Mergers, Entry, and Consumer Welfare

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Online Appendix

A Notes on Aggregative Games

In this appendix, we derive the aggregative games formulation of the Bertrand model with MNL, CES, NMNL, and NCES demand. We focus especially on the MNL and CES models in order to provide something of a "practitioner's guide" for those who previously have not studied aggregative games.

A.1 MNL Demand

We take as given the profit function and first order conditions of (1) and (2), the indirect utility of (3), and the market shares of (4). In this framework, it is well known that consumer surplus is given by

$$CS = \frac{1}{\alpha} \ln \left(1 + \sum_{j \in \mathcal{J}} \exp(v_j - \alpha p_j) \right). \tag{A.1}$$

The primitives of the aggregrative game reformulation are the vector of firm-specific types, $\{T^f\}\ \forall f \in \mathcal{F}$, and the price parameter, α. Equation (6) defines the type of each firm f as

$$T^f \equiv \sum_{j \in \mathcal{J}^f} \exp(v_j - \alpha c_j),$$

which represents the firm's contribution to consumer surplus if its prices equal its marginal costs. From these primitives, the Bertrand equilibrium can be characterized as a vector of "ι-markups," $\{\mu^f\}\ \forall f \in \mathcal{F}$, a vector of firm-level market shares, $\{s^f\}\ \forall f \in \mathcal{F}$, and a market aggregator, H. We define markups below, and let $s^f = \sum_{j \in \mathcal{J}^f} s_j$. The aggregator is defined as $H \equiv 1 + \sum_{j \in \mathcal{J}} \exp(v_j - \alpha p_j)$, which is the denominator from the market share formula of the product-level model (see (4)).

We first derive a relationship between the ι-markups and firm-level market shares. The product-specific price derivatives for logit demand are

$$\frac{\partial s_j}{\partial p_k} = \begin{cases} -\alpha s_j(1 - s_j) & \text{if } k = j \\ \alpha s_j s_k & \text{if } k \neq j. \end{cases}$$

Substituting these demand derivatives into the first order conditions of (2) for some product j and rearranging gives

$$\alpha(p_j - c_j) = 1 + \alpha \sum_{k \in \mathcal{J}^f} (p_k - c_k) s_k. \tag{A.2}$$

The right-hand side of this equation does not depend on the which product $j \in \mathcal{J}^f$ is referenced. This implies that the left-hand side is equivalent for all products sold by firm f, meaning each firm imposes a common markup (in levels) across all of its products. Following (7), define the ι-markup of firm f as $\mu^f \equiv \alpha(p_j - c_j)\ \forall j \in \mathcal{J}^f$. Substituting back into (A.2) obtains an equilibrium relationship between markups and shares:

$$\mu^f = \frac{1}{1 - s^f}. \tag{A.3}$$

We also have $s^f = (1/H)\sum_{j\in\mathcal{J}^f} \exp(v_j - \alpha p_j)$ from (4), after substituting in for the definition of the aggregator, H. Adding and subtracting αc_j inside the exponential and applying the definitions of μ^f and T^f gives

$$s^f = \frac{T^f}{H} \exp\left(-\mu_f\right) \tag{A.4}$$

$$\iff \frac{T^f}{H} = s^f \exp\left(\frac{1}{1 - s^f}\right). \tag{A.5}$$

Plugging (A.4) into (A.3), we obtain that equilibrium ι-markups satisfy (9):

$$\mu_f \left(1 - \frac{T^f}{H}\exp(-\mu_f)\right) = 1.$$

Let the unique solution for μ_f from this expression be written as $m(T^f/H)$. This *markup fitting-in function*, $m(\cdot)$, has the properties that $m(0) = 1$ and $m'(\cdot) > 0$. Plugging $\mu_f = m(T^f/H)$ into (A.4) yields the expression for equilibrium market shares provided in (10). Equilibrium market shares can be written $s^f = S(T^f/H)$, and thus equilibrium profit can be written $\Pi^f = \pi(T^f/H)$. To close the system, the aggregator satisfies an adding-up constraint of (11). The expressions for equilibrium profit and consumer surplus provided in (12) obtain immediately.

A.2 CES

Derivation of the CES aggregative game mirrors that of MNL case, except the CES demand derivatives and formula for shares must be used instead. With CES, the pricing first order condition for product j becomes

$$\sigma\frac{p_j - c_j}{p_j} = 1 + (\sigma - 1)\sum_{k\in\mathcal{J}^f} s_k \frac{p_k - c_k}{p_k}, \tag{A.6}$$

which is the counterpart to the MNL equation (A.2). We again see that the right-hand side of this equation does not depend on the identity of $j \in \mathcal{J}^f$, which in turn implies that each firm charges a constant percentage markup across all of its products.

Once we define the ι-markup as $\mu^f = \sigma(p_j - c_j)/p_j$ following (7), we obtain

$$\mu^f = \frac{1}{1 - \frac{\sigma-1}{\sigma}s^f} \tag{A.7}$$

after substituting into the pricing first order condition. Take the share equation (5) and multiply and divide it by $c_j^{1-\sigma}$. We can then substitute in the definitions of the aggregator H, μ^f, and the type T^f. Summing across the shares for the products sold by firm f gives

$$s^f = \frac{T^f}{H}\left(1 - \frac{\mu^f}{\sigma}\right)^{\sigma-1} \tag{A.8}$$

for firm-level revenue shares. Substituting this share into the markup expression in (A.7) gives the markup fitting-in function,

$$1 = \mu^f \left(1 - \frac{\sigma-1}{\sigma}\frac{T^f}{H}\left(1 - \frac{\mu^f}{\sigma}\right)^{\sigma-1}\right), \tag{A.9}$$

which appears in (9). The model is closed with the adding-up constraint given by (11).

A.3 NMNL

With NMNL demand, the following equations hold in Bertrand equilibrium:

$$\mu^f = \frac{1}{1-\rho s^{f|g} - (1-\rho)s^f} \tag{A.10}$$

$$1 = \mu^f \left(1 - \rho\frac{T^f}{H_g}\exp(-\mu^f) - (1-\rho)\frac{T^f}{H_g}\frac{H_g^{1-\rho}}{H}\exp(-\mu^f)\right) \tag{A.11}$$

$$\frac{T^f}{H_g} = s^{f|g}\exp\left(\frac{1}{1-\rho s^{f|g} - (1-\rho)s^f}\right) \tag{A.12}$$

$$\bar{s}_g = \frac{H_g^{1-\rho}}{H} \tag{A.13}$$

$$s^f = s^{f|g}\bar{s}_g \tag{A.14}$$

$$1 = \sum_{f\in\mathcal{F}_g} s^{f|g} \tag{A.15}$$

$$\frac{1}{H} = 1 - \sum_{f\in\mathcal{F}} s^f \tag{A.16}$$

$$\pi^f = \frac{1-\rho}{\alpha}\mu^f s^f \tag{A.17}$$

$$CS = \frac{1}{\alpha}\ln(H) \tag{A.18}$$

where T^f is the type of the firm, s^f is the share of the firm, $s^{f|g}$ is the share of the firm within its nest, $\bar{s}_g$ is the share of the nest, μ^f is the ι-markup of the firm, H_g is a nest aggregator, H is the market aggregator, π^f is the profit of the firm, and CS is consumer surplus.

Firm types are defined as in (23). Firm share is given by $s^f = \sum_{j\in\mathcal{J}^f} s_j$, as in the MNL and CES models. Firm share within its nest is given by $s^{f|g} = \sum_{j\in\mathcal{J}^f} s_{j|g}$, where the share of a product within a nest is

$$s_{j|g} = \frac{\exp\left(\frac{v_j - \alpha p_j}{1-\rho}\right)}{H_g}. \tag{A.19}$$

The aggregators are defined as $H_g \equiv \sum_{j\in\mathcal{J}_g} \exp((v_j - \alpha p_j)/(1-\rho))$ and $H \equiv 1 + \sum_{g\in\mathcal{G}} H_g^{1-\rho}$. The markup is defined as $\mu^f \equiv (\alpha/(1-\rho))(p_j - c_j)$ for all $j \in \mathcal{J}^f$.

The pricing first order condition for good j can be written as

$$\frac{\alpha}{1-\rho}(p_j - c_j) = 1 + \frac{\alpha\rho}{1-\rho}\sum_{k\in\mathcal{J}^f}(p_k - c_k)s_{k|g} + \alpha\sum_{k\in\mathcal{J}^f}(p_k - c_k)s_k, \tag{A.20}$$

under the assumption that firm f owns products only in nest g. We again see that the right-hand side of this condition does not depend on the identity of $j \in \mathcal{J}^f$. Substituting in for the definition of μ^f gives (A.10).

Next, adding and subtracting αc_j inside the exponential on the right-hand side of (A.19) and applying the definitions of μ^f, T^F, and H_g obtains

$$s^{f|g} = \frac{T^f}{H_g}\exp(-\mu^f), \tag{A.21}$$

which rearranges to (A.12). Then (A.11) can be obtained by plugging (A.12) and (A.13) back into (A.10). Next, (A.15) and (A.16) are adding-up constraints that close the model, (A.17) is obtained by plugging μ^f into the profit function, and (A.13), (A.14), and (A.18) follow directly from the NMNL functional form.

A.4 NCES

With NCES, the following equations hold in Bertrand equilibrium:

$$\mu^f = \frac{1}{1 - \frac{\gamma-1}{\sigma} s^f - \frac{\sigma-\gamma}{\sigma} s^{f|g}} \tag{A.22}$$

$$1 = \mu^f \left(1 - \frac{\gamma - 1}{\sigma} \frac{T^f}{H_g^{\frac{\sigma-\gamma}{\sigma-1}} H} \left(1 - \frac{\mu^f}{\sigma} \right)^{\sigma-1} - \frac{\sigma-\gamma}{\sigma} \frac{T^f}{H_g} \left(1 - \frac{\mu^f}{\sigma} \right)^{\sigma-1} \right) \tag{A.23}$$

$$\frac{T^f}{H_g} = s^{f|g} \left(1 - \frac{\mu^f}{\sigma} \right)^{1-\sigma} \tag{A.24}$$

$$s^f = \frac{T^f}{H_g^{\frac{\sigma-\gamma}{\sigma-1}} H} \left(1 - \frac{\mu^f}{\sigma} \right)^{\sigma-1} \tag{A.25}$$

$$1 = \sum_{f \in \mathcal{F}_g} s^{f|g} \tag{A.26}$$

$$\frac{1}{H} = 1 - \sum_{f \in \mathcal{F}} s^f \tag{A.27}$$

$$\pi^f = \frac{1}{\sigma} \mu^f s^f \tag{A.28}$$

$$CS = H^{1/(\gamma-1)} \tag{A.29}$$

where T^f is the type of the firm, s^f is the share of the firm, $s^{f|g}$ is the share of the firm within its nest, μ^f is the ι-markup of the firm, H_g is a nest aggregator, H is the market aggregator, π^f is the profit of the firm, and CS is consumer surplus.

