Online Appendix to "How Important Are Sectoral Shocks?"

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B Cross-sectional estimates

In this section, I will apply plant-level input price variation and materials usage to provide an alternate set of estimates of ε_Q. To do so, I pursue the following two-part strategy. For each industry, I estimate how easily individual plants substitute across their factors of production, by relating plants' materials purchases to their materials prices. Then, I apply the methods developed in Oberfield and Raval (2015), which allow me to combine information on a) the plant-level elasticity of substitution, b) the dispersion of materials cost shares, and c) the elasticity of plant scale to marginal costs so that I can ascertain the corroborating estimates of ε_Q.

To preview the main results of this section, the elasticity of substitution for the plant-level production function is approximately 0.65. Because within-industry variation in materials expenditure shares is small for each of the ten industries, the industry level production function's elasticity of substitution is only somewhat higher, 0.75. Moreover, across the industries in the sample, the industry-level elasticities of substitution are similar to one another.

B.1 Data source and sample

The data source, for this section, is the Census of Manufacturers. This dataset contains plant-level information for each manufacturer in the United States, and is collected once every five years, in years ending in a "2" or a "7." For certain industries, plants with greater than five employees are asked to provide information on each of the material inputs that they consume and each of the products that they produce. Critically, for the empirical analysis of this section, the Census Bureau elicits information on both the quantities and values of these inputs and outputs, allowing me to construct plant-level prices. Additionally, the Census Bureau records a plant identifier, which will allow me to compare the intermediate input purchases of the same plant across different time periods.

The sample in this section is identical to that which was used in an earlier paper (see Atalay 2014). The industries are those for which outputs and inputs are relatively homogeneous. This choice reflects a desire to, as much as possible, rule out heterogeneous quality as a source of input or output price variation. The ten industries that comprise the sample are

Sample	Units of Output	Material Inputs	N
Boxes, Year≤1987	Short Tons	Paper/Paperboard (90%)	1820
Boxes, Year≥1992	Square Feet	Paper/Paperboard (89%)	646
Ground Coffee	1000 Pounds	Green Coffee Beans (80%)	300
Ready-Mix Concrete	1000 Cubic Yards	Cement (53%), Sand/Gravel (28%)	3708
White Wheat Flour	50-Pound Sacks	Wheat (90%)	503
Gasoline	1000 Barrels	Crude Petroleum (84%)	692
Milk, Bulk	1000 Pounds	Unprocessed Whole Milk (88%)	127
Milk, Packaged	1000 Quarts	Unprocessed Whole Milk (72%)	2099
Raw Cane Sugar	Short Tons	Sugar Cane (93%)	177
Carded Cotton Yarn	1000 Pounds	Cotton Fibers (80%), Polyester Tow (10%)	431
Pooled	-	-	10,503

Table 8: Description of the 10 industries in the sample.
Notes: This table is a duplicate of Table 1 of Atalay (2014). The percentages that appear in the Material Inputs column are the fraction of materials expenditures that go to each particular material input. The Material Inputs column shows the inputs that represent greater than 6% of the average plant's total material purchases.

corrugated boxes (with the years 1972-1987 and 1992-1997 analyzed separately. The way in which material inputs are coded, for this industry, differs in the two parts of the sample), ground coffee, ready-mix concrete, white wheat flour, gasoline, bulk milk, packaged milk, raw cane sugar, and grey cotton yarn; see Table 8. For additional details regarding the sample, see Appendix B of Atalay (2014).

B.2 Environment and assumptions

Each industry, I, comprises a set of plants $i \in I$, who combine capital, labor, material inputs, and purchased services to produce a single product. The production function is constant-returns to scale; separable between material inputs, N, and other inputs, O; with constant elasticity of substitution, η_P:

$$\begin{aligned} Q_{it}(K_{it}, L_{it}, S_{it}, N_{it}) &= \left[(A_{it} \cdot O_{it})^{\frac{\eta_P - 1}{\eta_P}} + (B_{it} \cdot N_{it})^{\frac{\eta_P - 1}{\eta_P}} \right]^{\frac{\eta_P}{\eta_P - 1}}, \\ \text{where } O_{it} &= F\left(K_{it}, L_{it}, S_{it}\right) \end{aligned} \tag{19}$$

Also by assumption, F exhibits constant returns to scale. Plants are allowed to flexibly alter their input choices, including capital, each period. Furthermore, the factor prices that each plant faces, both for the material input and for the other input aggregate, are constant in the amount purchased. These assumptions serve a dual purpose. Not only do these assumptions greatly simplify the estimation of η_P, they also allow me to apply Oberfield and Raval (2015)'s methodology to estimate ε_Q from η_P.

Use P_{it}^{oth} and P_{it}^{mat} to denote the factor prices for a unit of the other input aggregate and the material input, respectively. Let A_{it} and B_{it} represent the two plant-level productivity measures (other-input-augmenting and materials augmenting).

The demand curve faced by each plant, i, has constant elasticity, ε_D:

$$Q_{it} = \exp\{\theta_{it}\} \cdot (P_{it})^{-\eta_D} \tag{20}$$

In Equation 20, θ_{it} represents a plant-year specific demand shifter. The assumption of a constant elasticity demand curve, while probably counterfactual, is again useful for multiple reasons. The constant-demand-elasticity assumption allows me to directly apply the Foster, Haltiwanger, and Syverson (2008) methodology to estimate η_D. Moreover, the same assumption is invoked by Oberfield and Raval (2015)—whose work I apply, here—in their aggregation of plant-level to industry level production functions.

The profit-maximizing levels of N_{it} and O_{it} yield the following expression for the material-output ratio:

$$\log\left[\frac{N_{it}}{Q_{it}}\right] = -\eta_P \cdot \log\left[\frac{P_{it}^{mat}}{P_{it}}\right] + \eta_P \cdot \log\left[\frac{\eta_D - 1}{\eta_D}\right] + (\eta_P - 1)\log B_{it} \tag{21}$$

This equation will form the basis of the estimation of η_P, a task to which I now turn.

B.3 The micro elasticity of substitution

In this subsection, I estimate the plant-level elasticity of substitution between purchased inputs and other inputs. The baseline regression that I run is:

$$n_{it} - q_{it} = -\eta_P \cdot \left(p_{it}^{mat} - p_{it}\right) + \epsilon_{it} \ . \tag{22}$$

In Equation 22, and throughout the remainder of the section, I use lower-case letters to denote the logged, de-meaned values of the variable of interest. In other words, η_P is estimated only using within industry-year variation. To emphasize, both n_{it} and q_{it} refer to the number of physical units, and not the values, of the material good that plant i purchases and the output that it produces.

Ordinary least squares results are presented in the first column of Table 9. For most industries, the estimate of η_P lies between 0.5 and 0.7, with concrete and flour having two of the lower estimates and bulk milk and raw cane sugar with two of the higher estimates.

There are at least two concerns regarding the interpretation of η_P—from an OLS estimate of Equation 22—as an estimate of the micro elasticity of substitution. First, to the extent that the constant elasticity of demand assumption—embodied in Equation 20—is violated, Equation 22 suffers from omitted variable bias. A positive correlation between $\log\left[\frac{\eta_D-1}{\eta_D}\right]$ and $(p_{it}^{mat}-p_{it})$ will engender a positive bias in η_P. Second, I have assumed that the materials supply curve that each i faces is flat. It is likely, however, that each plant's factor supply curve is upward sloping. This instance of simultaneity bias—whereby a high-B_{it} plant pays a high materials price—will also engender a positive bias in η_P

I offer two different approaches to circumvent these problems. Fist, I append plant-level fixed effects to Equation 22. These fixed effects aim to capture long-run cross-sectional variation in the conditions in output and factor markets. As Foster, Haltiwanger, and Syverson (2008, 2016) argue, the factor market conditions that a plant faces are substantially more persistent than its productivity.

In a second specification, I instrument plants' output and materials prices with the prices paid and charged by competitor plants. Specifically, the two instrumental variables, for $p_{it}^{mat}-p_{it}$, are a) the year-t average materials price for plants that are within 50 miles of plant i, and b) the year-t average output price for plants that are within 50 miles of plant i. The idea behind these instruments is that the price of materials in nearby markets is correlated with the price that i pays for its material inputs (if, for example, there is spatial correlation in the abundance of primary inputs used in the production of i's intermediate inputs, or if there is a very productive, low marginal-cost supplier nearby), but should not in any other way affect the propensity for i have exceptionally high or low materials expenditure shares.[32]

Results from the two sets of regressions are given in the second and third columns of Table 9. In the second column, estimates of η_P range from 0.40 to 0.92, with the two largest estimates corresponding to two of the smaller-sample industries, coffee and sugar. The pooled estimate of η_P is 0.68.

The instrumental variables are weak for the six smallest samples. For this reason, the IV specification is performed only on the samples of plants in the corrugated boxes, ready-mix concrete, packaged milk, and petroleum industries. In the third specification, the parameter estimates are smaller and much less precisely estimated. The biggest difference is for the ready-mix concrete industry, for which the estimate of η_P is essentially 0.

[32] Results from first-stage regressions indicate that these instruments are relevant, at least for the four largest subsamples: materials prices and output prices are each spatially correlated.

Sample	$\hat{\eta}^P$(OLS)	$\hat{\eta}^P$(FE)	$\hat{\eta}^P$(IV)	$\hat{\chi}$	$\hat{\eta}^D$	$\hat{\varepsilon}^Q$(OLS)	$\hat{\varepsilon}^Q$(FE)	$\hat{\varepsilon}^Q$(IV)	N(FE)	N(IV)
Boxes, Yr.≤'87	0.637	0.774	0.347	0.021	2.840	0.683	0.817	0.398	1500	1820
	(0.020)	(0.021)	(0.203)	[0.019,0.023]	(0.118)	[0.623,0.743]		[-0.531,0.769]		
Boxes, Yr.≥'92	0.644	0.776		0.022	1.825	0.670	0.799		549	646
	(0.029)	(0.038)		[0.018,0.025]	(0.148)	[0.599,0.742]				
Coffee	0.682	0.776		0.033	5.744	0.848	0.938		248	300
	(0.046)	(0.058)		[0.026,0.040]	(0.690)	[0.717,0.968]				
Concrete	0.371	0.491	-0.031	0.048	2.577	0.478	0.592	0.096	2047	3708
	(0.019)	(0.031)	(0.061)	[0.045,0.052]	(0.232)	[0.432,0.524]		[-0.027,0.212]		
Flour	0.312	0.652		0.019	1.151	0.328	0.661		452	503
	(0.035)	(0.038)		[0.016,0.021]	(0.351)	[0.224,0.434]				
Gasoline	0.482	0.403	0.496	0.020	3.466	0.540	0.464	0.555	635	692
	(0.054)	(0.065)	(0.184)	[0.016,0.023]	(1.308)	[0.365,0.685]		[0.077,1.005]		
Milk, Bulk	0.810	0.656		0.040	2.380	0.872	0.725		65	127
	(0.085)	(0.171)		[0.028,0.048]	(0.553)	[0.687,1.063]				
Milk, Packaged	0.605	0.731	0.420	0.035	2.555	0.674	0.797	0.496	1534	2099
	(0.020)	(0.025)	(0.161)	[0.032,0.039]	(0.145)	[0.629,0.728]		[0.078,0.874]		
Sugar	1.034	0.922		0.065	2.221	1.114	1.007		155	177
	(0.111)	(0.115)		[0.040,0.085]	(0.826)	[0.765,1.584]				
Yarn	0.551	0.629		0.029	1.525	0.580	0.655		332	431
	(0.037)	(0.050)		[0.025,0.032]	(0.187)	[0.497,0.657]				
Pooled	0.538	0.680	0.078	0.035	2.543	0.608	0.746	0.164	7517	10,503
	(0.010)	(0.012)	(0.053)	[0.033,0.036]	(0.080)	[0.582,0.636]		[0.048,0.279]		

Table 9: Components of the industry-level elasticity of substitution.

Notes: The first three columns present $\hat{\eta}^P$, as estimated using Equation 22. The values given in the fourth, sixth, seventh, and eighth, columns are computed as in Equation 23, while $\hat{\eta}^D$ is estimated using Equation 24. Robust standard errors are included in the first, second, and fourth columns, while bootstrapped confidence intervals are provided in the third, sixth, and eighth columns. "OLS," "FE," and "IV" refer, respectively, to the values corresponding to the ordinary least squares, fixed effects, and instrumental variables specifications for the estimate of η^P.

B.4 The industry-level elasticity of substitution

The previous subsection provided an estimate for the ease with which individual plants substitute between material inputs and other inputs. This is related to, but distinct from, how easily an industry substitutes between material inputs and other inputs.

Changes in the scale, across plants, potentially makes the industry-level elasticity of substitution larger than the corresponding plant-level elasticity. The difference between the plant-level and industry-level elasticities of substitution depends on a) the heterogeneity of materials shares, within the industry, and b) how much inputs shift across plants, in response to a change in relative factor prices.

Given the assumptions, specified in Section B.2, the industry-level elasticity of substitution has a simple expression:[33]

$$\begin{aligned}
\varepsilon_Q &= \chi_{tI} \cdot \eta_D + (1 - \chi_{tI}) \cdot \eta_P, \text{ where} \qquad (23)\\
\chi_{tI} &\equiv \underbrace{\frac{1}{S_{tI}\,(1 - S_{tI})}}_{①} \cdot \sum_{i \in I} \underbrace{\left(S_{tI} - \frac{M_{it}P_{it}^{in}}{M_{it}P_{it}^{in} + O_{it}P_{it}^{oth}}\right)^2}_{②} \cdot \underbrace{\frac{M_{it}P_{it}^{mat} + O_{it}P_{it}^{oth}}{\sum_{j \in I} M_{jt}P_{jt}^{mat} + O_{jt}P_{jt}^{oth}}}_{③}, \text{ and}\\
S_{tI} &\equiv \sum_{i \in I} \frac{M_{it}P_{it}^{mat}}{M_{it}P_{it}^{mat} + O_{it}P_{it}^{oth}}
\end{aligned}$$

In words, the industry-level elasticity of substitution is a convex combination of the plant-level elasticity of substitution and the plant-level elasticity of demand. The demand elasticity parameterizes how sensitive the scale of the plant is to changes in its marginal cost of production. Consider, for example, an increase in the price of the material input. The marginal cost of production will increase more for plants with relatively large materials cost shares. As a result, low-materials-share plants will produce relatively more of the total industry output following the increase of the materials price. The elasticity of demand determines how much less the high-materials-share plants will produce, following the increase in the materials price.

The scope for this across-plant factor substitution depends on the dispersion of materials intensities. According to Equation 23, the appropriate measure of the dispersion of materials intensity is a weighted, normalized variance of the materials cost shares. The fraction of total industry expenditures incurred by plant i (given in term ③) is the appropriate weight for summing over the within-industry deviation in materials cost shares (given in term ②). The normalization, given in term ①, ensures the χ_{tI} lies within the unit interval.

What remains, then, is to provide estimates for the normalized variance of materials

[33] A proof is given in Oberfield and Raval (2015). See Appendix A of that paper.

shares, χ, and the elasticity of demand, η_D, for the ten industries in my sample.

The normalized variance of materials shares, χ, ranges from 0.019 (for flour) to 0.065 (for sugar).[34] Given these low values, the industry elasticity of substitution will closely track the micro elasticity of substitution. In other words, the estimate of ε_Q will be, for the most part, insensitive to the way in which η_D is estimated.

I estimate η_D via the regression defined by the following equation:

$$q_{it} = \phi_t + \phi_1 \cdot \log INCOME_{\Upsilon t} + \eta_D \cdot p_{it} + \theta_{it} \tag{24}$$

This specification, and the variable definitions, follow Foster, Haltiwanger, and Syverson (2008). In Equation 24, $INCOME_{\Upsilon t}$ is the aggregate income in establishment i's market, Υ, at time t. This variable is included to account for any differences in establishment scale that may exist between areas of high and low density of economic activity.

A positive relationship between the demand shifter (θ_{it}) and output price (p_{it}) potentially induces a downward bias to the OLS estimates of η_D. Like Foster, Haltiwanger, and Syverson (2008), I instrument p_{it} with the marginal cost of plant i in year t. This instrumental variable is certainly relevant: plants with lower marginal costs have significantly lower output prices. Validity of the instrument rests, then, on the orthogonality of marginal costs and θ_{it}. Foster, Haltiwanger, and Syverson (2008) discuss two potential threats to the validity of the instrument (measurement error in plants' marginal costs, and a selection bias that induces a negative relationship between demand shocks and marginal costs), propose robustness checks to assess the salience of these two threats, and find that their results are similar across the different robustness checks.

The results of these regressions are presented in the fourth column of Table 9.[35] In each of the ten industries, the estimate for elasticity of demand is greater than 1, reassuringly indicating that plants are pricing on the elastic portion of their demand curve.

Combining the estimates of η_P, η_D, and χ yields the object of interest: the industry-level elasticity of substitution, ε_Q. Since there are three sets of estimates of η_P, there are also three sets of estimates of ε_Q. For the estimates corresponding to the fixed effects regression, ε_Q is 0.75 for the pooled sample.[36] Except for sugar and coffee (two of the smallest industries,

[34] To give the reader some idea, the (unnormalized) standard deviations of materials shares range from 4.3 percent to 11.4 percent across the ten industries, again lowest for bulk milk and highest for raw cane sugar.

[35] The results reported here are slightly different from those in Foster, Haltiwanger, and Syverson (2008): I restrict my sample to those plants for which I can observe materials prices, while Foster, Haltiwanger, and Syverson make no such restriction. Their estimate of η_D is lower for petroleum ($\hat{\eta}_D = 1.42$) and higher for ready-mix concrete ($\hat{\eta}_D = 5.93$). Again, because the normalized variances of materials shares are so small, these differences have will have only a moderate impact on the estimates of ε_Q.

[36] One dissimilarity between the analysis of the current section and that of Sections 2 to 4 concerns the industry definitions that I have used: to credibly compare the material purchases and material prices, I

representing only 5 percent of the sample), the industry-level elasticities of substitution range between 0.46 (for gasoline) and 0.82 (for corrugated boxes). For seven of the ten industries in the sample (with the exceptions being the smallest three subsamples), the data would reject a null hypothesis of $\varepsilon_Q = 1$.

The estimates of ε_Q that correspond to the instrumental-variables-based estimate of η_P are smaller, though again much less precisely estimated. The point estimate for ε_Q is 0.1 for the ready-mix concrete subsample, and is somewhat higher (between 0.40 and 0.55) for the other three industries.

In summation, micro data on plants' materials usage patterns indicate that material inputs are gross complements to other factors of production. For most specifications (all except for the IV specification for the ready-mix concrete subsample, or the fixed effects specification for the smaller industries), the data indicate that ε_Q ranges between 0.4 and 0.8.

C Details of the data from outside the U.S.

The data from other countries come from two sources. The flows of intermediate inputs, flows of goods output into final consumption expenditures, and industry-level prices are collected in the World Input Output Tables (WIOT). The data on industries' output are compiled in the European Union KLEMS Growth and Productivity Accounts (EUKLEMS).[37] The EUKLEMS data are reviewed, in detail, in Timmer et al. (2007) and O'Mahony and Timmer (2009). Flows of investment goods across industries are not available for other countries. For this set of variables, I imputed values using data from the U.S.

Of the thirty countries that are included in the EUKLEMS dataset, I restrict my analysis to six: Denmark, France, Italy, Japan, the Netherlands, and Spain. Many of the countries that I discarded are Eastern Bloc countries—such as Latvia, Lithuania, and Poland—for which pre-1990 data are unavailable. There are other countries, such as England, for which—for at least half of the sample period—intermediate input purchases and gross output are imputed from value added data. Data from all countries span 1970 to 2007, with the excep-

define products narrowly in this section. At the same time, limitations of the dataset necessitate a rather coarse industry definition in Sections 2 to 4. Going from a narrow to coarse industry classification should not systematically alter the estimates of η_P or η_D, but will cause an increase in the estimate for the within-industry variation in materials cost shares, χ. For this reason, a coarser industry classification would, in turn, lead to a larger estimate of ε_Q. As it turns out, the overall estimate of ε_Q is not particularly sensitive to the value of χ: Doubling the value of χ increases the OLS-based estimate of ε_Q from 0.61 to 0.67, and increases the fixed-effects-based estimate from 0.75 to 0.80.

[37] The data can be downloaded at http://www.euklems.net/ . In this section I use the ISIC Rev. 3 edition of the data.

