

Online Appendix for “Managerial Attention and Worker Performance”

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This Online Appendix contains the proof of our results for the undiscounted limit discussed in Section 2 of the paper, the proof of Proposition 2, details for the discussion of the forward-looking agent case described in Section 4, and an analysis of discontinuous equilibria.

A1 Undiscounted limit

Denote $\underline{a} \equiv \lim_{s \rightarrow \infty} a_s$. We prove the following result:

Proposition A1. *Fix any set of parameters (γ, μ, \bar{b}) and consider the limit as r goes to zero. Let $F < \bar{F}$ so that the equilibrium of Proposition 1 exists. As F approaches \bar{F} , \underline{a} vanishes, so the probability of returning to high performance goes to zero as time passes without recognition.*

Proof. Consider $\dot{\Psi}_s$ for $s \leq \bar{s}$, given by equation (26). In the limit as $r \rightarrow 0$, we have

$$\dot{\Psi}_s = \mu \bar{b} x_s - \gamma(F + \bar{\Psi} - \Psi_s) + \gamma \mu \bar{b} \int_s^{\bar{s}} x_t dt + \mu^2 \bar{b} x_s \Psi_s - \underline{a}, \quad (\text{A1})$$

where $\underline{a} = r \pi_s^L$. Using (27) and substituting with (22),

$$\underline{a} = \mu \bar{b} \bar{x} - \frac{\gamma^2 F}{\mu^2 \bar{b} \bar{x}} - (1 - \bar{x}) \gamma F. \quad (\text{A2})$$

Substituting (22) and (A2) in (A1) yields

$$\begin{aligned} \dot{\Psi}_s &= \mu \bar{b} x_s - \gamma \left(F + \frac{\gamma F}{\mu^2 \bar{b} \bar{x}} - \Psi_s \right) + \gamma \mu \bar{b} \int_s^{\bar{s}} x_t dt + \mu^2 \bar{b} x_s \Psi_s - \left[\mu \bar{b} \bar{x} - \frac{\gamma^2 F}{\mu^2 \bar{b} \bar{x}} - (1 - \bar{x}) \gamma F \right] \\ &= (\mu^2 \bar{b} x_s + \gamma) \Psi_s + \mu \bar{b} (x_s - \bar{x}) + \gamma \mu \bar{b} \int_s^{\bar{s}} x_t dt - \gamma F \bar{x}. \end{aligned}$$

Solving this differential equation with initial condition $\Psi_0 = 0$ gives that for $s \leq \bar{s}$,

$$\Psi_s = \int_0^s \left[\mu \bar{b} (x_\tau - \bar{x}) + \gamma \mu \bar{b} \int_\tau^{\bar{s}} x_t dt - \gamma F \bar{x} \right] e^{\int_\tau^s (\mu^2 \bar{b} x_i + \gamma) di} d\tau. \quad (\text{A3})$$

Following the same steps as in the proof of Proposition 1, an equilibrium is a value of $\bar{s} \in (0, \infty)$ such that (i) $\underline{a} \geq 0$ and (ii) $\Psi_{\bar{s}} = \bar{\Psi}$. For condition (i), note that the right-hand side of (A2) is increasing in \bar{x} and $\frac{\partial \bar{x}}{\partial \bar{s}} = -\gamma \bar{x} - \bar{x}^2(1 - \bar{x})\mu^2 \bar{b} < 0$, so \underline{a} is decreasing in \bar{s} . Note also that the value of \bar{s} that makes $\underline{a} = 0$ is finite. Hence, making the dependence of \underline{a} on \bar{s} explicit, $\underline{a}(\bar{s}) \geq 0$ is equivalent to $\bar{s} \leq \bar{s}^{\max}$ for \bar{s}^{\max} defined by $\underline{a}(\bar{s}^{\max}) = 0$. Note that \bar{s}^{\max} is a continuous and differentiable function of parameters.

Using (A3), condition (ii) is equivalent to

$$\bar{x} \int_0^{\bar{s}} \left[\mu \bar{b}(x_\tau - \bar{x}) + \gamma \mu \bar{b} \int_\tau^{\bar{s}} x_i dt - \gamma F \bar{x} \right] e^{\int_\tau^{\bar{s}} (\mu^2 \bar{b} x_i + \gamma) di} d\tau = \frac{\gamma F}{\mu^2 \bar{b}}. \quad (\text{A4})$$

Denote the left-hand side of (A4) by $lhs(\bar{s})$ and the right-hand side by rhs . We show that if \bar{s}' is an equilibrium, then $lhs(\bar{s})$ is strictly increasing at $\bar{s} = \bar{s}'$. The derivative of $lhs(\bar{s})$ with respect to \bar{s} is

$$\frac{\partial lhs(\bar{s})}{\partial \bar{s}} = \Psi_{\bar{s}} \frac{\partial \bar{x}}{\partial \bar{s}} + \bar{x} \left\{ \int_0^{\bar{s}} \left[-(\mu \bar{b} + \gamma F) \frac{\partial \bar{x}}{\partial \bar{s}} + \gamma \mu \bar{b} \bar{x} \right] e^{\int_\tau^{\bar{s}} (\mu^2 \bar{b} x_i + \gamma) di} d\tau \right. \\ \left. + \Psi_{\bar{s}} (\mu^2 \bar{b} \bar{x} + \gamma) - \gamma F \bar{x} \right\}.$$

