Online Supplement to

Lauermann and Wolinsky

“Bidder Solicitation, Adverse Selection, and the Failure of Competition”

Numerical Derivation of Full Equilibria

In this supplement, we provide the numerical calculations that verify the examples for full equilibria from the paper.

The Numerical Setup

Recall the example from the paper. The values are $v_\ell = 0$ and $v_h = 1$, with equal probability, $\rho_\ell = \rho_h = 1/2$. Signals are binary on $[\underline{x}, \bar{x}] = [0,1]$, with a jump at $\hat{x} = 1/2$. We consider the case with $\lambda = \frac{g_h(1)}{g_\ell(1)} = 3$, meaning,

$$g_\ell(x) = \begin{cases} \frac{2}{4} & \text{if} \quad x > \frac{1}{2}, \\ \frac{6}{4} & \text{if} \quad x \leq \frac{1}{2}, \end{cases} \quad \text{and} \quad g_h(x) = \begin{cases} \frac{6}{4} & \text{if} \quad x > \frac{1}{2}, \\ \frac{2}{4} & \text{if} \quad x \leq \frac{1}{2}, \end{cases}$$

and

$$G_\ell\left(\frac{1}{2}\right) = \frac{3}{4} \text{ and } G_h\left(\frac{1}{2}\right) = \frac{1}{4}.$$

Part 1: A Full Equilibrium with $(n_\ell, n_h) = (16, 5)$

We now show that for our numerical example, if

$$s = 0.0011 = 1.1 \times 10^{-3},$$

then the following numbers constitute a full equilibrium

$$\begin{aligned} \underline{b} &= 0.08 \text{ and } \bar{b} = 0.49 \\ n_\ell &= 16 \text{ and } n_h = 5 \end{aligned}$$

All calculations are done in MuPAD 3.1.

Seller’s optimality.

Choosing n_ω bidders is optimal given a two-step bidding function if and only if

$$(G_\omega)^{n_\omega - 1}(1 - G_\omega)\left(\bar{b} - \underline{b}\right) \geq s \geq (G_\omega)^{n_\omega - 1}(1 - G_\omega)\left(\bar{b} - \underline{b}\right),$$

with $G_\omega = G_\omega(\hat{x})$ here and in the following. Let $\Delta b = \left(\bar{b} - \underline{b}\right)$. In the example, $\Delta b = 0.41$. Substituting the numbers,

$$\begin{aligned}
\Delta b \left(G_h\right)^{n_h - 1}\left(1 - G_h\right) &= (0.41)\left(\frac{1}{4}\right)^{5-1}\left(1 - \frac{1}{4}\right) = 1.201\,171\,875 \times 10^{-3} \\
\Delta b \left(G_h\right)^{n_h}\left(1 - G_h\right) &= (0.41)\left(\frac{1}{4}\right)^{5}\left(1 - \frac{1}{4}\right) = 3.002\,929\,688 \times 10^{-4} \\
\Delta b \left(G_\ell\right)^{n_\ell - 1}\left(1 - G_\ell\right) &= (0.41)\left(\frac{3}{4}\right)^{16-1}\left(1 - \frac{3}{4}\right) = 1.369\,754\,754 \times 10^{-3} \\
\Delta b \left(G_\ell\right)^{n_\ell}\left(1 - G_\ell\right) &= (0.41)\left(\frac{3}{4}\right)^{16}\left(1 - \frac{3}{4}\right) = 1.027\,316\,065 \times 10^{-3}
\end{aligned}$$

Hence, the seller's optimality conditions hold with

$$s = 0.001\,1 = 1.1 \times 10^{-3}.$$

Bidder's Optimality.