#	Name	Denmark	France	Italy	Japan	Netherlands	Spain	U.S. Ind.
1	Agriculture	0.027	0.029	0.022	0.010	0.036	0.039	1
2	Mining	0.007	0.002	0.001	0.000	0.010	0.001	2,3
3	Food and Tobacco	0.103	0.061	0.065	0.069	0.098	0.085	5
4	Textiles and Leather	0.018	0.025	0.069	0.019	0.020	0.039	6, 7
5	Wood Products	0.007	0.002	0.003	0.002	0.004	0.002	8, 9
6	Paper and Publishing	0.017	0.014	0.015	0.002	0.022	0.015	10, 11
7	Petroleum Refining	0.012	0.011	0.013	0.010	0.024	0.016	13
8	Chemicals	0.040	0.047	0.037	0.016	0.073	0.038	12
9	Rubber and Plastics	0.013	0.009	0.012	0.005	0.012	0.008	14
10	Stone, Clay, and Glass	0.007	0.005	0.010	0.002	0.006	0.008	15
11	Metal products	0.021	0.018	0.024	0.015	0.029	0.020	16, 17
12	Non-Electrical Machinery	0.047	0.024	0.048	0.028	0.026	0.018	18
13	Electrical Machinery	0.038	0.039	0.027	0.042	0.040	0.024	19, 22
14	Transportation Equipment	0.026	0.065	0.040	0.040	0.034	0.074	20, 21
15	Misc. Manufacturing	0.024	0.013	0.024	0.007	0.018	0.020	23
16	Utilities	0.021	0.037	0.016	0.017	0.017	0.015	26
17	Construction	0.010	0.006	0.008	0.000	0.006	0.006	4
18	Wholesale and Retail	0.073	0.078	0.109	0.137	0.070	0.075	27
19	Hotels and Restaurants	0.020	0.031	0.051	0.048	0.021	0.106	28
20	Transport and Warehousing	0.060	0.038	0.048	0.043	0.062	0.037	24
21	Communications	0.010	0.010	0.012	0.014	0.012	0.012	25
22	Finance and Insurance	0.021	0.032	0.026	0.032	0.032	0.015	28
23	Real Estate	0.086	0.100	0.072	0.127	0.060	0.069	28
24	Business Services	0.015	0.030	0.018	0.009	0.052	0.017	29
25	Government	0.071	0.096	0.079	0.105	0.079	0.076	30
26	Education	0.058	0.056	0.053	0.060	0.035	0.052	29
27	Health and Social Work	0.111	0.087	0.065	0.082	0.070	0.066	29
28	Other Personal Services	0.038	0.035	0.032	0.058	0.030	0.048	29

Table 10: Industry definitions and consumption shares in the EUKLEMS dataset.

Notes: The final column shows the correspondence between the EUKLEMS industry definitions and the industry definitions for the U.S. data.

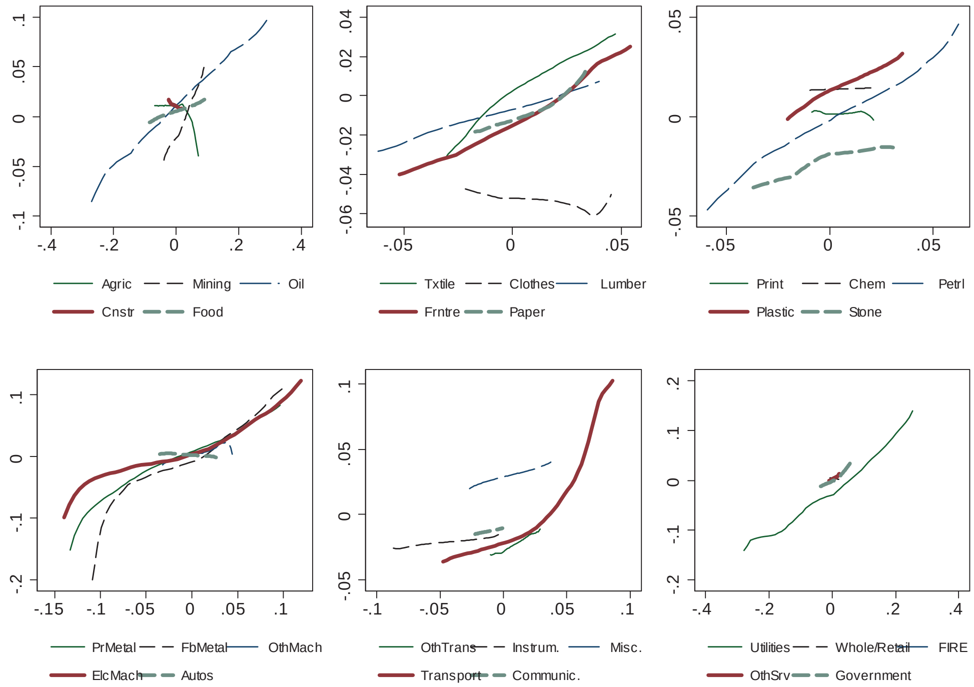

Figure 5: Relationship between changes in intermediate input purchases and intermediate input prices.

Notes: For each downstream industry, J, I take the most important (highest average intermediate input expenditure share) supplier industry, I. The x-axis of each panel gives $\Delta\log\left(\frac{P_{tI}}{P_{tJ}^{in}}\right)$. The y-axis gives, for each industry, changes in the fraction of industry J's intermediate input expenditures that go to industry I. I compute and plot a local polynomial curve of this relationship, for each industry.

tion of Japan, whose sample begins in 1973.

The industry definitions in the EUKLEMS database differ from those in the U.S. dataset. Service industries are more finely defined. For example, F.I.R.E. is now broken out between finance and insurance on the one hand and real estate on the other. Mining and manufacturing industries are more coarse. Table 10 describes the EUKLEMS industry classification, in addition to the consumption shares of each of the 28 industries. The main takeaway from this table is that the six countries are broadly similar in their industry compositions.

D Sensitivity analysis related to Section 3

D.1 Additional Plots

Figure 5 depicts the smoothed relationship between $\Delta\log\left(\frac{P_{tI}M_{t,I\rightarrow J}}{P_{tJ}^{in}M_{tJ}}\right)$ and $\Delta\log\left(\frac{P_{tI}}{P_{tJ}^{in}}\right)$, for each industry J and J's most important supplier industry. The takeaway from this figure

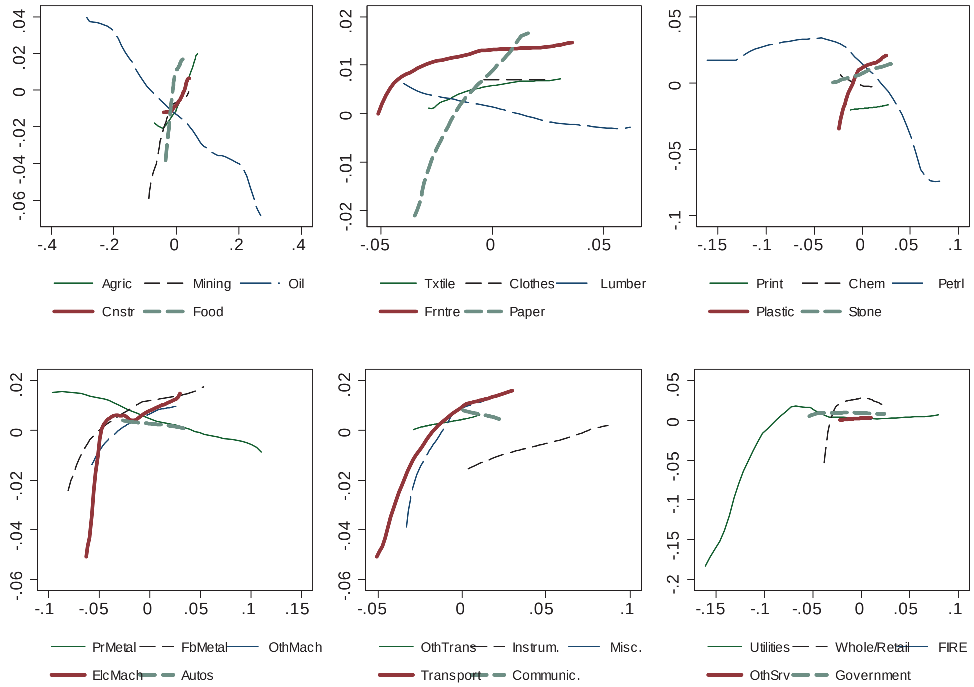

Figure 6: Relationship between changes in purchases of the intermediate input bundle and the relative price of the intermediate input bundle.
Notes: For each industry, J, I plot the relationship between changes in its cost share of intermediate inputs on the y-axis, and changes in the difference between the price of the intermediate input bundle and the marginal cost of production on the x-axis. I compute and plot a local polynomial curve of this relationship for each industry.

is that the relationships depicted in Figure 1 are broadly consistent with of the relationships within all 30 industries.

Figure 6 depicts the smoothed relationship between $\Delta \log \left(\frac{P^{in}_{tJ} M_{tJ}}{P_{tJ} Q_{tJ}} \right)$ and $\Delta \log \left(\frac{P^{in}_{tJ}}{P_{tJ}} \right)$. Here, the relationship between intermediate input cost shares and the price of intermediate inputs is positive for some industries, negative for others.

D.2 Different samples, changing the period length and industry classification

In this section, I re-estimate Equation 13 using different samples. First, in Table 11, I examine whether the estimates of the production elasticities, ε_Q and ε_M, differ according to the industry classification scheme or the period length. In the first four columns, the economy is broken up into nine industries; in the next four columns, a 67-industry classification is applied. The main takeaway from this table is that the estimates of ε_M, as in the original specifications, are close to 0, independent of how industries are defined. For longer period lengths, the estimated elasticity of substitution among intermediate inputs is somewhat higher; the elasticity of substitution between value added and intermediate inputs is somewhat lower. The IV results are unreported for this last robustness check, since the instruments are both weak and lead one to reject the Wu-Hausman test.

Next, in Table 12, I estimate the production elasticities separately for different broad sectors. The Primary sector consists of the first three industries in Table 7. The Manufacturing sector consists of the Construction and all manufacturing industries, the fourth through twenty-third industries according to Table 7. The remaining industries are in the Services sector. Estimates of ε_M are similar across sectors. Estimates of ε_Q, though less precisely estimated, are somewhat larger for the Primary sector and lower for the Services sector.

As a third set of robustness checks, I assess in Table 13 whether the number of upstream industries used in the sample alters the estimates of ε_M and ε_Q. In the benchmark regressions, in Table 1, the sample included the top ten upstream industries for each downstream industry J. There are no clear patterns, regarding the relationship between estimates of ε_Q and ε_M and the broadness of the sample.

D.3 Production elasticities of substitution in other countries

In this subsection, I report on results from other countries. I apply data from the World Input Output Tables, taking data from 1997 to 2011. The industry definitions, similar those used for the U.S. data, are given in Table 10. In Table 14, I report on regressions that relate changes in the inputs' cost shares with changes in the prices of individual inputs and prices of the intermediate input bundles. Unfortunately, for these countries, changes in military expenditures are not a sufficiently powerful source of variation to permit an IV regression. In this table, the slope of the relationship of changes in the intermediate input cost share on $\Delta \log P_{tJ}^{in} - \Delta \log P_{tI}$ is approximately 0.3 for France and between 0.6 and 0.8 for all other countries. In addition, the slope of the relationship of intermediate input purchases on $\Delta \log P_{tJ} - \Delta \log P_{tJ}^{in}$ is 0.10 for Denmark and between 0.4 and 0.8 for all other countries. While the coefficient estimates reported in 14 cannot identify ε_Q or ε_M on their own, they accord with the OLS estimates for the United States.

E Sensitivity analysis related to Section 4

In the first columns of Table 15, I re-estimate the correlations among shocks for different parts of the sample period. For the most part, the correlations among the ω productivity

Second stage regression results										
ε_M	0.17	0.13	0.20	0.22	0.16	0.12	-0.06	-0.03	0.29	0.29
	(0.05)	(0.05)	(0.28)	(0.27)	(0.03)	(0.03)	(0.22)	(0.24)	(0.07)	(0.07)
ε_Q	1.08	1.13	0.55	1.19	1.03	1.13	1.03	1.09	0.95	0.95
	(0.08)	(0.08)	(0.55)	(0.38)	(0.05)	(0.05)	(0.66)	(1.04)	(0.12)	(0.12)
First stage: Dependent variable is $\Delta \log P_{tJ}^{in} - \Delta \log P_{tI}$										
military spending			-0.56	-0.55			-0.44	-0.37		
shock$_{tI}$			(0.06)	(0.06)			(0.03)	(0.03)		
military spending			0.62	0.69			0.53	0.76		
shock$_{tJ\text{'s suppliers}}$			(0.09)	(0.12)			(0.04)	(0.06)		
military spending			-0.05	-0.06			0.01	-0.02		
shock$_{tJ}$			(0.07)	(0.07)			(0.03)	(0.03)		
F-statistic			32.29	14.92			88.57	29.78		
First stage: Dependent variable is $\Delta \log P_{tJ} - \Delta \log P_{tJ}^{in}$										
military spending			-0.11	0.01			-0.07	-0.05		
shock$_{tI}$			(0.04)	(0.04)			(0.02)	(0.02)		
military spending			-0.18	0.19			-0.09	0.00		
shock$_{tJ\text{'s suppliers}}$			(0.07)	(0.08)			(0.03)	(0.04)		
military spending			0.30	0.27			0.11	0.09		
shock$_{tJ}$			(0.05)	(0.05)			(0.02)	(0.02)		
F-statistic			17.83	13.36			21.37	26.92		
Cragg-Donald Statistic			15.34[i]	27.60[i]			21.37[i]	8.35		
Wu-Hausman test p-value			0.59	0.90			0.70	0.87		
Sample	Coarse Industries				Fine Industries				Period Length = 2 Years	
Year Fixed Effects	No	Yes	No	Yes	No	Yes	No	Yes	No	Yes
N	2400	2400	2296	2296	10720	10720	10496	10496	1296	1296

Table 11: Regression results related to Equation 13.

Notes: The overall sample includes pairs of industries J, and, for each industry J, the top ten supplying industries, I. In the row labeled "Cragg-Donald Statistic", an "i" indicates that the test for a weak instrument is rejected at the 10 percent threshold. The "military spending shock$_{tJ\text{'s suppliers}}$" term refers to the cost-weighted average of the "military spending shock$_{tI}$" term, averaging over industry J's suppliers.

Second stage regression results									
ε_M	0.02	-0.38	-0.35	-0.27	-0.27	-0.04	0.26	0.39	0.35
	(0.11)	(0.36)	(0.35)	(0.05)	(0.28)	(0.36)	(0.08)	(0.48)	(0.43)
ε_Q	1.41	1.44	1.64	1.35	1.55	0.49	0.02	0.93	0.90
	(0.10)	(0.36)	(0.42)	(0.09)	(1.15)	(0.74)	(0.15)	(0.91)	(0.89)
First stage: Dependent variable is $\Delta \log P_{tJ}^{in} - \Delta \log P_{tI}$									
military spending		-1.39	-1.43		-0.58	-0.48		-0.76	-0.77
shock$_{tI}$		(0.20)	(0.21)		(0.07)	(0.08)		(0.13)	(0.13)
military spending		1.55	-0.06		0.97	1.21		0.92	0.93
shock$_{tJ\text{'s suppliers}}$		(0.49)	(1.41)		(0.11)	(0.13)		(0.22)	(0.26)
military spending		-0.09	0.46		-0.25	-0.25		-0.22	-0.26
shock$_{tJ}$		(0.27)	(0.52)		(0.08)	(0.08)		(0.27)	(0.31)
F-statistic		16.78	4.59		38.36	11.41		13.13	3.20
First stage: Dependent variable is $\Delta \log P_{tJ} - \Delta \log P_{tJ}^{in}$									
military spending		0.01	0.02		-0.16	0.03		0.03	-0.03
shock$_{tI}$		(0.22)	(0.19)		(0.04)	(0.04)		(0.07)	(0.06)
military spending		-0.59	-3.03		0.12	0.55		-0.46	-0.73
shock$_{tJ\text{'s suppliers}}$		(0.52)	(1.31)		(0.06)	(0.07)		(0.12)	(0.12)
military spending		0.94	1.81		-0.04	-0.03		0.27	0.00
shock$_{tJ}$		(0.29)	(0.49)		(0.04)	(0.04)		(0.15)	(0.14)
F-statistic		15.36	13.36		7.99	21.98		12.15	26.54
Cragg-Donald Statistic		15.25[i]	12.79		6.02	11.42		11.39	12.73
Wu-Hausman test p-value		0.49	0.50		0.98	0.37		0.53	0.99
Sector		Primary			Manufacturing			Services	
Year Fixed Effects	No	No	Yes	No	No	Yes	No	No	Yes
N	480	480	480	3200	3200	3200	1120	912	912

Table 12: Regression results related to Equation 13.

Notes: The overall sample includes pairs of industries I-J that, for each industry J, I include J's top ten supplying industries, I. In the row labeled "Cragg-Donald Statistic", an "i" indicates that the test for a weak instrument is rejected at the 10 percent threshold. The "military spending shock$_{tJ\text{'s suppliers}}$" term refers to the cost-weighted average of the "military spending shock$_{tI}$" term, averaging over industry J's suppliers.

Second stage regression results						
ε_M	-0.14	-0.09	-0.05	-0.06	-0.10	-0.08
	(0.23)	(0.25)	(0.19)	(0.19)	(0.16)	(0.16)
ε_Q	0.82	0.42	0.73	0.88	0.98	0.88
	(0.56)	(0.46)	(0.45)	(0.36)	(0.41)	(0.33)
First stage: Dependent variable is $\Delta \log P^{in}_{tJ} - \Delta \log P_{tI}$						
military spending shock$_{tI}$	-0.89	-0.86	-0.86	-0.80	-0.81	-0.74
	(0.09)	(0.09)	(0.07)	(0.07)	(0.05)	(0.06)
military spending	0.82	0.95	1.02	1.22	1.15	1.35
shock$_{tJ\text{'s suppliers}}$	(0.13)	(0.17)	(0.10)	(0.13)	(0.09)	0.12)
military spending shock$_{tJ}$	0.11	0.11	-0.12	-0.13	-0.20	-0.22
	(0.09)	(0.09)	(0.07)	(0.07)	(0.06)	0.06)
F-statistic	41.09	10.84	63.80	15.01	95.92	20.11
First stage: Dependent variable is $\Delta \log P_{tJ} - \Delta \log P^{in}_{tJ}$						
military spending shock$_{tI}$	-0.17	-0.10	-0.14	-0.03	-0.12	0.02
	(0.07)	(0.07)	(0.05)	(0.05)	(0.03)	0.03)
military spending	-0.28	0.14	-0.29	0.11	-0.30	0.10
shock$_{tJ\text{'s suppliers}}$	(0.10)	(0.13)	(0.07)	(0.09)	(0.05)	0.07)
military spending shock$_{tJ}$	0.40	0.36	0.38	0.35	0.38	0.35
	(0.07)	(0.07)	(0.05)	(0.05)	(0.04)	0.04)
F-statistic	12.38	5.11	22.81	9.80	43.52	18.17
Cragg-Donald Statistic	9.43	13.55[i]	21.57[i]	32.33[i]	42.18[i]	62.21[i]
Wu-Hausman test p-value	0.85	0.32	0.56	0.56	0.37	0.16
Year Fixed Effects	No	Yes	No	Yes	No	Yes
Upstream Industries per downstream industry×year	4	4	8	8	15	15
N	1856	1856	3680	3680	6832	6832

Table 13: Regression results related to Equation 13.

Notes: The overall sample includes pairs of industries J, and, for each industry J, the top two supplying industries, I in the first four columns, and the top four supplying industries in the final four columns. In the row labeled "Cragg-Donald Statistic", an "i" indicates that the test for a weak instrument is rejected at the 10 percent threshold. The "military spending shock$_{tJ\text{'s suppliers}}$" term refers to the cost-weighted average of the "military spending shock$_{tI}$" term, averaging over industry J's suppliers.

ε_M	0.28	0.36	0.70	0.19	0.42	0.30
	(0.05)	(0.05)	(0.05)	(0.05)	(0.04)	(0.04)
ε_Q	0.11	0.56	0.71	0.81	0.79	0.47
	(0.06)	(0.05)	(0.04)	(0.05)	(0.04)	(0.05)
Year Fixed Effects	Yes	Yes	Yes	Yes	Yes	Yes
N	3920	3920	3920	3920	3920	3920
Country	DNK	ESP	FRA	ITA	JPN	NLD

Table 14: Regression results related to Equation 13.

Notes: This table contains OLS specifications, using ten input-supplying industries per downstream industry.

	Bench -mark	1960-1983	1984-2012	1960-2007	Period Length is 2 years	Durable Goods
R^2 (sectoral shocks)						
$(\varepsilon_M, \varepsilon_D, \varepsilon_Q) = (1, 1, 1)$	0.21$^{\text{kf}}$			0.32$^{\text{kf}}$		0.21
$(\varepsilon_M, \varepsilon_D, \varepsilon_Q) = \left(1, 1, \frac{4}{5}\right)$	0.19$^{\text{kf}}$			0.29$^{\text{kf}}$		0.19
$(\varepsilon_M, \varepsilon_D, \varepsilon_Q) = \left(\frac{1}{10}, 1, \frac{4}{5}\right)$	0.81			0.81		0.97$^{\text{kf}}$
$(\varepsilon_M, \varepsilon_D, \varepsilon_Q) = \left(\frac{1}{10}, \frac{2}{3}, 1\right)$	0.99$^{\text{kf}}$			1.00$^{\text{kf}}$		0.91$^{\text{kf}}$
$(\varepsilon_M, \varepsilon_D, \varepsilon_Q) = \left(\frac{1}{10}, 1, 1\right)$	0.83			0.94		0.98$^{\text{kf}}$
$(\varepsilon_M, \varepsilon_D, \varepsilon_Q) = \left(\frac{1}{10}, \frac{4}{3}, 1\right)$	0.59			0.71		0.92$^{\text{kf}}$
$\bar{\rho}\,(\omega)$						
$(\varepsilon_M, \varepsilon_D, \varepsilon_Q) = (1, 1, 1)$	0.19$^{\text{kf}}$	0.19$^{\text{kf}}$	0.19$^{\text{kf}}$	0.17$^{\text{kf}}$	0.18	0.20
$(\varepsilon_M, \varepsilon_D, \varepsilon_Q) = \left(1, 1, \frac{4}{5}\right)$	0.21$^{\text{kf}}$	0.21$^{\text{kf}}$	0.20$^{\text{kf}}$	0.18$^{\text{kf}}$	0.19	0.22
$(\varepsilon_M, \varepsilon_D, \varepsilon_Q) = \left(\frac{1}{10}, 1, \frac{4}{5}\right)$	0.06	0.06	0.09	0.06	0.05$^{\text{kf}}$	0.06$^{\text{kf}}$
$(\varepsilon_M, \varepsilon_D, \varepsilon_Q) = \left(\frac{1}{10}, \frac{2}{3}, 1\right)$	0.04$^{\text{kf}}$	0.03$^{\text{kf}}$	0.04$^{\text{kf}}$	0.03$^{\text{kf}}$	0.04$^{\text{kf}}$	0.05$^{\text{kf}}$
$(\varepsilon_M, \varepsilon_D, \varepsilon_Q) = \left(\frac{1}{10}, 1, 1\right)$	0.06	0.05	0.07	0.05	0.06$^{\text{kf}}$	0.06$^{\text{kf}}$
$(\varepsilon_M, \varepsilon_D, \varepsilon_Q) = \left(\frac{1}{10}, \frac{4}{3}, 1\right)$	0.08	0.07	0.07	0.07	0.08	0.06$^{\text{kf}}$

Table 15: Robustness checks: R^2(sectoral shocks) and $\bar{\rho}(\omega)$ for different values of ε_D, ε_M, and ε_Q.

