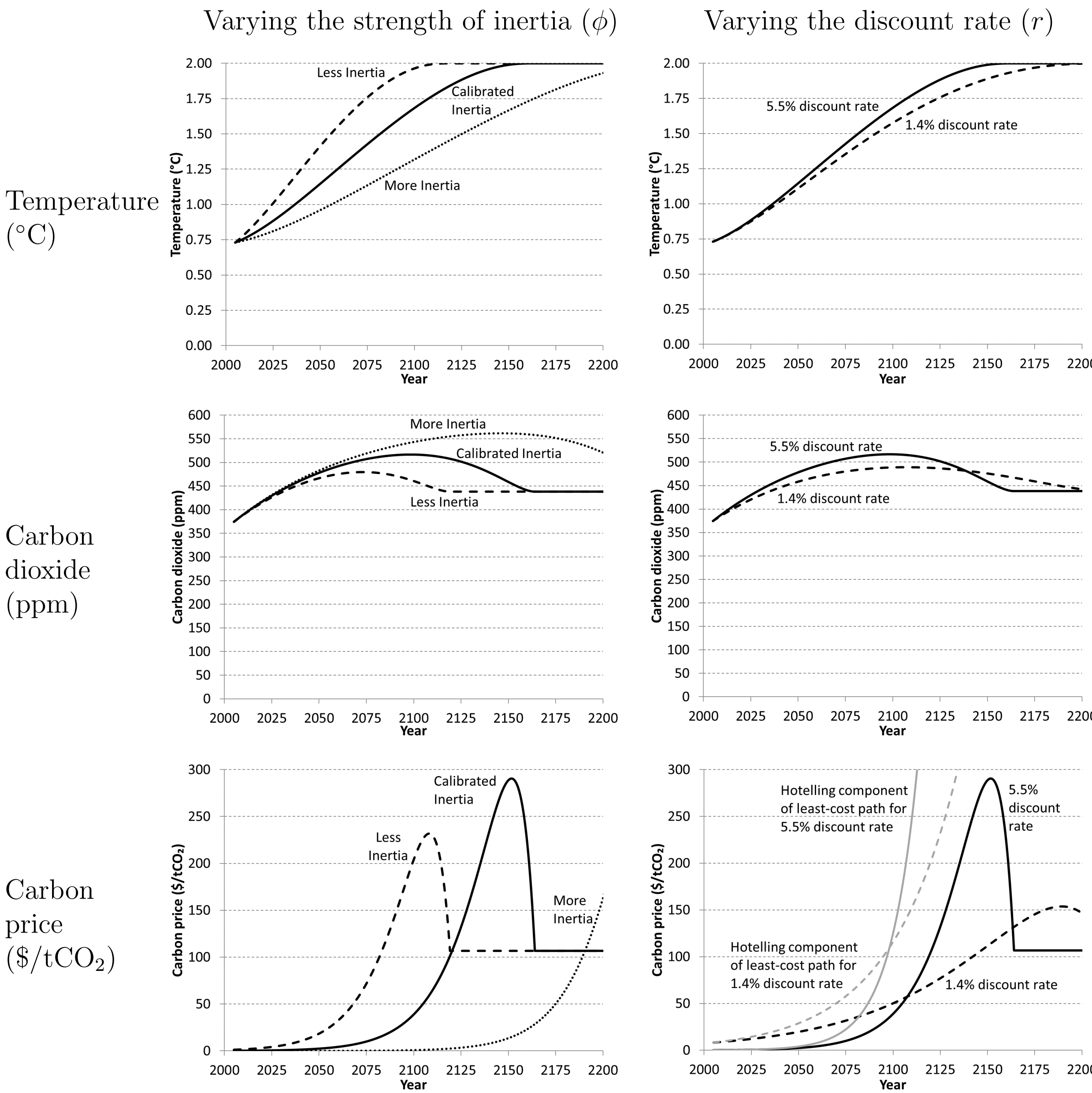
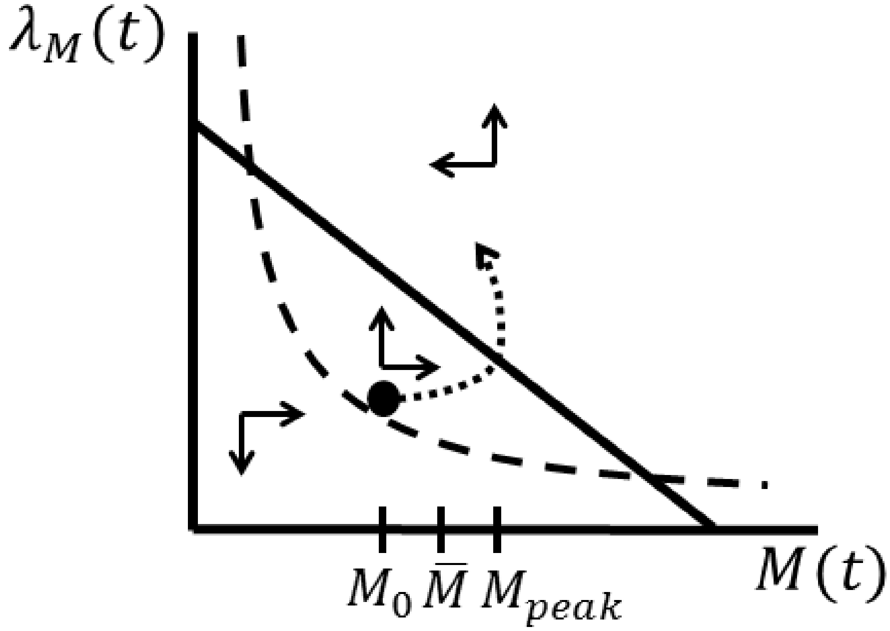
