The (human) sampler's curses

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Online Appendix

Part I

Simultaneous multiple expert consultation

In many contexts, multiple experts are consulted simultaneously rather than sequentially. It is common practice for the editor of an academic journal to simultaneously order reports on a given paper from multiple referees. The Amazon product page features a *reviews aggregator* summarizing grades awarded by anonymous reviewers.

As already argued in the main paper, a key problem is that expert incentives are often unknown. With regards to product reviews, while no doubt many benevolent individuals aim at helping others understand the value of the product, many firms use reviews as an advertizing method. As in our analysis of sequential consultation, we consider the case where experts either share the preferences of the decision maker (R) or wish to maximize the decision maker's action, the preference type of experts being unobservable.

We wish to understand what payoffs can be achieved by R through simultaneous consultation given these unknown sender incentives. This first entails identifying a satisfactory equilibrium prediction. Given that all senders are assumed to know the state perfectly there in principle exists an equilibrium involving full truthtelling. It is however not empirically compelling and lacks theoretical robustness (for example with respect to the presence of residual noise in communication, as we show). Once identified a robust scenario, we wish to establish whether the so-called *sampler's curses* identified in the sequential case apply here. Does R always benefit from a higher M, a higher β and a lower c? Our second main objective is to compare simultaneous and sequential consultation in terms of R's net payoffs, i.e. taking into account both learning and consultation costs.

We identify a robust communication strategy profile that extends the semi-revealing strategy proposed in Morgan and Stocken (2003) for the case of a *unique* sender. In the identified equilibrium, individual informativeness increases in the number of senders consulted and is strictly higher than in the semi-revealing equilibrium with a unique consultation. The trade-off between consultation extensiveness and individual informativeness characterizing the sequential case thus gives way to a complementarity. As a result, we find no counterpart of the *sampler's curses* in this environment. Last, we find (for a simple two experts example) that the optimal consultation protocol (sequential or simultaneous) depends on c and β. For intermediate β, sequential consultation is superior given high enough c. The intuition is as follows. A high enough c implies that avoiding ordering superfluous reports is important. An intermediate β implies that individual (semi-revealing) reports under simultaneous consultation are not radically more informative than individual (partitional) reports under sequential consultation. The result is strengthened if we bound the number of messages used under simultaneous consultation in line with the coarse grading grid in Amazon's reviews aggregator.

The key advantage of simultaneous consultation is that experts' messages do not affect R's consultation behavior because the consultation plan is chosen before messages are observed. Experts thus have no incentive to strategically inflect their messaging so as to influence R's consultation choices. As such incentives negatively affect the achievable individual informativeness, simultaneous consultation offers the prospect of higher informativeness. The downside of the protocol is that it does not optimize costs in contrast to sequential consultation which allows R to decide early if uncertainty is resolved early. An editor ordering three reports might realize after reading the first two that a paper is inadequate for publication, thus making the third report superfluous.

We proceed as follows. Section 1 presents the simultaneous consultation model. Section 2 contains a brief analysis of perfectly revealing and partitional communication and discards these. Section 3 focuses on our main prediction, semi-revealing communication. Section 4 provides a welfare comparison of sequential and simultaneous communication. Section 5 concludes. Proofs are mostly relegated to Section 6. Note that the numbering of items (e.g. Sections, Definitions) is independent of that used in the main paper.

1 The simultaneous consultation model

The basic elements of the model remain the same as in the main paper. R faces M experts (also called senders) and chooses a publicly observed number n of experts that are simultaneously consulted, each consultation costing c. Experts then simultaneously choose a costless message in $[0,1]$. If $n < M$, R observes a randomly picked subset of n messages among the M messages sent, any message having the same ex ante likelihood of being selected.

We focus on strategies featuring no randomization. A given n defines a simultaneous consultation strategy of R denoted by δ_n. A decision rule of R specifies an action for any given observable message profile (including those containing more or less messages than observed in equilibrium). A pure strategy for R is given by a consultation strategy and a decision rule. A pure communication strategy of sender S_i is a mapping $\{U, B\} \times \{0, .., M\} \times [0,1] \to [0,1]$. We restrict ourselves to symmetric equilibria featuring an identical communication strategy for all senders.

We examine Perfect Bayesian equilibria and simply call them equilibria. An equilibrium of the game is given by a communication strategy profile and a receiver strategy such that none of the parties has an advantageous deviation, i.e. 1) R has no incentive to consult either more or less experts than specified by her equilibrium strategy and simply chooses the ex post optimal action after observing messages given beliefs derived by Bayes' rule whenever possible 2) no individual sender has an incentive to deviate from his prescribed communication strategy.

If the message profile received by R is such that Bayes'rule cannot be applied, R assigns probability one to $\omega = 0$. Note that off the equilibrium path, R's beliefs conditional on messages can often be derived by Bayes'rule. To see this, assume that R is supposed to consult n senders in equilibrium and deviates to $n' \neq n$. Her beliefs given the obtained profile of n' messages are still derivable by Bayes'rule as long as the profile is compatible with the communication strategy specified by senders' equilibrium strategy given n'.

2 Preliminary observations

A Fully revealing communication

The first benchmark equilibrium prediction is one in which R uses δ_2 and both senders truthfully reveal the state if R consults at least two senders. If R uses δ_1, all senders babble. If reports are not identical, which is an out of equilibrium event, R attaches probability one to state 0. This equilibrium always exists but it is not robust, in a sense that we now specify.

Consider the following noisy version of our game. With a probability ε, the message sent by a sender is garbled, in which case the message received by R is drawn from a uniform distribution over $[0, 1]$. A *robust* equilibrium strategy profile is one that constitutes an equilibrium of the game with noise for any $\varepsilon \in [0, \varepsilon^*)$, for some $\varepsilon^* > 0$.

Consider now the above introduced fully revealing equilibrium profile in the game with noise. If the state is 0 a biased sender has a strictly advantageous deviation to any message $m' \in (0, 1]$. After such a deviation, the belief triggered by the observed report profile is strictly larger than 0. Indeed, when faced with two non-identical messages, one of which is strictly larger than 0, R does not know whose message was garbled and her posterior thus assigns positive probability to $\omega > 0$.

B Partitional communication

Note that partitional communication as studied in the sequential case is trivially feasible under simultaneous consultation. More precisely, given β and n, the unique N-partitions profile $(t_1(\beta, N, n), .., t_{N-1}(\beta, N, n))$ that is incentive compatible for unbiased senders in our main (sequential) setup given the sequential consultation strategy φ_n is also the unique incentive compatible N-partitions profile in the present game given δ_n. The key is that biased senders no longer have an incentive to deviate to m_{N-1} given that R's consultation decision is not conditional on received messages.

It follows that there is no upper bound on the number of achievable partitions given simultaneous multiple sender consultation. Recall that given a unique consultation, the semi-revealing profile $\theta^*(\beta, 1)$ corresponds to the limit for N tending to infinity of the partitional profile $\{t_r(\beta, N, 1)\}_{r=1}^{N-1}$. We prove in part II (section 6) of this online appendix that the same result holds for $n \geq 2$: The semi-revealing profile $\theta^*(\beta, n)$ corresponds to the limit of the partitional profile $\{t_r(\beta, N, n)\}_{r=1}^{N-1}$ for N tending to infinity. We furthermore know from Lemma 8 in the main paper that given any β, n, the payoff function $V_p(\beta, N, n)$ is increasing in N. In terms of R's gross expected payoff, an equilibrium in which R uses δ_n and senders respond by using $\theta^*(\beta, n)$ thus dominates any equilibrium in which R uses δ_n and senders respond with $\{t_r(\beta, N, n)\}_{r=1}^{N-1}$. Given the nature of expert utilities (see discussion in the main paper), it follows that R's gross payoff ranking of partitional and semi-revealing outcomes for a given δ_n is also a Pareto ranking.

3 Semi-revealing communication

A Exogenous consultation

Assume for now that R is exogenously forced to use δ_n. In this simplified game, a strategy of senders does not need to condition on different possible values of n. A given symmetric semi-revealing strategy profile is summarized by some $\theta \in (0, 1)$. Given threshold θ, an unbiased expert sends $m = \omega$ if $\omega < \theta$ and sends $m = \theta$ if $\omega \geq \theta$ while a biased expert always sends $m = \theta$. This communication strategy profile is a generalization to the case of multiple senders of a strategy first proposed in Morgan and Stocken (2003). We now define conditions under which a given semi-revealing profile θ is incentive compatible for experts given δ_n. Recall the following expression introduced in the analysis of semi-revealing communication in the context of sequential consultation:

$$\theta^*(\beta, n) = \frac{1}{\sqrt{(1-\beta)^n + 1}}. \tag{1}$$

$\theta^*(\beta, n)$ is the unique value of θ for which the expected value of ω conditional on the homogeneous message profile $(m_1 = \theta, .., m_n = \theta)$ is equal to θ, assuming that senders use the semi-revealing profile θ.

Lemma A.1 *Assume that R uses δ_n. The semi-revealing strategy θ is incentive compatible for both biased and unbiased senders if and only if $\theta = \theta^*(\beta, n)$.*

The (omitted) proof of the above result to a large extent builds on arguments appearing in the proof of Point a) of Lemma 1 in the main paper. The key here is that a biased sender does not affect the number of senders consulted through his message. In the sequential model, assuming an equilibrium featuring the semi-revealing profile $\theta^*(\beta, n)$, given ω small a biased sender has an incentive to deviate to $\theta^*(\beta, n) - \varepsilon$ in order to preempt further consultation that might lead to an encounter with an unbiased sender. This deviation incentive now disappears. Note that in the above equilibrium, the message $m = \theta^*(\beta, n)$ maximizes the expected belief of R for any value of ω.