Firm types are defined as in (23). Firm share is given by $s^f = \sum_{j \in \mathcal{J}^f} s_j$, as in the MNL and CES models. Firm share within its nest is given by $s^{f|g} = \sum_{j \in \mathcal{J}^f} s_{j|g}$, where the share of a product within a nest is

$$s_{j|g} = \frac{v_j p_j^{1-\sigma}}{\sum_{k \in \mathcal{J}_g} v_k p_k^{1-\sigma}}. \tag{A.30}$$

The aggregators are defined as $H_g \equiv \sum_{j \in \mathcal{J}_g} v_j p_j^{1-\sigma}$ and $H \equiv 1 + \sum_{g \in \mathcal{G}} H_g^{\gamma-1/\sigma-1}$. The markup is defined as $\mu^f \equiv \sigma(p_j - c_j)/p_j$ for all $j \in \mathcal{J}^f$, same as with CES demand.

The pricing first order condition for good j can be written as

$$\sigma \frac{p_j - c_j}{p_j} = 1 + \sum_{k \in \mathcal{J}^f} \frac{p_k - c_k}{p_k} [(\gamma - 1)s_k + (\sigma - \gamma)s_{k|g}] \tag{A.31}$$

under the assumption that firm f owns products only in nest g. We again see that the right-hand side of this condition does not depend on the identity of $j \in \mathcal{J}^f$. Substituting in for the definition of μ^f gives (A.22).

Next, multiplying and dividing by $c_j^{1-\sigma}$ on the right-hand side of (A.30) and applying the definitions of μ^f, T^F, and H_g obtains

$$s^{f|g} = \frac{T^f}{H_g}\left(1 - \frac{\mu^f}{\sigma}\right)^{\sigma-1}. \tag{A.32}$$

which rearranges to (A.24). Applying the same computation to (21) gives (A.25). Then (A.23) can be obtained by plugging (A.24) and (A.25) back into (A.22). Next, (A.26) and (A.27) are adding-up constraints that close the model, (A.28) is obtained by plugging μ^f into the profit function, and (A.29) follows directly from the NCES functional form.

B Numerical Extensions

B.1 Delayed and Probabilistic Entry

Our baseline model considers a three-stage game in which (1) firms decide to merge, (2) an outsider decides to enter, and (3) payoffs are realized according to a differentiated pricing game. In this appendix, we consider two variants. The first is a model of *delayed entry* in which incumbents obtain payoffs for N periods before entry occurs (if it does occur). The second is a model of *probabilistic entry* in which entry occurs in the second stage with some fixed probability p if it is profitable, and with probability zero otherwise.

With delayed and probabilistic entry, a merger that induces entry increases the net present value of the merging firms if and only if

$$\frac{1-\theta}{1-\delta}\pi^M_{m,ne} + \frac{\theta}{1-\delta}\pi^M_{m,e} \geq \sum_{i=1,2}\frac{1}{1-\delta}\pi^i_{nm,ne}, \tag{B.1}$$

where δ is a discount factor, $\theta = \delta^N$ with delayed entry, and $\theta = p$ with probabilistic entry. Similarly, a merger that induces entry increases the net present value of consumer surplus if and only if

$$\frac{1-\theta}{1-\delta}CS^{m,ne} + \frac{\theta}{1-\delta}CS^{m,e} \geq \frac{1}{1-\delta}CS^{nm,ne}. \tag{B.2}$$

As these equations nest both delayed and probabilistic entry, we proceed by analyzing mergers and entry in the two models jointly.

With $\theta = 1$, the analytical results from in the main body of the paper obtain, and with MNL or CES demands merger-induced entry sufficient to preserve consumer surplus renders the merger unprofitable. At the other end, entry is irrelevant with $\theta = 0$.

With $\theta \in (0, 1)$, our intuition is that Proposition 1 extends for most reasonable parameterizations. The reason is that as θ decreases from one, the profitability of the merger increases but so does the consumer surplus loss. Given the strict inequalities we obtain, the first of these effects would have to be considerably stronger than the second to generate a profitable, pro-competitive merger. Our examination of the implied relationships indicates this is unlikely to be the case.

In support of this conjecture, we conduct numerical simulations using a model with two incumbents and MNL demand. We consider market shares for the incumbents that range from 1% to 80%. After calibrating incumbent types, we examine entrants with types that range between that of the compensating entrant (Proposition 4) and ten times that of the merged firm. Finally, for each of these, we scale θ between zero and one in increments of 0.01. We find no cases in which a profitable merger increases consumer surplus.

This is not to claim that profitable, pro-competitive mergers cannot be found with *unreasonable* parameterizations. Indeed, for any initial set of incumbent types and MNL or CES demands, we can prove that there exists some θ and entrant type T^F for which a profitable merger improves consumer surplus. As one example, suppose that two incumbents each have of a market share of 40% initially. The implied types are $T^1 = T^2 = 10.59$. Further let $\theta = 0.099$, which obtains with 21.96 years of delay (given $\delta = 0.90$) or with a probability of post-merger entry just less than 10%. If, in addition, the entrant's type exceeds 3.59×10^{102}, then a profitable, pro-competitive merger obtains.[28] This entrant captures a market share of 99.6%; the incumbents' combined market share decreases to 0.4% and the share of the outside good is approximately zero.

We now formally state that with delayed and probabilistic entry, the model can generate profitable, pro-competitive mergers.

Proposition B.1. *Fix an initial market structure comprising $f = 1, \ldots, F-1$ incumbents and their types, and consider a merger of firms 1 and 2. With MNL and CES demands, there exists a θ and entrant type T^F such that the merger with induced entry increases the present value of consumer surplus and the merging firms' profit.*

Proof. See Appendix C. □

For intuition, if θ is small enough—i.e., entry is sufficiently delayed or unlikely—then a merger increases the present value of the merging firms' profit, even if this profit is approximately zero in every period after entry occurs. Thus for any baseline calibration, by choosing a small enough θ, the profit and surplus inequalities 'decouple,' in the sense that the profit inequality holds for *any* value of entrant type. However, as consumer surplus increases to infinity with the type of the entrant, one can then always find some sufficiently capable entrant such that the present value of consumer surplus increases. The numerical results we describe above suggest that this theoretical possibility is not practically relevant for merger review.

B.2 Random Coefficients Logit

The random coefficients logit (RCL) demand system is widely employed in modern empirical studies due to its flexibility. In this section, we explore whether the basic intuition that emerges from our analysis of NMNL and NCES demand—that entry by a distant competitor can offset the adverse effect of a profitable merger in SPE—extends to the RCL model. Throughout this section we rely on numerical analysis due to the RCL model's failure to exhibit the type aggregation or common markup properties underpinning our earlier analytic results.

We first consider whether merger-induced entry sufficient to eliminate consumer surplus loss from a profitable merger (without efficiencies) requires an entrant with products that are distant substitutes to those of the merging firms. We assume the indirect utility that consumer

[28] For comparison, there are approximately 2.40×10^{67} atoms in the Milky Way galaxy.

i receives from product j is

$$u_{ij} = (1 + \beta_i)v_j - \alpha p_j + \epsilon_{ij}, \tag{B.3}$$

where ϵ_{ij} is iid Type I extreme value and $\beta_i \sim N(0,1)$ is a consumer-specific valuation for quality. There are two single-product incumbents, each with $v_j = 4$ and $c_j = 2$. We consider four values of the price parameter: $\alpha = (1, 2, 3, 4)$. The larger values imply more elastic demand. With $\alpha = 4$, the pre-merger equilibrium features prices of 2.36, incumbent market shares of 6.4%, and a diversion ratio between incumbents of 45%. With $\alpha = 1$, these statistics are 3.72, 30%, and 72%, respectively. We consider entrants with marginal costs and qualities that range between -2 and 8. With a step size of 0.05, this yields 40,401 entrants. We simulate a merger between the incumbents under the assumption that it induces entry by one of the entrants. Iterating through the entrants, we determine whether consumer surplus and the merging firms' profit increase relative to the pre-merger baseline.

Figure B.1 summarizes the results. In each panel, the shaded gray region provides the entrant qualities and marginal costs for which the merger is both profitable and increases consumer surplus. In the top left panel ($\alpha = 4$), this region features entrant marginal costs that are close to zero or negative and entrant quality that is substantially less than that of the merging firms.[29] Comparing across panels, as demand becomes less elastic and incumbent market powers grows, the gray region requires even lower entrant marginal costs and qualities. In the bottom right panel ($\alpha = 1$), the region does not exist within the considered marginal cost and quality ranges.

We interpret these results as indicating that the intuition behind our results for the NMNL and NCES models extends to the RCL model: merger-induced entry sufficient to preserve consumer surplus can be compatible with a profitable merger, but only if the entrant's products are differentiated enough from those of the merging firms. The model also is informative of the entrant characteristics under which surplus-preserving merger-induced entry can arise in SPE. Thus in empirical work, knowledge of the production technologies could be paired with the model to determine whether merger-induced entry that restores consumer surplus in SPE is plausible. For example, the model might indicate that the entrant's marginal costs would have to be negative, or that its quality would have to be much higher than that of the incumbents.

C Section 3 Proofs

C.1 Proofs of Lemmas

C.1.1 Lemma 1

Proof. **(i)** $\implies$ **(ii)**: Suppose (i) holds, that is,

$$\pi^f_{nm,ne} = \pi^f_*.$$

[29] We suspect that a similar region exists for entrant costs and quality that are both much higher than the merging firms, but computing equilibrium in that parameter range is difficult for numerical reasons.

