# Price cutting and business stealing in imperfect cartels Online Appendix

B. Douglas Bernheim Erik Madsen

December 2016

## C.1 Proofs omitted from the main text

*Proof of Proposition 4.* We explicitly construct an equilibrium yielding zero lifetime profits for each firm. The equilibrium consists of two phases: Cooperation and Punishment. In the Punishment phase each firm enters his home market and posts a price $p^{PW} < \underline{p}_H$ characterized below, while in his away market he enters and posts price $\underline{p}_H$. If both firms post their assigned price in each market, they transit to Cooperation in the next period; otherwise they continue in the Punishment phase. In the Cooperation phase firms play the stage strategies characterized in the proof of Lemma B.11 supporting stage profits $(\Pi^\dagger, \Pi^\dagger)$, where $\Pi^\dagger$ is the unique solution in $[\Pi^C, \Pi^M]$ to

$$\Pi = (1-\delta)(2\Pi - \Delta c D(p^*(\Pi)))$$

when $\delta < 1/2$ and a solution exists, and otherwise $\Pi^\dagger = \Pi^M$. So long as no away firm posts a price at or below the floor price, the cartel continues in the Cooperation phase in the next period; otherwise they transit to the Punishment phase.

$p^{PW}$ is chosen so that beginning in the punishment phase yields zero lifetime profits for each firm, i.e.

$$(1-\delta)(D(p^{PW})(p^{PW} - c_H) - 2c) + \delta\Pi^\dagger = 0.$$

Suppose that $\Pi^\dagger \geq \frac{1-\delta}{\delta}c$. Then Assumption A4 ensures existence of a solution $p^{PW} \leq \underline{p}_H$ to this equation. If $\Pi^\dagger = \Pi^M$, then $\delta \geq \underline{\delta}$ combined with $\Pi^M > \Pi^0$ guarantees that this inequality is satisfied. So consider instead the case $\delta < 1/2$ and $\Pi^\dagger < \Pi^M$. Then by construction $\Pi^\dagger$ satisfies

$$\Pi^\dagger = (1-\delta)(2\Pi^\dagger - \Delta c D(p^*(\Pi^\dagger))),$$

which when re-arranged yields the bound

$$\Pi^\dagger = \frac{1-\delta}{1-2\delta}\Delta c D(p^*(\Pi^\dagger))) \geq \frac{1-\delta}{1-2\delta}\Delta c D(p_H^*) \geq \frac{1-\underline{\delta}}{1-2\underline{\delta}}\Delta c D(p_H^*).$$

Now, $\underline{\delta}$ satisfies the identity

$$\frac{1-\underline{\delta}}{1-2\underline{\delta}}\Delta c D(p_H^*) = \frac{1-\underline{\delta}}{\underline{\delta}}c,$$

so that

$$\Pi^\dagger \geq \frac{1-\underline{\delta}}{\underline{\delta}}c \geq \frac{1-\delta}{\delta}c.$$

Hence there always exists a solution $p^{PW} \leq \underline{p}_H$ to the zero-profit condition. (In the knife-edge case $\delta = \underline{\delta}$, some demand functions require $p^{PW} = \underline{p}_H$, which does not formally satisfy our construction. In this case a slightly modified equilibrium can be constructed in which the away firm enters just above $p^{PW}$ and plays a mixed strategy with enough support close to $\underline{p}_H$ that no price posted by the home firm achieves positive profits.)

We complete the proof by verifying incentive-compatibility in each phase. In the Punishment phase the most profitable stage deviation by each firm is to withdraw completely from each market, yielding stage and continuation profits of 0. So IC holds in the Punishment phase. Meanwhile in the Cooperation phase the most profitable stage deviation involves undercutting the floor price in the away market, yielding stage profits of $2\Pi^\dagger - \Delta c D(p^*(\Pi^\dagger))$ and zero continuation profits. So IC is equivalent to

$$\Pi^* \geq (1-\delta)(2\Pi^\dagger - \Delta c D(p^*(\Pi^\dagger))).$$

When $\delta < 1/2$, either $\Pi^\dagger$ satisfies this inequality exactly by construction, or else $\Pi^\dagger = \Pi^M$ and by construction the inequality is slack. Meanwhile when $\delta \geq 1/2$ the rhs is weakly smaller than $\Pi^\dagger$ for any $\Pi^\dagger \in [0, \Pi^M]$, and in particular when $\Pi^\dagger = \Pi^M$. So IC holds in the Cooperation phase as well.

