Appendix for Online Publication: Public debt and the political economy of reforms

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Appendix: Theory

A1. Computation of vote shares

We begin with the voters' second-period local utilities and the candidates' second-period expected vote shares. Then, we move back through the game tree to the calculation of the voters' first-period continuation utilities and the candidates' first-period expected vote shares.

Beginning in the second-period with any policy and debt state $(e, \delta(e)) \in \mathcal{S}_{pd}$ generated by the first-period's political process, in the event that candidate $i \in \{A, B\}$ wins the second-period election the second-period local utility for a generic voter z who, at the end of the second period, receives, from candidate i, the transfer $x_{i,2}(e, \delta(e))$ is:

$$u_{z,2}(x_{i,2}(e, \delta(e))|e) = x_{i,2}(e, \delta(e)) + \iota(e)(1 - \lambda)e. \tag{A1}$$

Note that the term $\iota(e)(1-\lambda)e$ in equation (A1) depends only on the policy state e and not a candidate identity.

Voter z casts a second-period vote for candidate i over candidate j if

$$u_{z,2}(x_{i,2}(e, \delta(e))|e) > u_{z,2}(x_{j,2}(e, \delta(e))|e) \quad \Longleftrightarrow \quad x_{i,2}(e, \delta(e)) > x_{j,2}(e, \delta(e))$$

with ties broken by fair randomization. At the beginning of the second period candidate i's net endowment offer of $x_{i,2}(e, \delta(e))$ to voter z is still a random variable, denoted $\widetilde{x}_{i,2}(e, \delta(e))$, that is distributed according to $F_{i,2}(\cdot|e, \delta(e))$. Given the state of the policy and debt $(e, \delta(e)) \in \mathcal{S}_{pd}$ generated by the first-period's political process, candidate A's second-period expected vote share is calculated as,

$$\begin{aligned} S_2^A(p_2^A(e, \delta(e)), p_2^B(e, \delta(e))|e, \delta(e)) = {} & \text{Prob}\left(\widetilde{x}_{A,2}(e, \delta(e)) > \widetilde{x}_{B,2}(e, \delta(e))\right) \\ & + \frac{1}{2}\text{Prob}\left(\widetilde{x}_{A,2}(e, \delta(e)) = \widetilde{x}_{B,2}(e, \delta(e))\right) \end{aligned} \tag{A2}$$

with $S_2^B(p_2^B(e, \delta(e)), p_2^A(e, \delta(e))|e, \delta(e))$ analogously defined.

Moving back to the first period, we now construct the voters' first-period continuation utilities at the end of the first period in the event that candidate $i \in \{A, B\}$ wins the first-period election. Given that candidate i has won the first-period election and that the policy state is $e \in \mathcal{E} \cup \emptyset$, the first-period local utility for a generic voter z who, at the end of the first period, receives, from candidate $i \in \{A, B\}$, the net endowment offer $x_{i,1}(e)$ is:

$$u_{z,1}(x_{i,1}(e)) = x_{i,1}(e).$$

Recall from equation (3) that second-period budget balancing requires that for each candidate i,

$$E(x_{i,2}(e,\delta(e))) = 1 + \iota(e)\lambda e - \delta(e).$$

If for each state $(e,\delta(e)) \in \mathcal{S}_{pd}$ both candidates use second-period budget-balancing platforms,[1] then from equations (3) and (A1) it follows that in policy state $e \in \mathcal{E} \cup \emptyset$ the continuation utility for a generic voter z who, at the end of the first period, receives a transfer of $x_{i,1}(e)$ from the candidate i that won the first-period election with a realized policy position of ι_i and debt level of $\delta_i(e)$ is:

$$U_z(x_{i,1}(e),\iota_i,\delta_i(e)|e) := x_{i,1}(e) + 1 + \iota_i e - \delta_i(e).$$

If there exists at least one candidate i with $\iota_i = 1$, then the draw of the policy state e from Γ_e is payoff relevant, and when voters cast their first-period votes they do not know the policy state $e \in \mathcal{E}$. Let $E_{e|\iota_i}\left(U_z(x_{i,1}(e),\iota_i,\delta_i(e)|e)\right)$ be defined as follows:

$$E_{e|\iota_i}\left(U_z(x_{i,1}(e),\iota_i,\delta_i(e)|e)\right) = \begin{cases} x_{i,1}(\emptyset) + 1 - \delta_i(\emptyset) & \text{if} \quad \iota_i = 0 \\ E_{\Gamma_e}\left(x_{i,1}(e) + 1 + e - \delta_i(e)\right) & \text{if} \quad \iota_i = 1 \end{cases}$$

where $E_{e|\iota_i}\left(U_z(x_{i,1}(e),\iota_i,\delta_i(e)|e)\right)$ denotes the expected continuation utility for a generic voter z who receives a net endowment offer of $x_{i,1}(\emptyset)$ from candidate i in the case that $\iota_i = 0$ and receives an $|\mathcal{E}|$-tuple of net endowment offers $(\{x_{i,1}(e)\}_{e\in\mathcal{E}})$ from candidate i in the case that $\iota_i = 1$. Voter z casts a first-period vote for candidate i over candidate j if

$$E_{e|\iota_i}\left(U_z(x_{i,1}(e,),\iota_i,\delta_i(e)|e)\right) > E_{e|\iota_j}\left(U_z(x_{j,1}(e),\iota_j,\delta_j(e)|e)\right),$$

with ties broken by fair randomization.

At the beginning of the first period, each candidate i announces a first-period platform of p_1^i and the expected continuation utility $E_{e|\iota_i}\left(U_z(x_{i,1}(e),\iota_i,\delta_i(e)|e)\right)$ that candidate i provides to an arbitrary voter z is a random variable, denoted $\widetilde{U}_z(p_1^i)$, where,

$$\text{(A3)} \quad \widetilde{U}_z(p_1^i) := \beta_i\left(\widetilde{x}_{i,1}^{\Gamma_e} + 1 + E_{\Gamma_e}(e - \delta_i(e))\right) + (1-\beta_i)\left(\widetilde{x}_{i,1}(\emptyset) + 1 - \delta_i(\emptyset)\right),$$

where $\widetilde{x}_{i,1}^{\Gamma_e}$ denotes the random variable corresponding to candidate i's average, with respect to the policy state e, first-period net endowment offer for an arbitrary $|\mathcal{E}|$-tuple drawn from $P_{i,1}^{\mathcal{E}}(\mathbf{x})$.

[1] Given our focus on subgame perfect Nash equilibrium, we focus here on the case that both candidates use second-period budget balancing platforms. However, it is straightforward to extend the continuation utilities to the case that one or both of the candidates do not use second-period budget-balancing platforms.

In period 1, we denote by $S_1^A(p_1^A, p_1^B)$ the first-period vote share that candidate A receives when she chooses the first-period platform p_1^A and candidate B chooses the first-period platform p_1^B, and both candidates use second-period budget-balancing platforms. Hence,

$$S_1^A(p_1^A, p_1^B) = \text{Prob}\left(\widetilde{U}_z(p_1^A) > \widetilde{U}_z(p_1^B)\right) + \frac{1}{2}\text{Prob}\left(\widetilde{U}_z(p_1^A) = \widetilde{U}_z(p_1^B)\right) \tag{A4}$$

and $S_1^B(p_1^B, p_1^A)$ is analogously defined.

A2. Proof of Theorem 1 and Corollary 1

We begin in the second period with any state $(e, \delta(e)) \in \mathcal{S}_{pd}$ and show that in the subgame arising in state $(e, \delta(e))$ the corresponding Theorem 1 and Corollary 1 second-period local strategies form a second-period local equilibrium and, furthermore, establish that this second-period local equilibrium is unique. Then, given the second-period local equilibrium strategies we move back through the game tree to the first period and characterize the remaining first-period component of the unique subgame-perfect equilibrium strategies.