Notes: I could not compute R^2(sectoral shocks) in the second, third, and fifth columns, as there are fewer time periods than there are industries in these samples. A "$^{\text{kf}}$" indicates the use of the Kalman filter, as opposed to direct applications of Equation 11, to infer the ω productivity shocks.

shocks are similar in the first half and the second half of the sample. (Unfortunately, since there are fewer time periods in either of the two halves of the sample than there are industries, I cannot compute the first factor of the industries' productivity shocks to assess the contribution of common productivity shocks to aggregate volatility.) In the fourth column, I exclude the Great-Recession period from the sample. Here, the assessed role of industry-specific shocks is somewhat larger. The fifth column applies biennial data. The final column incorporates good durability, in which I allow for certain industries' outputs to depreciate over a number of periods. In this column, I set $\delta_{C_J} = 1$ for all nondurable industries and $\delta_{C_J} = 0.4$ for durable industries. In this column, sectoral shocks now constitute a larger fraction of aggregate volatility when $\varepsilon_M = \frac{1}{10}$.[38]

A final set of robustness check considers the sensitivity of the main results to the de-trending procedure.[39] In the benchmark calculations, I had linearly de-trended each

[38] These depreciation rates are considerably larger than have been estimated elsewhere by, for example, Hulten and Wykoff (1981). Unfortunately, applying lower depreciation rates would lead to exceedingly large eigenvalues of $(\Pi_3)^{-1}\Pi_2$.

[39] In estimations of dynamic general equilibrium models, the choice of the de-trending procedure is potentially important; see Canova (2014). An alternative—intuitively appealing but unfortunately infeasible—way to deal with trends would be to include both transitory and permanent shocks in the model. This would obviate the need to de-trend the data before estimation; the parameters governing the permanent and transitory

De-trending Method	Benchmark	None	Hodrick-Prescott	Linear, Break in 1983	Linear, Censor Outliers
R^2 (sectoral shocks)					
$(\varepsilon_M, \varepsilon_D, \varepsilon_Q) = (1, 1, 1)$	0.21^{kf}	0.22^{kf}	0.22^{kf}	0.20^{kf}	0.22^{kf}
$(\varepsilon_M, \varepsilon_D, \varepsilon_Q) = \left(1, 1, \frac{4}{5}\right)$	0.19^{kf}	0.20^{kf}	0.19^{kf}	0.18^{kf}	0.20^{kf}
$(\varepsilon_M, \varepsilon_D, \varepsilon_Q) = \left(\frac{1}{10}, 1, \frac{4}{5}\right)$	0.81	0.80	0.70	0.79	0.85
$(\varepsilon_M, \varepsilon_D, \varepsilon_Q) = \left(\frac{1}{10}, \frac{2}{3}, 1\right)$	0.99^{kf}	0.99^{kf}	0.99^{kf}	0.99^{kf}	1.00^{kf}
$(\varepsilon_M, \varepsilon_D, \varepsilon_Q) = \left(\frac{1}{10}, 1, 1\right)$	0.83	0.83	0.76	0.82	0.77
$(\varepsilon_M, \varepsilon_D, \varepsilon_Q) = \left(\frac{1}{10}, \frac{4}{3}, 1\right)$	0.59	0.58	0.53	0.57	0.60
$\bar{\rho}(\omega)$					
$(\varepsilon_M, \varepsilon_D, \varepsilon_Q) = (1, 1, 1)$	0.19^{kf}	0.19^{kf}	0.20^{kf}	0.20^{kf}	0.20^{kf}
$(\varepsilon_M, \varepsilon_D, \varepsilon_Q) = \left(1, 1, \frac{4}{5}\right)$	0.21^{kf}	0.21^{kf}	0.21^{kf}	0.21^{kf}	0.21^{kf}
$(\varepsilon_M, \varepsilon_D, \varepsilon_Q) = \left(\frac{1}{10}, 1, \frac{4}{5}\right)$	0.06	0.05	0.05	0.06	0.06
$(\varepsilon_M, \varepsilon_D, \varepsilon_Q) = \left(\frac{1}{10}, \frac{2}{3}, 1\right)$	0.04^{kf}	0.04^{kf}	0.04^{kf}	0.04^{kf}	0.03^{kf}
$(\varepsilon_M, \varepsilon_D, \varepsilon_Q) = \left(\frac{1}{10}, 1, 1\right)$	0.06	0.05	0.06	0.06	0.05
$(\varepsilon_M, \varepsilon_D, \varepsilon_Q) = \left(\frac{1}{10}, \frac{4}{3}, 1\right)$	0.08	0.08	0.07	0.08	0.07

Table 16: Robustness checks: R^2(sectoral shocks) and $\bar{\rho}(\omega)$ for different values of ε_D, ε_M, and ε_Q.
Notes: A "$^{\text{kf}}$" indicates the use of the Kalman filter, as opposed to direct applications of Equation 11, to infer the ω productivity shocks.

industry-level observable before performing the filtering exercise. In Table 16, I consider three alternate de-trending procedures: not de-trending the data, a Hodrick-Prescott filter, and a linear trend with a break in the trend at 1983. These de-trending procedures have almost no quantitative impact on the relative contribution of sectoral vs. common shocks to aggregate volatility. Finally, censoring outlier observations (those industry-year output growth rates in the top or bottom centile) does not alter the estimated importance of sectoral shocks.

F Solution of the model filter

This section spells out the solution of the model. First, I write out the constrained maximization problem of a social planner. I take first-order conditions, write out the conditions that characterize the steady state, log-linearize around the steady state, solve for the policy functions, and then for the model filter. We allow not only for factor neutral productivity shocks (as used throughout the paper), but also labor-augmenting productivity shocks, as

shock processes would be jointly estimated in a single stage. I do not pursue this approach, mainly because of the difficulty of scaling the model by the permanent shocks. Doing so requires a clean characterization of the changes in the industry-level observable variables as functions of the permanent shocks, something that exists only for a few special cases of the model (such special cases can be found in, for example, Ngai and Pissarides 2007 and Acemoglu and Guerreri 2008).

in the specification in Table 4.

F.1 First order conditions and steady-state shares

Since this economy satisfies the welfare theorems, it will suffice to solve the social planner's problem. Begin with the Lagrangian:

$$\begin{aligned}\mathcal{L} = \mathbb{E}_0 \sum_{t=0}^{\infty} \beta^t \Bigg\{ \log \left[\left[\sum_{J=1}^{N} (\xi_J)^{\frac{1}{\varepsilon_D}} \left(\delta_{C_J} \cdot C_{tJ}\right)^{\frac{\varepsilon_D - 1}{\varepsilon_D}} \right]^{\frac{\varepsilon_D}{\varepsilon_D - 1}} \right] \\ - \frac{\varepsilon_{LS}}{\varepsilon_{LS}+1} \left(\sum_{J=1}^{N} L_{tJ} \right)^{\frac{\varepsilon_{LS}+1}{\varepsilon_{LS}}} + \sum_{J=1}^{N} P^{inv}_{tJ} \left[X_{tJ} + (1-\delta_K) K_{tJ} - K_{t+1,J} \right] \\ + \sum_{J=1}^{N} P_{tJ} \left[Q_{tJ} + \left(1-\delta_{C_J}\right) C_{tJ} - C_{t+1,J} - \sum_{I=1}^{N} \left[M_{t,J\to I} + X_{t,J\to I} \right] \right] \Bigg\} . \end{aligned} \tag{25}$$

Here, P^{inv}_{tJ} is the Lagrange multiplier on a unit of capital, and P_{tJ} is the Lagrange multiplier on the good-J market-clearing condition.

This Lagrangian incorporates durability for some consumption goods, something that was ignored in the body of the paper. The Lagrangian reflects a representative consumer who has preferences given by the following utility function:

$$\mathcal{U} = \sum_{t=0}^{\infty} \beta^t \log \left[\left[\sum_{J=1}^{N} (\xi_J)^{\frac{1}{\varepsilon_D}} \left(\delta_{C_J} \cdot C_{tJ}\right)^{\frac{\varepsilon_D - 1}{\varepsilon_D}} \right]^{\frac{\varepsilon_D}{\varepsilon_D - 1}} \right] - \frac{\varepsilon_{LS}}{\varepsilon_{LS}+1} \left(\sum_{J=1}^{N} L_{tJ} \right)^{\frac{\varepsilon_{LS}+1}{\varepsilon_{LS}}}$$

The demand parameters, ξ_J, again reflect time-invariant differences in the importance of industries' goods in the consumer's preferences. Now, C_{tJ} equals the stock of durable goods when J is a durable-good-producing industry and equals the expenditures on good/service J otherwise. For durable goods, J, the evolution of the stock of each consumption good C_{tJ} is given by

$$C_{t+1,J} = C_{tJ} \left(1-\delta_{C_J}\right) + \tilde{C}_{tJ} \ ,$$

where $\tilde{C}_{tJ}$ equals the consumer's new purchases on good J at time t and δ_{C_J} parameterizes the depreciation rate of good J.

I re-state the expression for Q_{tJ}:

$$Q_{tJ} = A_{tJ} \cdot \left[(1-\mu_J)^{\frac{1}{\varepsilon_Q}} \left(\left(\frac{K_{tJ}}{\alpha_J} \right)^{\alpha_J} \left(\frac{L_{tJ} \cdot B_{tJ}}{1-\alpha_J} \right)^{1-\alpha_J} \right)^{\frac{\varepsilon_Q - 1}{\varepsilon_Q}} + (\mu_J)^{\frac{1}{\varepsilon_Q}} \left(M_{tJ}\right)^{\frac{\varepsilon_Q - 1}{\varepsilon_Q}} \right]^{\frac{\varepsilon_Q}{\varepsilon_Q - 1}} . \tag{26}$$

The first-order conditions for the planner are:

$$\begin{aligned}
K_{t+1,J} &= X_{tJ}+(1-\delta_K)\cdot K_{tJ} \\
[C_{tJ}] &: P_{t-1,J} -\beta P_{tJ}\left(1-\delta_{C_J}\right)=\beta\left(\xi_J\right)^{\frac{1}{\varepsilon_D}}\left(\delta_{C_J}\right)^{\frac{\varepsilon_D-1}{\varepsilon_D}}\times \qquad (27)\\
& \left(C_{tJ}\right)^{-\frac{1}{\varepsilon_D}}\left(\sum_{I=1}^{N}\left(\xi_I\right)^{\frac{1}{\varepsilon_D}}\left(\delta_{C_I}\cdot C_{tI}\right)^{\frac{\varepsilon_D-1}{\varepsilon_D}}\right)^{-1}. \qquad (28)\\
[M_{t,I\to J}] &: \frac{P_{tI}}{P_{tJ}}=\left(A_{tJ}\right)^{\frac{\varepsilon_Q-1}{\varepsilon_Q}}\left(\frac{Q_{tJ}\cdot\mu_J}{M_{tJ}}\right)^{\frac{1}{\varepsilon_Q}}\left(\frac{M_{tJ}\cdot\Gamma^M_{IJ}}{M_{t,I\to J}}\right)^{\frac{1}{\varepsilon_M}}. \qquad (29)\\
[X_{t,I\to J}] &: P_{tI}=P^{inv}_{tJ}\left(\frac{X_{tJ}\cdot\Gamma^X_{IJ}}{X_{t,I\to J}}\right)^{\frac{1}{\varepsilon_X}}. \qquad (30)\\
[L_{tJ}] &: \left(\sum_{J'=1}^{N}L_{tJ'}\right)^{\frac{1}{\varepsilon_{LS}}}=P_{tJ}\cdot\left(A_{tJ}\right)^{\frac{\varepsilon_Q-1}{\varepsilon_Q}}B_{tJ}\left(Q_{tJ}\left(1-\mu_J\right)\right)^{\frac{1}{\varepsilon_Q}}\times \qquad (31)\\
& \left(\frac{K_{tJ}}{\alpha_J}\right)^{\alpha_J\frac{\varepsilon_Q-1}{\varepsilon_Q}}\left(\frac{L_{tJ}\cdot B_{tJ}}{1-\alpha_J}\right)^{\frac{\alpha_J-1-\alpha_J\varepsilon_Q}{\varepsilon_Q}}.\\
[K_{t+1,J}] &: P^{inv}_{tJ}=\beta\cdot\mathbb{E}_t\Bigg[P_{t+1,J}\left(Q_{t+1,J}\left(1-\mu_J\right)\right)^{\frac{1}{\varepsilon_Q}}\left(A_{t+1,J}\right)^{\frac{\varepsilon_Q-1}{\varepsilon_Q}} \qquad (32)\\
& \times\left(\frac{K_{t+1,J}}{\alpha_J}\right)^{-1+\alpha_J\cdot\frac{\varepsilon_Q-1}{\varepsilon_Q}}\left(\frac{L_{t+1,J}\cdot B_{t+1,J}}{1-\alpha_J}\right)^{(1-\alpha_J)\cdot\frac{\varepsilon_Q-1}{\varepsilon_Q}}\Bigg]\\
& +\beta(1-\delta_K)\mathbb{E}_t\left[P^{inv}_{t+1,J}\right].
\end{aligned}$$

Towards the goal of solving for the steady-state, drop time subscripts and re-arrange. Also, employ the normalization that steady-state labor is the numeraire good (so that

$\sum_{J'=1}^{N} L_{J'}^{\frac{1}{\varepsilon_{LS}}} = 1)$:

$$
\begin{aligned}
\delta_K K_J &= X_J \\
\frac{1-\beta\left(1-\delta_{C_J}\right)}{\beta} P_J &= \left(\xi_J\right)^{\frac{1}{\varepsilon_D}} \cdot \left(\delta_{C_J}\right)^{\frac{\varepsilon_D-1}{\varepsilon_D}} \left(C_J\right)^{-\frac{1}{\varepsilon_D}} \left(\sum_{I=1}^{N} \left(\xi_I\right)^{\frac{1}{\varepsilon_D}} \left(\delta_{C_I} \cdot C_I\right)^{\frac{\varepsilon_D-1}{\varepsilon_D}}\right)^{-1} \\
\frac{P_I}{P_J} &= \left(\frac{Q_J \cdot \mu_J}{M_J}\right)^{\frac{1}{\varepsilon_Q}} \left(\frac{M_J \cdot \Gamma_{IJ}^M}{M_{I\to J}}\right)^{\frac{1}{\varepsilon_M}} \\
P_J^{inv} &= P_I \left(\frac{X_J \cdot \Gamma_{IJ}^X}{X_{I\to J}}\right)^{-\frac{1}{\varepsilon_X}} \\
1 &= P_J \left(Q_J \left(1-\mu_J\right)\right)^{\frac{1}{\varepsilon_Q}} \left(\frac{K_J}{\alpha_J}\right)^{\alpha_J \cdot \frac{\varepsilon_Q-1}{\varepsilon_Q}} \left(\frac{L_J}{1-\alpha_J}\right)^{(1-\alpha_J)\cdot \frac{\varepsilon_Q-1}{\varepsilon_Q}-1} \\
Q_J &= \left[\left(1-\mu_J\right)^{\frac{1}{\varepsilon_Q}} \left(\left(\frac{K_J}{\alpha_J}\right)^{\alpha_J} \left(\frac{L_J}{1-\alpha_J}\right)^{1-\alpha_J}\right)^{\frac{\varepsilon_Q-1}{\varepsilon_Q}} + \left(\mu_J\right)^{\frac{1}{\varepsilon_Q}} \left(M_J\right)^{\frac{\varepsilon_Q-1}{\varepsilon_Q}}\right]^{\frac{\varepsilon_Q}{\varepsilon_Q-1}}
\end{aligned} \tag{33}
$$

First, I will solve for the prices of each industry's good, in the steady state, P_J. This will follow from each industry's cost-minimization condition.

Take the cost-minimization condition for capital, which equates the rental price of a unit of capital to the marginal revenue product of capital:

$$
\frac{1-\beta(1-\delta_K)}{\beta} \left[\sum \Gamma_{IJ}^X \left(P_I\right)^{1-\varepsilon_X}\right]^{1/(1-\varepsilon_X)} = P_J \left(Q_J \left(1-\mu_J\right)\right)^{\frac{1}{\varepsilon_Q}} \left(\frac{K_J}{\alpha_J}\right)^{\alpha_J \cdot \frac{\varepsilon_Q-1}{\varepsilon_Q}-1} \left(\frac{L_J}{1-\alpha_J}\right)^{(1-\alpha_J)\cdot \frac{\varepsilon_Q-1}{\varepsilon_Q}} \tag{34}
$$

Second, take cost-minimizing condition for industry J's intermediate input purchases:

$$
\left(\mu_J\right)^{\frac{1}{\varepsilon_Q}} \left(M_J\right)^{\frac{\varepsilon_Q-1}{\varepsilon_Q}} = \mu_J \left(Q_J\right)^{\frac{\varepsilon_Q-1}{\varepsilon_Q}} \left(\frac{P_J^{in}}{P_J}\right)^{1-\varepsilon_Q} \tag{35}
$$

And, third, the following equation takes the cost-minimizing choice of the capital-labor aggregate.

$$
\begin{aligned}
\left(1-\mu_J\right)^{\frac{1}{\varepsilon_Q}} \left(\left(\frac{K_J}{\alpha_J}\right)^{\alpha_J} \left(\frac{L_J}{1-\alpha_J}\right)^{1-\alpha_J}\right)^{\frac{\varepsilon_Q-1}{\varepsilon_Q}} &= \left(1-\mu_J\right) \left(Q_J\right)^{\frac{\varepsilon_Q-1}{\varepsilon_Q}} \times \\
&\left(\frac{\left(\frac{1-\beta(1-\delta_K)}{\beta}\right)^{\alpha_J} \left[\sum \Gamma_{IJ}^X \left(P_I\right)^{1-\varepsilon_X}\right]^{\alpha_J/(1-\varepsilon_X)}}{P_J}\right)^{1-\varepsilon_Q}
\end{aligned} \tag{36}
$$

Plug Equations 34-36 into Equation 33.

$$(P_J)^{1-\varepsilon_Q} = (1-\mu_J)\left(\beta^{-1}-(1-\delta_K)\right)^{\alpha_J\left(1-\varepsilon_Q\right)}\left[\sum_I \Gamma^X_{IJ}\,(P_I)^{1-\varepsilon_X}\right]^{\alpha_J\frac{1-\varepsilon_Q}{1-\varepsilon_X}} \tag{37}$$
$$+\mu_J\left[\sum_I \Gamma^M_{IJ}\,(P_I)^{1-\varepsilon_M}\right]^{\frac{1-\varepsilon_Q}{1-\varepsilon_M}}$$

Equation 37 describes and $N\times N$ system of equations for the N steady-state price levels. This completes the first part of the characterization of the steady state.

For the second part, consider the market clearing condition for good J:

$$Q_J = \delta_{C_J}C_J + \sum_{I=1}^{N}\left(M_{J\to I}+X_{J\to I}\right) \tag{38}$$

Below, I will write out the terms on the right-hand-side of Equation 38 in terms of the steady state prices (which have just been solved for):

First, write out the consumption of good J.

$$\frac{(1-\beta\,(1-\delta_{C_J}))}{\beta}P_J = (\xi_J)^{\frac{1}{\varepsilon_D}}\cdot(\delta_{C_J})^{\frac{\varepsilon_D-1}{\varepsilon_D}}\,(C_J)^{-\frac{1}{\varepsilon_D}}\left(\sum_{I=1}^{N}(\xi_I)^{\frac{1}{\varepsilon_D}}\,(\delta_{C_I}\cdot C_I)^{\frac{\varepsilon_D-1}{\varepsilon_D}}\right)^{-1}$$
$$\delta_{C_J}C_J = \xi_J\,(\delta_{C_J})^{\varepsilon_D}\left(\frac{1-\beta\,(1-\delta_{C_J})}{\beta}\right)^{-\varepsilon_D}(P_J)^{-\varepsilon_D}\,\bar{C}^{1-\varepsilon_D}, \tag{39}$$

where $\bar{C}$ is the aggregate consumption bundle, defined as the final term in parentheses on the first line raised to the $1/(1-\varepsilon_D)$ power.