We add some remarks on the comparative statics of $\theta^*(\beta, n)$ with respect to n and β. Inspection of the formula for $\theta^*(\beta, n)$ reveals two simple properties. First, it is increasing in n and tends to 1 for n tending to ∞. Second, it is continuous and increasing in β and tends to 1 for β tending to 1. The intuition behind the fact that $\theta^*(\beta, n)$ increases in n is as follows. In the considered equilibrium, though the profile of messages $(m_1 = \theta^*(\beta, n), .., m_n = \theta^*(\beta, n))$ leaves R uncertain about to the state because all messages may have been emitted by biased senders, the larger n the higher the probability that at least one of the messages was sent by an unbiased sender. It follows that the larger n, the more R updates her beliefs upwards by attaching an increasing probability to the state being located in $[\theta^*(\beta, n), 1]$. A similar intuition explains the effect of an increase in β.

Note that in contrast to the fully revealing equilibrium scenario, the above identified semi-revealing equilibrium scenario is furthermore robust according to the definition introduced above. Suppose that R is known to exogenously follow δ_n and consider the noisy version of the game. There is an $\varepsilon^* > 0$ such that the semi-revealing profile $\theta^*(\beta, n)$ is incentive compatible for senders for $\varepsilon \in [0, \varepsilon^*)$. We omit a formal proof of the result. The key intuition is that in the semi-revealing scenario $\theta^*(\beta, n)$, a biased sender cannot increase the expected beliefs of R by deviating from $m = \theta^*(\beta, n)$.

Denote by $V_{sr}(\beta, n)$ the gross expected payoff of R given that she uses δ_n for sure while experts use $\theta^*(\beta, n)$ for sure. We have

$$V_{sr}(\beta, n) = (1-\beta)^n \int_0^{\theta^*(\beta,n)} f(\omega)u(\theta^*(\beta, n), \omega)d\omega + \int_{\theta^*(\beta,n)}^1 f(\omega)u(\theta^*(\beta, n), \omega)d\omega.$$

B Endogenous consultation and welfare

We now consider the game with endogenous consultation. In a first stage of the game, R chooses how many senders to consult. Senders observe this decision and simultaneously pick a message. A communication strategy for any given sender now defines a mapping from the state to $[0, 1]$ for every possible number of senders consulted n in $\{1, .., M\}$. We consider a symmetric communication strategy profile that simply extends the semi-revealing profile introduced above.

Definition D.1 *An extended semi-revealing profile is given by* $\{\theta_r\}_{r=1}^M$*. For any* δ_n *in* $\{\delta_n, .., \delta_M\}$ *chosen by R, it specifies that experts apply the semi-revealing profile* θ_r*.*

We may now state the following necessary and sufficient conditions.

Lemma A.2 *The profile* $\left(\delta_n, \{\theta^*(\beta, r)\}_{r=1}^M\right)$ *constitutes an equilibrium of the game with endogenous consultation if and only if*

$$V_{sr}(\beta, n) - nc \geq V_{sr}(\beta, n') - n'c, \forall n' \neq n. \tag{2}$$

The above condition ensures that given experts using $\{\theta^*(\beta, r)\}_{r=1}^M$, R has no profitable deviation to some $\delta_{n'} \neq \delta_n$. Given (2), a key question is whether the marginal value of a consultation is always positive assuming that experts use $\{\theta^*(\beta, r)\}_{r=1}^M$, i.e. whether for any $n \in \{0, .., M-1\}$,

$$V_{sr}(\beta, n+1) - V_{sr}(\beta, n) \geq 0.$$

We know from for example Kawamura (2013) that there are simultaneous consultation games in which consulting more senders decreases the informativeness of individual reporting to the extent that the gross expected payoff of R at some point starts decreasing in the number of senders. The following lemma shows that the comparative statics of $V_{sr}(\beta, n)$ are very simple.

Lemma A.3 *a)* $V_{sr}(\beta, n)$ *is increasing in n and it is continuous and increasing in* β*.*

b) $V_{sr}(\beta, n+1) - V_{sr}(\beta, n)$ *is decreasing in n.*

Point a) proves that the gross marginal value of a consultation is always positive. Note that for any given δ_n, R always benefits from an increase in the threshold θ characterizing senders' semi-revealing strategy profile. We may immediately conclude that $V_{sr}(\beta, n)$ is increasing in n given that an increase in n affects $V_{sr}(\beta, n)$ through two positive channels, by leading to an increase in the semi-revealing threshold θ for a given n and by increasing the number n of reports gathered for a given threshold θ. Point b) states that the marginal value of information is decreasing. Our next lemma draws the immediate consequence of the above for the equilibrium characterization. Let Π_0 denote R's expected payoff in the absence of a consultation.

Lemma A.4 *Assume that $V_{sr}(\beta, 1) - \Pi_0 \geq c$. There exists a unique equilibrium featuring a pure simultaneous consultation strategy and an extended semi-revealing profile. Denote it by $\left(\delta_{n^*}, \{\theta^*(\beta, r)\}_{r=1}^{M}\right)$.*

a) If $c \geq V_{sr}(\beta, M+1) - V_{sr}(\beta, M)$, n^* is the unique integer in $\{1, .., M\}$ such that

$$V_{sr}(\beta, n^*) - V_{sr}(\beta, n^* - 1) \geq c \geq V_{sr}(\beta, n^* + 1) - V_{sr}(\beta, n^*). \tag{3}$$

b) If $c < V_{sr}(\beta, M+1) - V_{sr}(\beta, M)$, $n^* = M$.

The proof of the above lemma is immediate. We know that $\theta^*(\beta, n)$ is the unique incentive compatible symmetric semi-revealing strategy profile given any δ_n. Given that $V_{sr}(\beta, n)$ is increasing and concave in n, the maximizer n^* is unique and satisfies marginal condition (3) if c is high enough that R would not consult strictly more than M senders if more than M senders were available. We denote n^* by $n^*(\beta, M, c)$ in what follows to stress its dependence on exogenous parameters. We conclude with a characterization of the comparative statics of R's equilibrium welfare with respect to key exogenous variables.

Lemma A.5 ***No sampler's curses****.* $V_{sr}(\beta, n^*(\beta, M, c))$: *a) strictly increases in β, b) weakly increases in M and c) strictly decreases in c.*

The comparative statics of the simultaneous consultation model thus exhibit no counterpart of the *sampler's curses* identified in the sequential case. R benefits from an increase in average expert trustworthiness or in the number of available experts as well as from a decrease in consultation costs.

4 Comparing simultaneous and sequential consultation

R often can choose whether to consult simultaneously or sequentially, the consultation protocol being publicly known. An academic journal for example chooses a refereeing protocol which is then announced on its webpage and presumably known to referees. Comparing these two protocols from a welfare perspective thus appears worthwhile. For the case of two experts, we now numerically compare maximal payoffs obtainable under each. Recall that for a given communication strategy profile and consultation strategy, we call the *gross expected payoff* of R her expected payoff ignoring consultation costs. We call *net expected payoff* of R her expected payoff after substracting consultation costs. Recall also that $V_p(\beta, N, n)$ is the gross expected payoff of R conditional on using the sequential consultation strategy φ_n and senders using $\{t_r(\beta, N, n)\}_{r=1}^{N-1}$.

A Gross payoffs under each protocol

For any given β, we generically denote by $V_1(\beta)$ the gross expected payoff of R under simultaneous consultation in order to allow for different communication strategies under this protocol. While our primary focus is the case of $V_1(\beta) = V_{sr}(\beta, 2)$ corresponding to semi-revealing communication, we shall also consider the possibility that the number of equilibrium messages is exogenously bounded due to the use of a finite grading scale. In this case, V_1 becomes $V_p(\beta, N, 2)$, where N is the exogenous bound on the number of different messages used. Note that given $\{t_r(\beta, N, 2)\}_{r=1}^{N-1}$, R obtains the same gross expected payoff $V_p(\beta, N, 2)$ whether she uses the simultaneous consultation strategy δ_2 or the sequential consultation strategy φ_2. We exogenously assume that R consults 2 experts under simultaneous consultation, thus not allowing her to consult only one expert. We comment on this aspect in the analysis of our numerical examples below.

For any given β, the gross payoff attached to sequential consultation is $V_p(\beta, \overline{N}, 2)$. Variable $\overline{N}$ is shorthand for $\overline{N}(\beta, 2)$ which is the maximal achievable partitions number given β and φ_2. In the equilibrium featuring $\{t_r(\beta, N, 2)\}_{r=1}^{N-1}$ and φ_2, let $v_p^1(\beta, N)$ denote the increase in gross expected payoff achievable through the first consultation under the assumption that no subsequent consultation is done. Clearly, if consulting again is sometimes advantageous, the true value of the first consultation is weakly larger than $v_p^1(\beta, N)$. In the same equilibrium, let $v_p^2(\beta, N)$ denote the gross value of a second consultation conditional on having received m_N in the first consultation. For each β, we shall consider values of c satisfying $c \leq \min\left\{v_p^1(\beta, \overline{N}), v_p^2(\beta, \overline{N})\right\}$, i.e. such that given $\left\{t_r(\beta, \overline{N}, 2)\right\}_{r=1}^{\overline{N}-1}$ the sequential consultation strategy φ_2 is indeed incentive compatible. The last two subsections of the Proofs section provides formulas for $v_p^1(\beta, N)$ and $v_p^1(\beta, N)$ for $N = 2$ and $N = 3$, which are the two values of N that are relevant to our numerical analysis.