Let us calculate some critical conditional expected values. In particular,

$$E\left[v|\bar{x}, \text{sol}, \text{win at } b > \bar{b}\right] = \frac{\frac{g_h}{g_\ell}\frac{n_h}{n_\ell}}{\frac{g_h}{g_\ell}\frac{n_h}{n_\ell} + 1} = \frac{3\left(\frac{5}{16}\right)}{3\left(\frac{5}{16}\right) + 1} = 0.483\,870\,967\,7.$$

Furthermore,

$$E\left[v|\underline{x}, \text{sol}, \text{win at } \bar{b}\right] = \frac{\frac{\Pr\left(\text{win at } \bar{b}|h\right)}{\Pr\left(\text{win at } \bar{b}|\ell\right)}\left(\frac{5}{16}\right)\left(\frac{1}{3}\right)}{1 + \frac{\Pr\left(\text{win at } \bar{b}|h\right)}{\Pr\left(\text{win at } \bar{b}|\ell\right)}\left(\frac{5}{16}\right)\left(\frac{1}{3}\right)} = \frac{\frac{5}{16}\left(\frac{\frac{1-\left(\frac{1}{4}\right)^5}{5\left(1-\frac{1}{4}\right)}}{\frac{1-\left(\frac{3}{4}\right)^{16}}{16\left(1-\frac{3}{4}\right)}}\right)\left(\frac{1}{3}\right)}{1 + \frac{5}{16}\left(\frac{\frac{1-\left(\frac{1}{4}\right)^5}{5\left(1-\frac{1}{4}\right)}}{\frac{1-\left(\frac{3}{4}\right)^{16}}{16\left(1-\frac{3}{4}\right)}}\right)\left(\frac{1}{3}\right)} = 0.100\,821\,634\,7$$

and

$$E\left[v|\underline{x}, \text{sol}, \text{win at } \underline{b}\right] = \frac{\frac{\Pr(\text{win at } \underline{b}|h)}{\Pr(\text{win at } \underline{b}|\ell)}\frac{5}{16}\frac{1}{3}}{1 + \frac{\Pr(\text{win at } \underline{b}|h)}{\Pr(\text{win at } \underline{b}|\ell)}\frac{5}{16}\frac{1}{3}} = \frac{\frac{\frac{\left(\frac{1}{4}\right)^5}{(5)\frac{1}{4}}}{\frac{\left(\frac{3}{4}\right)^{16}}{(16)\frac{3}{4}}}\frac{5}{16}\frac{1}{3}}{1 + \frac{\frac{\left(\frac{1}{4}\right)^5}{(5)\frac{1}{4}}}{\frac{\left(\frac{3}{4}\right)^{16}}{(16)\frac{3}{4}}}\frac{5}{16}\frac{1}{3}} = 8.\,878\,520\,312 \times 10^{-2}$$

and for $b \in \left(\underline{b}, \bar{b}\right)$

$$E\left[v|\underline{x}, \text{sol}, \text{win at } b\right] = \frac{\frac{\Pr(\text{win at } b|h)}{\Pr(\text{win at } b|\ell)}\frac{5}{16}\frac{1}{3}}{1 + \frac{\Pr(\text{win at } b|h)}{\Pr(\text{win at } b|\ell)}\frac{5}{16}\frac{1}{3}} = \frac{\frac{\left(\frac{1}{4}\right)^4}{\left(\frac{3}{4}\right)^{15}}\frac{5}{16}\frac{1}{3}}{1 + \frac{\left(\frac{1}{4}\right)^4}{\left(\frac{3}{4}\right)^{15}}\frac{5}{16}\frac{1}{3}} = 2.\,954\,904\,5 \times 10^{-2}$$

We now show that bidding $\bar{b}$ is optimal for $\bar{x}$. We compare the payoff from bidding $\bar{b}$ to the payoff from bidding $b > \bar{b}$, $\underline{b}$, and from bidding $b \in \left(\underline{b}, \bar{b}\right)$. To do so, we derive the payoffs from each type of bid:

$$U\left(b > \bar{b}|\bar{x},\text{sol}\right) < E\left[v|\bar{x}, \text{win at } b > \bar{b}, \text{sol}\right] - \bar{b} = 0.483\,870\,967\,7 - 0.49 < 0$$