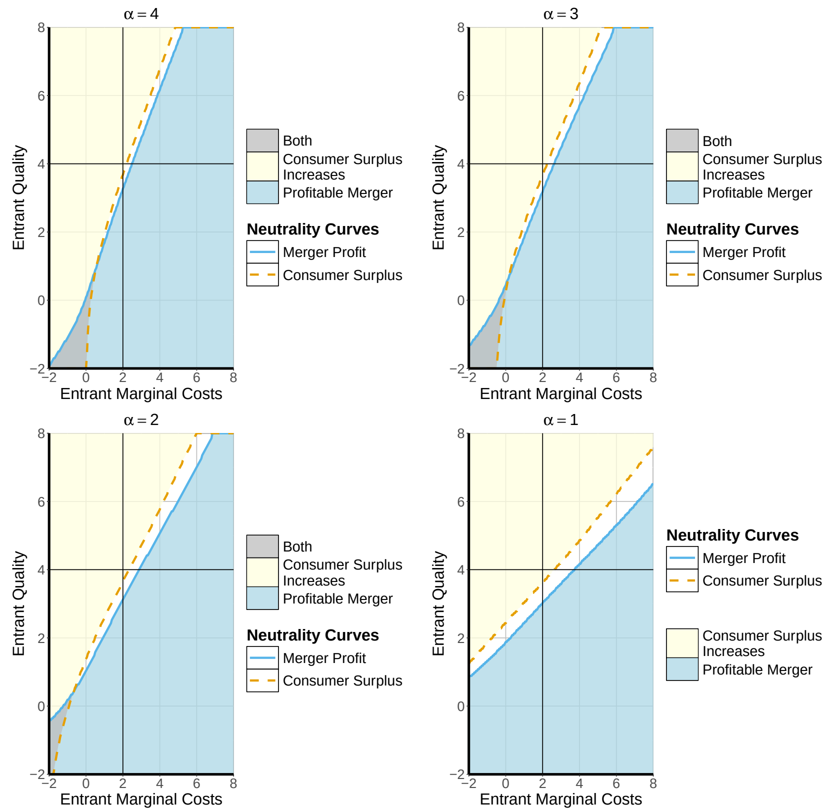

Figure B.1: Numerical Results for RCL Demand with $\alpha = (4, 3, 2, 1)$

Notes: The panels show the combinations of entrant quality and marginal cost for which a merger with entry increases consumer surplus (shaded yellow), increases the merging firms' profit (shaded blue), or both (shaded gray). The corresponding neutrality curves for merger profitability and consumer surplus are plotted as solid blue and dashed orange lines, respectively. The marginal cost and quality of the merging firms are plotted with the black vertical and horizontal lines.

By (12), $\mu^f_{nm,ne} = \mu^f_*$, and by (A.3), $s^f_{nm,ne} = s^f_*$. Because $T^f = T^f_{nm,ne} = T^f_*$ by hypothesis, (10) implies

$$\frac{T^f}{H_{nm,ne}} = s^f_{nm,ne} \exp\left(\frac{1}{1 - s^f_{nm,ne}}\right) = s^f_* \exp\left(\frac{1}{1 - s^f_*}\right) = \frac{T^f}{H_*},$$

and thus $H_{nm,ne} = H_{m,e}$, which implies (ii).

(ii) $\implies$ (i): Suppose now that $H_{nm,ne} = H_* = H$. By (10), we obtain $s^f_{nm,ne} = s^f_*$ for every $f \in \mathcal{F}_{nm,ne}$ immediately, and (i) follows by a chain of substitutions identical to the above.

(ii) $\implies$ (iii): Suppose now that $H_{nm,ne} = H_* = H$. From (11),

$$\frac{1}{H} + \sum_{f \in \mathcal{F}_{nm,ne}} s^f_{nm,ne} = \frac{1}{H} + \sum_{f \in \mathcal{F}_*} s^f_* \iff \sum_{f \in \mathcal{F}_{nm,ne}} s^f_{nm,ne} = \sum_{f \in \mathcal{F}_*} s^*_m,$$

which implies (iii) immediately upon cancelling terms (via appeal to (ii) implying (i) and hence to the shares also coinciding across scenarios).

(iii) $\implies$ **(ii)**: We proceed by contraposition. Thus suppose that the merger affects consumer surplus: $H_{nm,ne} \neq H_*$. Let f belong to both $\mathcal{F}_{nm,ne}$ and $\mathcal{F}_*$, i.e. let f denote any firm other than $1, 2$, M or potentially F. By (10), we have

$$\frac{T^f}{H_{nm,ne}} = s^f_{nm,ne} \exp\left(\frac{1}{1 - s^f_{nm,ne}}\right)$$

and

$$\frac{T^f}{H_*} = s^f_* \exp\left(\frac{1}{1 - s^f_*}\right).$$

For both equations, the right-hand side is strictly increasing in the relevant share, and thus for all such f,

$$\frac{1}{H_{nm,ne}} > \frac{1}{H_*} \iff s^f_{nm,ne} > s^f_*.$$

Thus,

$$\frac{1}{H_{nm,ne}} + \sum_{f \in \mathcal{F}_{nm,ne} \cap \mathcal{F}_*} s^f_{nm,ne} \neq \frac{1}{H_*} + \sum_{f \in \mathcal{F}_{nm} \cap \mathcal{F}_*} s^f_*,$$

and it follows by (11) that (iii) cannot hold. □

C.1.2 Other Lemmas

Lemma C.1. *In Bertrand equilibrium with MNL demand, all firms with positive share have markups such that $\mu^f \in (1, \infty)$. If we instead have CES demand, all firms with positive share have markups such that $\mu^f \in (1, \sigma)$.*

Proof. In equilibrium in the MNL case we have that

$$\mu^f = \frac{1}{1 - s^f}$$

from (A.3). There is an outside good with positive share, so $s^f < 1$ for all active firms. Thus we have that $\mu^f > 1$, since the denominator in the expression above, $1 - s^f$, is less than one for all positive values of s^f. We also have that μ^f approaches infinity as s^f approaches 1.

In equilibrium in the CES case we have that

$$\mu^f = \frac{1}{1 - \frac{\sigma - 1}{\sigma} s^f} = \frac{\sigma}{\sigma - s^f(\sigma - 1)}$$

from (A.7). Given that there is an outside good with positive share, $s^f < 1$ for all active firms. Thus, the first equality implies that $\mu^f > 1$, since the denominator $1 - ((\sigma - 1)/\sigma)s^f$ is less than one for all positive values of s^f. The second equality implies that μ^f is bounded above by σ as s^f approaches 1. □

Lemma C.2. *Define the function*

$$\phi(x) \equiv \begin{cases} xe^{-x} & \textit{(MNL or NMNL)} \\ x\left(1-\frac{x}{\sigma}\right)^{\sigma-1} & \textit{(CES or NCES)} \end{cases} \tag{C.1}$$

where the first specification applies to the MNL and NMNL models, and the second applies to the CES and NCES models. This function $\phi(\cdot)$ is decreasing on $(1,\infty)$ for the MNL/NMNL specification and decreasing on $(1,\sigma)$ for the CES/NCES specification.

Proof. The derivative for the MNL/NMNL specification is

$$\frac{d}{dx}\phi(x) = (1-x)\exp(-x).$$

This derivative is negative if and only if $1-x$ is negative. This in turn is true if $x > 1$.

For the CES/NCES specification, we employ a change of variables by defining $\tilde{x} = x/\sigma$. The derivative of the redefined function has the same sign as the original, since σ is positive. We have that $\phi(\tilde{x}) = \sigma\tilde{x}(1-\tilde{x})^{\sigma-1}$. Then the derivative is

$$\frac{d}{d\tilde{x}}\phi(\tilde{x}) = \sigma(1-\tilde{x})^{\sigma-1}\left[1-\frac{\tilde{x}(\sigma-1)}{1-\tilde{x}}\right].$$

This derivative is negative in the relevant range if and only if the term in brackets is negative, because $(1-\tilde{x})$ is positive for all $x \in (1,\sigma)$. The term in brackets is negative if and only if $\tilde{x} > 1/\sigma$. We know that $x > 1$, so this condition is met. □

C.2 Proof of Proposition 1

Proof. We first show that, for all choices of T^F, there is a unique efficiency E that makes the merger CS-neutral. Fix T^F and suppose that the merger is CS-neutral. Then $H_{nm,ne} = H_{m,e} = H$. Since types are unchanged across market structures, by (10) and (11) it follows that

$$s^1_{nm,ne} + s^2_{nm,ne} = s^F_{m,e} + s^M_{m,e}. \tag{C.2}$$

This establishes claim (iii). Clearly $s^1_{nm,ne}$ and $s^2_{nm,ne}$ do not depend upon E. Moreover, by (9) and (10), $s^F_{m,e}$ depends only on T^F and H, not E. Then, by appeal to (10) and (A.3), the only term in (C.2) that depends on E is pinned down by

$$\frac{T^1+T^2+E}{H} = s^M_{m,e}\exp\left(\frac{1}{1-s^M_{m,e}}\right), \tag{C.3}$$

the left-hand side of which is strictly increasing in E. However, by (C.2), the right-hand side does not depend on E and hence there can be only one such value for E.

We now establish that the CS-neutrality curve is downward-sloping. To this end, suppose consumer surplus is unchanged across the nm, ne and m, e equilibria, and hence that (C.3) obtains. By an identical argument, for the entrant F,

$$\frac{T^F}{H} = s^F_{m,e}\exp\left(\frac{1}{1-s^F_{m,e}}\right). \tag{C.4}$$

Suppose T^F is increased. This does not change H, as it is pinned down by its value in the nm, ne equilibrium (which does not depend upon T^F) and our hypothesis of CS neutrality. Then by (C.4), an increase in T^F leads to a higher equilibrium share $s^F_{m,e}$. But by (C.2) this implies a corresponding, equivalent decrease in $s^M_{m,e}$ as $s^1_{nm,ne}$ and $s^2_{nm,ne}$ do not depend upon T^F or E. By (C.3), we then conclude the CS neutrality curve is downward sloping.

Finally, claim (ii) follows immediately from the above, and the definitions of these objects.

□

C.3 Proof of Proposition 2

Proof. We first establish claim (iii). Suppose that the merger is profit-neutral:

$$\pi^1_{nm,ne} + \pi^2_{nm,ne} = \pi^M_{m,e}.$$

By (12), it follows that

$$\mu^M_{m,e} + 1 = \mu^1_{nm,ne} + \mu^2_{nm,ne}.$$

By substituting using (A.3) and solving for $s^M_{m,e}$ in terms of $s^1_{nm,ne}$ and $s^2_{nm,ne}$, we obtain

$$s^M_{m,e} = 1 - \frac{(1 - s^1_{nm,ne})(1 - s^1_{nm,ne})}{1 - s^1_{nm,ne} s^2_{nm,ne}}$$

as desired.

