Online Appendix for

Breakthroughs, Deadlines, and Self-Reported Progress: Contracting for Multistage Projects

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Overview

This supplemental appendix contains proofs of formal results in Section IV of Green and Taylor (2016). In this appendix, equations are numbered S.1, S.2, S.3,... References to equations, propositions, lemmas etc. not prefaced with S corresponds to ones in Green and Taylor (2016).

Omitted Proofs

Proof of Proposition 3. The proof takes several steps. In Steps 1 through 3 we assume that the principal *must* use the formal channel, but that she does so optimally. In Step 4 we prove that, for ρ sufficiently small, the principal indeed benefits by using the formal channel.

Step 1. First observe that the only possible reason for requiring a costly report is to relax the no-false-progress constraint so as to add more time to the clock following the first reported breakthrough. This implies that the formal communication channel should only be used to report progress (not lack of progress). Next, observe that it cannot be optimal for the principal to require the costly formal report at date t and take no action until date $t' > t$, because this is dominated by waiting until t' to require the report. (The project might be completed between t and t'.) Thus, it is optimal to putoff requiring a formal report as long as possible. Let y be the highest value of the low type's continuation utility at which the principal requires a formal report, and let $F_2(u_1, u_2; \rho)$ denote the principal's value function in the second stage. This value function is given by

$$F_2(u_1, u_2; \rho) = \left(\Pi - \frac{c}{\lambda}\right)\left(1 - e^{-\lambda(u_1+\rho)/\phi}\right) - u_2 - \mathbb{1}_{\{u_1 \leq y\}}\rho e^{-\lambda(u_1-y)/\phi}. \tag{S.1}$$

This is established using the same method of proof as for Proposition A.2 with two straight-forward alterations. First, to satisfy (B9), the termination policy must be such that

$\mathbb{E}[T] \leq (u_1 + \rho)/\phi$. Second, for $u_1 \leq y$, there is a chance that the high type will have to pay the reporting cost. Therefore, the promise keeping constraint necessitates

$$u_2 + \rho e^{-\lambda(u_1-y)/\phi} = \mathbb{E}^{a=1}\left[\int_0^{T \wedge \tau_2} dY_t \Big| s = 1\right],$$

Because $F_2(u_1, u_2; \rho)$ is concave in u_1 and linear in u_2, $\sigma = 0$ is optimal.

Step 2. We solve for the first-stage value function assuming $\sigma = 0$ and check for concavity. The HJB is

$$\lambda F_1(u_1; \rho) = \max_{y \geq 0} \lambda F_2(u_1, u_1 + \phi/\lambda; \rho) - c - \phi \frac{dF_1}{du_1}.$$

Because the principal prefers to delay formal reports as long as possible, $y = 0$ is optimal. Therefore we have

$$\frac{\lambda}{\phi} F_1(u_1; \rho) + F_1'(u_1; \rho) = \frac{\lambda}{\phi}\left[\Pi - \frac{2c}{\lambda} - \frac{\phi}{\lambda} - u_1 - \left(\rho + \left(\Pi - \frac{c}{\lambda}\right)e^{-\lambda\rho/\phi}\right)e^{-\lambda u_1/\phi}\right].$$

This has a solution of the form

$$F_1^c(u_1; \rho) = \Pi - \frac{2c}{\lambda} - u_1 - \left(\rho + \left(\Pi - \frac{c}{\lambda}\right)e^{-\lambda\rho/\phi}\right)\frac{\lambda u_1}{\phi}e^{-\lambda u_1/\phi} + Ke^{-\lambda u_1/\phi}. \tag{S.2}$$

Using the boundary condition $F_1(0; \rho) = 0$ gives

$$F_1^c(u_1; \rho) = \left(\Pi - \frac{2c}{\lambda}\right)\left(1 - e^{-\lambda u_1/\phi}\right) - u_1 - \left(\rho + \left(\Pi - \frac{c}{\lambda}\right)e^{-\lambda\rho/\phi}\right)\frac{\lambda u_1}{\phi}e^{-\lambda u_1/\phi}.$$

For small ρ, this is convex for u_1 sufficiently close to zero, necessitating random termination at some point $u_s(\rho)$.

