The Substitution Elasticity, Factor Shares, and the Low-Frequency Panel Model

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Online Appendix

Appendix I:
Specifying the Marginal Product of Capital with Neutral and Factor-Augmenting Technical Change

This appendix presents the details of the derivation of the marginal product of capital when there is both neutral and factor-augmenting technical change, and the derivation shows that the latter has no impact on the specification of the estimating equation used in this study. We assume that production possibilities are described by the following CES technology that relates output ($Y^*_{i,t}$) to capital ($K^*_{i,t}$), labor ($L^*_{i,t}$), and neutral technical progress ($A^*_{i,t}$), and factor-augmenting technical progress on capital and labor ($A^{K*}_{i,t}$ and $A^{L*}_{i,t}$, respectively) for industry i at time t,

$$Y^*_{i,t} = Y[K^*_{i,t}, L^*_{i,t}, A^*_{i,t}, A^{K*}_{i,t}, A^{L*}_{i,t}], \tag{I-1}$$

$$= A^*_{i,t}\left\{\phi(A^{K*}_{i,t}\,K^*_{i,t})^{[(\sigma-1)/\sigma]} + (1-\phi)(A^{L*}_{i,t}\,L^*_{i,t})^{[(\sigma-1)/\sigma]}\right\}^{[\sigma/(\sigma-1)]}$$

where ϕ is the capital distribution parameter and σ is the elasticity of substitution between labor and capital. We use the value-added form of the CES production function for convenience.

The derivative of $Y^*_{i,t}$ with respect to $K^*_{i,t}$, $Y'^*_{i,t}$, is computed from equation (I-1) as follows,

$$Y'^*_{i,t} = [\sigma/(\sigma-1)]A^*_{i,t}\left\{\phi(A^{K*}_{i,t}K^*_{i,t})^{[(\sigma-1)/\sigma]} + (1-\phi)(A^{L*}_{i,t}L^*_{i,t})^{[(\sigma-1)/\sigma]}\right\}^{[[\sigma/(\sigma-1)]-1]} \\ *[(\sigma-1)/\sigma]\phi(A^{K*}_{i,t}K^*_{i,t})^{[[(\sigma-1)/\sigma]-1]}A^{K*}_{i,t}. \quad \text{(I-2)}$$

Since $((\sigma-1)/\sigma)-1 = -1/\sigma$, we can rewrite equation (I-2) as follows,

$$Y'^*_{i,t} = \phi\, K^{*[-1/\sigma]}_{i,t} \\ A^*_{i,t}\left\{\phi(A^{K*}_{i,t}K^*_{i,t})^{[(\sigma-1)/\sigma]} + (1-\phi)(A^{L*}_{i,t}L^*_{i,t})^{[(\sigma-1)/\sigma]}\right\}^{[\sigma/(\sigma-1)]} \\ \left\{\phi(A^{K*}_{i,t}K^*_{i,t})^{[(\sigma-1)/\sigma]} + (1-\phi)(A^{L*}_{i,t}L^*_{i,t})^{[(\sigma-1)/\sigma]}\right\}^{-1} \\ A^{K*[(\sigma-1)/\sigma]}_{i,t}. \quad \text{(I-3)}$$

In equation (I-3), the second line equals $Y^*_{i,t}$ per equation (I-1), and the third line equals the product of $Y^*_{i,t}$ and $A^*_{i,t}$ raised to the appropriate powers,

$$Y'^*_{i,t} = \phi\, K^{*[-1/\sigma]}_{i,t} \\ Y^*_{i,t} \\ Y^{*[(1-\sigma)/\sigma]}_{i,t}\; A^{*[(\sigma-1)/\sigma]}_{i,t} \quad , \quad \text{(I-4)} \\ A^{K*[(\sigma-1)/\sigma]}_{i,t}$$

which can be rewritten as follows,

$$Y'^*_{i,t} = \phi\, K^{*[-1/\sigma]}_{i,t}\, Y^{*[1/\sigma]}_{i,t}\, A^{*[(\sigma-1)/\sigma]}_{i,t}\; A^{K*[(\sigma-1)/\sigma]}_{i,t} \quad . \quad \text{(I-5a)}$$

$$= \phi\,((Y_{i,t}/K_{i,t})^*)^{[1/\sigma]}\, U^{KY[1/\sigma]}_{i,t}, \quad \text{(I-5b)}$$

$$U^{KY[1/\sigma]}_{i,t} \equiv A^{*[\sigma-1]}_{i,t}\; A^{K*[\sigma-1]}_{i,t}. \quad \text{(I-5c)}$$

