A Behavioral Analysis of Stochastic Reference Dependence: Online Appendix*

Yusufcan Masatlioglu Collin Raymond

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Abstract

We examine the reference-dependent risk preferences of Kőszegi and Rabin (2007), focusing on their choice-acclimating personal equilibria. Although their model has only a trivial intersection (expected utility) with other reference-dependent models, it has very strong connections with models that rely on different psychological intuitions. We prove that the intersection of rank-dependent utility and quadratic utility, two well-known generalizations of expected utility, is exactly monotone linear gain-loss choice-acclimating personal equilibria. We use these relationships to identify parameters of the model, discuss loss and risk aversion, and demonstrate new applications.

*Masatlioglu: Department of Economics, University of Michigan, 611 Tappan Avenue, Ann Arbor, MI 48108 (e-mail: yusufcan@umich.edu); Raymond: Department of Economics, Amherst College, 100 Boltwood Avenue, Amherst, MA 01002 (e-mail: collinbraymond@gmail.com). Previous drafts circulated under the titles "Stochastic Reference Points, Loss Aversion and Choice under Risk" and "Drs. Kőszegi and Rabin or: How I Learned to Stop Worrying and Love Reference Dependence." For their helpful comments, we would like to thank Larry Samuelson; three anonymous referees; seminar participants at Amherst College; Boston University; BRIC; FUR; Harvard University; London Business School; London School of Economics; Ludwig-Maximilians-Universitat Munich; Monash University; Penn State University; Queen Mary University of London; University of California, Berkeley; University of Copenhagen; University of Michigan; University of Michigan; University of Oxford; University of Pennsylvania; and Johannes Abeler; Daniel Benjamin; Ian Crawford; Vince Crawford; David Dillenberger; Andrew Ellis; David Freeman; David Gill; David Huffman; Matthew Rabin; Neslihan Uler; and Jill Westfall. We would also like to thank Yang Lu for excellent assistance. Any remaining errors are ours. The authors declare that they have no relevant or material financial interests that relate to the research described in this paper.

Online Appendix A: Additional Results

Shapes of Indifference Curves

In this subsection, we investigate the structure of the indifference curves induced by $\mathbb{CPE}$. Quadratic functionals have simple graphical representations in the Marschak-Machina triangle; as pointed out by Chew, Epstein and Segal (1991), their indifference curves are conic sections (i.e. ellipses, parabolas or hyperbolas). Therefore, preferences in $\mathbb{CPE}$ must have indifference curves that are conic sections. Denote the probability attached to the worst outcome in the Marschak-Machina triangle as p, and the best q, the utility of the worst outcome to 0 and the best to 1, and the utility of the middle outcome as m. Lemma 1 demonstrates that the functional form of the indifference curves, giving utility level $\bar{u}$ is:

$$(1-m)\left[\frac{2-\lambda}{2(1-\lambda)}-q\right]^2+m\left[\frac{\lambda}{2(\lambda-1)}-p)\right]^2=h(\bar{u},\lambda,m)$$

where h is a real function. Thus, looking at this equation, it is apparent that indifference curves are concentric ellipses. The shared center of the ellipses lies on a line that passes through the best and worst degenerate outcomes. Moreover, the axes of ellipses are oriented parallel to the best-middle outcome edge and the worst-middle outcome edge.

The following figure demonstrates what the indifference curves appear like in the Marschak-Machina triangle for $\lambda = 2$ and when the utilities of three outcomes, b (best), m (middle), and w (worst), are equally spaced, showing the center of the concentric ellipses (C), and how the indifference curves extend beyond the unit simplex.

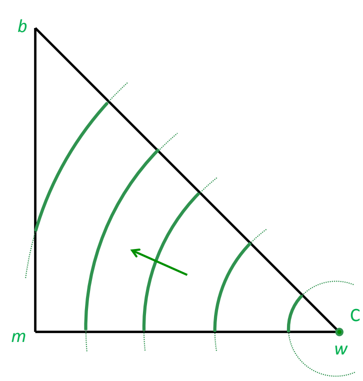

Figure 1: $\mathbb{CPE}$ Indifference Curves when $\lambda = 2$ and $u(b) - u(m) = u(m) - u(w)$

$\mathbb{CPE}$ representations have two factors that influence how lotteries are valued: the Bernoulli utility function u and the coefficient of loss aversion λ. Each of these determines a particular part of the structure of the ellipse.

First, the center's location along the best to worst outcome line varies with λ, but not with u. For example, if gain-loss utility is a strong component of $V_{\mathbb{CPE}}$, e.g. $\lambda > 2$, so that preferences are non-monotone, then the center is within the unit simplex (but below the point that represents a 50-50 mixture of the best and worst outcome) and is the minimal lottery in the simplex in terms of the preference ordering. As the loss aversion coefficient falls to 1, the center shifts down and to the right. $\mathbb{EU}$ is when the center is infinitely far from the best outcome. Figure 3 demonstrates how the center changes with λ, with C_i being the center of the concentric ellipses for individual i, where $i = A, B$, and C. It is the case that individual A is more loss-averse than individual B, who is more loss-averse than individual C: $\lambda_A > \lambda_B = 2 > \lambda_C > 1$.

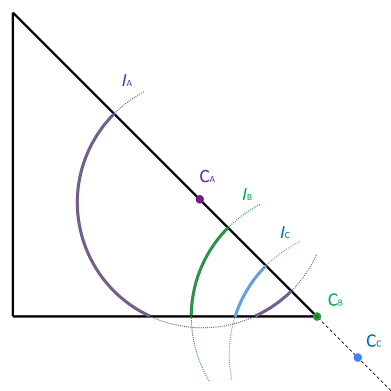

Figure 2: Changing Loss Aversion Coefficient (λ): $\lambda_A > \lambda_B = 2 > \lambda_C > 1$

Furthermore, as λ continues to fall (< 1), the center shifts to be on the the best to worst outcome line but above the best outcome. As it falls to 0, the center becomes the best outcome, and as λ becomes negative the center becomes within the unit simplex (but above the point that represents a 50-50 mixture of the best and worst outcome). In contrast to before, the center is now the best possible lottery in the simplex.