A direct consequence of this construction is the following result: if $\Pi^M > \Pi^0$, then $\delta^M < 1/2$. The proof simply observes that when $\delta \geq 1/2$, the equilibrium just constructed supports a monopoly division of the market in each period. □

*Proof of Proposition 7.* Lemma B.13, implied by the proof of Proposition 4, ensures that

$\delta < 1/2$. Then by Lemma B.9, at most one firm can earn positive profits in a given market, and by Lemma B.10 each firm makes positive profits in only one market. Then the payoff vector $(\Pi, \Pi)$ (streamlining the notation $\Pi^*$ to $\Pi$ for this proof) can only be supported by giving each firm $\Pi$ in his home market and 0 in his away market. To see this, observe that it cannot be that each firm makes $\Pi$ in his away market; for this leads to a deviation worth $\Pi + \Delta c\, D(p^*(\Pi)) > \Pi - \Delta c\, D(p^*(\Pi))$ in the home market, which violates the IC constraint. And we can rule out making negative profits in the away market, because this also increases the value of a deviation in that market and violates the IC constraint.

Now fix a market, say market 1. As firm 1 makes positive profits in this market, he enters w.p. 1. Further, firm 1 cannot have a profitable deviation in this market, else the IC constraint would be violated.

Let $F_i$ denote the price distribution of firm $i$ in market 1, and $\pi_i$ their entry probability. (We will suppress the dependence of these variables on the market to streamline notation.) We first claim that the support of $F^1$ is contained in $[p^*(\Pi), p^*_H]$. Below $p^*(\Pi)$ firm 1 cannot earn profits $\Pi$, so such prices can't be profit-maximizing. Meanwhile above $p^*_H$ he will earn weakly lower profits than at $p^*_H$, with profits strictly lower whenever profits at $p^*_H$ are positive. Thus his profits above $p^*_H$ are either non-positive, thus not optimal, or else strictly lower than at $p^*_H$, which would introduce a profitable devition if 1 did play above $p^*_H$ in equilibrium.

Let $p_1^L$ and $p_1^U$ be the infimum and supremum of 1's price support. We claim that $p_1^L = p^*(\Pi)$ and $p_1^U = p^*_H$. Suppose that $p_1^L > p^*(\Pi)$. Then firm 2 has a deviation worth at least $D(p_1^L)(p_1^L - c_A) - c > \Pi - \Delta c\, D(p^*(\Pi))$, violating the IC constraint. So the lower end of 1's support must be $p^*(\Pi)$. On the other hand, if $p_1^U < p^*_H$, then 2 must place an atom at $p_1^U$ to avoid giving 1 a profitable deviation up to $p^*_H$. If 1 doesn't place an atom at $p_1^U$, then 2 never wins at $p_1^U$ and thus makes negative profits there, which can't be profit-maximizing. But if he does place an atom at $p_1^U$, then he would have a profitable deviation to just below the atom, a contradiction. Hence $p_1^U = p^*_H$.

Now, suppose there exists an interval $[p_A, p_B] \subset [p^*(\Pi), p^*_H]$ such that $F_1((p_A, p_B)) = 0$. Let $\widehat{F}_1 = F_1(p)$ for $p \in (p_A, p_B)$, and enlarge $[p_A, p_B]$ if necessary so that $p_A = \inf\{p : F_1(p) = \widehat{F}_1\}$ and $p_B = \sup\{p : F_1(p) = \widehat{F}_1\}$. Given the support of $F_1^1$, we must have $\widehat{F}_1 \in (0, 1)$. Thus 1 has profit-maximizing prices arbitrarily close to both $p_A$ and $p_B$.