We now show that for all values of T^F, there is a unique efficiency E that makes the merger profit-neutral. Suppose then for some T^F, that there exists some efficiency E is such that the merger is profit-neutral. Then by (10) and (A.3), E satisfies

$$\frac{T^1 + T^2 + E}{H_{m,e}} = s^M_{m,e} \exp\left(\frac{1}{1 - s^M_{m,e}}\right).$$

However, (iii) implies the right-hand side is constant in E, as it is a function solely of the pre-merger equilibrium quantities $s^1_{nm,ne}$ and $s^2_{nm,ne}$, which do not depend on E. In the Online Appendix (p.110) of Nocke and Schutz (2018), it is shown that for any firm f, T^f/H is increasing in T^f. This implies that if there exists any such E, then it is necessarily unique. To show such an E exists, it suffices to show that the left-hand side (i.e. $T^M/H_{m,e}$) is unbounded above in T^M. Suppose, for sake of contradiction, this is not the case. Then as $T^M/H_{m,e}$ is increasing and bounded above, it converges to some limit $K < \infty$. Since $T^M \to \infty$, this implies $\lim_{T^M \to \infty} H_{m,e} = \infty$ as well. Thus for any $g \in \mathcal{F}_{m,e}$, $g \neq M$, (10) and (A.3) imply that $s^g_{m,e} \to 0$. As g was arbitrary, by (11), $s^M_{m,e} \to 1$ and hence by (10) and (A.3) $T^M/H_{m,e} \to \infty$, a contradiction. Thus $\lim_{T^M \to \infty} T^M/H_{m,e} = \infty$, and in particular, for any such T^F, there exists an E such that the merger is profit-neutral.

We now establish claim (i), that the merger profit-neutrality curve is upward sloping. Suppose that, for T^F, E is such that the merger is profit-neutral. Then, as noted prior, E must satisfy

$$\frac{T^1 + T^2 + E}{H_{m,e}} = s^M_{m,e} \exp\left(\frac{1}{1 - s^M_{m,e}}\right),$$

where, by (iii), the right-hand side is a constant function in E. Suppose T^F increases. This increases $H_{m,e}$. Since the right-hand side of the above is constant in T^F and $H_{m,e}$, for the equality to hold, the unique solution in E must increase (given the left-hand side is increasing and unbounded in E).

For (ii), the first claim follows immediately from the definitions of $\bar{T}^F$. For the latter claim, suppose that $E = \bar{E}$, and observe that if $T^F = 0$, then the merger is profitable. Conversely, suppose $T^F \to \infty$. Then, as shown above, $T^F/H_{m,e} \to \infty$ as well. By (10), $s^F_{m,e} \to 1$, and hence $s^M_{m,e}$ and $\pi^M_{m,e} \to 0$. Thus we conclude that as $T^F \to \infty$, the merged entrant's profits monotonically decreases to 0. Since pre-merger, the entrant is not in the market, the pre-merger profits of the merging entities are unaffected by T^F, there exists some T^F for which $(T^F, \bar{E})$ makes the merger profit-neutral; as $\pi^M_{m,e}$ is globally decreasing in T^F, this $\bar{T}^F$ is unique. □

C.4 Proof of Proposition 3

Proof. We first establish that, for all choices of $T^F > 0$, there is a unique efficiency E that makes the merger cause the entrant to be profit-neutral. Fix T^F and consider the associated nm, e and m, e equilibria. If the entrant's profits are equal across both equilibria, then by Lemma 1, $H_{nm,e} = H_{m,e} = H$, and

$$s^1_{nm,e} + s^2_{nm,e} = s^M_{m,e}.$$

In the m, e equilibrium we have that

$$\frac{T^1 + T^2 + E}{H} = s^M_{m,e} \exp\left(\frac{1}{1 - s^M_{m,e}}\right),$$

the left-hand side of which is strictly increasing in E. However, the right hand side is injective in $s^M_{m,e}$, and $s^M_{m,e}$ is fixed by the nm, e equilibrium and hence its equilibrium is fixed under the hypothesis of entrant profit-neutrality. Thus there can be only one E satisfying the above.[30]

We consider now claim (i), that the entrant profit neutrality curve is downward sloping. By Lemma 1, we know that $H_{nm,e} = H_{m,e} = H$ and $s^1_{nm,e} + s^2_{nm,e} = s^M_{m,e}$. In equilibrium,

$$\frac{T^1 + T^2 + E}{H} = s^M_{m,e} \exp\left(\frac{1}{1 - s^M_{m,e}}\right).$$

By Proposition 6 of Nocke and Schutz (2018), an increase in T^F for fixed E leads to a decrease in $s^1_{nm,e}$ and $s^2_{nm,e}$. But this implies a decrease in $s^M_{m,e}$ as it is the sum of these terms. Thus there must be a commensurate decrease in E.

We now establish claim (ii). Consider the following three market structures: $\mathcal{F}_{nm,ne}$, $\mathcal{F}_{nm,e}$, and $\mathcal{F}_{m,e}$. The entry neutrality line is determined by profit-neutrality across $\mathcal{F}_{nm,e}$ and $\mathcal{F}_{m,e}$; the CS neutrality line is determined by surplus remaining constant across $\mathcal{F}_{nm,ne}$ and $\mathcal{F}_{m,e}$. We first claim that if the two curves intersect for some (T^F, E) then $T^F = 0$. By Lemma 1, $CS_{nm,e} = CS_{m,e}$; by hypothesis, $CS_{m,e} = CS_{nm,ne}$. Hence in particular, $H_{nm,ne} = H_{nm,e} =$

[30] Here, as H is fixed by the nm, e equilibrium value, the left hand side is unbounded in E.

H. Then for each $f \in \mathcal{F}_{nm,e} \setminus \{F\}$, we have

$$s^f_{nm,e} \exp\left(\frac{1}{1-s^f_{nm,e}}\right) = \frac{T^f}{H} = s^f_{nm,ne} \exp\left(\frac{1}{1-s^f_{nm,ne}}\right),$$

and hence $s^f_{nm,e} = s^f_{nm,ne}$. By the adding up constraint,

$$\sum_{f \in \mathcal{F}_{nm,ne}} s^f = \sum_{f \in \mathcal{F}_{nm,e}} s^f,$$

and thus $s^F = 0$ and hence so too is T^F. Thus consider $T^F \to^+ 0$. If $T^F > 0$, then $CS_{nm,e} > CS_{nm,ne}$, however, $\lim_{T^F \to^+ 0} CS_{nm,e} = CS_{nm,ne}$. Thus as $T^F \to^+ 0$, the associated efficiency tends to $\bar{E}$ by definition.

Suppose now that $T^F > 0$. We will establish that the unique E such that (T^F, E) is entrant profit-neutral must be strictly positive. Suppose, for sake of contradiction, that $E = 0$. Since $T^M > \max\{T^1, T^2\}$, following the merger the markups for the merging firms increase. Given marginal costs remain fixed, the corresponding equilibrium prices increase and hence the effect of the merger on H is an unambiguous decrease. But this implies then $\pi^F_{m,e} > \pi^F_{nm,e} > 0$, a contradiction. By an argument analogous to that appearing in the proof of Proposition 2, an E such that the merger is profit-neutral for F must exist, thus we conclude $E > 0$.

Finally, claim (iii) follows from Proposition 1, and the immediate observation that, ceteris paribus, entry increases consumer surplus. □

C.5 Proof of Proposition 4

Proof. We begin by characterizing the implicit functions Υ^{MNL} and Υ^{CES}. With MNL demand, the type of the compensating entrant, $\tilde{T}^F$, satisfies

$$\frac{\tilde{T}^F}{T^1 + T^2} = \frac{(s^1 + s^2 - s^M) \exp\left(\frac{1}{1-s^1-s^2+s^M}\right)}{s^1 \exp\left(\frac{1}{1-s^1}\right) + s^2 \exp\left(\frac{1}{1-s^2}\right)}, \tag{C.5}$$

where s^M is the unique solution to

$$s^M \exp\left(\frac{1}{1-s^M}\right) = s^1 \exp\left(\frac{1}{1-s^1}\right) + s^2 \exp\left(\frac{1}{1-s^2}\right). \tag{C.6}$$

With CES demand, the type of the compensating entrant satisfies

$$\frac{\tilde{T}^F}{T^1 + T^2} = \frac{(s^1 + s^2 - s^M)\left(\sigma + \frac{s^1+s^2-s^M}{1-(s^1+s^2-s^M)}\right)^{\sigma-1}}{s^1\left(\sigma + \frac{s^1}{1-s^1}\right)^{\sigma-1} + s^2\left(\sigma + \frac{s^2}{1-s^2}\right)^{\sigma-1}}, \tag{C.7}$$

where s^M is the unique solution to

$$s^M \left(\sigma + \frac{s^M}{1-s^M}\right)^{\sigma-1} = s^1 \left(\sigma + \frac{s^1}{1-s^1}\right)^{\sigma-1} + s^2 \left(\sigma + \frac{s^2}{1-s^2}\right)^{\sigma-1}. \tag{C.8}$$

In order to derive the above relationships, begin by rearranging (10) to solve for firm type, giving

$$T^f = \begin{cases} Hs^f \exp\left(\frac{1}{1-s^f}\right) & \text{(MNL)} \\ Hs^f (\sigma-1)^{1-\sigma} \left(\sigma + \frac{s^f}{1-s^f}\right)^{\sigma-1} & \text{(CES)} \end{cases} \tag{C.9}$$

after substituting in for markups. Then evaluate this type equation for firm F after the merger and firms 1 and 2 before the merger, substituting in for the entrant share using $s^F = s^1 + s^2 - s^M$, which obtains from Lemma 1. Dividing the result for firm F by the sum of the results for firms 1 and 2 gives (C.5) and (C.7) for MNL and CES, respectively.

Without efficiencies, $T^M = T^1 + T^2$. Substituting into this sum for types using (C.9) gives (C.6) and (C.8). These two expressions have unique positive solutions because the expressions $x \exp(1/(1-x))$ and $x(\sigma + x/(1-x))^{\sigma-1}$ are increasing if $x \in [0, 1)$.