Second PeriodIn the second period with any state $(e, \delta(e))$, it follows from the second-period expected vote share calculation given in equation (A2) that candidate A's second-period expected vote share,

$$S_2^A(p_2^A(e, \delta(e)), p_2^{*B}(e, \delta(e))|e, \delta(e)),$$

from using the arbitrary second-period local strategy $p_2^A(e, \delta(e))$, given that candidate B uses the equilibrium second-period local strategy $p_2^{*B}(e, \delta(e))$ is:

$$S_2^A(p_2^A(e, \delta(e)), p_2^{*B}(e, \delta(e))|e, \delta(e)) = \int_{\text{Supp}F_{A,2}|e,\delta(e)} F_2^*(x|e, \delta(e))dF_{A,2}(x|e, \delta(e)). \tag{A5}$$

In any best response, it is clear that candidate A does not provide a voter z with a second-period utility level that is strictly greater than $2(1 + \iota(e)\lambda e - \delta(e))$. Thus, from equation (3)'s second-period budget-balancing condition (i.e. $E_{F_{A,2}|e,\delta(e)}(x) = 1 + \iota(e)\lambda e - \delta(e)$) it follows from equation (A5) that A's second-period expected vote share satisfies

$$\int_{\text{Supp}F_{A,2}|e,\delta(e)} \frac{x}{2(1 + \iota(e)\lambda e - \delta(e))} dF_{A,2}(x|e, \delta(e)) \leq \frac{1 + \iota(e)\lambda e - \delta(e)}{2(1 + \iota(e)\lambda e - \delta(e))} = \frac{1}{2}.$$

To complete the proof that for all states $(e, \delta(e)) \in \mathcal{S}_{pd}$ the Theorem 1 and Corollary 1 second-period local strategies form a second-period local equilibrium, ob-

serve that candidate A receives $\frac{1}{2}$ of the second-period vote share from any budget-balancing second-period local strategy $F_{A,2}(x|e,\delta(e))$ with $\text{Supp}\left(F_{A,2}|e,\delta(e)\right) \subseteq [0, 2(1+\iota(e)\lambda e-\delta(e))]$ and that candidate A has no profitable deviations. Because the second-period subgame for each state $(e,\delta(e)) \in \mathcal{S}_{pd}$ involves only redistribution, the proof of uniqueness of the second-period local equilibrium strategies follows from standard results on Myerson's formulation of the relaxed Colonel Blotto game (a.k.a. the General Lotto game, for further details see Kovenock and Roberson (2021)).

First PeriodGiven the second-period local equilibrium strategies specified by Theorem 1 and Corollary 1, we now move back through the game tree to the first period and characterize the remaining first-period component of the unique subgame-perfect equilibrium strategies. Note that in the second period, for any state $(e,\delta(e)) \in \mathcal{S}_{pd}$ it follows from the discussion above for the second period that each candidate's second-period local equilibrium expected vote share is $1/2$. Taking the second-period equilibrium expected vote shares as given, we begin with the first-period vote share calculation. Then, we turn to the proof that in Part (I.) the Theorem 1 first-period local strategies form the remaining first-period component of the unique subgame-perfect equilibrium strategies. Next, we perform the corresponding analysis for Part (II.). The proof that the subgame-perfect equilibrium unique is given in the appendix.

For the first-period vote share calculation, suppose, without loss of generality, that candidate A uses an arbitrary first-period local strategy p_1^A. Given that candidate B uses the equilibrium first-period platform p_1^*, candidate B's expected promise of continuation utility for an arbitrary voter z is the random variable $\widetilde{U}_z(p_1^*)$ defined by equation (A3) as:

$$\widetilde{U}_z(p_1^*) = \beta^* \left(\widetilde{x}_1^{\Gamma_e} + 1 + E_{\Gamma_e}(e - \delta_i(e))\right) + (1-\beta^*)\left(\widetilde{x}_1^*(\emptyset) + 1 - \delta^*(\emptyset)\right). \tag{A6}$$

For $u \in [0,4]$, let $G^*(u)$ denote the distribution of the random variable $\widetilde{U}_z(p_1^*)$, which we will examine in more detail below for cases (I.) and (II.) of Theorem 1. Similarly, let $G_{p_1^A}(u)$ denote the distribution of the random variable $\widetilde{U}_z(p_1^A)$ generated by the first-period platform p_1^A via equation (A3).

The probability that candidate A wins voter z's first-period vote when A provides voter z with a first-period continuation utility of $U_z(p_1^A)$ is $G^*(U_z(p_1^A))$. Thus, candidate A's first-period expected vote share when using an arbitrary first-period local strategy p_1^A and candidate B is using the the first-period platform p_1^* is

$$S_1^A(p_1^A, p_1^*) = \int_{\text{Supp}G_{p_1^A}} G^*(u) dG_{p_1^A}(u) \tag{A7}$$

We now use the equation (A7) first-period vote share calculation in the proof that in Part (I.) of Theorem 1 – where $H = 2c-(1+\lambda)E_{\Gamma_e}(e) \leq 0$ – the Theorem 1 first-period local strategies form a first-period local equilibrium. Given that candidate B is using the first-period local equilibrium strategy p_1^* specified by Part (I.) of Theorem 1, it follows that the random variable $\widetilde{U}_z(p_1^*)$ is distributed according to[2]

$$G^*(u) = \begin{cases} 0, & \text{if } u \leq (1-\lambda)E_{\Gamma_e}(e), \\ \frac{u-(1-\lambda)E_{\Gamma_e}(e)}{4+2\lambda E_{\Gamma_e}(e)-2c}, & \text{if } (1-\lambda)E_{\Gamma_e}(e) < u \leq 4-H, \\ 1, & \text{if } u > 4-H. \end{cases}$$

In any best-response, it is clear that candidate A does not provide voter z with a utility level $U_z(p_1^A)$ that is strictly greater than $4-H$.[3] Thus, given that B is using the first-period local equilibrium strategy p_1^*, it follows from equation (A7) that candidate A's first-period expected vote share in state e, from an arbitrary first-period local strategy p_1^A is

$$S_1^A(p_1^A, p_1^*) = \int_{\text{Supp}G_{p_1^A}} \frac{u-(1-\lambda)E_{\Gamma_e}(e)}{4+2\lambda E_{\Gamma_e}(e)-2c} dG_{p_1^A}(u). \tag{A8}$$

First we consider the case that $\iota_A = 1$. From equation (A3) with $\iota_A = 1$, it follows that

$$\widetilde{U}_z(p_1^A) = \widetilde{x}_{A,1}^{\Gamma_e} + 1 + E_{\Gamma_e}(e - \delta_A(e)). \tag{A9}$$

Then, from equation (1) we know that

$$E_{G_{p_1^A}}(\widetilde{x}_{A,1}^{\Gamma_e}) = E_{G_{p_1^A}}\left(\sum_{e\in\mathcal{E}} \Gamma_e(e)\widetilde{x}_{i,1}(e)\right) = \sum_{e\in\mathcal{E}} \Gamma_e(e) E_{F_{A,1}|e}(\widetilde{x}_{i,1}(e)) \leq 1+E_{\Gamma_e}(\delta_A(e))-c \tag{A10}$$

where the last inequality in equation (A10) follows from the first-period budget constraint given in equation (2). Inserting, equations (A9) and (A10) into equation (A8) we see that

$$S_1^A(p_1^A, p_1^*) \leq \frac{2+\lambda E_{\Gamma_e}(e)-c}{4+2\lambda E_{\Gamma_e}(e)-2c} = \frac{1}{2}. \tag{A11}$$

[2]Note that because $H = 2c-(1+\lambda)E_{\Gamma_e}(e)$ and $\delta^*(e) = 1+\lambda e$ when $\beta^* = 1$, it follows that $4-H = 2+2E_{\Gamma_e}(\delta^*(e))-2c+(1+E_{\Gamma_e}(e-\delta^*(e)))$ and $(1+E_{\Gamma_e}(e-\delta^*(e))) = (1-\lambda)E_{\Gamma_e}(e)$.

[3]Note that because $\delta^*(e)$ is the maximum level of debt, $\delta_A(e) \leq \delta^*(e)$ and $1+e-\delta_A(e) \geq 1+e-\delta^*(e)$. That is, if candidate A chooses $\iota_A = 1$, then candidate A is unable to provide voter z with a continuation utility below $1+e-\delta^*(e)$.

To summarize, if $H = 2c-(1+\lambda)E_{\Gamma_e}(e) \leq 0$ and candidate B uses the first-period local equilibrium strategy p_1^{*B} specified in Part (I.) of Theorem 1, then candidate A's first-period expected vote share from any arbitrary first-period platform p_1^A with $\iota_A = 1$ is less than or equal to $\frac{1}{2}$, where equation (A11) holds with equality if candidate A's strategy is first-period budget balancing as specified by equation (2).