Consider firm 2's strategy in $[p_A, p_B]$. He can set at most one price in $(p_A, p_B)$, since stage profits are strictly increasing in the interior of the interval. Say he plays some price $p_C \in (p_A, p_B)$ with positive probability. If he also places an atom at $p_A$, then 1 puts no atom there to avoid a profitable deviation. But then 2's stage profits at $p_C$ are strictly greater than at $p_A$, a contradiction. So 2 places no atom at $p_A$. But then $p_A$ must be profit-maximizing for 1, a contradiction given that his profits are strictly increasing on $[p_A, p_C)$. So 2 does not play in $(p_A, p_B)$.

Firm 2 *does* play an atom at $p_A$, else 1 would have a profitable deviation into the gap. It follows that $p_A > p^*(\Pi)$ and 1 plays no atom there and is profit-maximizing in the limit as $p \nearrow p_A$. Conversely, firm 2 does not place an atom at $p_B$, for otherwise 1 would have a profitable deviation just below it. It follows that 1 is profit-maximizing at $p_B$.

From these facts we can pin down the size of firm 2's atom at $p_A$. Firm 1's profits from playing just below $p_A$ are

$$\Pi = D(p_A)(p_A - c_H)\left(1 - \pi_2 F_2(p_A) + \frac{1}{2}\pi_2 \Delta F_2(p_A)\right) - c,$$

while his profits at $p_B$ are

$$\Pi = D(p_B)(p_B - c_H)(1 - \pi_2 F_2(p_A)) - c.$$

Using the second equation to eliminate $F_2(p_A)$ from the first, we find that

$$\pi_2 \Delta F_2(p_A) = (\Pi + c)\left(\frac{1}{D(p_A)(p_A - c_H)} - \frac{1}{D(p_B)(p_B - c_H)}\right).$$

This is the probability that 2 enters and plays in $[p_A, p_B]$. It is easy to check that this is equal to the probability that firm 2 plays in $[p_A, p_B]$ under equilibrium characterized in Proposition 6. (We will refer to this equilibrium as the "standard equilibrium" or the "no-gap case" in what follows.)

Similarly, we may calculate the size of firm 1's atom at $p_B$. Suppose $p_B < p_H^*$. As firm 2 places an atom at $p_A$, he must be profit-maximizing there. Then

$$0 = D(p_A)(p_A - c_A)(1 - F_1(p_A)) - c.$$

It is also true that firm 2 must be profit-maximizing arbitrarily close to $p_B$ from above. For otherwise he would not play in some interval above $p_B$, and firm 1 would have a profitable deviation upward from $p_B$. Then

$$0 = D(p_B)(p_B - c_A)\left(1 - F_1(p_A) - \frac{1}{2}\Delta F_1(p_B)\right) - c.$$

So

$$\Delta F_1(p_B) = c\left(\frac{1}{D(p_A)(p_A - c_A)} - \frac{1}{D(p_B)(p_B - c_A)}\right).$$

This is the same as the probability that firm 1 plays in $[p_A, p_B]$ under the standard equilibrium. If $p_B = p_H^*$ then we must modify the argument slightly: we still know $F_1(p_A)$, and now $F_1(p_B) = 1$. This again determines the atom, which is easily checked to give the same probability of playing in $[p_A, p_B]$ under the standard equilibrium.

We conclude that, for any gap in firm 1's mixing distribution, both firms play in the gap with the same frequency as in the no-gap case, except that the away firm concentrates all its support at the bottom of the gap, while the home firm prices only at the top. Hence business-stealing is strictly higher in regions where gaps have been added.