Furthermore, with MNL and CES demand, we can characterize the relationship between the entrant's type $\tilde{T}^F$ and an "average" type. Let s^a be the average of s^1 and s^2, calculated as $(s^1 + s^2)/2$. Let T^a be the type that generates a share of s^a given aggregator H, which can be found by solving (10) holding H fixed. (Note that if $s^1 \geq s^2$, then $s^1 \geq s^a \geq s^2$ and $T^1 \geq T^a \geq T^2$, the latter due to the monotonicity of shares in terms of T^f/H.) We can show that $\tilde{T}^F < T^a$ and $s^F < \frac{1}{2}\left(s^1 + s^2\right)$. In order for consumers to be unharmed, H must be unchanged due to the merger. Therefore, since $T^M > T^1$ and $T^M > T^2$, $T^M/H > T^1/H$ and $T^M/H > T^2/H$. In turn, this means that $s^M > s^1$ and $s^M > s^2$, since shares are increasing in T^f/H. Adding these inequalities gives $2s^M > s^1 + s^2$, and then dividing by two gives $s^M > s^a$. As shown by Lemma 1, if H remains the same, then $s^F + s^M = s^1 + s^2$, which also means that $s^F + s^M = 2s^a$. In order for this equality to hold when we also know that $s^M > s^a$, it must be that $s^F < s^a$. By the monotonicity of shares, this means that $\tilde{T}^F < T^a$. $\square$

C.6 Proof of Proposition 5

Proof. Suppose, for purposes of contradiction, there exists a SPE in which firms 1 and 2 merge, and consumers surplus does not decrease. Thus the merger must increase joint profits:

$$\pi^M \left(\frac{T^1 + T^2}{H_{m,e}}\right) \geq \pi^1 \left(\frac{T^1}{H_{nm,ne}}\right) + \pi^2 \left(\frac{T^2}{H_{nm,ne}}\right),$$

where $H_{nm,ne}$ denotes the aggregator with no merger and no entry. By hypothesis, consumers surplus does not fall, hence we have $H_{nm,e} \leq H_{m,e}$. Furthermore, by Nocke and Schutz (2018, Proposition 6), π^M is decreasing in H all else equal, meaning that

$$\pi^M \left(\frac{T^1 + T2}{H_{nm,ne}}\right) \geq \pi^1 \left(\frac{T^1}{H_{nm,ne}}\right) + \pi^2 \left(\frac{T^2}{H_{nm,ne}}\right). \tag{C.10}$$

Multiplying the markup and firm share shows that firm profit is given by

$$\pi^f = \begin{cases} \frac{1}{\alpha}\mu^f \frac{T^f}{H}\exp(-\mu^f) & \text{(MNL)} \\ \frac{1}{\sigma}\mu^f \frac{T^f}{H}\left(1-\frac{\mu^f}{\sigma}\right)^{\sigma-1} & \text{(CES).} \end{cases}$$

Then (C.10) is satisfied, after canceling certain constants, if and only if

$$(T^1+T^2)\phi\big(m(T^1+T^2, H^{nm,ne})\big) \geq T^1\phi\big(m(T^1, H^{nm,ne})\big) + T^2\phi\big(m(T^2, H^{nm,ne})\big),$$

where $\phi(\cdot)$ is defined as in C.1, and $m(\cdot)$ denotes the markup fitting-in function for the MNL or CES, as appropriate. This expression is equivalent to

$$\sum_{i\in\{1,2\}} T^i\big[\phi\big(m(T^i, H^{nm,ne})\big) - \phi\big(m(T^1+T^2, H^{nm,ne})\big)\big] \leq 0,$$

which is an impossibility. The function $\phi(\cdot)$ is decreasing for all possible markup values for both the MNL and CES cases according to Lemma C.2. Furthermore, for all i, $m(T^1+T^2) > m(T^i)$, since Nocke and Schutz (2018, Proposition 6) implies that markups are increasing in type for fixed H. Therefore, the sum above is component-wise strictly positive, which is a contradiction. □

C.7 Proof of Proposition 6

Proof. Immediate from Propositions 1 - 3. □

C.8 Proof of Proposition 7

Proof. Let $(\underline{E}, T^F)$ be such that (i) the merger is profit-neutral and (ii) consumer surplus is unchanged due to the merger. From (ii), we know that the aggregator is constant at some level H. From (C.9), we also have

$$T^M = T^1 + T^2 + \underline{E} = \begin{cases} Hs^M \exp\left(\frac{1}{1-s^M}\right) & \text{(MNL)} \\ Hs^M(\sigma-1)^{1-\sigma}\left(\sigma + \frac{s^M}{1-s^M}\right)^{\sigma-1} & \text{(CES).} \end{cases}$$

Plugging in for T^1 and T^2 again using (C.9) and solving for $\underline{E}$ yields (16) and (18). From (i), we obtain (17) and (19). We derive these expressions by evaluating the profit functions in (12) for the merged firms before and after the merger, plugging into $\pi^M = \pi^1 + \pi^2$, and substituting in for markups using (A.3) and (A.7), for MNL and CES, respectively (see the proof of Proposition 2, which works out the MNL case in more detail). □

C.9 Proof of Proposition 8

Proof. The proof mirrors that for Proposition 5, but within a nest. Suppose, for purposes of contradiction, there exists a SPE in which firms 1 and 2 merge, and consumers are unharmed.

Thus the merger must increase joint profits:

$$\pi^M(T^1+T^2,H_g^{m,e}) \geq \pi^1(T^1,H_g^{nm,ne}) + \pi^2(T^2,H_g^{nm,ne}),$$

where $H_g^{nm,ne}$ denotes the nest-level aggregator with no merger and no entry, while $H_g^{m,e}$ is the same object but for a merger with entry. The products in all other nests remain the same, meaning that the resulting overall aggregator is a function of activity from nest g, so we have dropped H in order to save on notation.

By hypothesis, consumers are unharmed, hence we have $H_g^{m,e} \geq H_g^{nm,ne}$. Furthermore, profits are decreasing in H_g according to Nocke and Schutz (2018, Proposition 6), extended to NMNL and NCES in their Appendix (pp. 104-106). Therefore, we have

$$\pi^M(T^1+T^2,H_g^{nm,ne}) \geq \pi^1(T^1,H_g^{nm,ne}) + \pi^2(T^2,H_g^{nm,ne}). \tag{C.11}$$

Multiplying the markup and firm share shows that firm profit is given by

$$\pi^f \equiv \begin{cases} \frac{1-\rho}{\alpha}\mu^f \frac{T^f}{H_g}\exp(-\mu^f)\bar{s}_g & \text{(NMNL)} \\ \frac{1}{\sigma}\mu^f \frac{T^f}{H_g^{\frac{\gamma-\sigma}{1-\sigma}}H}\left(1-\frac{\mu^f}{\sigma}\right)^{\sigma-1} & \text{(NCES).} \end{cases}$$

Substituting for profit in the inequality expression (C.11) with $\phi(\cdot)$ from (C.1) and canceling gives the condition

$$(T^1+T^2)\phi\big(m(T^1+T^2,H_g^{nm,ne})\big) \geq T^1\phi\big(m(T^1,H_g^{nm,ne})\big) + T^2\phi\big(m(T^2,H_g^{nm,ne})\big),$$

where $m(\cdot)$ denotes the markup fitting-in function for the NMNL or NCES, as appropriate. The profit inequality in (C.11) is satisfied if and only if this condition holds. Note that this condition is analogous to that in the non-nested proof for Proposition 5. Markups are also increasing in type, all else equal (again referencing Nocke and Schutz (2018, Proposition 6)). Thus, we also arrive at a contradiction in the nested case as well. □

C.10 Proof of Proposition 9

Traditionally, the continuity of a fixed point as a function of some set of parameters is established via an appeal to an appropriate form of the implicit function theorem. However, this requires one to consider parameters on the interior of their domain, whereas here we wish to establish continuity precisely on the boundary. Thus we instead employ an approach dating back to Mas-Colell (1974) utilizing a generalization of the implicit function theorem known as the regular value theorem (see Hirsch (2012), Theorem 1.4.1) which remains valid for problems on the boundary.

C.10.1 NMNL Preliminaries

We will prove Proposition 9 by first establishing two intermediate technical results. Define[31]

$$\Omega_g(H, H_g; \rho) = \frac{1}{H_g} \sum_{f \in \mathcal{F}_g} \sum_{j \in \mathcal{J}^f} \exp\left[\frac{\delta_j - \alpha c_j}{1-\rho} - \tilde{m}^f\left(\frac{\rho}{H_g} + (1-\rho)\frac{1}{H_g^\rho H}\,; \rho\right)\right].$$

where the the function $\tilde{m}^f(X; \rho)$ is defined as the solution in μ^f, for fixed ρ to

$$\frac{\mu^f - 1}{\mu^f} \frac{1}{T^f \exp\left(-\mu^f\right)} = X. \tag{C.12}$$

Let: $\Omega : \mathbb{R}_{++}^{G+1} \times [0, 1) \to \mathbb{R}^{G+1}$ via

$$\Omega\big((H_g)_{g \in \mathcal{G}}, H; \rho\big) = \begin{bmatrix} \Omega^1(H_1, H; \rho) - 1 \\ \vdots \\ \Omega^G(H_G, H; \rho) - 1 \\ 1 + \sum_{g \in \mathcal{G}} {H_g}^{1-\rho} - H \end{bmatrix}.$$

The set of equilibria, treating ρ as a free parameter, are precisely the solutions to

$$\Omega\big((H_g)_{g \in \mathcal{G}}, H; \rho\big) = \begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix}. \tag{C.13}$$

The differential of Ω, evaluated at a solution to (C.13), is of the form

$$D\Omega\big((H_g)_{g \in \mathcal{G}}, H; \rho\big) = \left(\begin{array}{ccc|c|c} & \Lambda & & \Theta & * \\ \hline (1-\rho)H_1^{-\rho} & \cdots & (1-\rho)H_G^{-\rho} & -1 & -\sum_{g \in \mathcal{G}} H_g^{1-\rho} \ln H_g \end{array}\right), \tag{C.14}$$

where Λ is a $G \times G$ diagonal matrix with

$$\Lambda_{gg} = \frac{1}{H_g}\left(\frac{\rho}{H_g} + \frac{\rho(1-\rho)}{H_g^\rho H}\right) B_g - \frac{1}{H_g},$$

and Θ is the $G \times 1$ matrix with

$$\Theta_g = \frac{\partial \Omega_g}{\partial H} = \frac{(1-\rho)}{H_g^\rho H^2} B_g,$$

[31] See equation (xxxi) in Nocke and Schutz (2018) Appendix (p. 70) for reference.

where the expression B_g is given by

$$B_g = \frac{1}{H_g} \sum_{f \in \mathcal{F}_g} \sum_{j \in \mathcal{J}^f} \exp\left[\frac{\delta_j - \alpha c_j}{1-\rho} - \tilde{m}^f\left(\frac{\rho}{H_g} + (1-\rho)\frac{1}{H_g^\rho H}\,;\rho\right)\right] \tilde{m}^{f'}\left(\frac{\rho}{H_g} + \frac{(1-\rho)}{H_g^\rho H}\right).$$

We now turn to our first technical lemma.