Step 3. Using the same analysis as in the proof of Lemma A.4 yields

$$F_1(u_1; \rho) = \begin{cases} \left(\Pi - \frac{2c}{\lambda}\right)\left[1 - \left(\dfrac{u_1 - u_s(\rho) + \frac{2\phi}{\lambda}}{u_s(\rho) + \frac{2\phi}{\lambda}}\right)e^{-\frac{\lambda}{\phi}(u_1 - u_s(\rho))}\right] - u_1 & u_1 \geq u_s(\rho), \quad \text{(S.3)} \\[3ex] \left(\Pi - \frac{2c}{\lambda}\right)\left[\dfrac{u_1}{u_s(\rho) + \frac{2\phi}{\lambda}}\right] - u_1 & u_1 \in [0, u_s(\rho)). \ \text{(S.4)} \end{cases}$$

where $u_s(\rho)$ is implicitly defined by

$$\Pi - \frac{2c}{\lambda} - \left(\rho + \left(\Pi - \frac{c}{\lambda}\right)^{-\lambda\rho/\phi}\right)\left(2 + \frac{\lambda u_s(\rho)}{\phi}\right)e^{-\lambda u_s(\rho)/\phi} \equiv 0.$$

Step 4. We wish to show that for ρ sufficiently small and any $u_1 > 0$, $F_1(u_1; \rho) > F_1(u_1; 0) =$

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$F_1(u_1)$. The claim will follow if we show $\frac{\partial F_1(u;0)}{\partial \rho} > 0$ for all $u > 0$. Note first that

$$u_s'(0) = -\frac{\left(\Pi - \frac{c}{\lambda} - \frac{\phi}{\lambda}\right)\left(2 + \frac{\lambda u_s}{\phi}\right)}{\left(\Pi - \frac{c}{\lambda}\right)\left(1 + \frac{\lambda u_s}{\phi}\right)} < 0,$$

and $F_1(u_1;\rho)$ depends on ρ only through $u_s(\rho)$. Therefore, for all $u > 0$,

$$\mathrm{sign}\left(\frac{\partial F_1(u;0)}{\partial \rho}\right) = -\mathrm{sign}\left(\frac{\partial F_1(u;0)}{\partial u_s}\right) > 0,$$

where the inequality follows from observing that both expressions in (S.3)-(S.4) are strictly decreasing in $u_s(\rho)$ for all $u_1 > 0$. $\qquad\square$

Proof of Proposition 4. First, we extend the analysis from Appendix A to characterize F_1^α for an arbitrary α. Using the same arguments as in Lemma A.2, we have that

$$F_2^\alpha(u_1, u_2) = \left(1 - e^{-\frac{\lambda u_1}{(1-\alpha)\phi}}\right)\left(\Pi - \frac{c(1-\alpha)}{\lambda}\right) - u_2$$

Replacing F_2 with F_2^α in (HJB) and the appropriately modified (binding) constraints, we find that F_1^α has the form

$$\hat{F}_1^\alpha(u_1) = \left(\Pi - \frac{2c}{\lambda} - u_1\right) + \left(\frac{1-\alpha}{2\alpha}\right)\left(\Pi - \frac{(1-\alpha)c}{\lambda}\right)e^{-\frac{\lambda u_1}{(1-\alpha)\phi}} + C_1^\alpha e^{-\frac{\lambda u_1}{(1+\alpha)\phi}}. \qquad (S.5)$$

If the terminal boundary condition is imposed ($\hat{F}_1^\alpha(0) = 0$) then $C_1^\alpha = \frac{-(1+\alpha)(\lambda\Pi - c(1+\alpha))}{2\alpha\lambda}$ and $\hat{F}_1^{\alpha\prime\prime}(0) = \frac{\lambda^2\Pi}{(1-\alpha^2)\phi} > 0$. Hence, there exists some $u_s(\alpha)$ such that random termination is optimal for $u \in (0, u_s(\alpha)]$. Let $c^\alpha(u)$ denote the constant in the principal's value function that satisfies the smooth-pasting condition (i.e., (A9)) at an arbitrary $u > 0$. That is,

$$c^\alpha(u) \equiv \frac{(\alpha+1)e^{\frac{2\alpha\lambda u}{(\alpha^2-1)\phi}}\left(c\left(\phi\left(\alpha^2 - 2\alpha + 4\alpha e^{\frac{\lambda u}{\phi-\alpha\phi}} + 1\right) + u(\lambda - \alpha\lambda)\right) - \lambda\Pi\left(-\alpha\phi + 2\alpha\phi e^{\frac{\lambda u}{\phi-\alpha\phi}} + \lambda u + \phi\right)\right)}{2\alpha\lambda(\alpha\phi + \lambda u + \phi)}. \qquad (S.6)$$