Finally, in any interval with no gap, both firms must play the entry-adjusted mixing distributions of the standard equilibrium. So business-stealing occurs at the same rate in these regions as in the standard equilibrium. Finally, sum the probability of business-stealing across all gap- and no-gap intervals. (Formally: there are at most a countable number of maximally-sized gaps, which can be well-ordered by their upper edges. The no-gap regions are then defined as the intervals between the upper edge of one gap and the lower edge of the next. These are also countable, so can be summed.) This sum is strictly higher than the standard equilibrium when gaps exist, and the standard equilibrium is the unique no-gap equilibrium. □

*Proof of Proposition 8.* By Lemma B.13, established in the proof of Proposition 4, $\delta < 1/2$. Consider a stationary equilibrium supporting profits $(\Pi^1, \Pi^2) > (\Pi^C, \Pi^C)$. Suppose wlog that in market 1, player 2 never wins the customer's business. Lemma B.10 ensures that player 1 earns positive profits only in that market, so $\Pi_1^1 \geq \Pi^1$. And as player 2 never wins the business of that market, it must be that $\Pi_2^2 \geq \Pi^2$.

Now, suppose player 2 does not enter market 1. Then player 1 can deviate upward to $p_H^*$ in his own market to earn stage profits $\Pi^M$, and can undercut player 2 in market 2 to earn $\Pi^2$. Thus the IC constraint

$$\Pi^1 \geq (1-\delta)(\Pi^M + \Pi^2 - \Delta c D(p^*(\Pi^2))) + \delta \underline{\Pi}(\delta)$$

must hold. (If $\Pi_2^2 > \Pi^2$ then an even stricter IC constraint holds.) Meanwhile, the usual IC constraint

$$\Pi^2 \geq (1-\delta)(\Pi^1 + \Pi^2 - \Delta c D(p^*(\Pi^1))) + \delta \underline{\Pi}(\delta)$$

holds for player 2. ($\Pi_2^2 > \Pi^2$ would imply that 2 makes negative profits in market 1, which would only increase the profitability of a deviation and tighten the IC constraint.) Because $\Pi^1 < \Pi^M$, the first constraint is violated at $(\Pi^1, \Pi^2) = (\Pi^*, \Pi^*)$. But the second constraint would be violated if $\Pi^2$ alone were lowered, as the lhs drops faster than the rhs and the constraint is saturated at $(\Pi^*, \Pi^*)$. Thus $(\Pi^1, \Pi^2)$ must be bounded below $(\Pi^M, \Pi^M)$ by continuity of $D(\cdot)$ and $p^*(\cdot)$ in order to satisfy both constraints.

On the other hand, suppose player 2 does enter market 1. As he never wins the market by assumption, his stage profits in that market are $-c$. Then he must enter w.p. 1, else he would not be optimizing by entering. Player 2's IC constraint is the same no matter what he plays in market 1. Meanwhile, to maximally relax player 1's IC constraint, 2 may mix just above the single price $p_1$ played by player 1 in that market with sufficient density close to $p_1$ to deter an upward deviation. (Because 1 always wins, he can be cannot be willing to mix between multiple prices.) In this case player 1's IC constraint is the usual

$$\Pi^1 \geq (1-\delta)(\Pi^1 + \Pi^2 - \Delta c D(p^*(\Pi^2))) + \delta \underline{\Pi}(\delta).$$

But now player 2's deviation to undercut player 1 in market 1 yields additional profits of $c$, so his IC constraint is tightened to

$$\Pi^2 \geq (1-\delta)(\Pi^1 + \Pi^2 + c - \Delta c D(p^*(\Pi^2))) + \delta \underline{\Pi}(\delta).$$

By a similar argument to the previous case, solutions to this pair of inequalities are bounded below $(\Pi^M, \Pi^M)$. □

*Proof of Proposition 12.* This proposition is a direct consequence of Lemma C.2 combined with Lemma B.13 (established in the proof of Proposition 4), which establishes that $\delta^M < 1/2$. Let $\mathcal{E}^B$ be the set of lifetime profit vectors supportable by balanced equilibria.

