Online Appendix

Does Household Finance Matter? Small Financial Errors with Large Social Costs

Harjoat S. Bhamra and Raman Uppal*

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*Bhamra: CEPR and Imperial College Business School, Tanaka Building, Exhibition Road, London SW7 2AZ (h.bhamra@imperial.ac.uk); Uppal: CEPR and Edhec Business School, 10 Fleet Place, Ludgate, London, United Kingdom EC4M 7RB (raman.uppal@edhec.edu).

A. Appendix

In this appendix, we provide full derivations for all the results in the main text. The title of each subsection below indicates the particular equation(s) derived in that subsection. To make it easier to read this appendix without having to go back and forth to the main text, we reproduce the key equations to be derived as propositions and also rewrite any equations from the main text that are needed; these equations are assigned the same numbers as in the main text.

A.1. The certainty equivalent in (6)

Definition A.1.1 *A certainty equivalent amount of a risky quantity is the equivalent risk-free amount in static utility terms, i.e.*

$$u_{\gamma_h}\left(\mu_{h,t}\left[U_{h,t+dt}\right]\right) = E_t[u_{\gamma_h}(U_{h,t+dt})], \tag{A1}$$

where $u_{\gamma_h}(\cdot)$ is the static utility index defined by the power utility function[2]

$$u_{\gamma_h}(x) = \begin{cases} \frac{x^{1-\gamma_h}}{1-\gamma_h}, & \gamma_h > 0,\, \gamma_h \neq 1 \\ \ln x, & \gamma_h = 1, \end{cases} \tag{A2}$$

and the conditional expectation $E_t[\cdot]$ is defined relative to a reference probability measure $\mathbb{P}$.

Proposition A.1.1 *The date-t certainty equivalent of household h's date-$t+dt$ utility is given by*

$$\mu_{h,t}[U_{h,t+dt}] = E_t[U_{h,t+dt}] - \frac{1}{2}\gamma_h\, U_{h,t}\, E_t\left[\left(\frac{dU_{h,t}}{U_{h,t}}\right)^2\right]. \tag{6}$$

Proof: The definition of the certainty equivalent in (A1) implies that

$$\mu_{h,t}[U_{h,t+dt}] = E_t\left[U_{h,t+dt}^{1-\gamma_h}\right]^{\frac{1}{1-\gamma_h}} = E_t\left[U_{h,t}^{1-\gamma_h} + d(U_{h,t}^{1-\gamma_h})\right]^{\frac{1}{1-\gamma_h}}.$$

Applying Ito's Lemma, we obtain

$$d(U_{h,t}^{1-\gamma_h}) = (1-\gamma_h)U_{h,t}^{-\gamma_h}dU_{h,t} - \frac{1}{2}(1-\gamma_h)\gamma U_{h,t}^{-\gamma_h-1}(dU_{h,t})^2$$

[2] In continuous time the more usual representation for utility is given by $J_{h,t}$, where $J_{h,t} = u_{\gamma_h}(U_{h,t})$, with the function u_{γ_h} defined in (A2).

$$= (1-\gamma_h) U_{h,t}^{1-\gamma_h} \left[\frac{dU_{h,t}}{U_{h,t}} - \frac{1}{2}\gamma_h \left(\frac{dU_{h,t}}{U_{h,t}} \right)^2 \right].$$

Therefore,

$$\begin{aligned}
\mu_{h,t}[U_{h,t+dt}] = E_t \left[U_{h,t+dt}^{1-\gamma_h} \right]^{\frac{1}{1-\gamma_h}} &= U_{h,t} \left(E_t \left[1 + (1-\gamma_h) \left[\frac{dU_{h,t}}{U_{h,t}} - \frac{1}{2}\gamma_h \left(\frac{dU_{h,t}}{U_{h,t}} \right)^2 \right] \right] \right)^{\frac{1}{1-\gamma_h}} \\
&= U_{h,t} \left(1 + (1-\gamma_h) \left[E_t \left[\frac{dU_{h,t}}{U_{h,t}} \right] - \frac{1}{2}\gamma_h E_t \left[\left(\frac{dU_{h,t}}{U_{h,t}} \right)^2 \right] \right] \right)^{\frac{1}{1-\gamma_h}}.
\end{aligned}$$

Hence, expanding the above expression, and using the notation $g = o(dt)$ to denote that $g/dt \to 0$ as $dt \to 0$, one obtains:

$$\mu_{h,t}[U_{h,t+dt}] = U_{h,t} \left(1 + E_t \left[\frac{dU_{h,t}}{U_{h,t}} \right] - \frac{1}{2}\gamma_h E_t \left[\left(\frac{dU_{h,t}}{U_{h,t}} \right)^2 \right] \right) + o(dt),$$

which, in the continuous-time limit, leads to the expression in (6). ■

A.2. The familiarity-biased certainty equivalent in (8)

While (8), giving the familiarity-biased certainty equivalent, is given as a definition within the main text of the paper, we can derive it from more primitive assumptions. To do so, we need additional definitions and lemmas.

We start by defining the measure $\mathbb{Q}^{\boldsymbol{\nu}_h}$.

Definition A.2.1 *The probability measure $\mathbb{Q}^{\boldsymbol{\nu}_h}$ is defined by*

$$\mathbb{Q}^{\boldsymbol{\nu}_h}(A) = E[1_A \xi_{h,T}],$$

where E is the expectation under $\mathbb{P}$, A is an event realized at date T, and $\xi_{h,t}$ is the exponential martingale (under the reference probability measure $\mathbb{P}$) given by

$$\frac{d\xi_{h,t}}{\xi_{h,t}} = \frac{1}{\sigma} \boldsymbol{\nu}_{h,t}^{\top} \Omega^{-1} d\boldsymbol{Z}_t,$$

where $\Omega = [\Omega_{nm}]$ is the $N \times N$ correlation matrix of returns on firms' capital stocks

$$\Omega_{nm} = \begin{cases} 1, & n = m, \\ \rho, & n \neq m, \end{cases}$$

and

$$\boldsymbol{Z}_t = (Z_{1,t}, \ldots, Z_{N,t})^\top.$$

Recall that when a household is less familiar with a particular firm, it adjusts its expected return, which is equivalent to changing the reference measure to a new measure, denoted by $\mathbb{Q}^{\boldsymbol{\nu}_h}$. Applying Girsanov's Theorem, we see that under the new measure $\mathbb{Q}^{\boldsymbol{\nu}_h}$, the evolution of firm n's capital stock is given by

$$dK_{n,t} = [(\alpha + \nu_{hn,t})K_{n,t} - D_{n,t}]dt + \sigma K_{n,t} dZ^{\boldsymbol{\nu}_h}_{n,t},$$

where $Z^{\boldsymbol{\nu}_h}_{n,t}$ is a standard Brownian motion under $\mathbb{Q}^{\boldsymbol{\nu}_h}$, such that

$$dZ^{\boldsymbol{\nu}_h}_{n,t} dZ^{\boldsymbol{\nu}_h}_{m,t} = \begin{cases} dt, & n = m. \\ \rho dt, & n \neq m. \end{cases}$$

Before motivating the definition of the penalty function, we make the following additional definition, so we can measure information losses stemming from biases with respect to a specific firm.

Definition A.2.2 *The probability measure $\mathbb{Q}^{\nu_{h,n}}$ is defined by*

$$\mathbb{Q}^{\nu_{h,n}}(A) = E[1_A \xi_{h,n,T}],$$

where E is the expectation under $\mathbb{P}$, A is an event realized at date T, and $\xi_{h,n,t}$ is the exponential martingale (under the reference probability measure $\mathbb{P}$) given by

$$\frac{d\xi_{h,n,t}}{\xi_{h,n,t}} = \frac{1}{\sigma} \nu_{h,n,t} dZ_{n,t}.$$

The probability measure $\mathbb{Q}^{\nu_{h,n}}$ is just the probability measure associated with familiarity bias with respect to firm n. Familiarity bias along this factor is equivalent to using $\mathbb{Q}^{\nu_{h,n}}$ instead of $\mathbb{P}$, which leads to a loss in information. The rate of information loss stemming from familiarity bias with respect to firm n can be quantified via the Kullback-Leibler divergence (per unit time) between $\mathbb{P}$ and $\mathbb{Q}^{\nu_{h,n}}$, given by

$$D^{KL}[\mathbb{P}|\mathbb{Q}^{\nu_{h,n}}] = \frac{1}{2} \frac{\nu^2_{h,n}}{\sigma^2}.$$

We can now think about how to measure the total rate of information loss from familiarity biases with respect to all N firms. We can form a simple weighted sum of the date-t

conditional Kullback-Leibler divergences for familiarity bias with respect to each individual firm, i.e.,

$$\hat{L}_{h,t} = \sum_{n=1}^{N} \mathcal{W}_{h,n} D^{KL}[\mathbb{P}|\mathbb{Q}^{\nu_{h,n}}],$$

in which $\mathcal{W}_{h,n}$ is a household-specific weighting matrix. We can think of the matrix $\mathcal{W}_{h,n}$ as a set of weights for information losses, analogous to the weights used in the generalized method of moments.

The choice of weighting matrix depends on how a household weights information losses, which we assume depends on the household's level of familiarity bias. For illustration, consider the simple case where $\mathcal{W}_{h,n} = \frac{f_{h,n}}{1-f_{h,n}}$, $\rho = 0$ so shocks to firm-level returns are mutually orthogonal, and the household h is completely unfamiliar with all firms save firm 1. In this case,

$$\mathcal{W}_{h,n} = \begin{cases} \frac{f_1}{1-f_1}, & n = 1 \\ 0, & n \neq 1. \end{cases}$$

Our expression for total rate of information loss from familiarity biases with respect to all N firms then reduces to

$$\hat{L}_{h,t} = \frac{f_1}{1-f_1} D^{KL}[\mathbb{P}|\mathbb{Q}^{\nu_{h,1}}].$$

So, we can see that if a household is completely unfamiliar with a particular firm, the information loss associated with deviating from the reference measure $\mathbb{P}$ is assigned a weight of zero. The more familiar a household is with a firm, the greater the weight on the information loss for that firm caused by deviating from the reference measure.

Motivated by the above discussion, we now define a penalty function for using the measure $\mathbb{Q}^{\boldsymbol{\nu}_h}$ instead of $\mathbb{P}$.

Definition A.2.3 *The penalty function for household h associated with its familiarity biases is given by*

$$\hat{L}_{h,t} = \frac{1}{2\sigma^2} \boldsymbol{\nu}_{h,t}^{\top} \Gamma_h^{-1} \boldsymbol{\nu}_{h,t}.$$

We can see that information losses linked to the firms with which the household is totally unfamiliar are not penalized in the penalty function. The household is penalized only for deviating from $\mathbb{P}$ with respect to a particular firm if it has some level of familiarity with that firm. If it has full familiarity with a firm, the associated penalty becomes infinitely large, so when making decisions involving this firm, the household will not deviate at all from the reference probability measure $\mathbb{P}$.

Theorem A.2.1 *The date-t familiarity-biased certainty equivalent of date-$t+dt$ household utility is given by*

$$\mu^{\boldsymbol{\nu}}_{h,t}[U_{h,t+dt}] = \widehat{\mu}^{\boldsymbol{\nu}}_{h,t}[U_{h,t+dt}] + U_{h,t}L_{h,t}dt, \tag{A3}$$

where $\widehat{\mu}^{\boldsymbol{\nu}}_{h,t}[U_{h,t+dt}]$ is defined by

$$u_\gamma\left(\widehat{\mu}^{\boldsymbol{\nu}}_{h,t}[U_{h,t+dt}]\right) = E^{\mathbb{Q}^{\boldsymbol{\nu}_h}}_t[u_\gamma\left(U_{h,t+dt}\right)], \tag{A4}$$

and

$$L_{h,t} = \frac{1}{\gamma}\frac{\boldsymbol{\nu}^\top_{h,t}\Gamma^{-1}_h\boldsymbol{\nu}_{h,t}}{\sigma^2} = \frac{1}{\gamma}\hat{L}_{h,t}, \tag{A5}$$

where $\boldsymbol{\nu}_{h,t} = (\nu_{h1,t},\ldots,\nu_{hN,t})^\top$ is the column vector of adjustments to expected returns, and $\Gamma_h = [\Gamma_{h,nm}]$ is the $N\times N$ diagonal matrix defined by

$$\Gamma_{h,nm} = \begin{cases} \frac{1-f_{hn}}{f_{hn}}, & n=m, \\ 0, & n\neq m, \end{cases}$$

and $f_{hn}\in[0,1]$ is a measure of how familiar the household is with firm, n, with $f_{hn}=1$ implying perfect familiarity, and $f_{hn}=0$ indicating no familiarity at all.

Proof: Using the penalty function given in Definition A.2.3, the construction of the familiarity-biased certainty equivalent of date-$t+dt$ utility is straightforward—it is merely the certainty-equivalent of date-$t+dt$ utility computed using the probability measure $\mathbb{Q}^{\boldsymbol{\nu}_h}$ plus a penalty. The household will choose its adjustment to expected returns by minimizing the familiarity-biased certainty equivalent of its date-$t+dt$ utility—the penalty stops the household from making the adjustment arbitarily large by penalizing it for larger adjustments. The size of the penalty is a measure of the information the household loses by deviating from the common reference measure, adjusted by its familiarity biases, and so

$$\mu^{\boldsymbol{\nu}}_{h,t}[U_{h,t+dt}] = \widehat{\mu}^{\boldsymbol{\nu}}_{h,t}[U_{h,t+dt}] + U_{h,t}L_{h,t}dt,$$

where $\widehat{\mu}^{\boldsymbol{\nu}}_{h,t}[U_{h,t+dt}]$ is defined by (A4) and $L_{h,t}$ is given in (A5). ∎

Equation (8) follows from Theorem A.2.1, so we restate the equation formally as the following corollary before giving a proof.

Corollary A.2.1 *The date-t familiarity-biased certainty equivalent of date-$t+dt$ household utility is given by*

$$(8) \quad \mu^{\boldsymbol{\nu}}_{h,t}[U_{h,t+dt}] = \mu_{h,t}[U_{h,t+dt}] + U_{h,t} \times \left(\frac{W_{h,t} U_{W_{h,t}}}{U_{h,t}} \boldsymbol{\nu}^{\top}_{h,t} \pi_{h,t} \boldsymbol{x}_{h,t} + \frac{1}{2\gamma_h} \frac{\boldsymbol{\nu}^{\top}_{h,t} \Gamma^{-1}_{h} \boldsymbol{\nu}_{h,t}}{\sigma^2} \right) dt,$$

where $U_{W_{h,t}} = \frac{\partial U_{h,t}}{\partial W_{h,t}}$ is the partial derivative of the utility of household h with respect to its wealth.

Proof: The date-t familiarity-biased certainty equivalent of date-$t + dt$ household utility is given by (A3), (A4), and (A5). We can see that $\widehat{\mu}^{\boldsymbol{\nu}}_{h,t}[U_{h,t+dt}]$ is like a certainty equivalent, but with the expectation taken under $\mathbb{Q}^{\boldsymbol{\nu}_h}$ in order to adjust for familiarity bias. From Lemma A.1.1, we know that

$$\widehat{\mu}^{\boldsymbol{\nu}}_{h,t}[U_{h,t+dt}] = U_{h,t} \left(1 + E^{\mathbb{Q}^{\boldsymbol{\nu}_h}}_{t} \left[\frac{dU_{h,t}}{U_{h,t}} \right] - \frac{1}{2} \gamma_h E_t \left[\left(\frac{dU_{h,t}}{U_{h,t}} \right)^2 \right] \right) + o(dt).$$

We therefore obtain from (A3)

$$(\text{A6}) \quad \mu^{\boldsymbol{\nu}}_{h,t}[U_{h,t+dt}] = U_{h,t} \left(1 + E^{\mathbb{Q}^{\boldsymbol{\nu}_h}}_{t} \left[\frac{dU_{h,t}}{U_{h,t}} \right] - \frac{1}{2} \gamma_h E_t \left[\left(\frac{dU_{h,t}}{U_{h,t}} \right)^2 \right] + L_{h,t} dt \right) + o(dt).$$

Applying Ito's Lemma, we see that under $\mathbb{Q}^{\boldsymbol{\nu}_h}$,

$$dU_{h,t} = W_{h,t} \frac{\partial U_{h,t}}{\partial W_{h,t}} \frac{dW_{h,t}}{W_{h,t}} + \frac{1}{2} W^2_{h,t} \frac{\partial^2 U_{h,t}}{\partial W^2_{h,t}} \left(\frac{dW_{h,t}}{W_{h,t}} \right)^2,$$

where

$$\frac{dW_{h,t}}{W_{h,t}} = \left(1 - \sum_{n=1}^{N} \omega_{hn,t} \right) i dt + \sum_{n=1}^{N} \omega_{hn,t} \left((\alpha + \nu_{h,t}) dt + \sigma dZ^{\mathbb{Q}^{\boldsymbol{\nu}_h}}_{n,t} \right) - c_h dt.$$

Hence, from Girsanov's Theorem, we have

$$E^{\mathbb{Q}^{\nu_h}}_{t} \left[\frac{dU_{h,t}}{U_{h,t}} \right] = E_t \left[\frac{dU_{h,t}}{U_{h,t}} \right] + \frac{W_{h,t}}{U_{h,t}} \frac{\partial U_{h,t}}{\partial W_{h,t}} \pi_{h,t} \boldsymbol{x}^{\top}_{h,t} \boldsymbol{\nu}_{h,t} dt.$$

We can therefore rewrite (A6) as

$$\mu^{\boldsymbol{\nu}}_{h,t}[U_{h,t+dt}] = U_{h,t} \left(1 + E_t \left[\frac{dU_{h,t}}{U_{h,t}} \right] - \frac{1}{2} \gamma_h E_t \left[\left(\frac{dU_{h,t}}{U_{h,t}} \right)^2 \right] + L_{h,t} dt + \frac{W_{h,t}}{U_{h,t}} \frac{\partial U_{h,t}}{\partial W_{h,t}} \pi_{h,t} \boldsymbol{x}^{\top}_{h,t} \boldsymbol{\nu}_{h,t} dt \right) + o(dt).$$

Using (6) we obtain

$$\mu^{\boldsymbol{\nu}}_{h,t}[U_{h,t+dt}] = \mu_{h,t}[U_{h,t+dt}] + U_{h,t}\left(\frac{W_{h,t}}{U_{h,t}}\frac{\partial U_{h,t}}{\partial W_{h,t}}\pi_{h,t}\boldsymbol{x}^\top_{h,t}\boldsymbol{\nu}_{h,t} + \frac{1}{2\gamma}\frac{\boldsymbol{\nu}^\top_{h,t}\Gamma^{-1}_h\boldsymbol{\nu}_{h,t}}{\sigma^2}\right)dt + o(dt),$$

and hence (8). ■

A.3. The Bellman Equation and Mean-Variance Choice in (11) and (12)

We state the Hamilton-Jacobi-Bellman equation as the following proposition.

Proposition A.3.1 *The utility function of a household with familiarity biases is given by the following Hamilton-Jacobi-Bellman equation:*

$$\text{(A7)} \qquad 0 = \sup_{C_{h,t}}\left(\delta_h\, u_{\psi_h}\left(\frac{C_{h,t}}{U_{h,t}}\right) + \sup_{\pi_{h,t},\,\boldsymbol{x}_{h,t}}\inf_{\nu_{h,t}}\frac{1}{U_{h,t}}\mu^{\boldsymbol{\nu}}_{h,t}\left[\frac{dU_{h,t}}{dt}\right]\right),$$

where the function

$$u_{\psi_h}(x) = \frac{x^{1-\frac{1}{\psi_h}} - 1}{1-\frac{1}{\psi_h}}, \psi_h > 0,$$

and

$$\mu^{\boldsymbol{\nu}}_{h,t}\left[dU_{h,t}\right] = \mu^{\boldsymbol{\nu}}_{h,t}\left[U_{h,t+dt} - U_{h,t}\right] = \mu^{\boldsymbol{\nu}}_{h,t}\left[U_{h,t+dt}\right] - U_{h,t},$$

with $\mu^{\boldsymbol{\nu}}_{h,t}\left[U_{h,t+dt}\right]$ *given in* (8).