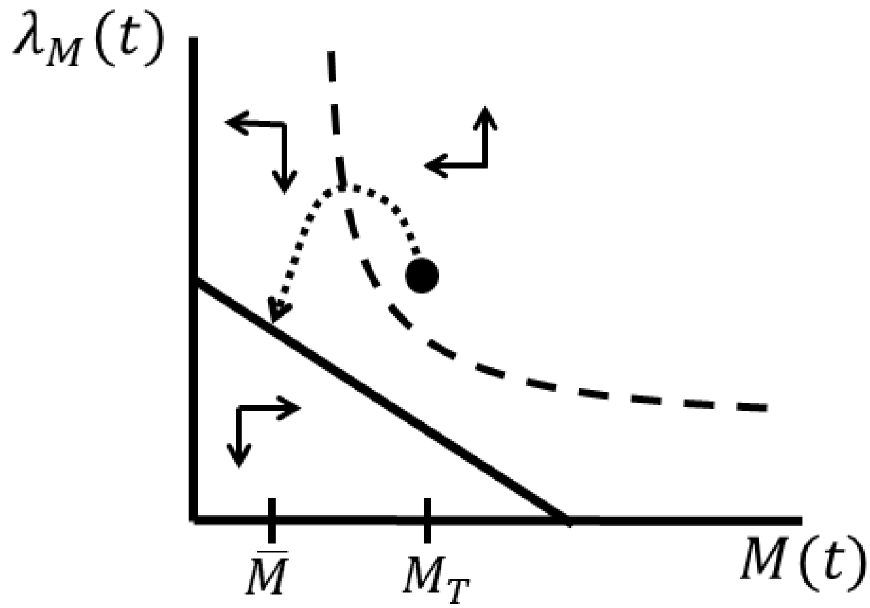