To complete the proof of existence for Part (I.) of Theorem 1, consider the remaining case in which candidate A chooses an arbitrary first-period strategy in which $\iota_A = 0$ with strictly positive probability. We now show that candidate A's payoff from a first-period platform with $\iota_A = 0$ is strictly less than if $\iota_A = 1$. Therefore, in any best response candidate A chooses $\iota_A = 1$ with probability one. From equation (A3) with $\iota_A = 0$ and the first-period budget constraint given in equation (2), it follows that

$$E_{G_{p_1^A}}(\widetilde{U}_z(p_1^A)) = E_{F_{A,1}|\emptyset}(\widetilde{x}_{A,1}(\emptyset)) + 1 - \delta_A(\emptyset) \leq 2. \tag{A12}$$

From equations (A8) and (A12), candidate A's first-period expected vote share, from such a strategy, is

$$S_1^A(p_1^A, p_1^*) \leq \frac{2 + \lambda E_{\Gamma_e}(e) - E_{\Gamma_e}(e)}{4 + 2\lambda E_{\Gamma_e}(e) - 2c} < \frac{1}{2} \tag{A13}$$

where the strict-inequality in equation (A13) follows from assumption (A1). Thus, candidate A's first-period expected vote share from deviating to any arbitrary first-period strategy with $\iota_A = 0$ is less than or equal to $\frac{1}{2}$. This completes the existence portion of the proof of Part (I.) of Theorem 1, and we will return to the uniqueness portion at the end of this subsection of the Appendix.

We now examine Part (II.) of Theorem 1, in which $H = 2c-(1+\lambda)E_{\Gamma_e}(e) > 0$. Given that candidate B is using the first-period equilibrium strategy specified by Part (II.) of Theorem 1, it follows that $\beta_B = \beta^* = 1 - \frac{1}{2}H < 1$ and for each realization of the policy benefit $e \in \mathcal{E} \cup \emptyset$ the debt is $\delta^*(e) = 1 + \iota(e)\lambda e$. In the event that $\iota_B = 0$, the random variable $\widetilde{x}_{B,1}(\emptyset)$ is distributed according to

$$F_1^*(x|e = \emptyset) = \begin{cases} 0, & \text{if } x \leq 0, \\ \frac{1}{2}\left(\frac{x}{H}\right), & \text{if } 0 < x \leq H, \\ \frac{1}{2}, & \text{if } H < x \leq 4 - H, \\ \frac{1}{2}\left(1 + \frac{x-4+H}{H}\right), & \text{if } 4 - H < x \leq 4, \\ 1, & \text{if } x > 4. \end{cases} \tag{A14}$$

and in the event that $\iota_B = 1$, the random variable $\widetilde{x}_1^{\Gamma_e}$ is uniformly distributed on the interval $[0, 4 + 2\lambda E_{\Gamma_e}(e) - 2c]$. Because $\iota_B = 1$ and $\iota_B = 0$ are mutually

exclusive events, the random variable $\widetilde{U}_z(p_1^*)$ is distributed according to[4]

(A15)
$$G_1^*(u) = \begin{cases} 0, & \text{if } x \leq 0, \\ \frac{u}{4}, & \text{if } 0 < u \leq H, \\ \frac{H}{4}, & \text{if } H < u \leq (1-\lambda)E_{\Gamma_e}(e), \\ \frac{H}{4} + \left(1 - \frac{H}{2}\right)\left(\frac{u-(1-\lambda)E_{\Gamma_e}(e)}{4+2\lambda E_{\Gamma_e}(e)-2c}\right), & \text{if } (1-\lambda)E_{\Gamma_e}(e) < u \leq 4-H, \\ \frac{u}{4}, & \text{if } 4-H < u \leq 4, \\ 1, & \text{if } x > 4. \end{cases}$$

If candidate A chooses a first-period platform p_1^A with $\iota_A = 1$ and $\text{Supp}(G_{p_1^A}(u)) \in [(1-\lambda)E_{\Gamma_e}(e), 4-H]$, then candidate A's expected vote share in state e, from such a strategy is

$$S_1^A(p_1^A, p_1^*) = \frac{H}{4} + \left(1 - \frac{H}{2}\right)\int_{\text{Supp}G_{p_1^A}} \frac{u-(1-\lambda)E_{\Gamma_e}(e)}{4+2\lambda E_{\Gamma_e}(e)-2c} dG_{p_1^A}(u) \tag{A16}$$

Inserting, equations (A9) and (A10) into equation (A16), we have that

$$S_1^A(p_1^A, p_1^*) \leq \frac{H}{4} + \left(1 - \frac{H}{2}\right)\frac{2+\lambda E_{\Gamma_e}(e)-c}{4+2\lambda E_{\Gamma_e}(e)-2c} = \frac{1}{2} \tag{A17}$$

Thus, candidate A's expected vote share from any first-period platform p_1^A with $\iota_A = 1$ and $\text{Supp}(G_{p_1^A}(u)) \in [(1-\lambda)E_{\Gamma_e}(e), 4-H]$ is less than or equal to $\frac{1}{2}$.

We now show that, given that candidate B is using the first-period equilibrium platform p_1^* specified by Part (II.) of Theorem 1, in any best-response by candidate A with $\iota_A = 1$ it must be the case that $\text{Supp}(G_{p_1^A}(u)) \in [(1-\lambda)E_{\Gamma_e}(e), 4-H]$. First, if candidate A uses a strategy with $\iota_A = 1$, then candidate A provides each voter with an expected utility of at least $(1-\lambda)E_{\Gamma_e}(e)$. Next, note that it is clearly suboptimal for candidate A to ever provide utility levels $U_z(p_1^A)$ above 4. The only remaining case with $\iota_A = 1$ is that there exists a measurable subset of $\text{Supp}(G_{p_1^A}(u))$ in the interval $[4-H, 4]$.

Because $\iota_A = 1$ all voters have a continuation utility offer of at least $(1-\lambda)E_{\Gamma_e}(e)$ from candidate A. Let M_1 denote the average of the continuation utility offers that candidate A makes in the interval $[(1-\lambda)E_{\Gamma_e}(e), 4-H]$, where $M_1 \geq (1-\lambda)E_{\Gamma_e}(e)$ and $G_{p_1^A}(4-H)$ voters receive such offers. Similarly, let M_2 denote the average of the continuation utility offers that candidate A makes in the interval $[4-H, 4]$, where $M_2 \geq 4-H$ and $1-G_{p_1^A}(4-H)$ voters receive such offers. From

[4]Note that when $\iota_B = 1$, $\delta^*(e) = 1+\lambda e$ and, thus, $4-H = 2+2E_{\Gamma_e}(\delta^*(e))-2c+(1+E_{\Gamma_e}(e-\delta^*(e)))$ and $(1+E_{\Gamma_e}(e-\delta^*(e))) = (1-\lambda)E_{\Gamma_e}(e)$.

equations (A9) and (A10) it follows that

$$G_{p_1^A}(4-H)M_1 + (1-G_{p_1^A}(4-H))M_2 \leq 2 + E_{\Gamma_e}(e) - c. \tag{A18}$$

Note that because $M_1 \geq (1-\lambda)E_{\Gamma_e}(e)$ and $M_2 \geq 4-H = 4-2c+(1+\lambda)E_{\Gamma_e}(e)$, it follows from equation (A18) that $G_{p_1^A}(4-H) \geq 1/2$, i.e. candidate A can offer at most half of the voters net endowments such that their continuation utility is at or above $4-H$.

Returning to candidate A's first period expected vote share which is given by:

$$S_1^A(p_1^A, p_1^*) = G_{p_1^A}\left[(4-H)\left(\frac{H}{4}\right) + \left(1-\frac{H}{2}\right)\frac{(M_1-(1-\lambda)E_{\Gamma_e}(e))}{4+2\lambda E_{\Gamma_e}(e)-2c}\right] + \frac{(1-G_{p_1^A}(4-H))M_2}{4}. \tag{A19}$$

Because $\frac{\left(1-\frac{H}{2}\right)}{4+2\lambda E_{\Gamma_e}(e)-2c} > \frac{1}{4}$, it follows that, for any $G_{p_1^A}(4-H) \geq 1/2$, candidate A's first period expected vote share in equation (A19) increases as M_2 decreases towards its lower bound of $4-H$ and M_1 increases subject to the constraint in equation (A18). This completes the proof that in any best-response by candidate A with $\iota_A = 1$ it must be the case that $\text{Supp}(G_{p_1^A}(u)) \in [(1-\lambda)E_{\Gamma_e}(e), 4-H]$.