Then write out the intermediate input purchases from industry J to industry I

$$\begin{aligned}
M_{J\to I} &= (Q_I\mu_I)^{\frac{\varepsilon_M}{\varepsilon_Q}}\cdot(M_I)^{\frac{\varepsilon_Q-\varepsilon_M}{\varepsilon_M}}\,\Gamma^M_{JI}\cdot\left(\frac{P_I}{P_J}\right)^{\varepsilon_M}\\
&= Q_I\mu_I\Gamma^M_{JI}\,(P_J)^{-\varepsilon_M}\left(P_I^{in}\right)^{\varepsilon_M-\varepsilon_Q}(P_I)^{\varepsilon_Q}\\
&= Q_I\mu_I\Gamma^M_{JI}\,(P_J)^{-\varepsilon_M}\left(\sum_{J'}\Gamma^M_{J'I}\,(P_{J'})^{1-\varepsilon_M}\right)^{\frac{\varepsilon_M-\varepsilon_Q}{1-\varepsilon_M}}(P_I)^{\varepsilon_Q}
\end{aligned} \tag{40}$$

And, finally, write out the investment input purchases from industry J sold to industry I. Begin by writing out the total investment purchases of industry J.

$$\left(\frac{K_J}{\alpha_J}\right) = \left(\frac{1-\beta(1-\delta_K)}{\beta}\left[\sum\Gamma^X_{IJ}\,(P_I)^{1-\varepsilon_X}\right]^{\frac{1}{1-\varepsilon_X}}\right)^{-1+\alpha_J\left(1-\varepsilon_Q\right)}(1-\mu_J)\,Q_J\,(P_J)^{\varepsilon_Q}$$
$$X_J = (1-\mu_J)\,Q_J\alpha_J\delta_K\left(\frac{1-\beta(1-\delta_K)}{\beta}\left[\sum\Gamma^X_{IJ}\,(P_I)^{1-\varepsilon_X}\right]^{\frac{1}{1-\varepsilon_X}}\right)^{-1+\alpha_J\left(1-\varepsilon_Q\right)}(P_J)^{\varepsilon_Q}$$

So:

$$
\begin{aligned}
X_{J\to I} &= X_I \cdot \Gamma^X_{JI} \cdot \left(\frac{P_J}{P_I^{inv}}\right)^{-\varepsilon_X} \\
&= X_I \cdot \Gamma^X_{JI} \cdot (P_J)^{-\varepsilon_X} \left(P_I^{inv}\right)^{\varepsilon_X} \\
&= Q_I \left(1-\mu_I\right) \alpha_I \delta_K \left(\frac{1-\beta(1-\delta_K)}{\beta}\right)^{-1+\alpha_I\left(1-\varepsilon_Q\right)} \Gamma^X_{JI} \times \\
&\left[\sum_{J'} \Gamma^X_{J'I} \left(P_{J'}\right)^{1-\varepsilon_X}\right]^{\frac{\varepsilon_X - 1 + \alpha_I\left(1-\varepsilon_Q\right)}{1-\varepsilon_X}} (P_J)^{-\varepsilon_X} (P_I)^{\varepsilon_Q} \qquad (41)
\end{aligned}
$$

Plug in the expressions (Equations 39-41) into the market clearing condition (Equation 38):

$$
Q_J - \sum_{I=1}^{N} \tilde{\Gamma}_{JI} Q_I = \xi_J \left(\delta_{C_J}\right)^{\varepsilon_D} \left(\frac{1-\beta\left(1-\delta_{C_J}\right)}{\beta}\right)^{-\varepsilon_D} (P_J)^{-\varepsilon_D} \bar{C}^{1-\varepsilon_D}
$$

where

$$
\begin{aligned}
\tilde{\Gamma}_{JI} = (P_I)^{\varepsilon_Q} \times \Bigg\{ & \mu_I \Gamma^M_{JI} \left[\sum_{J'} \Gamma^M_{J'I} \left(P_{J'}\right)^{1-\varepsilon_M}\right]^{\frac{\varepsilon_M-\varepsilon_Q}{1-\varepsilon_M}} (P_J)^{-\varepsilon_M} \\
& + \left(1-\mu_I\right) \alpha_I \delta_K \left(\frac{1-\beta(1-\delta_K)}{\beta}\right)^{-1+\alpha_I\left(1-\varepsilon_Q\right)} \Gamma^X_{JI} \left[\sum_{J'} \Gamma^X_{J'I} \left(P_{J'}\right)^{1-\varepsilon_X}\right]^{-1+\alpha_I \frac{1-\varepsilon_Q}{1-\varepsilon_X}} (P_J)^{-\varepsilon_X} \Bigg\}
\end{aligned}
$$

We can solve for the Q vector using linear algebra. From here, we can solve for the steady state shares:

$$L_J = Q_J(1-\alpha_J)(1-\mu_J)(P_J)^{\varepsilon_Q}\left(\frac{1-\beta(1-\delta_K)}{\beta}P_J^{inv}\right)^{\alpha_J\left(1-\varepsilon_Q\right)} \tag{42}$$

$$C_J = \xi_J\delta_{C_J}^{\varepsilon_D-1}\bar{C}^{1-\varepsilon_D}\left(\frac{1-\beta(1-\delta_{C_J})}{\beta}\right)^{-\varepsilon_D}(P_J)^{-\varepsilon_D}$$

$$S_I^C = \frac{(\xi_I)^{\frac{1}{\varepsilon_D}}(\delta_{C_I}\cdot C_I)^{\frac{\varepsilon_D-1}{\varepsilon_D}}}{\sum(\xi_{I'})^{\frac{1}{\varepsilon_D}}\left(\delta_{C_{I'}}C_{I'}\right)^{\frac{\varepsilon_D-1}{\varepsilon_D}}} \tag{43}$$

$$\frac{M_{J\to I}}{Q_J} = (Q_J)^{-1}Q_I\mu_I\Gamma_{JI}^M\left(P_I^{in}\right)^{\varepsilon_M-\varepsilon_Q}(P_J)^{-\varepsilon_M}(P_I)^{\varepsilon_Q}$$

$$\frac{X_{J\to I}}{Q_J} = (Q_J)^{-1}Q_I(1-\mu_I)\alpha_I\delta_K\left(\frac{1-\beta(1-\delta_K)}{\beta}\right)^{-1+\alpha_I\left(1-\varepsilon_Q\right)}\Gamma_{JI}^X\times$$
$$\left[\sum_{J'}\Gamma_{J'I}^X(P_{J'})^{1-\varepsilon_X}\right]^{\frac{\varepsilon_X-1+\alpha_J\left(1-\varepsilon_Q\right)}{1-\varepsilon_X}}(P_J)^{-\varepsilon_X}(P_I)^{\varepsilon_Q}$$

$$\left[S_X^1\right]_{IJ} = \Gamma_{IJ}^X\left(\frac{P_J^{inv}}{P_I}\right)^{\varepsilon_X-1} \tag{44}$$

$$\left[S_M^1\right]_{IJ} = \Gamma_{IJ}^M\left(\frac{P_J^{in}}{P_I}\right)^{\varepsilon_M-1} \tag{45}$$

Clearly, these equations depend on Q_J and the steady-state prices. Note that, however, these figures have already been solved for. For future reference, define $\tilde{S}_M^Q$ as the matrix that has, in its J, I entry, the fraction of good J that is sold to industry I as an intermediate input: $\left[\tilde{S}_M^Q\right]_{JI} \equiv \frac{M_{J\to I}}{Q_J}$. Similarly, define $\left[\tilde{S}_X^Q\right]_{JI} \equiv \frac{X_{J\to I}}{Q_J}$. Equation 42 characterizes the share of labor that is employed in industry J, in the steady state. Use $\tilde{S}^L$ as the $N\times N$ matrix that has, in its J^{th} column, this steady-state share. Also for future reference, define $\tilde{S}_I^C$ as the matrix that has S_I^C (as given in Equation 43) in its I^{th} column. And, finally, use $\left[\tilde{S}_C^Q\right]_J$ to denote the share of good J that is consumed (which can be computed by subtracting the sum of the $\left[\tilde{S}_M^Q\right]_{IJ}$ and $\left[\tilde{S}_X^Q\right]_{IJ}$ from 1.)

F.2 Log linearization

The log linearization of the first order conditions are rather straightforward to derive. Below, I will derive Equations 46 and 47. In all of these equations, a lower-case letter with the circumflex (^) denotes log-deviation from the steady state.

$$\begin{aligned}
\hat{x}_{tJ} &= \delta_K^{-1}\hat{k}_{t+1,J} + (1-\delta_K^{-1})\hat{k}_{tJ} \\
\hat{q}_{tJ} &= \delta_{C_J}^{-1} S_{CJ}^{Q}\hat{c}_{t+1J} + \left(1-\delta_{C_J}^{-1}\right) S_{CJ}^{Q}\hat{c}_{tJ} \qquad (46)\\
&+ \sum_{I=1}^{N}\left(S_{M,J\to I}^{Q}\hat{m}_{tJ\to I} + S_{X,J\to I}^{Q}\hat{x}_{tJ\to I}\right) \\
\frac{1}{1-\beta\left(1-\delta_{C_J}\right)}\hat{p}_{tJ} - \frac{\beta\left(1-\delta_{C_J}\right)}{1-\beta\left(1-\delta_{C_J}\right)}\hat{p}_{t+1,J} &\approx -\frac{1}{\varepsilon_D}\hat{c}_{t+1J} \qquad (47)\\
&- \sum_{I}\frac{\left(\xi_I\right)^{\frac{1}{\varepsilon_D}}\left(\delta_{C_I}\cdot C_I\right)^{\frac{\varepsilon_D-1}{\varepsilon_D}}}{\sum_{I'}\left(\xi_{I'}\right)^{\frac{1}{\varepsilon_D}}\left(\delta_{C_{I'}}C_{I'}\right)^{\frac{\varepsilon_D-1}{\varepsilon_D}}}\left[\frac{\varepsilon_D-1}{\varepsilon_D}\hat{c}_{t+1,I}\right] \\
\hat{p}_{tI} - \hat{p}_{tJ} &= \frac{\varepsilon_Q-1}{\varepsilon_Q}\hat{a}_{tJ} + \frac{1}{\varepsilon_Q}\hat{q}_{tJ} + \left(\frac{1}{\varepsilon_M}-\frac{1}{\varepsilon_Q}\right)\hat{m}_{tJ} - \frac{1}{\varepsilon_M}\hat{m}_{t,I\to J} \\
\hat{p}_{tI} &= \hat{p}_{tJ}^{inv} + \frac{1}{\varepsilon_X}\left(\hat{x}_{tJ} - \hat{x}_{t,I\to J}\right) \\
\frac{1}{\varepsilon_{LS}}\sum_{J'=1}^{N} S_J^L\hat{l}_{tJ} &= \hat{p}_{tJ} + \frac{\varepsilon_Q-1}{\varepsilon_Q}\hat{a}_{tJ} + \frac{\left(\varepsilon_Q-1\right)\left(1-\alpha_J\right)}{\varepsilon_Q}\hat{b}_{tJ} \\
&+ \frac{1}{\varepsilon_Q}\hat{q}_{tJ} + \alpha_J\frac{\varepsilon_Q-1}{\varepsilon_Q}\hat{k}_{tJ} + \frac{\alpha_J - 1 - \alpha_J\cdot\varepsilon_Q}{\varepsilon_Q}\hat{l}_{tJ} \\
\frac{1}{1-\beta\left(1-\delta_K\right)}\hat{p}_{tJ}^{inv} - \frac{\beta\left(1-\delta_K\right)}{1-\beta\left(1-\delta_K\right)}\hat{p}_{t+1J}^{inv} &= \hat{p}_{t+1,J} + \frac{1}{\varepsilon_Q}\hat{q}_{t+1,J} \\
&+ \frac{\varepsilon_Q-1}{\varepsilon_Q}\hat{a}_{t+1,J} + \frac{\left(\varepsilon_Q-1\right)\left(1-\alpha_J\right)}{\varepsilon_Q}\hat{b}_{t+1,J} \\
&+ \frac{\left(\varepsilon_Q-1\right)\left(1-\alpha_J\right)}{\varepsilon_Q}\hat{l}_{t+1,J} + \left[-1+\alpha_J\cdot\frac{\varepsilon_Q-1}{\varepsilon_Q}\right]\hat{k}_{t+1,J} \\
\hat{q}_{tJ} &= \hat{a}_{tJ} + \alpha_J\left(1-S_{M_J}\right)\hat{k}_{tJ} + \left(1-\alpha_J\right)\left(1-S_{M_J}\right)\hat{b}_{tJ} \\
&+ \left(1-\alpha_J\right)\left(1-S_{M_J}\right)\hat{l}_{tJ} + S_{M_J}\hat{m}_{tJ}
\end{aligned}$$

To derive Equation 46, take the market clearing condition for good J,

$$\begin{aligned}
\log\left[\exp\hat{q}_{tJ}\right] &= \log\left[-\left(1-\delta_{C_J}\right)S_{CJ}^{Q}\exp\hat{c}_{t,J} + S_{CJ}^{Q}\exp\hat{c}_{t+1,J} + \sum_{I=1}^{N} S_{M,J\to I}^{Q}\exp\hat{m}_{tJ\to I} + S_{X,J\to I}^{Q}\exp\hat{x}_{tJ\to I}\right] \\
&\approx S_{CJ}^{Q}\delta_{C_J}^{-1}\hat{c}_{t+1,J} + S_{CJ}^{Q}\left(1-\delta_{C_J}^{-1}\right)\hat{c}_{tJ} + \sum_{I=1}^{N} S_{M,J\to I}^{Q}\hat{m}_{tJ\to I} + S_{X,J\to I}^{Q}\hat{x}_{tJ\to I}
\end{aligned}$$

The following set of calculations yield Equation 47:

$$
\begin{aligned}
P_J\left[\frac{1}{\beta}\frac{P_{t-1,J}}{P_J}-\frac{P_{tJ}}{P_J}\left(1-\delta_{C_J}\right)\right] &= \left(\xi_J\right)^{\frac{1}{\varepsilon_D}}\left(\delta_{C_J}\right)^{\frac{\varepsilon_D-1}{\varepsilon_D}}\left(C_J\right)^{-\frac{1}{\varepsilon_D}}\left(\exp\left\{\hat{c}_{tJ}\right\}\right)^{-\frac{1}{\varepsilon_D}}\times \\
&\quad \left(\sum_{I=1}^{N}\left(\xi_I\right)^{\frac{1}{\varepsilon_D}}\left(\delta_{C_I}\cdot C_{tI}\right)^{\frac{\varepsilon_D-1}{\varepsilon_D}}\right)^{-1} \\
\frac{1}{1-\beta\left(1-\delta_{C_J}\right)}\exp\hat{p}_{t-1,J}-\frac{\beta\left(1-\delta_{C_J}\right)}{1-\beta\left(1-\delta_{C_J}\right)}\exp\hat{p}_{tJ} &= \left(\exp\hat{c}_{tJ}\right)^{-\frac{1}{\varepsilon_D}}\times \\
&\quad \left(\sum_{I=1}^{N}\frac{\left(\xi_I\right)^{\frac{1}{\varepsilon_D}}\left(\delta_{C_I}\cdot C_I\right)^{\frac{\varepsilon_D-1}{\varepsilon_D}}}{\sum\left(\xi_{I'}\right)^{\frac{1}{\varepsilon_D}}\left(\delta_{C_{I'}}C_{I'}\right)^{\frac{\varepsilon_D-1}{\varepsilon_D}}}\exp\left\{\hat{c}_{tI}\right\}^{\frac{\varepsilon_D-1}{\varepsilon_D}}\right)^{-1} \\
\frac{1}{1-\beta\left(1-\delta_{C_J}\right)}\hat{p}_{t-1J}-\frac{\beta\left(1-\delta_{C_J}\right)}{1-\beta\left(1-\delta_{C_J}\right)}\hat{p}_{tJ} &\approx -\frac{1}{\varepsilon_D}\hat{c}_{tJ} \\
&\quad -\sum\frac{\left(\xi_I\right)^{\frac{1}{\varepsilon_D}}\left(\delta_{C_I}\cdot C_I\right)^{\frac{\varepsilon_D-1}{\varepsilon_D}}}{\sum\left(\xi_{I'}\right)^{\frac{1}{\varepsilon_D}}\left(\delta_{C_{I'}}C_{I'}\right)^{\frac{\varepsilon_D-1}{\varepsilon_D}}}\left[\frac{\varepsilon_D-1}{\varepsilon_D}\hat{c}_{tI}\right]
\end{aligned}
$$

Write the log-linearized equations, as given in the beginning of the subsection, in matrix form.

$$
\begin{aligned}
\hat{k}_{t+1} &= \delta_K\hat{\mathbb{X}}_t+\left(1-\delta_K\right)\hat{k}_t \\
\hat{q}_t &= \delta_C^{-1}\tilde{S}_C^Q\hat{c}_{t+1}+\left(\mathbf{I}-\delta_C^{-1}\right)\tilde{S}_C^Q\hat{c}_t+\tilde{S_X^Q}\hat{x}_t+\tilde{S}_M^Q\hat{m}_t \\
\hat{p}_t &= \beta\left(\mathbf{I}-\delta_C\right)\hat{p}_{t+1}-\frac{1}{\varepsilon_D}\left(\mathbf{I}-\beta\left(\mathbf{I}-\delta_C\right)\right)\left[\mathbf{I}+S_I^C\left(\varepsilon_D-1\right)\right]\hat{c}_{t+1} \\
\hat{m}_t &= \frac{\varepsilon_M}{\varepsilon_Q}\left(\varepsilon_Q-1\right)T_1\hat{a}_t+\frac{\varepsilon_M}{\varepsilon_Q}T_1\hat{q}_t+\left(1-\frac{\varepsilon_M}{\varepsilon_Q}\right)T_1\hat{\mathbb{M}}_t+\varepsilon_M\left[T_1-T_2\right]\hat{p}_t \\
\hat{x}_t &= T_1\hat{\mathbb{X}}_t+\varepsilon_X T_1\hat{p}_t^{inv}-\varepsilon_X T_2\hat{p}_t \\
\frac{1}{\varepsilon_{LS}}S^L\hat{l}_t &= \hat{p}_t+\frac{\varepsilon_Q-1}{\varepsilon_Q}\hat{a}_t+\frac{\left(\varepsilon_Q-1\right)\left(\mathbf{I}-\alpha\right)}{\varepsilon_Q}\hat{b}_t+\frac{1}{\varepsilon_Q}\hat{q}_t+\frac{\varepsilon_Q-1}{\varepsilon_Q}\alpha\hat{k}_t+\frac{\alpha-\mathbf{I}-\alpha\varepsilon_Q}{\varepsilon_Q}\hat{l}_t \\
\hat{p}_t^{inv} &= \beta(1-\delta_K)\hat{p}_{t+1}^{inv}+(1-\beta(1-\delta_K))\left[\hat{p}_{t+1}+\frac{1}{\varepsilon_Q}\hat{q}_{t+1}+\frac{\varepsilon_Q-1}{\varepsilon_Q}\hat{a}_{t+1}\right. \\
&\quad \left.+\left(-\mathbf{I}+\alpha\frac{\varepsilon_Q-1}{\varepsilon_Q}\right)\hat{k}_{t+1}+\left(\mathbf{I}-\alpha\right)\frac{\varepsilon_Q-1}{\varepsilon_Q}\left(\hat{l}_{t+1}+\hat{b}_{t+1}\right)\right] \\
\hat{q}_t &= \hat{a}_t+\left(\mathbf{I}-\alpha\right)\left(\mathbf{I}-S_M\right)\hat{b}_t+\alpha\left(\mathbf{I}-S_M\right)\hat{k}_t+\left(\mathbf{I}-\alpha\right)\left(\mathbf{I}-S_M\right)\hat{l}_t+S_M\hat{\mathbb{M}}_t
\end{aligned}
$$

In these equations T_1 refers to the $N^2\times N$ matrix equal to $\mathbf{1}\otimes\mathbf{I}$, where $\mathbf{1}$ is an $N\times 1$ vector of 1s and $\otimes$ is the Kronecker product. Similarly, T_2 equals $\mathbf{I}\otimes\mathbf{1}$. Also, S_M is a diagonal matrix with the steady-state intermediate cost shares along the diagonal; δ_C is a matrix with δ_{C_J}s along the diagonal; and α is a diagonal matrix with the α_Js along the diagonal. Finally, $\hat{\mathbb{M}}_t$ and $\hat{\mathbb{X}}_t$ are the $N\times 1$ vectors which contain the intermediate input bundles and investment input bundles employed by each industry, whereas $\hat{m}_t$ and $\hat{x}_t$ refer to the $N^2 \times 1$ vectors which contain the flows of intermediate and investment inputs across pairs of industries.