Twice differentiability at $u_s(\alpha)$ (i.e., (A10)) is equivalent to $u_s(\alpha) = \max_u c^\alpha(u)$, which requires the first-order condition

$$\frac{e^{\lambda u_s(\alpha)/\phi(1-\alpha)}}{\lambda u_s(\alpha) + 2\phi} = \frac{\lambda\Pi - c(1-\alpha)}{\phi(1-\alpha)(\lambda\Pi - 2c)}. \qquad (S.7)$$

The right-hand side of the above expression is strictly greater than $1/2\phi$ for all $\alpha \in (-1, 1)$. The left-hand side is equal to $1/2\phi$ at $u_s(\alpha) = 0$, strictly increasing in $u_s(\alpha)$ and unbounded.

This guarantees the existence of a unique $u_s(\alpha)$ satisfying (S.7), which completes the characterization of F_1^α. To summarize,

- For $u \geq u_s(\alpha)$, F_1^α is of the form given in (S.5) with $C_1^\alpha = c^\alpha(u_s(\alpha))$, where u_s^α is the unique solution to (S.7).

- For $u \in [0, u_s(\alpha))$, $F_1^\alpha(u) = \frac{u}{u_s(\alpha)} F_1^\alpha(u_s(\alpha))$.

To prove (i), first note that by the envelope theorem $\frac{d}{d\alpha} c^\alpha(u_s(\alpha)) = \frac{\partial}{\partial\alpha} c^\alpha(u_s(\alpha))$. Using this fact, evaluating the derivative and taking the limit as $\alpha \to 0$, we get that for $u \geq u_s(0) = u_s$,

$$\lim_{\alpha \to 0} \left(\frac{d}{d\alpha} F_1^\alpha(u) \right) = \left(\frac{e^{-\frac{\lambda u}{\phi}} (u(\lambda u_s + \phi) - \lambda u_s^2)}{\phi^2(\lambda u_s + \phi)^2} \right) \times$$
$$\left(\Pi\lambda \left(\lambda^2 u_s^2 - \phi^2 \left(e^{\frac{\lambda u_s}{\phi}} - 1 \right) + \lambda u_s \phi \right) - c \left(\lambda^2 u_s^2 - 2\phi^2 \left(e^{\frac{\lambda u_s}{\phi}} - 1 \right) + 2\lambda u_s \phi \right) \right).$$

The first-term on the right hand side is clearly positive for $u \geq u_s$. Using (S.7), the second term reduces to $\lambda(\lambda u_s + \phi)((\lambda\Pi - c)u_s - \Pi\phi)$, which is also clearly positive if $\lambda u_s/\phi > \frac{\lambda\Pi/c}{\lambda\Pi/c-1}$. We now claim that if u_s solves (S.7) for $\alpha = 0$, then this latter inequality must hold. Let $x \equiv \lambda u_s/\phi \geq 0$, $y \equiv \lambda\Pi/c - 2 > 0$, and $\alpha = 0$. The claim is that

$$\frac{e^x}{x+2} = \frac{y+1}{y} \implies x > \frac{y+2}{y+1}.$$

To see that this is true, suppose that $\frac{e^x}{x+2} = \frac{y+1}{y}$ and $x \leq \frac{y+2}{y+1}$. Note that $\frac{e^x}{x+2}$ is strictly increasing. Therefore,

$$\frac{e^x}{x+2} \leq \frac{e^{\frac{y+2}{y+1}}}{\frac{y+2}{y+1}+2} < \frac{y+1}{y},$$

which gives the contradiction. We have thus shown that at $\alpha = 0$, the derivative of $F_1^\alpha(u)$ w.r.t. α is strictly positive for all $u \geq u_s(\alpha)$. That the same statement is true for $u \in (0, u_s)$ is immediate by the linearity of the value function below u_s. Since F_1^α is also continuously differentiable in both of its arguments, it must be strictly increasing in a neighborhood around $\alpha = 0$ for all $u > 0$, which completes the proof of (i).