For the case that candidate A chooses a first-period platform p_1^A with $\iota_A = 0$ and $\text{Supp}(G_{p_1^A}(u)) \in [0,H] \cup [4-H,4]$, it follows from equation (A15) that candidate A's expected vote share is

$$S_1^A(p_1^A, p_1^*) = \int_{\text{Supp}G_{p_1^A}} \frac{u}{4} dG_{p_1^A}(u) \tag{A20}$$

Given budget feasibility with $\iota_A = 0$, see equation (2), it follows from equation (A20) that candidate A's expected vote share from any such a strategy p_1^A is less than or equal to $1/2$, which holds with equality if p_1^A is budget balancing.

In the case of a strategy p_1^A with $\iota_A = 0$, it is clearly not payoff increasing for candidate A to offer continuation utilities in the interval $[H, 4-H]$. For the remaining case that of $\iota_A = 0$ with continuation utility offers in the interval $[4-H, 4]$, let $\widehat{M_1}$ denote the average of the continuation utility offers that candidate A makes in the interval $[0,H]$, where μ_1 voters receive such offers. Let $\widehat{M_2}$ and $\widehat{M_3}$ be similarly defined for the average of the continuation utility offers that candidate A makes in the intervals $[(1-\lambda)E_{\Gamma_e}(e), 4-H]$ and $[4-H,4]$ respectively, where μ_2 and μ_3 voters receive such offers, respectively. From equations (A9) and (A10) it follows that

$$\mu_1\widehat{M_1} + \mu_2\widehat{M_2} + \mu_3\widehat{M_3} \leq 2 \tag{A21}$$

where $\mu_1 + \mu_2 + \mu_3 = 1$.

Candidate A's first period expected vote share, with $\iota_A = 0$ and $\mu_2 \geq 0$ is given by:

(A22)
$$S_1^A(p_1^A, p_1^*) = \frac{\mu_1 \widehat{M}1 + \mu_3 \widehat{M}_3}{4} + \mu_2 \left[\frac{H}{4} + \left(1 - \frac{H}{2}\right) \frac{(\widehat{M}_2 - (1-\lambda)E_{\Gamma_e}(e))}{4 + 2\lambda E_{\Gamma_e}(e) - 2c} \right].$$

Inserting the constraint in equation (A21) into equation (A22), we have

$$S_1^A(p_1^A, p_1^*) \leq \frac{2 - \mu_2 \widehat{M}_2}{4} + \mu_2 \left[\frac{H}{4} + \left(1 - \frac{H}{2}\right) \frac{(\widehat{M}_2 - (1-\lambda)E_{\Gamma_e}(e))}{4 + 2\lambda E_{\Gamma_e}(e) - 2c} \right]. \tag{A23}$$

It follows from equation (A23) that $S_1^A(p_1^A, p_1^*)$ is strictly decreasing in μ_2,

$$\frac{\partial S_1^A(p_1^A, p_1^*)}{\partial \mu_2} = \left(\frac{H}{4}\right) + \left(1 - \frac{H}{2}\right) \frac{(\widehat{M}_2 - (1-\lambda)E_{\Gamma_e}(e))}{4 + 2\lambda E_{\Gamma_e}(e) - 2c} - \frac{\widehat{M}_2}{4} < 0 \tag{A24}$$

where the strict inequality in equation (A24) follows from the combination of $\widehat{M}_2 \in [(1-\lambda)E_{\Gamma_e}(e), 4-H]$ and $H < 2$. This completes the proof that in any best-response by candidate A with $\iota_A = 0$ it must be the case that $\text{Supp}(G_{p_1^A}(u)) \in [0, H] \cup [4 - H, 4]$.

Now we examine issue of subgame perfect equilibrium uniqueness for both parts (I.) and (II.) of Theorem 1. The discussion below focuses on the Part (II.) portion of the parameter space in which $H > 0$. The arguments for the Part (I.) portion of the parameter space ($H \leq 0$) follow along similar lines. Given that the second-stage local equilibrium payoffs are unique, the proof that the first-stage local equilibrium is unique follows from the fact that the first-stage local subgame is constant-sum and equilibria are interchangeable. In particular, note that it follows from standard arguments that in any first-stage local equilibrium each candidate i's equilibrium distribution of $\widetilde{U}_z(p_1^i)$ satisfies the following properties:

1) If $H > 0$, the distribution of $\widetilde{U}_z(p_1^i)$ has the same support as $G_1^*(u)$ defined in equation (A15).

2) If $H > 0$, the distribution of $\widetilde{U}_z(p_1^i)$ is strictly increasing on the intervals $[0, H]$ and $[(1-\lambda)E_{\Gamma_e}(e), 4]$.

3) If $H > 0$, there is no point in the distribution of $\widetilde{U}_z(p_1^i)$ at which player i places strictly positive mass.

4) If $H > 0$, the distribution of $\widetilde{U}_z(p_1^i)$ is equal to $G_1^*(u)$ for $u \in [0, H] \cup [(1-\lambda)E_{\Gamma_e}(e), 4]$.

This completes the uniqueness portion of the proofs of parts (I.) and (II.) of Theorem 1, and hence completes the proof of Theorem 1.

A3. Proof of Theorem 2 and Corollary 1: Hard limit on debt

We first state the complete characterization of the equilibrium.

Corollary 1 *Given a hard debt constraint of $\overline{\delta} > 0$, the set of subgame perfect equilibria is completely characterized as follows.*

First Period

*In the first period, there are two cases labeled **(I.)** and **(II.)**.*

(I.) *If $\widehat{H}^d \leq 0$, then in any subgame perfect equilibrium both candidates choose a first-period platform p_1^* that implements the policy with probability $\beta^* = 1$ and for each realization of the policy state $e \in \mathcal{E}$:*

- **(i)** *announce the maximum feasible debt: $\widehat{\delta}^d(e) = \min\{\overline{\delta}, 1 + \lambda e\}$, and*
- **(ii)** *choose an $(|\mathcal{E}| + 1)$-variate joint distribution $P_1^*(\mathbf{x})$ of first-period net endowments such that the random variable $\widetilde{x}_1^{\Gamma_e}$ is uniformly distributed on the interval $[0, 2B_P^d]$ and for each possible policy state e the random variable $\widetilde{x}_1^*(e)$ satisfies first-period budget balancing as defined in equation (7).*[5]

(II.) *If $\widehat{H}^d > 0$, then in the unique subgame perfect equilibrium both candidates choose a first-period platform p_1^* that implements the policy with probability $\beta^* = 1 - \frac{\widehat{H}^d}{B_{NP}^d} (< 1)$ and for each realization of the policy state $e \in \mathcal{E} \cup \emptyset$:*

- **(i)** *announce the maximum feasible debt: $\widehat{\delta}^d(e) = \min\{\overline{\delta}, 1 + \iota(e)\lambda e\}$, and*
- **(ii)** *choose an $(|\mathcal{E}| + 1)$-variate joint distribution $P_{i,1}^*(\mathbf{x})$ of first-period net endowments such that:*

(A25)

$$F_1^*(x|e = \emptyset) = \begin{cases} 0, & \text{if } x \leq 0, \\ \frac{1}{2}\left(\frac{x}{\widehat{H}^d}\right), & \text{if } 0 \leq x \leq \widehat{H}^d, \\ \frac{1}{2}, & \text{if } \widehat{H}^d \leq x \leq 2B_{NP}^d - \widehat{H}^d, \\ \frac{1}{2}\left(1 + \frac{x - 2B_{NP}^d + \widehat{H}^d}{\widehat{H}^d}\right), & \text{if } 2B_{NP}^d - \widehat{H}^d \leq x \leq 2B_{NP}^d, \\ 1, & \text{if } x \geq 2B_{NP}^d. \end{cases}$$

[5] Because $e = \emptyset$ arises with probability 0 when $\beta^* = 1$, in case (I.) any feasible specification of first-period transfers may be used to complete the specification of a strategy for the policy state $e = \emptyset$.

and for $e \neq \emptyset$, the random variable $\widetilde{x}_1^{\Gamma_e}$ is uniformly distributed on the interval $[0, 2B_p^d]$ such that for each possible policy state e the random variable $\widetilde{x}_1^*(e)$ satisfies first-period budget balancing as defined in equation (7).

Second Period

Given any second-period state $(e, \delta(e)) \in \mathcal{S}_{pd}$, the unique subgame perfect second-period local equilibrium is for each candidate to choose the second-period platform $p_2^*(e, \delta(e))$ that uniformly distributes net endowments on the interval $[0, 2(1 + \iota(e)\lambda e - \delta(e))]$.

Along any equilibrium path, the equilibrium debt level is $\widehat{\delta}^d(e) = \min\{\overline{\delta}, 1 + \iota(e)\lambda e\}$ and the second-period local equilibrium net endowments are uniformly distributed on the interval $[0, 2(1 + \iota(e)\lambda e - \widehat{\delta}^d(e))]$.