While λ governs the location of the center of the ellipses, it does not affect their orientation (i.e. the orientation and relative length of the axes of the ellipses). Instead, this is governed solely by the consumption utility functional u. The two axes of the ellipses are always aligned with the horizontal and vertical axes of the unit simplex — the edge connecting the middle and worst outcomes and the edge connecting the middle and best outcome respectively. If the individual is consumption risk neutral (i.e. u is linear) and the outcomes are equally spaced (i.e. $w = 0$, $m = 1$ and $b = 2$), then their indifference curves are circles. As the individual becomes more consumption risk-averse, i.e. u becomes more concave, the vertical axis becomes relatively longer than the horizontal axis (and vice versa for less consumption risk-averse). Figure 4 demonstrates what happens as consumption risk aversion changes. Individual B is more consumption risk-averse than individuals A — u_B is more concave than u_A.

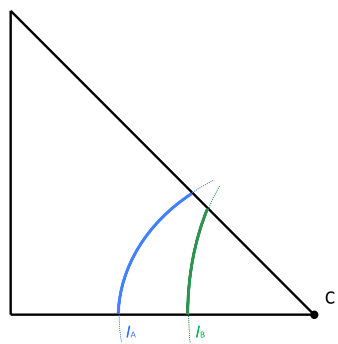

Figure 3: Changing Consumption Risk Aversion (u)

Later in this Appendix we formalize these graphical intuitions relating the indifference curves, u and λ and so relate the values of u and λ, to observable preferences.

Comparing Individuals

In the body of the paper we provided statements that linked u and λ to risk aversion and first-order risk aversion, providing some characterization results. Here, we extend our analysis, demonstrating how to compare preferences, and the parameters u and λ, across individuals.

First, we will consider comparative risk aversion. We can extend Proposition 6 in order to compare risk aversion across individuals. Following Machina (1982) we define comparative risk aversion.

Definition: *Individual A is more risk-averse than Individual B if $g \succsim_A f$ whenever $f \sim_B g$ and there exists an $x_0 \in X$ such that $F(x) \geq G(x)$ for all $x < x_0$ and $F(x) \leq G(x)$ for all $x \geq x_0$.*

Given the results of Proposition 6 we will restrict ourselves to considering preferences in monotone $\mathbb{CPE}$. As Proposition 6 would suggest, it is the case that the relative curvature of u and value of λ jointly determine comparative risk aversion.

Proposition 1 *Let (u_i, λ_i) be a monotone $\mathbb{CPE}$ representation of Individual i's preference for $i \in \{A, B\}$. Then Individual A is more risk-averse than Individual B if and only if u_A is a concave transformation of u_B and $\lambda_A \geq \lambda_B$.*

Thus, it is not the case that there is a tradeoff between the concavity of u and the amount of loss aversion. Proposition 1 tells us that for Individual A to be more risk-averse than Individual B it must be the case that u_A is more concave than u_B and Individual A is more loss-averse than Individual B. Regardless of how much more concave u_A is than u_B, if Individual B is even slightly more loss-averse than Individual A, then Individual A cannot be more risk-averse than Individual B.

Next, we will discuss how to extend our analysis of first-order risk aversion in order to accommodate comparative first-order risk aversion. In order to do so we will first define comparative loss aversion between two individuals using the relative sizes of λ. Recall we related the parameter λ to risk preferences over small-stakes lotteries, showing that the intuitive relationship between λ and small-stakes risk preferences holds. Moreover, recall that we also linked λ to attitudes towards the mixing of otherwise indifferent lotteries. Thus, we are also able to link values of λ to comparative mixture aversion.

We can extend the analysis in the body of the paper by ordering individuals' degree of first-order risk aversion by the absolute size of $\frac{\partial \pi(w+\epsilon f)}{\partial \epsilon}|_{\epsilon=0^+}$.

Definition: *Individual A is more first-order risk-averse than Individual B at wealth level w if $\frac{\partial \pi_B(w+\epsilon f)}{\partial \epsilon}|_{\epsilon=0^+} \leq \frac{\partial \pi_A(w+\epsilon f)}{\partial \epsilon}|_{\epsilon=0^+}$.*

This definition allows us to relate λ to preferences over small stakes lotteries (as captured by the risk premium attached to those lotteries).

Proposition 2 *Let (u_i, λ_i) be a monotone $\mathbb{CPE}$ representation of Individual i's preference for $i \in \{A, B\}$ with u_i everywhere differentiable. Then Individual A is more first-order risk-averse than Individual B at all wealth levels if and only if $\lambda_A \geq \lambda_B$.*

An Alternative Characterization: Above we provide a characterization for comparing loss aversion parameters across individuals. However, the above exercise utilizes first-order risk aversion which heavily relies on differentiability and behavior at the limit. We now provide a second characterization without these shortcomings.

Note that if preferences are represented by monotone $\mathbb{CPE}$ then $\lambda \geq 1$ if and only if preferences are mixture-averse. We will leverage this fact. In doing so we will restrict ourselves to individuals whose preferences are represented by monotone $\mathbb{CPE}$ and satisfy mixture aversion. Extending the analysis to non-monotone, and/or mixture loving, preferences is straightforward, but at the cost of expositional ease.

A key factor in understanding the behavioral content of loss aversion is the fact that observed choices over lotteries (i.e. observed risk aversion) is generated by both λ and u. We want to observe choices that relate only to the value of λ and not to u. In order to understand what choices these might be we first must develop intuition regarding the relaxation of Independence that holds for $\mathbb{CPE}$ preferences. This will be useful not only for understanding the results of this subsection but also the next.