We prove (ii) and (iii) for the case of $\alpha \to 1$, the proof for $\alpha \to -1$ follows a similar argument. We first show that $\lim_{\alpha \to 1} u_s(\alpha) = 0$. To do so, rewrite (S.7) as

$$\phi(1-\alpha)e^{\lambda u_s(\alpha)/\phi(1-\alpha)} = \frac{(\lambda\Pi - c(1-\alpha))(\lambda u_s(\alpha) + 2\phi)}{\lambda\Pi - 2c}.$$

Suppose $u_s(1) \equiv \lim_{\alpha \to 1} u_s(\alpha) \in (0, \infty)$. Then $\lim_{\alpha \to 1} \phi(1-\alpha)e^{\lambda u_s(\alpha)/\phi(1-\alpha)} = \infty > \frac{(\lambda\Pi - c(1-\alpha))(\lambda u_s(1)+2\phi)}{\lambda\Pi-2c}$, a contradiction. Also, clearly $u_s(1) < \infty$ otherwise the principal's

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value function would be arbitrarily negative. The only remaining possibility is $u_s(1) = 0$.

From (S.5), we have that for $u \geq u_s^\alpha$,

$$\lim_{\alpha \to 1} \left(\frac{d}{d\alpha} F_1^\alpha(u) \right) = e^{-\frac{\lambda u}{2\phi}} \lim_{\alpha \to 1} \left(\frac{\lambda u}{4\phi} c^\alpha(u_s(\alpha)) + \frac{\partial}{\partial \alpha} c^\alpha(u_s(\alpha)) \right).$$

Notice from (S.6) that $\lim_{\alpha \to 1} (\lim_{u \to 0} c^\alpha(u)) = \lim_{u \to 0} (\lim_{\alpha \to 1} c^\alpha(u)) = -(\Pi - 2c/\lambda)$. There-fore, we can conclude that $\lim_{\alpha \to 1} c^\alpha(u_s(\alpha)) = -(\Pi - 2c/\lambda)$. Hence, to prove (ii), it is sufficient to show that $\lim_{\alpha \to 1} \frac{\partial}{\partial \alpha} c^\alpha(u_s(\alpha)) = 0$, for this implies $\frac{d}{d\alpha} F_1^\alpha(u) \approx -e^{-\frac{\lambda u}{2\phi}} (\Pi - 2c/\lambda) \frac{\lambda u}{4\phi} < 0$ for $u \geq u_s(\alpha)$ and α sufficiently close to 1. To see that $\lim_{\alpha \to 1} \frac{\partial}{\partial \alpha} c^\alpha(u_s(\alpha)) = 0$, first notice from (S.7) that

$$\lim_{\alpha \to 1} \frac{e^{-\frac{\lambda u_s(\alpha)}{\phi(1-\alpha)}}}{(1-\alpha)} \in (0, \infty), \tag{S.8}$$

implying that $u_s(\alpha)$ is $O((1-\alpha) \ln(1-\alpha))$ as $\alpha \to 1$. Differentiating (S.6) with respect to α and omitting the argument of $u_s(\alpha)$, we get that

$$\frac{\partial}{\partial \alpha} c^\alpha(u_s(\alpha)) =$$

$$\frac{e^{\frac{2\alpha\lambda u_s}{(\alpha^2-1)\phi}}}{2(1-\alpha)^2\alpha^2(\alpha+1)\lambda\phi(\alpha\phi+\lambda u_s+\phi)^2} \times \bigg[\lambda\Pi\Big((1-\alpha)^2(\alpha+1)^3\phi^3 + 2\lambda^3 u_s^3\alpha(\alpha^2+1)$$