F.3 System reduction

Step 1: Substitute out $\hat{x}_t$ and $\hat{m}_t$:

$$\begin{aligned}
\hat{m}_t &= \frac{\varepsilon_M}{\varepsilon_Q}\left(\varepsilon_Q - 1\right) T_1 \hat{a}_t + \frac{\varepsilon_M}{\varepsilon_Q} T_1 \hat{q}_t + \left(1 - \frac{\varepsilon_M}{\varepsilon_Q}\right) T_1 \hat{\mathbb{M}}_t + \varepsilon_M \left[T_1 - T_2\right] \hat{p}_t \\
\hat{x}_t &= T_1 \hat{\mathbb{X}}_t + \varepsilon_X T_1 \hat{p}_t^{inv} - \varepsilon_X T_2 \hat{p}_t
\end{aligned}$$

to get:

$$\begin{aligned}
\hat{k}_{t+1} &= \delta_K \hat{\mathbb{X}}_t + \left(1 - \delta_K\right) \hat{k}_t \\
\left(\mathbf{I} - \frac{\varepsilon_M}{\varepsilon_Q} \tilde{S}_M^Q T_1\right) \hat{q}_t &= \delta_C^{-1} \tilde{S}_C^Q \hat{c}_{t+1} + \left(\mathbf{I} - \delta_C^{-1}\right) \tilde{S}_C^Q \hat{c}_t + \tilde{S_X^Q} T_1 \hat{\mathbb{X}}_t + \frac{\varepsilon_M}{\varepsilon_Q}\left(\varepsilon_Q - 1\right) \tilde{S}_M^Q T_1 \hat{a}_t \\
&\quad + \left(1 - \frac{\varepsilon_M}{\varepsilon_Q}\right) \tilde{S}_M^Q T_1 \hat{\mathbb{M}}_t + \varepsilon_X \tilde{S_X^Q} T_1 \hat{p}_t^{inv} + \left[\varepsilon_M \tilde{S}_M^Q \left[T_1 - T_2\right] - \varepsilon_X \tilde{S_X^Q} T_2\right] \hat{p}_t \\
\hat{p}_t &= \beta\left(\mathbf{I} - \delta_C\right) \hat{p}_{t+1} - \frac{1}{\varepsilon_D}\left(\mathbf{I} - \beta\left(\mathbf{I} - \delta_C\right)\right)\left[\mathbf{I} + S_I^C\left(\varepsilon_D - 1\right)\right] \hat{c}_{t+1} \\
\frac{1}{\varepsilon_{LS}} S^L \hat{l}_t &= \hat{p}_t + \frac{\varepsilon_Q - 1}{\varepsilon_Q} \hat{a}_t + \frac{\left(\varepsilon_Q - 1\right)\left(\mathbf{I} - \alpha\right)}{\varepsilon_Q} \hat{b}_t + \frac{1}{\varepsilon_Q} \hat{q}_t + \frac{\varepsilon_Q - 1}{\varepsilon_Q} \alpha \hat{k}_t + \frac{\alpha - \mathbf{I} - \alpha\varepsilon_Q}{\varepsilon_Q} \hat{l}_t \\
\hat{p}_t^{inv} &= \beta(1 - \delta_K) \hat{p}_{t+1}^{inv} + \left(1 - \beta(1 - \delta_K)\right)\left[\hat{p}_{t+1} + \frac{1}{\varepsilon_Q} \hat{q}_{t+1} + \frac{\varepsilon_Q - 1}{\varepsilon_Q} \hat{a}_{t+1}\right. \\
&\quad \left. + \left(-\mathbf{I} + \alpha \frac{\varepsilon_Q - 1}{\varepsilon_Q}\right) \hat{k}_{t+1} + \left(\mathbf{I} - \alpha\right) \frac{\varepsilon_Q - 1}{\varepsilon_Q}\left(\hat{l}_{t+1} + \hat{b}_{t+1}\right)\right] \\
\hat{q}_t &= \hat{a}_t + \left(\mathbf{I} - \alpha\right)\left(\mathbf{I} - S_M\right) \hat{b}_t + \alpha\left(\mathbf{I} - S_M\right) \hat{k}_t + \left(\mathbf{I} - \alpha\right)\left(\mathbf{I} - S_M\right) \hat{l}_t + S_M \hat{\mathbb{M}}_t
\end{aligned}$$

Step 2: Use $S_1^X \hat{p}_t = \hat{p}_t^{inv}$ (S_1^X is the matrix that gives the share of different industries' outputs in the investment input bundle) and $\hat{\mathbb{X}}_t = \delta_K^{-1} \hat{k}_{t+1} + \left(1 - \delta_K^{-1}\right) \hat{k}_t$; and define $\tilde{\beta} \equiv 1 - \beta\left(1 - \delta_K\right)$

to get:

$$\begin{aligned}
\left(\mathbf{I}-\frac{\varepsilon_M}{\varepsilon_Q}\tilde{S}^Q_M T_1\right)\hat{q}_t &= \delta_C^{-1}\tilde{S}^Q_C\hat{c}_{t+1}+\left(\mathbf{I}-\delta_C^{-1}\right)\tilde{S}^Q_C\hat{c}_t+\frac{\varepsilon_M}{\varepsilon_Q}\left(\varepsilon_Q-1\right)\tilde{S}^Q_M T_1\hat{a}_t+\left(1-\frac{\varepsilon_M}{\varepsilon_Q}\right)\tilde{S}^Q_M T_1\hat{\mathbb{M}}_t\\
&+\left[\varepsilon_X\tilde{S^Q_X}T_1S^X_1+\varepsilon_M\tilde{S}^Q_M\left[T_1-T_2\right]-\varepsilon_X\tilde{S^Q_X}T_2\right]\hat{p}_t+\tilde{S^Q_X}T_1\delta_K^{-1}\hat{k}_{t+1}+\tilde{S^Q_X}T_1\left(1-\delta_K^{-1}\right)\hat{k}_t\\
\hat{p}_t &= \beta\left(\mathbf{I}-\delta_C\right)\hat{p}_{t+1}-\frac{1}{\varepsilon_D}\left(\mathbf{I}-\beta\left(\mathbf{I}-\delta_C\right)\right)\left[\mathbf{I}+S^C_I\left(\varepsilon_D-1\right)\right]\hat{c}_{t+1}\\
\frac{1}{\varepsilon_{LS}}S^L\hat{l}_t &= \hat{p}_t+\frac{\varepsilon_Q-1}{\varepsilon_Q}\hat{a}_t+\frac{\left(\varepsilon_Q-1\right)\left(\mathbf{I}-\alpha\right)}{\varepsilon_Q}\hat{b}_t+\frac{1}{\varepsilon_Q}\hat{q}_t+\frac{\varepsilon_Q-1}{\varepsilon_Q}\alpha\hat{k}_t+\frac{\alpha-\mathbf{I}-\alpha\varepsilon_Q}{\varepsilon_Q}\hat{l}_t\\
S^X_1\hat{p}_t &= \left[\tilde{\beta}\mathbf{I}+\beta(1-\delta_K)S^X_1\right]\hat{p}_{t+1}+\tilde{\beta}\left[\frac{1}{\varepsilon_Q}\hat{q}_{t+1}+\frac{\varepsilon_Q-1}{\varepsilon_Q}\hat{a}_{t+1}\right.\\
&\left.+\left(-\mathbf{I}+\alpha\frac{\varepsilon_Q-1}{\varepsilon_Q}\right)\hat{k}_{t+1}+\left(\mathbf{I}-\alpha\right)\frac{\varepsilon_Q-1}{\varepsilon_Q}\left(\hat{l}_{t+1}+\hat{b}_{t+1}\right)\right]\\
\hat{q}_t &= \hat{a}_t+\left(\mathbf{I}-\alpha\right)\left(\mathbf{I}-S_M\right)\hat{b}_t+\alpha\left(\mathbf{I}-S_M\right)\hat{k}_t+\left(\mathbf{I}-\alpha\right)\left(\mathbf{I}-S_M\right)\hat{l}_t+S_M\hat{\mathbb{M}}_t
\end{aligned}$$

Step 3: Use

$$\begin{aligned}
\hat{\mathbb{M}}_t &= \left(\varepsilon_Q-1\right)\hat{a}_t+\hat{q}_t+\varepsilon_Q\left(\mathbf{I}-S^M_1\right)\hat{p}_t\\
\text{where } S^M_1\hat{p}_t &= \hat{p}^{in}_t
\end{aligned}$$

$$\begin{aligned}
\left(\mathbf{I}-\tilde{S}^Q_M T_1\right)\hat{q}_t &= \delta_C^{-1}\tilde{S}^Q_C\hat{c}_{t+1}+\left(\mathbf{I}-\delta_C^{-1}\right)\tilde{S}^Q_C\hat{c}_t+\left(\varepsilon_Q-1\right)\tilde{S}^Q_M T_1\hat{a}_t\\
&+\tilde{S^Q_X}T_1\delta_K^{-1}\hat{k}_{t+1}+\tilde{S^Q_X}T_1\left(1-\delta_K^{-1}\right)\hat{k}_t\\
&+\left[\varepsilon_Q\tilde{S}^Q_M T_1\left(\mathbf{I}-S^M_1\right)+\varepsilon_M\tilde{S}^Q_M\left[T_1S^M_1-T_2\right]+\varepsilon_X\tilde{S^Q_X}\left[T_1S^X_1-T_2\right]\right]\hat{p}_t\\
\hat{p}_t &= \beta\left(\mathbf{I}-\delta_C\right)\hat{p}_{t+1}-\frac{1}{\varepsilon_D}\left(\mathbf{I}-\beta\left(\mathbf{I}-\delta_C\right)\right)\left[\mathbf{I}+S^C_I\left(\varepsilon_D-1\right)\right]\hat{c}_{t+1}\\
\frac{1}{\varepsilon_{LS}}S^L\hat{l}_t &= \hat{p}_t+\frac{\varepsilon_Q-1}{\varepsilon_Q}\hat{a}_t+\frac{\left(\varepsilon_Q-1\right)\left(\mathbf{I}-\alpha\right)}{\varepsilon_Q}\hat{b}_t+\frac{1}{\varepsilon_Q}\hat{q}_t+\frac{\varepsilon_Q-1}{\varepsilon_Q}\alpha\hat{k}_t+\frac{\alpha-\mathbf{I}-\alpha\varepsilon_Q}{\varepsilon_Q}\hat{l}_t \qquad (48)\\
S^X_1\hat{p}_t &= \left[\tilde{\beta}\mathbf{I}+\beta(1-\delta_K)S^X_1\right]\hat{p}_{t+1}+\tilde{\beta}\frac{1}{\varepsilon_Q}\hat{q}_{t+1}\\
&+\tilde{\beta}\frac{\varepsilon_Q-1}{\varepsilon_Q}\hat{a}_{t+1}+\tilde{\beta}\left(-\mathbf{I}+\alpha\frac{\varepsilon_Q-1}{\varepsilon_Q}\right)\hat{k}_{t+1}+\tilde{\beta}\left(\mathbf{I}-\alpha\right)\frac{\varepsilon_Q-1}{\varepsilon_Q}\left(\hat{l}_{t+1}+\hat{b}_{t+1}\right)\\
\hat{q}_t &= \left(\mathbf{I}-S_M\right)^{-1}\left(\mathbf{I}+S_M\left(\varepsilon_Q-1\right)\right)\hat{a}_t+\left(\mathbf{I}-\alpha\right)\hat{b}_t+\alpha\hat{k}_t \qquad (49)\\
&+\left(\mathbf{I}-\alpha\right)\hat{l}_t+\left(\mathbf{I}-S_M\right)^{-1}S_M\varepsilon_Q\left(\mathbf{I}-S^M_1\right)\hat{p}_t
\end{aligned}$$

Step 4: Use the production function, given in Equation 49, to substitute $\hat{q}_t$ out of the first, third, and fourth equations:

$$
\begin{aligned}
\frac{1}{\varepsilon_Q}\hat{q}_t &= \frac{1}{\varepsilon_Q}\left(\mathbf{I}-S_M\right)^{-1}\left(\mathbf{I}+S_M\left(\varepsilon_Q-1\right)\right)\hat{a}_t+\frac{1}{\varepsilon_Q}\left(\mathbf{I}-\alpha\right)\hat{b}_t+\frac{1}{\varepsilon_Q}\alpha\hat{k}_t \qquad (50)\\
&+\frac{1}{\varepsilon_Q}\left(\mathbf{I}-\alpha\right)\hat{l}_t+\left(\mathbf{I}-S_M\right)^{-1}S_M\left(\mathbf{I}-S_1^M\right)\hat{p}_t\\
\left(\mathbf{I}-\tilde{S}_M^Q T_1\right)\hat{q}_t &= \left(\mathbf{I}-\tilde{S}_M^Q T_1\right)\left(\mathbf{I}-S_M\right)^{-1}\left(\mathbf{I}+S_M\left(\varepsilon_Q-1\right)\right)\hat{a}_t+\left(\mathbf{I}-\tilde{S}_M^Q T_1\right)\left(\mathbf{I}-\alpha\right)\hat{b}_t\\
&+\left(\mathbf{I}-\tilde{S}_M^Q T_1\right)\alpha\hat{k}_t+\left(\mathbf{I}-\tilde{S}_M^Q T_1\right)\left(\mathbf{I}-\alpha\right)\hat{l}_t\\
&+\left(\mathbf{I}-\tilde{S}_M^Q T_1\right)\left(\mathbf{I}-S_M\right)^{-1}S_M\left(\mathbf{I}-S_1^M\right)\varepsilon_Q\hat{p}_t
\end{aligned}
$$

to get

$$
\begin{aligned}
0 &= \delta_C^{-1}\tilde{S}_C^Q\hat{c}_{t+1}+\left(\mathbf{I}-\delta_C^{-1}\right)\tilde{S}_C^Q\hat{c}_t\\
&+\left[\left(\varepsilon_Q-1\right)\tilde{S}_M^Q T_1-\left(\mathbf{I}-\tilde{S}_M^Q T_1\right)\left(\mathbf{I}-S_M\right)^{-1}\left(\mathbf{I}+S_M\left(\varepsilon_Q-1\right)\right)\right]\hat{a}_t\\
&+\tilde{S}_X^Q T_1\delta_K^{-1}\hat{k}_{t+1}+\left[\tilde{S}_X^Q T_1\left(1-\delta_K^{-1}\right)-\left(\mathbf{I}-\tilde{S}_M^Q T_1\right)\alpha\right]\hat{k}_t\\
&-\left(\mathbf{I}-\tilde{S}_M^Q T_1\right)\left(\mathbf{I}-\alpha\right)\hat{b}_t-\left(\mathbf{I}-\tilde{S}_M^Q T_1\right)\left(\mathbf{I}-\alpha\right)\hat{l}_t\\
&+\left[\varepsilon_X\tilde{S}_X^Q\left[T_1S_1^X-T_2\right]-\left(\mathbf{I}-\tilde{S}_M^Q T_1\right)\varepsilon_Q\left(\mathbf{I}-S_M\right)^{-1}S_M\left(\mathbf{I}-S_1^M\right)\right]\hat{p}_t\\
&+\left[\varepsilon_Q\tilde{S}_M^Q T_1\left(\mathbf{I}-S_1^M\right)+\varepsilon_M\tilde{S}_M^Q\left[T_1S_1^M-T_2\right]\right]\hat{p}_t\\
\hat{p}_t &= \beta\left(\mathbf{I}-\delta_C\right)\hat{p}_{t+1}-\frac{1}{\varepsilon_D}\left(\mathbf{I}-\beta\left(\mathbf{I}-\delta_C\right)\right)\left[\mathbf{I}+S_I^C\left(\varepsilon_D-1\right)\right]\hat{c}_{t+1}\\
S_1^X\hat{p}_t &= \left[\tilde{\beta}\mathbf{I}+\beta(1-\delta_K)S_1^X+\tilde{\beta}\left(I-S_M\right)^{-1}S_M\left(\mathbf{I}-S_1^M\right)\right]\hat{p}_{t+1}\\
&+\tilde{\beta}\left[\frac{1}{\varepsilon_Q}\left(\mathbf{I}-S_M\right)^{-1}\left(\mathbf{I}+S_M\left(\varepsilon_Q-1\right)\right)+\frac{\varepsilon_Q-1}{\varepsilon_Q}\mathbf{I}\right]\hat{a}_{t+1}\\
&+\tilde{\beta}\left(\mathbf{I}-\alpha\right)\hat{b}_{t+1}+\tilde{\beta}\left(-\mathbf{I}+\alpha\right)\hat{k}_{t+1}+\tilde{\beta}\left(\mathbf{I}-\alpha\right)\hat{l}_{t+1}
\end{aligned}
$$

Step 5: Use the following equation:

$$
\begin{aligned}
\left(\mathbf{I}-\alpha\right)\hat{l}_t &= \vartheta\left(\mathbf{I}-\alpha\right)\hat{b}_t+\vartheta\alpha\hat{k}_t+\vartheta\left[\frac{\varepsilon_Q-1}{\varepsilon_Q}\mathbf{I}+\frac{1}{\varepsilon_Q}\left(\mathbf{I}-S_M\right)^{-1}\left(\mathbf{I}+S_M\left(\varepsilon_Q-1\right)\right)\right]\hat{a}_t \qquad (51)\\
&+\vartheta\left[\left(\mathbf{I}-S_M\right)^{-1}S_M\left(\mathbf{I}-S_1^M\right)+\mathbf{I}\right]\hat{p}_t
\end{aligned}
$$

where $\vartheta=\left(\mathbf{I}-\alpha\right)\left[\frac{1}{\varepsilon_{LS}}S^L+\alpha\right]^{-1}$

(this equation comes from plugging Equation 49 into Equation 48 and re-arranging) to get

$$
\begin{aligned}
0 &= \delta_C^{-1}\tilde{S}_C^Q\hat{c}_{t+1} + \left(\mathbf{I}-\delta_C^{-1}\right)\tilde{S}_C^Q\hat{c}_t \\
&+ \left[(\varepsilon_Q-1)\tilde{S}_M^Q T_1 - \left(\mathbf{I}-\tilde{S}_M^Q T_1\right)\left(\mathbf{I}+\vartheta\varepsilon_Q^{-1}\right)(\mathbf{I}-S_M)^{-1}(\mathbf{I}+S_M(\varepsilon_Q-1))\right]\hat{a}_t \\
&- \left(\mathbf{I}-\tilde{S}_M^Q T_1\right)\vartheta\frac{\varepsilon_Q-1}{\varepsilon_Q}\hat{a}_t + \tilde{S_X^Q}T_1\delta_K^{-1}\hat{k}_{t+1} \\
&+ \left[\tilde{S_X^Q}T_1\left(1-\delta_K^{-1}\right) - \left(\mathbf{I}-\tilde{S}_M^Q T_1\right)(\mathbf{I}+\vartheta)\alpha\right]\hat{k}_t - \left(\mathbf{I}-\tilde{S}_M^Q T_1\right)(\mathbf{I}+\vartheta)(\mathbf{I}-\alpha)\hat{b}_t \\
&+ \left[\varepsilon_Q\tilde{S}_M^Q T_1\left(\mathbf{I}-S_1^M\right) + \varepsilon_M\tilde{S}_M^Q\left[T_1S_1^M - T_2\right] + \varepsilon_X\tilde{S_X^Q}\left[T_1S_1^X - T_2\right]\right]\hat{p}_t \\
&+ \left[-\left(\mathbf{I}-\tilde{S}_M^Q T_1\right)\left[\varepsilon_Q(\mathbf{I}-S_M)^{-1}S_M\left(\mathbf{I}-S_1^M\right) + \vartheta\left[(\mathbf{I}-S_M)^{-1}S_M\left(\mathbf{I}-S_1^M\right)+\mathbf{I}\right]\right]\right]\hat{p}_t
\end{aligned}
\tag{52}
$$

$$
\hat{p}_t = \beta(\mathbf{I}-\delta_C)\hat{p}_{t+1} - \frac{1}{\varepsilon_D}(\mathbf{I}-\beta(\mathbf{I}-\delta_C))\left[\mathbf{I}+S_I^C(\varepsilon_D-1)\right]\hat{c}_{t+1}
$$

$$
\begin{aligned}
S_1^X\hat{p}_t &= \left[\beta(1-\delta_K)S_1^X + \tilde{\beta}(\mathbf{I}+\vartheta)\left(\mathbf{I}+(\mathbf{I}-S_M)^{-1}S_M\left(\mathbf{I}-S_1^M\right)\right)\right]\hat{p}_{t+1} \\
&+ \tilde{\beta}(\mathbf{I}+\vartheta)\left[\frac{1}{\varepsilon_Q}(\mathbf{I}-S_M)^{-1}(\mathbf{I}+S_M(\varepsilon_Q-1)) + \frac{\varepsilon_Q-1}{\varepsilon_Q}\mathbf{I}\right]\hat{a}_{t+1} + \tilde{\beta}(\mathbf{I}+\vartheta)(\mathbf{I}-\alpha)\hat{b}_{t+1} \\
&+ \tilde{\beta}(-\mathbf{I}+\alpha+\vartheta\alpha)\hat{k}_{t+1}
\end{aligned}
$$

So now we are down to three equations and three sets of endogenous unknowns ($\hat{p}_t$, $\hat{k}_t$, and $\hat{c}_t$). How we proceed will depend on whether we allow for consumption to be durable or not.
Case 1: No Durables
Plug

$$
\hat{c}_t = -\varepsilon_D\left[\mathbf{I}+S_I^C(\varepsilon_D-1)\right]^{-1}\hat{p}_t
$$

in to the other two equations, above, to substitute out the $\hat{c}_t$ vector.