A profit-maximizing firm will equate the marginal product of capital in equation (I-5a) to the user cost of capital, the price of capital divided by the price of output,

$$(P^K_{i,t}/P^Y_{i,t})^* = \phi\,((Y_{i,t}/K_{i,t})^*)^{[1/\sigma]}\; U^{KY[1/\sigma]}_{i,t}. \quad \text{(I-6)}$$

Equation (I-6) can be rearranged to isolate the capital/output ratio on the left-side,

$$(K_{i,t} / Y_{i,t})^{*} = \phi^{\sigma} \left((P^{K}_{i,t} / P^{Y}_{i,t})^{*}\right)^{-\sigma} U^{KY}_{i,t}, \qquad \text{(I-7)}$$

which is equation (2) in the text except for the inclusion of $A^{K*}_{i,t}$ in the error term (cf. equation (I-5c). Thus, factor-augmenting technical change has no impact on the specification of the estimating equation for σ, and the only implication is a warning about possible correlation between the error term and the regressor.

Appendix II:
Data Transformations and the Frequency Response Scalars

Our estimation strategy is designed to emphasize long-run variation, and Section II uses spectral analysis to evaluate our approach and the choices of $\tilde{\omega}$ and q. This appendix provides some analytic details underlying the results stated and used in Section II.

In analyzing the spectral properties of our estimator, it is convenient to write the LPF transformation (for a finite q), the logarithmic transformation, and the first-difference transformation as follows,

$$x^{*}_{i,t}[\tilde{\omega},q] = \sum_{h=-q}^{q} d_h[\tilde{\omega}]\ x_{i,t-h}, \tag{II-1a}$$

$$y^{*}_{i,t}[\tilde{\omega},q] = \ln[x^{*}_{i,t}[\tilde{\omega},q]], \tag{II-1b}$$

$$z^{*}_{i,t}[\tilde{\omega},q] = \Delta y^{*}_{i,t}[\tilde{\omega},q], \tag{II-1c}$$

where $x_{i,t}$ represents the raw data series, either $(K_{i,t} / Y_{i,t})$ or $(P^{K}_{i,t} / P^{Y}_{i,t})$. The spectra corresponding to the $x^{*}_{i,t}[.]$, $y^{*}_{i,t}[.]$, and $z^{*}_{i,t}[.]$ output series in equations (II-1) are defined over the interval $\omega = [0,\pi]$ as the product of the spectrum for an input series and a scalar that is nonnegative, real, and may be depend on ω, $\tilde{\omega}$, or q,

$$g_{x*}[\omega,\tilde{\omega},q] = \alpha[\omega,\tilde{\omega},q]\ g_x[\omega], \tag{II-2a}$$

$$g_{y*}[\omega,\tilde{\omega},q] = \beta\ g_{x*}[\omega], \tag{II-2b}$$

$$g_{z*}[\omega,\tilde{\omega},q] = \gamma[\omega]\ g_{y*}[\omega], \tag{II-2c}$$

where $g_x[\omega]$ is the spectrum for the raw series and the scalars are defined as follows,

$$\alpha[\omega,\tilde{\omega},q] = a[\tilde{\omega},q] \left\{ \begin{array}{l} (\tilde{\omega}/\pi) + 2\sum_{h=1}^{q} \cos[h\,\omega]\ d'_h[\tilde{\omega}] \\ +\ \theta[\tilde{\omega},q]\ \{(1-\cos[\omega(2q+1)])/(1-\cos[\omega])\}^{1/2} \end{array} \right\}^{2} \tag{II-3a}$$