Substituting with $\frac{\partial \bar{x}}{\partial \bar{s}} = -\gamma \bar{x} - \bar{x}^2(1 - \bar{x})\mu^2 \bar{b}$ and $\Psi_{\bar{s}} = \bar{\Psi}$ and canceling and rearranging terms yields

$$\frac{\partial lhs(\bar{s})}{\partial \bar{s}} = \bar{x} \left\{ (\mu \bar{b} + \gamma F) [\gamma \bar{x} + \bar{x}^2(1 - \bar{x})\mu^2 \bar{b}] + \gamma \mu \bar{b} \bar{x} \right\} \int_0^{\bar{s}} e^{\int_\tau^{\bar{s}} (\mu^2 \bar{b} x_i + \gamma) di} d\tau > 0.$$

Since both $lhs(\bar{s})$ and rhs are continuous, this implies that the equilibrium threshold time \bar{s} is unique: there exists a unique point \bar{s}' where $lhs(\bar{s}') = rhs$. Moreover, the fact that the derivative of $lhs(\bar{s})$ is bounded away from zero allows to apply the Implicit Function Theorem and obtain that the equilibrium is continuous (in fact differentiable) in the parameters. Hence, given an original equilibrium \bar{s}' with $\bar{s}' < \bar{s}^{\max'}$, a new equilibrium with $\bar{s}'' < \bar{s}^{\max''}$ exists for any local change of parameters.

We next show that increasing F reduces \underline{a} . Together with the results above, this implies that starting from any given continuous equilibrium with investment, one can increase F until \underline{a} becomes arbitrarily close to zero in equilibrium. Note that for a fixed \bar{s} , rhs increases when F increases whereas $lhs(\bar{s})$ decreases pointwise. Therefore, the point \bar{s} at which $lhs(\bar{s}) = rhs$ increases when F increases. Note that \bar{x} depends on F only through \bar{s} , and $\frac{\partial \bar{s}}{\partial F} > 0$ implies

$\frac{\partial \bar{x}}{\partial F} < 0$. Hence, using (A2),

$$\frac{\partial \underline{a}}{\partial F} = \left(\mu \bar{b} + \frac{\gamma^2 F}{\mu^2 \bar{b} \bar{x}^2} + \gamma F \right) \frac{\partial \bar{x}}{\partial F} - \frac{\gamma^2}{\mu^2 \bar{b} \bar{x}} - (1 - \bar{x})\gamma < 0.$$

Q.E.D.

A2 Proof of Proposition 2

The case with $\mu > \nu$ is analogous to that studied in Proposition 1 and thus omitted. Consider $\mu \leq \nu$.

The construction of the equilibrium is simple. Given a threshold time $\hat{s} \in (0, \infty)$, the law of motion for the agent's belief on $[0, \hat{s}]$ is given by (9). The solution to this differential equation with initial condition $x_0 = 1$ yields the agent's belief x_s and the agent's effort $a_s = (\mu \bar{b} - \nu \underline{b})x_s$ for $s \in [0, \hat{s}]$. Since the right-hand side of (9) is Lipschitz continuous in x , it follows from the Picard-Lindelöf theorem that there exists a solution and it is unique. Note that this solution does not depend on \hat{s} , and that both x_s and a_s are decreasing for all $s < \hat{s}$. Denote the values at \hat{s} by $x_{\hat{s}} \equiv \hat{x}(\hat{s})$ and $a_{\hat{s}} \equiv \hat{a}(\hat{s})$; we omit the dependence on \hat{s} in what follows.

As explained in the text, (10) implies that the agent's effort must be constant for all $s \geq \hat{s}$. Therefore, given continuity of x_s , the agent's belief and effort must be $x_s = \hat{x}$ and $a_s = \hat{a}$ for all $s \geq \hat{s}$. Moreover, given these constant values, the principal's investment is also pinned down: setting $\dot{x}_s = 0$ in (8), we obtain that for all $s \geq \hat{s}$, q_s must be equal to

$$\hat{q} = \frac{\gamma \hat{x}}{1 - \hat{x}} + \hat{x}[\mu \hat{a} + \nu(1 - \hat{a})].$$

Consider now the claim that any continuous equilibrium with positive investment must take this form. First, note that this equilibrium is the unique continuous equilibrium with positive investment where, at each time $s \geq 0$, the principal either is indifferent or does not have incentives to invest. This follows from (10), which implies that in any continuous equilibrium, the agent's effort and the principal's value of recognition must be constant for all times $s \geq \tilde{s}$ if the principal is indifferent between investing and not investing at \tilde{s} . Next, consider continuous equilibria in which the principal has strict incentives to invest over some time interval. By the same reasoning as in the proof of Proposition 1, there exists $\Delta > 0$ such that the principal has strict incentives to invest at $s \in [0, \Delta]$. However, this requires $[\mu a_0 + \nu(1 - a_0)]\Psi_0 \geq (\gamma + r)F$, which cannot be satisfied since $\Psi_0 = 0$. Thus, a continuous

equilibrium in which the principal has strict incentives to invest does not exist, and the claim follows.