Proof: Writing out (10) explicitly gives

$$U_{h,t}^{1-\frac{1}{\psi_h}} = (1-e^{-\delta_h dt})C_{h,t}^{1-\frac{1}{\psi_h}} + e^{-\delta_h dt}\left(\mu^{\boldsymbol{\nu}}_{h,t}[U_{h,t+dt}]\right)^{1-\frac{1}{\psi_h}},$$

where for ease of notation sup and inf have been suppressed. Now,

$$\begin{aligned}\left(\mu^{\boldsymbol{\nu}}_{h,t}[U_{h,t+dt}]\right)^{1-\frac{1}{\psi_h}} &= \left(U_{h,t} + \mu^{\boldsymbol{\nu}}_{h,t}[dU_{h,t}]\right)^{1-\frac{1}{\psi_h}} \\ &= U_{h,t}^{1-\frac{1}{\psi_h}}\left(1 + \mu^{\boldsymbol{\nu}}_{h,t}\left[\frac{dU_{h,t}}{U_{h,t}}\right]\right)^{1-\frac{1}{\psi_h}} \\ &= U_{h,t}^{1-\frac{1}{\psi_h}}\left(1 + \left(1-\frac{1}{\psi_h}\right)\mu^{\boldsymbol{\nu}}_{h,t}\left[\frac{dU_{h,t}}{U_{h,t}}\right]\right) + o(dt).\end{aligned}$$

Hence,

$$U_{h,t}^{1-\frac{1}{\psi_h}} = \delta_h C_{h,t}^{1-\frac{1}{\psi_h}} dt + U_{h,t}^{1-\frac{1}{\psi_h}} \left(1 + \left(1 - \frac{1}{\psi_h}\right) \mu^{\boldsymbol{\nu}}_{h,t} \left[\frac{dU_{h,t}}{U_{h,t}}\right]\right) - \delta_h U_{h,t}^{1-\frac{1}{\psi_h}} dt + o(dt),$$

from which we obtain (A7). ■

Equations (11) and (12) are obtained from the following proposition by setting $\rho = 0$.

Proposition A.3.2 *The household's optimization problem consists of two parts, a mean-variance optimization*

$$\sup_{\pi_{h,t},\, \boldsymbol{\omega}_{h,t}} \inf_{\boldsymbol{\nu}_{h,t}} MV_h(\pi_{h,t}, \boldsymbol{\omega}_{h,t}, \boldsymbol{\nu}_{h,t}),$$

and an intertemporal consumption choice problem

$$0 = \sup_{C_{h,t}} \left(\delta_h\, u_{\psi_h}\left(\frac{C_{h,t}}{U_{h,t}}\right) - \frac{C_{h,t}}{W_{h,t}} + \sup_{\pi_{h,t},\, \boldsymbol{\omega}_{h,t}} \inf_{\boldsymbol{\nu}_{h,t}} MV_h(\pi_{h,t}, \boldsymbol{\omega}_{h,t}, \boldsymbol{\nu}_{h,t}) \right), \tag{11}$$

where

(12)

$$MV(\pi_{h,t}, \boldsymbol{\omega}_{h,t}, \boldsymbol{\nu}_{h,t}) = i + (\alpha - i)\pi_{h,t} - \frac{1}{2}\gamma_h \sigma^2 \pi_{h,t}^2 \boldsymbol{x}_{h,t}^\top \Omega \boldsymbol{x}_{h,t} + \boldsymbol{\nu}_{h,t}^\top \pi_{h,t} \boldsymbol{x}_{h,t} + \frac{1}{2\gamma_h} \frac{\boldsymbol{\nu}_{h,t}^\top \Gamma_h^{-1} \boldsymbol{\nu}_{h,t}}{\sigma^2}.$$

Proof: Assuming a constant risk-free rate, homotheticity of preferences combined with constant returns to scale for production implies that we have $U_{h,t} = \kappa_h W_{h,t}$, for some constant κ_h. Equations (11) and (12) are then direct consequences of (8) and (A7). ■

A.4. Adjustment to expected returns and portfolio choice in (13)–(16)

Proposition A.4.1 *For a given portfolio, $\boldsymbol{\omega}_{h,t} = \pi_{h,t}\boldsymbol{x}_{h,t}$, adjustments to firm n's expected return are given by*

$$\nu_{hn,t} = -\frac{W_{h,t} U_{W_{h,t}}}{U_{h,t}} \Big(\frac{1}{f_{hn}} - 1\Big) \sigma^2\, \gamma_h\, \pi_{h,t} x_{hn,t},\ n \in \{1, \ldots, N\}. \tag{A8}$$

Proof: From (8), we can see that

$$\inf_{\boldsymbol{\nu}_{h,t}} \mu^{\boldsymbol{\nu}}_{h,t}[U_{h,t+dt}]$$

is equivalent to

$$\inf_{\boldsymbol{\nu}_{h,t}} \frac{W_{h,t} U_{W_{h,t}}}{U_{h,t}} \boldsymbol{\nu}_{h,t}^\top \pi_{h,t} \boldsymbol{x}_{h,t} + \frac{1}{2\gamma_h \sigma^2} \boldsymbol{\nu}_{h,t}^\top \Gamma_h^{-1} \boldsymbol{\nu}_{h,t}.$$

The minimum exists and is given by the first-order condition,

$$\frac{\partial}{\partial \boldsymbol{\nu}_{h,t}} \left[\frac{W_{h,t} U_{W_{h,t}}}{U_{h,t}} \boldsymbol{\nu}_{h,t}^{\top} \pi_{h,t} \boldsymbol{x}_{h,t} + \frac{1}{2\gamma_h \sigma^2} \boldsymbol{\nu}_{h,t}^{\top} \Gamma_h^{-1} \boldsymbol{\nu}_{h,t} \right] = 0.$$

Carrying out the differentiation and exploiting the fact that Γ_h^{-1} is symmetric, we obtain

$$0 = \frac{W_{h,t} U_{W_{h,t}}}{U_{h,t}} \pi_{h,t} \boldsymbol{x}_{h,t} + \frac{1}{\gamma_h \sigma^2} \Gamma_h^{-1} \boldsymbol{\nu}_{h,t}.$$

Hence,

$$\boldsymbol{\nu}_{h,t} = -\gamma_h \sigma^2 \frac{W_{h,t} U_{W_{h,t}}}{U_{h,t}} \Gamma_h \, \pi_{h,t} \boldsymbol{x}_{h,t}.$$

Therefore, we obtain (A8). ■

Proposition A.4.2 *For a given portfolio decision, the optimal adjustment to firm-level expected returns is given by*

(A9)
$$\boldsymbol{\nu}_{h,t} = -\gamma_h \sigma^2 \, \Gamma_h \, \pi_{h,t} \boldsymbol{x}_{h,t}.$$

Each household then faces the following mean-variance portfolio problem:

(A10)
$$\sup_{\pi_{h,t}, \boldsymbol{x}_{h,t}} \inf_{\boldsymbol{\nu}_{h,t}} MV(\pi_{h,t}, \boldsymbol{x}_{h,t}, \boldsymbol{\nu}_{h,t}) = \Big(i + \big(\alpha + \frac{1}{2} \boldsymbol{\nu}_{h,t}^{\top} \boldsymbol{x}_{h,t} - i\big) \pi_{h,t} \Big) - \frac{1}{2} \gamma_h \sigma^2 \pi_{h,t}^2 \boldsymbol{x}_{h,t}^{\top} \Omega \boldsymbol{x}_{h,t}.$$

Proof: Because household utility is a constant multiple of wealth, the expression for the optimal adjustment to expected returns in (A8) simplifies to (A9). Substituting (A9) into (12), we see that each household faces the mean-variance portfolio problem in (A10).

For the special case in which a household is fully familiar with all firms, Γ_h is the zero matrix, and from (A9) we can see the adjustment to expected returns is zero and the portfolio weights are exactly the standard mean-variance portfolio weights. For the special case in which the household is completely unfamiliar with all firms, each $\Gamma_{h,nn}$ becomes infinitely large and $\pi_h = 0$: complete unfamiliarity leads the household to avoid any investment in risky firms, in which case we get non-participation in the stock market in this partial-equilibrium setting. ■

Proposition A.4.3 *The optimal adjustment to expected returns made by a household with familiarity biases is*

(A11)
$$\boldsymbol{\nu}_h = -(\alpha - i) \, [I + \Omega \Gamma_h^{-1}]^{-1} \mathbf{1},$$

where $\mathbf{1}$ *is the* N *by 1 vector of ones. The optimal vector of optimal portfolio weights is* $\boldsymbol{\omega}_h = \pi_h \boldsymbol{x}_h$*, where*

$$\pi_h = \frac{\mu_{qh}}{\gamma_h} \frac{\alpha - i}{\sigma^2_{1/N}}, \tag{A12}$$

$$\boldsymbol{x}_h = \frac{1}{\mu_{qh}} \frac{1}{N} \boldsymbol{q}_h, \tag{A13}$$

$\sigma^2_{1/N}$ *is the variance of the fully diversified portfolio i.e.*

$$\sigma^2_{1/N} = \sigma^2 (\boldsymbol{x}_h^U)^\top \Omega \, \boldsymbol{x}_h^U = \frac{\sigma^2}{N}[1 + (N-1)\rho],$$

and $\boldsymbol{q}_h$ *is the following* N *by 1 vector,*

$$\boldsymbol{q}_h = (1 + (N-1)\rho)(\Omega + \Gamma_h)^{-1}\mathbf{1}, \tag{A14}$$

the entries of which have the following arithmetic mean

$$\mu_{qh} = \frac{1}{N} \mathbf{1}^\top \boldsymbol{q}_h.$$

For the special case of $\rho = 0$ *used in the main text, we obtain equations* (13)*,* (14)*, and* (16) *in the main text:*

$$\boldsymbol{\nu}_h = -(\alpha - i)\,(\mathbf{1} - \boldsymbol{f}_h), \tag{13}$$

$$\boldsymbol{x}_h = \frac{\boldsymbol{f}_h}{\mu_{fh}} \frac{1}{N} \mathbf{1}, \tag{14}$$

$$\pi_h = \frac{\mu_{fh}}{\gamma_h} \frac{\alpha - i}{\sigma^2_{1/N}}, \tag{16}$$

where $\mu_{fh} = \frac{1}{N} \mathbf{1}^\top \boldsymbol{f}_h$*.*

Proof: Minimizing (12) with respect to $\nu_{h,t}$ gives (A9). Substituting (A9) into (12) and simplifying gives

$$MV_h = i + (\alpha - i)\pi_h - \frac{1}{2}\gamma_h \sigma^2 \pi_h^2 \boldsymbol{x}_h^\top (\Omega + \Gamma_h)\boldsymbol{x}_h. \tag{A15}$$

We find $\boldsymbol{x}_h$ by minimizing $\sigma^2 \boldsymbol{x}_h^\top (\Omega + \Gamma_h)\boldsymbol{x}_h$, so we can see that $\boldsymbol{x}_h$ is household h's minimum-variance portfolio adjusted for familiarity bias. The minimization we wish to perform is

$$\min_{\boldsymbol{x}_h} \frac{1}{2} \boldsymbol{x}_h^\top (\Omega + \Gamma_h)\, \boldsymbol{x}_h,$$

subject to the constraint

$$\mathbf{1}^\top \boldsymbol{x}_h = 1.$$

The Lagrangian for this problem is

$$\mathcal{L}_h = \frac{1}{2}\boldsymbol{x}_h^\top (\Omega + \Gamma_h)\, \boldsymbol{x}_h + \lambda_h (1 - \mathbf{1}^\top \boldsymbol{x}_h),$$

where λ_h is the Lagrange multiplier. The first-order condition with respect to $\boldsymbol{x}_h$ is

$$(\Omega + \Gamma_h)\, \boldsymbol{x}_h = \lambda_h \mathbf{1}.$$

Hence

$$\boldsymbol{x}_h = \lambda_h (\Omega + \Gamma_h)^{-1}\mathbf{1}.$$

The first order condition with respect to λ_h gives us the constraint

$$\mathbf{1}^\top \boldsymbol{x}_h = 1,$$

which implies that

$$\lambda_h = \left[\mathbf{1}^\top (\Omega + \Gamma_h)^{-1}\mathbf{1}\right]^{-1}.$$

Therefore, we have

$$\boldsymbol{x}_h = \frac{(\Omega + \Gamma_h)^{-1}\mathbf{1}}{\mathbf{1}^\top (\Omega + \Gamma_h)^{-1}\mathbf{1}} = \frac{\boldsymbol{q}_h}{\mathbf{1}^\top \boldsymbol{q}_h},$$

where $\boldsymbol{q}_h$ is defined in (A14). Hence

$$\lambda_h = \frac{1 + (N-1)\rho}{\mathbf{1}^\top \boldsymbol{q}_h}.$$

Substituting the optimal choice of $\boldsymbol{x}_h$ back into $\boldsymbol{x}_h^\top (\Omega + \Gamma_h)\boldsymbol{x}_h$ implies that

$$\boldsymbol{x}_h^\top (\Omega + \Gamma_h)\boldsymbol{x}_h = \lambda_h.$$

Therefore, to find the optimal π_h, we need to minimize

$$MV_h = i + (\alpha - i)\pi_h - \frac{1}{2}\gamma_h \sigma^2 \pi_h^2 \lambda_h.$$

Hence,

$$\pi_h = \frac{1}{\lambda_h}\frac{1}{\gamma_h}\frac{\alpha - i}{\sigma^2} = \frac{\mathbf{1}^\top \boldsymbol{q}_h}{\gamma_h}\frac{\alpha - i}{\sigma^2[1+(N-1)\rho]} = \frac{\frac{\mathbf{1}^\top \boldsymbol{q}_h}{N}}{\gamma_h}\frac{\alpha - i}{\frac{\sigma^2}{N}[1+(N-1)\rho]},$$

which gives us the result in (A12). Substituting (A12) and (A13) into (A9) and simplifying gives (A11). Settinng $\rho = 0$ in these expressions gives us the results in the main text.

We can express $\boldsymbol{\omega}_h = \pi_h \boldsymbol{x}_h$ in terms of the familiarity-biased adjustment made to expected returns:

$$\boldsymbol{\omega}_h = \frac{1}{\gamma_h}\Omega^{-1}\frac{\alpha\mathbf{1} + \boldsymbol{\nu}_h - i\mathbf{1}}{\sigma^2}.$$

Substituting the expressions for the portfolio choices and the Lagrange multiplier λ_h into the mean-variance objective function with familiarity biases gives:

$$MV_h = i + \frac{1}{2}\frac{1}{\gamma_h}\left(\frac{\alpha - i}{\sigma_{1/N}}\right)^2 \frac{\mathbf{1}^\top \boldsymbol{q}_h}{N}.$$

Hence

$$MV_h = i + \frac{1}{2}\frac{1}{\gamma_h}\left(\frac{\alpha - i}{\sigma_{1/N}}\right)^2 \mu_{qh}. \tag{A16}$$

■

A.5. Mean-Variance Welfare in (19)

The following proposition summarizes results on how familiarity biases impact a household's mean-variance welfare.

Proposition A.5.1 *Mean-variance welfare evaluated using the portfolio policy which is optimal in the presence of familiarity biases is given by*

$$i + \frac{1}{2\gamma_h}\left(\frac{\alpha - i}{\sigma_{1/N}}\right)^2 \left(1 - (\mu_{qh} - 1)^2 - \sigma_{qh}^2\right). \tag{A17}$$

The increase in mean-variance welfare from removing familiarity biases is given by

$$\frac{1}{2\gamma_h}\left(\frac{\alpha - i}{\sigma_{1/N}}\right)^2 \left((\mu_{qh} - 1)^2 + \sigma_{qh}^2\right), \tag{A18}$$

where $\frac{1}{2\gamma_h}\left(\frac{\alpha-i}{\sigma_{1/N}}\right)^2\sigma_{qh}^2$ *is the increase in mean-variance welfare obtained by first removing familiarity biases in the choice of composition of the subportfolio of risky assets, and* $\frac{1}{2\gamma_h}\left(\frac{\alpha-i}{\sigma_{1/N}}\right)^2(\mu_{qh}-1)^2$ *is the subsequent increase in mean-variance welfare obtained by removing also familiarity biases in the capital allocation decision, i.e. the choice of which proportion of wealth to invest in risky assets.*

Proof: We start by giving both the mean-variance objective function in the presence of familiarity biases and the mean-variance welfare function in terms of general, not necessarily optimal, portfolio choices.

Mean-variance welfare is given as a function of the proportion of wealth invested in risky assets, π_h, and the subportfolio of risky assets $\boldsymbol{x}_h$ by (A15). Substituting in the household's decisions, given in (A12) and (A13) into the above expression and simplifying gives

$$U_h^{MV}(\pi_h,\boldsymbol{x}_h) = i + \frac{1}{2\gamma}\left(\frac{\alpha-i}{\sigma_{1/N}}\right)^2\left[2\mu_{qh} - \frac{\frac{1}{N}\boldsymbol{q}_h^\top\Omega\boldsymbol{q}_h}{\frac{1}{N}\mathbf{1}^\top\Omega\mathbf{1}}\right],$$

where

$$\frac{1}{N}\mathbf{1}^\top\Omega\mathbf{1} = 1 + (N-1)\rho.$$

Defining

$$\sigma_{qh}^2 = \frac{\frac{1}{N}\boldsymbol{q}_h^\top\Omega\boldsymbol{q}_h}{\frac{1}{N}\mathbf{1}^\top\Omega\mathbf{1}} - \mu_{qh}^2,$$

we obtain

$$U_h^{MV}(\pi_h,\boldsymbol{x}_h) = i + \frac{1}{2\gamma}\left(\frac{\alpha-i}{\sigma_{1/N}}\right)^2\left[1-(1-\mu_{qh})^2-\sigma_{qh}^2\right].$$

Setting $\rho = 0$ gives expression (19) in the main text.

Without familiarity biases, mean-variance welfare is given by

$$U_h^{MV}(\pi_h^U,\boldsymbol{x}_h^U) = i + \frac{1}{2\gamma_h}\left(\frac{\alpha-i}{\sigma_{1/N}}\right)^2.$$

Hence, the increase in mean-variance welfare obtained from removing familiarity biases is given by (A18).

We now study how mean-variance welfare changes when the biases in the subportfolio of risky assets are eliminated, followed by eliminating the biases in the proportion of wealth invested in risky assets. Denote the biased portfolio choices by $\pi_h, \boldsymbol{x}_h$ and the unbiased portfolio choices by $\pi_h^U = \pi_h + \Delta\pi_h$, $\boldsymbol{x}_h^U = \boldsymbol{x}_h + \Delta\boldsymbol{x}_h$, i.e.

$$\pi_h = \frac{1}{\gamma_h}\frac{\alpha - i}{\sigma_{1/N}^2}\mu_{qh}, \tag{A19}$$

$$\boldsymbol{x}_h = \frac{1}{\mu_{qh}}\frac{1}{N}\boldsymbol{q}_h, \tag{A20}$$

$$\pi_h^U = \pi_h + \Delta\pi_h = \frac{1}{\gamma_h}\frac{\alpha - i}{\sigma_{1/N}^2}, \tag{A21}$$

$$\boldsymbol{x}_h^U = \boldsymbol{x}_h + \Delta\boldsymbol{x}_h = \frac{1}{N}\mathbf{1}. \tag{A22}$$

Observe that

$$\begin{aligned}
&MV^e(\pi_h + \Delta\pi_h, \boldsymbol{x}_h + \Delta\boldsymbol{x}_h) - MV^e(\pi_h, \boldsymbol{x}_h) \\
&= -\frac{1}{2}\gamma_h\sigma^2\pi_h^2\left[(\boldsymbol{x}_h + \Delta\boldsymbol{x}_h)^\top\Omega(\boldsymbol{x}_h + \Delta\boldsymbol{x}_h) - \boldsymbol{x}_h^\top\Omega\boldsymbol{x}_h\right] \\
&\quad + (\alpha - i)\Delta\pi_h - \frac{1}{2}\gamma_h\sigma^2\left[[(\pi_h + \Delta\pi_h)^2 - \pi^2](\boldsymbol{x}_h + \Delta\boldsymbol{x}_h)^\top\Omega(\boldsymbol{x}_h + \Delta\boldsymbol{x}_h)\right],
\end{aligned}$$

where $\frac{1}{2}\gamma_h\sigma^2\pi_h^2\left[(\boldsymbol{x}_h + \Delta\boldsymbol{x}_h)^\top\Omega(\boldsymbol{x}_h + \Delta\boldsymbol{x}_h) - \boldsymbol{x}_h^\top\Omega\boldsymbol{x}_h\right]$ is the change in mean-variance welfare when the biases in the subportfolio of risky assets are eliminated, and $(\alpha - i)\Delta\pi_h - \frac{1}{2}\gamma_h\sigma^2\left[[(\pi_h + \Delta\pi_h)^2 - \pi_h^2](\boldsymbol{x}_h + \Delta\boldsymbol{x}_h)^\top\Omega(\boldsymbol{x}_h + \Delta\boldsymbol{x}_h)\right]$ is the change in mean-variance welfare by then eliminating the biases in the proportion of wealth invested in risky assets. Using the expressions in (A19), (A20), (A21), and (A22), it follows that

$$-\frac{1}{2}\gamma_h\sigma^2\left[\pi_h^2[(\boldsymbol{x}_h + \Delta\boldsymbol{x}_h)^\top\Omega(\boldsymbol{x}_h + \Delta\boldsymbol{x}_h) - \boldsymbol{x}_h^\top\boldsymbol{x}_h]\right] = \frac{1}{2\gamma_h}\left(\frac{\alpha - i}{\sigma_{1/N}}\right)^2\sigma_{qh}^2,$$

and

$$(\alpha - i)\Delta\pi_h - \frac{1}{2}\gamma_h\sigma^2\left[[(\pi_h + \Delta\pi_h)^2 - \pi_h^2](\boldsymbol{x}_h + \Delta\boldsymbol{x}_h)^\top\Omega(\boldsymbol{x}_h + \Delta\boldsymbol{x}_h)\right] = \frac{1}{2\gamma_h}\left(\frac{\alpha - i}{\sigma_{1/N}}\right)^2(\mu_{qh} - 1)^2.$$

■

A.6. Optimal consumption in (20)

The following proposition summarizes results on optimal consumption choice.