$$\beta = b\ (\mu_{x*})^{-2}, \tag{II-3b}$$

$$\gamma[\omega] = c\ 2\ (1-\cos[\omega]), \tag{II-3c}$$

where μ_{x*} equals the unconditional expectation of $x^{*}_{i,t}[.]$. To ensure comparability in the analyses to follow that vary $\tilde{\omega}$ and q, the areas under the spectra from 0 to π are normalized to one by an appropriate choice of normalizing constants, $a[\tilde{\omega}, q]$, b, and c in equations (II-3).

The three scalars -- $\alpha[\omega, \tilde{\omega}, q]$, β, and $\gamma[\omega]$ -- correspond to the LPF, logarithmic, and first-difference transformations, respectively, and they are derived as follows. The $\alpha[\omega, \tilde{\omega}, q]$ scalar is based on Sargent (1987, Chapter XI, equation (33)),

$$\alpha[\omega, \tilde{\omega}, q] = a[\tilde{\omega}, q] \left\{ \sum_{h=-q}^{q} e^{-ih\omega} d_h[\tilde{\omega}] \right\} \left\{ \sum_{h=-q}^{q} e^{ih\omega} d_h[\tilde{\omega}] \right\}. \qquad \text{(II-4)}$$

The two-sided summations are symmetric about zero and only differ by the minus sign in the exponential terms. Hence, the two sums in braces are nearly identical. The $d_h[.]$'s appearing in the summations are separated into $\theta[.]$ and the $d'_h[.]$'s (cf. equations (6)). For the latter terms, a further distinction is made between the term at h=0 and the remaining terms (h=±1,±q) that are symmetric about h=0. Equation (II-4) can be written as follows,

$$\alpha[\omega, \tilde{\omega}, q] = a[\tilde{\omega}, q] \left\{ \theta[\tilde{\omega}, q] \sum_{h=-q}^{q} e^{ih\omega} + (\tilde{\omega}/\pi) + \sum_{h=1}^{q} (e^{-ih\omega} + e^{ih\omega})\, d'_h[\tilde{\omega}] \right\}^2. \qquad \text{(II-5)}$$

The first sum of exponential terms in equation (II-5) is evaluated based on Sargent (1987, p. 275),

$$\sum_{h=-q}^{q} e^{ih\omega} = \left\{ \left(\sum_{h=-q}^{q} e^{ih\omega} \right)^2 \right\}^{1/2} = \left\{ \left((1 - \cos[(2q+1)\omega] / (1 - \cos[\omega]) \right) \right\}^{1/2}. \qquad \text{(II-6)}$$

The second sum of exponential terms in equation (II-5) is evaluated with the Euler relations, $e^{\pm ih\omega} = \cos[h\omega] \pm i \sin[h\omega]$,

$$\sum_{h=1}^{q} (e^{-ih\omega} + e^{ih\omega})\, d'_h[\tilde{\omega}] = 2 \sum_{h=1}^{q} \cos[h\omega]\, d'_h[\tilde{\omega}]. \qquad \text{(II-7)}$$

The β scalar is based on the approximation in Granger (1964, p. 48, equation 3.7.6), which states that the approximation will be accurate if the mean is much larger than the standard deviation of the input series ($x^{*}_{i,t}[.]$).

The $\gamma[\omega]$ scalar is based on the well-known formula for the first-difference transformation (Hamilton 1994, equation 6.4.8).

The importance of the above analytical results is that the combined effects of the three transformations are captured by three scalars that multiply the spectrum of the raw series,

$$g_{Z^*}[\omega, \tilde{\omega}, q] = \alpha[\omega, \tilde{\omega}, q]\ \beta\ \gamma[\omega]\ g_X[\omega]. \qquad \text{(II-8)}$$

Equation (II-8) allows us to examine the extent to which our estimation strategy emphasizes long-run frequencies. Since the spectra for the raw series ($g_X[\omega]$) and the scalars associated with the logarithmic and first-difference transformations (β and $\gamma[\omega]$, respectively) do not depend on $\tilde{\omega}$ or q, their impacts on the data will be absorbed in the normalizing constants ($a[\tilde{\omega}, q]$, b, and c), and hence they will not affect relative comparisons. Alternative values of $\tilde{\omega}$ or q, will only affect the $\mathrm{LPF}[\tilde{\omega}, q]$ and the associated scalar, $\alpha[\omega, \tilde{\omega}, q]$.