B Net payoffs under each protocol

To obtain the net expected payoff of R under each of the two protocols, we substract the expected consultation cost from the gross payoff. The sequential consultation strategy φ_2, given the partitional profile $\{t_r(\beta, N, 2)\}_{r=1}^{N-1}$, specifies that R consults a second time if and only if she receives message m_N in the first round, which occurs with ex ante probability

$$t_{N-1}(\beta, N, 2)(1-\beta) + 1 - t_{N-1}(\beta, N, 2).$$

The first element $t_{N-1}(.)(1-\beta)$ corresponds to the probability that $\omega < t_{N-1}(.)$ and the first consulted expert is biased. The second element corresponds to the probability that $\omega \geq t_{N-1}(.)$. Under simultaneous consultation, we simply substract $2c$ from R's gross payoff $V_1(\beta)$.

C A numerical comparison of net payoffs

In order to compare the two protocols, we thus examine the inequality

$$V_1(\beta) - 2c \leq V_p(\beta, \overline{N}, 2) - \left[1 + t_{\overline{N}-1}\left(\beta, \overline{N}, 2\right)(1-\beta) + 1 - t_{\overline{N}-1}\left(\beta, \overline{N}, 2\right)\right] c, \tag{4}$$

which simplifies to

$$V_1(\beta) - V_p(\beta, \overline{N}, 2) \leq \beta t_{\overline{N}-1}\left(\beta, \overline{N}, 2\right) c.$$

In Figures 1 and 2 below, dashed curves correspond to $V_1(\beta) - V_p(\beta, \overline{N}, 2)$ while solid curves correspond to $\beta t_{\overline{N}-1}\left(\beta, \overline{N}, 2\right) c$.

Figure 1 considers $\beta \in \left(\frac{1}{2}, \frac{2}{3}\right]$, implying $\overline{N}(\beta, 2) = 3$. Different dashed curves correspond to different instances of V_1, which we set equal to respectively $V_p(\beta, 4, 2)$, $V_p(\beta, 5, 2)$, $V_p(\beta, 6, 2)$ and $V_{sr}(\beta, 2)$. Recall that for any β,

$$V_p(\beta, 4, 2) < V_p(\beta, 5, 2) < V_p(\beta, 6, 2) < V_{sr}(\beta, 2).$$

It can be shown that for c, β s.t. $c \leq .02$ and $\beta \in \left(\frac{1}{2}, \frac{2}{3}\right]$, the consultation strategy φ_2 is incentive compatible given $\{t_r(\beta, 3, 2)\}_{r=1}^{2}$ while the simultaneous consultation strategy δ_2 is incentive compatible given the extended semi-revealing profile $\{\theta^*(\beta, r)\}_{r=1}^{2}$. The solid curves correspond to different values of c given by respectively .005, .01, .015, .02. The figure reveals that for $c \geq .015$, sequential consultation always dominates simultaneous consultation even if we assume the semi-revealing outcome under simultaneous consultation. The intuition is as follows. A relatively high c implies that avoiding ordering superfluous reports is important. An intermediate β implies that individual (semi-revealing) reports under simultaneous consultation are not radically more informative than individual (partitional) reports under sequential consultation.

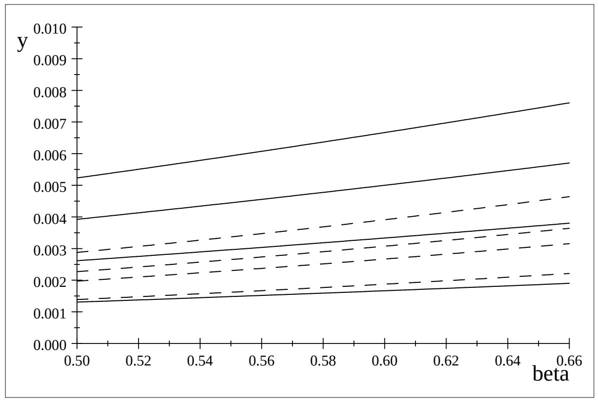

Figure 1.

Figure 2 considers the case of $\beta \in \left(\frac{2}{3}, 1\right)$, implying $\overline{N}(\beta, 2) = 2$. Different dashed curves correspond to different instances of V_1, which we set equal to respectively $V_p(\beta, 3, 2)$, $V_p(\beta, 4, 2)$, $V_{sr}(\beta, 5, 2)$ and $V_{sr}(\beta, 2)$. Conditional on experts using the two-partitions profile $t_1(\beta, 2, 2)$, it can be shown that $v_p^1(\beta, 2) > v_p^2(\beta, 2)$, i.e. the marginal value of the second consultation is always smaller than that of the first. Instead of assuming a constant value of c across βs as in Figure A, we set $c(\beta) = v_p^2(\beta, 2)$ so that the solid curve corresponds to $t_{N-1}(\beta, 2, 2)\,\beta v_p^2(\beta, 2)$. An important caveat is that given this cost, the simultaneous strategy δ_2 is typically not incentive compatible given the extended semi-revealing profile $\{\theta^*(\beta, r)\}_{r=1}^2$: R would consult only one sender given this profile. The figure reveals that given the assumed cost for any β (i.e. $c(\beta)$), sequential consultation dominates simultaneous consultation for $\beta \leq .8$ if we allow for no more than three partitions under simultaneous consultation. Given more partitions under simultaneous consultation, the latter dominates sequential consultation for all $\beta \in \left(\frac{2}{3}, 1\right)$.

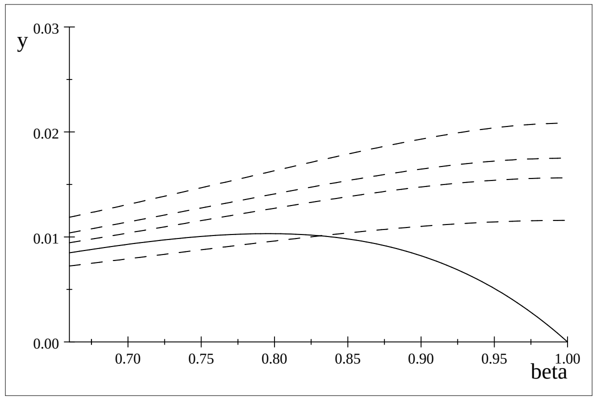

Figure 2.

5 Conclusion

We find that simultaneous consultation allows for more informative communication than sequential consultation because experts have no incentive to inflect their messaging so as to affect R's consultation choice. As a consequence, the trade-offs underlying the sampler's curses in the sequential case break down and no counterparts of the sampler's curses arise. Numerical analysis shows that sequential consultation may nonetheless dominate simultaneous consultation because the former gives R the beneficial option to terminate consultation early if uncertainty is resolved after few reports. As a last remark, note that there also exist hybrid consultation protocols. A classical refereeing process is to first simultaneously order two reports and consult a third referee only in case the first two disagree, in order to resolve the tie. Whether such a mixed protocol outperforms each of the two pure protocols remains to be clarified.

6 Proofs

A Proof of Point a) of Lemma A.3

This appears in the analysis of $V_{sr}(\beta, n)$ given in part II of the online appendix. ■

B Proof of Point b) of Lemma A.3

Step 1 Steps 1-5 develop an explicit expression for $\frac{\partial V_{sr}(\beta,n)}{\partial n}$, which allows us to obtain a simple expression for $V_{sr}(\beta, n+1) - V_{sr}(\beta, n)$. Remaining steps use this to show that $V_{sr}(\beta, n+1) - V_{sr}(\beta, n)$ is decreasing in n. In what follows, we abusively replace $\theta^*(\beta, n)$ by $\theta^*(n)$ for the sake of notational simplicity. We sometimes simply write θ^* when no ambiguity arises. Recall that:

$$V_{sr}(\beta, n) = 1 - (1-\beta)^n \int_0^{\theta^*(n)} f(\omega)(\theta^*(n)-\omega)^2 d\omega - \int_{\theta^*(n)}^1 f(\omega)(\theta^*(n)-\omega)^2 d\omega. \tag{5}$$

Step 2

$$\begin{aligned}
\frac{\partial V_{sr}(\beta,n)}{\partial n} &= -\frac{\partial\left((1-\beta)^n\right)}{\partial n}\left(\int_0^{\theta^*(n)} f(\omega)(\theta^*(n)-\omega)^2 d\omega\right) \\
&\quad - (1-\beta)^n \frac{\partial\left(\int_0^{\theta^*(n)} f(\omega)(\theta^*(n)-\omega)^2 d\omega\right)}{\partial n} \\
&\quad - \frac{\partial\left(\int_{\theta^*(n)}^1 f(\omega)(\theta^*(n)-\omega)^2 d\omega\right)}{\partial n} \\
&= -\frac{\partial\left((1-\beta)^n\right)}{\partial n}\left(\int_0^{\theta^*(n)} f(\omega)(\theta^*(n)-\omega)^2 d\omega\right) \\
&\quad - (1-\beta)^n \int_0^{\theta^*(n)} \frac{\partial(f(\omega)(\theta^*(n)-\omega)^2)}{\partial n} d\omega \\
&\quad - \int_{\theta^*(n)}^1 \frac{\partial\left(f(\omega)(\theta^*(n)-\omega)^2\right)}{\partial n} d\omega.
\end{aligned}$$