Proposition A.6.1 *A household's optimal consumption-to-wealth ratio is given by*

$$\frac{C_{h,t}}{W_{h,t}} = c_h = \psi_h \delta_h + (1-\psi_h)\left(i + \frac{1}{2}\frac{1}{\gamma_h}\left(\frac{\alpha - i}{\sigma_{1/N}}\right)^2 \mu_{qh}\right). \tag{20}$$

Proof: Mean-variance utility subject to familiarity biases and with the household's decisions is given by (A16). Hence, we can rewrite (11) as

$$0 = \sup_{C_{h,t}} \left(\delta_h u_{\psi_h}\left(\frac{C_{h,t}}{U_{h,t}}\right) - c_h + MV_h(\pi_h, \boldsymbol{x}_h, \boldsymbol{\nu}_h)\right). \tag{A23}$$

The first-order condition with respect to consumption is

$$\delta_h \left(\frac{C_{h,t}}{U_{h,t}}\right)^{-\frac{1}{\psi_h}} \frac{1}{U_{h,t}} - \frac{1}{W_{h,t}} = 0.$$

Hence, we obtain

$$c_h = \delta_h^{\psi_h}\left(\frac{U_{h,t}}{W_{h,t}}\right)^{1-\psi_h},$$

which implies that

$$\sup_{C_{h,t}} \left(\delta_h\, u_{\psi_h}\left(\frac{C_{h,t}}{U_{h,t}}\right) - c_h + MV_h(\pi_h, \boldsymbol{x}_h, \boldsymbol{\nu}_h)\right) = \frac{c_h - \psi_h\delta_h}{\psi_h - 1} + MV_h,$$

where $C_{h,t}/W_{h,t}$ is the consumption-wealth ratio chosen by household h and MV_h is her resulting mean-variance utility subject to familiarity biases. It follows from (A23) that

$$c_h = \psi_h\delta_h + (1-\psi_h)MV_h = \psi_h\delta_h + (1-\psi_h)\left(i + \frac{1}{2}\frac{1}{\gamma_h}\left(\frac{\alpha - i}{\sigma_{1/N}}\right)^2 \mu_{qh}\right)$$

from which one can get the expression in the text by setting $\rho = 0$. ■

A.7. Welfare in (21)

Proposition A.7.1 *Welfare is given by a function of the proportion of wealth invested in risky assets, π_h, the subportfolio of risky assets $\boldsymbol{x}_h$, and the consumption-wealth ratio, $c_h = C_{h,t}/W_{h,t}$ by*

$$U_{h,t} = \kappa_h\left(c_h, \pi_h, \boldsymbol{x}_h\right) W_{h,t},$$

where

$$(21) \qquad \kappa_h\left(c_h, \pi_h, \boldsymbol{x}_h\right) = \kappa_h = \left[\frac{\delta_h \psi_h}{\psi_h \delta_h + (1-\psi_h)\left(U^{MV}(\pi_h, \boldsymbol{x}_h) - c_h\right)}\right]^{\frac{1}{1-\frac{1}{\psi_h}}} c_h,$$

in which $c_h = C_{h,t}/W_{h,t}$. *The impact of a one percent change in* $U_h^{MV}(\pi_h, \boldsymbol{x}_h)$ *on the percentage change in welfare is given by the following elasticity*

$$\frac{\partial \ln \kappa_h}{\partial \ln U_h^{MV}(\pi_h, \boldsymbol{x}_h)} = U_h^{MV}(\pi_h, \boldsymbol{x}_h) \frac{1}{\delta_h - \left(1 - \frac{1}{\psi_h}\right)\left(U_h^{MV}(\pi, \boldsymbol{x}) - c_h\right)} > 0.$$

The size of the above elasticity beyond one captures the size of the additional intertemporal effect of a change in mean-variance utility on lifetime welfare.

The impact of a one percent change in the consumption-wealth ratio on the percentage change in welfare is given by the following elasticity

$$\frac{\partial \ln \kappa_h}{\partial \ln(c_h)} = \frac{\psi_h \delta_h + (1-\psi_h) U_h^{MV}(\pi_h, \boldsymbol{x}_h) - c_h}{\psi_h \delta_h + (1-\psi_h)(U_h^{MV}(\pi_h, \boldsymbol{x}_h) - c_h)}.$$

When removing familiarity biases the resulting percentage change in welfare is always positive, i.e. $\frac{\partial \ln \kappa_h}{\partial \ln(c_h)} \frac{\Delta(c_h)}{(c_h)} > 0$, *where* $\frac{\Delta(c_h)}{(c_h)}$ *is the percentage change in the consumption-wealth ratio.*

Proof: We start from the recursive equation for welfare

$$U_{h,t} = \mathcal{A}(C_{h,t}, \mu_{h,t}[U_{h,t+dt}]).$$

Hence

$$\begin{aligned}
(U_{h,t})^{1-\frac{1}{\psi_h}} &= (1 - e^{-\delta_h dt}) C_{h,t}^{1-\frac{1}{\psi_h}} + e^{-\delta_h dt}\left(\mu_{h,t}\left[U_{h,t+dt}\right]\right)^{1-\frac{1}{\psi_h}} \\
&= \delta_h dt (C_{h,t})^{1-\frac{1}{\psi_h}} + (1-\delta_h dt)(U_{h,t} + \mu_{h,t}[dU_{h,t}])^{1-\frac{1}{\psi_h}} \\
&= \delta_h dt (C_{h,t})^{1-\frac{1}{\psi_h}} + (1-\delta_h dt)(U_{h,t})^{1-\frac{1}{\psi_h}}\left(1 + \mu_{h,t}\left[\frac{dU_{h,t}}{U_{h,t}}\right]\right)^{1-\frac{1}{\psi_h}} \\
&= \delta_h dt (C_{h,t})^{1-\frac{1}{\psi_h}} + (1-\delta_h dt)(U_{h,t})^{1-\frac{1}{\psi_h}}\left(1 + \left(1 - \frac{1}{\psi_h}\right)\mu_{h,t}\left[\frac{dU_{h,t}}{U_{h,t}}\right] + o(dt)\right) \\
&= \delta_h C_{h,t}^{1-\frac{1}{\psi_h}} dt + (U_{h,t})^{1-\frac{1}{\psi_h}}\left(1 + \left(1 - \frac{1}{\psi_h}\right)\mu_{h,t}\left[\frac{dU_{h,t}}{U_{h,t}}\right] - \delta_h dt\right) - (U_{h,t})^{1-\frac{1}{\psi_h}} + o(dt) \\
0 &= \delta_h C_{h,t}^{1-\frac{1}{\psi_h}} dt + (U_{h,t})^{1-\frac{1}{\psi_h}}\left(1 + \left(1 - \frac{1}{\psi_h}\right)\mu_{h,t}\left[\frac{dU_{h,t}}{U_{h,t}}\right] - \delta_h dt\right) + o(dt).
\end{aligned}$$

Hence, in the continuous time limit, we obtain

$$0 = \delta_h C_{h,t}^{1-\frac{1}{\psi_h}} + (U_{h,t})^{1-\frac{1}{\psi_h}} \left[\left(1 - \frac{1}{\psi_h}\right) \frac{1}{U_{h,t}} \mu_{h,t} \left[\frac{dU_{h,t}}{dt} \right] - \delta_h \right].$$

Treating $U_{h,t}$ as a function of $W_{h,t}$, we have via Ito's Lemma

$$\frac{dU_{h,t}}{U_{h,t}} = \frac{W_{h,t} \frac{\partial U_{h,t}}{\partial W_{h,t}}}{U_t} \frac{dW_{h,t}}{W_{h,t}} + \frac{1}{2} \frac{W_{h,t}^2 \frac{\partial^2 U_{h,t}}{\partial W_{h,t}^2}}{U_{h,t}} \left(\frac{dW_{h,t}}{W_{h,t}} \right)^2.$$

Assuming that

$$U_{h,t} = \kappa_h W_{h,t},$$

where κ_h is a constant, we obtain

$$\frac{dU_{h,t}}{U_{h,t}} = \frac{dW_{h,t}}{W_{h,t}}.$$

Hence

$$\begin{aligned} \mu_{h,t} \left[\frac{dU_{h,t}}{U_{h,t}} \right] &= \mu_{h,t} \left[\frac{dW_{h,t}}{W_{h,t}} \right] \\ &= \left[i + (\alpha - i)\pi_h - \frac{1}{2} \gamma_h \sigma^2 \pi_h^2 \boldsymbol{x}_h^\top \Omega \boldsymbol{x}_h - c_h \right] dt, \end{aligned}$$

where we assume that i, π_h, $\boldsymbol{x}_h$, and c_h are constants. Therefore

$$\begin{aligned} 0 &= \delta_h C_{h,t}^{1-\frac{1}{\psi_h}} + (U_{h,t})^{1-\frac{1}{\psi_h}} \left[\left(1 - \frac{1}{\psi_h}\right) \left(i + (\alpha - i)\pi_h - \frac{1}{2} \gamma_h \sigma^2 \pi_h^2 \boldsymbol{x}_h^\top \Omega \boldsymbol{x}_h - c_h \right) - \delta_h \right] \\ 0 &= \delta_h (c_h)^{1-\frac{1}{\psi_h}} + (\kappa_h)^{1-\frac{1}{\psi_h}} \left[\left(1 - \frac{1}{\psi_h}\right) \left(U_h^{MV}(\pi_h, \boldsymbol{x}_h) - c_h \right) - \delta_h \right] \\ 0 &= \psi_h \delta_h (c_h)^{1-\frac{1}{\psi_h}} - (\kappa_h)^{1-\frac{1}{\psi_h}} \left[\psi_h \delta_h + (1 - \psi_h) \left(U_h^{MV}(\pi_h, \boldsymbol{x}_h) - c_h \right) \right] \end{aligned}$$

$$\begin{aligned} (\kappa_h)^{1-\frac{1}{\psi_h}} &= \frac{\psi_h \delta_h (c_h)^{1-\frac{1}{\psi_h}}}{\psi_h \delta_h + (1 - \psi_h) \left(U_h^{MV}(\pi_h, \boldsymbol{x}_h) - c_h \right)} \\ \kappa_h &= \left[\frac{\psi_h \delta_h}{\psi_h \delta_h + (1 - \psi_h) (U^{MV}(\pi_h, \boldsymbol{x}_h) - c_h)} \right]^{\frac{1}{1-\frac{1}{\psi_h}}} c_h. \end{aligned}$$

Therefore, we obtain (21).

For a given consumption-wealth ratio, we now consider the impact of changes in $\boldsymbol{x}_h$ and π_h on percentage changes in the utility-wealth ratio and hence on κ_h, that is we compute $\frac{\partial \ln \kappa_h}{\partial \ln U^{MV}(\pi_h, \boldsymbol{x}_h)}$:

$$\frac{\partial \ln \kappa_h}{\partial \ln U_h^{MV}(\pi_h, \boldsymbol{x}_h)} = U_h^{MV}(\pi_h, \boldsymbol{x}_h) \frac{1}{\delta_h - \left(1 - \frac{1}{\psi_h}\right)\left(U_h^{MV}(\pi_h, \boldsymbol{x}_h) - c_h\right)}.$$

Observe that a necessary condition for κ_h to be well-defined is that

$$\delta_h - \left(1 - \frac{1}{\psi_h}\right)\left(U_h^{MV}(\pi_h, \boldsymbol{x}_h) - c_h\right) > 0.$$

Hence, we can see that a percentage decrease in $U_h^{MV}(\pi_h, \boldsymbol{x}_h)$ is multiplied by the factor $U_h^{MV}(\pi_h, \boldsymbol{x}_h) \frac{1}{\delta_h - \left(1 - \frac{1}{\psi_h}\right)\left(U_h^{MV}(\pi_h, \boldsymbol{x}_h) - c_h\right)} > 0$. The size of this elasticity beyond one captures the size of the additional intertemporal effect of a change in mean-variance utility on lifetime welfare.

Now note that

$$\begin{aligned}
\frac{\Delta \kappa_h^e}{\kappa_h^e} &\approx \frac{\partial \ln \kappa_h}{\partial \ln(c_h)} \frac{\Delta(c_h)}{c_h} + \frac{c_h^2}{\kappa_h^e} \frac{1}{2} \frac{\partial^2 \kappa_h^e}{\partial c_h^2} \left(\frac{\Delta c_h}{c_h}\right)^2 \\
&= \frac{\psi_h \delta_h + (1 - \psi_h) U_h^{MV}(\pi, \boldsymbol{x}) - c_h}{\psi_h \delta_h + (1 - \psi_h)(U^{MV}(\pi_h, \boldsymbol{x}_h) - c_h)} \frac{\Delta c_h}{c_h} \\
&\quad + \psi_h \left(\psi_h - \frac{1}{2}\right) \frac{c_h^2}{\left[\psi_h \delta_h + (1 - \psi_h)(U_h^{MV}(\pi_h, \boldsymbol{x}_h) - c_h)\right]^2} \left(\frac{\Delta c_h}{c_h}\right)^2.
\end{aligned}$$

Hence, we can see that to first order, increasing c_h increases utility if $c_h < \psi_h \delta_h + (1 - \psi_h) U_h^{MV}(\pi_h, \boldsymbol{x}_h)$. ■

A.8. Condition for no-aggregate-biases across households in (22)

We start by formally stating the "no aggregate bias" condition.

Definition A.8.1 *Suppose household h's risky portfolio weight for firm n is given by*

$$x_{hn} = \frac{1}{N} + \epsilon_{hn},$$

where $\frac{1}{N}$ is the unbiased portfolio weight and ϵ_{hn} is the bias of household h's portfolio when investing in firm n. The biases ϵ_{hn} "cancel out in aggregate" if

$$\forall n, \quad \frac{1}{H} \sum_{h=1}^{H} \epsilon_{hn} = 0.$$

Proposition A.8.1 *The following condition holds*

$$\forall n,\ \frac{1}{H}\sum_{h=1}^{H}\frac{1}{\mu_{qh}}q_{hn}=1,$$

if and only if portfolio biases cancel out in aggregate. Observe that the above condition reduces to (22) *for the special case of* $\rho = 0$*.*

Proof: The no aggregate bias condition is equivalent to

$$\sum_{h=1}^{H}\boldsymbol{x}_h=\sum_{h=1}^{H}\frac{1}{N}\mathbf{1},$$

which is equivalent to

$$\frac{1}{H}\sum_{h=1}^{H}\boldsymbol{x}_h=\frac{1}{N}\mathbf{1}.$$

Because the optimal risky portfolio with familiarity biases is given by (A13), the above condition can be rewritten as

$$\frac{1}{H}\sum_{h=1}^{H}\frac{1}{\mu_{qh}}\boldsymbol{q}_h=\mathbf{1},$$

i.e.

$$\forall\, n\in\{1,\ldots,N\},\ \frac{1}{H}\sum_{h=1}^{H}\frac{1}{\mu_{qh}}q_{hn}=1.$$

Now suppose that

$$\frac{1}{H}\sum_{h=1}^{H}\frac{1}{\mu_{qh}}\boldsymbol{q}_h=\mathbf{1}.$$

It follows that

$$\frac{1}{H}\sum_{h=1}^{H}\boldsymbol{x}_h=\frac{1}{N}\mathbf{1},$$

which is equivalent to the no aggregate bias condition.

Therefore,

$$\forall\, n\in\{1,\ldots,N\},\ \frac{1}{H}\sum_{h=1}^{H}\frac{1}{\mu_{qh}}q_{hn}=1.$$

holds if and only if the no aggregate bias condition holds. ■

A.9. The symmetry condition in (23)

In order to derive a closed-form expression for the equilibrium interest rate, we impose the following "symmetry condition".

Definition A.9.1 *The "symmetry condition" states that for distinct households, h and j, we have*

$$\mu_{qh} = \mu_{qj} = \mu_q.$$

Observe that for the special case of $\rho = 0$ used in the text, the symmetry condition reduces to (23).

Proposition A.9.1 *The following condition is equivalent to the combination of the symmetry condition and the no aggregate bias condition:*

$$\frac{1}{H}\sum_{h=1}^{H} q_{hn} = \frac{1}{N}\sum_{n=1}^{N} q_{hn}. \tag{A24}$$

Proof: Because the LHS of (A24) is independent of h, it follows that $\mu_{qh} = \frac{1}{N}\sum_{n=1}^{N} q_{hn}$ is independent of h, which is the symmetry condition. Hence,

$$\frac{1}{H}\sum_{h=1}^{H} q_{hn} = \mu_q,$$

which implies that the no aggregate bias condition holds.

Now suppose that both the symmetry condition and the no aggregate bias condition hold. No aggregate bias implies that

$$\forall\, n \in \{1, \ldots, N\},\ \frac{1}{H}\sum_{h=1}^{H} \frac{1}{\mu_{qh}} q_{hn} = 1.$$

Using the symmetry condition, the above expression becomes

$$\forall\, n \in \{1, \ldots, N\},\ \frac{1}{H}\sum_{h=1}^{H} \frac{1}{\mu_q} q_{hn} = 1,$$

which reduces to

$$\forall\, n \in \{1, \ldots, N\},\ \frac{1}{H}\sum_{h=1}^{H} q_{hn} = \mu_q,$$

which is equivalant to (A24). ■

A.10. Equilibrium interest rate in (24)

The following proposition summarizes the equilibrium interest rate.

Proposition A.10.1 *The equilibrium risk-free interest rate is given by the constant*

$$i = \alpha - \gamma \frac{\sigma^2_{1/N}}{\mu_q}. \tag{24}$$

Proof: Market clearing in the bond market implies that

$$\sum_{h=1}^{H} B_{h,t} = 0, \tag{A25}$$

where the amount of wealth held in the bond by household h is given by

$$B_{h,t} = (1 - \pi_{h,t}) W_{h,t}.$$

Using the expression for $\pi_{h,t}$ given in (A12), we can rewrite the market clearing condition (A25) as

$$\sum_{h=1}^{H} \left(1 - \frac{\mu_{qh}}{\gamma} \frac{\alpha - i}{\sigma^2_{1/N}} \right) W_{h,t} = 0.$$

Hence,

$$\begin{aligned} \sum_{h=1}^{H} W_{h,t} &= \frac{1}{\gamma} \frac{\alpha - i}{\sigma^2_{1/N}} \sum_{h=1}^{H} \mu_{qh} W_{h,t} \\ i &= \alpha - \frac{\sum_{h=1}^{H} W_{h,t}}{\sum_{h=1}^{H} \mu_{qh} W_{h,t}} \gamma \sigma^2_{1/N}, \end{aligned}$$

which reduces to

$$i = \alpha - \gamma \frac{\sigma^2_{1/N}}{\mu_q},$$

if the symmetry condition holds, and upon setting $\rho = 0$ it gives the expression for the interest rate in (24) in the main text. ■

A.11. Equilibrium macroeconomic quantities in (26) and (27)

Proposition A.11.1 *The general equilibrium economy-wide consumption-wealth ratio is given by*

$$\text{(26)} \qquad \frac{C_t^{agg}}{W_t^{agg}} = c = \alpha - g = \psi\delta + (1-\psi)\Big(\alpha - \frac{1}{2}\gamma\sigma_{\boldsymbol{x}}^2\Big),$$

where g, the aggregate growth rate of the economy, is equal to the aggregate investment-capital ratio, which is given by

$$g = \frac{I_t^{agg}}{K_t^{agg}} = \alpha - c = \psi(\alpha - \delta) - \frac{1}{2}(\psi - 1)\gamma\sigma_{\boldsymbol{x}}^2.$$

Proof: Substituting the equilibrium interest rate in (24) into the expression in (20) for the consumption-wealth ratio for each individual gives the general-equilibrium consumption-wealth ratio:

$$\text{(A26)} \qquad c_h = c = \psi\delta + (1-\psi)\Big(\alpha - \frac{1}{2}\gamma\frac{\sigma_{1/N}^2}{\mu_q}\Big),$$

where μ_q is constant across households because of the symmetry condition. Observe that

$$\begin{aligned}
\sigma_{\boldsymbol{x}_h}^2 &= \sigma^2 \boldsymbol{x}_h^\top \Omega \boldsymbol{x}_h \\
&= \sigma^2 \frac{\boldsymbol{q}_h^\top \Omega \boldsymbol{q}_h}{(\mathbf{1}^\top \boldsymbol{q}_h)^2} \\
&= \sigma^2 \frac{1}{N\mu_q^2} \frac{\mathbf{1}^\top \Omega \mathbf{1}}{N} \frac{\frac{\boldsymbol{q}_h^\top \Omega \boldsymbol{q}_h}{N}}{\frac{\mathbf{1}^\top \Omega \mathbf{1}}{N}} \\
&= \sigma^2 \frac{1}{N\mu_q^2} \frac{\mathbf{1}^\top \Omega \mathbf{1}}{N} (\sigma_{qh}^2 + \mu_q^2) \\
&= \sigma_{1/N}^2 \left(1 + \left(\frac{\sigma_{qh}}{\mu_q}\right)^2\right).
\end{aligned}$$

For the case $\rho = 0$ and under the condition that each familiarity coefficient f_{hn} can be either 1 or 0, we have that $\mu_{fh} = \frac{1}{N}\sum_{n=1}^N f_{hn} = \frac{1}{N}\sum_{n=1}^N f_{hn}^2$, implying that $\sigma_{fh}^2 = \frac{1}{N}\sum_{n=1}^N f_{hn}^2 - \mu_{fh}^2 = \mu_{fh} - \mu_{fh}^2$. Therefore, using (15), we get:

$$\text{(A27)} \qquad \sigma_{\boldsymbol{x}_h}^2 = \frac{\sigma_{1/N}^2}{\mu_{fh}},$$

and, under the symmetry condition that $\mu_{fh} = \mu_f$, (A27) implies that $\sigma^2_{\boldsymbol{x}_h} = \sigma^2_{\boldsymbol{x}}$ is identical across all households, leading to

$$c_h = c = \psi\delta + (1-\psi)\Big(\alpha - \frac{1}{2}\gamma\sigma^2_{\boldsymbol{x}}\Big).$$

Observe that in the expression above, all the terms on the right-hand side of the second equality are constants, implying that the consumption-wealth ratio is the same across households. Exploiting the fact that the consumption-wealth ratio is constant across households allows us to obtain the ratio of aggregate consumption-to-wealth ratio, where aggregate consumption is $C_t^{\text{agg}} = \sum_{h=1}^{H} C_{h,t}$ and aggregate wealth is $W_t^{\text{agg}} = \sum_{h=1}^{H} W_{h,t}$:

$$\frac{C_t^{\text{agg}}}{W_t^{\text{agg}}} = c,$$

which is the second equality in (26).