Appendix III:

Appendix Table 3: OLS Estimates of Equation (5): Various Critical Periodicities ($\tilde{\zeta}$) and Fixed Window (q=3)

		Reordered Transformations	Cross-Sectional Dependence	Value-Added Output	Split-Sample 1960-1982	Split-Sample 1983-2005	K/L Model
		(1)	(2)	(3)	(4)	(5)	(6)
$\tilde{\zeta} = 2$	σ [NW s.e.] {R^2}	0.229 [0.022] {0.405}	0.409 (0.115)	0.335 [0.028] {0.493}	0.219 [0.035] {0.419}	0.242 [0.036] {0.369}	0.120 [0.050] {0.154}
$\tilde{\zeta} = 4$	σ [NW s.e.] {R^2}	0.323 [0.024] {0.514}	0.375 (0.124)	0.442 [0.031] {0.611}	0.291 [0.030] {0.539}	0.350 [0.033] {0.446}	0.262 [0.051] {0.274}
$\tilde{\zeta} = 6$	σ [NW s.e.] {R^2}	0.361 [0.026] {0.503}	0.370 (0.119)	0.503 [0.035] {0.641}	0.316 [0.029] {0.510}	0.413 [0.040] {0.472}	0.304 [0.052] {0.315}
$\tilde{\zeta} = 8$	**σ [NW s.e.] {R^2}**	**0.395 [0.032] {0.500}**	**0.391 (0.060)**	**0.543 [0.038] {0.652}**	**0.336 [0.034] {0.472}**	**0.460 [0.047] {0.496}**	**0.333 [0.053] {0.339}**
$\tilde{\zeta} = 10$	σ [NW s.e.] {R^2}	0.406 [0.035] {0.494}	0.291 (0.171)	0.550 [0.039] {0.641}	0.341 [0.037] {0.442}	0.472 [0.051] {0.499}	0.333 [0.051] {0.333}
$\tilde{\zeta} = 20$	σ [NW s.e.] {R^2}	0.401 [0.035] {0.479}	0.415 (0.095)	0.534 [0.038] {0.607}	0.335 [0.036] {0.407}	0.464 [0.052] {0.484}	0.310 [0.046] {0.301}
$\tilde{\zeta} \to \infty$	σ [NW s.e.] {R^2}	0.398 [0.035] {0.476}	0.376 (0.069)	0.528 [0.037] {0.600}	0.332 [0.036] {0.403}	0.460 [0.052] {0.480}	0.303 [0.046] {0.294}

Table notes are placed after the table in the published paper.

Appendix Table 4: IV and OLS Estimates of Equation (5): Various Critical Periodicities ($\tilde{\zeta}$) and Fixed Window (q=3)