Furthermore,

$$\begin{aligned} U\left(\bar{b}|\bar{x},\text{sol}\right) &= \frac{\frac{g_h}{g_\ell}\frac{n_h}{n_\ell}}{\frac{g_h}{g_\ell}\frac{n_h}{n_\ell}+1}\Pr\left(\text{win at } \bar{b}|h\right)\left(1-\bar{b}\right) + \frac{1}{\frac{g_h}{g_\ell}\frac{n_h}{n_\ell}+1}\Pr\left(win \text{ at } \bar{b}|\ell\right)\left(-\bar{b}\right) \\ &= \frac{3\left(\frac{5}{16}\right)}{3\left(\frac{5}{16}\right)+1}\frac{1-\left(\frac{1}{4}\right)^5}{5\left(1-\frac{1}{4}\right)}(1-0.49) + \frac{1}{3\left(\frac{5}{16}\right)+1}\frac{1-\left(\frac{3}{4}\right)^{16}}{16\left(1-\frac{3}{4}\right)}(-0.49) \\ &= 3.\,150\,067\,748 \times 10^{-3} \end{aligned}$$

and

$$\begin{aligned} U\left(\underline{b}|\bar{x},\text{sol}\right) &= \frac{\frac{g_h}{g_\ell}\frac{n_h}{n_\ell}}{\frac{g_h}{g_\ell}\frac{n_h}{n_\ell}+1}\Pr\left(\text{win at } \underline{b}|h\right)\left(1-\underline{b}\right) + \frac{1}{\frac{g_h}{g_\ell}\frac{n_h}{n_\ell}+1}\Pr\left(win \text{ at } \underline{b}|\ell\right)\left(-\underline{b}\right) \\ &= \frac{3\left(\frac{5}{16}\right)}{3\left(\frac{5}{16}\right)+1}\frac{\left(\frac{1}{4}\right)^5}{5\left(\frac{1}{4}\right)}(1-0.08) + \frac{1}{3\left(\frac{5}{16}\right)+1}\frac{\left(\frac{3}{4}\right)^{16}}{16\left(\frac{3}{4}\right)}(-0.08) \\ &= 3.\,132\,959\,071 \times 10^{-4} \end{aligned}$$

and for $b \in \left(\underline{b}, \bar{b}\right)$

$$\begin{aligned} U\left(b|\bar{x},\text{sol}\right) &\leq \frac{\frac{g_h}{g_\ell}\frac{n_h}{n_\ell}}{\frac{g_h}{g_\ell}\frac{n_h}{n_\ell}+1}\Pr\left(\text{win at } b|h\right)\left(1-\underline{b}\right) + \frac{1}{\frac{g_h}{g_\ell}\frac{n_h}{n_\ell}+1}\Pr\left(win \text{ at } b|\ell\right)\left(-\underline{b}\right) \\ &= \frac{3\left(\frac{5}{16}\right)}{3\left(\frac{5}{16}\right)+1}\left(\frac{1}{4}\right)^4(1-0.08) + \frac{1}{3\left(\frac{5}{16}\right)+1}\left(\frac{3}{4}\right)^{15}(-0.08) \\ &= 1.\,187\,129\,674 \times 10^{-3} \end{aligned}$$

Comparing the profit at these four candidate bids shows that it is optimal to bid $\bar{b}$.