With a few modifications, the proof of Theorem 2 and Corollary 1 follows along the lines of the proof of Theorem 1 and Corollary 1. Beginning in the second period with any state $(e, \delta(e)) \in \mathcal{S}_{pd}$, note that borrowing is not possible in the second period and so the debt limit $\overline{\delta}$ does not change the Theorem 1 second-period local equilibrium strategies. Given the second-period local equilibrium strategies, we move back through the game tree and examine the effect of the debt limit on the first-period local equilibrium strategies. We begin by examining the first-period vote share calculation. Then, we turn to the proof that in Part (I.) the Theorem 2 first-period local strategies form a first-period local equilibrium. Next, we perform the corresponding analysis for Part (II.).

We now use the equation (A7) first-period vote share calculation in the proof that in Part (I.) of Theorem 2 – where $\widehat{H}^d \equiv 2B_{NP}^d - 2B_P^d - 1 - E_{\Gamma_e}(e - \widehat{\delta}^d(e))$ – the Theorem 2 first-period local strategies form a first-period local equilibrium. Given that candidate B is using the first-period local equilibrium strategy p_1^* specified by Part (I.) of Theorem 1, it follows that the random variable $\widetilde{U}_z(p_1^*)$ is distributed according to

$$G^*(u) = \begin{cases} 0, & \text{if } u \leq 1 + E_{\Gamma_e}(e - \widehat{\delta}^d(e)), \\ \frac{u - 1 - E_{\Gamma_e}(e - \widehat{\delta}^d(e))}{2B_P^d}, & \text{if } 1 + E_{\Gamma_e}(e - \widehat{\delta}^d(e)) \leq u \leq 2B_P^d + 1 + E_{\Gamma_e}(e - \widehat{\delta}^d(e)), \\ 1, & \text{if } u \geq 2B_P^d + 1 + E_{\Gamma_e}(e - \widehat{\delta}^d(e)). \end{cases}$$

In any best-response, it is clear that candidate A does not provide voter z with a utility level $U_z(p_1^A)$ that is strictly greater than $2B_P^d + 1 + E_{\Gamma_e}(e - \widehat{\delta}^d(e))$. Thus, given that B is using the first-period local equilibrium strategy p_1^*, it follows from equation (A7) that candidate A's first-period expected vote share in state e, from

an arbitrary first-period local strategy p_1^A with $\iota_A = 1$, is

$$S_1^A(p_1^A, p_1^*) = \int_{\text{Supp}G_{p_1^A}} \frac{u - 1 - E_{\Gamma_e}(e - \widehat{\delta}^d(e))}{2B_P^d} dG_{p_1^A}(u). \tag{A26}$$

First we consider the case that $\iota_{A,1} = 1$. From equation (A3) with $\iota_{A,1} = 1$, it follows that

$$\widetilde{U}_z(p_1^A) = \widetilde{x}_{A,1}^{\Gamma_e} + 1 + E_{\Gamma_e}(e - \widehat{\delta}^d(e)). \tag{A27}$$

Then, from equations (1) and (7) we know that

$$E_{G_{p_1^A}}(\widetilde{x}_{A,1}^{\Gamma_e}) \leq B_P^d \tag{A28}$$

where the inequality in equation (A28) follows from the first-period budget constraint given in equation (7). Inserting, equations (A27) and (A28) into equation (A26) we see that

$$S_1^A(p_1^A, p_1^*) \leq \frac{B_P^d}{2B_P^d} = \frac{1}{2}. \tag{A29}$$

To summarize, if $\widehat{H}^d \leq 0$ and candidate B uses the first-stage local equilibrium strategy p_1^{*B} specified in Part (I.) of Theorem 2, then candidate A's first-period expected vote share from any arbitrary first-period platform p_1^A is less than or equal $\frac{1}{2}$, where equation (A29) holds with equality if candidate A's strategy is first-period budget balancing as specified by equation (7).

The proof of the remaining case in which candidate A chooses an arbitrary first-period strategy in which $\iota_A = 0$ with strictly positive probability, follows along the lines of the corresponding part of the Theorem 1 proof.

We now examine Part (II.) of Theorem 1, in which $\widehat{H}^d > 0$. Given that candidate B is using the first-period equilibrium strategy specified by Part (II.) of Theorem 2, it follows that $\beta_B = \beta^* = 1 - \frac{1}{2}\widehat{H}^d < 1$ and for each realization of the policy benefit $e \in \mathcal{E} \cup \emptyset$ the debt is $\widehat{\delta}^d(e) = \min\{\overline{\delta}, 1 + \lambda e\}$. In the event that $\iota_B = 0$, the random variable $\widetilde{x}_{B,1}(\emptyset)$ is distributed according to

$$F_1^*(x|e = \emptyset) = \begin{cases} 0, & \text{if } x \leq 0, \\ \frac{1}{2}\left(\frac{x}{\widehat{H}^d}\right), & \text{if } 0 \leq x \leq \widehat{H}^d, \\ \frac{1}{2}, & \text{if } \widehat{H}^d \leq x \leq 2B_{NP}^d - \widehat{H}^d, \\ \frac{1}{2}\left(1 + \frac{x - 2B_{NP}^d + \widehat{H}^d}{\widehat{H}^d}\right), & \text{if } 2B_{NP}^d - \widehat{H}^d \leq x \leq 2B_{NP}^d, \\ 1, & \text{if } x \geq 2B_{NP}^d. \end{cases} \tag{A30}$$

and for $e \neq \emptyset$, the random variable $\widetilde{x}_1^{\Gamma_e}$ is uniformly distributed on the interval $[0, 2B_p^d]$. Because $\iota_B = 1$ and $\iota_B = 0$ are mutually exclusive events, the random variable $\widetilde{U}_z(p_1^*)$ is distributed according to

(A31)

$$G_1^*(u) = \begin{cases} 0, & \text{if } x \leq 0, \\ \frac{u}{2B_{NP}^d}, & \text{if } 0 \leq u \leq \widehat{H}^d, \\ \frac{\widehat{H}^d}{2B_{NP}^d}, & \text{if } \widehat{H}^d \leq u \leq 1 + E_{\Gamma_e}(e - \widehat{\delta}^d(e)), \\ \frac{\widehat{H}^d}{2B_{NP}^d} + \left(1 - \frac{\widehat{H}^d}{B_{NP}^d}\right)\left(\frac{u-1-E_{\Gamma_e}(e-\widehat{\delta}^d(e))}{2B_P^d}\right), & \text{if } 1 + E_{\Gamma_e}(e - \widehat{\delta}^d(e)) \leq u \leq 2B_{NP}^d - \widehat{H}^d, \\ \frac{u}{2B_{NP}^d}, & \text{if } 2B_{NP}^d - \widehat{H}^d \leq u \leq 2B_{NP}^d, \\ 1, & \text{if } x \geq 4. \end{cases}$$

If candidate A chooses a first-period platform p_1^A with $\iota_A = 1$ and $\text{Supp}(G_{p_1^A}(u)) \in [1 + E_{\Gamma_e}(e - \widehat{\delta}^d(e)), 2B_{NP}^d - \widehat{H}^d]$, then candidate A's expected vote share in state e, from such a strategy is

(A32)

$$S_1^A(p_1^A, p_1^*) = \frac{\widehat{H}^d}{2B_{NP}^d} + \left(1 - \frac{\widehat{H}^d}{B_{NP}^d}\right) \int_{\text{Supp}G_{p_1^A}} \frac{u - 1 - E_{\Gamma_e}(e - \widehat{\delta}^d(e))}{2B_P^d} dG_{p_1^A}(u)$$

Inserting, equations (A27) and (A28) into equation (A32), we have that

$$S_1^A(p_1^A, p_1^*) \leq \frac{\widehat{H}^d}{2B_{NP}^d} + \left(1 - \frac{\widehat{H}^d}{B_{NP}^d}\right) \frac{B_P^d}{2B_P^d} = \frac{1}{2} \tag{A33}$$

Thus, candidate A's expected vote share from any strategy with $\iota_A = 1$ and $\text{Supp}(G_{p_1^A}(u)) \in [1 + E_{\Gamma_e}(e - \widehat{\delta}^d(e)), 2B_{NP}^d - \widehat{H}^d]$ is less than or equal to $\frac{1}{2}$. The proof that in any best-response by candidate A with $\iota_A = 1$ it must be the case that $\text{Supp}(G_{p_1^A}(u)) \in [1 + E_{\Gamma_e}(e - \widehat{\delta}^d(e)), 2B_{NP}^d - \widehat{H}^d]$ follows along the same lines as the corresponding part of the proof of Theorem 1.