Lemma C.3. *For some $\varepsilon > 0$, the differential $D\Omega$, evaluated at any solution to (C.13) with $\rho \in [0, \varepsilon)$, is of rank $G+1$.*

Proof. We break down the proof into steps.

1. **Rank at least** G: Firstly, by direct observation, the upper-left $G \times G$ block Λ is diagonal. Moreover, each diagonal element is strictly negative (see Nocke and Schutz (2018) Online Appendix, Lemma XXIII proof). Hence the first G columns of $D\Omega$ are linearly independent, evaluated at any solution to (C.13).

2. **Removal of Nuisance Terms**: Suppose we evaluate $D\Omega$ at the unique solution to (C.13) with $\rho = 0$. Then, in particular, we have that

$$\Lambda_{gg}\big|_{\rho=0} = -\frac{1}{H_g},$$

and

$$\Theta_g\big|_{\rho=0} = \frac{1}{H^2} B_g\big|_{\rho=0}.$$

3. **Contradiction Hypothesis**: Suppose, for sake of contradiction, that the $G+1$st column of $D\Omega$ evaluated at the unique solution to (C.13) where $\rho = 0$ is a linear combination of the first G columns. Then there exist $(a_g)_{g=1}^G$ such that

$$(\forall g) \quad \Lambda_{gg}|_{\rho=0} a_g = \Theta_g|_{\rho=0},$$

and which satisfy

$$\sum_{g=1}^G a_g = -1. \tag{$*$}$$

Using the results of the preceding step, we can back out these weights

$$(\forall g) \quad a_g = -\frac{H_g}{H^2} B_g|_{\rho=0}.$$

4. **Algebra**: Then, plugging in to $(*)$, we obtain

$$\sum_{g \in \mathcal{G}} \sum_{f \in \mathcal{F}_g} \sum_{j \in \mathcal{J}^f} \exp\left[\delta_j - \alpha c_j - \tilde{m}^f\big(1/H\big)\right] \tilde{m}^{f'}\big(1/H\big) = H^2.$$

Since we're at an equilibrium (i.e. a solution to (C.13)) we can simplify this using the usual system of equations that hold in an equilibrium. In particular, we have that

$$\sum_{g\in\mathcal{G}}\sum_{f\in\mathcal{F}_g} T^f \exp\left(-\mu^f\right)\tilde{m}^{f'}\big(1/H\big) = H^2.$$

5. **Dealing with** $\tilde{m}^{f'}$: Recall $\tilde{m}^f$ is the implicit solution to (C.12). In particular,

$$\frac{d\tilde{m}^f}{dX} = \frac{T^f\tilde{m}^f \exp\left(-\tilde{m}^f\right)}{1 - XT^f\left[\exp\left(-\tilde{m}^f\right) - \tilde{m}^f\exp\left(-\tilde{m}^f\right)\right]}.$$

For the H under consideration, let us define $\mu^f = \tilde{m}^f(1/H)$. Then this derivative, evaluated at $X = 1/H$, is

$$\frac{T^f\mu^f \exp\left(-\mu^f\right)}{1 - \frac{1}{H}T^f\left[\exp\left(-\mu^f\right) - \mu^f\exp\left(-\mu^f\right)\right]}.$$

Now, as we are working at an equilibrium, it must be the case that $T^f\mu^f\exp\left(-\mu^f\right) = H(\mu^f - 1)$, hence our expression for the derivative at $1/H$ may be simplified to

$$\frac{H\mu^f(\mu^f-1)}{1+\mu^f(\mu^f-1)}.$$

6. **Simplifying** Plugging in the result of Step 5 into that of Step 4 and dividing both sides by H yields

$$\sum_{g\in\mathcal{G}}\sum_{f\in\mathcal{F}_g} T^f \exp\left(-\mu^f\right)\left[\frac{\mu^f(\mu^f-1)}{1+H\mu^f(\mu^f-1)}\right] = H.$$

Note that the square bracketed term lies strictly within $[0,1)$ for all $\mu > 1$. Thus,

$$\begin{aligned}
\sum_{g\in\mathcal{G}}\sum_{f\in\mathcal{F}_g} T^f \exp\left(-\mu^f\right)\left[\frac{\mu^f(\mu^f-1)}{1+\mu^f(\mu^f-1)}\right] &< \sum_{g\in\mathcal{G}}\sum_{f\in\mathcal{F}_g} T^f\exp\left(-\mu^f\right)\\
&= \sum_{g\in\mathcal{G}}\sum_{f\in\mathcal{F}_g} H_g s^{f|g}\\
&= \sum_{g\in\mathcal{G}} H_g \sum_{f\in\mathcal{F}_g} s^{f|g}\\
&= \sum_{g\in\mathcal{G}} H_g\\
&< 1 + \sum_{g\in\mathcal{G}} H_g\\
&= H.
\end{aligned}$$

Thus $(*)$ can never hold for any (a_g), and the first $G+1$ columns of $D\Omega$, at the solution to (21) where $\rho = 0$, are linearly independent. By continuity of these terms in ρ, the same must be true for some small enough open set of ρ's containing 0, and the result follows.

□

We now establish the following immediate corollary:

Lemma C.4. *Let $\hat{\Omega} : \mathbb{R}^{G+1}_{++} \to \mathbb{R}^{G+1}$ denote the restriction of Ω to the (relatively) open set $\mathbb{R}^{G+1}_{++} \times \{0\}$. Then $D\hat{\Omega}$ is of full rank at the unique solution to (C.13) in this domain.*

Proof. By direct calculation,

$$D\hat{\Omega} = \left(\begin{array}{ccc|c} & \hat{\Lambda} & & \hat{\Theta} \\ & & & \\ \hline 1 & \cdots & 1 & -1 \end{array}\right),$$

where $\hat{\Lambda}_{gg} = -1/H_g$ and $\hat{\Theta}_g = (1/H^2)B_g|_{\rho=0}$. Thus an identical argument to the prior lemma yields the result. □

C.10.2 NCES Preliminaries

For NCES we define the function $\tilde{m}^f(X;\sigma)$ as the solution in μ^f, for fixed σ to

$$\frac{\mu^f - 1}{\mu^f} \frac{1}{T^f\left(1 - \frac{\mu^f}{\sigma}\right)^{\sigma-1}} = X. \tag{C.15}$$

Define:

$$\Omega_g(H, H_g; \sigma) = \frac{1}{H_g} \sum_{f \in \mathbb{F}_g} \sum_{j \in \mathbb{J}^f} \delta_j c_j^{1-\sigma} \left[1 - \frac{1}{\sigma}\tilde{m}^f\left(\frac{\gamma - \sigma}{\sigma}\frac{1}{H_g} + \frac{\gamma - 1}{\sigma}\frac{1}{H_g^{\frac{\sigma-\gamma}{\sigma-1}} H}\right)\right]^{\sigma-1}. \tag{C.16}$$

The solutions to (C.16) are equivalent to solving $\Omega_g(H_g, H; \sigma) = 1$. Let: $\Omega : \mathbb{R}^{G+1}_{++} \times [\gamma, \infty) \to \mathbb{R}^{G+1}$ via

$$\Omega\big((H_g)_{g\in\mathbb{G}}, H; \sigma\big) = \begin{bmatrix} \Omega^1(H_1, H; \sigma) - 1 \\ \vdots \\ \Omega^G(H_G, H; \sigma) - 1 \\ \sum_{g\in\mathbb{G}} H_g^{\frac{\gamma-1}{\sigma-1}} - H \end{bmatrix}.$$

The set of equilibria, treating σ as a free parameter, are precisely the solutions to

$$\Omega\big((H_g)_{g\in\mathbb{G}}, H; \sigma\big) = \begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix}. \tag{C.17}$$

The differential of Ω is of the form

$$D\Omega\big((H_g)_{g\in\mathbb{G}}, H;\sigma\big) = \left(\begin{array}{ccc|c|c} & \Lambda & & \Theta & * \\ \hline \frac{\gamma-1}{\sigma-1}H_1^{\frac{\gamma-\sigma}{\sigma-1}} & \cdots & \frac{\gamma-1}{\sigma-1}H_G^{\frac{\gamma-\sigma}{\sigma-1}} & -1 & * \end{array}\right), \tag{C.18}$$

where Λ is a $G\times G$ diagonal matrix with

$$\Lambda_{gg} = \frac{-1}{H_g} + \frac{1-\sigma}{\sigma}\left[-\frac{\gamma-\sigma}{\sigma}\frac{1}{H_g^2} + \frac{\gamma-1}{\sigma}\frac{\gamma-\sigma}{\sigma-1}\frac{1}{HH_g^{\frac{1-\gamma+2\sigma}{\sigma-1}}}\right]B_g \tag{C.19}$$

at any solution to (C.17), where

$$\begin{aligned} B_g =& \frac{1}{H_g}\sum_{f\in\mathbb{F}_g}\sum_{j\in\mathbb{J}^f}\delta_j c_j^{1-\sigma}\left[1-\frac{1}{\sigma}\,\tilde{m}^f\left(\frac{\gamma-\sigma}{\sigma}\frac{1}{H_g}+\frac{\gamma-1}{\sigma}\frac{1}{H_g^{\frac{\sigma-\gamma}{\sigma-1}}H}\right)\right]^{\sigma-2}\times \\ & \tilde{m}^{f\prime}\left(\frac{\gamma-\sigma}{\sigma}\frac{1}{H_g}+\frac{\gamma-1}{\sigma}\frac{1}{H_g^{\frac{\sigma-\gamma}{\sigma-1}}H}\right), \end{aligned} \tag{C.20}$$

and Θ is a $G\times 1$ matrix with

$$\Theta_g = \frac{\partial\Omega_g}{\partial H} = -\frac{1-\sigma}{\sigma}\frac{\gamma-1}{\sigma}\frac{1}{H_g^{\frac{\sigma-\gamma}{\sigma-1}}H^2}B_g. \tag{C.21}$$

Lemma C.5. *For some $\varepsilon > 0$, the differential $D\Omega$, evaluated at any solution to* (C.17) *with $\sigma \in [\gamma, \gamma+\varepsilon)$ is of rank $G+1$.*

Proof. We again break down the proof into steps.