Equation (1) implies

$$\sum_{n=1}^{N} Y_{n,t} = \alpha \sum_{n=1}^{N} K_{n,t},$$

and Equation (2) implies

$$d\left(E_t\left[\sum_{n=1}^{N} K_{n,t}\right]\right) = E_t\left[d\sum_{n=1}^{N} K_{n,t}\right] = \alpha\sum_{n=1}^{N} K_{n,t} - \sum_{n=1}^{N} D_{n,t}dt.$$

In equilibrium $\sum_{n=1}^{N} K_{n,t} = W_t^{\text{agg}}$ and $\sum_{n=1}^{N} D_{n,t} = C_t^{\text{agg}}$. Therefore,

$$\frac{dW_t^{\text{agg}}}{W_t^{\text{agg}}} = \left(\alpha - \frac{C_t^{\text{agg}}}{W_t^{\text{agg}}}\right) dt.$$

We also know that

$$\frac{dW_t^{\text{agg}}}{W_t^{\text{agg}}} = \frac{dY_t^{\text{agg}}}{Y_t^{\text{agg}}},$$

so

$$g\,dt = E_t\left[\frac{dY_t^{\text{agg}}}{Y_t^{\text{agg}}}\right] = \left(\alpha - \frac{C_t^{\text{agg}}}{W_t^{\text{agg}}}\right) dt = (\alpha - c)dt.$$

Rearranging terms leads to the first equality in (26):

$$c = \alpha - g. \tag{26}$$

We now derive the aggregate investment-capital ratio. The aggregate investment flow must be equal to aggregate output flow less the aggregate consumption flow:

$$I_t^{\text{agg}} = \alpha K_t^{\text{agg}} - C_t^{\text{agg}}.$$

It follows that the aggregate investment-capital ratio is given by

$$\frac{I_t^{\text{agg}}}{K_t^{\text{agg}}} = \alpha - \frac{C_t^{\text{agg}}}{K_t^{\text{agg}}} = \alpha - c = \psi\delta - (\psi - 1)\frac{1}{2}\gamma\sigma_{\boldsymbol{x}}^2.$$

Finally, we relate trend output growth to aggregate investment. Firms all have constant returns to scale and differ only because of shocks to their capital stocks. Therefore, the aggregate growth rate of the economy is the aggregate investment-capital ratio:

$$g = \frac{I_t^{\text{agg}}}{K_t^{\text{agg}}},$$

which gives us the expression for g in (27). ■

A.12. Social welfare per unit of aggregate capital in (29) and (30)

Proposition A.12.1 *Welfare is given in terms of the endogenous growth rate of the economy, g, by*

$$\text{(A28)} \quad \kappa_h = \left[\frac{\delta}{\delta - \left(1 - \frac{1}{\psi}\right)\left(g - \frac{1}{2}\gamma\sigma_{1/N}^2\left(1 + \frac{\sigma_{qh}^2}{\mu_q^2}\right)\right)}\right]^{\frac{1}{1-\frac{1}{\psi}}} \left[\delta - \left(1 - \frac{1}{\psi}\right)\left(g - \frac{1}{2}\gamma\frac{\sigma_{1/N}^2}{\mu_q}\right)\right],$$

where

$$\text{(A29)} \qquad g = \psi(\alpha - \delta) - \frac{1}{2}(\psi - 1)\gamma\frac{\sigma_{1/N}^2}{\mu_q}.$$

Proof: We impose the symmetry condition. Because

$$g = \psi(\alpha - \delta) - \frac{1}{2}(\psi - 1)\gamma\frac{\sigma_{1/N}^2}{\mu_q},$$

and

$$\alpha - i = \gamma\frac{\sigma_{1/N}^2}{\mu_q},$$

it follows that

$$c_h = \psi\delta + (1-\psi)\left[i + \frac{1}{2\gamma}\left(\frac{\alpha - i}{\sigma_{1/N}}\right)^2 \mu_{qh}\right] = \delta - \left(1 - \frac{1}{\psi}\right)\left(g - \frac{1}{2}\gamma\frac{\sigma_{1/N}^2}{\mu_q}\right).$$

Hence,

$$\begin{aligned}
& c_h - \Big(1 - \frac{1}{\psi}\Big)\frac{1}{2\gamma}\Big(\frac{\alpha - i}{\sigma_{1/N}}\Big)^2 (\mu_q - \sigma_{qh}^2 - \mu_q^2) \\
& = \delta - \Big(1 - \frac{1}{\psi}\Big)\Big(g - \frac{1}{2}\frac{\sigma_{1/N}^2}{\mu_q}\Big) - \Big(1 - \frac{1}{\psi}\Big)\frac{1}{2\gamma}\Big(\frac{\sigma_{1/N}}{\mu_q}\Big)^2 (\mu_q - \sigma_{qh}^2 - \mu_q^2) \\
& = \delta - \Big(1 - \frac{1}{\psi}\Big)\Big(g - \frac{1}{2}\gamma\sigma_{1/N}^2\big(1 + \frac{\sigma_{qh}^2}{\mu_q^2}\big)\Big)
\end{aligned}$$

Hence, we obtain (A28). ■

We now look at the special case where the familiarity coefficients f_{hn} are restricted to be either 0 or 1.

Proposition A.12.2 *If $\rho = 0$, σ_{fh} is independent of h and the familiarity coefficients f_{hn} are restricted to be either 0 or 1, then κ_h is independent of h and is given by*

$$\kappa = \begin{cases} \left[\frac{\psi\delta + (1-\psi)U^{MV}(\sigma_{\boldsymbol{x}})}{\delta^{\psi}}\right]^{\frac{1}{1-\psi}} & \psi \neq 0, \\ U^{MV}(\sigma_{\boldsymbol{x}}) & \psi = 0, \end{cases} \tag{29}$$

and

$$U^{MV}(\sigma_{\boldsymbol{x}}) = \delta + \frac{1}{\psi}\left(g - \frac{1}{2}\gamma\sigma_{\boldsymbol{x}}^2\right) = \alpha - \frac{\gamma}{2}\sigma_{\boldsymbol{x}}^2,$$

with the endogenous aggregate growth rate g given in (27) *and where we use $U^{MV}(\sigma_{\boldsymbol{x}})$ to denote the utility of a mean-variance household after imposing market clearing, which is obtained by substituting into* (18) *the equilibrium interest rate from* (24) *and the condition that $\pi_h = 1$ for each household.*

Proof: We assume that $\rho = 0$, σ_{fh} is independent of h and the familiarity coefficients f_{hn} are restricted to be either 0 or 1. Consequently, (A26) reduces to

$$c_h = c = \psi\delta + (1-\psi)\Big(\alpha - \frac{1}{2}\gamma\sigma_{\boldsymbol{x}}^2\Big).$$

Furthermore, substituting the equilibrium interest rate from (24) into (A17) and simplifying gives

$$U^{MV}(\sigma_{\boldsymbol{x}}) = \alpha - \frac{1}{2}\gamma\sigma_{\boldsymbol{x}}^2.$$

Therefore (21) reduces to (29). ■

A.13. Disentangling the micro- and macro-level effects in (32)

Proposition A.13.1 *Suppose that σ_{qh} is independent of h. A reduction in familiarity biases changes social welfare per unit capital stock as follows:*

$$d\ln\left(U_t^{social}/K_t^{agg}\right) = d\ln\kappa = d\ln\kappa|_{\textit{micro-level}} + d\ln\kappa|_{\textit{macro-level}},$$

where $d\ln\kappa|_{\textit{micro-level}}$ captures the effect of a reduction in familiarity biases at the micro-level, that is, a reduction in σ_q^2 and an increase in μ_q for individual households, whereas $d\ln\kappa|_{\textit{macro-level}}$ gives the macro-level effect of a change in the equilibrium growth rate driven by an increase in μ_q for individual households.

The micro-level effect of a reduction in familiarity biases on social welfare is given by

$$d\ln\kappa|_{\textit{micro-level}} = \frac{1}{2}\gamma\,\sigma_{1/N}^2\left[v_q^2 k_i(-d\ln\sigma_{qh}^2 + 2d\ln\mu_q) - \left(1-\frac{1}{\psi}\right)\frac{k_c}{\mu_q}\,d\ln\mu_q\right],$$

where k_i captures the intertemporal effects and k_c captures the effects arising from current consumption:

$$k_i = \frac{1}{\delta - \left(1-\frac{1}{\psi}\right)\left(g - \frac{1}{2}\gamma\,\sigma_{1/N}^2(1+v_q^2)\right)}$$

$$k_c = \frac{1}{\delta - \left(1-\frac{1}{\psi}\right)\left(g - \frac{1}{2}\gamma\,\frac{\sigma_{1/N}^2}{\mu_q}\right)},$$

where

$$v_q^2 = \left(\frac{\sigma_q}{\mu_q}\right)^2.$$

The macro-level effect of a reduction in familiarity biases on social welfare is given by

$$d\ln\kappa|_{\textit{macro-level}} = \left[k_i - \left(1-\frac{1}{\psi}\right)k_c\right]dg,$$

where

$$dg = \frac{1}{2}(\psi - 1)\gamma \frac{\sigma_{1/N}^2}{\mu_q} d\ln \mu_q. \tag{A30}$$

If $\rho = 0$ and $f_{hn} \in \{0, 1\}$, then the percentage change in social welfare per unit of aggregate capital stock stemming from a change in familiarity biases is given by

$$\frac{d\ln\kappa}{d\sigma_{\boldsymbol{x}}^2} = \frac{1}{c}\frac{dU^{MV}(\sigma_{\boldsymbol{x}})}{d\sigma_{\boldsymbol{x}}^2},$$

where

$$\begin{aligned}
\frac{dU^{MV}(\sigma_{\boldsymbol{x}})}{d\sigma_{\boldsymbol{x}}^2} &= \frac{\partial U^{MV}(\sigma_{\boldsymbol{x}})}{\partial g}\frac{\partial g}{\partial \sigma_{\boldsymbol{x}}^2} + \frac{\partial U^{MV}(\sigma_{\boldsymbol{x}})}{\partial \sigma_{\boldsymbol{x}}^2} \\
&= -\frac{1}{2}\gamma\Bigg[\underbrace{\left(1 - \frac{1}{\psi}\right)}_{\textit{macro-level effect}} + \underbrace{\frac{1}{\psi}}_{\textit{micro-level effect}}\Bigg].
\end{aligned}$$

Proof: Define the square of the coefficient of variation

$$v_{qh}^2 = \left(\frac{\sigma_{qh}}{\mu_q}\right)^2.$$

From (A28), we can see that

$$\kappa_h = (\delta k_i)^{\frac{1}{1-\frac{1}{\psi}}}\frac{1}{k_c}.$$

Therefore

$$d\ln\kappa_h = \frac{1}{1-\frac{1}{\psi}} d\ln k_i - d\ln k_c,$$

and

$$\begin{aligned}
\frac{\partial \ln k_i}{\partial \ln v_{qh}^2} &= -\frac{1}{2}\gamma v_{qh}^2 \sigma_{1/N}^2 \left(1 - \frac{1}{\psi}\right) k_i, \\
\frac{\partial \ln k_c}{\partial \ln \mu_q} &= \frac{1}{2}\frac{\gamma}{\mu_q}\sigma_{1/N}^2 \left(1 - \frac{1}{\psi}\right) k_c, \\
\frac{\partial \ln k_i}{\partial \ln g} &= \left(1 - \frac{1}{\psi}\right) g k_i, \\
\frac{\partial \ln k_c}{\partial \ln g} &= \left(1 - \frac{1}{\psi}\right) g k_c.
\end{aligned}$$

Hence

$$\frac{\partial \ln \kappa_h}{\partial \ln v_{qh}^2} = \frac{1}{1-\frac{1}{\psi}} \frac{\partial \ln k_i}{\partial \ln v_{qh}^2} = -\frac{1}{2}\gamma v_{qh}^2 \sigma_{1/N}^2 k_i$$

$$\frac{\partial \ln \kappa_h}{\partial \ln \mu_q} = -\frac{1}{2}\frac{\gamma}{\mu_q}\sigma_{1/N}^2 \left(1-\frac{1}{\psi}\right) k_c.$$

Therefore

$$\begin{aligned}
d\ln \kappa_h|_{\text{micro-level}} &= \frac{\partial \ln \kappa_h}{\partial \ln v_{qh}^2} d\ln v_{qh}^2 + \frac{\partial \ln \kappa_h}{\partial \ln \mu_q} d\ln \mu_q \\
&= -\frac{1}{2}\gamma v_{qh}^2 \sigma_{1/N}^2 k_i d\ln v_{qh}^2 - \frac{1}{2}\frac{\gamma}{\mu_q}\sigma_{1/N}^2 \left(1-\frac{1}{\psi}\right) k_c d\ln \mu_q \\
&= -\frac{1}{2}\gamma \sigma_{1/N}^2 \left[v_{qh}^2 k_i \, d\ln v_q^2 + \frac{k_c}{\mu_q}\left(1-\frac{1}{\psi}\right) d\ln \mu_q \right] \\
&= -\frac{1}{2}\gamma \sigma_{1/N}^2 \left[v_{qh}^2 k_i \left(d\ln \sigma_q^2 - 2d\ln \mu_q \right) + \frac{k_c}{\mu_q}\left(1-\frac{1}{\psi}\right) d\ln \mu_q \right] \\
&= \frac{1}{2}\gamma \sigma_{1/N}^2 \left[v_{qh}^2 k_i \left(-d\ln \sigma_q^2 + 2d\ln \mu_q \right) - \frac{k_c}{\mu_q}\left(1-\frac{1}{\psi}\right) d\ln \mu_q \right].
\end{aligned}$$

Also

$$\begin{aligned}
d\ln \kappa_h|_{\text{macro-level}} &= \frac{\partial \ln \kappa_h}{\partial \ln g} d\ln g \\
&= \frac{1}{1-\frac{1}{\psi}} \frac{\partial \ln k_i}{\partial \ln g} d\ln g - \frac{\partial \ln k_c}{\partial \ln g} d\ln g \\
&= g\left[k_i - \left(1-\frac{1}{\psi_h}\right) k_c \right] d\ln g \\
&= \left[k_i - \left(1-\frac{1}{\psi_h}\right) k_c \right] dg.
\end{aligned}$$

Equation (A30) then follows from (A29).

If σ_{qh} is independent of h, then κ_h is independent of h and social welfare per unit wealth is given by $\kappa = \kappa_h = \sum_{h=1}^{H} \frac{U_{h,t}}{W_{h,t}} = \frac{U_t^{\text{social}}}{W_t^{\text{agg}}}$. Hence, the increase in social welfare per unit wealth from infinitesimally small changes in familiarity biases is given by $d\kappa = d\kappa_h$.

We now impose the assumptions that $\rho = 0$ and $f_{hn} \in \{0, 1\}$, which implies that

$$1 + v_{fh}^2 = \frac{1}{\mu_f}.$$

The above expression tells us that the independence of μ_f from h implies that v_{fh} and hence σ_{fh} is independent of h. We can thus see from that (A28) that κ_h becomes independent of h and

$$\kappa = \begin{cases} \left[\frac{\psi\delta+(1-\psi)U^{MV}(\sigma_{\boldsymbol{x}})}{\delta^{\psi}}\right]^{\frac{1}{1-\psi}} & \psi \neq 0, \\ U^{MV}(\sigma_{\boldsymbol{x}}) & \psi = 0, \end{cases}$$

in which

$$U^{MV}(\sigma_{\boldsymbol{x}}) = \delta + \frac{1}{\psi}\left(g - \frac{1}{2}\gamma\sigma_{\boldsymbol{x}}^2\right) = \alpha - \frac{\gamma}{2}\sigma_{\boldsymbol{x}}^2,$$

where

$$\sigma_{\boldsymbol{x}}^2 = \frac{\sigma_{1/N}^2}{\mu_f} = \sigma_{1/N}^2(1 + v_f^2).$$

Therefore

$$\begin{aligned} \frac{d\ln\kappa}{d\sigma_{\boldsymbol{x}}^2} &= \frac{1}{\psi\delta + (1-\psi)U^{MV}(\sigma_{\boldsymbol{x}})}\frac{dU^{MV}(\sigma_{\boldsymbol{x}})}{d\sigma_{\boldsymbol{x}}^2} \\ &= \frac{1}{c}\frac{dU^{MV}(\sigma_{\boldsymbol{x}})}{d\sigma_{\boldsymbol{x}}^2}, \end{aligned}$$

where

$$\begin{aligned} \frac{dU^{MV}(\sigma_{\boldsymbol{x}})}{d\sigma_{\boldsymbol{x}}^2} &= \frac{\partial U^{MV}(\sigma_{\boldsymbol{x}})}{\partial g}\frac{\partial g}{\partial\sigma_{\boldsymbol{x}}^2} + \frac{\partial U^{MV}(\sigma_{\boldsymbol{x}})}{\partial\sigma_{\boldsymbol{x}}^2} \\ &= -\frac{1}{2}\gamma\Bigg[\underbrace{\left(1 - \frac{1}{\psi}\right)}_{\text{macro-level effect}} + \underbrace{\frac{1}{\psi}}_{\text{micro-level effect}}\Bigg]. \end{aligned}$$

■

A.14. Social welfare with preference heterogeneity in (33)

The proposition below shows that when households are heterogeneous, then social welfare per unit of aggregate wealth is given by κ_h *averaged* across all households, in contrast to the case where households had identical preferences and social welfare per unit of aggregate wealth was given by (the common) κ.