For the case that candidate A chooses a first-period platform p_1^A with $\iota_A = 0$ and $\text{Supp}(G_{p_1^A}(u)) \in [0, \widehat{H}^d] \cup [2B_{NP}^d - \widehat{H}^d, 2B_{NP}^d]$, it follows from equation (A31) candidate A's expected vote share is

$$S_1^A(p_1^A, p_1^*) = \int_{\text{Supp}G_{p_1^A}} \frac{u}{2B_{NP}^d} dG_{p_1^A}(u) \tag{A34}$$

Given budget feasibility with $\iota_A = 0$, see equation (7), it follows from equation (A34) that candidate A's expected vote share from any such a strategy p_1^A is less than or equal to 1/2, which holds with equality if p_1^A is budget balancing. The

proof that in any best-response by candidate A with $\iota_A = 0$ it must be the case that $\text{Supp}(G_{p_1^A}(u)) \in [0, \widehat{H}^d] \cup [2B^d_{NP} - \widehat{H}^d, 2B^d_{NP}]$ follows along the same lines as the corresponding part of the proof of Theorem 1.

A4. Proof of Theorem 3: Soft limit on debt

We first state the complete characterization of the equilibrium.

Corollary 2 *Given a soft debt constraint of $\overline{\delta} > 0$, the set of subgame perfect equilibria is completely characterized as follows.*

First Period

*In the first period, there are two cases labeled **(I.)** and **(II.)**.*

(I.) *If $\widehat{H}^{sd} \leq 0$, then in any subgame perfect equilibrium both candidates choose a first-period platform p_1^* that implements the policy with probability $\beta^* = 1$ and:*

- **(i)** *announce the maximum feasible average debt: $\min\{\overline{\delta}, 1 + \lambda E_{\Gamma_e}(e)\}$, and*
- **(ii)** *choose an $(|\mathcal{E}| + 1)$-variate joint distribution $P_1^*(\mathbf{x})$ of first-period net endowments such that the random variable $\widetilde{x}_1^{\Gamma_e}$ is uniformly distributed on the interval $[0, 2B_P^{sd}]$ and satisfies the constraint on the average first-period budget as defined in equation (13).*[6]

(II.) *If $\widehat{H}^{sd} > 0$, then in the unique subgame perfect equilibrium both candidates choose a first-period platform p_1^* that implements the policy with probability $\beta^* = 1 - \frac{\widehat{H}^{sd}}{B^{sd}_{NP}} (< 1)$ and:*

- **(i)** *announce the maximum feasible average debt: $\min\{\overline{\delta}, 1 + \iota(e)\lambda E_{\Gamma_e}(e)\}$, and*
- **(ii)** *choose an $(|\mathcal{E}| + 1)$-variate joint distribution $P^*_{i,1}(\mathbf{x})$ of first-period net endowments such that:*

(A35)

$$F_1^*(x|e = \emptyset) = \begin{cases} 0, & \text{if } x \leq 0, \\ \frac{1}{2}\left(\frac{x}{\widehat{H}^{sd}}\right), & \text{if } 0 \leq x \leq \widehat{H}^{sd}, \\ \frac{1}{2}, & \text{if } \widehat{H}^{sd} \leq x \leq 2B^{sd}_{NP} - \widehat{H}^{sd}, \\ \frac{1}{2}\left(1 + \frac{x - 2B^{sd}_{NP} + \widehat{H}^{sd}}{\widehat{H}^{sd}}\right), & \text{if } 2B^{sd}_{NP} - \widehat{H}^{sd} \leq x \leq 2B^{sd}_{NP}, \\ 1, & \text{if } x \geq 2B^{sd}_{NP}. \end{cases}$$

[6] Because $e = \emptyset$ arises with probability 0 when $\beta^* = 1$, in case (I.) any feasible specification of first-period transfers may be used to complete the specification of a strategy for the policy state $e = \emptyset$.

and for $e \neq \emptyset$, the random variable $\widetilde{x}_1^{\Gamma_e}$ is uniformly distributed on the interval $[0, 2B_p^{sd}]$ and satisfies the constraint on the average first-period budget as defined in equation (13).

Second Period

Given any second-period state $(e, \delta(e)) \in \mathcal{S}_{pd}$, the unique subgame perfect second-period local equilibrium is for each candidate to choose the second-period platform $p_2^*(e, \delta(e))$ that uniformly distributes net endowments on the interval $[0, 2(1 + \iota(e)\lambda e - \delta(e))]$. Along any equilibrium path, the equilibrium average debt level is $\min\{\overline{\delta}, 1 + \iota(e)\lambda E_{\Gamma_e}(e)\}$.

The proof of Theorem 3 and Corollary 2 follows along the same lines as the proof of Theorem 2 and Corollary 1, with the caveat that unlike the case of a hard debt limit, when the policy is implemented a soft debt limit does not directly impose conditions on the first-period budget for each of the individual policy states $e \in \mathcal{E}$. Instead the soft debt limit only imposes a constraint on the expectation of the first-period budget, across the set of policy states, when the policy is implemented, B_P^{sd}. Thus, the set of policy-state contingent public debt levels $\{\widehat{\delta}_i^{\eta^*}(e)\}_{e \in \mathcal{E}}$, given by equation (11) with η equal to η^*, provide one set of equilibrium policy-state contingent public debt levels, but the equilibrium debt level for each policy state is not pinned down by the soft debt constraint.

A5. Proof of Propositions 1 and 2

We begin with the proofs of the results on the probability with which the policy is implemented in equilibrium, parts (I.) and (III.) of Proposition 1 along with subpart (i) of parts (I.) and (II.) of Proposition 2, and then move on to the results on equilibrium inequality, parts (II.) and (IV.) of Proposition 1 along with subpart (ii) of parts (I.) and (II.) of Proposition 2.

For part (I.) of Proposition 1, given that (i) $\min\{\overline{\delta}, 1+\lambda E_{\Gamma_e}(e)\} \geq E_{\Gamma_e}(\min\{\overline{\delta}, 1+\lambda e\})$ for all $\overline{\delta} > 0$ and (ii) from equations (6) and (12) we know that $B_{NP}^d = B_{NP}^{sd} = 1 + \min\{\overline{\delta}, 1\}$, it follows directly from parts (I.) and (II.) of Theorems 2 and 3 that the equilibrium probability that the policy is implemented under the soft debt limit is at least as high as under the hard debt limit if and only if $\widehat{H}^{sd} \leq \widehat{H}^d$ for all $\overline{\delta} > 0$. Then, from the definitions of $\widehat{H}^d$ and $\widehat{H}^{sd}$ in equations (8) and (14) respectively and recalling that $B_P^d = 1 + E_{\Gamma_e}(\min\{\overline{\delta}, 1 + \lambda e\}) - c$, it follows that $\widehat{H}^{sd} \leq \widehat{H}^d$ requires that

$$\min\{\overline{\delta}, 1 + \lambda E_{\Gamma_e}(e)\} - E_{\Gamma_e}(\min\{\overline{\delta}, 1 + \lambda e\}) \leq 2\left(\min\{\overline{\delta}, 1 + \lambda E_{\Gamma_e}(e)\} - E_{\Gamma_e}(\min\{\overline{\delta}, 1 + \lambda e\})\right). \tag{A36}$$

Because $\min\{\overline{\delta}, 1 + \lambda E_{\Gamma_e}(e)\} \geq E_{\Gamma_e}(\min\{\overline{\delta}, 1 + \lambda e\})$ for all $\overline{\delta} > 0$, it follows that $\widehat{H}^{sd} \leq \widehat{H}^d$ for all $\overline{\delta} > 0$, and thus, the equilibrium probability that the policy is

implemented under the soft debt limit is at least as high as under the hard debt limit.