Finally, it is optimal to bid $\underline{b} = 0.08$ for $\underline{x}$. To see this, recall the expected values conditional on winning at $\underline{b}$ and at candidates for deviations:

$$\begin{aligned} E\left[v|\underline{x},\text{sol,win at } b = \underline{b}\right] &= 8.\,878\,520\,312 \times 10^{-2} > \underline{b} \\ E\left[v|\underline{x},\text{sol,win at } b \in \left(\underline{b}, \bar{b}\right)\right] &= 2.\,954\,904\,5 \times 10^{-2} < \underline{b} \\ E\left[v|\underline{x},\text{sol,win at } b = \bar{b}\right] &= 0.100\,821\,634\,7 < \bar{b} \\ E\left[v|\underline{x},\text{sol,win at } b > \bar{b}\right] &= \frac{\frac{g_h}{g_\ell}\frac{n_h}{n_\ell}}{\frac{g_h}{g_\ell}\frac{n_h}{n_\ell}+1} = \frac{\frac{1}{3}\left(\frac{5}{16}\right)}{\frac{1}{3}\left(\frac{5}{16}\right)+1} = 9.\,434\,0 \times 10^{-2} < \bar{b}. \end{aligned}$$

Part 2: A Full Equilibrium with $(n_\ell, n_h) = (40, 10)$

We now show that for our numerical example, if

$$s = 0.0000011 = 1.1 \times 10^{-6},$$

then the following numbers constitute a full equilibrium

$$\begin{aligned} \underline{b} &= 0.08 \text{ and } \bar{b} = 0.49, \\ n_\ell &= 40 \text{ and } n_h = 10. \end{aligned}$$

Seller's optimality.

Choosing n_ω bidders is optimal given a two-step bidding function if and only if

$$(G_\omega)^{n_\omega - 1} (1 - G_\omega) \left(\bar{b} - \underline{b}\right) \geq s \geq (G_\omega)^{n_\omega - 1} (1 - G_\omega) \left(\bar{b} - \underline{b}\right).$$

Let $\Delta b = \left(\bar{b} - \underline{b}\right)$. In the example, $\Delta b = 0.41$. Substituting the numbers,

$$\begin{aligned} \Delta b \, (G_h)^{n_h - 1} (1 - G_h) &= (0.41) \left(\frac{1}{4}\right)^{10-1} \left(1 - \frac{1}{4}\right) = 1.173\,019\,4 \times 10^{-6} \\ \Delta b \, (G_h)^{n_h} (1 - G_h) &= (0.41) \left(\frac{1}{4}\right)^{10} \left(1 - \frac{1}{4}\right) = 2.932\,548\,5 \times 10^{-7} \\ \Delta b \, (G_\ell)^{n_\ell - 1} (1 - G_\ell) &= (0.41) \left(\frac{3}{4}\right)^{40-1} \left(1 - \frac{3}{4}\right) = 1.374\,400\,0 \times 10^{-6} \\ \Delta b \, (G_\ell)^{n_\ell} (1 - G_\ell) &= (0.41) \left(\frac{3}{4}\right)^{40} \left(1 - \frac{3}{4}\right) = 1.030\,800\,0 \times 10^{-6} \end{aligned}$$

Hence, the seller's optimality conditions hold with

$$s = 1.1 \times 10^{-6}.$$

Bidder's Optimality.

Let us calculate the critical conditional expected values. In particular,

$$E\left[v|\bar{x}, \text{sol}, \text{win at } b > \bar{b}\right] = \frac{\frac{g_h}{g_\ell}\frac{n_h}{n_\ell}}{\frac{g_h}{g_\ell}\frac{n_h}{n_\ell} + 1} = \frac{3\left(\frac{10}{40}\right)}{3\left(\frac{10}{40}\right) + 1} = 0.428\,571\,428\,6$$