We now relate λ in $\mathbb{CPE}$ to 'expansion paths' which is a fairly unknown concept introduced by Chew, Epstein and Segal (1991, 1994). They consider the expansion paths and supporting lines of the indifference curves. Two points f and g lie on the same expansion path if there is a common sub-gradient (supporting lines have the same slope) to the indifference curves at f and g (Figure 4). Chew, Epstein and Segal (1991) shows that for quadratic utility functionals, all expansion paths are straight lines and have a common intersection point.[1]

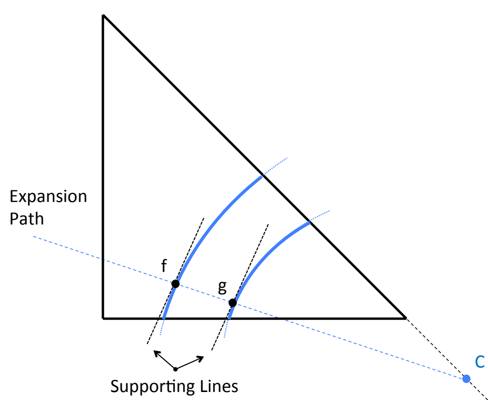

Figure 4: Constructing An Expansion Path

Recall that the center of the ellipses which define the indifference curves must lie on the line that connects the best and the worst outcome for any Δ_3. Therefore, one of the expansion paths must lie on the best to worst outcome edge if preferences are represented by $\mathbb{CPE}$. The construction we use to behaviorally compare λ relies on this fact. In the Marschak-Machina triangle the expansions paths

[1]They could be parallel straight lines (never intersect) when the decision-maker is an expected utility maximizer.

for individual i all intersect at $(\frac{\lambda_i}{2(\lambda_i-1)}, \frac{2-\lambda_i}{2(1-\lambda_i)})$. Since one expansion path is the line connecting the best to worst outcomes, only a second expansion path is needed to locate the center of the indifference curves — which is the single common point of intersection of the expansion paths. This identifies λ_i. Of course, if preferences are represented by $\mathbb{EU}$ then all points lie on a single expansion path (because all indifference curves have the same slope). In this case indifference curves are linear, and although the intuition in the paragraph fails, it is easy to identify that $\lambda = 1$ in this scenario (and similarly, if $\lambda > 1$ then indifference curves cannot be linear).

We can use this construction to compare loss aversion across individuals. Individual A is more loss-averse than B if the center of A's indifference curves is closer to the best degenerate outcome. Because the center of A's indifference curves is closer to the best degenerate outcome than the center of B's indifference curves, for any point f, A's expansion path through f will be steeper than B's.

Consider an individual i whose preferences over Δ_3 are represented by monotone $\mathbb{CPE}$. Let $f' = \alpha_1\bar{\delta} + (1-\alpha_1)\underline{\delta}$, and $g = \alpha_2\hat{\delta} + (1-\alpha_2)\underline{\delta}$, for some $\alpha_1, \alpha_2 \in (0,1)$ so that $\alpha_1 > \alpha_2$. Denote $g'_i = \alpha'_i\bar{\delta} + (1-\alpha'_i)\underline{\delta}$, so that $g'_i \sim_i g$. g'_i always exists since individual i has monotone preferences.

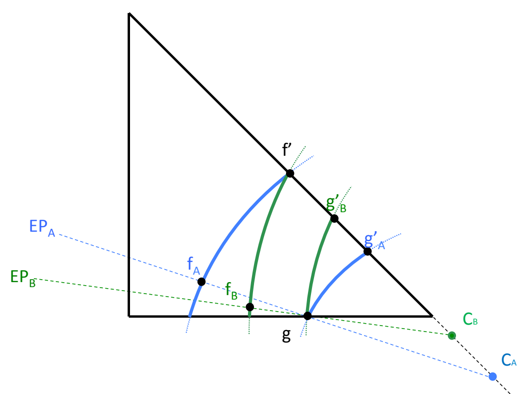

Figure 5: Loss Aversion Across Individuals

For some combinations of f' and g we can define an f_i such that $f_i \sim_i f'$ and $\beta_i f_i + (1-\beta_i)g \sim \beta_i f' + (1-\beta_i)g'_i$ for all $\beta_i \in (0,1)$. Denote the weight applied to the high outcome in f_i as h_i and to the middle outcome as m_i. Importantly, so long as $\lambda_i \neq 1$, then if $\frac{h_A}{m_A} \geq \frac{h_B}{m_B}$ for a particular f', g combination that define an f_i, then the inequality will also hold for all f', g combinations. Figure 5 demonstrates the construction of f', g', g_i and f_i, along with the steepness of the respective expansion paths EP_i. Importantly, because λ alone governs the steepness of the expansion paths, if A and B both have monotone $\mathbb{CPE}$ representations and A exhibits steeper expansion paths than B

in a single Δ_3 then A exhibits steeper expansion paths than B in all Δ_3. Because full independence holds if preferences are represented by $\mathbb{EU}$, the procedure we just defined is not well defined. Thus, we will define the ratio $\frac{h_i}{m_i}$ to be equal to ∞ if the decision-maker is an expected utility maximizer.

Definition: *Individual A has steeper expansion paths than individual B if $\frac{h_A}{m_A} \geq \frac{h_B}{m_B}$ for an f, g that define an f_i for a given Δ_3.*

The steepness of the expansions paths reflects the location of the center of the ellipses that define the indifference curves. Therefore, the steepness of the expansion paths characterizes the loss aversion of an individual.