		With Time Fixed Effects		Without Time Fixed Effects			
		IV		OLS	IV		
		$p^{\&}_{i,t-2}$	$p^{@}_{i,t-2}$	Benchmark Model	Corporate Tax Rate	Three Corporate Tax Rates	Three Corporate Tax Rates $p^{\&}_{i,t-2}$
		(1)	(2)	(3)	(4)	(5)	(6)
$\tilde{\zeta}=2$	σ [NW s.e.] /F/ (J *p*-value)	0.359 [0.055] /7.14/	0.316 [0.069] /6.63/	0.254 [0.021]	0.440 [0.070] /29.49/	0.513 [0.057] /15.74/ (0.001)	0.456 [0.039] /24.64/ (0.002)
$\tilde{\zeta}=4$	σ [NW s.e.] /F/ (J *p*-value)	0.367 [0.049] /10.57/	0.314 [0.062] /11.93/	0.353 [0.023]	0.433 [0.085] /43.35/	0.478 [0.053] /28.09/ (0.078)	0.417 [0.037] /63.68/ (0.082)
$\tilde{\zeta}=6$	σ [NW s.e.] /F/ (J *p*-value)	0.412 [0.058] /12.99/	0.333 [0.077] /13.94/	0.387 [0.024]	0.445 [0.095] /70.56/	0.481 [0.064] /36.34/ (0.381)	0.440 [0.045] /88.33/ (0.244)
$\tilde{\zeta}=8$	**σ [NW s.e.] /F/ (J *p*-value)**	**0.438 [0.062] /14.16/**	**0.345 [0.087] /15.16/**	**0.415 [0.029]**	**0.452 [0.103] /93.56/**	**0.483 [0.080] /40.80/ (0.704)**	**0.455 [0.050] /111.80/ (0.554)**
$\tilde{\zeta}=10$	σ [NW s.e.] /F/ (J *p*-value)	0.450 [0.063] /14.30/	0.349 [0.091] /15.12/	0.423 [0.031]	0.456 [0.108] /99.65/	0.485 [0.091] /40.26/ (0.455)	0.462 [0.053] /117.58/ (0.594)
$\tilde{\zeta}=20$	σ [NW s.e.] /F/ (J *p*-value)	0.462 [0.065] /13.90/	0.354 [0.094] /14.00/	0.417 [0.031]	0.460 [0.115] /94.40/	0.492 [0.104] /35.57/ (0.120)	0.472 [0.055] /109.47/ (0.195)
$\tilde{\zeta}\to\infty$	σ [NW s.e.] /F/ (J *p*-value)	0.464 [0.065] /13.77/	0.355 [0.095] /13.69/	0.414 [0.016]	0.460 [0.116] /92.09/	0.493 [0.107] /34.40/ (0.088)	0.459 [0.073] /57.37/ (0.141)

Table notes are placed after the table in the published paper.

Appendix IV:
Stationarity Properties of the Model Variables

To assess stationarity of the variables used in the regression models, we use the panel unit root test proposed by Pesaran (2007) that extends the standard augmented Dickey-Fuller test to allow for cross-sectional dependence in panel data. For a given variable $X_{i,t} = \left\{(K_{i,t} / Y_{i,t}),\ (P^K_{i,t} / P^Y_{i,t}),\ \Delta k_{i,t},\ \Delta p_{i,t},\right\}$, we estimate the following auxiliary equation,

$$\Delta X_{i,t} = a_i + b_i X_{i,t-1} + \bar{b}_i \bar{X}_{t-1} + \sum_{j=1}^{J'} d_{i,j} \Delta X_{i,t-j} + \sum_{j=0}^{J'} \bar{d}_{i,j} \overline{\Delta X}_{t-j} + g_i t + u_{i,t}, \quad \text{(IV-1a)}$$

$$u_{i,t} = r_i\ u_{i,t-1} + \varepsilon_{i,t}, \quad \text{(IV-1b)}$$

$$\mu_b = \sum_{i=1}^{35} b_i / 35, \quad \text{(IV-1c)}$$

where $\bar{X}_t$ is a cross-section average of $X_{i,t}$, t is a time index, and the remaining lower case roman letters are parameters to be estimated. The lag length (J') for the lagged dependent variable ($\Delta X_{i,t-j}$) and lagged difference in cross-section averages ($\overline{\Delta X}_{t-j}$) is determined by the need to absorb any serial correlation in the errors. The null hypothesis that $X_{i,t}$ has a unit root is evaluated by μ_b, the average of the estimated b_i coefficients. The critical values for μ_b are provided in Pesaran's Tables II.b and II.c for tests without and with a time trend ($g_i t$ in the above equation), respectively; they are reported in columns (3) to (5) in Appendix Table IV below.