**Definition C.1.** *A balanced equilibrium $\sigma$ with lifetime payoffs $U = (U^1, U^2)$ is* B-optimal *if, for $(\widetilde{U}^1, U^2) \neq U$ in $\mathcal{E}^B$, $U^i > \widetilde{U}^i$ for some $i$.*

**Lemma C.1.** *Suppose $\delta \leq 1/2$. Let $\sigma$ be an B-optimal equilibrium $\sigma$ with lifetime payoffs $(U^1, U^2)$. Then there exist constants $\Pi^1, \Pi^2 \in [\Pi^C, \Pi^M]$ and $(\widetilde{U}^1, U^2) \in \mathcal{E}^B$ such that for each $i$, $\Pi^i$ and $\widetilde{U}^i$ are firm $i$'s first-period expected stage and continuation profits, respectively, so that $U^i = (1-\delta)\Pi^i + \delta\widetilde{U}^i$; and the IC constraint*

$$U^i \geq (1-\delta)\left(\Pi^1 + \Pi^2 - \Delta c D(p^*(\Pi^{-i}))\right) + \delta \underline{\Pi}(\delta)$$

*holds. Conversely, given constants $\Pi^1, \Pi^2 \in [\Pi^C, \Pi^M]$ and payoffs $(\widetilde{U}^1, \widetilde{U}^2) \in \mathcal{E}^B$ satisfying* *the above inequalities, there exists a balanced equilibrium with initial-period expected stage payoffs* $\Pi^i$ *and continuation payoffs* $\widetilde{U}^i$ *for each firm* $i$.

*Proof.* Fix a B-optimal equilibrium $\sigma$ with lifetime payoffs $(U^1, U^2)$ and period-0 stage-game strategy profile $\sigma(h^0) = \tau$. Then there exist constants $\Pi^i_m$ such that the period-0 stage profits of any on-path action by firm $i$ in market $m$ are $\Pi^i_m$.[1] Let $\Pi^i = \Pi^i_1 + \Pi^i_2$ for each $i$. Then we can decompose each $U^i$ as

$$U^i = (1-\delta)\Pi^i + \delta\widetilde{U}^i$$

for some constants $(\widetilde{U}^1, \widetilde{U}^2) \in \mathcal{E}^B$.

Suppose first that $\Pi^i_m > 0$ for all $i$ and $m$. Then by the argument in the proof of Lemma B.8, there exists a deviation for each firm yielding expected stage profits of at least $2\Pi^i + c$. As the harshest possible punishment continuation following a deviation yields profits $\underline{\Pi}(\delta)$, the unprofitability of this deviation implies the IC constraint

$$(1-\delta)\Pi^i + \delta\widetilde{U}^i \geq (1-\delta)(2\Pi^i + c) + \delta\underline{\Pi}(\delta)$$

for each $i$. Re-arranging yields

$$\widetilde{U}^i \geq \frac{1-\delta}{\delta} 2(\Pi^i + c) + \underline{\Pi}(\delta).$$

Now, by definition of B-optimality, for some $i$ we must have $U^i \geq \widetilde{U}^i$. For this firm we have $U^i = (1-\delta)\Pi^i + \delta\widetilde{U}^i \geq \widetilde{U}^i$, or $\Pi^i \geq \widetilde{U}^i$. Combining this restriction with the IC constraint produces

$$\widetilde{U}^i \geq \frac{1-\delta}{\delta}(\widetilde{U}^i + c) + \underline{\Pi}(\delta),$$

which in turn implies $\delta > 1/2$.

It must therefore be the case that $\Pi^i_i > 0$ for each $i$, with all other stage profits non-positive. Note further that each $\Pi^i_i \geq \Pi^C$, as otherwise we could construct an equilibrium with strictly higher lifetime profits for some firms by playing the stage-game Nash equilibrium in the first period for all markets $m$ such that $\Pi^m_m < \Pi^C$. (This cannot introduce additional profitable deviations and thus must still be supportable as an equilibrium.) Then $p^*(\Pi^i_i)$ is well-defined for each $i$, and firm $i$ must play prices no lower than $p^*(\Pi^i_i)$ to achieve expected stage profits $\Pi^i_i$ in his home market. Then each firm $i$ has a deviation yielding stage-game profits of at least $\Pi^1_1 + \Pi^2_2 - \Delta c D(p^*(\Pi^{-i}_{-i}))$, achieved by just undercutting the infimum of

[1] This is a basic property of balanced equilibria; details and a complete proof may be found in the Online Appendix.

the home firm's price support in firm $i$'s away market.