Proposition 3 *Let (u_i, λ_i) be a $\mathbb{CPE}$ representation of Individual i's preference for $i \in \{A, B\}$ with $1 \leq \lambda_i \leq 2$. Then $\lambda_A \leq \lambda_B$ if and only if Individual A has steeper expansion paths than Individual B for all Δ_3.*

Our last exercise is focused on identifying the relative curvature of the other component of $\mathbb{CPE}$ functional: u. We will again focus on cases where $1 \leq \lambda \leq 2$.[2] To gain intuition for our next results we will fix three outcomes and look at the induced Marschak-Machina triangle.

For a given $f = \alpha\bar{\delta} + (1-\alpha)\underline{\delta}$, and Individual i we consider the supporting line of i at f, denoted by $S_{i,f}$. We then denote $g_{i,f}$ as the *binary* lottery that is on $S_{i,f}$ and places some weight on the middle outcome (it must be on the edge of Marschak-Machina triangle). We can compare the slopes of the $S_{i,f}$ across different individuals by comparing $g_{i,f}$. Figure 6 demonstrates the construction of f, $S_{i,f}$ and $g_{i,f}$.

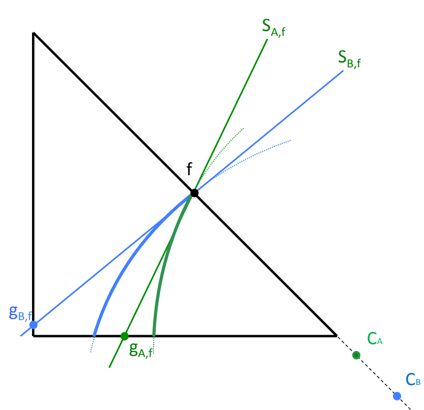

Figure 6: Consumption Risk Aversion Across Individuals

[2] Again, the analysis easily extends to mixture loving and non-monotone preferences.

Definition: *Individual A has steeper supporting lines than Individual B if $g_{B,f}$ first-order stochastically dominates $g_{A,f}$ for all f for a given Δ_3.*

Note that $S_{i,f}$ is determined by the tangency of the indifference curves of i passing through the best to worst outcome line. Moreover, the center of the ellipses that define the indifference curves are also always on this line. Because the center of the ellipses is determined only by λ and the shape of the ellipses only by u, the tangency condition on this edge is determined only by u, not λ. In the Marschak-Machina triangle the slope of $S_{i,f}$ is simply $\frac{1-u_i(\hat{\delta})}{u_i(\hat{\delta})}$. Thus we can easily recover the utility value of $\hat{\delta}$ for any individual (after normalizing $u_i(\bar{\delta}) = 1$ and $u_i(\underline{\delta}) = 0$). Thus changing the center of an ellipse, but not the relative length of its axes, will leave the tangency condition unchanged. This allows us to compare the curvature of u across individuals. The next proposition formalizes the intuition that the steepness of the supporting lines characterizes consumption risk aversion.

Proposition 4 *Let (u_i, λ_i) be a $\mathbb{CPE}$ representation of Individual i's preference for $i \in \{A, B\}$ with $1 \leq \lambda_i \leq 2$. Then u_A is more concave than u_B if and only if Individual A has steeper budget constraints than Individual B in all Δ_3.*

Online Appendix B: Proofs

Proposition 4 *Any preference with a monotone $\mathbb{CPE}$ representation (u, λ) has also a $\mathbb{RDU}$ representation (u, w_λ) where $w_\lambda(z) = (2 - \lambda)z + (\lambda - 1)z^2$.*

Proof: The proof is demonstrated as part of the proof of Theorem 1. ■

Proposition 5 *If a decision-maker's preference can be represented by both $V_{\mathbb{CPE}}$ and $V_{\mathbb{BLS}}$ (or $V_{\mathbb{B}}$), she must be an expected utility maximizer.*

Proof: For the first part, recall that Chew and Epstein (1989) point out that there is no preference (other than $\mathbb{EU}$) which can be represented by both $\mathbb{RDU}$ and $\mathbb{B}$.[3]

We will next show that $\mathbb{BLS}$ have no quadratic utility representation. Consider lotteries over three outcomes; $x > y > z$. Normalize $u(x) = 1$ and $u(z) = 0$ and $u(y) = m$. Divide the simplex into two disjoint sets X_1 and X_2 such that $f \in X_1$ and $f' \in X_2$ and $m < E_u(f)$ and $m > E_u(f')$. Denote the probability assigned to the best/worst outcome in f and f' as q/p and q'/p'.

[3] One could also use the fact in Chew, Epstein and Segal (1991) that there is no preference (other than $\mathbb{EU}$) which can be represented by both $\mathbb{Q}$ and $\mathbb{B}$).

Then we consider all lotteries in X_1:

$$\begin{aligned} V^1_{\mathbb{BLS}}(f) &= E_u(f) + (1 - E_u(f))q + \lambda(m - E_u(f))(1 - p - q) + \lambda(0 - E_u(f))p \\ &= E_u(f)(1 - \lambda)(1 - q) + q + \lambda m(1 - p - q) \end{aligned}$$

Similarly, consider all lotteries in X_2:

$$\begin{aligned} V^2_{\mathbb{BLS}}(f') &= E_u(f') + (1 - E_u(f'))q' + (m - E_u(f'))(1 - p' - q') + \lambda(0 - E_u(f'))p' \\ &= E_u(f')(1 - \lambda)p' + q' + m(1 - p' - q') \end{aligned}$$

Notice that each of them includes at most one of either p^2 or q^2 term since $E_u(f) = q + m(1 - p - q)$ and $E_u(f') = q' + m(1 - p' - q')$. Hence, even though each of them has a quadratic representation, they are two distinct quadratic representations. Hence, a preference with a $\mathbb{BLS}$ representation does not have a $\mathbb{Q}$ representation except when it has an $\mathbb{EU}$ representation. Moreover, it is also clear that because a preference with a $\mathbb{BLS}$ has preferences that are two proper quadratic functionals "stitched" together, they only also have a $\mathbb{B}$ representation if they are expected utility, since quadratic functionals intersect with betweenness preferences only at expected utility. ■