Proposition A.14.1 *We assume that the following symmetry condition holds*

$$\frac{1}{\mathcal{R}} = \frac{\mu_{qh}}{\gamma_h}, \forall h \in \{1, \ldots, H\}.$$

Social welfare per unit of aggregate wealth at date t is given by the wealth-weighted average of κ_h:

$$\frac{U_t^{social}}{W_t^{agg}} = \frac{\sum_{h=1}^{H} U_{h,t}}{\sum_{h=1}^{H} W_{h,t}} = \sum_{h=1}^{H} \kappa_h \frac{W_{h,t}}{\sum_{j=1}^{H} W_{j,t}},$$

where

$$\kappa_h = \left[\frac{\psi_h \delta_h}{\psi_h \delta_h + (1-\psi_h)\left[g_h - \frac{1}{2}\gamma_h \sigma_{1/N}^2 \left(1 + \frac{\sigma_{qh}^2}{\mu_{qh}^2}\right)\right]}\right]^{\frac{1}{1-\frac{1}{\psi_h}}} \left[\delta_h - \left(1 - \frac{1}{\psi_h}\right)\left(g_h - \frac{1}{2}\mathcal{R}\sigma_{1/N}^2\right)\right],$$

and

$$g_h = \psi_h(\alpha - \delta_h) - \frac{1}{2}(\psi_h - 1)\mathcal{R}\sigma_{1/N}^2.$$

For the special case of $\rho = 0$ and with the assumption that $f_{hn} \in \{0, 1\}$, κ_h is given by

$$\kappa_h = \begin{cases} \left[\frac{\psi_h \delta_h + (1-\psi_h) U^{MV}}{\delta_h^{\psi_h}}\right]^{\frac{1}{1-\psi_h}} & \psi_h \neq 0, \\ U^{MV} & \psi_h = 0, \end{cases},$$

where

$$U^{MV} = \alpha - \frac{\mathcal{R}}{2}\sigma_{1/N}^2.$$

If we assume that all households have equal date-t wealth, we obtain

$$\frac{U_t^{social}}{W_t^{agg}} = \left(\frac{1}{H}\sum_{h=1}^{H} \kappa_h\right),$$

where date-t aggregate wealth is given by

$$W_t^{agg} = \sum_{h=1}^{H} W_{h,t} = H\, W_{h,t}.$$

Proof: If the investment opportunity set is constant, that is, the interest rate is constant (below we will specify the condition that ensures this indeed is the case), then the vector of optimal portfolio weights of household h is given by

$$\boldsymbol{\omega}_h = \frac{\alpha - i}{\gamma_h \sigma^2_{1/N}} \frac{\boldsymbol{q_h}}{N}.$$

Furthermore, the date-t optimal consumption rate of household h is given by

$$\frac{C_{h,t}}{W_{h,t}} = \psi_h \delta_h + (1 - \psi_h) \left(i + \frac{1}{2\gamma_h} \left(\frac{\alpha - i}{\sigma_{1/N}} \right)^2 \mu_{qh} \right). \tag{A31}$$

In equilibrium the bond market clears and so

$$\sum_{h=1}^{H} (1 - \pi_h) W_{h,t} = 0.$$

Therefore,

$$\sum_{h=1}^{H} \pi_h W_{h,t} = \sum_{h=1}^{H} W_{h,t}.$$

Hence,

$$\frac{\alpha - i}{\sigma^2_{1/N}} \sum_{h=1}^{H} \frac{\mu_{qh}}{\gamma_h} W_{h,t} = \sum_{h=1}^{H} W_{h,t},$$

and so

$$i = \alpha - \mathcal{R}\, \sigma^2_{1/N}, \tag{A32}$$

where

$$\mathcal{R} = \left(\frac{\sum_{h=1}^{H} \frac{\mu_{qh}}{\gamma_h} W_{h,t}}{\sum_{h=1}^{H} W_{h,t}} \right)^{-1}.$$

We now impose the symmetry condition that for distinct households h and j:

$$\frac{\mu_{qh}}{\gamma_h} = \frac{\mu_{qj}}{\gamma_j}.$$

We hence obtain

$$\mathcal{R} = \frac{\gamma_h}{\mu_{qh}}. \tag{A33}$$

Substituting (A32) into (A31) and using (A33) gives

$$c_h = \frac{C_{h,t}}{W_{h,t}} = \psi_h \delta_h + (1-\psi_h)\left(\alpha - \frac{1}{2}\mathcal{R}\,\sigma^2_{1/N}\right).$$

Observe that the symmetry condition implies that for every household$h$:

$$\pi_h = 1.$$

Household h's experienced utility level is given by (A15). It follows that in equilibrium with the symmetry condition, we have

$$U_h^{MV}(\sigma_{\boldsymbol{x}_h}) = \alpha - \frac{\gamma_h}{2}\sigma^2_{\boldsymbol{x}_h},$$

where

$$\sigma^2_{\boldsymbol{x}_h} = \sigma^2_{1/N}\left(1 + \frac{\sigma^2_{qh}}{\mu^2_{qh}}\right).$$

Therefore $\kappa_h = \frac{U_{h,t}}{W_{h,t}}$ is given by

$$\kappa_h = \left[\frac{\psi_h \delta_h}{\psi_h \delta_h + (1-\psi_h)\left(U^{MV}(\sigma_{\boldsymbol{x}_h}) - c_h\right)}\right]^{\frac{1}{1-\frac{1}{\psi_h}}} c_h.$$

Now observe that

$$U^{MV}(\sigma_{\boldsymbol{x}_h}) - c_h = g_h - \frac{1}{2}\gamma_h \sigma^2_{1/N}\left(1 + \frac{\sigma^2_{qh}}{\mu^2_{qh}}\right),$$

where

$$g_h = \frac{1}{W_{h,t}} E_t\left[\frac{dW_{h,t}}{dt}\right] = \psi_h(\alpha - \delta_h) - \frac{1}{2}(\psi_h - 1)\mathcal{R}\sigma^2_{1/N}.$$

We also have

$$g_h - \frac{1}{2}\mathcal{R}\sigma^2_{1/N} = \psi_h\left(\alpha - \delta_h - \frac{1}{2}\mathcal{R}\sigma^2_{1/N}\right),$$

and so

$$c_h = \delta_h - \left(1 - \frac{1}{\psi_h}\right)\left(g_h - \frac{1}{2}\mathcal{R}\sigma^2_{1/N}\right).$$

Therefore

$$\kappa_h = \left[\frac{\psi_h \delta_h}{\psi_h \delta_h + (1-\psi_h) \left[g_h - \frac{1}{2}\gamma_h \sigma^2_{1/N} \left(1 + \frac{\sigma^2_{gh}}{\mu^2_{qh}}\right)\right]} \right]^{\frac{1}{1-\frac{1}{\psi_h}}} \left[\delta_h - \left(1 - \frac{1}{\psi_h}\right)\left(g_h - \frac{1}{2}\mathcal{R}\sigma^2_{1/N}\right)\right]$$

For the special case of $\rho = 0$ and with the assumption that $f_{hn} \in \{0,1\}$, we have $\sigma^2_{f,h} = \mu_{f,h} - \mu^2_{f,h}$, and so

$$U^{MV} = U^{MV}_h(\sigma_{\boldsymbol{x}_h}) = \alpha - \frac{\mathcal{R}}{2}\sigma^2_{1/N},$$

while $U^{MV}(\sigma_{\boldsymbol{x}_h}) - c_h$ simplifies to give

$$U^{MV} - c_h = \psi_h \left(\alpha - \delta_h - \frac{1}{2}\mathcal{R}\sigma^2_{1/N}\right).$$

Therefore

$$\frac{\psi_h \delta_h}{\psi_h \delta_h + (1-\psi_h)\left(U^{MV}(\sigma_{\boldsymbol{x}_h}) - c_h\right)} = \frac{\delta_h}{c_h},$$

where

$$c_h = \psi_h \delta_h + (1-\psi_h)\left(\alpha - \frac{1}{2}\mathcal{R}\,\sigma^2_{1/N}\right).$$

Hence, we obtain

$$\kappa_h = \left[\frac{\psi\delta + (1-\psi)U^{MV}}{\delta\psi}\right]^{\frac{1}{1-\psi}},$$

where

$$U^{MV} = \alpha - \frac{\mathcal{R}}{2}\sigma^2_{1/N}.$$

■

B. Labor Income

In this section, we first provide the details of the model with labor income and then provide the proofs for all the propositions.

B.1. Labor Income: Details

A household's dynamic budget constraint in the presence of labor income is given by

$$\frac{dW_{h,t}}{W_{h,t}} = \left(1 - \pi_{h,t}\right) i\, dt + \pi_{h,t} \sum_{n=1}^{N} x_{hn,t}\left(\alpha\, dt + \sigma\, dZ_{n,t}\right) - c_{h,t}dt + \frac{Y_{h,t}}{W_{h,t}}dt,$$

where $c_{h,t} = C_{h,t}/W_{h,t}$, $Y_{h,t}$ is the date-t labor income flow of household h, and

$$\frac{dY_{h,t}}{Y_{h,t}} = \theta_Y \left(m_Y - \ln \frac{W_{h,t}}{Y_{h,t}}\right) dt + \sigma_Y dZ_{Y,h,t},$$

where $Z_{Y,h}$ is a standard Brownian motion under the reference measure $\mathbb{P}$ such that

$$\begin{aligned} dZ_{Y,h,t}dZ_{h'\,t} &= \rho_{YK}(1 + \epsilon\delta_{hh'})dt, \\ dZ_{Y,h,t}dZ_{Y,h',t} &= \delta_{hh'}dt. \end{aligned}$$

We can define a new vector of Brownian motions, consisting of the Brownian motions driving labor income shocks, i.e.

$$\boldsymbol{Z}_{Y,t} = (Z_{Y,1,t}, \ldots, Z_{Y,H,t})^\top.$$

The correlation matrix for the combined vector of Brownian shocks $(d\boldsymbol{Z}_t^\top, (d\boldsymbol{Z}_{Y,t})^\top)^\top$ is denoted by Ω_A, that is

$$(d\boldsymbol{Z}_t^\top, (d\boldsymbol{Z}_{Y,t})^\top)^\top (d\boldsymbol{Z}_t^\top, (d\boldsymbol{Z}_{Y,t})^\top) = \Omega_A dt,$$

where

$$\Omega_A = \begin{pmatrix} \Omega & \rho_{YK}(J_N + \epsilon I_N) \\ \rho_{YK}(J_N + \epsilon I_N) & I_N \end{pmatrix},$$

and J_N is the $N \times N$ matrix in which every element is a one.

B.2. Labor Income: Propositions and Proofs

We start by extending the definition of the probability measure $\mathbb{Q}^{\boldsymbol{\nu}_h}$ to make clear that the expected labor income flow to a household is unaffected by familiarity biases.

Definition B.2.1 *The probability measure $\mathbb{Q}^{\boldsymbol{\nu}_h}$ is defined by*

$$\mathbb{Q}^{\boldsymbol{\nu}_h}(A) = E[1_A \xi_{h,T}],$$

where E is the expectation under $\mathbb{P}$, A is an event, and $\xi_{h,t}$ is an exponential martingale (under the reference probability measure $\mathbb{P}$)

$$\frac{d\xi_{h,t}}{\xi_{h,t}} = \frac{1}{\sigma}(\boldsymbol{\nu}_{h,t}^\top, \mathbf{0}_H^\top)\Omega_A^{-1}(d\boldsymbol{Z}_t^\top, (d\boldsymbol{Z}_{Y,t})^\top)^\top,$$

where $\mathbf{0}_H$ is the $H \times 1$ vector of zeros.

Proposition B.2.1 *The stochastic optimal control problem for a household with familiarity biases and exogenous labor income can be solved via the following Hamilton-Jacobi-Bellman equation*

$$\begin{aligned}
&\sup_{\widehat{C}_{h,t},\, \pi_{h,t},\, \boldsymbol{x}_{h,t}} \inf_{\boldsymbol{\nu}_{h,t}} \delta_h u_{\psi_h}\left(\frac{\widehat{C}_{h,t}}{\widehat{U}_{h,t}}\right) + \frac{\delta_h - k_{1h,t}}{1-\frac{1}{\psi_h}} \\
&+ \frac{\widehat{W}_{h,t}\widehat{U}_{ht,\widehat{W}_h}}{\widehat{U}_{h,t}}\left[k_{2h,t} + \pi_{h,t}(\alpha + \boldsymbol{x}_h^\top \boldsymbol{\nu}_h - \gamma_h\rho_{YK}\sigma\sigma_Y(1+\epsilon x_{hh,t}) - i) - \frac{\widehat{C}_{h,t}}{\widehat{W}_{h,t}} + \frac{1}{\widehat{W}_{h,t}}\right] \\
&+ \frac{1}{2}\left(\frac{\widehat{W}_{h,t}^2\widehat{U}_{ht,\widehat{W}_h\widehat{W}_h}}{\widehat{U}_{h,t}} - \gamma_h\left(\frac{\widehat{W}_{h,t}\widehat{U}_{ht,\widehat{W}_h}}{\widehat{U}_{h,t}}\right)^2\right)(\sigma^2\pi_{h,t}^2\boldsymbol{x}_{h,t}^\top\Omega\boldsymbol{x}_{h,t} - 2\rho_{YK}\sigma_Y\sigma\pi_{h,t}(1+\epsilon x_{hh,t}) + \sigma_Y^2) \\
&+ \frac{1}{2\gamma_h}\frac{\boldsymbol{\nu}_{h,t}^\top\Gamma_h^{-1}\boldsymbol{\nu}_{h,t}}{\sigma^2},
\end{aligned} \tag{B1}$$

where $\widehat{U}_{h,t} = U_{h,t}/Y_{h,t}$, $\widehat{W}_{h,t} = W_{h,t}/Y_{h,t}$, and

$$\begin{aligned}
k_{1h,t} &= \delta_h + \frac{1}{\psi_h}\mu_{Y,h,t} - \frac{1}{2}\left(1+\frac{1}{\psi_h}\right)\gamma_h\sigma_Y^2 + \gamma_h\sigma_Y^2 - \mu_{Y,h,t}, \\
k_{2h,t} &= i + \gamma_h\sigma_Y^2 - \mu_{Y,h,t}, \\
\mu_{Y,h,t} &= \theta_Y\left(m_Y - \ln\frac{W_{h,t}}{Y_{h,t}}\right).
\end{aligned}$$

Proof: We now define

$$\widehat{W}_{h,t} = \frac{W_{h,t}}{Y_{h,t}}.$$

Hence, using Ito's Lemma

$$\begin{aligned}
\frac{d\widehat{W}_{h,t}}{\widehat{W}_{h,t}} &= \frac{dW_{h,t}}{W_{h,t}} + \frac{d(Y_{h,t}^{-1})}{Y_{h,t}^{-1}} + \frac{dW_{h,t}}{W_{h,t}}\frac{d(Y_{h,t}^{-1})}{Y_{h,t}^{-1}} \\
&= \frac{dW_{h,t}}{W_{h,t}} + \frac{-Y_{h,t}^{-2}dY_{h,t} + \frac{1}{2}2Y_{h,t}^{-3}(dY_{h,t})^2}{Y_{h,t}^{-1}} + \frac{dW_{h,t}}{W_{h,t}}\frac{(-Y_{h,t}^{-2})dY_{h,t}}{Y_{h,t}^{-1}} \\
&= \frac{dW_{h,t}}{W_{h,t}} - \frac{dY_{h,t}}{Y_{h,t}} + \left(\frac{dY_{h,t}}{Y_{h,t}}\right)^2 - \frac{dW_{h,t}}{W_{h,t}}\frac{dY_{h,t}}{Y_{h,t}} \\
&= idt + \pi_{h,t}(\alpha - i)dt - c_{h,t}dt + \frac{Y_{h,t}}{W_{h,t}}dt + \sigma\pi_{h,t}\sum_{n=1}^{N} x_{hn,t}\, dZ_{n,t} \\
&- \mu_{Y,h,t}dt - \sigma_Y dZ_{Y,h,t} + \sigma_Y^2 dt \\
&- \sigma\sigma_Y\pi_{h,t}dZ_{Y,h,t}\sum_{n=1}^{N} x_{hn,t}dZ_{n,t} \\
&= \pi_{h,t}(\alpha - i)dt - c_{h,t}dt + \frac{Y_{h,t}}{W_{h,t}}dt + \sigma\pi_{h,t}\sum_{n=1}^{N} x_{hn,t}\, dZ_{n,t} \\
&- \mu_{Y,h,t}dt - \sigma_Y dZ_{Y,h,t} + \sigma_Y^2 dt - \sigma\sigma_Y\pi_{h,t}\left(\rho_{YK}\sum_{n=1}^{N} x_{hn,t} + \rho_{YK}\epsilon x_{hh,t}\right)dt \\
\frac{d\widehat{W}_{h,t}}{\widehat{W}_{h,t}} &= \left[i + \pi_{h,t}[\alpha - \rho_{YK}\sigma_Y\sigma(1+\epsilon x_{hh,t}) - i] - \frac{\widehat{C}_{h,t}}{\widehat{W}_{h,t}} + \frac{1}{\widehat{W}_{h,t}} - \left(\mu_{Y,h,t} - \sigma_Y^2\right)\right]dt \\
&+ \sigma\pi_{h,t}\sum_{n=1}^{N} x_{hn,t}\, dZ_{n,t} - \sigma_Y dZ_{Y,h,t} \\
&= \left[i + \pi_{h,t}[\alpha - \rho_{YK}\sigma_Y\sigma(1+\epsilon \boldsymbol{e}_h^\top \boldsymbol{x}_{h,t}) - i] - \frac{\widehat{C}_{h,t}}{\widehat{W}_{h,t}} + \frac{1}{\widehat{W}_{h,t}} - \left(\mu_{Y,h,t} - \sigma_Y^2\right)\right]dt \\
&+ (\sigma\pi_{h,t}\boldsymbol{x}_{h,t}^\top, -\sigma_Y\boldsymbol{e}_h^\top)(d\boldsymbol{Z}_t^\top, d\boldsymbol{Z}_{Y,t}^\top)^\top \\
&= \left[i + \pi_{h,t}(\widehat{\alpha} - \epsilon\rho_{YK}\sigma_Y\sigma\boldsymbol{e}_h^\top\boldsymbol{x}_{h,t} - i) - \frac{\widehat{C}_{h,t}}{\widehat{W}_{h,t}} + \frac{1}{\widehat{W}_{h,t}} - \left(\mu_{Y,h,t} - \sigma_Y^2\right)\right]dt \\
&+ (\sigma\pi_{h,t}\boldsymbol{x}_{h,t}^\top, -\sigma_Y\boldsymbol{e}_h^\top)(d\boldsymbol{Z}_t^\top, d\boldsymbol{Z}_{Y,t}^\top)^\top
\end{aligned}$$

where

$$\begin{aligned}
\mu_{Y,h,t} &= \theta_Y\left(m_Y - \ln\frac{W_{h,t}}{Y_{h,t}}\right) = \theta_Y\left(m_Y - \ln\widehat{W}_{h,t}\right) \\
\widehat{\alpha} &= \alpha - \rho_{YK}\sigma_Y\sigma.
\end{aligned}$$

For algebraic simplicity, we define

$$\tau_0 = \theta_Y m_Y$$
$$\tau_1 = -\theta_Y.$$

We now derive the dynamics of $\widehat{W}_{h,t}$ under the probability measure $\mathbb{Q}^{\boldsymbol{\nu}_h}$ by using Girsanov's Theorem. Hence, we obtain

$$\frac{d\widehat{W}_{h,t}}{\widehat{W}_{h,t}} = \left[i + \pi_{h,t}[\widehat{\alpha} + \boldsymbol{x}_{h,t}^\top(\boldsymbol{\nu}_{h,t} - \epsilon\rho_{YK}\sigma_Y\sigma\boldsymbol{e}_h) - i] - \frac{\widehat{C}_{h,t}}{\widehat{W}_{h,t}} + \frac{1}{\widehat{W}_{h,t}} - \left(\mu_{Y,h,t} - \sigma_Y^2\right) \right] dt$$
$$+ (\sigma\pi_{h,t}\boldsymbol{x}_{h,t}^\top, -\sigma_Y\boldsymbol{e}_h^\top)((d\boldsymbol{Z}_t^{\boldsymbol{\nu}})^\top, d\boldsymbol{Z}_{Y,t}^\top)^\top,$$

where

$$d\boldsymbol{Z}_t^{\boldsymbol{\nu}} = d\boldsymbol{Z}_t - \pi_{h,t}\boldsymbol{x}_{h,t}^\top\boldsymbol{\nu}_{h,t}.$$

The dynamics of $Y_{h,t}$ remain the same under $\mathbb{Q}^{\boldsymbol{\nu}_h}$, because shocks to labor income are orthogonal to shocks to the exponential martingale $\xi_{h,t}$.

We start from the recursive definition of the utility function

$$U_{h,t} = \mathcal{A}(C_{h,t}, \mu_{h,t}^{\boldsymbol{\nu}}[U_{h,t+dt}]).$$

Defining

$$\widehat{U}_{h,t} = \frac{U_{h,t}}{Y_{h,t}},$$

we obtain

$$\widehat{U}_{h,t} = \mathcal{A}\left(\widehat{C}_{h,t}, \mu_{h,t}^{\boldsymbol{\nu}}\left[\frac{Y_{h,t+dt}}{Y_{h,t}}\widehat{U}_{h,t+dt}\right]\right).$$

Observe that under both the reference probability measure $\mathbb{P}$ and $\mathbb{Q}^{\boldsymbol{\nu}_h}$

$$\frac{dY_{h,t}}{Y_{h,t}} = \mu_{Y,h,t}dt + \sigma_Y dZ_{Y,h,t}.$$

and

$$Y_{h,u} = Y_{h,t}e^{\int_t^u \mu_{Y,h,s}ds}e^{-\frac{1}{2}\sigma_Y^2(u-t)+\sigma_Y(Z_{Y,h,u}-Z_{Y,h,t})},$$

and so

$$\left(\frac{Y_{h,u}}{Y_{h,t}}\right)^{1-\gamma} = e^{(1-\gamma)\left(\int_t^u \mu_{Y,s}ds - \frac{1}{2}\gamma_h\sigma_Y^2(u-t)\right)}e^{-\frac{1}{2}(1-\gamma_h)^2\sigma_Y^2(u-t)+(1-\gamma_h)\left[\sigma_Y(Z_{Y,h,u}-Z_{Y,h,t})\right]}$$

$$= e^{(1-\gamma_h)\left(\int_t^u \mu_{Y,s} ds - \frac{1}{2}\gamma_h \sigma_Y^2 (u-t)\right)} \frac{M_{Y,h,u}}{M_{Y,h,t}},$$

where

$$M_{Y,h,t} = e^{-\frac{1}{2}(1-\gamma_h)^2 \sigma_Y^2 t + (1-\gamma_h)\sigma_Y Z_{Y,h,t}},$$

is an exponential martingale with respect to both the reference probability measure $\mathbb{P}$ and $\mathbb{Q}^{\boldsymbol{\nu}_h}$.