For part (III.) of Proposition 1, note that because $\widehat{H}^{sd} \leq \widehat{H}^{d}$ for all $\overline{\delta} > 0$ it follows that if $\widehat{H}^{sd} > 0$ then $\widehat{H}^{d} > 0$, and, as a result, the equilibrium probabilities of the policy being implemented under the hard debt limit and the soft debt limit are specified in Part (II.) of Theorems 2 and 3, respectively. Then, recalling that $B^{d}_{NP} = B^{sd}_{NP}$, it follows from Part (II.) of Theorems 2 and 3, that the equilibrium probability that the policy is implemented under the soft debt limit is strictly higher than under the hard debt limit if and only if $0 < \widehat{H}^{sd} < \widehat{H}^{d}$, which from equation (A36) requires that $\min\{\overline{\delta}, 1 + \lambda E_{\Gamma_e}(e)\} > E_{\Gamma_e}(\min\{\overline{\delta}, 1 + \lambda e\})$, which only holds in the case that the distribution of policy states, Γ_e, is non-degenerate and is equivalent to $\delta \in (1+\lambda\underline{e}, 1+\lambda\overline{e})$. Note that this corresponds to the portion of the parameter region in which the set of policy-state contingent public debt levels $\{\widehat{\delta}^{\eta*}_i(e)\}_{e\in\mathcal{E}}$ that are feasible under the soft debt limit differ from the set of feasible policy-state contingent debt levels under the hard debt limit.

The last two results on the probability with which the policy is implemented in equilibrium — subpart (i) of parts (I.) and (II.) of Proposition 2 — follow directly from equation (A36) and are illustrated in Figure 1.

Now we turn to the proofs of the results on equilibrium inequality. We begin by calculating the expected Gini coefficients in all three cases: a hard debt limit, a soft debt limit, and no debt limit. The proofs of parts (II.) and (IV.) of Proposition 1 along with subpart (ii) of parts (I.) and (II.) of Proposition 2 follow directly from the Gini coefficient calculations. Given that second-stage inequality is the same for all three cases, we focus on first-period inequality. To construct the Gini-coefficient, we begin with the calculation of the Lorenz curve. Given an equilibrium redistribution schedule F_1, the Lorenz curve is calculated as:

$$L(y) = \frac{\int_0^y F_1^{-1}(x)dx}{\int_0^1 F_1^{-1}(x)dx} \quad \text{for } y \in [0,1]. \tag{A37}$$

Then, the Gini Coefficient is calculated as:

$$G = 1 - 2\int_0^1 L(x)dx. \tag{A38}$$

Now we calculate the expected Gini coefficient in the case of a hard debt limit. If $\widehat{H}^{d} \leq 0$, then the redistribution schedule F_1 is uniform on $[0, 2B^{d}_{NP}]$ and the Gini Coefficient is $G^{d} = \frac{1}{3}$.

If $\widehat{H}^{d} \in [0,1]$, then:

1) With probability $(1 - \frac{\widehat{H}^{d}}{B^{d}_{NP}})$, the redistribution schedule F_1 is uniform on

$[0, 2B^d_{NP}]$ and the Gini Coefficient is $G^d = \frac{1}{3}$.

2) With probability $\frac{\widehat{H}^d}{B^d_{NP}}$, the redistribution schedule is:

(A39)

$$F_1 = \begin{cases} 0 & \text{if } x < 0 \\ \frac{1}{2}\left(\frac{x}{\widehat{H}^d}\right) & \text{if } x \in \left[0, \widehat{H}^d\right] \\ \frac{1}{2} & \text{if } x \in \left(\widehat{H}^d, 2B^d_{NP} - \widehat{H}^d\right) \\ \frac{1}{2}\left(1 + \frac{x - 2B^d_{NP} + \widehat{H}^d}{\widehat{H}^d}\right) & \text{if } x \in \left[2B^d_{NP} - \widehat{H}^d, 2B^d_{NP}\right] \\ 1 & \text{if } x > 2B^d_{NP} \end{cases}$$

in which case the full inverse F_1^{-1} is the set-valued function:

(A40)

$$F_1^{-1}(x) = \begin{cases} 2\widehat{H}^d x & x \in \left[0, \frac{1}{2}\right) \\ \left[\widehat{H}^d, 2B^d_{NP} - \widehat{H}^d\right] & x = \frac{1}{2} \\ 2\widehat{H}^d x - 2\widehat{H}^d + 2B^d_{NP} & x \in \left(\frac{1}{2}, 1\right] \end{cases}$$

Given the expression for the full inverse $F_1^{-1}(x)$ from equation (A40), the equation (A37) formula for the Lorenz curve may be written as:

(A41)

$$\begin{aligned} L(y) &= \frac{\int_0^y F_1^{-1}(x)dx}{\int_0^1 F_1^{-1}(x)dx} \\ &= \frac{\int_0^y 2\widehat{H}^d x dx + \int_{0.5}^{\min\{0.5,y\}} (2B^d_{NP} - 2\widehat{H}^d)dx}{\int_0^1 2\widehat{H}^d x dx + \int_{0.5}^1 (2B^d_{NP} - 2\widehat{H}^d)dx} \\ &= \frac{y^2\widehat{H}^d + \int_{0.5}^{\min\{0.5,y\}} (2B^d_{NP} - 2\widehat{H}^d)dx}{B^d_{NP}} \\ &= \begin{cases} \frac{y^2\widehat{H}^d}{B^d_{NP}} & \text{if } y \in \left[0, \frac{1}{2}\right) \\ \frac{y^2\widehat{H}^d}{B^d_{NP}} + \frac{\left(2B^d_{NP} - 2\widehat{H}^d\right)\left(y - \frac{1}{2}\right)}{B^d_{NP}} & \text{if } y \in \left[\frac{1}{2}, 1\right] \end{cases} \end{aligned}$$

Using the expression for the Lorenz curve $L(y)$ from the last equality in

equation (A41), the equation (A38) formula for the Gini Coefficient may be written as:

$$\begin{aligned} G^d &= 1 - 2\int_0^1 L(x)dx \\ &= 1 - 2\left[\int_0^1 \frac{y^2\widehat{H}^d}{B^d_{NP}}dy + \int_{0.5}^1 \frac{(2B^d_{NP} - 2\widehat{H}^d)(y-\frac{1}{2})}{B^d_{NP}}dy\right] \\ &= 1 - 2\left[\left(\frac{y^3}{3}\frac{\widehat{H}^d}{B^d_{NP}}\right)\Big|_0^1 + \left(\frac{(B^d_{NP} - \widehat{H}^d)(y^2-y)}{B^d_{NP}}\right)\Big|_{0.5}^1\right] \\ &= 1 - 2\left[\frac{1}{3}\frac{\widehat{H}^d}{B^d_{NP}} - \frac{1}{4}\frac{(B^d_{NP} - \widehat{H}^d)}{B^d_{NP}}\right] \\ &= \frac{1}{2} - \frac{\widehat{H}^d}{6B^d_{NP}} \end{aligned} \tag{A42}$$

To summarize, in the case that $\widehat{H}^d \in [0,1]$, we have that with probability $\frac{\widehat{H}^d}{B^d_{NP}}$ the redistribution schedule F_1 is given by equation (A39) and the corresponding Gini Coefficient is given by the last equality in equation (A42), while with probability $(1 - \frac{\widehat{H}^d}{B^d_{NP}})$ the redistribution schedule F_1 is uniform on $[0, 2B^d_{NP}]$ with Gini Coefficient $\frac{1}{3}$. Thus, the expected Gini Coefficient $E(G^d)$ is:

$$E(G^d) = \left(\frac{\widehat{H}^d}{B^d_{NP}}\right)\left[\frac{1}{2} - \frac{\widehat{H}^d}{6B^d_{NP}}\right] + \left(1 - \frac{\widehat{H}^d}{B^d_{NP}}\right)\left[\frac{1}{3}\right] = \frac{1}{3} + \frac{1}{6}\frac{\widehat{H}^d}{B^d_{NP}} - \frac{1}{6}\left(\frac{\widehat{H}^d}{B^d_{NP}}\right)^2 \tag{A43}$$

The cases of a soft debt limit and no debt limit follow along similar lines, and we now address the case of a soft debt limit. If $\widehat{H}^{sd} \leq 0$, then the redistribution schedule F_1 is uniform on $[0, 2B^{sd}_{NP}]$, and the Gini Coefficient is $G^{sd} = \frac{1}{3}$.

If $\widehat{H}^{sd} \in [0,1]$, then:

1) With probability $(1 - \frac{\widehat{H}^{sd}}{B^{sd}_{NP}})$, the redistribution schedule F_1 is uniform on $[0, 2B^{sd}_{NP}]$ with the Gini Coefficient is $G^{sd} = \frac{1}{3}$.