$$+ \lambda^2 u_s^2\phi \Big[3\alpha + 2\alpha^4 e^{\frac{\lambda u_s}{\phi(1-\alpha)}} + \alpha^3\Big(5 - 4e^{\frac{\lambda u_s}{\phi(1-\alpha)}}\Big) + \alpha^2\Big(2e^{\frac{\lambda u_s}{\phi(1-\alpha)}} - 1\Big) + 1\Big] + 2(1-\alpha)(\alpha+1)^2\lambda u_s\phi^2\Big)$$

$$- (1-\alpha)c\Big[(\alpha-1)^2(\alpha+1)^4\phi^3 + 2\alpha(\alpha^2+1)\lambda^3 u_s^3 - \lambda^2 u_s^2\phi\Big(\alpha^4 - 4\alpha + 4\alpha^3\Big(e^{\frac{\lambda u_s}{\phi(1-\alpha)}} - 1\Big)$$

$$- 4\alpha^2 e^{\frac{\lambda u_s}{\phi(1-\alpha)}} - 1\Big) + 2(1-\alpha)(\alpha+1)^3\lambda u_s\phi^2\Big]\bigg].$$

Using (S.8), we know that $e^{\frac{2\alpha\lambda u_s(\alpha)}{(\alpha^2-1)\phi}}$ is $O(1-\alpha)$ as $\alpha \to 1$. Therefore, any term inside the outermost brackets that goes to zero faster than $O(1-\alpha)$ will converge to zero when scaled by the fraction outside the brackets. By inspection, the only terms that do not clearly go to zero faster than $O(1-\alpha)$ are

$$\lambda^2 u_s(\alpha)^2\phi \Big[2\alpha^4 e^{\frac{\lambda u_s(\alpha)}{\phi(1-\alpha)}} - 4\alpha^3 e^{\frac{\lambda u_s(\alpha)}{\phi(1-\alpha)}} + 2\alpha^2 e^{\frac{\lambda u_s(\alpha)}{\phi(1-\alpha)}}\Big].$$

Thus, we get have that

$$\lim_{\alpha \to 1} \left(\frac{\partial}{\partial \alpha} c^\alpha(u_s(\alpha)) \right) = \lim_{\alpha \to 1} \left(\frac{e^{\frac{2\alpha\lambda u_s(\alpha)}{(\alpha^2-1)\phi}} e^{\frac{\lambda u_s(\alpha)}{\phi(1-\alpha)}}}{(1-\alpha)^2(\alpha+1)(\alpha\phi + \lambda u_s(\alpha) + \phi)^2} \lambda u_s(\alpha)^2 \left[\alpha^2 - 2\alpha + 1 \right] \right)$$

$$= \lim_{\alpha \to 1} \left(\frac{e^{\frac{\lambda u_s(\alpha)}{(1+\alpha)\phi}}}{(\alpha+1)(\alpha\phi + \lambda u_s(\alpha) + \phi)^2} \lambda u_s(\alpha)^2 \right)$$

$$= \frac{\lambda}{4\phi^2} \left(\lim_{\alpha \to 1} u_s(\alpha) \right)^2$$

$$= 0,$$

which completes the proof of (ii). For (iii), we have

$$\hat{F}_1^\alpha(u) - \bar{V}^{\frac{\lambda}{2}}(u) = \left(\frac{1-\alpha}{2\alpha} \right) \left(\Pi - \frac{(1-\alpha)c}{\lambda} \right) e^{-\frac{\lambda u}{(1-\alpha)\phi}} + C_1^\alpha e^{-\frac{\lambda u}{(1+\alpha)\phi}} - \left(\Pi - \frac{2c}{\lambda} \right) e^{-\frac{\lambda u}{2\phi}}$$

$$\leq \left(\frac{1-\alpha}{2\alpha} \right) \left(\Pi - \frac{(1-\alpha)c}{\lambda} \right) + \left| C_1^\alpha - \left(\Pi - \frac{2c}{\lambda} \right) \right| + \left| C_1^\alpha \left(e^{-\frac{\lambda u}{\phi(1+\alpha)}} - e^{-\frac{\lambda u}{2\phi}} \right) \right|$$