Further

$$E\left[v|\underline{x}, \text{sol}, \text{win at } \bar{b}\right] = \frac{\frac{\Pr\left(win \text{ at } \bar{b}|h\right)}{\Pr\left(win \text{ at } \bar{b}|\ell\right)} \frac{10}{40}\left(\frac{1}{3}\right)}{1 + \frac{\Pr\left(win \text{ at } \bar{b}|h\right)}{\Pr\left(win \text{ at } \bar{b}|\ell\right)} \frac{10}{40}\left(\frac{1}{3}\right)} = \frac{\frac{10}{40}\left(\frac{\frac{1-\left(\frac{1}{4}\right)^{10}}{10\left(1-\frac{1}{4}\right)}}{\frac{1-\left(\frac{3}{4}\right)^{40}}{40\left(1-\frac{3}{4}\right)}}\right)\left(\frac{1}{3}\right)}{1 + \frac{10}{40}\left(\frac{\frac{1-\left(\frac{1}{4}\right)^{10}}{10\left(1-\frac{1}{4}\right)}}{\frac{1-\left(\frac{3}{4}\right)^{40}}{40\left(1-\frac{3}{4}\right)}}\right)\left(\frac{1}{3}\right)} = 0.100\,000\,819\,3$$

and

$$E\left[v|\underline{x},\text{win at }\underline{b},\text{sol}\right]=\frac{\frac{\Pr(\text{win at }\underline{b}|h)}{\Pr(\text{win at }\underline{b}|\ell)}\frac{10}{40}\frac{1}{3}}{1+\frac{\Pr(\text{win at }\underline{b}|h)}{\Pr(\text{win at }\underline{b}|\ell)}\frac{10}{40}\frac{1}{3}}=\frac{\frac{\frac{\left(\frac{1}{4}\right)^{10}}{(10)\frac{1}{4}}}{\frac{\left(\frac{3}{4}\right)^{40}}{(40)\frac{3}{4}}}\frac{10}{40}\frac{1}{3}}{1+\frac{\frac{\left(\frac{1}{4}\right)^{10}}{(10)\frac{1}{4}}}{\frac{\left(\frac{3}{4}\right)^{40}}{(40)\frac{3}{4}}}\frac{10}{40}\frac{1}{3}}=8.661\,687\,9\times10^{-2}$$

and for $b\in\left(\underline{b},\bar{b}\right)$

$$E\left[v|\underline{x},\text{win at }b,\text{sol}\right]=\frac{\frac{\Pr(win\text{ at }\underline{b}|h)}{\Pr(win\text{ at }\underline{b}|\ell)}\frac{10}{40}\frac{1}{3}}{1+\frac{\Pr(win\text{ at }\underline{b}|h)}{\Pr(win\text{ at }\underline{b}|\ell)}\frac{10}{40}\frac{1}{3}}=\frac{\frac{\left(\frac{1}{4}\right)^{10}}{\left(\frac{3}{4}\right)^{40}}\frac{10}{40}\frac{1}{3}}{1+\frac{\left(\frac{1}{4}\right)^{10}}{\left(\frac{3}{4}\right)^{40}}\frac{10}{40}\frac{1}{3}}=7.840\,608\,206\times10^{-3}$$

and

$$E\left[v|\bar{x},\text{win at }b,\text{sol}\right]=\frac{\frac{\Pr(win\text{ at }\underline{b}|h)}{\Pr(win\text{ at }\underline{b}|\ell)}\frac{10}{40}3}{1+\frac{\Pr(win\text{ at }\underline{b}|h)}{\Pr(win\text{ at }\underline{b}|\ell)}\frac{10}{40}3}=\frac{\frac{\left(\frac{1}{4}\right)^{10}}{\left(\frac{3}{4}\right)^{40}}\frac{10}{40}3}{1+\frac{\left(\frac{1}{4}\right)^{10}}{\left(\frac{3}{4}\right)^{40}}\frac{10}{40}3}=6.640\,051\,074\times10^{-2}$$

We now show that bidding $\bar{b}$ is optimal for $\bar{x}$. We compare the payoff from bidding $\bar{b}$ to the payoff from bidding $b>\bar{b}$, $\underline{b}$, and from bidding $b\in\left(\underline{b},\bar{b}\right)$.