2) With probability $\frac{\widehat{H}^{sd}}{B^{sd}_{NP}}$, the redistribution schedule is:

(A44)
$$F_1 = \begin{cases} 0 & \text{if } x < 0 \\ \frac{1}{2}\left(\frac{x}{\widehat{H}^{sd}}\right) & \text{if } x \in \left[0, \widehat{H}^{sd}\right] \\ \frac{1}{2} & \text{if } x \in \left(\widehat{H}^{sd}, 2B^{sd}_{NP} - \widehat{H}^{sd}\right) \\ \frac{1}{2}\left(1 + \frac{x - 2B^{sd}_{NP} + \widehat{H}^{sd}}{\widehat{H}^{sd}}\right) & \text{if } x \in \left[2B^{sd}_{NP} - \widehat{H}^{sd}, 2B^{sd}_{NP}\right] \\ 1 & \text{if } x > 2B^{sd}_{NP} \end{cases}$$

The full inverse and Lorenz curve follow similar lines as the Hard limit. In the case that $\widehat{H}^{sd} \in [0, 1]$ the expected Gini Coefficient is:

$$E(G^{sd}) = \frac{1}{3} + \frac{1}{6}\frac{\widehat{H}^{sd}}{B^{sd}_{NP}} - \frac{1}{6}\left(\frac{\widehat{H}^{sd}}{B^{sd}_{NP}}\right)^2. \tag{A45}$$

In the final case of no debt limit, if $H \leq 0$, then the redistribution schedule F_1 is uniform on $[0, 4 + 2\lambda E_{\Gamma_e}(e) - 2c]$ and the Gini Coefficient is $G = \frac{1}{3}$.

If $H \in [0, 1]$, then:

1) With probability $(1 - \frac{H}{2})$, the redistribution schedule F_1 is uniform on $[0, 4 + 2\lambda E_{\Gamma_e}(e) - 2c]$ with $G = \frac{1}{3}$.

2) With probability $\frac{H}{2}$, the redistribution schedule is:

$$F_1 = \begin{cases} 0 & \text{if } x < 0 \\ \frac{1}{2}\left(\frac{x}{H}\right) & \text{if } x \in [0, H] \\ \frac{1}{2} & \text{if } x \in \left(\widehat{H}, 4 - H\right) \\ \frac{1}{2}\left(1 + \frac{x - 4 + H}{H}\right) & \text{if } x \in [4 - H, 4] \\ 1 & \text{if } x > 4 \end{cases} \tag{A46}$$

The full inverse and Lorenz curve follow similar lines as the Hard limit. The expected Gini Coefficient is:

$$E(G) = \frac{1}{3} + \frac{H}{12} - \frac{H^2}{24}. \tag{A47}$$

Given the expected Gini coefficients for all three cases, we can now compare the equilibrium inequality that arises in the three cases. Beginning with the comparison between hard and soft limits, recall that as noted above in the proof of Part (I.) of Proposition 1 that because $\min\{\overline{\delta}, 1+\lambda E_{\Gamma_e}(e)\} \geq E_{\Gamma_e}(\min\{\overline{\delta}, 1+\lambda e\})$ for all $\overline{\delta} > 0$, it follows that $\widehat{H}^{sd} \leq \widehat{H}^{d}$ for all $\overline{\delta} > 0$. Thus, it follows from equations (A43) and (A45) that

$$E(G^{sd}) \leq E(G^{d}). \tag{A48}$$

That is, hard limits result in weakly higher inequality than soft limits. This completes the proof of part (II) of Proposition 1. If $\min\{\overline{\delta}, 1 + \lambda E_{\Gamma_e}(e)\} > E_{\Gamma_e}(\min\{\overline{\delta}, 1 + \lambda e\})$ then $\widehat{H}^{sd} \leq \widehat{H}^{d}$ and hard limits result in strictly higher inequality than soft limits. This completes the proof of part (IV) of Proposition 1. The proofs of subpart (ii) of parts (I.) and (II.) of Proposition 2 follow directly from equations (A43) and (A45).

Regarding the comparison between hard limits, soft limits and no limits, note that debt limits may raise or lower inequality relative to the no limits case, depending on the parameter configuration. For example as $E_{\Gamma}(e) \to c$, $c \to 1$, $\lambda \to 0$, and $\overline{\delta} \to 0$ limits may lower inequality. However, as $E_{\Gamma}(e) \to c$, $c \to 1$, $\lambda \to 1$, and $\overline{\delta} \to 1$ limits may raise inequality.

Appendix: Empirical analysis

We describe the indicators used in our empirical approach. A summary of the variables outlined below is provided in Table B1 at the end of the section.

General Public Investment: Government Investment as a percentage of GDP. This indicator is constructed from several relevant OECD variables to obtain a measure of investment undertaken by government authorities as a percentage of GDP. Specifically, we combine OECD (2021e) and OECD (2021d). Our indicator displays the annual amount of Gross Fixed Capital Formation (GFCF) carried out by all levels of government for a given country and year. As per the OECD's variable description, government investment "typically means investment in R&D, military weapons systems, transport infrastructure and public buildings such as schools and hospitals" (OECD (2021d)). This indicator should therefore encompass forms of public investment with targeted benefits as well as those that are similar to a public good in nature.

Targeted Transfers: Government Spending on Housing and Community Amenities as a percentage of GDP. This indicator is provided by OECD data concerning the composition of government spending and is collected by national governments according to the 2008 System of National Accounts criteria (OECD (2021b)). Corresponding to COFOG division 06 as per United Nations (2000), it is comprised of government spending in the following areas: Housing

development (06.1), Community development (06.2), Water supply (06.3), Street lighting (06.4), R&D housing and community amenities (06.5), and Housing and community amenities not elsewhere classified (06.6). We chose this variable as a measure of targetable government spending/investment as we would expect this category of expenditure to be easily directed towards specific localities or constituencies, with the resulting benefits largely confined to these groups of intended recipients. Alternatively, in the Supplementary material, we consider two government expenditure functions within the COFOG division 08 (Recreation, Culture and Religion): Recreational and sporting services (08.1), and Cultural services (08.2).

Debt: Gross Government Debt as a percentage of GDP. This variable is self-explanatory and is taken from an OECD database reporting levels of government debt from 1995 onward (OECD (2021a)).

Gini coefficient. Series are sourced from the World Inequality Database (see World Inequality Lab (2017)). We consider the Gini coefficient for the distribution of post-tax national income among adults (individuals aged over 20) with an equal split of resources within couples. In the database, post-tax national income corresponds to the "sum of primary incomes over all sectors (private and public), minus taxes". Relatively to post-tax disposable income, it includes the income sources that cannot be individualized (e.g., collective consumption expenditures), which are distributed equally across individuals.

Eurozone Membership. This variable is used to investigate whether differential relationships exist between debt and our dependent variables of interest for Eurozone and non-Eurozone states, which may be partially explained by the fiscal rules of the Eurozone. Eurozone members are sourced from European Commission (2023).

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Table B1—: Description of variables and sources

Variable	Description	Source
Investment (GFCF)	Annual level of total investment (public & private) carried out at the national level, at current prices in millions of USD (PPP)	OECD (2021e)
Investment by sector	Annual share of total investment carried out by sector (government, household, or corporate) at the national level	OECD (2021d)
Gross domestic product (GDP)	Annual level of GDP at current prices in millions of USD (PPP)	OECD (2021c)
Government expenditure on housing and community amenities	Annual share of government spending as a percentage of GDP on: Housing development, Community development, Water supply, Street lighting, R&D for housing and community amenities, and Housing and community amenities not elsewhere classified*	OECD (2021b)
General government debt	Annual level of gross government debt as a percentage of GDP	OECD (2021a)
Gini coefficient	Gini coefficient for the distribution of post-tax national income among adults (individuals aged above 20) with an equal split of resources within couples. Among OECD member states, the Gini coefficient is available for 32 countries over different time windows. These data thus exclude six OECD members: *Canada, Israel, Japan, Korea, New Zealand, Turkey*	World Inequality Lab (2017)
Eurozone membership	The list of countries categorized as belonging to the Eurozone in our analysis is: *Austria, Belgium, Estonia, Finland, France, Germany, Greece, Ireland, Italy, Latvia, Lithuania, Luxembourg, Netherlands, Portugal, Slovakia, Slovenia, Spain, Denmark**. Data required in the analysis were not available for remaining Eurozone members, Cyprus and Malta	European Commission (2023)

Note: First, the series of government expenditure on housing and community amenities displays an outlying value for the Netherlands in 1995, due to the so-called "grossing and balancing operation" operated that year (European Commission (2012)); we exclude this one-off peak from our analysis. Second, Denmark does not formally belong to the Eurozone; however, it was a signatory to the 2012 European Fiscal Compact and we have thus chosen to consider it as part of the Eurozone for our purposes.

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