Therefore

$$\begin{aligned}
\mu^{\boldsymbol{\nu}}_{h,t}\left[\frac{Y_{h,t+dt}}{Y_{h,t}}\widehat{U}_{h,t+dt}\right] &= \left(E_t^{\mathbb{Q}^{\boldsymbol{\nu}_h}}\left[\left(\frac{Y_{h,t+dt}}{Y_{h,t}}\right)^{1-\gamma_h}\widehat{U}_{h,t+dt}^{1-\gamma_h}\right]\right)^{\frac{1}{1-\gamma_h}} + \widehat{U}_{h,t}L_{h,t}dt \\
&= \left(E_t^{\mathbb{Q}^{\boldsymbol{\nu}_h}}\left[e^{(1-\gamma_h)\left(\mu_{Y,h,t}dt - \frac{1}{2}\gamma_h\sigma_Y^2 dt\right)}\frac{M_{Y,h,t+dt}}{M_{Y,h,t}}\widehat{U}_{h,t+dt}^{1-\gamma_h}\right]\right)^{\frac{1}{1-\gamma_h}} + \widehat{U}_{h,t}L_{h,t}dt \\
&= e^{\left(\mu_{Y,h,t} - \frac{1}{2}\gamma_h\sigma_Y^2\right)dt}\left(E_t^{\mathbb{Q}^{\boldsymbol{\nu}_h}}\left[\frac{M_{Y,h,t+dt}}{M_{Y,h,t}}\widehat{U}_{h,t+dt}^{1-\gamma_h}\right]\right)^{\frac{1}{1-\gamma_h}} + \widehat{U}_{h,t}L_{h,t}dt \\
&= e^{\left(\mu_{Y,h,t} - \frac{1}{2}\gamma_h\sigma_Y^2\right)dt}\left(E_t^{\mathbb{Q}_Y^{\boldsymbol{\nu}_h}}\left[\widehat{U}_{h,t+dt}^{1-\gamma_h}\right]\right)^{\frac{1}{1-\gamma_h}} + \widehat{U}_{h,t}L_{h,t}dt,
\end{aligned}$$

where the probability measure $\mathbb{Q}_Y^{\boldsymbol{\nu}_h}$ is defined by the martingale $M_{Y,h}$. Now

$$\begin{aligned}
\left(E_t^{\mathbb{Q}_Y^{\boldsymbol{\nu}_h}}\left[\widehat{U}_{h,t+dt}^{1-\gamma_h}\right]\right)^{\frac{1}{1-\gamma_h}} &= \left(E_t^{\mathbb{Q}_Y^{\boldsymbol{\nu}_h}}\left[(\widehat{U}_{h,t} + d\widehat{U}_{h,t})^{1-\gamma_h}\right]\right)^{\frac{1}{1-\gamma_h}} \\
&= \widehat{U}_{h,t}\left(E_t^{\mathbb{Q}_Y^{\boldsymbol{\nu}_h}}\left[\left(1 + \frac{d\widehat{U}_{h,t}}{\widehat{U}_{h,t}}\right)^{1-\gamma_h}\right]\right)^{\frac{1}{1-\gamma_h}} \\
&= \widehat{U}_{h,t}\left(E_t^{\mathbb{Q}_Y^{\boldsymbol{\nu}_h}}\left[\left(1 + (1-\gamma_h)\frac{d\widehat{U}_{h,t}}{\widehat{U}_{h,t}} - \frac{1}{2}(1-\gamma_h)\gamma_h\left(\frac{d\widehat{U}_{h,t}}{\widehat{U}_{h,t}}\right)^2\right)\right] + o(dt)\right)^{\frac{1}{1-\gamma_h}} \\
&= \widehat{U}_{h,t}\left(1 + E_t^{\mathbb{Q}_Y^{\boldsymbol{\nu}_h}}\left[\frac{d\widehat{U}_{h,t}}{\widehat{U}_{h,t}}\right] - \frac{1}{2}\gamma_h E_t^{\mathbb{Q}_Y^{\boldsymbol{\nu}_h}}\left[\left(\frac{d\widehat{U}_{h,t}}{\widehat{U}_{h,t}}\right)^2\right] + o(dt)\right).
\end{aligned}$$

Therefore

$$\widehat{U}_{h,t}^{1-\frac{1}{\psi_h}} = (1 - e^{-\delta_h dt})\widehat{C}_{h,t}^{1-\frac{1}{\psi_h}}$$

$$+ e^{-\delta_h dt}\widehat{U}_{h,t}^{1-\frac{1}{\psi_h}} e^{\left(1-\frac{1}{\psi_h}\right)\left(\mu_{Y,h,t}-\frac{1}{2}\gamma_h\sigma_Y^2\right)dt}\left(1+E_t^{\mathbb{Q}_Y^{\boldsymbol{\nu}_h}}\left[\frac{d\widehat{U}_{h,t}}{\widehat{U}_{h,t}}\right]-\frac{1}{2}\gamma_h E_t^{\mathbb{Q}_Y^{\boldsymbol{\nu}_h}}\left[\left(\frac{d\widehat{U}_{h,t}}{\widehat{U}_{h,t}}\right)^2\right]+o(dt)\right)^{1-\frac{1}{\psi_h}}$$

$$0 = \delta_h \left(\frac{\widehat{C}_{h,t}}{\widehat{U}_{h,t}}\right)^{1-\frac{1}{\psi_h}} dt$$

$$- k_{1h,t}dt + \left(1-\frac{1}{\psi_h}\right)\left(E_t^{\mathbb{Q}_Y^{\boldsymbol{\nu}_h}}\left[\frac{d\widehat{U}_{h,t}}{\widehat{U}_{h,t}}\right]-\frac{1}{2}\gamma_h E_t^{\mathbb{Q}_Y^{\boldsymbol{\nu}_h}}\left[\left(\frac{d\widehat{U}_{h,t}}{\widehat{U}_{h,t}}\right)^2\right]+L_{h,t}dt\right)$$

$$+ o(dt),$$

where

$$k_{1h,t} = \delta_h + \frac{1}{\psi_h}\mu_{Y,h,t} - \frac{1}{2}\left(1+\frac{1}{\psi_h}\right)\gamma_h\sigma_Y^2 + \gamma_h\sigma_Y^2 - \mu_{Y,h,t}.$$

Hence, in the continuous time limit, we obtain

$$0 = \delta_h \left(\frac{\widehat{C}_{h,t}}{\widehat{U}_{h,t}}\right)^{1-\frac{1}{\psi_h}}$$

$$- k_{1h,t} + \left(1-\frac{1}{\psi_h}\right)\frac{1}{dt}\left(E_t^{\mathbb{Q}_Y^{\boldsymbol{\nu}_h}}\left[\frac{d\widehat{U}_{h,t}}{\widehat{U}_{h,t}}\right]-\frac{1}{2}\gamma_h E_t^{\mathbb{Q}_Y^{\boldsymbol{\nu}_h}}\left[\left(\frac{d\widehat{U}_{h,t}}{\widehat{U}_{h,t}}\right)^2\right]+L_{h,t}dt\right),$$

which can be rewritten as

$$\delta_h u_h\left(\frac{\widehat{C}_{h,t}}{\widehat{U}_{h,t}}\right) + \frac{\delta_h - k_{1h,t}}{1-\frac{1}{\psi}_h} + \frac{1}{dt}\left(E_t^{\mathbb{Q}_Y^{\boldsymbol{\nu}_h}}\left[\frac{d\widehat{U}_{h,t}}{\widehat{U}_{h,t}}\right]-\frac{1}{2}\gamma_h E_t^{\mathbb{Q}_Y^{\boldsymbol{\nu}_h}}\left[\left(\frac{d\widehat{U}_{h,t}}{\widehat{U}_{h,t}}\right)^2\right]+L_{h,t}dt\right).$$

It follows from Girsanov's Theorem that under probability measure $\mathbb{Q}_Y^{\boldsymbol{\nu}_h}$, we have

$$\frac{d\widehat{W}_{h,t}}{\widehat{W}_{h,t}} = \left[k_{2h,t} + \pi_{h,t}(\alpha + \boldsymbol{x}_h^\top\boldsymbol{\nu}_h - \gamma_h\rho_{YK}\sigma\sigma_Y(1+\epsilon x_{hh,t}) - i) - \frac{\widehat{C}_{h,t}}{\widehat{W}_{h,t}} + \frac{1}{\widehat{W}_{h,t}}\right]dt$$
$$+ (\sigma\pi_{h,t}\boldsymbol{x}_{h,t}^\top, -\sigma_Y\boldsymbol{e}_h^\top)((d\boldsymbol{Z}_t^{\boldsymbol{\nu}})^\top, d\boldsymbol{Z}_{Y,t}^\top)^\top,$$

where

$$k_{2h,t} = i + \gamma_h\sigma_Y^2 - \mu_{Y,h,t}.$$

Hence

$$\frac{1}{dt}E_t^{\mathbb{Q}_Y^{\boldsymbol{\nu}_h}}\left[\left(\frac{d\widehat{W}_{h,t}}{\widehat{W}_{h,t}}\right)^2\right] = \sigma^2\pi_{h,t}^2\boldsymbol{x}_{h,t}^\top\Omega\boldsymbol{x}_{h,t} - 2\rho_{YK}\sigma_Y\sigma\pi_{h,t}\boldsymbol{x}_{h,t}^\top(J_N+\epsilon I_N)\boldsymbol{e}_h + \sigma_Y^2\boldsymbol{e}_h^\top I_N\boldsymbol{e}_h$$
$$= \sigma^2\pi_{h,t}^2\boldsymbol{x}_{h,t}^\top\Omega\boldsymbol{x}_{h,t} - 2\rho_{YK}\sigma_Y\sigma\pi_{h,t}(1+\epsilon x_{hh,t}) + \sigma_Y^2.$$

Also,

$$E_t^{\mathbb{Q}_Y^{\boldsymbol{\nu}_h}}\left[\frac{d\widehat{U}_{h,t}}{\widehat{U}_{h,t}}\right] = \frac{\widehat{W}_{h,t}\widehat{U}_{ht,\widehat{W}_h}}{\widehat{U}_{h,t}}E_t^{\mathbb{Q}_Y^{\boldsymbol{\nu}_h}}\left[\frac{d\widehat{W}_{h,t}}{\widehat{W}_{h,t}}\right] + \frac{1}{2}\frac{\widehat{W}_{h,t}^2\widehat{U}_{ht,\widehat{W}_h\widehat{W}_h}}{\widehat{U}_{h,t}}E_t^{\mathbb{Q}_Y^{\boldsymbol{\nu}_h}}\left[\left(\frac{d\widehat{W}_{h,t}}{\widehat{W}_{h,t}}\right)^2\right]$$

$$E_t^{\mathbb{Q}_Y^{\boldsymbol{\nu}_h}}\left[\left(\frac{d\widehat{U}_{h,t}}{\widehat{U}_{h,t}}\right)^2\right] = \left(\frac{\widehat{W}_{h,t}\widehat{U}_{ht,\widehat{W}_h}}{\widehat{U}_{h,t}}\right)^2 E_t^{\mathbb{Q}_Y^{\boldsymbol{\nu}_h}}\left[\left(\frac{d\widehat{W}_{h,t}}{\widehat{W}_{h,t}}\right)^2\right],$$

and so

$$E_t^{\mathbb{Q}_Y^{\boldsymbol{\nu}_h}}\left[\frac{d\widehat{U}_{h,t}}{\widehat{U}_{h,t}}\right] - \frac{1}{2}\gamma_h E_t^{\mathbb{Q}_Y^{\boldsymbol{\nu}_h}}\left[\left(\frac{d\widehat{U}_{h,t}}{\widehat{U}_{h,t}}\right)^2\right]$$

$$= \frac{\widehat{W}_{h,t}\widehat{U}_{ht,\widehat{W}_h}}{\widehat{U}_{h,t}}E_t^{\mathbb{Q}_Y^{\boldsymbol{\nu}_h}}\left[\frac{d\widehat{W}_{h,t}}{\widehat{W}_{h,t}}\right] + \frac{1}{2}\left(\frac{\widehat{W}_{h,t}^2\widehat{U}_{ht,\widehat{W}_h\widehat{W}_h}}{\widehat{U}_{h,t}} - \gamma_h\left(\frac{\widehat{W}_{h,t}\widehat{U}_{ht,\widehat{W}_h}}{\widehat{U}_{h,t}}\right)^2\right)E_t^{\mathbb{Q}_Y^{\boldsymbol{\nu}_h}}\left[\left(\frac{d\widehat{W}_{h,t}}{\widehat{W}_{h,t}}\right)^2\right]$$

$$= \frac{\widehat{W}_{h,t}\widehat{U}_{ht,\widehat{W}_h}}{\widehat{U}_{h,t}}\left[k_{2h,t} + \pi_{h,t}(\alpha + \boldsymbol{x}_h^\top\boldsymbol{\nu}_h - \gamma_h\rho_{YK}\sigma\sigma_Y(1+\epsilon x_{hh,t}) - i) - \frac{\widehat{C}_{h,t}}{\widehat{W}_{h,t}} + \frac{1}{\widehat{W}_{h,t}}\right]dt$$

$$+ \frac{1}{2}\left(\frac{\widehat{W}_{h,t}^2\widehat{U}_{ht,\widehat{W}_h\widehat{W}_h}}{\widehat{U}_{h,t}} - \gamma_h\left(\frac{\widehat{W}_{h,t}\widehat{U}_{ht,\widehat{W}_h}}{\widehat{U}_{h,t}}\right)^2\right)(\sigma^2\pi_{h,t}^2\boldsymbol{x}_{h,t}^\top\Omega\boldsymbol{x}_{h,t} - 2\rho_{YK}\sigma_Y\sigma\pi_{h,t}(1+\epsilon x_{hh,t}) + \sigma_Y^2)dt.$$

Therefore, we obtain (B1). ■

Proposition B.2.2 *The FOC's of the Hamilton-Jacobi-Bellman equation* (B1) *give the following expressions for the optimal controls in terms of the normalized value function* $\widehat{U}_{h,t}$*:*

1. *The optimal consumption-wealth ratio* $\widehat{c}_{h,t}$ *is given by*

$$\widehat{c}_{h,t} = \frac{\widehat{C}_{h,t}}{\widehat{W}_{h,t}} = \delta_h^{\psi_h}\left(\frac{\widehat{U}_{h,t}}{\widehat{W}_{h,t}}\right)^{1-\psi_h}\left(\frac{\widehat{W}_{h,t}\widehat{U}_{ht,\widehat{W}_h}}{\widehat{U}_{h,t}}\right)^{-\psi_h}.$$

2. *The optimal portfolio policy is given by $\boldsymbol{\omega}_{h,t} = \pi_{h,t}\boldsymbol{x}_{h,t}$, where*

$$\boldsymbol{\omega}_{h,t} = \Psi_{h,t}^{-1}\left[\frac{1}{\widehat{\gamma}_{h,t}}\frac{\alpha - i}{\sigma^2}\mathbf{1} - \beta\left(\frac{\gamma_h}{\widehat{\gamma}_{h,t}} - 1\right)(\mathbf{1} + \epsilon\boldsymbol{e}_h)\right],$$

and

$$\beta = \frac{\rho_{YK}\sigma_Y\sigma}{\sigma^2}$$

$$\widehat{\gamma}_{h,t} = \gamma_h\frac{\widehat{W}_{h,t}\widehat{U}_{ht,\widehat{W}_h}}{\widehat{U}_{h,t}} + \frac{-\frac{\widehat{W}_{h,t}^2\widehat{U}_{ht,\widehat{W}_h\widehat{W}_h}}{\widehat{U}_{h,t}}}{\frac{\widehat{W}_{h,t}\widehat{U}_{ht,\widehat{W}_h}}{\widehat{U}_{h,t}}},$$

$$\Psi_{h,t} = \Omega + \frac{\gamma_h}{\widehat{\gamma}_{h,t}}\frac{\widehat{W}_{h,t}\widehat{U}_{ht,\widehat{W}_h}}{\widehat{U}_{h,t}}\Gamma_h.$$

3. *The optimal adjustment to the vector of expected returns is given by $\boldsymbol{\nu}_{h,t}$, where*

$$\boldsymbol{\nu}_{h,t} = -\left(I + \widehat{\gamma}_{h,t}\left(\gamma_h\frac{\widehat{W}_h\widehat{U}_{ht,\widehat{W}_h}}{\widehat{U}_{h,t}}\right)^{-1}\Omega\Gamma_h^{-1}\right)^{-1}\left[(\alpha - i)\mathbf{1} - \rho_{YK}\sigma\sigma_K\left(\gamma_h - \widehat{\gamma}_{h,t}\right)(\mathbf{1} + \epsilon\boldsymbol{e}_h)\right].$$

Proof: The FOC for consumption can be solved to give

$$\widehat{c}_{h,t} = \frac{\widehat{C}_{h,t}}{\widehat{W}_{h,t}} = \delta_h^{\psi_h}\left(\frac{\widehat{U}_{h,t}}{\widehat{W}_{h,t}}\right)^{1-\psi_h}\left(\frac{\widehat{W}_{h,t}\widehat{U}_{ht,\widehat{W}_h}}{\widehat{U}_{h,t}}\right)^{-\psi_h},$$

and so using the optimal consumption choice we obtain

$$\delta_h u_{\psi_h}\left(\frac{\widehat{C}_{h,t}}{\widehat{U}_{h,t}}\right) - \frac{\widehat{W}_{h,t}\widehat{U}_{ht,\widehat{W}_h}}{\widehat{U}_{h,t}}\frac{\widehat{C}_t}{\widehat{W}_t} = \frac{\frac{1}{\psi_h}\delta^{\psi_h}(\widehat{U}_{ht,\widehat{W}_h})^{1-\psi_h} - \delta_h}{1 - \frac{1}{\psi_h}}.$$

Hence

$$\frac{\delta^{\psi_h}(\widehat{U}_{ht,\widehat{W}_h})^{1-\psi_h} - \psi_h\delta_h}{\psi_h - 1} + \frac{\psi_h\delta_h - \psi_h k_{1h,t}}{\psi_h - 1}$$

$$+ \frac{\widehat{W}_{h,t}\widehat{U}_{ht,\widehat{W}_h}}{\widehat{U}_{h,t}}\left[k_{2h,t} + \pi_{h,t}(\alpha + \boldsymbol{x}_h^\top\boldsymbol{\nu}_h - \gamma_h\rho_{YK}\sigma\sigma_Y(1 + \epsilon x_{hh,t}) - i)) + \frac{1}{\widehat{W}_{h,t}}\right]$$

$$+\frac{1}{2}\left(\frac{\widehat{W}_{h,t}^2\widehat{U}_{ht,\widehat{W}_h\widehat{W}_h}}{\widehat{U}_{h,t}}-\gamma_h\left(\frac{\widehat{W}_{h,t}\widehat{U}_{ht,\widehat{W}_h}}{\widehat{U}_{h,t}}\right)^2\right)(\sigma^2\pi_{h,t}^2\boldsymbol{x}_{h,t}^\top\Omega\boldsymbol{x}_{h,t}-2\rho_{YK}\sigma_Y\sigma\pi_{h,t}(1+\epsilon x_{hh,t})+\sigma_Y^2)$$

$$+\frac{1}{2\gamma_h}\frac{\boldsymbol{\nu}_{h,t}^\top\Gamma_h^{-1}\boldsymbol{\nu}_{h,t}}{\sigma^2}.$$

The FOC for $\boldsymbol{\nu}_{h,t}$ can be solved to give

$$\boldsymbol{\nu}_{h,t}=-\gamma_h\sigma^2\frac{\widehat{W}_h\widehat{U}_{ht,\widehat{W}_h}}{\widehat{U}_{h,t}}\pi_{h,t}\Gamma_h\boldsymbol{x}_{h,t},$$

and so

$$\begin{aligned}
&\frac{\delta^{\psi_h}(\widehat{U}_{ht,\widehat{W}_h})^{1-\psi_h}-\psi_h\delta_h}{\psi_h-1}+\frac{\psi_h\delta_h-\psi_hk_{1h,t}}{\psi_h-1}\\
&+\frac{\widehat{W}_{h,t}\widehat{U}_{ht,\widehat{W}_h}}{\widehat{U}_{h,t}}\left[k_{2h,t}+\pi_{h,t}(\alpha-\gamma_h\rho_{YK}\sigma_Y\sigma\pi_{h,t}(1+\epsilon x_{hh,t})-i)+\frac{1}{\widehat{W}_{h,t}}\right]\\
&+\frac{1}{2}\frac{\widehat{W}_{h,t}^2\widehat{U}_{ht,\widehat{W}_h\widehat{W}_h}}{\widehat{U}_{h,t}}(\sigma^2\pi_{h,t}^2\boldsymbol{x}_{h,t}^\top\Omega\boldsymbol{x}_{h,t}-2\rho_{YK}\sigma_Y\sigma\pi_{h,t}(1+\epsilon x_{hh,t})+\sigma_Y^2)\\
&-\frac{1}{2}\gamma_h\left(\frac{\widehat{W}_{h,t}\widehat{U}_{ht,\widehat{W}_h}}{\widehat{U}_{h,t}}\right)^2(\sigma^2\pi_{h,t}^2\boldsymbol{x}_{h,t}^\top(\Omega+\Gamma_h)\boldsymbol{x}_{h,t}-2\rho_{YK}\sigma_Y\sigma\pi_{h,t}(1+\epsilon x_{hh,t})+\sigma_Y^2).
\end{aligned}$$