$$\leq \left(\frac{1-\alpha}{2\alpha} \right) \Pi + |C_1^\alpha - (\Pi - 2c/\lambda)| + \left| C_1^\alpha \left(e^{-\frac{x^*(\alpha)}{2}} - e^{-\frac{x^*(\alpha)}{(1+\alpha)}} \right) \right|,$$

where the first inequality uses the triangle inequality and $e^{-|x|} \leq 1$ and the second uses the fact that $e^{-\frac{x}{2}} - e^{-\frac{x}{(1+\alpha)}}$ is hump-shaped in x and achieves its maximum at $x^*(\alpha) \equiv 2(1+\alpha)\ln(\frac{1+\alpha}{2})/(\alpha-1)$. Clearly all three terms converge to 0 as $\alpha \to 1$. $\square$

Proof of Proposition 5. Consider first any simple contract with deadline T. If the agent does not shirk,[1] then the probability that the project succeeds at $t \in [0, T]$ is given by $\lambda^2 t e^{-\lambda t}$ and the probability that the project does not succeed prior to T is given by $e^{-\lambda T}(1 + \lambda T)$. Therefore, the total surplus is given by

$$S(T) \equiv \int_0^T \lambda^2 t e^{-\lambda t}(\Pi - ct)dt + e^{-\lambda T}(1 + \lambda T)(-cT).$$

In order to induce the agent to work, he must be given rents in the amount of at least $u = \phi T$, otherwise he can do better by shirking. Making the change of variables from T to u, we have that the principal's ex-ante expected payoff under a simple contract with deadline $T = u/\phi$ is bounded above by

$$G(u) \equiv S(u/\phi) - u = \left(1 - e^{-\lambda u/\phi}(1 + \lambda u/\phi) \right) \Pi - \frac{2c}{\lambda} \left(1 - e^{-\lambda u/\phi}(1 + \lambda u/2\phi) \right) - u. \quad \text{(S.9)}$$

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[1] Arguments similar to those made for a single-stage project can be used to confirm shirking is suboptimal.

Note that $G(u)$ is not the principal's value function under the optimal contract, since her belief about the project stage changes over time. We will construct the value function shortly. To prove the proposition, it suffices to show that

(i) The principal's ex-ante payoff for a project with unobservable progress under *any contract* is bounded above by $\max_u G(u)$.

(ii) There exists an incentive-compatible simple contract under which the principal's ex-ante expected payoff is $\max_u G(u)$.

(iii) For all $u > 0$, $G(u) < F_1(u)$. Therefore, the principal does strictly better with intangible progress than she does with unobservable progress.

For (i), let $w^* = \arg\max_u G(u)$, which is generically unique. We have already argued that the principal's maximal payoff under a simple contract is bounded above by $G(w^*)$. Given that neither player has any information about the status of the project, the only possibility is that the principal randomizes over the termination date. It therefore suffices to show that the principal cannot benefit from such randomization, or equivalently, that the principal's value function (under this simple contract with the optimally chosen deadline) is globally concave in the agent's continuation value.

Suppose that the principal can implement a simple contract in which the incentive compatibility condition holds with equality for all t (this is the best possible case for the principal). It is most intuitive to construct this value function from the pair of value functions that are conditional on s (i.e., whether a breakthrough has been made) and weight them appropriately by the probability that the principal assigns to each. Given u, the principal's payoff conditional on being in the first stage ($s = 1$) is $G(u)$; i.e.,

$$
\begin{aligned}
F^{unobs}(u|s=1) &= \int_0^{u/\phi} \lambda^2 t e^{-\lambda t}(\Pi - ct)dt - e^{-\lambda u/\phi}(1 + \lambda u/\phi)cu/\phi - u \\
&= (1 - e^{-\lambda u/\phi})(\Pi - 2c/\lambda) - \frac{\lambda u}{\phi}e^{-\lambda u/\phi}(\Pi - c/\lambda) - u.
\end{aligned}
$$

Conditional on being in the second stage, the principal's value function is the benchmark payoff $\bar{V}(u)$:

$$
F^{unobs}(u|s=2) = \int_0^{u/\phi} \lambda e^{-\lambda t}(\Pi - ct) + e^{-\lambda u/\phi}(-cu/\phi) - u = (1 - e^{-\lambda u/\phi})(\Pi - c/\lambda) - u.
$$