Proposition 6 *Suppose (u, λ) represents a decision-maker's preference. Then:*

1. *if $\lambda < 1$ then the decision-maker is not risk-averse,*

2. *if $1 \leq \lambda \leq 2$ then the decision-maker is risk-averse if and only if u is concave,*

3. *if $2 < \lambda$ then the decision-maker is risk-averse if and only if u is linear.*

Proof: First, consider preferences with a monotone $\mathbb{CPE}$ representation (so $0 \leq \lambda \leq 2$). Recall that Proposition 4 implies that a monotone $\mathbb{CPE}$ representation (u, λ) has a concave u and $\lambda \geq 1$ ($\lambda \leq 1$) if and only if the corresponding $\mathbb{RDU}$ representation (v, w) has a concave v and a convex (concave) w. Chew, Karni and Safra (1987) show that preferences in $\mathbb{RDU}$ with a representation (v, w) are risk-averse if and only if v is concave and w is convex. Thus, if $0 \leq \lambda \leq 2$ then the decision-maker is risk-averse if and only if $1 \leq \lambda \leq 2$ and u is concave

Next, consider non-monotone $\mathbb{CPE}$. Chew, Epstein and Segal (1991) show that necessary and sufficient conditions for quadratic preferences to be risk-averse are that $\phi(x, y)$ is concave in x for all y. If $\lambda < 0$ then $\phi(x, y)$ is locally convex around $x = y$, and so preferences cannot be risk-averse. Thus, if $\lambda < 0$ then the decision-maker is not risk-averse.

In contrast, if $\lambda > 2$ then $\phi(x, y)$ is locally concave around $x = y$. Moreover, there are two additional conditions to check that $\phi(x, y)$ concave in x for all y are $u'' + (1 - \lambda)u'' \leq 0$ and $u'' - (1 - \lambda)u'' \leq 0$ (i.e. ϕ must be concave if $x \geq y$ and ϕ must be concave if $y \geq x$). Since we are focusing on $\lambda > 2$, the first inequality implies $u'' \geq 0$ and the second implies $u'' \leq 0$. Therefore, it is the case that $u'' = 0$.

If u is linear then fix a y and normalize the utility so that $u(y) = 0$ and $u(x) = x - y$. Denote $z = x - y$. Then $\phi(x, y) = \frac{z-(\lambda-1)|z|}{2}$. If $x \leq y$ then $\phi(x, y) = \frac{\lambda z}{2}$. If $x \geq y$ then $\phi(x, y) = \frac{(2-\lambda)z}{2}$. Clearly ϕ is concave in z, and so concave in x. ■

Proposition 7 *Suppose* (u, λ) *represents a decision-maker's preference with u everywhere differentiable. Then the decision-maker exhibits first-order risk-averse (loving) at all wealth levels if and only if* $\lambda > 1$ $(\lambda < 1)$.

Proof: For monotone preferences, Proposition 4 of Segal and Spivak (1990) demonstrate that if $\lambda > (<)1$ then preferences exhibit first-order risk aversion (loving). Moreover, as Segal and Spivak (1990) note if preferences are in $\mathbb{EU}$ (i.e. $\lambda = 1$), then preferences do not exhibit first-order risk aversion.

Furthermore, for non-monotone preferences, looking at the proof of Theorem 1, we can observe that we can represent any preferences with a $\mathbb{CPE}$ representation (u, λ) with a representation that has the same functional form as $\mathbb{RDU}$ with representation (u, w) but is not in the actual class $\mathbb{RDU}$, because w is not monotone. Moreover, w is still convex (concave) if and only if $\lambda > (<)1$. The proof of Proposition 4 of Segal and Spivak (1990) depends not on the monotonicity of w but rather just the convexity or concavity. Thus, if $\lambda > 2$ preferences are first-order risk-averse, and if $\lambda < 0$ preferences are first-order risk loving. ■

Proposition 10 *If a preference has a* $\mathbb{GCPE}$ *representation, then, for any* Δ_3, *(i) indifference curves are ellipses and (ii) preferences are mixture-averse (loving) if and only if* $\lambda \geq 1$ $(\lambda \leq 1)$.

Proof: Consider Δ_3, assume without loss of generality three distinct outcomes. Normalize the value of the high outcome to 1 and of the low outcome to 0 (due to the uniqueness results regarding quadratic functionals in Chew, Epstein and Segal, 1991). Denote the utility of middle outcome as $m \in (0, 1)$. The value of a lottery f that assigns weight p to the low outcome and q to the high outcome is:

$$(1-p-q)m + q + (1-\lambda)pq\nu(1) + (1-\lambda)p(1-p-q)\nu(m) + (1-\lambda)q(1-p-q)\nu(1-m)$$

Recall that the canonical form of indifference curves in Δ_3 taking on the shape of conic sections is:

$$Aq^2 + Bpq + Cp^2 + Dq + Ep + J = 0$$

where A through E and J depend on ϕ.[4]

A conic section is an ellipse if and only if $B^2 - 4AC < 0$. Substituting in, we obtain:

$$B^2 - 4AC \quad = \quad (1-\lambda)^2[(\nu(1) - \nu(m) - \nu(1-m))^2 - 4\nu(1-m)\nu(m)]$$

To proceed, we will normalize $\nu(1) = 1$ (remember $\nu(0) = 0$). Observe that this is possible by simply rescaling the loss aversion coefficient.[5] The sign of this is the same as the sign of H_ν

$$H_\nu \equiv (1 - \nu(m) - \nu(1-m))^2 - 4\nu(1-m)\nu(m)$$

Recall that ν is increasing. Hence, $\nu(m)$ and $\nu(1-m)$ are between 0 and 1. Let ν_L be the linear function with $\nu_L(1) = 1$ and $\nu_L(0) = 0$, then

$$\begin{aligned} H_{\nu_L} &= (1 - m - (1-m))^2 - 4m(1-m) \\ &= -4m(1-m) < 0 \text{ for all } m \in (0,1) \end{aligned}$$

Let ν be a concave function with $\nu(1) = 1$ and $\nu(0) = 0$. Concavity implies $\nu(m) \geq m$ and $\nu(1-m) \geq 1-m$. It is routine to show that $H_\nu < H_{\nu_L} < 0$ for all $m \in (0,1)$.