1. **Rank at least G**: Firstly, by direct observation, the upper-left $G\times G$ block Λ is diagonal. Moreover, each diagonal element is strictly negative (see Nocke and Schutz (2018) Online Appendix, Lemma XXIII proof). Hence the first G columns of $D\Omega$ are linearly independent, evaluated at a solution to (C.17).

2. **Removal of Nuisance Terms**: Suppose we evaluate $D\Omega$ at the unique solution to (C.17) with $\sigma = \gamma$. Then (C.19) becomes

$$\Lambda_{gg}\big|_{\sigma=\gamma} = -\frac{1}{H_g}$$

   and (C.21),

$$\Theta_g\big|_{\sigma=\gamma} = \frac{(\gamma-1)^2}{\gamma^2}\frac{1}{H^2}B_g\big|_{\sigma=\gamma}.$$

3. **Contradiction Hypothesis**: Suppose, for sake of contradiction, that the $G+1$st column of $D\Omega$ is a linear combination of the first G columns when evaluated at the unique solution with $\sigma = \gamma$. Since Λ is diagonal, this means that there exist real numbers

$\{a_g\}_{g\in\mathbb{G}}$ such that $a_g\Lambda_{gg}\big|_{\sigma=\gamma} = \Theta_g\big|_{\sigma=\gamma}$ (from the first G rows), and $\sum_g a_g = -1$ (the $G+1$st row). From these equations we can solve for a_g,

$$\begin{aligned} a_g &= \frac{\Theta_g\big|_{\sigma=\gamma}}{\Lambda_{gg}\big|_{\sigma=\gamma}} \\ &= -\frac{(\gamma-1)^2}{\gamma^2}\frac{H_g}{H^2}B_g\big|_{\sigma=\gamma}. \end{aligned} \tag{C.22}$$

4. **Algebra**: Plugging in (C.22) for the contradiction hypothesis that $\sum_g a_g = -1$, we obtain

$$\frac{(\gamma-1)^2}{\gamma^2}\sum_{g\in\mathbb{G}}\sum_{f\in\mathbb{F}_g}\sum_{j\in\mathbb{J}^f}\delta_j c_j^{1-\sigma}\left[1-\frac{1}{\gamma}\tilde{m}^f\left(\frac{\gamma-1}{\gamma}\frac{1}{H}\right)\right]^{\sigma-2}\tilde{m}^{f\prime}\left(\frac{\gamma-1}{\gamma}\frac{1}{H}\right) = H^2. \tag{$*$}$$

5. **Dealing with $\tilde{m}^{f\prime}$**: Consider the $\tilde{m}^{f\prime}$ term now. We know $\tilde{m}^f(X)$ is the solution (in μ^f) to (C.15). Thus, by direct computation we have that

$$\frac{d\tilde{m}^f}{dX} = \frac{\tilde{m}^f T^f\left(1-\frac{\tilde{m}^f}{\gamma}\right)^{\gamma-1}}{1-XT^f\left[\left(1-\frac{\tilde{m}^f}{\gamma}\right)^{\gamma-1}-\tilde{m}^f\frac{\gamma-1}{\gamma}\left(1-\frac{\tilde{m}^f}{\gamma}\right)^{\gamma-2}\right]}. \tag{C.23}$$

Considering some fixed solution to (C.17) at $\sigma=\gamma$, define $\mu^f = \tilde{m}^f\big(((\gamma-1)/\gamma)(1/H)\big)$, and let $X = ((\gamma-1)/\gamma)(1/H)$. Then (C.23) becomes

$$\frac{\mu^f T^f\left(1-\frac{\mu^f}{\gamma}\right)^{\gamma-1}}{1-\frac{\gamma-1}{\gamma}\frac{1}{H}T^f\left[\left(1-\frac{\mu^f}{\gamma}\right)^{\gamma-1}-\mu^f\frac{\gamma-1}{\gamma}\left(1-\frac{\mu^f}{\gamma}\right)^{\gamma-2}\right]}$$

which, given we are at a solution to (C.17), simplifies to

$$\left(\frac{\gamma}{\gamma-1}\right)\frac{H(\mu^f-1)}{1-(\mu^f-1)\left[\frac{1}{\mu^f}-\frac{\gamma-1}{\gamma}\frac{1}{1-\mu^f/\gamma}\right]}.$$

6. Simplifying: Plugging in to $(*)$ we obtain

$$\sum_{g\in\mathbb{G}}\sum_{f\in\mathbb{F}_g}\sum_{j\in\mathbb{J}^f}\delta_j c_j^{1-\sigma}\left[1-\frac{\mu^f}{\gamma}\right]^{\sigma-2}\frac{(\gamma-1)}{\gamma}\frac{(\mu^f-1)}{1-(\mu^f-1)\left[\frac{1}{\mu^f}-\frac{\gamma-1}{\gamma}\frac{1}{1-\mu^f/\gamma}\right]} = H. \tag{C.24}$$

Simplifying yields

$$\sum_{g\in\mathbb{G}}\sum_{f\in\mathbb{F}_g}T^f\left[1-\frac{\mu^f}{\gamma}\right]^{\sigma-1}\underbrace{\left(\frac{\gamma-1}{\gamma}\right)\left(\frac{1}{1-\frac{\mu^f}{\gamma}}\right)\frac{(\mu^f-1)}{1-(\mu^f-1)\left[\frac{1}{\mu^f}-\frac{\gamma-1}{\gamma}\frac{1}{1-\mu^f/\gamma}\right]}}_{\equiv\chi^f} = H. \tag{C.25}$$

By Lemma C.1, in any solution $\mu^f\in[1,\gamma)$ and hence for all g and all $f\in\mathcal{F}_g$, χ^f lives within $[0, 1)$. Thus, considering the left-hand side of (C.25),

$$\begin{aligned}\sum_{g\in\mathbb{G}}\sum_{f\in\mathbb{F}_g} T^f\left[1-\frac{\mu^f}{\gamma}\right]^{\sigma-1}\left(\frac{\gamma-1}{\gamma}\right)\left(\frac{1}{1-\frac{\mu^f}{\gamma}}\right)&\frac{(\mu^f-1)}{1-(\mu^f-1)\left[\frac{1}{\mu^f}-\frac{\gamma-1}{\gamma}\frac{1}{1-\mu^f/\gamma}\right]}\\
&<\sum_{g\in\mathbb{G}}\sum_{f\in\mathbb{F}_g} T^f\left[1-\frac{\mu^f}{\gamma}\right]^{\sigma-1}\\
&=\sum_{g\in\mathbb{G}}\sum_{f\in\mathbb{F}_g} H_g s^{f|g}\\
&=\sum_{g\in\mathbb{G}} H_g\\
&=H,\end{aligned}$$

a contradiction of (C.25). Thus the Jacobian $D\Omega$, evaluated at any solution to (C.17) with $\sigma=\gamma$, is of full rank.

□

Lemma C.6. *Let $\hat{\Omega}:\mathbb{R}^{G+1}_{++}\to\mathbb{R}^{G+1}$ denote the restriction of Ω to the (relatively) open set $\mathbb{R}^{G+1}_{++}\times\{\gamma\}$. Then $D\hat{\Omega}$ is of full rank at the unique solution to* (C.17) *in this domain.*

Proof. By direct calculation we have that

$$D\hat{\Omega}=\left(\begin{array}{ccc|c} & \hat{\Lambda} & & \hat{\Theta}\\ \hline 1 & \cdots & 1 & -1\end{array}\right),$$

where $\hat{\Lambda}_{gg}=-1/H_g$ and $\hat{\Theta}_g=((\gamma-1)^2/\gamma^2)(1/H^2)B_g\big|_{\sigma=\gamma}$. Thus, an identical argument to the prior lemma yields the result. □

C.10.3 Proof of Proposition 9

Proof. We state the proof for the NMNL case; the NCES case follows, *mutatis mutandis*, using Lemmas C.5 and C.6. Let $\varepsilon>0$ be any such value such that the conclusions of Lemmas C.3 and C.4 hold, and by abuse of notation, denote the restriction of Ω to $\mathbb{R}^{G+1}_{++}\times[0,\varepsilon')$ for any $0<\varepsilon'<\varepsilon$ simply by Ω. By Lemma C.3, 0 is a regular value of Ω on this domain, and by Lemma C.4, 0 is also a regular value of Ω restricted to the boundary of this domain. Thus by the Regular Value Theorem (see Hirsch (2012) Theorem 1.4.1, see also Mas-Colell (1974) Theorem 2), $\Omega^{-1}(0)$ is a C^1 submanifold of $\mathbb{R}^{G+1}_{++}\times[0,\varepsilon')$, with boundary precisely equal to the unique equilibrium at $\rho=0$. Consider the (necessarily unique) connected component of $\Omega^{-1}(0)$ that intersects $\mathbb{R}^{G+1}_{++}\times\{0\}$. Since this component is a connected C^1 manifold with boundary, it is C^1-diffeomorphic to $[0,1)$ (Hirsch (2012) Exercise 1.5.9).[32] Since the

[32]It cannot be diffeomorphic to $[0,1]$ as from the Regular Value theorem, its boundary is given precisely by its intersection with the boundary of the domain, and at $\rho=0$ the equilibrium is unique.

Regular Value Theorem guarantees its intersection with the slice $\mathbb{R}^{G+1}_{++} \times \{0\}$ is transverse, the restriction of this component to $\mathbb{R}^{G+1}_{++} \times [0, \varepsilon'']$ for some $0 < \varepsilon'' < \varepsilon'$ is diffeomorphic to $[0, 1]$, and hence is compact.