Indeed, applying the Leibniz rule,

$$\begin{aligned}
\frac{\partial\left(\int_0^{\theta^*(n)} f(\omega)(\theta^*(n)-\omega)^2 d\omega\right)}{\partial n} &= f(\theta^*(n))(\theta^*(n)-\theta^*(n))^2 \frac{\partial\theta^*(n)}{\partial n} - f(0)(\theta^*(n)-0)^2 \frac{\partial\,(0)}{\partial n} \\
&\quad + \int_0^{\theta^*(n)} \frac{\partial(f(\omega)(\theta^*(n)-\omega)^2)}{\partial n} d\omega \\
&= \int_0^{\theta^*(n)} \frac{\partial(f(\omega)(\theta^*(n)-\omega)^2)}{\partial n} d\omega.
\end{aligned}$$

Similarly,

$$\begin{aligned}
\frac{\partial\left(\int_{\theta^*(n)}^1 f(\theta)(\theta^*(n)-\omega)^2 d\omega\right)}{\partial n} &= f(1)(\theta^*(n)-1)^2 \frac{\partial\,(1)}{\partial n} - f(\theta^*(n))(\theta^*(n)-\theta^*(n))^2 \frac{\partial\left(\theta^*(n)\right)}{\partial n} \\
&\quad + \int_{\theta^*(n)}^1 \frac{\partial\left(f(\omega)(\theta^*(n)-\omega)^2\right)}{\partial n} d\omega.
\end{aligned}$$

Note furthermore that

$$\begin{aligned} & \int_0^{\theta^*(n)} \frac{\partial (f(\omega)(\theta^*(n)-\omega)^2)}{\partial n} d\omega \\ = \;& 2\frac{\partial\left(\theta^*(n)\right)}{\partial n} \int_0^{\theta^*(n)} f(\omega)\left(\theta^*(n)-\omega\right) d\omega \\ = \;& 2\frac{\partial\left(\theta^*(n)\right)}{\partial n} F(\theta^*(n))\left(\theta^*(n) - E\left(\omega \,|\omega \le \theta^*(n)\right)\right) \end{aligned}$$

and

$$\begin{aligned} & \int_{\theta^*(n)}^1 \frac{\partial\left(f(\omega)(\theta^*(n)-\omega)^2\right)}{\partial n} d\omega \\ = \;& 2\frac{\partial\left(\theta^*(n)\right)}{\partial n} \int_{\theta^*(n)}^1 f(\omega)\left(\theta^*(n)-\omega\right) d\omega \\ = \;& 2\frac{\partial\left(\theta^*(n)\right)}{\partial n} (1-F(\theta^*(n)))\left(\theta^*(n) - E\left(\omega \,|\omega > \theta^*(n)\right)\right). \end{aligned}$$

Step 3 It follows that

$$\begin{aligned} & -(1-\beta)^n \int_0^{\theta^*(n)} \frac{\partial (f(\theta)(\theta^*(n)-\omega)^2)}{\partial n} d\omega - \int_{\theta^*(n)}^1 \frac{\partial\left(f(\theta)(\theta^*(n)-\omega)^2\right)}{\partial n} d\omega \\ = \;& -2\frac{\partial\theta^*(n)}{\partial n}\left[(1-\beta)^n F(\theta^*)\left(\theta^* - E(\omega\,|\omega \le \theta^*)\right) + (1-F(\theta^*)\left(\theta^* - E\left(\omega\,|\omega > \theta^*\right)\right)\right]. \end{aligned} \tag{6}$$

Now, note that using the fact that

$$E(\theta) = F(\theta^*(n))E\left(\omega\,|\omega \le \theta^*(n)\right) + (1-F(\theta^*(n))E\left(\omega\,|\omega \ge \theta^*(n)\right)),$$

the equilibrium equality $B(\theta^*(n),\beta,n) = \theta^*(n)$ implies that (6) is equal to 0. Indeed, $B(\theta^*(n),\beta,n) = \theta^*(n)$ can be rearranged to obtain

$$\begin{aligned} & \theta^*(n)\left(\left(1-(1-\beta)^n\right)(1-F(\theta^*(n))) + (1-\beta)^n\right) \\ = \;& \left(1-(1-\beta)^n\right)(1-F(\theta^*(n)))\,E\left(\omega\,|\omega \ge \theta^*(n)\right) \\ & + (1-\beta)^n F(\theta^*(n))E\left(\omega\,|\omega \le \theta^*(n)\right) \\ & + (1-\beta)^n (1-F(\theta^*(n)))E\left(\omega\,|\omega \le \theta^*(n)\right). \end{aligned}$$

Simplifying both sides of the equality, this is equivalent to

$$\begin{aligned} & \theta^*(n)\left[1 - F(\theta^*(n) + (1-\beta)^n F(\theta^*(n))\right] \\ = \;& E\left(\omega\,|\omega \ge \theta^*(n)\right)(1-F(\theta^*(n))) + (1-\beta)^n F(\theta^*(n))E\left(\omega\,|\omega \le \theta^*(n)\right) \end{aligned}$$

which is equivalent to

$$(1-\beta)^n F(\theta^*(n)) \left[\theta^*(n) - E\left(\omega \left|\omega \le \theta^*(n)\right.\right)\right] + (1 - F(\theta^*(n)) \left[\theta^*(n) - E\left(\omega \left|\omega \ge \theta^*(n)\right.\right)\right] = 0$$

which is in turn equivalent to

$$(1-\beta)^n F(\theta^*(n)) \left[\theta^*(n) - E\left(\omega \left|\omega \le \theta^*(n)\right.\right)\right] = (1 - F(\theta^*(n)) \left[E\left(\omega \left|\omega \ge \theta^*(n)\right.\right) - \theta^*(n)\right].$$

Step 4 It follows from the above arguments that

$$\frac{\partial V_{sr}(\beta, n)}{\partial n} = \frac{\partial (1-\beta)^n}{\partial n} \left(\int_0^{\theta^*(n)} f(\omega)(\theta^*(n) - \omega)^2 d\omega \right) > 0.$$

Step 5 Clearly,

$$V_{sr}(\beta, n+1) - V_{sr}(\beta, n) = \int_n^{n+1} \frac{\partial V_{sr}(\beta, n)}{\partial n} dn.$$

Step 6 Assuming a uniform distribution, we thus obtain

$$\frac{\partial V_{sr}(\beta, n)}{\partial n} = \frac{1}{3} \left(\ln(1-\beta)\right) \frac{(1-\beta)^n}{\left(\sqrt{(1-\beta)^n} + 1\right)^3} > 0$$

and

$$\frac{\partial^2 V_{sr}(\beta, n)}{\partial^2 n} = -\frac{1}{6} \left(\ln^2(1-\beta)\right) \frac{\sqrt{(1-\beta)^n}}{(1-\beta)^n} \frac{(1-\beta)^{2n} - 2\sqrt{(1-\beta)^n}\,(1-\beta)^n}{\left(\sqrt{(1-\beta)^n} + 1\right)^4} < 0.$$

Note that it follows from the sign of $\frac{\partial^2 V_{sr}(\beta,n)}{\partial^2 n}$ that

$$\int_n^{n+1} \frac{\partial V_{sr}(\beta, x)}{\partial x} dx < \int_{n-1}^{n} \frac{\partial V_{sr}(\beta, x)}{\partial x} dx$$

which implies

$$V_{sr}(\beta, n+1) - V_{sr}(\beta, n) < V_{sr}(\beta, n) - V_{sr}(\beta, n-1).$$

■

C Proof of Lemma A.5

Step 1 This proves Point a). We know that $\frac{\partial V_{sr}(\beta,n)}{\partial \beta} > 0$. We thus know that for any n, $V_{sr}(\beta',n) > V_{sr}(\beta,n)$ if $\beta' > \beta$. Now, it follows immediately that

$$V_{sr}(\beta', n^*\left(\beta', M, c\right)) > V_{sr}(\beta', n^*\left(\beta, M, c\right)) > V_{sr}(\beta, n^*\left(\beta, M, c\right)).$$

Step 2 Point b) follows from the fact that adding more senders by going from M to $M+1$ simply adds an accessible net expected payoff for R given by $V_{sr}(\beta_1, M+1) - (M+1)\,c$. Point c) is trivial. ■

D Two-partitions equilibrium

Step 1 Consider the putative equilibrium ς_2 featuring the two partitions strategy $t_1(\beta,2,2)$ and φ_2. In what follows, we show how to calculate $v_p^1(\beta,2)$ and $v_p^2(\beta,2)$. In the putative equilibrium ς_2, we denote by $\widehat{\mu}_H$ the belief of R after the first consultation if she received m_2. We denote by $\widehat{\mu}_{HH}$ her belief after receiving m_2 in the first and the second consultation. Note that

$$\widehat{\mu}_H = \frac{(1-(1-\beta))\,(1-t_1(\beta,2,2))\left(\frac{1+t_1(\beta,2,2)}{2}\right) + (1-\beta)\,\frac{1}{2}}{(1-(1-\beta))\,(1-t_1(\beta,2,2)) + (1-\beta)},$$

$$\begin{aligned} \widehat{\mu}_{HH} &= B(t_1(\beta,2,2),\beta,2) \\ &= t_1(\beta,2,2)\left(\frac{1+2}{2}\right). \end{aligned}$$

Step 2 A lower bound on the gross marginal value of the first consultation is:

$$\begin{aligned} v_p^1(\beta,2) &= 1-(1-\beta)\int_0^{t_1(\beta,2,2)} (\widehat{\mu}_H-\omega)^2\,d\omega - \beta\int_0^{t_1(\beta,2,2)}\left(\frac{t_1(\beta,2,2)}{2}-\omega\right)^2 d\omega \\ &\quad -\int_{t_1(\beta,2,2)}^1 (\widehat{\mu}_H-\omega)^2\,d\omega - \left(1-\int_0^1\left(\frac{1}{2}-\omega\right)^2\right). \end{aligned}$$