We now rewrite the above expression as

$$\begin{aligned}
&\frac{\delta^{\psi_h}(\widehat{U}_{ht,\widehat{W}_h})^{1-\psi_h}-\psi_hk_{1h,t}}{\psi_h-1}\\
&+\frac{\widehat{W}_{h,t}\widehat{U}_{ht,\widehat{W}_h}}{\widehat{U}_{h,t}}\Bigg[k_{2h,t}+\frac{1}{\widehat{W}_{h,t}}+\pi_{h,t}(\alpha-i)-(\gamma_h-\widehat{\gamma}_{h,t})\rho_{YK}\sigma_Y\sigma\pi_{h,t}(1+\epsilon x_{hh,t})\\
&-\frac{1}{2}\widehat{\gamma}_{h,t}(\sigma^2\pi_{h,t}^2\boldsymbol{x}_{h,t}^\top\Psi_{h,t}\boldsymbol{x}_{h,t}+\sigma_Y^2)\Bigg],
\end{aligned}$$

where

$$\widehat{\gamma}_{h,t}=\gamma_h\frac{\widehat{W}_{h,t}\widehat{U}_{ht,\widehat{W}_h}}{\widehat{U}_{h,t}}+\frac{-\frac{\widehat{W}_{h,t}^2\widehat{U}_{ht,\widehat{W}_h\widehat{W}_h}}{\widehat{U}_{h,t}}}{\frac{\widehat{W}_{h,t}\widehat{U}_{ht,\widehat{W}_h}}{\widehat{U}_{h,t}}},$$

$$\Psi_{h,t}=\Omega+\frac{\gamma_h}{\widehat{\gamma}_{h,t}}\frac{\widehat{W}_{h,t}\widehat{U}_{ht,\widehat{W}_h}}{\widehat{U}_{h,t}}\Gamma_h.$$

If $\widehat{U}_{h,t}$ is a concave function of $\widehat{W}_{h,t}$, then the optimal portfolio is given by the following optimization problem:

$$\sup_{\boldsymbol{\omega}_{h,t}} (\alpha - i)\mathbf{1}^\top \boldsymbol{\omega}_{h,t} - (\gamma_h - \widehat{\gamma}_{h,t})\rho_{YK}\sigma_Y\sigma(\mathbf{1}^\top \boldsymbol{\omega}_{h,t} + \epsilon\omega_{hh,t}) - \frac{1}{2}\widehat{\gamma}_{h,t}\sigma^2\boldsymbol{\omega}_{h,t}^\top \Psi_{h,t}\boldsymbol{\omega}_{h,t}.$$

The FOC for the optimal portfolio policy $\boldsymbol{\omega}_{h,t}$ is therefore

$$\sigma^2\widehat{\gamma}_{h,t}\Psi_{h,t}\boldsymbol{\omega}_{h,t} = (\alpha - i)\mathbf{1} - (\gamma_h - \widehat{\gamma}_{h,t})\rho_{YK}\sigma_Y\sigma(\mathbf{1} + \epsilon\boldsymbol{e}_h),$$

and so

$$\boldsymbol{\omega}_{h,t} = \Psi_{h,t}^{-1}\left[\frac{1}{\widehat{\gamma}_{h,t}}\frac{\alpha - i}{\sigma^2}\mathbf{1} - \beta\left(\frac{\gamma_h}{\widehat{\gamma}_{h,t}} - 1\right)(\mathbf{1} + \epsilon\boldsymbol{e}_h)\right],$$

where

$$\beta = \frac{\rho_{YK}\sigma_Y\sigma}{\sigma^2}.$$

It follows that

$$\begin{aligned}
&\sup_{\boldsymbol{\omega}_{h,t}} (\alpha - i)\mathbf{1}^\top \boldsymbol{\omega}_{h,t} - (\gamma_h - \widehat{\gamma}_{h,t})\rho_{YK}\sigma_Y\sigma(\mathbf{1}^\top \boldsymbol{\omega}_{h,t} + \epsilon\omega_{hh,t}) - \frac{1}{2}\widehat{\gamma}_{h,t}\sigma^2\boldsymbol{\omega}_{h,t}^\top \Psi_{h,t}\boldsymbol{\omega}_{h,t} \\
&= \frac{1}{2}\widehat{\gamma}_{h,t}\sigma^2\boldsymbol{\omega}_{h,t}^\top \Psi_{h,t}\boldsymbol{\omega}_{h,t}.
\end{aligned}$$

Therefore at the optimum

$$0 = \frac{\delta^{\psi_h}(\widehat{U}_{ht,\widehat{W}_h})^{1-\psi_h} - \psi_h k_{1h,t}}{\psi_h - 1} + \frac{\widehat{W}_{h,t}\widehat{U}_{ht,\widehat{W}_h}}{\widehat{U}_{h,t}}\left[k_{2h,t} + \frac{1}{\widehat{W}_{h,t}} + \frac{1}{2}\widehat{\gamma}_{h,t}(\sigma^2\pi_{h,t}^2\boldsymbol{x}_{h,t}^\top\Psi_{h,t}\boldsymbol{x}_{h,t} - \sigma_Y^2)\right].$$

Consequently,

$$\begin{aligned}
\boldsymbol{\nu}_{h,t} &= -\gamma_h\sigma^2\frac{\widehat{W}_h\widehat{U}_{ht,\widehat{W}_h}}{\widehat{U}_{h,t}}\Gamma_h\Psi_{h,t}^{-1}\left[\frac{1}{\widehat{\gamma}_{h,t}}\frac{\alpha - i}{\sigma^2}\mathbf{1} - \beta\left(\frac{\gamma_h}{\widehat{\gamma}_{h,t}} - 1\right)(\mathbf{1} + \epsilon\boldsymbol{e}_h)\right] \\
&= -\gamma_h\sigma^2\frac{\widehat{W}_h\widehat{U}_{ht,\widehat{W}_h}}{\widehat{U}_{h,t}}\left(\frac{\gamma_h}{\widehat{\gamma}_{h,t}}\frac{\widehat{W}_{h,t}\widehat{U}_{ht,\widehat{W}_h}}{\widehat{U}_{h,t}}I + \Omega\Gamma_h^{-1}\right)^{-1}\left[\frac{1}{\widehat{\gamma}_{h,t}}\frac{\alpha - i}{\sigma^2}\mathbf{1} - \beta\left(\frac{\gamma_h}{\widehat{\gamma}_{h,t}} - 1\right)(\mathbf{1} + \epsilon\boldsymbol{e}_h)\right] \\
&= -\gamma_h\sigma^2\left(\frac{\gamma_h}{\widehat{\gamma}_{h,t}}I + \left(\frac{\widehat{W}_h\widehat{U}_{ht,\widehat{W}_h}}{\widehat{U}_{h,t}}\right)^{-1}\Omega\Gamma_h^{-1}\right)^{-1}\left[\frac{1}{\widehat{\gamma}_{h,t}}\frac{\alpha - i}{\sigma^2}\mathbf{1} - \beta\left(\frac{\gamma_h}{\widehat{\gamma}_{h,t}} - 1\right)(\mathbf{1} + \epsilon\boldsymbol{e}_h)\right]
\end{aligned}$$

$$= -\left(I + \widehat{\gamma}_{h,t}\left(\gamma_h \frac{\widehat{W}_h \widehat{U}_{ht,\widehat{W}_h}}{\widehat{U}_{h,t}}\right)^{-1} \Omega\Gamma_h^{-1}\right)^{-1} \left[(\alpha - i)\mathbf{1} - \rho_{YK}\sigma\sigma_K\left(\gamma_h - \widehat{\gamma}_{h,t}\right)\left(\mathbf{1} + \epsilon \boldsymbol{e}_h\right)\right].$$

■

Proposition B.2.3 *If we look for an approximate loglinear solution of the form*

$$\widehat{U}_{h,t} = \kappa_h \widehat{W}_{h,t}^{a_h},$$

then the optimal consumption-wealth ratio is given by

$$\widehat{c}_{h,t} = \frac{\widehat{C}_{h,t}}{\widehat{W}_{h,t}} = \delta_h^{\psi_h} \kappa_h^{1-\psi_h} a_h^{-\psi_h} \widehat{W}_{h,t}^{(1-a_h)(\psi_h - 1)},$$

and the optimal portfolio policy by

$$\boldsymbol{\omega}_{h,t} = \boldsymbol{\omega}_h$$

$$= \Psi_h^{-1}\left[\frac{1}{\widehat{\gamma}_h}\frac{\alpha - i}{\sigma^2}\mathbf{1} - \beta\left(\frac{\gamma_h}{\widehat{\gamma}_h} - 1\right)\left(\mathbf{1} + \epsilon \boldsymbol{e}_h\right)\right], \tag{B2}$$

where

$$\widehat{\gamma}_h = a_h \gamma_h + 1 - a_h,$$

$$\Psi_h = \Omega + \frac{\gamma_h a_h}{\widehat{\gamma}_h}\Gamma_h.$$

Also the vector of adjustments to expected returns is given by

$$\boldsymbol{\nu}_{h,t} = \boldsymbol{\nu}_h$$

$$= -\left(I + \frac{\widehat{\gamma}_h}{\gamma_h a_h}\Omega\Gamma_h^{-1}\right)^{-1} \left[(\alpha - i)\mathbf{1} - \rho_{YK}\sigma_Y\sigma\left(\gamma_h - \widehat{\gamma}_h\right)\left(\mathbf{1} + \epsilon \boldsymbol{e}_h\right)\right].$$

Furthermore,

$$\kappa_h = \left[\left(\frac{\delta_h}{a_h}\right)^{\psi_h} \frac{1}{\widehat{c}_h^*}\right]^{\frac{1}{\psi_h - 1}} (\widehat{W}_h^*)^{1-a_h},$$

where

$$\widehat{c}_h^* = \frac{-\tau_1}{a_h} + \frac{1}{1-a_h}(\widehat{W}_h^*)^{-1}.$$

and a_h and $\widehat{W}_h^*$ can be determined in terms of exogenous variables by solving

$$\frac{-\tau_1}{a_h} + \frac{1}{1-a_h}(W_h^*)^{-1} = \frac{1}{\widehat{W}_h^*} + k_{2h}^*$$

$$
\begin{aligned}
&\quad + \widehat{\gamma}_h \left[\sigma^2 \pi_h^2 \boldsymbol{x}_h^\top \Omega \boldsymbol{x}_h - \rho_{YK}\sigma_Y \sigma \pi_h (1+\epsilon x_{hh})\right], \\
-\tau_1 + \frac{a_h}{1-a_h}(\widehat{W}_h^*)^{-1} &= \psi_h k_{1h}^* \\
&\quad + (1-\psi_h) a_h \left[k_{2h}^* + (\widehat{W}_h^*)^{-1} + \frac{1}{2}\widehat{\gamma}_h(\pi_h^2 \sigma^2 \boldsymbol{x}_h^\top \Psi_h \boldsymbol{x}_h - \sigma_Y^2)\right],
\end{aligned}
$$

where $k_{ih}^* = k_{ih,t}|_{\widehat{W}_{h,t}=\widehat{W}_h^*}$, $i \in \{1,2\}$.

The ratio of utility to wealth when $\widehat{W}_{h,t} = \widehat{W}_h^*$ *is given by*

$$
\frac{U_h^*}{W_h^*} = \left[\left(\frac{\delta_h}{a_h}\right)^{\psi_h} \frac{1}{\widehat{c}_h^*}\right]^{\frac{1}{\psi_h - 1}}.
$$

Note that $\widehat{W}_h^*$ *is the level of the wealth-income ratio such that*

$$
E_t^{\mathbb{Q}_Y^{\boldsymbol{\nu}_h}}\left[\frac{d\widehat{W}_{h,t}}{\widehat{W}_{h,t}}\right] = 0,
$$

where for an event A *realized at date* T

$$
\mathbb{Q}_Y^{\boldsymbol{\nu}_h}(A|\mathcal{F}_t) = E_t^{\mathbb{Q}^{\boldsymbol{\nu}_h}}\left[\frac{M_{Y,h,T}}{M_{Y,h,t}} 1_A\right],
$$

and

$$
M_{Y,h,t} = e^{-\frac{1}{2}(1-\gamma_h)^2 \sigma_Y^2 t + (1-\gamma_h)\sigma_Y Z_{Y,h,t}}.
$$

Proof: If we look for an approximate loglinear solution of the form

$$
\widehat{U}_{h,t} = \kappa_h \widehat{W}_{h,t}^{a_h},
$$

we see that the optimal consumption policy is given by

$$
\widehat{c}_{h,t} = \frac{\widehat{C}_{h,t}}{\widehat{W}_{h,t}} = \delta_h^{\psi_h} \kappa_h^{1-\psi_h} a_h^{-\psi_h} \widehat{W}_{h,t}^{(1-a_h)(\psi_h - 1)},
$$

and the optimal portfolio policy by

$$
\begin{aligned}
\boldsymbol{\omega}_{h,t} &= \boldsymbol{\omega}_h \\
&= \Psi_h^{-1}\left[\frac{1}{\widehat{\gamma}_h}\frac{\alpha - i}{\sigma^2}\mathbf{1} - \beta\left(\frac{\gamma_h}{\widehat{\gamma}_h} - 1\right)(\mathbf{1} + \epsilon \boldsymbol{e}_h)\right].
\end{aligned}
$$

where

$$
\widehat{\gamma}_h = a_h \gamma_h + 1 - a_h,
$$

$$\Psi_h = \Omega + \frac{\gamma_h a_h}{\widehat{\gamma}_h}\Gamma_h.$$

If we define

$$\boldsymbol{b}_h = \Psi_h^{-1}\mathbf{1}$$
$$\overline{\boldsymbol{b}}_h = \Psi_h^{-1}\boldsymbol{e}_h,$$

then

$$\boldsymbol{\omega}_h = \frac{1}{\widehat{\gamma}_h}\frac{\alpha - i}{\sigma^2}\boldsymbol{b}_h - \beta\left(\frac{\gamma_h}{\widehat{\gamma}_h} - 1\right)(\boldsymbol{b}_h + \epsilon\overline{\boldsymbol{b}}_h).$$

Also

$$\begin{aligned}
\boldsymbol{\nu}_{h,t} &= \boldsymbol{\nu}_h \\
&= -a_h\gamma_h\sigma^2\Gamma_h\boldsymbol{\omega}_h \\
&= -a_h\gamma_h\sigma^2\Gamma_h\Psi_h^{-1}\left[\frac{1}{\widehat{\gamma}_h}\frac{\alpha - i}{\sigma^2}\mathbf{1} - \beta\left(\frac{\gamma_h}{\widehat{\gamma}_h} - 1\right)(\mathbf{1} + \epsilon\boldsymbol{e}_h)\right] \\
&= -\gamma_h a_h\sigma^2\left(\frac{\gamma_h}{\widehat{\gamma}_h}a_h I + \Omega\Gamma_h^{-1}\right)^{-1}\left[\frac{1}{\widehat{\gamma}_h}\frac{\alpha - i}{\sigma^2}\mathbf{1} - \beta\left(\frac{\gamma_h}{\widehat{\gamma}_h} - 1\right)(\mathbf{1} + \epsilon\boldsymbol{e}_h)\right] \\
&= -\sigma^2\left(\frac{1}{\widehat{\gamma}_h}I + \frac{1}{\gamma_h a_h}\Omega\Gamma_h^{-1}\right)^{-1}\left[\frac{1}{\widehat{\gamma}_h}\frac{\alpha - i}{\sigma^2}\mathbf{1} - \beta\left(\frac{\gamma_h}{\widehat{\gamma}_h} - 1\right)(\mathbf{1} + \epsilon\boldsymbol{e}_h)\right] \\
&= -\widehat{\gamma}_h\sigma^2\left(I + \frac{\widehat{\gamma}_h}{\gamma_h a_h}\Omega\Gamma_h^{-1}\right)^{-1}\left[\frac{1}{\widehat{\gamma}_h}\frac{\alpha - i}{\sigma^2}\mathbf{1} - \beta\left(\frac{\gamma_h}{\widehat{\gamma}_h} - 1\right)(\mathbf{1} + \epsilon\boldsymbol{e}_h)\right] \\
&= -\left(I + \frac{\widehat{\gamma}_h}{\gamma_h a_h}\Omega\Gamma_h^{-1}\right)^{-1}\left[(\alpha - i)\mathbf{1} - \rho_{YK}\sigma_Y\sigma\left(\gamma_h - \widehat{\gamma}_h\right)(\mathbf{1} + \epsilon\boldsymbol{e}_h)\right].
\end{aligned}$$

Hence,

$$a_h\widehat{c}_h = \psi_h k_{1h,t} + (1 - \psi_h)a_h\left\{k_{2h,t} + \frac{1}{\widehat{W}_{h,t}} + \frac{1}{2}\widehat{\gamma}_h(\sigma^2\pi_{h,t}^2\boldsymbol{x}_{h,t}^\top\Psi_h\boldsymbol{x}_{h,t} - \sigma_Y^2)\right\}.$$

The steady-state value of the wealth-labor income ratio, $\widehat{W}_h^*$, is defined by

$$E_t^{\mathbb{Q}_Y^{\boldsymbol{\nu}_h}}\left[\frac{d\widehat{W}_{h,t}}{\widehat{W}_{h,t}}\right]\Bigg|_{\widehat{W}_{h,t} = \widehat{W}_h^*} = 0,$$

so we see that at the steady-state, where variables are denoted by an $*$, we have

$$\hat{c}_h^*(\widehat{W}_h^*, a_h) = k_{2h}^* + \pi_h(a_h)(\alpha + \boldsymbol{x}_h(a_h)^\top\boldsymbol{\nu}_h(a_h) - \gamma_h\rho_{YK}\sigma\sigma_Y(1 + x_{hh}(a_h)) - i) + \frac{1}{\widehat{W}_h^*},$$

which using the expression for the optimal portfolio vector in (B2), can be rewritten as

(B3)

$$\hat{c}_h^*(\widehat{W}_h^*, a_h) = k_{2h}^* + \frac{1}{\widehat{W}_h^*} + \widehat{\gamma}_h(a_h)\left[\sigma^2\pi_h(a)^2\boldsymbol{x}_h(a_h)^\top\Omega\boldsymbol{x}_h(a_h) - \rho_{YK}\sigma_Y\sigma\pi_h(a_h)(1+\epsilon x_{hh}(a_h))\right]$$

Note also that

$$\hat{c}_h^*(\widehat{W}_h^*, a_h) = \frac{\widehat{C}_h^*}{\widehat{W}_h^*} = \delta_h^{\psi_h}\kappa_h^{1-\psi_h}a_h^{-\psi_h}(W_h^*(a_h))^{(1-a_h)(\psi_h-1)} \tag{B4}$$

$$k_{2h}^*(\widehat{W}_h^*) = i + \gamma_h\sigma_Y^2 - (\tau_0 + \tau_1\ln\widehat{W}_h^*)$$

Defining

$$\widehat{z}_{h,t} = \ln\left(\frac{\widehat{W}_{h,t}}{\widehat{W}_h^*}\right),$$

we see that at the steady state $\widehat{z}_{h,t} = 0$. At the optimum, we have

(B5)

$$a_h\hat{c}_{h,t}e^{(\psi_h-1)(1-a_h)\hat{z}_{h,t}} = \psi_h k_{1h,t} + (1-\psi_h)a_h\left[k_{2h,t} + \frac{1}{\widehat{W}_{h,t}} + \frac{1}{2}\widehat{\gamma}_{h,t}(\sigma^2\pi_{h,t}^2\boldsymbol{x}_{h,t}^\top\Psi_{h,t}\boldsymbol{x}_{h,t} - \sigma_Y^2)\right].$$

If we expand (B5) around $\widehat{z}_{h,t} = 0$, we obtain

$$a_h\hat{c}_h^*(\widehat{W}_h^*, a_h)[1 + (1-a_h)(\psi_h - 1)\widehat{z}_{h,t}]$$

$$= \psi_h k_{1h,t} + (1-\psi_h)a_h\left[k_{2h,t} + (W_h^*)^{-1}(1-\widehat{z}_{h,t}) + \frac{1}{2}\widehat{\gamma}_h(\pi_h^2\sigma^2\boldsymbol{x}_h(a_h)^\top\Psi_h\boldsymbol{x}_h(a_h) - \sigma_Y^2)\right] + O(\widehat{z}_h^2).$$

By comparing coefficients of $\widehat{z}_h$, we obtain

$$\begin{aligned} a_h\hat{c}_h^*(\widehat{W}_h^*, a_h) &= \psi_h k_{1h}^*(W_h^*) \\ &\quad + (1-\psi_h)a_h\left[k_{2h}^*(W_h^*) + (W_h^*)^{-1} + \frac{1}{2}\widehat{\gamma}_h(\pi_h^2\sigma^2\boldsymbol{x}_h(a_h)^\top\Psi_h\boldsymbol{x}_h(a_h) - \sigma_Y^2)\right], \end{aligned} \tag{B6}$$

(B7)

$$-(1-a_h)a_h\hat{c}_h^*(\widehat{W}_h^*, a_h) = \tau_1(1-a_h) - a_h(W_h^*)^{-1}$$

Rearranging (B7), we have

$$\hat{c}_h^*(\widehat{W}_h^*, a_h) = \frac{-\tau_1}{a_h} + \frac{1}{1-a_h}(\widehat{W}_h^*)^{-1}. \tag{B8}$$

Using (B8), we rewrite (B3) and (B6) as

$$\frac{-\tau_1}{a_h} + \frac{1}{1-a_h}(W_h^*)^{-1} = \frac{1}{\widehat{W}^*} + k_2^*(\widehat{W}_h^*)$$

$$+ \widehat{\gamma}_h(a_h)\left[\sigma^2\pi_h(a)^2\boldsymbol{x}_h(a_h)^\top\Omega\boldsymbol{x}_h(a_h) - \rho_{YK}\sigma_Y\sigma\pi_h(a_h)(1+\epsilon x_{hh}(a_h))\right] \tag{B9}$$

$$-\tau_1 + \frac{a_h}{1-a_h}(\widehat{W}_h^*)^{-1} = \psi_h k_{1h}^*(\widehat{W}_h^*)$$

$$+ (1-\psi_h)a_h\left[k_{2h}^*(\widehat{W}_h^*) + (\widehat{W}_h^*)^{-1} + \frac{1}{2}\widehat{\gamma}_h(a)(\pi_h^2\sigma^2\boldsymbol{x}_h(a_h)^\top\Psi_h\boldsymbol{x}_h(a_h) - \sigma_Y^2)\right] \tag{B10}$$

To summarize, we can find an approximate loglinear solution by solving (B9) and (B10) numerically to obtain a and W^*. We can then use (B8) to obtain $\widehat{c}^*$, the steady-state consumption-wealth ratio. We can rearrange (B4) to obtain

$$\kappa_h = \left[\left(\frac{\delta_h}{a_h}\right)^{\psi_h}\frac{1}{\widehat{c}_h^*}\right]^{\frac{1}{\psi_h - 1}}(\widehat{W}_h^*)^{1-a_h}.$$

It follows that

$$\begin{aligned}\frac{U_h^*}{W_h^*} &= \frac{\widehat{U}_h^*}{\widehat{W}_h^*} \\ &= \kappa_h(\widehat{W}_h^*)^{1-a_h} \\ &= \left[\left(\frac{\delta_h}{a_h}\right)^{\psi_h}\frac{1}{\widehat{c}_h^*}\right]^{\frac{1}{\psi_h-1}}.\end{aligned}$$

■

Observe that $\widehat{W}_h^*$ is a stochastic steady state level of household wealth, which accounts for the long term pricing of risk as in Hansen and Scheinkman (2009), because the expected rate of change of wealth is zero at $\widehat{W}_{h,t} = \widehat{W}_h^*$ under the probability measure $\mathbb{Q}_Y^{\boldsymbol{\nu}_h}$, as opposed to under the reference probability measure $\mathbb{P}$. That is, it is the probability measure $\mathbb{Q}_Y^{\boldsymbol{\nu}_h}$ that adjusts for the long term pricing of risk.