Because indifference curves are always ellipses, they must be universally mixture loving or mixture-averse. Observe that the utility function is quadratic in p and q, so combined with the fact that the indifference curves form ellipses, there must be a unique maximum or minimum — the center of the ellipses. Preferences are mixture-averse (loving) if and only if the center of the

[4] Notice that in the case of our preferences $E = 2 + A + D - C$. In other words, E is not independent.

[5] To see this, observe that we can do the following normalization. We have the value of a lottery f is $EU_u(f) + \sum_{z \in Z}(1-\lambda)\nu(z)p(z)$, where Z is the set of utility comparisons between outcomes in the support of f in absolute value and $p(z)$ is the joint probability of that comparison occurring (i.e. if $z = |u(x) - u(y)|$ then $p(z) = f(y)f(x)$). Denote $\hat{\lambda} = 1 - \lambda$. Define an $\hat{\nu}$ such that $\hat{\nu}(z) = \frac{\nu(z)}{\nu(1)}$, and $\widetilde{\lambda} = \nu(1)\hat{\lambda}$. Then the value of f can be written equivalently as $EU_u(f) + \sum_{z \in Z} \widetilde{\lambda}\hat{\nu}(z)p(z)$, where $\hat{\nu}(1) = 1$.

ellipses is the minimum (maximum). Checking the second order conditions, we find that

$$\frac{\partial^2 V_{\mathbb{CPE}}}{\partial p^2} = 2(\lambda-1)\nu(m), \quad \frac{\partial^2 V_{\mathbb{CPE}}}{\partial q^2} = 2(\lambda-1)\nu(1-m),$$

and

$$\frac{\partial^2 V_{\mathbb{CPE}}}{\partial p \partial q} = \frac{\partial^2 V_{\mathbb{CPE}}}{\partial q \partial p} = (1-\lambda)(\nu(1)-\nu(m)-\nu(1-m))$$

Then, the center is a minimum if $2(\lambda-1)\nu(m) > 0$ and $4(\lambda-1)^2\nu(1-m)\nu(m) - (1-\lambda)^2(\nu(1) - \nu(m) - \nu(1-m))^2 > 0$. The center is a maximum if $2(\lambda-1)\nu(m) < 0$ and $4(\lambda-1)^2\nu(1-m)\nu(m) - (1-\lambda)^2(\nu(1)-\nu(m)-\nu(1-m))^2 > 0$. The second condition for both cases is the same, and is equivalent to $4\nu(1-m)\nu(m) - (\nu(1)-\nu(m)-\nu(1-m))^2 > 0$, which we showed above must be true (when we showed that indifference curves are always ellipses). Thus, preferences are mixture-averse if and only if $2(\lambda-1)\nu(m) > 0$, or $\lambda - 1 \geq 0$. ■

We next prove a lemma which describes the shape of the indifference curves in Δ_3 for $V_{\mathbb{CPE}}$.

Lemma 1 *Given any Δ_3, with the utility (probability) of the worst outcome denoted 0 (p), the middle outcome m ($1-p-q$) and the best outcome 1 (q), the indifference curves take the shape*

$$(1-m)\left[\frac{2-\lambda}{2(1-\lambda)} - q\right]^2 + m\left[\frac{\lambda}{2(\lambda-1)} - p)\right]^2 = \hat{r}.$$

Proof: The utility of a lottery f, which places weight q on the high outcome, p on the low outcome, and the rest of the weight on the middle outcome is:

$$V_{\mathbb{CPE}}(f) = (1-p-q)m + q + (1-\lambda)pq + (1-\lambda)p(1-p-q)m + (1-\lambda)q(1-p-q)(1-m)$$

We can rewrite the value as

$$V_{\mathbb{CPE}}(f) \quad = \quad (1-p-q)m + q + (1-\lambda)p(1-p)m + (1-\lambda)q(1-q)(1-m)$$

Given utility level c, we can rewrite the equation of an indifference curve as

$$(1-m)[(2-\lambda)q - (1-\lambda)q^2] + m[-\lambda p - (1-\lambda)p^2] = c - m$$

Divide both sides by $-(1-\lambda)$ to get

$$(1-m)\left[q^2 - \frac{2-\lambda}{1-\lambda}q\right] + m\left[p^2 - \frac{\lambda}{\lambda-1}p\right] = -\frac{c-m}{1-\lambda}$$

We now add $(1-m)\frac{(2-\lambda)^2}{4(1-\lambda)^2} + m\frac{\lambda^2}{4(1-\lambda)^2}$ and factor to obtain

$$(1-m)\left[\frac{2-\lambda}{2(1-\lambda)} - q\right]^2 + m\left[\frac{\lambda}{2(\lambda-1)} - p)\right]^2 = \hat{r}$$

where $\hat{r} = -\frac{c-m}{1-\lambda} + (1-m)\frac{(2-\lambda)^2}{4(1-\lambda)^2} + m\frac{\lambda^2}{4(1-\lambda)^2}$. ■