Figure F3: The least-cost trajectories for temperature, CO_2, and the carbon price for a temperature limit of $\bar{T} = 2°C$. The solid lines show the paths under the baseline calibration. In the left column, dashed lines double ϕ to 0.0182 and dotted lines halve ϕ to 0.0046 (from the baseline value of 0.0091). In the right column, dashed lines lower r to 0.014 (from the baseline value of 0.055).



(a) Near-term



(b) Long-term

Figure G4: Phase portraits conditional on λ_T and $t < \tau$. Solid curves give the M-nullclines, dashed curves give the λ_M-nullclines, dotted curves depict least-cost trajectories, and arrows give the direction of motion in each sector. The top panel corresponds to a case with $T(t)$ sufficiently far below $\bar{T}$, and the bottom panel corresponds to a case with $T(t)$ closer to $\bar{T}$.

each panel depicts, from equation (5), the λ_M-nullcline:

$$\lambda_M(t)|_{\dot{\lambda}_M(t)=0} = \frac{\phi\, s}{r+\delta} F'(M(t))\, \lambda_T(t) = e^{(r+\phi)(t-t_0)} \frac{\phi\, s}{r+\delta} F'(M(t))\, \lambda_T(t_0).$$

At these combinations of CO_2 and abatement, a least-cost trajectory holds abatement constant. The nullcline's convexity arises from using the scientific result that $F''(M(t)) < 0$, and the nullcline shifts out as the shadow cost of temperature increases. The arrows describe the direction of motion in each sector. They follow from recognizing that

$$\frac{\partial \dot{M}(t)}{\partial \lambda_M(t)} < 0 \text{ and } \frac{\partial \dot{\lambda}_M(t)}{\partial M(t)} > 0,$$

where we again use $F''(M(t)) < 0$. In sectors above (below) the M-nullcline, the direction of motion is to the left (right). In sectors to the right (left) of the λ_M-nullcline, the direction of motion is upward (downward).

The top panel depicts a case in which the nullclines intersect: business-as-usual emissions are sufficiently great that the M-nullcline is pushed out, and temperature is sufficiently far below $\bar{T}$ that its shadow cost is low and the λ_M-nullcline is pushed in. This case corresponds to the present day for a sufficiently lax temperature target. The point M_0 depicts a typical starting point, and $\bar{M} > M_0$ indicates the steady-state level of CO_2 corresponding to $\bar{T}$. The optimal emission price begins by following the dotted curve. It starts at a relatively low level in the space between the two nullclines, and it increases along with CO_2. It eventually crosses the M-nullcline at M_{peak}, at which point CO_2 begins to fall even as abatement continues increasing. This crossing illustrates how the least-cost CO_2 trajectory temporarily overshoots the terminal level $\bar{M}$.

As time passes, the shadow cost of temperature increases and the λ_M-nullcline shifts out.[19] Eventually we reach a situation such as the bottom panel, where the two nullclines no longer intersect. This corresponds to a world like that in the next century, once temperatures are closer to the chosen limit and once technological change has potentially lowered business-as-usual emissions. It also corresponds to the present world under a sufficiently stringent temperature target. In this panel, CO_2 has already peaked. The story from the last panel finished at a point such as M_T, where we pick up in this panel. As already noted, abatement is increasing and CO_2 is decreasing. The terminal condition has the policymaker hitting the M-nullcline at $\bar{M}$. As CO_2 falls, the system crosses the λ_M-nullcline. At this point, abatement peaks. As the policymaker steers the system towards $\bar{T}$, she decreases abatement towards the level compatible with steady-state $\bar{M}$.

In sum, we have seen that the type of CO_2 trajectory depends on the stringency of the temperature limit. For a sufficiently lax limit, least-cost policy increases CO_2 past its terminal level, relying on the climate system's inertia to avoid crossing $\bar{T}$. It then decreases CO_2

[19] And if business-as-usual emissions exogenously decrease, then the M-nullcline shifts in.

back towards its terminal level, using both abatement and natural decay. For a sufficiently stringent target, CO_2 begins far enough past its terminal level that abatement policy immediately begins decreasing CO_2. In either case, least-cost abatement policy generally increases before decreasing. This least-cost abatement trajectory looks quite different from the conventionally assumed, monotonically increasing Hotelling-like trajectory, and the least-cost CO_2 trajectory looks quite different from the CO_2 trajectory implied by capping concentrations at the terminal level $\bar{M}$.