Step 3 Assume that R has observed m_2 in the first consultation. At this stage, her gross expected payoff conditional on a second consultation is:

$$\widehat{\Pi}_2 = \begin{array}{c} 1-\frac{(1-\beta)}{1+(1-\beta)}\,(1-\beta)\,\frac{1}{t_1(\beta,2,2)}\int_0^{t_1(\beta,2,2)}\left(\widehat{\mu}_{HH}-\omega\right)^2 d\omega \\ -\frac{(1-\beta)}{1+(1-\beta)}\beta\frac{1}{t_1(\beta,2,2)}\int_0^{t_1(\beta,2,2)}\left(\frac{t_1(\beta,2,2)}{2}-\omega\right)^2 d\omega - \frac{1}{1-t_1(\beta,2,2)}\frac{1}{1+(1-\beta)}\int_{t_1(\beta,2,2)}^1\left(\widehat{\mu}_{HH}-\omega\right)^2 d\omega. \end{array}$$

At this same stage, her gross expected payoff of deciding without a second consultation is:

$$\widehat{\Pi}_1 = 1 - \frac{(1-\beta)}{1+(1-\beta)} \frac{1}{t_1(\beta,2,2)} \int_0^{t_1(\beta,2,2)} (\widehat{\mu}_H - \omega)^2 \, d\omega - \frac{1}{1+(1-\beta)} \frac{1}{1-t_1(\beta,2,2)} \int_{t_1(\beta,2,2)}^1 (\widehat{\mu}_H - \omega)^2 \, d\omega.$$

The gross marginal value of the second consultation is given by $v_p^2(\beta,2) = \widehat{\Pi}_2 - \widehat{\Pi}_1$. ■

E Three-partitions equilibrium

Step 1 Consider the putative equilibrium ς_3 featuring the three partitions strategy $(t_1(\beta,3,2), t_2(\beta,3,2))$ and φ_2. In what follows, we show how to calculate $v_p^1(\beta,3)$ and $v_p^2(\beta,3)$. In the putative equilibrium ς_3, we denote by $\widetilde{\mu}_H$ the belief of R after the first consultation if she received m_3. We denote by $\widetilde{\mu}_{HH}$ her belief after receiving m_3 in the first and the second consultation. Note that

$$\widetilde{\mu}_H = \frac{(1-(1-\beta))(1-t_2(\beta,3,2))\left(\frac{1+t_2(\beta,3,2)}{2}\right) + (1-\beta)\frac{1}{2}}{(1-(1-\beta))(1-t_2(\beta,3,2)) + (1-\beta)},$$

$$\begin{aligned} \widetilde{\mu}_{HH} &= B(t_2(\beta,3,2),\beta,2) \\ &= t_2(\beta,3,2)\left(\frac{1+4}{4}\right). \end{aligned}$$

Step 2 A lower bound on the gross marginal value of the first consultation is:

$$\begin{aligned} v_p^1(\beta,3) &= 1-(1-\beta)\int_0^{t_2(\beta,3,2)} (\widetilde{\mu}_H-\omega)^2 \, d\omega - 2\beta \int_0^{\frac{t_2(\beta,3,2)}{2}} \left(\frac{t_2(\beta,3,2)}{4} - \omega\right)^2 d\omega \\ &\quad - \int_{t_2(\beta,3,2)}^1 (\widetilde{\mu}_H-\omega)^2 \, d\omega - \left(1 - \int_0^1 \left(\frac{1}{2}-\omega\right)^2\right). \end{aligned}$$

Step 3 Assume that R has observed message m_3 in the first consultation. At this stage, her gross expected payoff conditional on a second consultation is:

$$\widetilde{\Pi}_2 = \begin{array}{c} 1-(1-\beta)\frac{(1-\beta)}{1+(1-\beta)}\frac{1}{t_2(\beta,3,2)}\int_0^{t_2(\beta,3,2)} (\widetilde{\mu}_{HH}-\omega)^2 \, d\omega \\ -2\beta\frac{(1-\beta)}{1+(1-\beta)}\frac{1}{t_2(\beta,3,2)}\int_0^{\frac{t_2(\beta,3,2)}{2}} \left(\frac{t_2(\beta,3,2)}{4}-\omega\right)^2 d\omega - \frac{1}{1-t_2(\beta,3,2)}\frac{1}{1+(1-\beta)}\int_{t_2(\beta,3,2)}^1 (\widetilde{\mu}_{HH}-\omega)^2 \, d\omega. \end{array}$$

At this same stage, her gross expected payoff of deciding without a second consultation is:

$$\widetilde{\Pi}_1 = 1 - \frac{(1-\beta)}{1+(1-\beta)} \frac{1}{t_2(\beta,3,2)} \int_0^{t_2(\beta,3,2)} (\widetilde{\mu}_H - \omega)^2 \, d\omega - \frac{1}{1+(1-\beta)} \frac{1}{1-t_2(\beta,3,2)} \int_{t_2(\beta,3,2)}^1 (\widetilde{\mu}_H - \omega)^2 \, d\omega.$$

The gross marginal value of the second consultation is thus given by $v_p^2(\beta,3) = \widetilde{\Pi}_2 - \widetilde{\Pi}_1$. ■

Part II

Proof of Lemmas 6 and 8

The following contains a proof of Lemmas 6 and 8 appearing in the main paper. We analyze the general case of $M \geq 2$ experts. Let $V_{sr}(\beta, n)$ denote R's gross expected payoff given that experts use the semi-revealing strategy $\theta^*(\beta, n)$ and that R simultaneously consults n experts. Let $\{t_r(\beta, N, n)\}_{r=1}^{N-1}$ be the unique N-partitions strategy profile that is incentive compatible for unbiased senders conditional on R using the sequential consultation strategy φ_n. The latter specifies that R stops consulting as soon as she receives a message $m_i \neq m_N$ while she continues consulting for a maximum of n rounds as long as she receives m_N. We denote by $V_p(\beta, N, n)$ the gross expected payoff of R in a scenario involving $\{t_r(\beta, N, n)\}_{r=1}^{N-1}$ and φ_n.

Section 1 analyzes $V_{sr}(\beta, n)$ and thereby proves Point a) of Lemma 6. Sections 2-5 analyze $V_p(\beta, N, n)$. Section 2 obtains a simplified expression for $V_p(\beta, N, n)$. Section 3 shows that $V_p(\beta, N, n)$ is continuous in β. Section 4 shows that it is increasing in β and n. Section 5 shows that it is increasing in N. Sections 2-5 thus prove Point b) of Lemma 6 as well as Lemma .8 Section 6 compares $V_p(\beta, N, n)$ and $V_{sr}(\beta, n)$ and thereby proves Point c) of Lemma 6.

7 Analysis of $V_{sr}(\beta, n)$

Step 1 Assume that R simultaneously samples n experts who use the semi-revealing profile $\theta^*(\beta, n)$. In such a scenario her gross expected payoff is

$$V_{sr}(\beta, n) = 1 - (1-\beta)^n \int_0^{\theta^*(\beta,n)} f(\omega)(\theta^*(\beta, n) - \omega)^2 d\omega + \int_{\theta^*(\beta,n)}^1 f(\omega)(\theta^*(\beta, n) - \omega)^2 d\omega.$$

It is easily shown that

$$V_{sr}(\beta, n) = 1 - \left(((1-\beta)^n - 1)\frac{1}{3}(\theta^*(\beta, n))^3 + (\theta^*(\beta, n))^2 - (\theta^*(\beta, n)) + \frac{1}{3} \right). \tag{7}$$

Step 2 Define

$$\widehat{V}_{sr}(\theta, \beta, n) = 1 - (1-\beta)^n \int_0^{\theta} f(\omega)(B(\theta, \beta, n) - \omega)^2 d\omega - \int_{\theta}^1 f(\omega)(B(\theta, \beta, n) - \omega)^2 d\omega,$$

which is the gross expected payoff of R in a hypothetical outcome in which R simultaneously consults n senders who follow the arbitrary semi-revealing strategy profile θ. Note that $V_{sr}(\beta, n) =$

$\widehat{V}_{sr}(\theta^*(\beta,n),\beta,n)$. It can be shown that:

$$\frac{\partial \widehat{V}_{sr}(\theta,\beta,n)}{\partial \theta} = \frac{1}{4}\frac{(1-\beta)^n - 1}{\left(\theta\,(1-\beta)^n - \theta + 1\right)^2}\left(2\theta + \theta^2\,(1-\beta)^n - \theta^2 - 1\right)^2 > 0.$$

Step 3 We now prove that $V_{sr}(\beta,n)$ is continuous and increasing in β for any $n \geq 1$. Inspection of (7) immediately reveals that $V_{sr}(\beta,n)$ is continuous in β given that $(1-\beta)^n$ and $\theta^*(\beta,n)$ are continuous in β. Note that $\theta^*(\beta,n)$ is increasing in β. Note also that $\widehat{V}_{sr}(\theta,\beta',n) - \widehat{V}_{sr}(\theta,\beta,n) > 0$, $\forall \theta,\beta',\beta,n$ s.t. $\beta' > \beta$. The latter fact trivially follows from the fact that as β increases, for fixed θ and n, R gains access to an experiment that is more informative in the sense of Blackwell. It immediately follows from the above facts that $V_{sr}(\beta,n)$ is increasing in β.