Finally, we prove the claims in fn. 18 of the paper. As explained above, the solution to (9) (with initial condition $x_0 = 1$) uniquely determines x_s , and thus $a_s = (\mu\bar{b} - \nu\underline{b})x_s$, for $s \leq \hat{s}$, independently of the value of \hat{s} . Moreover, note that for any given \hat{s} , the values of a_s , π_s^L and π_s^H are pinned down for $s \geq \hat{s}$ —these values are $a_s = \hat{a}$, $\pi_s^L = \frac{\hat{a}}{r}$, and $\pi_s^H = \pi_s^L + F$ —and as a result the values of π_s^L and π_s^H are also pinned for $s \leq \hat{s}$:

$$\begin{aligned}\pi_s^L &= \int_s^{\hat{s}} e^{-r(\tau-s)} a_\tau d\tau + e^{-r(\hat{s}-s)} \frac{\hat{a}}{r}, \\ \pi_s^H &= \int_s^{\hat{s}} e^{-(\gamma+r)(\tau-s) - \int_s^\tau [\mu a_z + \nu(1-a_z)] dz} \left\{ a_\tau + \gamma \pi_\tau^L + [\mu a_\tau + \nu(1-a_\tau)] \pi_0^H \right\} d\tau \\ &\quad + e^{-(\gamma+r)(\hat{s}-s) - \int_s^{\hat{s}} [\mu a_z + \nu(1-a_z)] dz} \left(\frac{\hat{a}}{r} + F \right).\end{aligned}\tag{A5}$$

Using (A5), it follows that for any given \hat{s} , $\Psi_{\hat{s}} = \pi_0^H - \pi_{\hat{s}}^H$ is given by

$$\Psi_{\hat{s}} = \int_0^{\hat{s}} e^{-(\gamma+r)\tau - \int_0^\tau [\mu a_z + \nu(1-a_z)] dz} \left\{ \begin{array}{l} (a_\tau - \hat{a}) + \gamma(\pi_\tau^L - \pi_{\hat{s}}^L) \\ + [\mu a_\tau + \nu(1-a_\tau)] \Psi_{\hat{s}} - (\gamma+r)F \end{array} \right\} d\tau.$$

It is immediate to verify that $\Psi_{\hat{s}}$ is strictly increasing in \hat{s} and is thus bounded above by $\lim_{\hat{s} \rightarrow \infty} \Psi_{\hat{s}}$, which is finite. Note that $\mu\hat{a} + \nu(1-\hat{a})$ is also strictly increasing in \hat{s} and is bounded above by ν . Therefore, it follows that there exists $\hat{F} > 0$ such that a time \hat{s} at which (10) is satisfied (i.e., $\Psi_{\hat{s}}[\mu a_{\hat{s}} + \nu(1-a_{\hat{s}})] = (\gamma+r)F$) exists if and only if $F \leq \hat{F}$, and such a time \hat{s} is unique. Given the construction and claims above, this proves that a continuous equilibrium with positive investment exists if and only if F is small enough, and such an equilibrium is unique.

A3 Details for Section 4

In this section, we describe how we solve numerically the case of a forward-looking agent discussed in Section 4.

Agent's problem. The agent's expected payoff at $s = 0$ is

$$U_0 = \int_0^\infty e^{-rs} (1 - R_s) \left[\mu x_s a_s (\bar{b} + U_0) - \frac{1}{2} a_s^2 \right] ds,\tag{BC0}$$

where $R_s \equiv 1 - e^{-\int_0^s \mu x_\tau a_\tau d\tau}$. The agent's optimization problem is

$$\max_{a_s} U_0 = \int_0^\infty e^{-rs} (1 - R_s) \left[\mu x_s a_s (\bar{b} + U_0) - \frac{1}{2} a_s^2 \right] ds$$

$$\text{subject to } \dot{x}_s = -\gamma x_s - x_s(1 - x_s)\mu a_s + (1 - x_s)q_s, \quad (\text{A6})$$

$$\dot{R}_s = (1 - R_s)\mu a_s x_s, \quad (\text{A7})$$

$$x_0 = 1, \quad R_0 = 0. \quad (\text{BC1})$$

Note that given the principal's equilibrium strategy, the agent faces a relatively simple single-agent experimentation problem where the evolution of the underlying state (the principal's type) depends only on recognition, as the principal's investment is only a function of the time that has passed since recognition (and her type). The agent's action affects both the payoff process and the learning process, and the forward-looking agent takes this into account when choosing effort. In particular, we solve for the agent's optimal sequence of effort taking into account how his belief will evolve and affect effort choices depending on the effort he chooses. This computation ensures that no deviation (including double deviations) is profitable for the agent.