$$
\begin{aligned}
0 &= \left[(\varepsilon_Q-1)\tilde{S}_M^Q T_1 - \left(\mathbf{I}-\tilde{S}_M^Q T_1\right)\left(\mathbf{I}+\vartheta\varepsilon_Q^{-1}\right)(\mathbf{I}-S_M)^{-1}(\mathbf{I}+S_M(\varepsilon_Q-1)) - \left(\mathbf{I}-\tilde{S}_M^Q T_1\right)\vartheta\frac{\varepsilon_Q-1}{\varepsilon_Q}\right]\hat{a}_t \\
&+ \tilde{S_X^Q}T_1\delta_K^{-1}\hat{k}_{t+1} + \left[\tilde{S_X^Q}T_1\left(1-\delta_K^{-1}\right) - \left(\mathbf{I}-\tilde{S}_M^Q T_1\right)(\mathbf{I}+\vartheta)\alpha\right]\hat{k}_t - \left(\mathbf{I}-\tilde{S}_M^Q T_1\right)(\mathbf{I}+\vartheta)(\mathbf{I}-\alpha)\hat{b}_t \\
&+ \left[-\varepsilon_D\tilde{S}_C^Q\left[\mathbf{I}+S_I^C(\varepsilon_D-1)\right]^{-1} + \varepsilon_Q\tilde{S}_M^Q T_1\left(\mathbf{I}-S_1^M\right) + \varepsilon_M\tilde{S}_M^Q\left[T_1S_1^M-T_2\right] + \varepsilon_X\tilde{S_X^Q}\left[T_1S_1^X-T_2\right]\right]\hat{p}_t \\
&- \left(\mathbf{I}-\tilde{S}_M^Q T_1\right)\left[\varepsilon_Q(\mathbf{I}-S_M)^{-1}S_M\left(\mathbf{I}-S_1^M\right) + \vartheta\left[(\mathbf{I}-S_M)^{-1}S_M\left(\mathbf{I}-S_1^M\right)+\mathbf{I}\right]\right]\hat{p}_t \\
0 &= -S_1^X\hat{p}_t + \left[\beta(1-\delta_K)S_1^X + \tilde{\beta}(\mathbf{I}+\vartheta)\left(\mathbf{I}+(I-S_M)^{-1}S_M\left(\mathbf{I}-S_1^M\right)\right)\right]\hat{p}_{t+1} \\
&+ \tilde{\beta}(\mathbf{I}+\vartheta)\left[\frac{1}{\varepsilon_Q}(I-S_M)^{-1}(\mathbf{I}+S_M(\varepsilon_Q-1)) + \frac{\varepsilon_Q-1}{\varepsilon_Q}\mathbf{I}\right]\hat{a}_{t+1} \\
&+ \tilde{\beta}(\mathbf{I}+\vartheta)(\mathbf{I}-\alpha)\hat{b}_{t+1} + \tilde{\beta}(-\mathbf{I}+\alpha+\vartheta\alpha)\hat{k}_{t+1}
\end{aligned}
$$

Case 2: Durables

Combine the final two equations in the line before "So now..."

$$
\begin{aligned}
& -S_1^X \frac{1}{\varepsilon_D}\left(\mathbf{I}-\beta\left(\mathbf{I}-\delta_C\right)\right) \\
\times\left[\mathbf{I}+S_I^C\left(\varepsilon_D-1\right)\right] \hat{c}_{t+1} = & \left[\tilde{\beta}\left(\mathbf{I}+\vartheta\right)\left(\mathbf{I}+\left(\mathbf{I}-S_M\right)^{-1} S_M\left(\mathbf{I}-S_1^M\right)\right)+S_1^X \beta\left(\delta_C-\delta_K \mathbf{I}\right)\right] \hat{p}_{t+1} \\
& +\tilde{\beta}\left(\mathbf{I}+\vartheta\right)\left[\frac{1}{\varepsilon_Q}\left(\mathbf{I}-S_M\right)^{-1}\left(\mathbf{I}+S_M\left(\varepsilon_Q-1\right)\right)+\frac{\varepsilon_Q-1}{\varepsilon_Q} \mathbf{I}\right] \hat{a}_{t+1} \\
& +\tilde{\beta}\left(\mathbf{I}+\vartheta\right)\left(\mathbf{I}-\alpha\right) \hat{b}_{t+1}+\tilde{\beta}\left(-\mathbf{I}+\alpha+\vartheta \alpha\right) \hat{k}_{t+1}
\end{aligned}
$$

to get.

$$
\begin{aligned}
\hat{c}_t = & \tilde{\vartheta}\left[\tilde{\beta}\left(\mathbf{I}+\vartheta\right)\left(\mathbf{I}+\left(\mathbf{I}-S_M\right)^{-1} S_M\left(\mathbf{I}-S_1^M\right)\right)+S_1^X \beta\left(\delta_C-\mathbf{I} \delta_K\right)\right] \hat{p}_t \\
& +\tilde{\vartheta} \tilde{\beta}\left(\mathbf{I}+\vartheta\right)\left[\frac{1}{\varepsilon_Q}\left(\mathbf{I}-S_M\right)^{-1}\left(\mathbf{I}+S_M\left(\varepsilon_Q-1\right)\right)+\frac{\varepsilon_Q-1}{\varepsilon_Q} \mathbf{I}\right] \hat{a}_t \\
& +\tilde{\vartheta} \tilde{\beta}\left(\mathbf{I}+\vartheta\right)\left(\mathbf{I}-\alpha\right) \hat{b}_t+\tilde{\beta} \tilde{\vartheta}\left(-\mathbf{I}+\alpha+\vartheta \alpha\right) \hat{k}_t
\end{aligned}
$$

where

$$
\tilde{\vartheta} \equiv\left[-S_1^X \frac{1}{\varepsilon_D}\left(\mathbf{I}-\beta\left(\mathbf{I}-\delta_C\right)\right)\left[\mathbf{I}+S_I^C\left(\varepsilon_D-1\right)\right]\right]^{-1}
$$

Plug this in:

$$
\begin{aligned}
0 &= \tilde{S}_C^Q \tilde{\vartheta}\tilde{\beta}\left(\mathbf{I}+\vartheta\right)\left[\frac{1}{\varepsilon_Q}\left(\mathbf{I}-S_M\right)^{-1}\left(\mathbf{I}+S_M\left(\varepsilon_Q-1\right)\right)+\frac{\varepsilon_Q-1}{\varepsilon_Q}\mathbf{I}\right]\hat{a}_t \qquad (53)\\
&+\left[\left(\varepsilon_Q-1\right)\tilde{S}_M^Q T_1-\left(\mathbf{I}-\tilde{S}_M^Q T_1\right)\left(\mathbf{I}+\vartheta\varepsilon_Q^{-1}\right)\left(\mathbf{I}-S_M\right)^{-1}\left(\mathbf{I}+S_M\left(\varepsilon_Q-1\right)\right)-\left(\mathbf{I}-\tilde{S}_M^Q T_1\right)\vartheta\frac{\varepsilon_Q-1}{\varepsilon_Q}\right]\hat{a}_t\\
&+\delta_C^{-1}\tilde{S}_C^Q\tilde{\vartheta}\left[\tilde{\beta}\left(\mathbf{I}+\vartheta\right)\left(\mathbf{I}+\left(\mathbf{I}-S_M\right)^{-1}S_M\left(\mathbf{I}-S_1^M\right)\right)+S_1^X\beta\left(\delta_C-\mathbf{I}\delta_K\right)\right]\hat{p}_{t+1}\\
&+\left(\mathbf{I}-\delta_C^{-1}\right)\tilde{S}_C^Q\tilde{\vartheta}\left[\tilde{\beta}\left(\mathbf{I}+\vartheta\right)\left(\mathbf{I}+\left(\mathbf{I}-S_M\right)^{-1}S_M\left(\mathbf{I}-S_1^M\right)\right)+S_1^X\beta\left(\delta_C-\mathbf{I}\delta_K\right)\right]\hat{p}_t\\
&+\left[\varepsilon_Q\tilde{S}_M^Q T_1\left(\mathbf{I}-S_1^M\right)+\varepsilon_M\tilde{S}_M^Q\left[T_1S_1^M-T_2\right]+\varepsilon_X\tilde{S_X^Q}\left[T_1S_1^X-T_2\right]\right]\hat{p}_t\\
&+\left[-\left(\mathbf{I}-\tilde{S}_M^Q T_1\right)\left[\left(\mathbf{I}-S_M\right)^{-1}S_M\left(\mathbf{I}-S_1^M\right)\varepsilon_Q+\vartheta\left[\left(\mathbf{I}-S_M\right)^{-1}S_M\left(I-S_1^M\right)+\mathbf{I}\right]\right]\right]\hat{p}_t\\
&+\left[\tilde{S_X^Q}T_1\delta_K^{-1}+\delta_C^{-1}\tilde{S}_C^Q\tilde{\beta}\tilde{\vartheta}\left(-\mathbf{I}+\alpha+\vartheta\alpha\right)\right]\hat{k}_{t+1}\\
&+\left[\tilde{S_X^Q}T_1\left(1-\delta_K^{-1}\right)-\left(\mathbf{I}-\tilde{S}_M^Q T_1\right)\left(\mathbf{I}+\vartheta\right)\alpha+\left(\mathbf{I}-\delta_C^{-1}\right)\tilde{S}_C^Q\tilde{\beta}\tilde{\vartheta}\left(-\mathbf{I}+\alpha+\vartheta\alpha\right)\right]\hat{k}_t\\
&+\left[\tilde{S}_C^Q\tilde{\vartheta}\tilde{\beta}\left(\mathbf{I}+\vartheta\right)\left(\mathbf{I}-\alpha\right)-\left(\mathbf{I}-\tilde{S}_M^Q T_1\right)\left(\mathbf{I}+\vartheta\right)\left(\mathbf{I}-\alpha\right)\right]\hat{b}_t\\
0 &= -S_1^X\hat{p}_t+\left[\beta(1-\delta_K)S_1^X+\tilde{\beta}\left(\mathbf{I}+\vartheta\right)\left(\mathbf{I}+\left(\mathbf{I}-S_M\right)^{-1}S_M\left(\mathbf{I}-S_1^M\right)\right)\right]\hat{p}_{t+1} \qquad (54)\\
&+\tilde{\beta}\left(\mathbf{I}+\vartheta\right)\left[\frac{1}{\varepsilon_Q}\left(I-S_M\right)^{-1}\left(\mathbf{I}+S_M\left(\varepsilon_Q-1\right)\right)+\mathbf{I}\frac{\varepsilon_Q-1}{\varepsilon_Q}\right]\hat{a}_{t+1}+\tilde{\beta}\left(\mathbf{I}+\vartheta\right)\left(\mathbf{I}-\alpha\right)\hat{b}_{t+1}\\
&+\tilde{\beta}\left(-\mathbf{I}+\alpha+\vartheta\alpha\right)\hat{k}_{t+1}
\end{aligned}
$$

F.4 Blanchard-Kahn

In Equations 53 and 54, we have expressed the reduced system as

$$
\begin{bmatrix}\mathbb{E}_t[\hat{p}_{t+1}]\\ \hat{k}_{t+1}\end{bmatrix}=\mathbf{\Psi}\begin{bmatrix}\hat{p}_t\\ \hat{k}_t\end{bmatrix}+\mathbf{\Phi}\begin{bmatrix}\hat{a}_t\\ \hat{b}_t\end{bmatrix}
$$

Here, $\mathbf{\Psi}$ has N stable and N unstable eigenvalues.

Using a Jordan decomposition, write $\mathbf{\Psi}=\mathbf{VDV}^{-1}$ where $\mathbf{D}$ is diagonal and is ordered such that the N explosive eigenvalues are ordered first and the N stable eigenvalues are ordered last. Re-write:

$$
\begin{aligned}
\Upsilon_{t+1} &\equiv \mathbf{V}^{-1}\begin{bmatrix}\mathbb{E}_t[\hat{p}_{t+1}]\\ \hat{k}_{t+1}\end{bmatrix}=\mathbf{DV}^{-1}\begin{bmatrix}\hat{p}_t\\ \hat{k}_t\end{bmatrix}+\mathbf{V}^{-1}\mathbf{\Phi}\begin{bmatrix}\hat{a}_t\\ \hat{b}_t\end{bmatrix}\\
&\equiv \mathbf{D}\Upsilon_t+\tilde{\mathbf{\Phi}}\begin{bmatrix}\hat{a}_t\\ \hat{b}_t\end{bmatrix}
\end{aligned}
$$

Partition Υ_t into the first $N\times 1$ block, Υ_{1t}, and the lower $N\times 1$ block, Υ_{2t}. Similarly partition $\tilde{\mathbf{\Phi}}$ and $\mathbf{D}$.

$$\Upsilon_{1,t} = \mathbf{D}_1^{-1}\mathbb{E}_t[\Upsilon_{1,t+1}] - \mathbf{D}_1^{-1}\tilde{\mathbf{\Phi}} \begin{bmatrix} \hat{a}_t \\ \hat{b}_t \end{bmatrix}$$

Substitute recursively

$$\Upsilon_{1,t} = -\mathbf{D}_1^{-1}\sum_{s=0}^{\infty}\mathbf{D}_1^{-s}\tilde{\mathbf{\Phi}}_1 \begin{bmatrix} \hat{a}_t \\ \hat{b}_t \end{bmatrix} = -\mathbf{D}_1^{-1}(I - \mathbf{D}_1^{-1})^{-1}\tilde{\mathbf{\Phi}}_1 \begin{bmatrix} \hat{a}_t \\ \hat{b}_t \end{bmatrix} \tag{55}$$

For $Y_{2,t}$:

$$\Upsilon_{2,t} = \mathbf{D}_2\Upsilon_{2,t-1} + \tilde{\mathbf{\Phi}}_2 \cdot \begin{bmatrix} \hat{a}_t \\ \hat{b}_t \end{bmatrix}$$

Remember that

$$\begin{bmatrix} \Upsilon_{1,t} \\ \Upsilon_{2,t} \end{bmatrix} = \mathbf{V}^{-1}\begin{bmatrix} \hat{p}_t \\ \hat{k}_t \end{bmatrix},$$

and therefore, from Equation 55

$$\begin{aligned} \hat{p}_t &= -(\mathbf{V}_{11}^{-1})^{-1}\mathbf{V}_{12}^{-1}\hat{k}_t + (\mathbf{V}_{11}^{-1})^{-1}\Upsilon_{1t} \qquad (56) \\ &= -(\mathbf{V}_{11}^{-1})^{-1}\mathbf{V}_{12}^{-1}\hat{k}_t - (\mathbf{V}_{11}^{-1})^{-1}\mathbf{D}_1^{-1}(\mathbf{I} - \mathbf{D}_1^{-1})^{-1}\tilde{\mathbf{\Phi}}_1 \begin{bmatrix} \hat{a}_t \\ \hat{b}_t \end{bmatrix} \end{aligned}$$

The endogenous state evolves as follows:

$$\begin{aligned} \hat{k}_{t+1} &= \mathbf{\Psi}_{22}\hat{k}_t + \mathbf{\Psi}_{21}\hat{p}_t + \mathbf{\Phi}_2 \begin{bmatrix} \hat{a}_t \\ \hat{b}_t \end{bmatrix} \\ &= \underbrace{(\mathbf{\Psi}_{22} - \mathbf{\Psi}_{21}(\mathbf{V}_{11}^{-1})^{-1}\mathbf{V}_{12}^{-1})}_{\equiv M_{kk}}\hat{k}_t + \underbrace{\left(-\mathbf{\Psi}_{21}(\mathbf{V}_{11}^{-1})^{-1}\mathbf{D}_1^{-1}(\mathbf{I} - \mathbf{D}_1^{-1})^{-1}\tilde{\Phi}_1 + \mathbf{\Phi}_2\right)}_{\equiv[M_{ka},\, M_{kb}]} \begin{bmatrix} \hat{a}_t \\ \hat{b}_t \end{bmatrix} \qquad (57) \end{aligned}$$

For future reference:

$$\hat{p}_t = \mathbf{\Psi}_{21}^{-1}\hat{k}_{t+1} - \mathbf{\Psi}_{21}^{-1}\mathbf{\Psi}_{22}\hat{k}_t - \mathbf{\Psi}_{21}^{-1}\mathbf{\Phi}_2 \begin{bmatrix} \hat{a}_t \\ \hat{b}_t \end{bmatrix} \tag{58}$$

F.5 Obtaining the model filter

Combine Equations 50 and 51 to write $\hat{q}_t$ as a function of the exogenous variables, $\hat{k}$, and $\hat{p}$

$$\begin{aligned} \hat{q}_t = {} & (\mathbf{I} + \vartheta)(\mathbf{I} - \alpha)\hat{b}_t + (\mathbf{I} + \vartheta)\alpha\hat{k}_t \qquad (59) \\ & + \left[\frac{\varepsilon_Q - 1}{\varepsilon_Q}\vartheta + \left(\frac{\vartheta}{\varepsilon_Q} + \mathbf{I}\right)(\mathbf{I} - S_M)^{-1}(\mathbf{I} + S_M(\varepsilon_Q - 1))\right]\hat{a}_t \\ & + \left[(\vartheta + \varepsilon_Q\mathbf{I})(\mathbf{I} - S_M)^{-1}S_M\left(\mathbf{I} - S_1^M\right) + \vartheta\right]\hat{p}_t \end{aligned}$$

Plug Equation 57 and 58 in so that we may write:

$$\hat{q}_t = \Phi_{kq}\hat{k}_t + \Phi_{bq}\hat{b}_t + \Phi_{aq}\hat{a}_t, \tag{60}$$

where the Φ_{kq}, Φ_{bq}, and Φ_{aq} are matrices that collect the appropriate terms.[40]

So long as Φ_{kq} is invertible, Equation 60 is equivalent to

$$\hat{k}_t = \Phi_{kq}^{-1}\hat{q}_t - \Phi_{kq}^{-1}\Phi_{bq}\hat{b}_t - \Phi_{kq}^{-1}\Phi_{aq}\hat{a}_t$$

Equation 60, one period ahead, is

$$\hat{q}_{t+1} = \Phi_{kq}\hat{k}_{t+1} + \Phi_{bq}\hat{b}_{t+1} + \Phi_{aq}\hat{a}_{t+1}$$

Apply Equation 57 to this previous equation

$$\begin{aligned}
\hat{q}_{t+1} &= \Phi_{bq}\hat{b}_{t+1} + \Phi_{aq}\hat{a}_{t+1} \\
&\quad +\Phi_{kq}\left(M_{kk}\hat{k}_t + M_{ka}\hat{a}_t + M_{kb}\hat{b}_t\right) \\
&= \Phi_{bq}\hat{b}_{t+1} + \Phi_{aq}\hat{a}_{t+1} \\
&\quad +\Phi_{kq}M_{ka}\hat{a}_t + \Phi_{kq}M_{kb}\hat{b}_t \\
&\quad +\Phi_{kq}M_{kk}\Phi_{kq}^{-1}\hat{q}_t - \Phi_{kq}M_{kk}\Phi_{kq}^{-1}\Phi_{bq}\hat{b}_t - \Phi_{kq}M_{kk}\Phi_{kq}^{-1}\Phi_{aq}\hat{a}_t \\
&= \Phi_{bq}\hat{b}_{t+1} + \Phi_{aq}\hat{a}_{t+1} + \Phi_{kq}M_{kk}\Phi_{kq}^{-1}\hat{q}_t \\
&\quad + \left[\Phi_{kq}M_{ka} - \Phi_{kq}M_{kk}\Phi_{kq}^{-1}\Phi_{aq}\right]\hat{a}_t + \left[\Phi_{kq}M_{kb} - \Phi_{kq}M_{kk}\Phi_{kq}^{-1}\Phi_{bq}\right]\hat{b}_t
\end{aligned}$$

Finally, take two adjacent periods, and use the definitions of $\omega_{t+1}^{A}\,(\equiv \hat{a}_{t+1} - \hat{a}_t)$ and

[40] Combine Equations 57 and 58:

$$\begin{aligned}
\hat{p}_t &= \boldsymbol{\Psi}_{21}^{-1}\hat{k}_{t+1} - \boldsymbol{\Psi}_{21}^{-1}\boldsymbol{\Psi}_{22}\hat{k}_t - \boldsymbol{\Psi}_{21}^{-1}\boldsymbol{\Phi}_2\begin{bmatrix}\hat{a}_t\\ \hat{b}_t\end{bmatrix} \\
&= \boldsymbol{\Psi}_{21}^{-1}\left[\boldsymbol{\Psi}_{22} - \boldsymbol{\Psi}_{21}(\mathbf{V}_{11}^{-1})^{-1}\mathbf{V}_{12}^{-1}\right]\hat{k}_t - \boldsymbol{\Psi}_{21}^{-1}\boldsymbol{\Psi}_{22}\hat{k}_t \\
&\quad -\boldsymbol{\Psi}_{21}^{-1}\boldsymbol{\Phi}_2\begin{bmatrix}\hat{a}_t\\ \hat{b}_t\end{bmatrix} + \boldsymbol{\Psi}_{21}^{-1}\left[\left(-\boldsymbol{\Psi}_{21}(\mathbf{V}_{11}^{-1})^{-1}\mathbf{D}_1^{-1}(\mathbf{I}-\mathbf{D}_1^{-1})^{-1}\tilde{\Phi}_1 + \boldsymbol{\Phi}_2\right)\right]\begin{bmatrix}\hat{a}_t\\ \hat{b}_t\end{bmatrix} \\
&= -(\mathbf{V}_{11}^{-1})^{-1}\mathbf{V}_{12}^{-1}\hat{k}_t - (\mathbf{V}_{11}^{-1})^{-1}\mathbf{D}_1^{-1}(\mathbf{I}-\mathbf{D}_1^{-1})^{-1}\tilde{\Phi}_1\begin{bmatrix}\hat{a}_t\\ \hat{b}_t\end{bmatrix}
\end{aligned}$$

So:

$$\begin{aligned}
\hat{q}_t = &\left\{(\mathbf{I}+\vartheta)\,\alpha - \left[(\vartheta + \varepsilon_Q\mathbf{I})\,(\mathbf{I}-S_M)^{-1}\,S_M\left(\mathbf{I}-S_1^M\right) + \vartheta\right](\mathbf{V}_{11}^{-1})^{-1}\mathbf{V}_{12}^{-1}\right\}\hat{k}_t \\
&+ \left[\frac{\varepsilon_Q - 1}{\varepsilon_Q}\vartheta + \left(\frac{\vartheta}{\varepsilon_Q} + \mathbf{I}\right)(\mathbf{I}-S_M)^{-1}\left(\mathbf{I} + S_M\left(\varepsilon_Q - 1\right)\right)\right]\hat{a}_t + (\mathbf{I}+\vartheta)\,(\mathbf{I}-\alpha)\,\hat{b}_t \\
&- \left[(\vartheta + \varepsilon_Q\mathbf{I})\,(\mathbf{I}-S_M)^{-1}\,S_M\left(\mathbf{I}-S_1^M\right) + \vartheta\right](\mathbf{V}_{11}^{-1})^{-1}\mathbf{D}_1^{-1}(\mathbf{I}-\mathbf{D}_1^{-1})^{-1}\tilde{\Phi}_1\begin{bmatrix}\hat{a}_t\\ \hat{b}_t\end{bmatrix}
\end{aligned}$$

$\omega^B_{t+1}\left(\equiv \hat{b}_{t+1}-\hat{b}_t\right)$ so that

$$\begin{aligned}\Delta\hat{q}_{t+1} &= \Phi_{kq}M_{kk}\Phi_{kq}^{-1}\Delta\hat{q}_t+\Phi_{bq}\omega^B_{t+1}+\Phi_{aq}\omega^A_{t+1}+\\ &\quad +\left[\Phi_{kq}M_{ka}-\Phi_{kq}M_{kk}\Phi_{kq}^{-1}\Phi_{aq}\right]\omega^A_t+\left[\Phi_{kq}M_{kb}-\Phi_{kq}M_{kk}\Phi_{kq}^{-1}\Phi_{bq}\right]\omega^B_t\end{aligned}$$

Parsing out the factor-neutral productivity shocks yields Equation 10 of the paper. This equation also describes how one can recover labor-augmenting productivity shocks using data on industries' output growth rates.