To do so, we derive the payoffs from each type of bid:

$$U\left(b>\bar{b}|\bar{x}\text{,sol}\right)<E\left[v|\bar{x},\text{win at }b>\bar{b},\text{sol}\right]-\bar{b}=0.428\,571\,428\,6-0.49<0$$

Furthermore,

$$\begin{aligned}
U\left(\bar{b}|\bar{x}\text{,sol}\right) &= \frac{\frac{g_h}{g_\ell}\frac{n_h}{n_\ell}}{\frac{g_h}{g_\ell}\frac{n_h}{n_\ell}+1}\Pr\left(win\text{ at }\bar{b}|h\right)\left(1-\bar{b}\right)+\frac{1}{\frac{g_h}{g_\ell}\frac{n_h}{n_\ell}+1}\Pr\left(win\text{ at }\bar{b}|\ell\right)\left(-\bar{b}\right)\\
&= \frac{3\left(\frac{10}{40}\right)}{3\left(\frac{10}{40}\right)+1}\frac{1-\left(\frac{1}{4}\right)^{10}}{10\left(1-\frac{1}{4}\right)}(1-0.49)+\frac{1}{3\left(\frac{10}{40}\right)+1}\frac{1-\left(\frac{3}{4}\right)^{40}}{40\left(1-\frac{3}{4}\right)}(-0.49)\\
&= 1.143\,110\,9\times10^{-3}
\end{aligned}$$

and

$$\begin{aligned}
U\left(\underline{b}|\bar{x}\text{,sol}\right) &= \frac{\frac{g_h}{g_\ell}\frac{n_h}{n_\ell}}{\frac{g_h}{g_\ell}\frac{n_h}{n_\ell}+1}\Pr\left(win\text{ at }\underline{b}|h\right)\left(1-\underline{b}\right)+\frac{1}{\frac{g_h}{g_\ell}\frac{n_h}{n_\ell}+1}\Pr\left(win\text{ at }\underline{b}|\ell\right)\left(-\underline{b}\right)\\
&= \frac{3\left(\frac{10}{40}\right)}{3\left(\frac{10}{40}\right)+1}\frac{\left(\frac{1}{4}\right)^{10}}{10\left(\frac{1}{4}\right)}(1-0.08)+\frac{1}{3\left(\frac{10}{40}\right)+1}\frac{\left(\frac{3}{4}\right)^{40}}{40\left(\frac{3}{4}\right)}(-0.08)\\
&= 1.350\,837\,434\times10^{-7}
\end{aligned}$$

and

$$U\left(b\in\left(\underline{b},\bar{b}\right)|\bar{x},\text{sol}\right)<\left(E\left[v|\bar{x},\text{win at } b,\text{sol}\right]-\underline{b}\right)<0$$

Thus, $U\left(b|\bar{x},\text{sol}\right)$ is maximal at $\bar{b}$.

Finally, it is optimal to bid $\underline{b}$ for $\underline{x}$. This follows from

$$\begin{aligned}
E\left[v|\underline{x},\text{win at } \underline{b},\text{sol}\right] &= 8.661\,687\,931\times10^{-2}>\underline{b}\\
E\left[v|\underline{x},\text{win at } \in\left(\underline{b},\bar{b}\right),\text{sol}\right] &= 7.840\,608\,206\times10^{-3}<\underline{b}\\
E\left[v|\underline{x},\text{win at } \bar{b},\text{sol}\right] &= 0.100\,000\,819\,3<\bar{b}\\
E\left[v|\underline{x},\text{win at } b>\bar{b},\text{sol}\right] &= \frac{\frac{g_h}{g_\ell}\frac{n_h}{n_\ell}}{\frac{g_h}{g_\ell}\frac{n_h}{n_\ell}+1}=\frac{\frac{1}{3}\left(\frac{10}{40}\right)}{\frac{1}{3}\left(\frac{10}{40}\right)+1}<\bar{b}
\end{aligned}$$