For multipliers $\lambda_{1s}, \lambda_{2s}$, the Hamiltonian is:

$$H = \int_0^\infty \left\{ e^{-rs} (1 - R_s) \left[\mu x_s a_s (\bar{b} + U_0) - \frac{1}{2} a_s^2 \right] + \lambda_{1s} \dot{x}_s + \lambda_{2s} \dot{R}_s \right\} ds.$$

The first order conditions with respect to a_s, λ_{1s} and λ_{2s} yield

$$\begin{aligned} 0 &= e^{-rs} (1 - R_s) [\mu x_s (\bar{b} + U_0) - a_s] - \lambda_{1s} x_s (1 - x_s) \mu + \lambda_{2s} (1 - R_s) \mu x_s, \\ -\dot{\lambda}_{1s} &= e^{-rs} (1 - R_s) \mu a_s (\bar{b} + U_0) - \lambda_{1s} [\gamma + (1 - 2x_s)\mu a_s + q_s] + \lambda_{2s} (1 - R_s) \mu a_s, \\ -\dot{\lambda}_{2s} &= -e^{-rs} \left[\mu x_s a_s (\bar{b} + U_0) - \frac{1}{2} a_s^2 \right] - \lambda_{2s} \mu a_s x_s. \end{aligned}$$

Replacing with $m_{1s} = \lambda_{1s} e^{rs}$ and $m_{2s} = \lambda_{2s} e^{rs}$,

$$0 = (1 - R_s) [\mu x_s (\bar{b} + U_0) - a_s] - m_{1s} x_s (1 - x_s) \mu + m_{2s} (1 - R_s) \mu x_s, \quad (\text{A8})$$

$$-\dot{m}_{1s} + r m_{1s} = (1 - R_s) \mu a_s (\bar{b} + U_0) - m_{1s} [\gamma + (1 - 2x_s)\mu a_s + q_s] + m_{2s} (1 - R_s) \mu a_s, \quad (\text{A9})$$

$$-\dot{m}_{2s} + r m_{2s} = - \left[\mu x_s a_s (\bar{b} + U_0) - \frac{1}{2} a_s^2 \right] - m_{2s} \mu a_s x_s. \quad (\text{A10})$$

The transversality condition on m_{1s} and m_{2s} is

$$\lim_{s \rightarrow \infty} e^{-rs} m_{1s} = \lim_{s \rightarrow \infty} e^{-rs} m_{2s} = 0. \quad (\text{BC2})$$

Given the principal's investment q_s , the agent's effort and belief are determined by equations (A6)-(A10) and boundary conditions (BC0)-(BC2).

Equilibrium dynamics and smooth pasting. We consider an equilibrium with threshold time $\bar{s} \in (0, \infty)$ so that the principal does not invest at $s < \bar{s}$ and she mixes between investing and not investing at $s \geq \bar{s}$. As in the case of a myopic agent, we have that for $s < \bar{s}$,

$$\dot{\Lambda}_s = (\gamma + r)\Lambda_s - \mu a_s \Psi_s, \quad (\text{A11})$$

$$\dot{\Psi}_s = -\dot{\Lambda}_s - \dot{\pi}_s^L, \quad (\text{A12})$$

$$\dot{\pi}_s^L = -a_s + r\pi_s^L, \quad (\text{A13})$$

with boundary conditions

$$\Lambda_{\bar{s}} = F, \quad \Psi_0 = 0, \quad \text{and} \quad \pi_{\bar{s}}^L = \bar{\pi}^L, \quad (\text{BC3})$$

where $\bar{\Psi} = \frac{(\gamma+r)F}{\mu a_{\bar{s}}}$ and $\bar{\pi}^L$ is derived below.

For $s \geq \bar{s}$, $\Lambda_s = F$, so the system is

$$\dot{\Lambda}_s = 0, \quad (\text{A14})$$

$$\dot{\Psi}_s = -\dot{\pi}_s^L, \quad (\text{A15})$$

$$\dot{\pi}_s^L = -a_s + r\pi_s^L, \quad (\text{A16})$$

with boundary conditions

$$\Lambda_{\bar{s}} = F, \quad \Psi_{\bar{s}} = \bar{\Psi}, \quad \text{and} \quad \pi_{\bar{s}}^L = \bar{\pi}^L. \quad (\text{BC4})$$

The value of $\bar{\pi}^L$ is obtained from smooth pasting: we require that a_s and x_s be continuously differentiable at \bar{s} . This gives

$$\bar{\pi}^L = \frac{a_{\bar{s}} - \dot{\Psi}_{\bar{s}}}{r} = \frac{1}{r} \left(\frac{(r + \gamma)F}{\mu \Psi_{\bar{s}}} + \frac{\dot{a}_{\bar{s}} \mu \Psi_{\bar{s}}^2}{(\gamma + r)F} \right).$$

Solving for the equilibrium. We find values $(\bar{s}, U_0, m_{10}, m_{20})$ such that equations (A6)-(A16) and boundary conditions (BC0)-(BC4) are satisfied. Begin by fixing a set of initial values $(\bar{s}, U_0, m_{10}, m_{20})$. We proceed as follows:

1. Solve the agent's problem for $s < \bar{s}$. Given $(\bar{s}, U_0, m_{10}, m_{20})$ and initial conditions (BC1), and setting $q_s = 0$, we can solve (A6)-(A10) on $[0, \bar{s}]$. We obtain a_s, x_s, R_s, m_{1s} , and m_{2s} for $s < \bar{s}$.
2. Solve the system characterizing equilibrium dynamics for $s \geq \bar{s}$. We solve (A14)-(A16) given the boundary conditions (BC4). We obtain a_s, π_s^L, Ψ_s , and Λ_s for $s \geq \bar{s}$.
3. Solve for the agent's belief and the principal's investment for $s \geq \bar{s}$. We obtain the belief x_s on $[\bar{s}, \infty)$ by inputting the effort path a_s obtained in step 2 into the agent's problem (A6)-(A10). Then having solved for x_s and a_s , we can solve for the investment q_s on $[\bar{s}, \infty)$. We obtain q_s, x_s, R_s, m_{1s} , and m_{2s} for $s \geq \bar{s}$.
4. Solve the system characterizing equilibrium dynamics for $s < \bar{s}$. We solve (A11)-(A13) given boundary conditions (BC3). Note that the value of Λ_s is unknown here but we can solve the system because a_s is pinned down at this point. We obtain π_s^L, Ψ_s , and Λ_s for $s \leq \bar{s}$.
5. Compare solution to initial values. Having solved for all variables, compute now the resulting values for the value of recognition and the agent's expected payoff at time $s = 0$, which we can denote by $\tilde{\Psi}_0$ and \tilde{U}_0 respectively, and the limits $\lim_{s \rightarrow \infty} e^{-rs} m_{1s}$ and $\lim_{s \rightarrow \infty} e^{-rs} m_{2s}$. If given initial values $(\bar{s}, U_0, m_{10}, m_{20})$, we obtain $\tilde{\Psi}_0 = 0, \tilde{U}_0 = U_0, \lim_{s \rightarrow \infty} e^{-rs} m_{1s} = 0$, and $\lim_{s \rightarrow \infty} e^{-rs} m_{2s} = 0$, then we have found an equilibrium. Otherwise we change the initial values, searching on a grid of $(\bar{s}, U_0, m_{10}, m_{20})$, until these four conditions are satisfied up to some precision target.

A4 Discontinuous equilibria

Consider the setting of Section 1. Our analysis in the paper restricted attention to equilibria in which the agent's belief as a function of the time since recognition, x_s , is continuous. In this section, we study equilibria in which this belief can jump. Because such equilibria can in principle take many arbitrary forms, we focus on a simple class of discontinuous equilibria that are stationary. We show that the principal prefers the continuous equilibrium characterized in Proposition 1 to any discontinuous equilibrium in this class.

We define a stationary discontinuous equilibrium as an equilibrium in which the principal does not invest except in countably many points s^1, s^2, \dots such that, for all $n \in \mathbb{N} = \{1, 2, \dots\}$, (i) $s^{n+1} = s^n + \Delta$ for some $\Delta > 0$, and (ii) the principal invests with a mass probability $\kappa > 0$ at s^n . Denote the set of times at which the principal invests by $J = \{s^1, s^1 + \Delta, s^1 + 2\Delta, \dots\}$. s^1 , Δ , and κ are such that for some $0 \leq x^- < x^+ \leq 1$, the agent's belief that the principal is a high type satisfies $x_{s^-} = x^-$ and $x_{s^+} = x^+$ for all $s \in J$. Let $s^0 \equiv \min\{s : x_s = x^+\}$; note that $s^1 - s^0 = \Delta$.

Figure A1 depicts a discontinuous equilibrium. (While the scale makes it difficult to see, the values of all variables shown in the figure are strictly positive at all $s \geq 0$.) At each time $s \in J$ at which the principal invests, the agent's belief x_s jumps from x^- to x^+ , and so effort a_s jumps from $a^- = \mu \bar{b} x^-$ to $a^+ = \mu \bar{b} x^+$. At all other times $s \notin J$, the evolution of x_s is given by the law of motion (3), the same one that describes the agent's belief over $[0, \bar{s}]$ in the continuous equilibrium. Note that since the principal has incentives to invest only at the instants $s \in J$, she must be indifferent between investing and not investing at these times.¹ Also, by construction, the low type and high type's expected payoffs are the same at each $s \in J \cup s^0$, and hence the principal is also indifferent at s^0 . Analogous to (6) and (7), it follows that $\Lambda_s = F$ and $\mu a^+ \Psi_s = (\gamma + r)F$ at all $s \in J \cup s^0$.

It is worth noting that in any stationary discontinuous equilibrium, Δ must be bounded from below by a strictly positive value.² Although the smooth pasting condition need not be satisfied in a discontinuous equilibrium, roughly speaking the intuition is related to that for smooth pasting in the continuous equilibrium: if Δ is too small, the principal's indifference between investing and not when she invests would imply that she has strict incentives to invest at a previous point. Thus, as discussed in the paper, an equilibrium where the agent's belief is constant from (approximately) the time at which the principal starts investing does not exist.

Comparing with the continuous equilibrium of Proposition 1, we find:³

Proposition A2. *The principal's expected payoff at $s = 0$, π_0^H , is higher in the continuous equilibrium of Proposition 1 than in any stationary discontinuous equilibrium.*

¹If the principal had strict incentives to invest at $s \in J$, she would invest over a time interval $[s - \varepsilon, s + \varepsilon]$ for some $\varepsilon > 0$.