In the remainder of this subsection, we work out the expression for industries' value added growth rates. Begin with the first-order condition for industries' intermediate input purchases

$$\begin{aligned}M_{tJ}P^{in}_{tJ} &= \mu_J A_{tJ}^{\varepsilon_Q-1}\left(\frac{P^{in}_{tJ}}{P_{tJ}}\right)^{1-\varepsilon_Q}P_{tJ}Q_{tJ}\\ VA_{tJ} &= P_{tJ}Q_{tJ}-M_{tJ}P^{in}_{tJ}\\ &= P_{tJ}Q_{tJ}\left[1-\mu A_{tJ}^{\varepsilon_Q-1}\left(\frac{P^{in}_{tJ}}{P_{tJ}}\right)^{1-\varepsilon_Q}\right]\\ \frac{VA_{tJ}}{P_{tJ}} &= Q_{tJ}\cdot\left[1-\mu A_{tJ}^{\varepsilon_Q-1}\left(\frac{P^{in}_{tJ}}{P_{tJ}}\right)^{1-\varepsilon_Q}\right]\end{aligned}$$

So, the log-linearized expression for real value added is

$$\hat{v}_t=\hat{q}_t-S^M\cdot(\varepsilon_Q-1)\cdot\hat{a}_t-S^M\cdot(\varepsilon_Q-1)\cdot\left(\mathbf{I}-S^M_1\right)\hat{p}_t$$

Substituting out the expression for $\hat{q}_t$:

$$\begin{aligned}\hat{v}_t &= \Phi_{kq}\hat{k}_t+\Phi_{bq}\hat{b}_t+\left[\Phi_{aq}-S^M\cdot(\varepsilon_Q-1)\right]\hat{a}_t\\ &\quad -S^M\cdot(\varepsilon_Q-1)\cdot\left(\mathbf{I}-S^M_1\right)\hat{p}_t\end{aligned}$$

And then substituting out the expression for $\hat{p}_t$:

$$\begin{aligned}\Delta\hat{v}_t &= \left[\Phi_{kq}+S^M\cdot(\varepsilon_Q-1)\cdot\left(\mathbf{I}-S^M_1\right)\cdot(\mathbf{V}^{-1}_{11})^{-1}\mathbf{V}^{-1}_{12}\right]\Delta\hat{k}_t\\ &\quad +\Phi_{bq}\omega^B_t+\left[\Phi_{aq}-S^M\cdot(\varepsilon_Q-1)\right]\omega^A_t\\ &\quad +S^M\cdot(\varepsilon_Q-1)\cdot\left(\mathbf{I}-S^M_1\right)\cdot(\mathbf{V}^{-1}_{11})^{-1}\mathbf{D}^{-1}_1(\mathbf{I}-\mathbf{D}^{-1}_1)^{-1}\tilde{\mathbf{\Phi}}_1\begin{bmatrix}\omega^A_t\\ \omega^B_t\end{bmatrix}\end{aligned}$$

Equation 57 allows one to recursively compute the variance-covariance matrix of $\Delta\hat{k}_t$. From here, in combination with the last equation, one can write the covariance matrix of value added as a function of the covariance matrix of sectoral productivity shocks.

F.6 Calculations related to Section 2.4

In this section, I solve for covariance of industries' output as functions of the model parameters and the exogenous TFP terms. The solution involves three steps. First, I solve for the wage. Second, I solve for the relative prices and intermediate input cost shares. Third, I solve for real sales. As there is no capital or durable goods, the decisions within each period are independent of those made in other periods. As such, I will omit time subscripts in this section.

Step 1: For later use, I will first solve for the wage in each period. For this portion of the analysis, it will be sufficient to examine how much the consumer wants to work and how much she wants to consume. Since the consumer's problems are separable across periods, the objective function for the consumer is

$$\begin{aligned} \mathcal{U} &= \log C - \frac{\varepsilon_{LS}}{\varepsilon_{LS}+1} L^{\frac{\varepsilon_{LS}+1}{\varepsilon_{LS}}} \text{ subject to} \\ P \cdot C &= W \cdot L \ . \end{aligned}$$

The solution to this constrained optimization problem is:

$$W = L^{\frac{1}{\varepsilon_{LS}}} \text{ and } C = \frac{1}{P} \ . \tag{61}$$

Invoking the budget constraint of the representative consumer:

$$L^{\frac{\varepsilon_{LS}+1}{\varepsilon_{LS}}} = 1,$$

implying $W = 1$.

Step 2: Now consider the cost-minimization problem of the representative firm in industry J. As I argued in the text, the cost-minimization problem implies the following recursive equation for the marginal cost (equivalently, price) of industry J's good:

$$P_J = \frac{1}{A_J} \left[1 - \mu + \mu \left[\sum_{I=1}^{N} \frac{1}{N} \left(P_I\right)^{1-\varepsilon_M} \right]^{\frac{1-\varepsilon_Q}{1-\varepsilon_M}} \right]^{\frac{1}{1-\varepsilon_Q}} \quad \text{for } J = \{1, ...N\}. \tag{62}$$

The log-linear approximation to the previous equation is:

$$\log P_J \approx -\log A_J + \frac{\mu}{N} \sum_{I=1}^{N} \log P_I \ . \tag{63}$$

for all pairs of industries, so that Equation 63 implies:

$$\log P_J \approx -\log A_J + \frac{\mu}{N} \sum_{I=1}^{N} \left[\log P_J + \log A_J - \log A_I \right] \ .$$

Re-arranging:

$$\log P_J \approx -\log A_J - \frac{\mu}{N(1-\mu)} \sum_{I=1}^{N} \log A_I \ .$$

Because all industries' cost shares are identical (both in the consumer's preferences and in the production of each industry's intermediate input bundle):

$$\log P_J^{in} \approx \log P \approx -\frac{1}{N(1-\mu)} \sum_{J=1}^{N} \log A_J \ .$$

Step 3: The last task is to solve for Q_J. To do so, apply the market clearing condition for good I, plug in the intermediate input demand by customers of I, then re-arrange:

$$\begin{aligned} Q_I &= C_I + \sum_{J=1}^{N} M_{I \to J} \ . \\ Q_I &= C_I + \frac{\mu}{N} (P_I)^{-\varepsilon_M} \sum_{J=1}^{N} Q_J (P_J)^{\varepsilon_Q - 1} \left(P_J^{in}\right)^{\varepsilon_M - \varepsilon_Q} \end{aligned}$$

Next, take the log-linear approximation around the point at which all of the A's equal 1:

$$\begin{aligned} \log Q_I &\approx \log\left(\frac{1}{1-\mu}\right) + (1-\mu)\log C_I - \mu\varepsilon_M \log P_I + \frac{\mu}{N}\sum_J \log Q_J \\ &\quad + \frac{\mu}{N}\sum_{J=1}^{N} (\varepsilon_Q - 1)\log P_J + (\varepsilon_M - \varepsilon_Q)\log P_J^{in} \ . \\ \log Q_I - \frac{\mu}{N}\sum_J \log Q_J &\approx \log\left(\frac{1}{1-\mu}\right) + (1-\mu)\log C_I - \mu\varepsilon_M \log P_I + \frac{\mu}{N}\sum_{J=1}^{N} (\varepsilon_M - 1)\log P_J \\ &\approx \log\left(\frac{1}{1-\mu}\right) + (1-\mu)\log C_I \qquad (64) \\ &\quad + \mu\varepsilon_M \log A_I + \frac{\mu\left[\varepsilon_M(\mu-1)+1\right]}{N(1-\mu)} \sum_{I=1}^{N} \log A_J \end{aligned}$$

Given the preferences of the representative consumer, the demand function for good I is:

$$\begin{aligned} \log C_I &= \log\frac{1}{N} - \varepsilon^D \log\left(\frac{P_I}{P}\right) - \log P \ . \\ &\approx \log\frac{1}{N} + \varepsilon^D \frac{1}{N}\sum_{J=1}^{N} \log\left(\frac{A_I}{A_J}\right) + \frac{1}{N(1-\mu)}\sum_{J=1}^{N} \log A_J \\ &\approx \log\frac{1}{N} + \varepsilon^D \log A_I + \frac{1-(1-\mu)\varepsilon_D}{N(1-\mu)} \sum_{J=1}^{N} \log A_J \end{aligned}$$

Plug this expression back into Equation 64 and combine terms:

$$\begin{aligned}
\log Q_I - \frac{\mu}{N}\sum_J \log Q_J &\approx (1-\mu)\log\frac{1}{N} + \log\left(\frac{1}{1-\mu}\right) + (\mu\varepsilon_M + (1-\mu)\varepsilon_D)\log A_I \\
&\quad + \left[\frac{(1-\mu)(1-(1-\mu)\varepsilon_D) + \mu\left[\varepsilon_M(\mu-1)+1\right]}{N}\right]\sum_{J=1}^{N}\frac{\log A_J}{1-\mu} \\
&\approx (1-\mu)\log\frac{1}{N} + \log\left(\frac{1}{1-\mu}\right) + (\mu\varepsilon_M + (1-\mu)\varepsilon_D)\log A_I \\
&\quad + \frac{1-(1-\mu)(\mu\varepsilon_M + (1-\mu)\varepsilon_D)}{N}\sum_{J=1}^{N}\frac{\log A_J}{1-\mu} \qquad (65)
\end{aligned}$$

Equation 65 is a system of N linear equations. The solution to these equations are

$$\begin{aligned}
\log Q_I &\approx \log\frac{1}{N} + \frac{1}{1-\mu}\log\left(\frac{1}{1-\mu}\right) + (\mu\varepsilon_M + (1-\mu)\varepsilon_D)\log A_I \\
&\quad + \frac{1}{N}\left[\left(\frac{1}{1-\mu}\right)^2 - (\mu\varepsilon_M + (1-\mu)\varepsilon_D)\right]\sum_{J=1}^{N}\log A_J \qquad (66)
\end{aligned}$$

Equation 66 is equivalent to the expression given in the body of the paper.

F.7 Calculations related to Section 3

In this appendix, I demonstrate that the instrumental variable strategy outlined in Acemoglu, Akcigit, and Kerr (2016) extends to a set-up in which sectoral production functions are CES rather than Cobb-Douglas. To do so, I will extend the benchmark model to explicitly accommodate demand shocks. As in Acemoglu, Akcigit, and Kerr (2016), the model will be static, with neither capital nor durable consumption goods. Also as in Acemoglu, Akcigit, and Kerr (2016), I impose that the logarithm of productivity equals zero: $\log A_I = \log B_I = 0$ for all industries, I.

The goal of this exercise is to examine how a demand shock in one industry—in particular the Government industry, which would be directly affected by an exogenous increase in military spending—impacts output in other industries. In particular, I wish to show that a linear relationship exists irrespective of the values of ε_M and ε_Q.

Begin with the Lagrangian of the social planner's problem, dropping t subscripts:

$$\begin{aligned}
\mathcal{L} &= \sum_{I'}(D_{I'}\xi_{I'})^{\frac{1}{\varepsilon_D}}\cdot\log\left[\left[\sum_{J=1}^{N}(D_J\xi_J)^{\frac{1}{\varepsilon_D}}(C_{tJ})^{\frac{\varepsilon_D-1}{\varepsilon_D}}\right]^{\frac{\varepsilon_D}{\varepsilon_D-1}}\right] \\
&\quad - \frac{\varepsilon_{LS}}{\varepsilon_{LS}+1}\left(\sum_{J=1}^{N}L_J\right)^{\frac{\varepsilon_{LS}+1}{\varepsilon_{LS}}} + P_J\left[Q_J - C_J - \sum_{I=1}^{N}M_{J\to I}\right]
\end{aligned}$$

The production function is, as before:

$$\begin{aligned}
Q_J &= A_J \cdot \left[(1-\mu_J)^{\frac{1}{\varepsilon_Q}}(L_J \cdot B_{tJ})^{\frac{\varepsilon_Q-1}{\varepsilon_Q}} + (\mu_J)^{\frac{1}{\varepsilon_Q}}(M_J)^{\frac{\varepsilon_Q-1}{\varepsilon_Q}}\right]^{\frac{\varepsilon_Q}{\varepsilon_Q-1}}, \text{ where} \\
M_J &\equiv \left[\sum_I \left(\Gamma_{IJ}^M\right)^{\frac{1}{\varepsilon_M}}(M_{IJ})^{\frac{\varepsilon_M-1}{\varepsilon_M}}\right]^{\frac{\varepsilon_M}{\varepsilon_M-1}}
\end{aligned}$$

The first-order conditions associated with the planner's problem are:

$$\begin{aligned}
P_J &= (D_J\xi_J)^{\frac{1}{\varepsilon_D}}(C_J)^{-\frac{1}{\varepsilon_D}}\left(\sum_{I=1}^N \frac{(\xi_I D_I)^{\frac{1}{\varepsilon_D}}}{\sum_{I'}(D_{I'}\xi_{I'})^{\frac{1}{\varepsilon_D}}}(C_I)^{\frac{\varepsilon_D-1}{\varepsilon_D}}\right)^{-1} \qquad (67) \\
\frac{P_I}{P_J} &= (A_J)^{\frac{\varepsilon_Q-1}{\varepsilon_Q}}\left(\frac{Q_J \cdot \mu_J}{M_J}\right)^{\frac{1}{\varepsilon_Q}}\left(\frac{M_J \cdot \Gamma_{IJ}^M}{M_{I\to J}}\right)^{\frac{1}{\varepsilon_M}}. \\
\left(\sum_{J'=1}^N L_{J'}\right)^{\frac{1}{\varepsilon_{LS}}} &= P_J (A_J)^{\frac{\varepsilon_Q-1}{\varepsilon_Q}} B_J (Q_J(1-\mu_J))^{\frac{1}{\varepsilon_Q}}(L_J \cdot B_J)^{-\frac{1}{\varepsilon_Q}}. \\
(1-\mu_J)^{\frac{1}{\varepsilon_Q}}(L_J \cdot B_J)^{\frac{\varepsilon_Q-1}{\varepsilon_Q}} &= (P_J)^{\varepsilon_Q-1}(A_J)^{\frac{(\varepsilon_Q-1)^2}{\varepsilon_Q}}(B_J)^{\varepsilon_Q-1}(Q_J)^{\frac{\varepsilon_Q-1}{\varepsilon_Q}}(1-\mu_J)\left(\sum_{J'=1}^N L_{J'}\right)^{\frac{1-\varepsilon_Q}{\varepsilon_{LS}}} \\
\mu_J^{\frac{1}{\varepsilon_Q}}(M_J)^{\frac{\varepsilon_Q-1}{\varepsilon_Q}} &= (P_J)^{\varepsilon_Q-1}(A_J)^{\frac{(\varepsilon_Q-1)^2}{\varepsilon_Q}}(Q_J)^{\frac{\varepsilon_Q-1}{\varepsilon_Q}}\mu_J\left(P_J^{in}\right)^{1-\varepsilon_Q}
\end{aligned}$$

Combining the appropriate first-order conditions, and setting labor as the numeraire good (so that $\left(\sum_{J'=1}^N L_{J'}\right)^{\frac{1}{\varepsilon_{LS}}} = 1$) yields the following expression for industries' prices:

$$P_J^{1-\varepsilon_Q} = A_J^{\varepsilon_Q-1} \cdot \left[(1-\mu_J)B_J^{\varepsilon_Q-1} + \mu_J\left(\sum_I \Gamma_{IJ}^M P_I^{1-\varepsilon_M}\right)^{\frac{1-\varepsilon_Q}{1-\varepsilon_M}}\right]$$

Importantly, sectoral prices do not depend on the demand shocks. Also, with $\log A_I = \log B_I = 0$, all sectoral prices equal 1.

As a second step, manipulating Equation 67 and invoking the fact that sectoral prices are all equal 1, yields

$$C_I = D_I\xi_I$$

Plugging this expression into the market clearing condition

$$Q_I = D_I\xi_I + \sum_J M_{I\to J}$$

Since prices are constant, $d\log Q_I = \frac{d(Q_I P_I)}{Q_I P_I} = \frac{dQ_I}{Q_I}$ and

$$\begin{aligned} Q_I P_I &= D_I \xi_I P_I + \sum_J M_{I\to J} P_I \\ &= D_I \xi_I P_I + \sum_J P_J \Gamma_{IJ} Q_J \mu_J \end{aligned}$$

So

$$\begin{aligned} \frac{d\left(Q_I P_I\right)}{Q_I P_I} &= \xi_I \frac{dD_I}{Q_I} + \sum_J \Gamma_{IJ}\mu_J \frac{d\left(Q_J P_J\right)}{Q_I P_I} \\ &= \xi_I \frac{dD_I}{Q_I} + \sum_J \frac{\Gamma_{IJ}\mu_J Q_J P_J}{Q_I P_I} \frac{d\left(Q_J P_J\right)}{Q_J P_J} \end{aligned}$$

In matrix form, industries' output levels are given by:

$$\mathbf{d}\log\mathbf{Q} = \left(\mathbf{I} - \tilde{\mathbf{\Gamma}}\right)^{-1}\mathbf{dD}, \tag{68}$$

where the elements of $\tilde{\mathbf{\Gamma}}$ are given by $\frac{\Gamma_{IJ}\mu_J Q_J P_J}{Q_I P_I}$. Equation 68 is equivalent to Equation A8 from Acemoglu, Akcigit, and Kerr (2016).[41] Briefly, the reason why the result from Acemoglu, Akcigit, and Kerr (2016) extends to the current environment is that i) the impact of a demand shock on industries' sales depends on the production elasticities of substitution only if industries' prices react to demand shocks, but ii) demand shocks do not alter industries' prices.

F.8 Solution of the model filter with government demand shocks

In this subsection I work through a version of the model filter in which the government industry is not subject to productivity shocks. Instead, there are demand shocks in the government industry. For this robustness check, I assume that all goods are nondurable. I begin with the first-order condition from Equation 67

$$P_{tJ} = \left(D_{tJ}\xi_{tJ}\right)^{\frac{1}{\varepsilon_D}} \left(C_{tJ}\right)^{-\frac{1}{\varepsilon_D}} \left(\sum_{I=1}^{N} \frac{\left(D_{tI}\xi_I\right)^{\frac{1}{\varepsilon_D}}}{\sum_{I'}\left(D_{tI'}\xi_{I'}\right)^{\frac{1}{\varepsilon_D}}} \left(C_{tI}\right)^{\frac{\varepsilon_D - 1}{\varepsilon_D}}\right)^{-1}$$

Notice that demand shocks do not appear in any of the other first-order conditions. Nor do they enter in the market-clearing conditions. To compute the log-linear approximation

[41] A parameter, λ, from Acemoglu, Akcigit, and Kerr (2016) describes the labor supply response from a change in government spending. The equation, here, is consistent with $\lambda \to \infty$.