Finally, consider how the least-cost CO_2 trajectory changes with properties of the climate system. In the top panel, whether CO_2 initially increases or decreases depends on how M_0 corresponds to the gap between the nullclines. For sufficiently large M_0, abatement begins at a sufficiently high level to decrease CO_2. This case is more likely the larger are ϕ, s, $F'(M(t))$, and $\lambda_T(t_0)$. For a given temperature, larger ϕ (i.e., lower inertia) increases the speed with which warming responds to any CO_2 in excess of $\bar{M}$. Larger s and $F'(M(t))$ increase the effect of CO_2 on temperature, which decreases $\bar{M}$ and so increases the degree to which M_0 is overshooting $\bar{M}$. Finally, greater $\lambda_T(t_0)$ corresponds to a more stringent temperature target, which also decreases $\bar{M}$ and increases the degree of overshoot from M_0.

H Least-cost geoengineering trajectory

The only way to achieve a CO_2 target is to reduce emissions or, perhaps, to suck CO_2 directly out of the atmosphere, but a temperature target could be achieved by directly reducing forcing. Geoengineering methods for reducing forcing typically involve "solar radiation management": if we reflect incoming solar radiation by injecting particles into the atmosphere, by placing mirrors in space, or by brightening the tops of clouds, then we can reduce forcing without reducing greenhouse gases. These methods are drawing increasing attention because they are potentially cheap but also potentially full of surprises and side-effects (Keith, 2000; Shepherd, 2012; Caldeira et al., 2013).

We here extend the theoretical setting by allowing for a geoengineering control in the form of solar radiation management. The time t level of the control is $G(t) \geq 0$, and the cost of exercising the control is a strictly increasing, convex function $D(G)$, where $D(0) = 0$. The geoengineering control reduces contemporaneous forcing, which changes the temperature transition to

$$\dot{T}(t) = \phi \left[s \left\{ F(M(t)) - G(t) \right\} - T(t) \right]. \tag{H-18}$$

The policymaker's objective is to select abatement and geoengineering trajectories in order to minimize the present cost of maintaining temperature weakly below $\bar{T}$:

$$
\begin{aligned}
V(M(t_0), T(t_0), t_0) = \min_{A(t), G(t)} \int_{t_0}^{\infty} e^{-r(t-t_0)} \; [C(A(t)) + D(G(t))] \; dt \\
\text{subject to equations (1) and (H-18),} \\
T(t) \leq & \bar{T}, \\
A(t) \geq & 0, \\
G(t) \geq & 0, \\
M(t_0) = M_0, \; & T(t_0) = T_0.
\end{aligned}
$$

The current-value Hamiltonian becomes:

$$
\begin{aligned}
H(M(t), T(t), A(t), G(t), \lambda_M(t), \lambda_T(t)) \; =& C(A(t)) + D(G(t)) \\
& + \lambda_M(t) \left[E - A(t) - \delta \; (M(t) - M_{pre}) \right] \\
& + \lambda_T(t) \, \phi \left[s \; \{F(M(t)) - G(t)\} - T(t) \right].
\end{aligned}
$$

The necessary conditions are unchanged, except that the new temperature transition equation must be obeyed and there is now an additional condition:

$$
D'(G(t)) = \lambda_T(t) \, \phi \, s - \nu(t) \, \phi^2 s.
$$

For times $t < \tau$, we have $\nu(t) = 0$. Therefore, for $t < \tau$, the marginal cost of geoengineering along a least-cost path increases with the shadow cost of temperature, which we have seen increases exponentially at rate $r + \phi$. Intuitively, the goengineering control directly affects temperature, so an efficient policy pathway equates its marginal cost to the shadow cost of temperature. And we have already seen that the shadow cost of temperature grows at rate $r + \phi$, reflecting both the time benefit and the inertial benefit of delaying a unit of warming.

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