Step 4 We now prove that $V_{sr}(\beta,n)$ is increasing in n. Recall that $\theta^*(\beta,n)$ is increasing in n. Also, $\widehat{V}_{sr}(\theta,\beta,n+1) - \widehat{V}_{sr}(\theta,\beta,n) > 0\ \forall \theta,\beta,n$. This latter fact trivially follows from the fact that individual consultations are i.i.d. signals, for a fixed threshold θ. By increasing n, R gains access to an experiment that is more informative in the sense of Blackwell. It immediately follows from the stated facts that $V_{sr}(\beta,n)$ is increasing in n.■

8 Obtaining a simple expression for $V_p\,(\beta,N,n)$

Step 1 Given the sequential consultation strategy φ_n, the incentive compatibility conditions characterizing the N-partitions profile $\{t_r(\beta,N,n)\}_{r=1}^{N-1}$ imply that

$$\frac{B\,(t_{N-1}(\beta,N,n),\beta,n)}{t_{N-1}(\beta,N,n)} = \frac{2(N-1)+1}{2(N-1)}.$$

It follows that for given n and β,

$$t_{N-1}(\beta,N,n) = -\frac{2\,(N-1) - \sqrt{4\,(N-1)\,(1-\beta)^n + 4\,(N-1)^2\,(1-\beta)^n + 1} + 1}{-2\,(N-1) + 2\,(1-\beta)^n + 2\,(N-1)\,(1-\beta)^n - 2}.$$

Letting $\widetilde{N} = N-1$, we shall denote $t_{N-1}(\beta,N,n)$ as $\widehat{\theta}\left(\beta,\widetilde{N},n\right)$ in what follows. Thus,

$$\widehat{\theta}\left(\beta,\widetilde{N},n\right) = -\frac{2\widetilde{N} - \sqrt{4\widetilde{N}\,(-\beta+1)^n + 4\widetilde{N}^2\,(-\beta+1)^n + 1} + 1}{-2\widetilde{N} + 2\,(-\beta+1)^n + 2\widetilde{N}\,(-\beta+1)^n - 2}.$$

We mostly simply write $\widehat{\theta}$ in what follows for convenience. In this section, we show that

$$\begin{aligned} V_p(\beta, N, n) &= \frac{2}{3} + \widehat{\theta}^3 \frac{1}{3}(1-(1-\beta)^n)\left(\left(\frac{1+2\widetilde{N}}{2\widetilde{N}}\right)^3 - \left(\frac{1}{2\widetilde{N}}\right)^2 - \left(\frac{1}{2\widetilde{N}}\right)^3\right) \\ &\quad - \left(\widehat{\theta}\frac{1+2\widetilde{N}}{2\widetilde{N}}\right)^2 + \widehat{\theta}\frac{1+2\widetilde{N}}{2\widetilde{N}}. \end{aligned}$$

Step 2 Abbreviating $t_i(\beta, N, n)$ by t_i and recalling that receiving n times in a row message m_N gives rise to belief $B\left(\widehat{\theta}, \beta, n\right)$,

$$\begin{aligned} V_p(\beta, N, n) &= \\ & 1-(1-(1-\beta)^n)\sum_{i=1}^{i=N-2}\int_{t_i}^{t_{i+1}}\left(\omega - \frac{t_i+t_{i+1}}{2}\right)^2 d\omega \\ & -(1-\beta)^n \int_0^1 \left(\omega - B\left(\widehat{\theta},\beta,n\right)\right)^2 d\omega \\ & -(1-(1-\beta)^n)\int_{B\left(\widehat{\theta},\beta,n\right)}^1 \left(\omega - B\left(\widehat{\theta},\beta,n\right)\right)^2 d\omega \\ & -(1-(1-\beta)^n)\int_{\widehat{\theta}}^{B\left(\widehat{\theta},\beta,n\right)} \left(\omega - B\left(\widehat{\theta},\beta,n\right)\right)^2 d\omega. \end{aligned}$$

We now use the following simple facts: $\int_b^a \left(x - \frac{a+b}{2}\right)^2 dx = \frac{1}{12}(a-b)^3$; $\int_0^1 (x-\mu)^2 dx = \mu^2 - \mu + \frac{1}{3}$; $\int_\mu^1 (x-\mu)^2 dx = -\frac{1}{3}(\mu-1)^3$; $\int_c^\mu (x-\mu)^2 dx = \frac{1}{3}(\mu-c)^3$. We can thus write

$$\begin{aligned} & V_p(\beta, N, n) = \\ & 1-(1-(1-\beta)^n)\widetilde{N}\frac{1}{12}\left(2\left(B\left(\widehat{\theta},\beta,n\right)-\widehat{\theta}\right)\right)^3 \\ & -(1-\beta)^n\left(B\left(\widehat{\theta},\beta,n\right)^2 - B\left(\widehat{\theta},\beta,n\right) + \frac{1}{3}\right) \\ & -(1-(1-\beta)^n)\frac{1}{3}\left(1-B\left(\widehat{\theta},\beta,n\right)\right)^3 \\ & -(1-(1-\beta)^n)\frac{1}{3}\left(B\left(\widehat{\theta},\beta,n\right)-\widehat{\theta}\right)^3. \end{aligned}$$

Replacing $\widetilde{N}$ by $\frac{\widehat{\theta}}{2\left(B\left(\widehat{\theta},\beta,n\right)-\widehat{\theta}\right)}$, these expressions being equal, we obtain

$$\begin{aligned}
&V_p(\beta, N, n) = \\
&1-(1-(1-\beta)^n)\frac{1}{12}\widehat{\theta}\left(2\left(B\left(\widehat{\theta},\beta,n\right)-\widehat{\theta}\right)\right)^2 \\
&-(1-\beta)^n\left(B\left(\widehat{\theta},\beta,n\right)^2-B\left(\widehat{\theta},\beta,n\right)+\frac{1}{3}\right) \\
&-(1-(1-\beta)^n)\frac{1}{3}\left(1-B\left(\widehat{\theta},\beta,n\right)\right)^3 \\
&-(1-(1-\beta)^n)\frac{1}{3}\left(B\left(\widehat{\theta},\beta,n\right)-\widehat{\theta}\right)^3.
\end{aligned}$$

Now, note that

$$\begin{aligned}
\frac{2\left(B\left(\widehat{\theta},\beta,n\right)-\widehat{\theta}\right)}{\widehat{\theta}} &= \frac{1}{\widetilde{N}} \Leftrightarrow 2\left(B\left(\widehat{\theta},\beta,n\right)-\widehat{\theta}\right)=\frac{\widehat{\theta}}{\widetilde{N}} \Leftrightarrow \\
B\left(\widehat{\theta},\beta,n\right) &= \frac{\widehat{\theta}}{2\widetilde{N}}+\widehat{\theta}=\widehat{\theta}\frac{1+2\widetilde{N}}{2\widetilde{N}} \Leftrightarrow 1-B\left(\widehat{\theta},\beta,n\right)=1-\widehat{\theta}\frac{1+2\widetilde{N}}{2\widetilde{N}}.
\end{aligned}$$

We may thus rewrite

$$\begin{aligned}
&V_p(\beta, N, n) = \\
&1-(1-(1-\beta)^n)\frac{1}{12}\widehat{\theta}\left(\frac{\widehat{\theta}}{\widetilde{N}}\right)^2 \\
&-(1-\beta)^n\left(\left(\widehat{\theta}\frac{2\widetilde{N}+1}{2\widetilde{N}}\right)^2-\left(\widehat{\theta}\frac{2\widetilde{N}+1}{2\widetilde{N}}\right)+\frac{1}{3}\right) \\
&-(1-(1-\beta)^n)\frac{1}{3}\left(1-\widehat{\theta}\frac{1+2\widetilde{N}}{2\widetilde{N}}\right)^3 \\
&-(1-(1-\beta)^n)\frac{1}{3}\left(\frac{\widehat{\theta}}{2\widetilde{N}}\right)^3.
\end{aligned}$$

Using the fact that $\frac{1}{12}\widehat{\theta}\left(\frac{\widehat{\theta}}{N}\right)^2 = \frac{1}{3}\widehat{\theta}\left(\frac{\widehat{\theta}}{2\widetilde{N}}\right)^2$ and the fact that $(1-x)^3 = -x^3+3x^2-3x+1$, we may write

$$\begin{aligned}
&V_p\left(\beta, N, n\right) = \\
&1-(1-(1-\beta)^n)\frac{1}{3}\widehat{\theta}\left(\frac{\widehat{\theta}}{2\widetilde{N}}\right)^2 \\
&-(1-\beta)^n\left(\left(\widehat{\theta}\frac{1+2\widetilde{N}}{2\widetilde{N}}\right)^2-\left(\widehat{\theta}\frac{1+2\widetilde{N}}{2\widetilde{N}}\right)+\frac{1}{3}\right) \\
&-(1-(1-\beta)^n)\frac{1}{3}\left(-\left(\widehat{\theta}\frac{1+2\widetilde{N}}{2\widetilde{N}}\right)^3+3\left(\widehat{\theta}\frac{1+2\widetilde{N}}{2\widetilde{N}}\right)^2-3\widehat{\theta}\frac{1+2\widetilde{N}}{2\widetilde{N}}+1\right) \\
&-(1-(1-\beta)^n)\frac{1}{3}\left(\frac{\widehat{\theta}}{2\widetilde{N}}\right)^3
\end{aligned}$$

which simplifies to

$$V_p\left(\beta, N, n\right) = \frac{2}{3}+\widehat{\theta}^3\frac{1}{3}(1-(1-\beta)^n)\left(\left(\frac{1+2\widetilde{N}}{2\widetilde{N}}\right)^3-\left(\frac{1}{2\widetilde{N}}\right)^2-\left(\frac{1}{2\widetilde{N}}\right)^3\right)-\left(\widehat{\theta}\frac{1+2\widetilde{N}}{2\widetilde{N}}\right)^2+\widehat{\theta}\frac{1+2\widetilde{N}}{2\widetilde{N}}.\blacksquare \tag{8}$$