Proposition B.2.4 *The utility of a household making biased consumption-portfolio choices is given by*

$$u_{e,h}^* = \frac{\widehat{U}_{e,h}^*}{\widehat{W}_{e,h}^*} = \left(\frac{\delta_h\psi_h}{\psi_h k_{e,1h}^* + (1-\psi_h)a_{e,h}\left(LQ_{e,h}^* - \widehat{c}_{e,h}^*\right)}\right)^{\frac{1}{1-\frac{1}{\psi_h}}}\widehat{c}_{e,h}^*,$$

where

$$
\begin{aligned}
LQ^*_{e,h} &= k^*_{e,2h} \\
&\quad + \pi_{d,h}(\alpha - \gamma_h \rho_{YK}\sigma\sigma_Y(1+\epsilon x_{d,hh}) - i) \\
&\quad + (\widehat{W}^*_{e,h})^{-1} - \frac{1}{2}\left[a_{e,h}\gamma_h + (1-a_{e,h})\right](\sigma^2\pi^2_{d,h}\boldsymbol{x}^\top_{d,h}\Omega\boldsymbol{x}_{d,h} - 2\rho_{YK}\sigma_Y\sigma\pi_{d,h}(1+\epsilon x_{d,hh}) + \sigma^2_Y).
\end{aligned}
$$

The constant $a_{e,h}$ is the solution of

$$
\begin{aligned}
&\delta_h\psi_h b_{e,h} \\
&= \psi_h k^*_{e,1h} \\
&+ (1-\psi_h)a_{e,h}\Big\{k^*_{e,2h} + \pi_{d,h}(\alpha - \gamma_h\rho_{YK}\sigma\sigma_Y(1+\epsilon x_{d,hh}) - i) - \widehat{c}^*_{e,h} + (\widehat{W}^*_{e,h})^{-1} \\
&-\frac{1}{2}\left[a_{e,h}\gamma_h + (1-a_{e,h})\right](\sigma^2\pi^2_{d,h}\boldsymbol{x}^\top_{d,h}\Omega\boldsymbol{x}_{d,h} - 2\rho_{YK}\sigma_Y\sigma\pi_{d,h}(1+\epsilon x_{d,hh}) + \sigma^2_Y)\Big\},
\end{aligned}
$$

where $b_{e,h}$. The constant $a_{d,h}$ together with $\pi_{d,h}$, $\boldsymbol{x}_{d,h}$ define the biased decisions of the household.

The asterisk $*$ indicates that all values are computed at the steady state $\widehat{W}_{h,t} = \widehat{W}_{h,t}$ defined by

$$
E_t^{\mathbb{P}_Y}\left[\frac{d\widehat{W}_{e,h,t}}{\widehat{W}_{e,h,t}}\right]\Bigg|_{\widehat{W}_{e,h,t}=\widehat{W}^*_{e,h}} = 0,
$$

where for an event A realized at date T

$$
\mathbb{P}_Y(A|\mathcal{F}_t) = E_t^{\mathbb{P}}\left[\frac{M_{Y,h,T}}{M_{Y,h,t}}1_A\right],
$$

and

$$
M_{Y,h,t} = e^{-\frac{1}{2}(1-\gamma_h)^2\sigma_Y^2 t + (1-\gamma_h)\sigma_Y Z_{Y,h,t}}.
$$

Proof: We start by computing welfare for a household without familiarity biases, using the consumption and portfolio policy for the household with familiarity biases. The consumption policy is of the form

$$
\widehat{c}_{e,h,t} = \frac{\widehat{C}_{e,h,t}}{\widehat{W}_{e,h,t}} = \delta_h^{\psi_h}\kappa_{d,h}^{1-\psi_h}a_{d,h}^{-\psi_h}\widehat{W}_{e,h,t}^{(1-a_{d,h})(\psi_h-1)},
$$

where $\kappa_{d,h}$ and $a_{d,h}$ have subscripts d to make it clear that they pin down the approximate optimal controls for a household with familiarity biases, but not for a household without such biases. Furthermore, $\widehat{c}_{e,h,t}$, $\widehat{C}_{e,h,t}$, and $\widehat{W}_{e,h,t}$ contain the subscript e to make it clear that they apply to a household without familiarity biases. Hence,

$$\widehat{c}_{e,h,t} = \frac{\widehat{C}_{e,h,t}}{\widehat{W}_{e,h,t}} = \delta_h^{\psi_h} \kappa_{d,h}^{1-\psi_h} a_{d,h}^{-\psi_h} (\widehat{W}^*_{e,h})^{(1-a_{d,h})(\psi_h-1)} e^{(1-a_{d,h})(\psi_h-1)\widehat{z}_{e,h,t}},$$

where $\widehat{W}^*_{e,h}$ is the steady-state value of the wealth-labor income ratio for a household without familiarity biases, but whose controls are taken from the optimal problem for a household with familiarity biases, and also,

$$\widehat{z}_{e,h,t} = \ln \frac{\widehat{W}_{e,h,t}}{\widehat{W}^*_{e,h}}.$$

Consequently, when $\widehat{W}_{e,h,t} = \widehat{W}^*_{e,h}$, we have $\widehat{z}_{e,h,t} = 0$. Expanding around $\widehat{z}_{e,h,t} = 0$, we see that

$$\widehat{c}_{e,h,t} = \frac{\widehat{C}_{e,h,t}}{\widehat{W}_{e,h,t}} = \widehat{c}^*_{e,h} e^{(1-a_{d,h})(\psi_h-1)\widehat{z}_{e,h,t}},$$

where

$$\widehat{c}^*_{e,h} = \delta_h^{\psi_h} \kappa_{d,h}^{1-\psi_h} a_{d,h}^{-\psi_h} (\widehat{W}^*_{e,h})^{(1-a_{d,h})(\psi_h-1)}. \tag{B11}$$

We assume the utility-labor income ratio is given by

$$\widehat{U}_{e,h,t} = \kappa_{e,h} \widehat{W}^{a_{e,h}}_{e,h,t},$$

where $\kappa_{e,h}$ and $a_{e,h}$ are endogenous constants we need to determine. Therefore

$$\begin{aligned}
\widehat{u}_{e,h,t} &= \frac{\widehat{U}_{e,h,t}}{\widehat{W}_{e,h,t}} \\
&= \kappa_{e,h} \widehat{W}^{a_{e,h}-1}_{e,h,t} \\
&= \kappa_{e,h} (\widehat{W}^*_{e,h})^{a_{e,h}-1} e^{(a_{e,h}-1)\widehat{z}_{e,h,t}} \\
&= \widehat{u}^*_{e,h} e^{(a_{e,h}-1)\widehat{z}_{e,h,t}},
\end{aligned}$$

where

$$\widehat{u}^*_{e,h} = \kappa_{e,h} (\widehat{W}^*_{e,h})^{a_{e,h}-1}.$$

Hence

$$\frac{\widehat{C}_{e,h,t}}{\widehat{U}_{e,h,t}} = \frac{\widehat{c}^*_{e,h}}{\widehat{u}^*_{e,h}} e^{[(1-a_{d,h})(\psi_h-1))-(a_{e,h}-1)]\widehat{z}_{e,h,t}}.$$

The evolution of the wealth-labor income ratio is determined by the consumption-portfolio policies, which are biased. Consequently, the steady-state shall be as before, that is, given by

$$\hat{c}^*_{d,h} = k^*_{2h} + \pi_{d,h}(\alpha + \boldsymbol{x}^\top_{d,h}\boldsymbol{\nu}_h - \gamma_h \rho_{YK}\sigma\sigma_Y(1 + x_{d,hh}) - i) + \frac{1}{\widehat{W}^*_h},$$

The steady-state value of the wealth-labor income ratio, $\widehat{W}^*_{e,h}$, is defined by

$$E^{\mathbb{P}_Y}_t \left[\frac{d\widehat{W}_{e,h,t}}{\widehat{W}_{e,h,t}} \right] \Bigg|_{\widehat{W}_{e,h,t} = \widehat{W}^*_{e,h}} = 0,$$

where for an event A realized at date T

$$\mathbb{P}_Y(A|\mathcal{F}_t) = E^{\mathbb{P}}_t \left[\frac{M_{Y,h,T}}{M_{Y,h,t}} 1_A \right],$$

and

$$M_{Y,h,t} = e^{-\frac{1}{2}(1-\gamma_h)^2\sigma_Y^2 t + (1-\gamma_h)\sigma_Y Z_{Y,h,t}}.$$

So we see that at the steady-state, we have

$$\hat{c}^*_{e,h}(\widehat{W}^*_{e,h}) = k^*_{e,2h} + \pi_{d,h}(\alpha - \gamma_h \rho_{YK}\sigma\sigma_Y(1 + \epsilon x_{d,hh}) - i) + \frac{1}{\widehat{W}^*_{e,h}}.$$

Therefore, using (B11), we obtain

$$\delta_h^{\psi_h} \kappa_{d,h}^{1-\psi_h} a_{d,h}^{-\psi_h} (\widehat{W}^*_{e,h})^{(1-a_{d,h})(\psi_h - 1)} = k^*_{e,2h} + \pi_{d,h}(\alpha - \gamma_h \rho_{YK}\sigma\sigma_Y(1 + \epsilon x_{d,hh}) - i) + \frac{1}{\widehat{W}^*_{e,h}}.$$

We can solve the above equation numerically to obtain $\widehat{W}^*_{e,h}$ in terms of exogenous constants. We can then use (B11) to obtain $\hat{c}^*_{e,h}$ in terms of exogenous constants. From (B1), we can see that

$$\frac{\delta_h \left(\frac{\widehat{c}^*_{e,h}}{\widehat{u}^*_{e,h}} \right)^{1-\frac{1}{\psi_h}} e^{\left(1-\frac{1}{\psi_h}\right)[(1-a_{d,h})(\psi_h-1)) - (a_{e,h}-1)]\widehat{z}_{e,h,t}} - k_{1h,t}}{1 - \frac{1}{\psi_h}}$$

$$+ a_{e,h} \left[k_{2h,t} + \pi_{d,h,t}(\alpha - \gamma_h \rho_{YK}\sigma\sigma_Y(1 + \epsilon x_{d,hh,t}) - i) - \widehat{c}^*_{e,h} e^{(1-a_{d,h})(\psi_h - 1)\widehat{z}_{e,h,t}} + (\widehat{W}^*_{e,h})^{-1} e^{-\widehat{z}_{e,h,t}} \right]$$

$$- \frac{1}{2} a_{e,h} \left[a_{e,h}\gamma_h + (1 - a_{e,h}) \right] (\sigma^2 \pi^2_{d,h,t} \boldsymbol{x}^\top_{d,h,t} \Omega \boldsymbol{x}_{d,h,t} - 2\rho_{YK}\sigma_Y \sigma \pi_{d,h,t}(1 + \epsilon x_{d,hh,t}) + \sigma_Y^2),$$

which we can rewrite as

$$\delta_h \psi_h \left(\frac{\widehat{c}^*_{e,h}}{\widehat{u}^*_{e,h}} \right)^{1-\frac{1}{\psi_h}} e^{\left(1-\frac{1}{\psi_h}\right)[(1-a_{d,h})(\psi_h-1))-(a_{e,h}-1)]\widehat{z}_{e,h,t}}$$

$$= \psi_h k_{1h,t}$$

$$+ (1-\psi_h) a_{e,h} \left\{ k_{2h,t} + \pi_{d,h,t}(\alpha - \gamma_h \rho_{YK} \sigma \sigma_Y (1 + \epsilon x_{d,hh,t}) - i) - \widehat{c}^*_{e,h} e^{(1-a_{d,h})(\psi_h-1)\widehat{z}_{e,h,t}} + (\widehat{W}^*_{e,h})^{-1} e^{-\widehat{z}_{e,h,t}} \right.$$

$$\left. -\frac{1}{2} \left[a_{e,h}\gamma_h + (1-a_{e,h}) \right] \left(\sigma^2 \pi^2_{d,h,t} \boldsymbol{x}^\top_{d,h,t} \Omega \boldsymbol{x}_{d,h,t} - 2\rho_{YK}\sigma_Y \sigma \pi_{d,h,t}(1+\epsilon x_{d,hh,t}) + \sigma^2_Y \right) \right\}$$

For ease of notation, define

$$b_{e,h} = \left(\frac{\widehat{c}^*_{e,h}}{\widehat{u}^*_{e,h}} \right)^{1-\frac{1}{\psi_h}},$$

and so

$$\delta_h \psi_h b_{e,h} e^{\left(1-\frac{1}{\psi_h}\right)[(1-a_{d,h})(\psi_h-1))-(a_{e,h}-1)]\widehat{z}_{e,h,t}}$$

$$= \psi_h k_{1h,t}$$

$$+ (1-\psi_h) a_{e,h} \left\{ k_{2h,t} + \pi_{d,h,t}(\alpha - \gamma_h \rho_{YK} \sigma \sigma_Y (1 + \epsilon x_{d,hh,t}) - i) - \widehat{c}^*_{e,h} e^{(1-a_{d,h})(\psi_h-1)\widehat{z}_{e,h,t}} + (\widehat{W}^*_{e,h})^{-1} e^{-\widehat{z}_{e,h,t}} \right.$$

$$\left. -\frac{1}{2} \left[a_{e,h}\gamma_h + (1-a_{e,h}) \right] \left(\sigma^2 \pi^2_{d,h,t} \boldsymbol{x}^\top_{d,h,t} \Omega \boldsymbol{x}_{d,h,t} - 2\rho_{YK}\sigma_Y \sigma \pi_{d,h,t}(1+\epsilon x_{d,hh,t}) + \sigma^2_Y \right) \right\}.$$

Expanding the above equation around $\widehat{z}_{e,h,t} = 0$ up to first order and comparing coefficients gives

$$\begin{aligned}
\text{(B12) } & \delta_h \psi_h b_{e,h} \\
&= \psi_h k^*_{e,1h} \\
&+ (1-\psi_h) a_{e,h} \left\{ k^*_{e,2h} + \pi_{d,h}(\alpha - \gamma_h \rho_{YK} \sigma \sigma_Y (1 + \epsilon x_{d,hh}) - i) - \widehat{c}^*_{e,h} + (\widehat{W}^*_{e,h})^{-1} \right. \\
&\left. -\frac{1}{2} \left[a_{e,h}\gamma_h + (1-a_{e,h}) \right] \left(\sigma^2 \pi^2_{d,h} \boldsymbol{x}^\top_{d,h} \Omega \boldsymbol{x}_{d,h} - 2\rho_{YK}\sigma_Y \sigma \pi_{d,h}(1+\epsilon x_{d,hh}) + \sigma^2_Y \right) \right\},
\end{aligned}$$

and

$$\text{(B13)} \qquad \delta_h \psi_h b_{e,h}\left(1-\frac{1}{\psi_h}\right)\left[(1-a_{d,h})(\psi_h-1))-(a_{e,h}-1)\right]$$
$$= \psi_h\left(\frac{1}{\psi_h}-1\right)\tau_1$$
$$+ (1-\psi_h)a_{e,h}\left\{-\tau_1-\widehat{c}^*_{e,h}(1-a_{d,h})(\psi_h-1)-(\widehat{W}^*_{e,h})^{-1}\right\},$$

where

$$k^*_{e,1,h} = \delta_h + \left(\frac{1}{\psi_h}-1\right)(\tau_0+\tau_1\ln \widehat{W}^*_{e,h}) - \frac{1}{2}\left(1+\frac{1}{\psi_h}\right)\gamma_h\sigma_Y^2 + \gamma_h\sigma_Y^2$$
$$k^*_{e,2,h} = i + \gamma_h\sigma_Y^2 - (\tau_0+\tau_1\ln\widehat{W}^*_{e,h}).$$

We can make $b_{e,h}$ the subject of (B13) as follows:

$$\delta_h b_{e,h}\left(\psi_h-1\right)\left[(1-a_{d,h})(\psi_h-1))-(a_{e,h}-1)\right]$$
$$= (1-\psi_h)\,\tau_1$$
$$+ (1-\psi_h)a_{e,h}\left\{-\tau_1-\widehat{c}^*_{e,h}(1-a_{d,h})(\psi_h-1)-(\widehat{W}^*_{e,h})^{-1}\right\}$$
$$\delta_h b_{e,h}[(1-a_{d,h})(\psi_h-1))-(a_{e,h}-1)]$$
$$= -\tau_1 - a_{e,h}\left\{-\tau_1-\widehat{c}^*_{e,h}(1-a_{d,h})(\psi_h-1)-(\widehat{W}^*_{e,h})^{-1}\right\}$$

$$\text{(B14)} \qquad b_{e,h} = \frac{-\tau_1 - a_{e,h}\left\{-\tau_1-\widehat{c}^*_{e,h}(1-a_{d,h})(\psi_h-1)-(\widehat{W}^*_{e,h})^{-1}\right\}}{\delta_h[(1-a_{d,h})(\psi_h-1)-(a_{e,h}-1)]}.$$

Substituting (B14) into (B12) gives a nonlinear algebraic equation for a_e, which can be solved numerically.

It follows from (B12) that

$$u^*_{e,h} = \frac{\widehat{U}^*_{e,h}}{\widehat{W}^*_{e,h}} = \left(\frac{\delta_h\psi_h}{\psi_h k^*_{e,1h} + (1-\psi_h)a_{e,h}\left(LQ^*_{e,h}-\widehat{c}^*_{e,h}\right)}\right)^{\frac{1}{1-\frac{1}{\psi_h}}}\widehat{c}^*_{e,h},$$

where

$$LQ^*_{e,h} = k^*_{e,2h}$$
$$+ \pi_{d,h}(\alpha - \gamma_h\rho_{YK}\sigma\sigma_Y(1+\epsilon x_{d,hh})-i)$$
$$+ (\widehat{W}^*_{e,h})^{-1} - \frac{1}{2}\left[a_{e,h}\gamma_h + (1-a_{e,h})\right](\sigma^2\pi^2_{d,h}\boldsymbol{x}^\top_{d,h}\Omega\boldsymbol{x}_{d,h} - 2\rho_{YK}\sigma_Y\sigma\pi_{d,h}(1+\epsilon x_{d,hh})+\sigma_Y^2).$$

■