²To prove this, we can show that $s^0 \leq \bar{s} \leq s^1$, which implies that if Δ (and thus κ) were to go to zero, then s^0 and s^1 would go to \bar{s} . However, in this limit, the discontinuous equilibrium would yield a higher payoff for the principal at $s = 0$, π_0^H , than the continuous equilibrium of Proposition 1, contradicting Proposition A2 below. A formal proof for the claim that $s^0 \leq \bar{s} \leq s^1$ is available from the authors upon request.

³A welfare analysis of the agent is uninteresting because the agent is myopic. A myopic agent is indifferent between the continuous and discontinuous equilibria at time $s = 0$; at any other time, he prefers the equilibrium that induces higher effort.

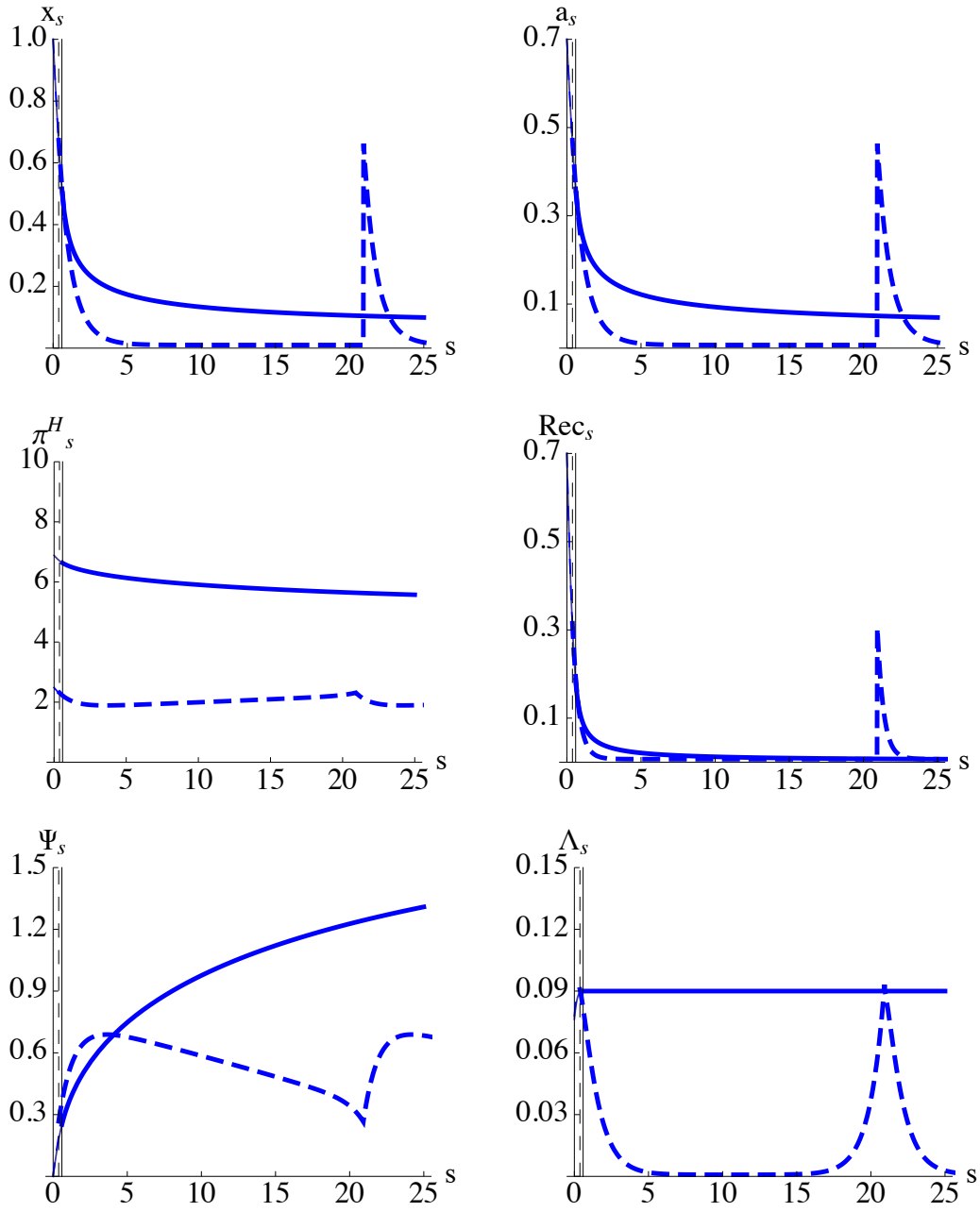


Figure A1: Dynamics in the continuous equilibrium (solid lines) and the discontinuous equilibrium (dashed lines). Parameters are the same as in Figure 1. Rec_s is the unconditional instantaneous probability of recognition, given by $\mu x_s a_s$. The vertical lines indicate the times \bar{s} and s^0 .