Proposition 1 *Let (u_i, λ_i) be a monotone $\mathbb{CPE}$ representation of Individual i's preference for $i \in \{A, B\}$. Then Individual A is more risk-averse than Individual B if and only if u_A is a concave transformation of u_B and $\lambda_A \geq \lambda_B$.*

Proof: Chew, Karni and Safra (1987) show that the analogous comparative static holds for $\mathbb{RDU}$ if and only if u_A is a concave transformation of u_B and w_A is a convex transformation of w_B. Therefore, we want to consider what values of λ_A and λ_B induce the corresponding w_A to be a convex transformation of the corresponding w_B. Recall that

$$w_\lambda(z) = (\lambda-1)z^2 + (2-\lambda)z.$$

Using the equivalence of Arrow-Pratt measures of risk aversion and convex and concave transformations, we know that $\frac{-w''_\lambda}{w'_\lambda} = \frac{-2(\lambda-1)}{2-\lambda+2(\lambda-1)z}$.

Then

$$\begin{aligned}
\frac{-w''_A}{w'_A} = \frac{-2(\lambda_A-1)}{2-\lambda_A+2(1-\lambda_A)z} \quad &\leq \quad \frac{-2(\lambda_B-1)}{2-\lambda_B+2(1-\lambda_B)z} = \frac{-w''_B}{w'_B} \\
&\Longleftrightarrow \\
2(\lambda_A-1)(2-\lambda_B+2(1-\lambda_B)z) \quad &\geq \quad 2(\lambda_B-1)(2-\lambda_A+2(1-\lambda_A)z) \\
&\Longleftrightarrow \\
(\lambda_A-1)(2-\lambda_B) \quad &\geq \quad (\lambda_B-1)(2-\lambda_A) \\
&\Longleftrightarrow \\
2\lambda_A+\lambda_B \quad &\geq \quad 2\lambda_B+\lambda_A \\
&\Longleftrightarrow \\
\lambda_A \quad &\geq \quad \lambda_B
\end{aligned}$$

■

Proposition 2 *Let* (u_i, λ_i) *be a monotone* $\mathbb{CPE}$ *representation of Individual* i*'s preference for* $i \in \{A, B\}$ *with* u_i *everywhere differentiable. Then Individual* A *is more first-order risk-averse than Individual* B *at all wealth levels if and only if* $\lambda_A \geq \lambda_B$*.*

Proof: Assume $\lambda_A \geq \lambda_B$. Denote the support (in increasing order of value) of the lottery of f as $x_1, x_2, \ldots, x_n$. By Proposition 4 of Segal and Spivak (1990) if preferences are in $\mathbb{RDU}$ then

$$\frac{\partial \pi(w+\epsilon f)}{\partial \epsilon}|_{\epsilon=0^+} = -\sum_i x_i \left[w\left(\sum_{j\geq i} f(x_j)\right) - w\left(\sum_{j>i} f(x_j)\right)\right] = -x_1 - \sum_{i\geq 2}(x_i - x_{i-1}) w\left(\sum_{j\geq i} f(x_j)\right)$$

Recall that $w_\lambda(z) = (2-\lambda)z + (\lambda-1)z^2$. In this case the derivative of

$$-\sum_i x_i \left[w\left(\sum_{j\geq i} f(x_j)\right) - w\left(\sum_{j>i} f(x_j)\right)\right]$$

with respect to λ is

$$-\sum_{i\geq 2}(x_i - x_{i-1}) \sum_{j\geq i} f(x_j) (\sum_{j\geq i} f(x_j) - 1)$$

Observe that $(x_i - x_{i-1})$ is positive but $\sum_{j\geq i} f(x_j)(\sum_{j\geq i} f(x_j) - 1)$ is negative, so the whole term must be positive. Therefore first-order risk aversion is increasing in λ. Since values of λ are just real numbers, this completes the proof. ■

Proposition 3 *Let* (u_i, λ_i) *be a* $\mathbb{CPE}$ *representation of Individual* i*'s preference for* $i \in \{A, B\}$ *with* $1 \leq \lambda_i \leq 2$*. Then* $\lambda_A \leq \lambda_B$ *if and only if Individual* A *has steeper expansion paths than Individual* B *for all* Δ_3*.*

Proof: We will prove this in a series of claims.

Claim 1 $\lambda_A \leq \lambda_B$ *if and only if the center of* A*'s indifference curves is farther from the best outcome than the center of* B*'s indifference curves.*

Proof: Recall that the equations for the indifference curves is

$$(1-m)(\frac{2-\lambda}{2(1-\lambda)} - q)^2 + m(\frac{\lambda}{2(\lambda-1)} - p)^2 = \hat{r}$$

Therefore the center of the indifference curves in (p, q) space is $(\frac{\lambda}{2(\lambda-1)}, \frac{2-\lambda}{2(1-\lambda)})$, and so the claim is true. $\square$

Claim 2 *The center of A's indifference curves is farther from the best outcome than the center of B's indifference curves if and only if A has steeper expansion paths than B.*

This claim is a direct implication of Chew, Epstein and Segal (1991), since all expansion paths must cross only once, and at the center of the ellipses. $\square$

Claim 3 *Suppose the preference $\succsim$ has a $\mathbb{Q}$ representation but cannot be written as $\mathbb{EU}$, respects first-order stochastic dominance and suppose $f \sim f'$ and $g \sim g'$, where all lotteries are in the same Δ_3 and f' and g' are lotteries whose support is only the best and worst outcome. Then there exists $\alpha \in (0, 1)$ such that $\alpha f + (1-\alpha)g \sim \alpha f' + (1-\alpha)g'$ if and only if f and g lie on the same expansion path and f' and g' lie on the same expansion path.*

Proof: O bserve that because the indifference curve of $\succsim$ must be homothetic with respect to a particular point — the center of the elliptical indifference curves. Translate all points in the vector space of lotteries so this center is denoted (0,0). Thus, two points f and f' are indifferent if and only if the mixture of them in proportion α with (0,0) are also indifferent. Moreover, if g is on the same expansion path as f, and g' is on the same expansion path as f', and $g \sim g'$ then there exists a β such that $\alpha f + (1-\alpha)(0,0) = \beta f + (1-\beta)g$ and $\alpha f' + (1-\alpha)(0,0) = \beta f' + (1-\beta)g'$.