9 Proof that $V_p\left(\beta, N, n\right)$ is continuous in β

Consider the expression (8) obtained for $V_p\left(\beta, N, n\right)$. Note that both $\widehat{\theta}\left(\beta, \widetilde{N}, n\right)$ and $(1-\beta)^n$ are continuous in β. It follows that $V_p\left(\beta, N, n\right)$ is continuous in β.$\blacksquare$

10 Proof that $V_p\left(\beta, N, n\right)$ is increasing in β and n

Step 1 $V_p\left(\beta, N, n\right)$ can be rewritten as $Q_1\left(\beta, \widetilde{N}, n\right)+Q_2\left(\beta, \widetilde{N}, n\right)$, Q_1 and Q_1 being defined below. We shall show that each of these expressions is increasing in β and n. We have

$$Q_1\left(\beta, \widetilde{N}, n\right) = \frac{2}{3}+\frac{1}{3}\left(\widehat{\theta}\left(\beta, \widetilde{N}, n\right)\frac{1+2\widetilde{N}}{2\widetilde{N}}\right)^3-\left(\widehat{\theta}\left(\beta, \widetilde{N}, n\right)\frac{1+2\widetilde{N}}{2\widetilde{N}}\right)^2+\left(\widehat{\theta}\left(\beta, \widetilde{N}, n\right)\frac{1+2\widetilde{N}}{2\widetilde{N}}\right).$$

Now, note that $\frac{\partial\left(\frac{2}{3}+(x)^3\frac{1}{3}-(x)^2+x\right)}{\partial x} > 0$ and also that $\widehat{\theta}\left(\beta, \widetilde{N}, n\right)$ decreases in $(1-\beta)^n$. It follows that the above expression is increasing in β and n.

Step 2 We denote $(1-\beta)^n$ by X in what follows. We have

$$Q_2\left(\beta,\widetilde{N},n\right)=-\left(\widehat{\theta}\left(\beta,\widetilde{N},n\right)\right)^3\frac{1}{3}X\left(\frac{1+2\widetilde{N}}{2\widetilde{N}}\right)^3+\left(\widehat{\theta}\left(\beta,\widetilde{N},n\right)\right)^3\frac{1}{3}(1-X)\left(-\left(\frac{1}{2\widetilde{N}}\right)^2-\left(\frac{1}{2\widetilde{N}}\right)^3\right).$$

We shall now show that this expression decreases in X for $\widetilde{N}\geq 2$, which is equivalent to showing that the expression is increasing in β and n given that $(1-\beta)^n$ decreases in β and n. Let us first rewrite explicitly the above expression as a function of X and $\widetilde{N}$, recalling that $\widehat{\theta}\left(\beta,\widetilde{N},n\right)$ can also be rewritten as a function of these two variables. We obtain

$$\begin{aligned}G(\widetilde{N},X) &= \left(-\frac{2\widetilde{N}-\sqrt{4\widetilde{N}X+4\widetilde{N}^2X+1}+1}{-2\widetilde{N}+2X+2\widetilde{N}X-2}\right)^3\left(-\frac{1}{3}X\left(\frac{1+2\widetilde{N}}{2\widetilde{N}}\right)^3+\frac{1}{3}(1-X)\left(-\left(\frac{1}{2\widetilde{N}}\right)^2-\left(\frac{1}{2\widetilde{N}}\right)^3\right)\right)\\ &= \frac{1}{192\widetilde{N}^3}\frac{2\widetilde{N}+1}{\left(\widetilde{N}+1\right)^3(X-1)^3}\left(2\widetilde{N}-\sqrt{4X\widetilde{N}^2+4X\widetilde{N}+1}+1\right)^3\left(4X\widetilde{N}^2+4X\widetilde{N}+1\right).\end{aligned}$$

Now, note that

$$\begin{aligned}\frac{\partial G(\widetilde{N},X)}{\partial X} &= -\frac{1}{(X-1)^4}\frac{1}{192\widetilde{N}^3}\frac{2\widetilde{N}+1}{\left(\widetilde{N}+1\right)^3}\left(2\widetilde{N}-\sqrt{4X\widetilde{N}^2+4X\widetilde{N}+1}+1\right)^2\\ &\quad\times\left(\begin{array}{c}10\widetilde{N}-10\widetilde{N}\sqrt{4X\widetilde{N}^2+4X\widetilde{N}+1}+24\widetilde{N}^2X+16\widetilde{N}^3X\\ -10\widetilde{N}^2\sqrt{4X\widetilde{N}^2+4X\widetilde{N}+1}-3\left(4X\widetilde{N}^2+4X\widetilde{N}+1\right)^{\frac{3}{2}}+8\widetilde{N}X+12\widetilde{N}^2\\ +8\widetilde{N}^3+10\widetilde{N}X\sqrt{4X\widetilde{N}^2+4X\widetilde{N}+1}+10\widetilde{N}^2X\sqrt{4X\widetilde{N}^2+4X\widetilde{N}+1}+3\end{array}\right).\end{aligned}$$

To show that $\frac{\partial G(\widetilde{N},X)}{\partial X}<0$, we simply need to show that the large expression in parenthesis above is positive. So we study

$$\begin{aligned}&H(\widetilde{N},X)\\ =\ &10\widetilde{N}-10\widetilde{N}\sqrt{4X\widetilde{N}^2+4X\widetilde{N}+1}+24\widetilde{N}^2X+16\widetilde{N}^3X-10\widetilde{N}^2\sqrt{4X\widetilde{N}^2+4X\widetilde{N}+1}-3\left(4X\widetilde{N}^2+4X\widetilde{N}+1\right)^{\frac{3}{2}}\\ &+8\widetilde{N}X+12\widetilde{N}^2+8\widetilde{N}^3+10\widetilde{N}X\sqrt{4X\widetilde{N}^2+4X\widetilde{N}+1}+10\widetilde{N}^2X\sqrt{4X\widetilde{N}^2+4X\widetilde{N}+1}+3.\end{aligned}$$

and show that $H(\widetilde{N},X)$ is positive for any $\widetilde{N}\geq 2$ and X. Note the following boundary conditions

$$H(\widetilde{N},0)=2\widetilde{N}^2\left(4\widetilde{N}+1\right);\ H(\widetilde{N},1)=0.$$

Furthermore

$$\frac{\partial H(\widetilde{N},X)}{\partial X}$$
$$= \frac{-4\widetilde{N}\left(\widetilde{N}+1\right)}{\sqrt{4X\widetilde{N}^2+4X\widetilde{N}+1}}\left(5\widetilde{N}-4\widetilde{N}\sqrt{4X\widetilde{N}^2+4X\widetilde{N}+1}+3\widetilde{N}^2X-2\sqrt{4X\widetilde{N}^2+4X\widetilde{N}+1}+3\widetilde{N}X+5\widetilde{N}^2+2\right).$$

Proving that $\frac{\partial H(\widetilde{N},X)}{\partial X}$ is negative is equivalent to proving that in the above expression, the large expression in parenthesis is positive. This is in turn equivalent to showing that

$$\left(5\widetilde{N}+3\widetilde{N}^2X+3\widetilde{N}X+5\widetilde{N}^2+2\right)^2-\left(4\widetilde{N}+2\right)^2\left(4X\widetilde{N}^2+4X\widetilde{N}+1\right)>0.$$

The above inequality is equivalent to

$$-\widetilde{N}\left(\widetilde{N}+1\right)(X-1)\left(25\widetilde{N}-9\widetilde{N}X+25\widetilde{N}^2-9\widetilde{N}^2X+4\right)>0,$$

which is trivially always true. Given the boundary values obtained for $H(\widetilde{N},X)$ and the negative sign of $\frac{\partial H(\widetilde{N},X)}{\partial X}$, we may thus conclude that $H(\widetilde{N},X)$ is positive for any $\widetilde{N}\geq 2$ and $X\in(0,1)$. It follows that $\frac{\partial G(\widetilde{N},X)}{\partial X}<0$ for any $\widetilde{N}\geq 2$ and $X\in(0,1)$.■

11 Proof that $V_p(\beta,N,n)$ is increasing in N

Step 1 Recall that we showed in section 2 of part II of this online appendix that $V_p(\beta,N,n)$ can be rewritten as:

$$\frac{2}{3}+\frac{1}{3}\left(\widehat{\theta}\frac{1+2\widetilde{N}}{2\widetilde{N}}\right)^3-\left(\widehat{\theta}\frac{1+2\widetilde{N}}{2\widetilde{N}}\right)^2+\widehat{\theta}\left(\frac{1+2\widetilde{N}}{2\widetilde{N}}\right)$$
$$-\frac{1}{3}\widehat{\theta}^3\left((1-\beta)^n\left(\frac{1+2\widetilde{N}}{2\widetilde{N}}\right)^3+(1-(1-\beta)^n)\left(\left(\frac{1}{2\widetilde{N}}\right)^2+\left(\frac{1}{2\widetilde{N}}\right)^3\right)\right).$$

We first study the component:

$$\frac{1}{3}\left(\widehat{\theta}\frac{1+2\widetilde{N}}{2\widetilde{N}}\right)^3-\left(\widehat{\theta}\frac{1+2\widetilde{N}}{2\widetilde{N}}\right)^2+\widehat{\theta}\left(\frac{1+2\widetilde{N}}{2\widetilde{N}}\right) \tag{9}$$

and show that it is monotonously increasing in $\widetilde{N}$. We subsequently show a similar result for the second component of $V_p(\beta,N,n)$. In what follows, we replace $(1-\beta)^n$ by X.