The test statistics for the raw series are presented in panel A of Appendix Table IV, and they are all negative. For $(K_{i,t} / Y_{i,t})$, the test statistics are greater than the (negative) critical values, even at the 10% level. For $(P^K_{i,t} / P^Y_{i,t})$, the test statistics are less than the (negative) critical at the 10% level. These results suggest that the relative price term is stationary, while the capital/output ratio is nonstationary.

As discussed in sub-section IV.A. , the LPF is not strictly valid when applied to nonstationary data, and we exploit the commutative property of the filters (equation (7d)). In this case, we take logs and first-difference the data before applying the LPF. As shown in panel B of Appendix Table IV, the unit root null hypothesis is rejected for all four model variables with or without a deterministic trend. Thus, it is appropriate to apply the LPF to the logarithmically differenced data.

Appendix Table IV is on the next page.

Appendix Table IV: Test Statistics for Stationarity of the Model Variables μ_b in Equation (IV-1c)

Panel A: Raw Series

Variable	With Deterministic Trend ($g_i \neq 0$)	Critical Values 10%	5%	1%	Without Deterministic Trend ($g_i = 0$)	Critical Values 10%	5%	1%
	(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)
$K_{i,t} / Y_{i,t}$	-2.494	-2.550	-2.600	-2.720	-1.725	-2.050	-2.110	-2.230
$P^K_{i,t} / P^Y_{i,t}$	-2.852	-2.550	-2.600	-2.720	-2.085	-2.050	-2.110	-2.230
$K_{i,t} / L_{i,t}$	-1.856	-2.550	-2.600	-2.720	-1.722	-2.050	-2.110	-2.230
$P^K_{i,t} / P^L_{i,t}$	-2.730	-2.550	-2.600	-2.720	-2.448	-2.050	-2.110	-2.230

Panel B: Transformed Series; First-Differences of Logs of Raw Series

Variable	With Deterministic Trend ($g_i \neq 0$)	Critical Values 10%	5%	1%	Without Deterministic Trend ($g_i = 0$)	Critical Values 10%	5%	1%
	(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)
$\Delta k_{i,t}$	-5.342	-2.550	-2.600	-2.720	-5.118	-2.050	-2.110	-2.230
$\Delta p_{i,t}$	-6.133	-2.550	-2.600	-2.720	-5.974	-2.050	-2.110	-2.230
$\Delta kl_{i,t}$	-5.702	-2.550	-2.600	-2.720	-5.495	-2.050	-2.110	-2.230
$\Delta p^{kl}_{i,t}$	-6.243	-2.550	-2.600	-2.720	-6.059	-2.050	-2.110	-2.230

Appendix V:
The Common Correlated Effects (CCE) Estimator

Cross-sectional dependence may not be fully captured by time fixed effects and hence becomes part of the error term. Correlation between these shocks and the regressors would lead to inconsistent estimates of σ. Even absent such correlation, the shocks will lead to biased standard errors. The common correlated effects (CCE) estimator introduced by Pesaran (2006) is feasible for panels with a large number of cross-section units (unlike the Seemingly Unrelated Regression framework), and it accounts for the effects of cross-sectional dependence by including cross-section averages (CSA's) of the dependent and independent variables as additional right-hand side variables,

$$\Delta k^{*}_{i,t} = \xi - \sigma \Delta p^{*}_{i,t} + e_{i,t} + \gamma_i \left(\mathrm{CSA}\left[\Delta k^{*}_{i,t} \right] - \sigma \, \mathrm{CSA}\left[\Delta p^{*}_{i,t} \right] \right), \qquad \text{(V-1)}$$

where $\mathrm{CSA}[.]$ is the cross-section average operator and the γ_i's are 35 additional parameters to be estimated. If the γ_i's in equation (V-1) are constrained to be 1.0 for all i, the specification would be equivalent to transforming the data by demeaning each variable with respect to its CSA, the standard way of controlling for time fixed effects with the least squares dummy variables (LSDV) estimator.