Proof. Using superscripts d and c to denote variables in the discontinuous equilibrium and the continuous equilibrium respectively, we have $x_s^d = x_s^c$ and $a_s^d = a_s^c$ at all $0 \leq s \leq \min\{\bar{s}, s^1\}$, and $x_{s^+}^d = x_{s^0}^d$ and $a_{s^+}^d = a_{s^0}^d$ for all $s \in J$. As for the principal's incentives, as noted, indifference implies $\Lambda_s^d = F$ and $\mu a_{s^0}^d \Psi_s^d = (\gamma + r)F$ at each $s \in J \cup s^0$, and at any $s \notin J \cup s^0$ we must have $\Lambda_s^d < F$. Finally, we will use the fact that in the continuous equilibrium, $\mu a_s^c \Psi_s^c \leq (\gamma + r)F$ for all $s \leq \bar{s}$. This follows from the proof of Proposition 1, where we show that the equilibrium threshold time \bar{s} is such that the left-hand side of (32) is less than the right-hand side at all $s \leq \bar{s}$.

We now proceed by proving two claims.

Claim 1. *If $s^0 > \bar{s}$, then $\pi_0^{Hc} \geq \pi_0^{Hd}$.*

Proof of Claim 1. Suppose by contradiction that $s^0 > \bar{s}$ and $\pi_0^{Hd} > \pi_0^{Hc}$. Note that $s^0 > \bar{s}$ implies

$$\mu a_{s^0}^d \Psi_{s^0}^d = (\gamma + r)F = \mu a_{s^0}^c \Psi_{s^0}^c$$

where $a_{s^0}^d < a_{s^0}^c$. Therefore,

$$\Psi_{s^0}^d = \pi_0^{Hd} - \pi_{s^0}^{Hd} > \pi_0^{Hc} - \pi_{s^0}^{Hc} = \Psi_{s^0}^c. \quad (\text{A17})$$

Now note that we can write

$$\begin{aligned} \pi_0^{Hd} &= \int_0^{s^0} e^{-(\gamma+r)\tau - \int_0^\tau \mu a_\tau^d d\tilde{\tau}} \left[a_\tau^d + \gamma \pi_\tau^{Ld} + \mu a_\tau^d \pi_0^{Hd} \right] d\tau + e^{-(\gamma+r)s^0 - \int_0^{s^0} \mu a_\tau^d d\tilde{\tau}} \pi_{s^0}^{Hd} \\ &< \int_0^{s^0} e^{-(\gamma+r)\tau - \int_0^\tau \mu a_\tau^c d\tilde{\tau}} \left[a_\tau^c + \gamma \left(\int_\tau^{s^0} e^{-r(\tilde{\tau}-\tau)} a_\tau^c d\tilde{\tau} + e^{-r(s^0-\tau)} \pi_{s^0}^{Ld} \right) + \mu a_\tau^c \pi_0^{Hd} \right] d\tau \\ &\quad + e^{-(\gamma+r)s^0 - \int_0^{s^0} \mu a_\tau^c d\tilde{\tau}} \pi_{s^0}^{Hd}, \end{aligned}$$

where the inequality follows from the fact that $a_s^d = a_s^c$ for $s \in [0, \bar{s}]$, $a_s^d < a_s^c$ for $s \in [\bar{s}, s^0]$, and $\pi_0^{Hd} > \pi_{s^0}^{Hd}$ for $s \in (0, s^0]$. It then follows that

$$\begin{aligned} \pi_0^{Hd} - \pi_0^{Hc} &< \int_0^{s^0} e^{-(\gamma+r)\tau - \int_0^\tau \mu a_\tau^c d\tilde{\tau}} \left[\gamma e^{-r(s^0-\tau)} (\pi_{s^0}^{Ld} - \pi_{s^0}^{Lc}) + \mu a_\tau^c (\pi_0^{Hd} - \pi_0^{Hc}) \right] d\tau \\ &\quad + e^{-(\gamma+r)s^0 - \int_0^{s^0} \mu a_\tau^c d\tilde{\tau}} (\pi_{s^0}^{Hd} - \pi_{s^0}^{Hc}). \end{aligned} \quad (\text{A18})$$

Note that $\pi_{s^0}^{Ld} = \pi_{s^0}^{Hd} - F$ and $\pi_{s^0}^{Lc} = \pi_{s^0}^{Hc} - F$; hence, substituting,

$$\begin{aligned} \pi_0^{Hd} - \pi_0^{Hc} &< \int_0^{s^0} e^{-(\gamma+r)\tau - \int_0^\tau \mu a_\tau^c d\bar{\tau}} \left[\gamma e^{-r(s^0-\tau)} (\pi_{s^0}^{Hd} - \pi_{s^0}^{Hc}) + \mu a_\tau^c (\pi_0^{Hd} - \pi_0^{Hc}) \right] d\tau \\ &+ e^{-(\gamma+r)s^0 - \int_0^{s^0} \mu a_\tau^c d\bar{\tau}} (\pi_{s^0}^{Hd} - \pi_{s^0}^{Hc}). \end{aligned} \quad (\text{A19})$$