However, $\Omega^{-1}(0)|_{\mathbb{R}^{G+1}_{++} \times [0,\varepsilon'']}$ is also the graph of the function $e : [0, \varepsilon''] \to \mathbb{R}^{G+1}_{++}$ that takes a nesting parameter value and maps it to the unique equilibrium of the associated differentiated Bertrand-Nash pricing game. By the preceding argument, we may without loss restrict the codomain of e to be some compacta $K \subseteq \mathbb{R}^{G+1}_{++}$ such that (i) $K \times [0, \varepsilon'']$ contains $\Omega^{-1}(0)|_{\mathbb{R}^{G+1}_{++} \times [0,\varepsilon'']}$, and (ii) the graph of e is a closed subset of $K \times [0, \varepsilon'']$.[33] But then by the Closed Graph Theorem (Aliprantis and Border (2006) Theorem 2.58), this map is continuous on $[0, \varepsilon'']$. □

C.11 Proof of Proposition 10

Proof. We know from Proposition 5 that in the MNL and CES cases, if consumer surplus remains unchanged after a merger, then the profits of the merging firms must fall. Thus, in the NMNL (resp. NCES) model, for $\rho = 0$ (resp. $\sigma = \gamma$), if consumer surplus is unharmed then,

$$\pi^M(T^1 + T^2, H^{m,e}_g, H) < \pi^1(T^1, H^{nm,ne}_g, H) + \pi^2(T^2, H^{nm,ne}_g, H).$$

Suppose then that we consider a sequence of nesting parameter values $(\rho_n)_{n\in\mathbb{N}}$ such that $\rho_n \to 0$ (resp. $(\sigma_n)_{n\in\mathbb{N}}$ such that $\sigma_n \to \gamma$). By Proposition 9, and the continuous dependence of profits and markups on the underlying equilibrium variables $(H_g)_{g\in\mathcal{G}}$ and H, we obtain a sequence of profits for the individual merging parties and the merged entity which converge to their $\rho = 0$ values as $\rho_n \to 0$ (resp. $\sigma = \gamma$ values as $\sigma_n \to \gamma$). In the NMNL model, for n large enough it then must be the case that

$$\pi^M(T^1 + T^2, H^{m,e}_g(\rho_n), H(\rho_n)) < \pi^1(T^1, H^{nm,ne}_g(\rho_n), H(\rho_n)) + \pi^2(T^2, H^{nm,ne}_g(\rho_n), H(\rho_n)),$$

establishing the result. The sequence of profits would generate an analogous inequality in the NCES model. □

C.12 Proof of Proposition B.1

For brevity we focus on MNL demand. An analogous proof for CES demand can be provided upon request by the authors. We first show that as the type of any one firm goes to infinity, so too does the market aggregator.

Lemma C.7. *Fix any market structure $\mathcal{F}_*$ and vector of model primitives. For any $f \in \mathcal{F}_*$,*

$$\lim_{T^f \to \infty} H_* = \infty.$$

[33] It suffices to let K be the projection of $\Omega^{-1}(0)|_{\mathbb{R}^{G+1}_{++} \times [0,\varepsilon'']}$ onto $\mathbb{R}^{G+1}_{++}$ to satisfy both these properties. In particular, this set is compact by continuity of the projection, and the graph of e is closed in $\mathbb{R}^{G+1}_{++} \times [0, \varepsilon'']$, hence it is closed in the subspace topology on $K \times [0, \varepsilon”]$.

Proof. First note that $\lim_{T^f \to \infty} T^f / H_* = \infty$, as established in the proof of Proposition 2. Thus, as

$$s^f = \frac{T^f}{H_*} \exp\left[\frac{-1}{1 - s^f}\right],$$

as T^f goes to infinity, s^f goes to one. But as:

$$\frac{1}{H_*} + \sum_{f \in \mathcal{F}_*} s^f = 1,$$

it follows that $H_* \to \infty$. □

We now prove the proposition.

Proof. Fix an arbitrary market structure $\mathcal{F}_{nm,ne}$ and associated $\mathcal{F}_{m,ne}$. The merger is profitable with delayed or probabilistic entry if and only if

$$(1 - \theta)\left[\frac{1}{1 - s^M_{m,ne}}\right] + \theta\left[\frac{1}{1 - s^M_{m,e}}\right] \geq \frac{1}{1 - \underline{s}^M_{m,e}}, \tag{C.26}$$

where

$$\underline{s}^M_{m,e} = 1 - \frac{(1 - s^1)(1 - s^2)}{1 - s^1 s^2}$$

is the market share of the merged firm in a counterfactual with entry that makes the merger exactly neutral for stage-game profit.[34] We obtain (C.26) by substituting in for profit using (12) and (A.3). Note that $f(x) = \frac{1}{1-x}$ is increasing, and as $s^M_{m,ne} > \underline{s}^M_{m,e}$,

$$\frac{1}{1 - s^M_{m,ne}} > \frac{1}{1 - \underline{s}^M_{m,e}}.$$

Define

$$(1 - \theta^*) \equiv \frac{\frac{1}{1 - \underline{s}^M_{m,e}}}{\frac{1}{1 - s^M_{m,ne}}}.$$

Thus for the choice $\theta = \theta^*$, the profit inequality reduces to

$$\theta^*\left[\frac{1}{1 - s^M_{m,e}}\right] \geq 0,$$

which always holds strictly. The definition of θ^* does not depend on T^F. Thus, for $\theta = \theta^*$, the type of the entrant does not affect whether the merger is profitable. Assuming that $\theta = \theta^*$, we turn to consumer surplus, which weakly increases if and only if

$$(1 - \theta^*) \ln H_{m,ne} + \theta^* \ln H_{m,e} \geq \ln H_{nm,ne}.$$

The only term in this inequality that depends on T^F is $H_{m,e}$. Furthermore, if we send T^F to

[34] We derive the expression for $\underline{s}^M_{m,e}$ in Proposition 2.

infinity, then $H_{m,e}$ also goes to infinity, by Lemma C.7. Therefore, for some large enough T^F, and for $\theta = \theta^*$, the merger is profitable and consumer surplus strictly increases. $\square$

D Numerical Methods

In this appendix, we describe how a model of Bertrand competition with MNL demand can be calibrated based on data on market shares, and then simulated to obtain the percentage changes in markups, profit, and consumer surplus due to a merger. The NMNL is analogous if one has knowledge of the nesting parameter. We then detail the data sources and methods that are used in the application to the T-Mobile/Sprint merger that is presented in Section 4.

D.1 Calibration and Simulation

With MNL demand, it is possible to recover types from market shares, and vice-versa. To implement the former—a calibration step—first obtain the market aggregator from (11), and the ι-markups from (A.3). Firm types then are given by a rearranged (10),

$$T^f = \frac{s^f H}{\exp\left(-\mu^f\right)}.$$

To implement the latter—a simulation step—use a nonlinear equation solver to recover the shares and the market aggregator, given a set of types. There are $F+1$ nonlinear equations that must be solved simultaneously. One of these is the adding-up constraint of (11), and the others are obtained by plugging (A.3) into (10), which yields

$$s^f = \frac{T^f}{H} \exp\left(-\frac{1}{1-s^f}\right).$$

If one knows the types, and thus also the aggregator, then markups, profit, and consumer surplus are identified up to a multiplicative constant (see (9), (12), and (13)). An implication is that the outcomes that arise with different firm types can be meaningfully compared—the ratio of outcomes is identified because the multiplicative constant cancels.

A full calibration also recovers the multiplicative constant—the price parameter, α. This can be accomplished with data on one margin, for example. See also the Nocke and Schutz (2018) Online Appendix. Then markups, profit, and consumer surplus also are obtained (not just up to a multiplicative constant). However, these objects are not necessary for our purposes, so we use partial calibration.

An observation is that our market shares, $\{s^f\}\ \forall f \in \mathcal{F}$, assign a positive share to the outside good. Thus, they differ from the antitrust market shares described in the US Merger Guidelines, which assign zero weight to products that are outside the relevant market.[35] Nonetheless, it is possible to convert antitrust market shares into our market shares using information that often is available during merger review. For example, suppose one has information on the diversion ratio that characterizes substitution from firm k to firm j. Then,

[35] See the US DOJ/FTC Merger Guidelines §4.4 for a discussion of market shares.

in the context of MNL (and CES) we have

$$\frac{\frac{\partial s^j}{\partial p^k}}{\frac{\partial s^k}{\partial p^k}} \equiv DIV_{k\to j} = \frac{s_j}{1-s_k}. \tag{D.1}$$

Letting the relevant antitrust market comprise the products of firms $f \in \mathcal{F}$, we have

$$\hat{s}^f = \frac{s^f}{1-s^0}, \tag{D.2}$$

where $\hat{s}^f$ is the antitrust market share and s^0 is the outside good share in our context. The system of equations in (D.1) and (D.2) identifies s^0 and $\{s^f\}\ \forall f \in \mathcal{F}$ from data on diversion, $DIV_{k\to j}$, for some $j \neq k$, and the antitrust market shares, $\{\hat{s}^f\}\ \forall f \in \mathcal{F}$.

D.2 Application to T-Mobile/Sprint

Our primary source of data is the 2017 Annual Report of the FCC on competition in the mobile wireless sector.[36] We obtain the following information:

- Among national providers, Verizon, AT&T, T-Mobile, and Sprint account for 35.0%, 32.4%, 17.1%, and 14.3% of total connections at end-of-year 2016, respectively. See Figure II.B.1 on page 15.
- The average revenue per user (ARPU) in 2016:Q4 for Verizon, AT&T, T-Mobile, and Sprint is 37.52, 36.58, 33.80, and 32.03, respectively. See Figure III.A.1 on page 42. Following common practice, we use the ARPU as a measure of price.
- The EBITDA per subscriber in 2016 for Verizon, AT&T, T-Mobile, and Sprint is 22.71, 18.30, 11.80, and 13.00, respectively. See Figure II.D.1 on page 24. We interpret the EBITDA as providing the markup.

Finally, we obtain a market elasticity of -0.3 from regulatory filings.[37] The market elasticity is defined theoretically as $\epsilon = -\alpha s_0 \bar{p}$, where $\bar{p}$ is the weighted-average price.

The main distinction between the T-Mobile/Sprint application and our other numerical results is that we do not observe pre-merger market shares. The reason is that the FCC data on total connections does not incorporate the consumer option to purchase the outside good. Thus, we use a full calibration approach with the market elasticity and a markup (specifically that of T-Mobile) to recover the outside good share and the price coefficient. We obtain an outside good share of 8.4%. With this in hand, the pre-merger market share for T-Mobile, for example, is $17.1/(1-0.084)$. With the pre-merger market shares, Figure 5 can be created using the methods described above.

[36] *Annual Report and Analysis of Competitive Market Conditions With Respect to Mobile Wireless, Including Commercial Mobile Services*, FCC-17-126.

[37] Specifically, we reference Appendix F of the 2018 Joint Opposition Filing by T-Mobile and Sprint in FCC WT Docket No. 18-197.