Step 2 Note first that

$$\begin{aligned}
& \frac{\partial\left(\widehat{\theta}\left(\frac{1+2\widetilde{N}}{2\widetilde{N}}\right)\right)}{\partial\widetilde{N}} \\
= & -\frac{1}{4\widetilde{N}^2\left(\widetilde{N}+1\right)^2(X-1)\sqrt{4X\widetilde{N}^2+4X\widetilde{N}+1}} \\
& \left(2\widetilde{N}-2\widetilde{N}\sqrt{4X\widetilde{N}^2+4X\widetilde{N}+1}+2\widetilde{N}^2X-\sqrt{4X\widetilde{N}^2+4X\widetilde{N}+1}+2\widetilde{N}X+2\widetilde{N}^2+1\right).
\end{aligned}$$

The above expression can be shown to be always positive. Indeed,

$$\begin{aligned}
\left(2\widetilde{N}+1+2X\widetilde{N}^2+2X\widetilde{N}+2\widetilde{N}^2\right)^2 &> (2\widetilde{N}+1)^2\left(4X\widetilde{N}^2+4X\widetilde{N}+1\right) \Leftrightarrow \\
4\widetilde{N}^2\left(\widetilde{N}+1\right)^2(X-1)^2 &> 0.
\end{aligned}$$

Second, note that the polynomial $\frac{1}{3}x^3-x^2+x$ is always increasing in x, for $x>0$. Combining the above facts shows that (9) is monotonously increasing in $\widetilde{N}$.

Step 3 We now study the second component of $V_p(\beta,N,n)$, which is given by

$$-\frac{1}{3}\widehat{\theta}^3\left((1-\beta)^n\left(\frac{1+2\widetilde{N}}{2\widetilde{N}}\right)^3+(1-(1-\beta)^n)\left(\left(\frac{1}{2\widetilde{N}}\right)^2+\left(\frac{1}{2\widetilde{N}}\right)^3\right)\right). \tag{10}$$

We show that the expression is monotonously increasing in $\widetilde{N}$. Let us first rewrite explicitly the above expression as a function of X and $\widetilde{N}$, recalling that $\widehat{\theta}\left(\beta,\widetilde{N},n\right)$ can also be rewritten as a function of these two variables. Ignoring the multiplicative constant $\frac{1}{3}$, we obtain

$$\begin{aligned}
T(\widetilde{N},X) &= \left(-\frac{2\widetilde{\widetilde{N}}-\sqrt{4\widetilde{\widetilde{N}}X+4\widetilde{\widetilde{N}}^2X+1}+1}{-2\widetilde{\widetilde{N}}+2X+2\widetilde{\widetilde{N}}X-2}\right)^3\left(-X\left(\frac{1+2\widetilde{N}}{2\widetilde{N}}\right)^3-(1-X)\left(\frac{1}{2\widetilde{\widetilde{N}}}\right)^2-(1-X)\left(\frac{1}{2\widetilde{\widetilde{N}}}\right)^3\right) \\
&= \frac{1}{64\widetilde{N}^3}\frac{2\widetilde{N}+1}{\left(\widetilde{N}+1\right)^3(X-1)^3}\left(2\widetilde{N}-\sqrt{4X\widetilde{N}^2+4X\widetilde{N}+1}+1\right)^3\left(4X\widetilde{N}^2+4X\widetilde{N}+1\right).
\end{aligned}$$

Now, note that

$$
\begin{aligned}
\frac{\partial\left(T(\widetilde{N},X)\right)}{\partial\widetilde{N}} &= -\frac{1}{64\widetilde{N}^4\left(\widetilde{N}+1\right)^4(X-1)^3}\left(2\widetilde{N}-\sqrt{4X\widetilde{N}^2+4X\widetilde{N}+1}+1\right)^2 \\
&\quad\times\left(\begin{array}{c}
10\widetilde{N}-10\widetilde{N}\left(4X\widetilde{N}^2+4X\widetilde{N}+1\right)^{\frac{3}{2}}+24\widetilde{N}^2X+16\widetilde{N}^3X \\
-10\widetilde{N}^2\left(4X\widetilde{N}^2+4X\widetilde{N}+1\right)^{\frac{3}{2}}-3\left(4X\widetilde{N}^2+4X\widetilde{N}+1\right)^{\frac{3}{2}} \\
+8\widetilde{N}X+12\widetilde{N}^2+8\widetilde{N}^3+10\widetilde{N}X\sqrt{4X\widetilde{N}^2+4X\widetilde{N}+1}+50\widetilde{N}^2X\sqrt{4X\widetilde{N}^2+4X\widetilde{N}+1} \\
+80\widetilde{N}^3X\sqrt{4X\widetilde{N}^2+4X\widetilde{N}+1}+40\widetilde{N}^4X\sqrt{4X\widetilde{N}^2+4X\widetilde{N}+1}+3
\end{array}\right).
\end{aligned}
$$

We want to show that $\frac{\partial\left(T(\widetilde{N},X)\right)}{\partial\widetilde{N}} > 0$ for any $X \in (0,1)$. This is equivalent to showing that the expression in the large parenthesis is positive for any $X \in (0,1)$. Define thus

$$
\begin{aligned}
W(\widetilde{N},X) &= 10\widetilde{N}-10\widetilde{N}\left(4X\widetilde{N}^2+4X\widetilde{N}+1\right)^{\frac{3}{2}}+24\widetilde{N}^2X+16\widetilde{N}^3X \\
&\quad -10\widetilde{N}^2\left(4X\widetilde{N}^2+4X\widetilde{N}+1\right)^{\frac{3}{2}}-3\left(4X\widetilde{N}^2+4X\widetilde{N}+1\right)^{\frac{3}{2}} \\
&\quad +8\widetilde{N}X+12\widetilde{N}^2+8\widetilde{N}^3+10\widetilde{N}X\sqrt{4X\widetilde{N}^2+4X\widetilde{N}+1}+50\widetilde{N}^2X\sqrt{4X\widetilde{N}^2+4X\widetilde{N}+1} \\
&\quad +80\widetilde{N}^3X\sqrt{4X\widetilde{N}^2+4X\widetilde{N}+1}+40\widetilde{N}^4X\sqrt{4X\widetilde{N}^2+4X\widetilde{N}+1}+3.
\end{aligned}
$$

Note the following boundary conditions

$$
W(\widetilde{N},0) = 2\widetilde{N}^2\left(4\widetilde{N}+1\right); W(\widetilde{N},1) = 0.
$$

Note furthermore that

$$
\begin{aligned}
&\frac{\partial W(\widetilde{N},X)}{\partial X} \\
&= \frac{-4\widetilde{N}\left(\widetilde{N}+1\right)}{\sqrt{4X\widetilde{N}^2+4X\widetilde{N}+1}}\left(5\widetilde{N}-4\widetilde{N}\sqrt{4X\widetilde{N}^2+4X\widetilde{N}+1}+3\widetilde{N}^2X-2\sqrt{4X\widetilde{N}^2+4X\widetilde{N}+1}+3\widetilde{N}X+5\widetilde{N}^2+2\right).
\end{aligned}
$$

We want to show that $\frac{\partial W(\widetilde{N},X)}{\partial X} < 0$ for any $X \in (0,1)$, thus implying that $W(\widetilde{N},0) > 0$ for any $X \in (0,1)$ given the obtained boundary conditions. Showing $\frac{\partial W(\widetilde{N},X)}{\partial X} < 0$ is equivalent to showing that

$$
5\widetilde{N}-4\widetilde{N}\sqrt{4X\widetilde{N}^2+4X\widetilde{N}+1}+3\widetilde{N}^2X-2\sqrt{4X\widetilde{N}^2+4X\widetilde{N}+1}+3\widetilde{N}X+5\widetilde{N}^2+2 > 0,
$$

which is in turn equivalent to

$$-\widetilde{N}\left(\widetilde{N}+1\right)(X-1)\left(25\widetilde{N}-9\widetilde{N}X+25\widetilde{N}^2-9\widetilde{N}^2X+4\right)>0.$$

The latter inequality is trivially always true. We may thus conclude that $\frac{\partial\left(T(\widetilde{N},X)\right)}{\partial\widetilde{N}}>0$ for any $X\in(0,1)$.■

12 Comparing $V_p\left(\beta,N,n\right)$ and $V_{sr}(\beta,n)$

We know that given n and β, $\{t_r(\beta,N,n)\}_{r=1}^{N-1}$ is s.t.

$$t_{N-1}(\beta,N,n)=-\frac{2\left(N-1\right)-\sqrt{4\left(N-1\right)\left(1-\beta\right)^n+4\left(N-1\right)^2\left(1-\beta\right)^n+1}+1}{-2\left(N-1\right)+2\left(1-\beta\right)^n+2\left(N-1\right)\left(1-\beta\right)^n-2}.$$

It is easily checked that

$$\lim_{N\to\infty}t_{N-1}(\beta,N,1)=\theta^*(\beta,n)=\frac{1}{\sqrt{(1-\beta)^n+1}}.$$

Recall furthermore that as N tends to infinity, the number of partitions to the left of $t_{N-1}(\beta,N,n)$ also tends to infinity. It follows that for any $n\geq 1$, $\lim_{N\to\infty}V_p(\beta,N,n)=V_{sr}(\beta,n)$.■