Recall that by the contradiction assumption, $\pi_0^{Hd} > \pi_0^{Hc}$. But then (A19) requires $\pi_0^{Hd} - \pi_0^{Hc} < \pi_{s^0}^{Hd} - \pi_{s^0}^{Hc}$, contradicting (A17).⁴

Claim 2. *If $s^0 \leq \bar{s}$, then $\pi_0^{Hc} \geq \pi_0^{Hd}$.*

Proof of Claim 2. Suppose by contradiction that $s^0 \leq \bar{s}$ and $\pi_0^{Hd} > \pi_0^{Hc}$. Note that $s^0 \leq \bar{s}$ implies

$$\mu a_{s^0}^d \Psi_{s^0}^d = (\gamma + r)F \geq \mu a_{s^0}^c \Psi_{s^0}^c.$$

Note that $a_s^d = a_s^c$ for $s \in [0, s^0]$. Hence, we obtain

$$\Psi_{s^0}^d = \pi_0^{Hd} - \pi_{s^0}^{Hd} \geq \pi_0^{Hc} - \pi_{s^0}^{Hc} = \Psi_{s^0}^c. \quad (\text{A20})$$

Now note that given $a_s^d = a_s^c$ for $s \in [0, s^0]$, we can write

$$\begin{aligned} \pi_0^{Hd} - \pi_0^{Hc} &= \int_0^{s^0} e^{-(\gamma+r)\tau - \int_0^\tau \mu a_\tau^c d\bar{\tau}} \left[\gamma (\pi_\tau^{Ld} - \pi_\tau^{Lc}) + \mu a_\tau^c (\pi_0^{Hd} - \pi_0^{Hc}) \right] d\tau \\ &+ e^{-(\gamma+r)s^0 - \int_0^{s^0} \mu a_\tau^c d\bar{\tau}} (\pi_{s^0}^{Hd} - \pi_{s^0}^{Hc}). \end{aligned} \quad (\text{A21})$$

For any $\tau \leq s^0$,

$$\pi_\tau^{Ld} - \pi_\tau^{Lc} = e^{-r(s^0-\tau)} (\pi_{s^0}^{Ld} - \pi_{s^0}^{Lc}) \leq e^{-r(s^0-\tau)} (\pi_{s^0}^{Hd} - \pi_{s^0}^{Hc}),$$

where the last inequality follows from the fact that $\pi_{s^0}^{Ld} = \pi_{s^0}^{Hd} - F$ whereas $\pi_{s^0}^{Lc} \geq \pi_{s^0}^{Hc} - F$.

⁴To see why (A19) requires $\pi_0^{Hd} - \pi_0^{Hc} < \pi_{s^0}^{Hd} - \pi_{s^0}^{Hc}$, divide both sides by $\pi_0^{Hd} - \pi_0^{Hc}$ under the assumption that $\pi_0^{Hd} - \pi_0^{Hc} > 0$:

$$1 < \int_0^{s^0} e^{-(\gamma+r)\tau - \int_0^\tau \mu a_\tau^c d\bar{\tau}} \left[\gamma e^{-r(s^0-\tau)} \left(\frac{\pi_{s^0}^{Hd} - \pi_{s^0}^{Hc}}{\pi_0^{Hd} - \pi_0^{Hc}} \right) + \mu a_\tau^c \right] d\tau + e^{-(\gamma+r)s^0 - \int_0^{s^0} \mu a_\tau^c d\bar{\tau}} \left(\frac{\pi_{s^0}^{Hd} - \pi_{s^0}^{Hc}}{\pi_0^{Hd} - \pi_0^{Hc}} \right).$$

The claim follows from the fact that

$$1 = \int_0^{s^0} e^{-(\gamma+r)\tau - \int_0^\tau \mu a_\tau^c d\bar{\tau}} [r + \gamma + \mu a_\tau^c] d\tau + e^{-(\gamma+r)s^0 - \int_0^{s^0} \mu a_\tau^c d\bar{\tau}}.$$

Hence, substituting $(\pi_\tau^{Ld} - \pi_\tau^{Lc})$ in (A21), we obtain

$$\begin{aligned} \pi_0^{Hd} - \pi_0^{Hc} &\leq \int_0^{s^0} e^{-(\gamma+r)\tau - \int_0^\tau \mu a_\tau^c d\bar{\tau}} \left[\gamma e^{-r(s^0 - \tau)} (\pi_{s^0}^{Hd} - \pi_{s^0}^{Hc}) + \mu a_\tau^c (\pi_0^{Hd} - \pi_0^{Hc}) \right] d\tau \\ &\quad + e^{-(\gamma+r)s^0 - \int_0^{s^0} \mu a_\tau^c d\bar{\tau}} (\pi_{s^0}^{Hd} - \pi_{s^0}^{Hc}). \end{aligned} \tag{A22}$$

Recall that by the contradiction assumption, $\pi_0^{Hd} - \pi_0^{Hc} > 0$. But then (A22) requires $\pi_0^{Hd} - \pi_0^{Hc} < \pi_{s^0}^{Hd} - \pi_{s^0}^{Hc}$, contradicting (A20).⁵ *Q.E.D.*

⁵This can be verified following analogous steps to those in fn. 4.