To prove the other direction, assume, without loss of generality, that $\alpha f + (1-\alpha)g' \sim \alpha f' + (1-\alpha)g'$ and $f \neq f'$.[6] By construction f is not on the same expansion path as g'. Pick out the point $\hat{g}$ such that $\hat{g} \sim g'$ and f and $\hat{g}$ are on the same expansion path. Because expansion paths are all rays from the center of the ellipses, and preferences are homothetic with respect to the center of the ellipses, the line connecting $\hat{g}$ and g', denoted $[g', \hat{g}]$ must be parallel to the line between f and f', denoted $[f, f']$. Moreover, since 'standard' Independence implies that lines are preserved by mixing, it must be the case that the line between $\alpha f + (1-\alpha)g'$ and $\alpha f' + (1-\alpha)f'$, denoted $[\alpha f + (1-\alpha)g', \alpha f' + (1-\alpha)g']$ is also parallel to $[g', \hat{g}]$. Moreover, by our previous paragraph, $[\alpha f + (1-\alpha)g', \alpha f' + (1-\alpha)g']$ must pass through $\alpha f + (1-\alpha)\hat{g}$.

It is clear that if preferences are not in $\mathbb{EU}$ then they must be strictly mixture-averse. But $\alpha f + (1-\alpha)g'$ is a convex combination of $\alpha f' + (1-\alpha)g'$ and $\alpha f + (1-\alpha)\hat{g}$, and $\alpha f' + (1-\alpha)g' \sim$

[6] To see why this is without loss of generality, pick out an arbitrary g. Then also select an f'' such that $f'' \sim f$ and f'' and g are on the same expansion path. Then we simply relabel g as g' and f'' as f'.

$\alpha f + (1-\alpha)\hat{g} \sim \alpha f + (1-\alpha)g'$. This is a contradiction. $\square$

Claim 4 *If for f' and g we can define an f_i and g_i' such that $f_i \sim_i f'$ and $\beta_i f_i + (1-\beta_i)g \sim \beta_i f' + (1-\beta_i)g_i'$ for all $\beta_i \in (0,1)$ if and only if f_i and g lie on the same expansion path.*

Proof: This is an immediate implication of Claim 3. $\square$

Thus, if preferences are not in $\mathbb{EU}$ the proposition is proven for any Δ_3. If one (or both) of the preferences is in $\mathbb{EU}$, which is immediately behaviorally identifiable, then by definition the steepness of the expansion paths are equal to ∞, and so the proposition also holds for any Δ_3. This proves the proposition for any Δ_3. Clearly, if A is less loss-averse than B then A has steeper expansion paths than B for all Δ_3. Next, assume A has steeper expansion paths than B for a particular Δ_3. Then by construction A must be more loss-averse than B for lotteries over those 3 outcomes, and so for all possible Δ_3. $\blacksquare$

Proposition 4 *Let (u_i, λ_i) be a $\mathbb{CPE}$ representation of Individual i's preference for $i \in \{A, B\}$ with $1 \leq \lambda_i \leq 2$. Then u_A is more concave than u_B if and only if Individual A has steeper budget constraints than Individual B in all Δ_3.*

Proof: Recall that the equation for the indifference curves in Δ_3 is:

$$(1-m)(\frac{2-\lambda}{2(1-\lambda)} - q)^2 + m(\frac{\lambda}{2(\lambda-1)} - p)^2 = \hat{r}$$

Denote the left hand side of the equation $\iota(p, q, m, \lambda)$.

First, we will consider the tangency conditions of the indifference curves along best to worst outcome edge. Taking the partials of ι gives:

$$\frac{\partial \iota}{\partial q} = -2(\frac{2-\lambda}{2(1-\lambda)} - q)(1-m)$$

and

$$\frac{\partial \iota}{\partial p} = -2(\frac{\lambda}{2(\lambda-1)} - p)m$$

Therefore, the tangency condition for any point is simply

$$\frac{-2(\frac{2-\lambda}{2(1-\lambda)} - q)(1-m)}{-2(\frac{\lambda}{2(\lambda-1)} - p)m}$$

Next we restrict ourselves to the best to worst outcome edge, so that $p = 1 - q$. Then the condition becomes $\frac{1-m}{m}$. For both decision-makers fix the value of the worst outcome in Δ_3 at 0 and the best at 1 and assume u_A is a concave transformation of u_B. Then $m_A \leq m_B$. Importantly the tangency condition along that edge does not depend on the center of the circle, only on the foci — which are determined by $\frac{1}{1-m}$ and $\frac{1}{m}$, as shown in the elliptical representation of the indifference curves.

Since this is a quadratic utility functional, by Chew, Epstein and Segal (1991) the indifference curves are tangent to linear budget constraints along the expansion paths, one of which, as we have just shown, is the best to worst outcome edge.

Assume that A is more consumption risk-averse than B. Then m_A is smaller than m_B. As m gets bigger, the tangency slope falls, so A has steeper budget constraints than B.

Now assume that A has steeper budget constraints than B. This implies that m_A must be smaller than m_B. Then that A is more consumption risk-averse than B.

This proves the proposition for any Δ_3. Observe that if A is more risk-averse, then the proof works for any Δ_3. If there exists some Δ_3 where A has steeper budget constraints than B then on that Δ_3 A must be less risk-averse. $\blacksquare$