(around the point at which all productivity and demand shocks equals 1), begin with

$$\frac{P_{tJ}}{P_J}\cdot P_J = \left(D_{tJ}\xi_J\right)^{\frac{1}{\varepsilon_D}}\left(C_J\right)^{-\frac{1}{\varepsilon_D}}\left(\exp\left\{\hat{c}_{tJ}\right\}\right)^{-\frac{1}{\varepsilon_D}}\times$$
$$\left(\sum_{I=1}^{N}\frac{\left(D_{tI}\xi_I\right)^{\frac{1}{\varepsilon_D}}}{\sum_{I'}\left(D_{tI'}\xi_{I'}\right)^{\frac{1}{\varepsilon_D}}}\left(\frac{C_{tI}}{C_I}\right)^{\frac{\varepsilon_D-1}{\varepsilon_D}}\left(C_I\right)^{\frac{\varepsilon_D-1}{\varepsilon_D}}\right)^{-1}$$

and substitute in the steady-state relationship between consumption and prices

$$\exp\left\{\hat{p}_{tJ}\right\} = \exp\left\{\hat{d}_{tJ}\right\}^{\frac{1}{\varepsilon_D}}\exp\left\{\hat{c}_{tJ}\right\}^{-\frac{1}{\varepsilon_D}}\times$$
$$\left(\sum_{I=1}^{N}\frac{\exp\left\{\hat{d}_{tI}\right\}^{\frac{1}{\varepsilon_D}}}{\sum_{I'}\exp\left\{\hat{d}_{tI'}\right\}^{\frac{1}{\varepsilon_D}}\left(\xi_{I'}\right)^{\frac{1}{\varepsilon_D}}}\frac{\left(\xi_I\right)^{\frac{1}{\varepsilon_D}}\left(C_I\right)^{\frac{\varepsilon_D-1}{\varepsilon_D}}}{\sum_{I'=1}^{N}\left(\xi_I'\right)^{\frac{1}{\varepsilon_D}}\left(C_{I'}\right)^{\frac{\varepsilon_D-1}{\varepsilon_D}}}\exp\left\{\hat{c}_{tI}\right\}^{\frac{\varepsilon_D-1}{\varepsilon_D}}\right)^{-1}$$

Taking derivatives of the logarithm of each side of the previous equation, around the point at which $\hat{p}_{tJ}=0$, $\hat{d}_{tJ}=0$, and $\hat{c}_{tJ}=0$ yields:

$$\hat{p}_{tJ} = \frac{1}{\varepsilon_D}\hat{d}_{tJ} - \frac{1}{\varepsilon_D}\hat{c}_{tJ} + \sum_{I}\frac{\left(\xi_I\right)^{\frac{1}{\varepsilon_D}}\left(C_I\right)^{\frac{\varepsilon_D-1}{\varepsilon_D}}}{\sum_{I'=1}^{N}\left(\xi_I'\right)^{\frac{1}{\varepsilon_D}}\left(C_{I'}\right)^{\frac{\varepsilon_D-1}{\varepsilon_D}}}\left[1-\varepsilon_D\right]\left[\hat{c}_{tI}-\hat{d}_{tI}\right]$$

In vector form, the log-linearized equation for consumption as a function of prices and demand shocks is:

$$\begin{aligned}
\hat{p}_t &= \left[\mathbf{I}+S_I^C\left(\varepsilon_D-1\right)\right]\left[-\frac{1}{\varepsilon_D}\hat{c}_t+\frac{1}{\varepsilon_D}\hat{d}_t\right] \quad\Rightarrow\\
\hat{c}_t &= \hat{d}_t - \varepsilon_D\left[\mathbf{I}+S_I^C\left(\varepsilon_D-1\right)\right]^{-1}\hat{p}_t
\end{aligned}$$

Plug this log-linearized equation into Equation 52 to substitute out the $\hat{c}_t$ vector.

$$\begin{aligned}
0 = &\left[\left(\varepsilon_Q-1\right)\tilde{S}_M^QT_1-\left(\mathbf{I}-\tilde{S}_M^QT_1\right)\left(\mathbf{I}+\vartheta\varepsilon_Q^{-1}\right)\left(\mathbf{I}-S_M\right)^{-1}\left(\mathbf{I}+S_M\left(\varepsilon_Q-1\right)\right)-\left(\mathbf{I}-\tilde{S}_M^QT_1\right)\vartheta\frac{\varepsilon_Q-1}{\varepsilon_Q}\right]\hat{a}_t\\
&+\tilde{S}_X^QT_1\delta_K^{-1}\hat{k}_{t+1}+\left[\tilde{S}_X^QT_1\left(1-\delta_K^{-1}\right)-\left(\mathbf{I}-\tilde{S}_M^QT_1\right)\left(\mathbf{I}+\vartheta\right)\alpha\right]\hat{k}_t-\left(\mathbf{I}-\tilde{S}_M^QT_1\right)\left(\mathbf{I}+\vartheta\right)\left(\mathbf{I}-\alpha\right)\hat{b}_t+\tilde{S}_C^Q\hat{d}_t\\
&+\left[-\varepsilon_D\tilde{S}_C^Q\left[\mathbf{I}+S_I^C\left(\varepsilon_D-1\right)\right]^{-1}+\varepsilon_Q\tilde{S}_M^QT_1\left(\mathbf{I}-S_1^M\right)+\varepsilon_M\tilde{S}_M^Q\left[T_1S_1^M-T_2\right]+\varepsilon_X\tilde{S}_X^Q\left[T_1S_1^X-T_2\right]\right]\hat{p}_t\\
&-\left(\mathbf{I}-\tilde{S}_M^QT_1\right)\left[\varepsilon_Q\left(\mathbf{I}-S_M\right)^{-1}S_M\left(\mathbf{I}-S_1^M\right)+\vartheta\left[\left(\mathbf{I}-S_M\right)^{-1}S_M\left(\mathbf{I}-S_1^M\right)+\mathbf{I}\right]\right]\hat{p}_t\\
0 = &-S_1^X\hat{p}_t+\left[\beta(1-\delta_K)S_1^X+\tilde{\beta}\left(\mathbf{I}+\vartheta\right)\left(\mathbf{I}+\left(I-S_M\right)^{-1}S_M\left(\mathbf{I}-S_1^M\right)\right)\right]\hat{p}_{t+1}\\
&+\tilde{\beta}\left(\mathbf{I}+\vartheta\right)\left[\frac{1}{\varepsilon_Q}\left(I-S_M\right)^{-1}\left(\mathbf{I}+S_M\left(\varepsilon_Q-1\right)\right)+\frac{\varepsilon_Q-1}{\varepsilon_Q}\mathbf{I}\right]\hat{a}_{t+1}\\
&+\tilde{\beta}\left(\mathbf{I}+\vartheta\right)\left(\mathbf{I}-\alpha\right)\hat{b}_{t+1}+\tilde{\beta}\left(-\mathbf{I}+\alpha+\vartheta\alpha\right)\hat{k}_{t+1}
\end{aligned}$$

Again, now with demand shocks, we have expressed the reduced system as

$$\begin{bmatrix} \mathbb{E}_t[\hat{p}_{t+1}] \\ \hat{k}_{t+1} \end{bmatrix} = \boldsymbol{\Psi} \begin{bmatrix} \hat{p}_t \\ \hat{k}_t \end{bmatrix} + \overset{d}{\boldsymbol{\Phi}} \begin{bmatrix} \hat{a}_t \\ \hat{b}_t \\ \hat{d}_t \end{bmatrix}$$

As in Appendix F.4, $\boldsymbol{\Psi}$ has N stable and N unstable eigenvalues. Here, the "d" in $\overset{d}{\boldsymbol{\Phi}}$ refers to the modification of Φ to allow for demand shocks. Using a similar set of calculations as in Appendix F.4, we arrive at the following equation for the evolution of the endogenous state:

$$\begin{aligned} \hat{k}_{t+1} \quad = \quad & \underbrace{(\boldsymbol{\Psi}_{22} - \boldsymbol{\Psi}_{21}(\mathbf{V}_{11}^{-1})^{-1}\mathbf{V}_{12}^{-1})}_{\equiv M_{kk}}\hat{k}_t \\ & + \underbrace{\left(-\boldsymbol{\Psi}_{21}(\mathbf{V}_{11}^{-1})^{-1}\mathbf{D}_1^{-1}(\mathbf{I} - \mathbf{D}_1^{-1})^{-1}\overset{d}{\tilde{\Phi}}_1 + \overset{d}{\boldsymbol{\Phi}}_2 \right)}_{\equiv \left[\overset{d}{\mathbf{M}}_{ka},\, \overset{d}{\mathbf{M}}_{kb},\, \overset{d}{\mathbf{M}}_{kd} \right]} \begin{bmatrix} \hat{a}_t \\ \hat{b}_t \\ \hat{d}_t \end{bmatrix} \end{aligned} \tag{69}$$

As before

$$\hat{p}_t = \boldsymbol{\Psi}_{21}^{-1}\hat{k}_{t+1} - \boldsymbol{\Psi}_{21}^{-1}\boldsymbol{\Psi}_{22}\hat{k}_t - \boldsymbol{\Psi}_{21}^{-1}\overset{d}{\boldsymbol{\Phi}}_2 \begin{bmatrix} \hat{a}_t \\ \hat{b}_t \\ \hat{d}_t \end{bmatrix} \tag{70}$$

As before, the following equation describes $\hat{q}$ as a function of the exogenous variables, $\hat{k}$, and $\hat{p}$:

$$\begin{aligned} \hat{q}_t = & \left(\mathbf{I} + \vartheta\right)\left(\mathbf{I} - \alpha\right)\hat{b}_t + \left(\mathbf{I} + \vartheta\right)\alpha\hat{k}_t \\ & + \left[\frac{\varepsilon_Q - 1}{\varepsilon_Q}\vartheta + \left(\frac{\vartheta}{\varepsilon_Q} + \mathbf{I} \right) \left(\mathbf{I} - S_M\right)^{-1}\left(\mathbf{I} + S_M\left(\varepsilon_Q - 1\right)\right) \right] \hat{a}_t \\ & + \left[\left(\vartheta + \varepsilon_Q \mathbf{I}\right)\left(\mathbf{I} - S_M\right)^{-1} S_M \left(\mathbf{I} - S_1^M\right) + \vartheta \right] \hat{p}_t \end{aligned} \tag{71}$$

Plug Equation 69 and 70 in to 71 so that we may write:

$$\hat{q}_t = \Phi_{kq}\hat{k}_t + \Phi_{bq}\hat{b}_t + \Phi_{aq}\hat{a}_t + \Phi_{dq}\hat{d}_t, \tag{72}$$

where the Φ_{kq}, Φ_{bq},Φ_{aq}, and Φ_{dq} are matrices that collect the appropriate terms.[42]

So long as Φ_{kq} is invertible, Equation 72 is equivalent to

$$\hat{k}_t = \Phi_{kq}^{-1}\hat{q}_t - \Phi_{kq}^{-1}\Phi_{bq}\hat{b}_t - \Phi_{kq}^{-1}\Phi_{aq}\hat{a}_t - \Phi_{kq}^{-1}\Phi_{dq}\hat{d}_t$$

Equation 72, one period ahead, is

$$\hat{q}_{t+1} = \Phi_{kq}\hat{k}_{t+1} + \Phi_{bq}\hat{b}_{t+1} + \Phi_{aq}\hat{a}_{t+1} + \Phi_{dq}\hat{d}_{t+1}$$

Apply Equation 69 to this previous equation

$$\begin{aligned}
\hat{q}_{t+1} &= \Phi_{bq}\hat{b}_{t+1} + \Phi_{aq}\hat{a}_{t+1} + \Phi_{dq}\hat{d}_{t+1} \\
&\quad +\Phi_{kq}\left(M_{kk}\hat{k}_t + \overset{d}{\mathbf{M}}_{ka}\hat{a}_t + \overset{d}{\mathbf{M}}_{kb}\hat{b}_t + \overset{d}{\mathbf{M}}_{kd}\hat{d}_t \right) \\
&= \Phi_{bq}\hat{b}_{t+1} + \Phi_{aq}\hat{a}_{t+1} + \Phi_{dq}\hat{d}_{t+1} \\
&\quad +\Phi_{kq}\overset{d}{\mathbf{M}}_{ka}\hat{a}_t + \Phi_{kq}\overset{d}{\mathbf{M}}_{kb}\hat{b}_t + \Phi_{kq}M_{kd}\hat{d}_t + \Phi_{kq}M_{kk}\Phi_{kq}^{-1}\hat{q}_t \\
&\quad -\Phi_{kq}M_{kk}\Phi_{kq}^{-1}\Phi_{bq}\hat{b}_t - \Phi_{kq}M_{kk}\Phi_{kq}^{-1}\Phi_{aq}\hat{a}_t - \Phi_{kq}M_{kk}\Phi_{kq}^{-1}\Phi_{dq}\hat{d}_t \\
&= \Phi_{dq}\hat{d}_{t+1} + \Phi_{bq}\hat{b}_{t+1} + \Phi_{aq}\hat{a}_{t+1} + \Phi_{kq}M_{kk}\Phi_{kq}^{-1}\hat{q}_t + \left[\Phi_{kq}\overset{d}{\mathbf{M}}_{ka} - \Phi_{kq}M_{kk}\Phi_{kq}^{-1}\Phi_{aq}\right]\hat{a}_t \\
&\quad + \left[\Phi_{kq}\overset{d}{\mathbf{M}}_{kb} - \Phi_{kq}M_{kk}\Phi_{kq}^{-1}\Phi_{bq}\right]\hat{b}_t + \left[\Phi_{kq}\overset{d}{\mathbf{M}}_{kd} - \Phi_{kq}M_{kk}\Phi_{kq}^{-1}\Phi_{dq}\right]\hat{d}_t
\end{aligned}$$

[42] Combine Equations 69 and 70:

$$\begin{aligned}
\hat{p}_t &= \mathbf{\Psi}_{21}^{-1}\hat{k}_{t+1} - \mathbf{\Psi}_{21}^{-1}\mathbf{\Psi}_{22}\hat{k}_t - \mathbf{\Psi}_{21}^{-1}\overset{d}{\mathbf{\Phi}}_2 \begin{bmatrix}\hat{a}_t\\ \hat{b}_t\\ \hat{d}_t\end{bmatrix} \\
&= \mathbf{\Psi}_{21}^{-1}\left[\mathbf{\Psi}_{22} - \mathbf{\Psi}_{21}(\mathbf{V}_{11}^{-1})^{-1}\mathbf{V}_{12}^{-1}\right]\hat{k}_t - \mathbf{\Psi}_{21}^{-1}\mathbf{\Psi}_{22}\hat{k}_t \\
&\quad -\mathbf{\Psi}_{21}^{-1}\overset{d}{\mathbf{\Phi}}_2 \begin{bmatrix}\hat{a}_t\\ \hat{b}_t\\ \hat{d}_t\end{bmatrix} + \mathbf{\Psi}_{21}^{-1}\left[\left(-\mathbf{\Psi}_{21}(\mathbf{V}_{11}^{-1})^{-1}\mathbf{D}_1^{-1}(\mathbf{I}-\mathbf{D}_1^{-1})^{-1}\overset{d}{\tilde{\Phi}}_1 + \overset{d}{\mathbf{\Phi}}_2\right)\right]\begin{bmatrix}\hat{a}_t\\ \hat{b}_t\\ \hat{d}_t\end{bmatrix} \\
&= -(\mathbf{V}_{11}^{-1})^{-1}\mathbf{V}_{12}^{-1}\hat{k}_t - (\mathbf{V}_{11}^{-1})^{-1}\mathbf{D}_1^{-1}(\mathbf{I}-\mathbf{D}_1^{-1})^{-1}\overset{d}{\tilde{\Phi}}_1 \begin{bmatrix}\hat{a}_t\\ \hat{b}_t\\ \hat{d}_t\end{bmatrix}
\end{aligned}$$

So:

$$\begin{aligned}
\hat{q}_t = &\left\{(\mathbf{I}+\vartheta)\,\alpha - \left[(\vartheta+\varepsilon_Q\mathbf{I})\,(\mathbf{I}-S_M)^{-1}\,S_M\left(\mathbf{I}-S_1^M\right)+\vartheta\right](\mathbf{V}_{11}^{-1})^{-1}\mathbf{V}_{12}^{-1}\right\}\hat{k}_t \\
&+ \left[\frac{\varepsilon_Q-1}{\varepsilon_Q}\vartheta + \left(\frac{\vartheta}{\varepsilon_Q}+\mathbf{I}\right)(\mathbf{I}-S_M)^{-1}\left(\mathbf{I}+S_M\left(\varepsilon_Q-1\right)\right)\right]\hat{a}_t + (\mathbf{I}+\vartheta)\,(\mathbf{I}-\alpha)\,\hat{b}_t \\
&- \left[(\vartheta+\varepsilon_Q\mathbf{I})\,(\mathbf{I}-S_M)^{-1}\,S_M\left(\mathbf{I}-S_1^M\right)+\vartheta\right](\mathbf{V}_{11}^{-1})^{-1}\mathbf{D}_1^{-1}(\mathbf{I}-\mathbf{D}_1^{-1})^{-1}\overset{d}{\tilde{\Phi}}_1\begin{bmatrix}\hat{a}_t\\ \hat{b}_t\\ \hat{d}_t\end{bmatrix}
\end{aligned}$$

Finally, take two adjacent periods, and use the definitions of ω^A_{t+1} ($\equiv \hat{a}_{t+1} - \hat{a}_t$), ω^B_{t+1} $\left(\equiv \hat{b}_{t+1} - \hat{b}_t\right)$, and ω^D_{t+1} $\left(\equiv \hat{d}_{t+1} - \hat{d}_t\right)$ so that

$$\begin{aligned}\Delta\hat{q}_{t+1} &= \Phi_{kq} M_{kk} \Phi_{kq}^{-1} \Delta\hat{q}_t + \Phi_{dq}\omega^D_{t+1} + \Phi_{bq}\omega^B_{t+1} + \Phi_{aq}\omega^A_{t+1} + \\ &+ \left[\Phi_{kq}\overset{d}{\mathbf{M}}_{ka} - \Phi_{kq} M_{kk}\Phi_{kq}^{-1}\Phi_{aq}\right]\omega^A_t + \left[\Phi_{kq}\overset{d}{\mathbf{M}}_{kb} - \Phi_{kq} M_{kk}\Phi_{kq}^{-1}\Phi_{bq}\right]\omega^B_t \\ &+ \left[\Phi_{kq}\overset{d}{\mathbf{M}}_{kd} - \Phi_{kq} M_{kk}\Phi_{kq}^{-1}\Phi_{dq}\right]\omega^D_t\end{aligned}$$

In the robustness check with productivity shocks in all non-government industries in combination with demand shocks in the government sector, the filter is given by inverting the following equation:

$$\begin{aligned}\Delta\hat{q}_{t+1} &= \Phi_{kq} M_{kk} \Phi_{kq}^{-1} \Delta\hat{q}_t \\ &+ \left[\left[\Phi_{aq}\right]_{[1:N,1:N-1]} \vdots \Phi_{dq[1:N,N]}\right] \cdot \left[\begin{array}{c}\left[\omega^A_{t+1}\right]_{[1:N-1]} \\ \left[\omega^D_{t+1}\right]_N\end{array}\right] \\ &+ \left[\left[\Phi_{kq}\overset{d}{\mathbf{M}}_{ka} - \Phi_{kq} M_{kk}\Phi_{kq}^{-1}\Phi_{aq}\right]_{[1:N,1:N-1]} \vdots \left[\Phi_{kq}\overset{d}{\mathbf{M}}_{kd} - \Phi_{kq} M_{kk}\Phi_{kq}^{-1}\Phi_{dq}\right]_{[1:N,N]}\right] \\ &\cdot \left[\begin{array}{c}\left[\omega^A_t\right]_{[1:N-1]} \\ \left[\omega^D_t\right]_N\end{array}\right]\end{aligned} \tag{73}$$

Since the government sector is the final (N^{th}) industry, the filter recovers an $N-1$ dimensional productivity vector along with a single, final element of the demand shock vector. In Equation 73, a $[1:N-1]$ subscript refers to the first $N-1$ elements of a vector; a $[1:N, 1:N-1]$ subscript refers to the first $N-1$ columns of a given matrix; and a $[1:N,N]$ subscript refers to the final column. Here, we have removed labor-augmenting productivity shocks and factor-neutral productivity shocks in the N^{th} (governmental) sector as a source of output fluctuations.

Additional references

Acemoglu, Daron, and Veronica Guerrieri. 2008. "Capital Deepening and Nonbalanced Economic Growth." *Journal of Political Economy*, 116(3): 467-498.

Atalay, Enghin. 2014. "Materials Prices and Productivity." *Journal of the European Economic Association*, 12(3): 575-611.

Canova, Fabio. 2014. "Bridging DSGE Models and the Raw Data." *Journal of Monetary Economics* 67(1): 1-15.

Foster, Lucia, John Haltiwanger, and Chad Syverson. 2016. "The Slow Growth of New

Plants: Learning about Demand?" *Economica*, 83(329): 91-129.

Hulten, Charles R., and Frank C. Wykoff. 1981. "The Measurement of Economic Depreciation." In *Depreciation, Inflation, and the Taxation of Income from Capital*, edited by Charles Hulten, Washington, DC: Urban Institute.

O' Mahony, Mary, and Marcel P. Timmer. 2009. "Output, Input and Productivity Measures at the Industry Level: The EU KLEMS Database." *Economic Journal*, 119(538): 374-403.

Timmer, Marcel, Ton van Moergastel, Edwin Stuivenwold, Gerard Ypma, Mary O' Mahony, and Mari Kangasniemi. 2007. "EU KLEMS Growth and Productivity Accounts Version 1.0." mimeo.