Over time, the principal's beliefs will evolve about the state of the project. Conditional on reaching state $u < w^*$ prior to project success, a period of time of length $t(u; w^*) = \frac{w^* - u}{\phi}$ has

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elapsed. Therefore, the principal's beliefs are given by

$$\mu(u; w^*) = \Pr(\tau_1 \le t(u; w^*) | \tau_2 > t(u; w^*)) = \frac{\lambda\left(\frac{w^* - u}{\phi}\right)}{1 + \lambda\left(\frac{w^* - u}{\phi}\right)}.$$

The principal's value function for $u \le w^*$ is therefore given by

$$F^{unobs}(u; w^*) = \mu(u; w^*) F^{unobs}(u|s = 2) + (1 - \mu(u; w^*)) F^{unobs}(u|s = 1).$$

We will now verify this value function is concave for all $u \le w^*$. Using the functional forms given above and twice differentiating $F^{unobs}(u; w^*)$, we get that

$$\frac{d^2}{du^2} F^{unobs}(u; w^*) = \frac{-\lambda e^{-\lambda u/\phi}}{\phi^2 \left(\lambda(w^* - u) + \phi\right)^3} \Bigg[(\lambda^3 w^*(w^* - u)^2 + \lambda w^* \phi^2)(\lambda \Pi - c)$$
$$+ \lambda^2 \phi(w^* - u)^2(\lambda \Pi - 2c) + \phi^3 \left(\lambda \Pi + 2(e^{\lambda u} - 1)\right) \Bigg].$$

All three terms inside the brackets are clearly positive, implying the value function is concave in u for all $u \le w^*$, which completes the proof of (i).

We prove (ii) by showing that for any T, there exists a $w : [0, T] \to \mathbb{R}_+$ such that (1) it is incentive compatible for the agent to work for all $t \in [0, T]$, and (2) the agent's continuation utility at date t is $u(t) = \phi(T - t)$. Let $u_2(t)$ be the promised continuation value conditional on being in the second stage and $u(t)$ be the unconditional continuation value at t. Promise keeping requires that

$$\lambda u(t) = \lambda\left(\mu(t)w(t) + (1 - \mu(t))u_2(t)\right) + u'(t).$$

Conditional on progress, the evolution of u_2 is given by

$$\lambda u_2(t) = \lambda w(t) + u_2'(t). \tag{S.10}$$

We want to find $w(t)$ such that $u(t) = \phi(T - t)$ for all $t \in [0, T]$. Note that this implies that $u'(t) = -\phi$. Using the promise keeping condition,

$$\phi(T - t + 1/\lambda) = (1 - \mu(t))u_2(t) + \mu(t)w(t).$$

Substituting for $w(t)$ from (S.10), we get that

$$\phi\left(T - t + \frac{1}{\lambda}\right) = (1 - \mu(t))u_2(t) + \mu(t)\left(u_2(t) - \frac{u_2'(t)}{\lambda}\right)$$

$$= u_2(t) - \frac{t}{1 + \lambda t}u_2'(t).$$

Imposing the boundary condition $u_2(T) = 0$, we arrive at a unique solution for $u_2(t)$, which we can then substitute back into (S.10), to arrive at

$$w(t) = \phi\left(T - t + \frac{1}{\lambda} + \frac{e^{-\lambda(T-t)}}{\lambda^2 T} + \frac{e^{\lambda t}}{\lambda}\left(q(-\lambda T) - q(-\lambda t)\right)\right),$$

where $q(z) = -\int_{-z}^{\infty} e^{-x}/x\,dx$. It is straightforward to check that $w(t) > 0$ for all $t \in [0, T]$, which completes the proof of (ii). For (iii), note that $G(u)$ has the same form as $F_1(u)$ (see (A8)) with $H_1 = \frac{2c}{\lambda} - \Pi$. The result then follows from the fact that F_1 has a constant strictly larger than $\frac{2c}{\lambda} - \Pi$. $\qquad\square$

References

Green, Brett, and Curtis Taylor. 2016. "Breakthroughs, Deadlines, and Self-Reported Progress: Contracting for Multistage Projects." *American Economic Review (forthcoming)*.