In general, the CSA's in the CCE estimator are formed with a set of state weights, v_j for j = 1,…,J, such that,

$$\bar{x}_t \equiv \sum_{j=1}^{J} v_j x_{j,t}, \qquad \sum_{j=1}^{J} v_j = 1. \qquad \text{(V-2)}$$

As shown by Pesaran (2006), the asymptotic properties of the CCE estimator are invariant to the choice of the v_j weights. The empirical work reported here is based on equal weighting ($v_j = 1/J$ for all j).

Appendix VI:

Relating Heterogeneous Industry and Aggregated σ's

This appendix develops the formula for relating the aggregated σ (σ_{agg}) to heterogeneous industry σ's (σ_i). It then considers two alternative weighting schemes.

We begin with the definitions of σ_{agg} and σ_i that follow from equation (4) and are stated in terms of percentage changes in the aggregate and industry capital/output ratios ((K_{agg}/Y_{agg}) and (K_i/Y_i), respectively) and the aggregate and industry relative prices of capital (P_{agg}^{KY} and P_i^{KY}, respectively),

$$\sigma_{agg} \equiv \frac{d\left(K_{agg}/Y_{agg}\right)/\left(K_{agg}/Y_{agg}\right)}{d\,P_{agg}^{KY}/P_{agg}^{KY}}, \qquad Y_{agg} = \text{constant}, \tag{VI-1}$$

$$\sigma_i \equiv \frac{d\left(K_i/Y_i\right)/\left(K_i/Y_i\right)}{d\,P_i^{KY}/P_i^{KY}}, \qquad Y_i = \text{constant}. \tag{VI-2}$$

While the relative price of capital varies by industry, we consider a percentage change that is equal across all industries (e.g., a change in the nominal or relative price of investment),

$$d\,P_{agg}^{KY}/P_{agg}^{KY} = d\,P_i^{KY}/P_i^{KY} = d\,P/P, \qquad \forall i. \tag{VI-3}$$

We begin with identities relating changes in the aggregated to industry capital, and changes in the aggregated and industry capital/output ratios, respectively,

$$\frac{dK_{agg}}{d\,P/P} \equiv \sum_i \frac{dK_i}{d\,P/P}, \tag{VI-4}$$

$$\frac{dK_{agg}}{d\,P/P} \equiv \frac{d\left(K_{agg}/Y_{agg}\right)}{d\,P/P} * Y_{agg}, \qquad Y_{agg} = \text{constant}, \tag{VI-5}$$

$$\frac{dK_i}{d\,P/P} \equiv \frac{d\left(K_i/Y_i\right)}{d\,P/P} * Y_i, \qquad Y_i = \text{constant}. \tag{VI-6}$$

Substituting equations (VI-5) and (VI-6) into equation (VI-4), dividing both sides by K_{agg}, and rearranging, we obtain the following equation,

$$\frac{d\left(K_{agg}/Y_{agg}\right)/\left(K_{agg}/Y_{agg}\right)}{d\,P/P} = \sum_i \frac{d\left(K_i/Y_i\right)}{d\,P/P} * \left(Y_i/K_{agg}\right). \tag{VI-7}$$

The left-side equals σ_{agg} by equation (VI-1). Multiplying the right-side by $\left(K_i / K_i\right)$ and rearranging, we obtain the following equation,

$$\sigma_{agg} = \sum_i \frac{d\left(K_i / Y_i\right) * \left(K_i / Y_i\right)}{d\,P / P} * \left(K_i / K_{agg}\right). \qquad \text{(VI-8)}$$

Using the definition of σ_i from equation (VI-2) and defining the latter object in equation (VI-8) as an industry weight, we obtain the following equation,[1]

$$\sigma_{agg} = \sum_i \sigma_i * w_i, \qquad w_i \equiv \left(K_i / K_{agg}\right). \qquad \text{(VI-9)}$$

In equation (VI-9), the aggregated σ (σ_{agg}) is a weighted average of the industry σ's (σ_i's), where the w_i's are industry weights defined in terms of industry capital ratios.[2]

Alternative aggregation procedures have been developed in two papers. First, Oberfield and Raval (2014; hereafter, OR) present a scheme that estimates σ's at the plant level, and then aggregates them to the industry and manufacturing levels. Their procedure allows for reallocation of resources as a result of the change in factor prices.

The aggregation procedure developed in this Appendix and used in Section V differs from that of OR in that we evaluate our derivatives at the initial, pre-change point (see equations (VI-5) and (VI-6)), while OR evaluate derivatives at the post-change point. See the equation between equations (5) and (6) in their paper. Their $d\theta_i / d\ln(w / r)$ term represents the change in industry weights that is non-zero in OR and zero in our paper. (In a loose sense, the difference in aggregation procedures is akin to the difference between using a Paasche (OR) or Laspeyres (our paper) price indices.) The benefit of the OR procedure is that it allows for reallocations of factors and economic activity. However, their procedure also requires estimates of the own-industry and cross-industries demand elasticities and the elasticity of substitution between

[1] Note that σ can be defined in terms of the capital/labor ratio analyzed in Section IV.F. In this case, the derivation presented in this appendix is unaffected (merely replace Y_{agg} with L_{agg} and Y_i with L_i in equations (VI-5) to (VI-8) and P_{agg}^{KY} with P_{agg}^{KL} and P_i^{KY} with P_i^{KL} in equation (VI-3).

[2] It should be noted that the σ from the benchmark model is also effectively a weighted-average estimate. The heterogeneous model analyzed in this appendix weights the σ_i's by industry capital shares, while the homogeneous model effectively weights the σ_i's by relative industry variances.

material and non-material factors. Also, once changes in product demand are introduced, the price elasticity of demand for a factor of production is no longer equal to just σ. Rather, per Hicks formula for the derived demand of a factor of production (see Chirinko and Mallick 2011, equations (10) or (11) for a recent statement of the Hicks formula), the price elasticity also depends on the capital income share and the own-demand elasticity.

A less important difference is that OR analyze σ's in terms of the capital/labor ratio, while we analyze the capital/output ratio.

Second, in a provocative paper, Jones (2005) formally relates industry and aggregate (global) production functions to the distribution of alternative production techniques (APT's) for combining capital and labor. His striking result is that the industry and aggregate production functions will be Cobb-Douglas in the long-run. This approach has the benefit of developing solid microfoundations for production functions but is sensitive to the assumed distribution of ideas. When APT's are distributed according to a Pareto distribution with independence between marginal APT distributions, the Cobb-Douglas result obtains. However, when APT's are distributed according to a Weibull distribution (Growiec 2008a) or a Pareto distribution with dependence between marginal APT distributions (Growiec 2008b), the industry and aggregate production functions are CES. These theoretical results, coupled with the empirical results presented in this paper, suggest the need for further study of aggregation procedures and the underlying distribution of ideas.

Appendix VII:

Relating the Capital Share of Income and σ_{agg}

This appendix formally relates the capital share of income to σ_{agg} in order to evaluate the impact of a rising σ_{agg} in Figure 4 (from 0.63 to 0.67) on a rising capital share of income (from 0.25 to 0.40; **Chirinko, Wilson and Zidar 2015, Slide 5**). Time and firm subscripts, "*" superscripts, and technology shocks are omitted for notational convenience.

The capital share of income (KS) is defined as follows,

$$KS \equiv \frac{MPK * K}{Y} \ . \qquad \text{(VII-1)}$$

Equating the MPK to the relative price of capital (P^K / P^Y), denoting this relative price by the the Jorgensonian user cost of capital (C), and substituting equation (2) for the K/Y ratio, we obtain the following equation,

$$KS \ = \ C \ \phi^{\sigma} \ C^{-\sigma} \ = \ C^{1-\sigma} \ \phi^{\sigma} \ . \qquad \text{(VII-2)}$$

Calibrating equation (VII-2) by assuming that KS = 0.25 (the value in 1980 for the gross capital share), C = 0.15, and $\sigma = 0.63$ (the value in 1980 from Figure 4), $\phi = 0.34$. If σ rises to 0.67, KS rises by only one percentage point to 0.26.

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