In fact, the usual partially collusive structure yields no deviations more profitable than this one, and yields zero profits to all away firms in each market. Thus it must be the case that $\Pi^i_m = 0$ for all $i$ and $m \neq i$ (else we could strictly improve the equilibrium for some firms by playing a partially collusive structure in the first period and then reverting to $\sigma$). So $\Pi^i = \Pi^i_i$ for all $i$, and the IC constraints

$$(1-\delta)\Pi^i + \delta\widetilde{U}^i \geq (1-\delta)\left(\Pi^1 + \Pi^2 - \Delta c D(p^*(\Pi^{-i}))\right) + \delta\underline{\Pi}(\delta)$$

must hold for all $i$. Conversely, given any balanced equilibrium payoffs $(\widetilde{U}^1, \widetilde{U}^2) \in \mathcal{E}^B$, any constants $\Pi^i \in [\Pi^C, \Pi^M]$ satisfying the above IC constraints yield a balanced equilibrium with first-period profits $\Pi^i$ and continuation profits $\widetilde{U}^i$. □

**Lemma C.2.** *Suppose $\delta \leq 1/2$. Then there exists a unique B-optimal equilibrium payoff vector $(U, U)$, which is supportable by a symmetric stationary equilibrium.*

*Proof.* Let $U^* \in R$ be the supremum of all payoffs $U_1$ such that for some $U_2$ the payoff vector $(U_1, U_2)$ is supportable by a balanced equilibrium. Let $(U_1^{(n)}, U_2^{(n)})$ be a sequence of balanced equilibrium-supportable payoff vectors such that $U_1^{(n)} \uparrow U^*$. (If $U^*$ is itself supportable as a balanced equilibrium, this could be a constant sequence.) For each $n$, let $\Pi_1^{(n)}, \Pi_2^{(n)}$ and $(\widetilde{U}_1^{(n)}, \widetilde{U}_2^{(n)})$ be the corresponding constants whose existence is ensured by lemma C.1. Passing to a subsequence if necessary, suppose that $U_2^{(n)} \to U_2^\infty$, and similarly for $\Pi_1^{(n)}, \Pi_2^{(n)}, \widetilde{U}_1^{(n)}, \widetilde{U}_2^{(n)}$. (All of these sequences exist in compact subsets of the real line, so such subsequences exist.)

Because $U^* \geq \widetilde{U}_1^{(n)}$ for all $n$, we must have $U^* \geq \widetilde{U}_1^\infty$. Also given $U_1^{(n)} = (1-\delta)\Pi_1^{(n)} + \delta\widetilde{U}_1^{(n)}$ for all $n$ we have

$$U^* = (1-\delta)\Pi_1^\infty + \delta U_1^\infty \leq (1-\delta)\Pi_1^\infty + \delta U^*,$$

or $\Pi_1^\infty \geq U^*$. And since $U^* \geq \widetilde{U}_2^{(n)}$ for all $n$ (by symmetry of the game) we must have $U^* \geq \widetilde{U}_2^\infty$. Then from the IC constraints

$$(1-\delta)\Pi_2^{(n)} + \delta\widetilde{U}_2^{(n)} \geq (1-\delta)\left(\Pi_1^{(n)} + \Pi_2^{(n)} - \Delta c D(p^*(\Pi_1^{(n)}))\right) + \delta\underline{\Pi}(\delta; N)$$

implied by lemma C.1, we conclude that

$$(1-\delta)\Pi_2^\infty + \delta\Pi_1^\infty \geq (1-\delta)\left(\Pi_1^\infty + \Pi_2^\infty - \Delta c D(p^*(\Pi_1^\infty)\right) + \delta\underline{\Pi}(\delta; N),$$

or equivalently

$$\Pi_1^\infty \geq (1-\delta)(2\Pi_1^\infty - \Delta c D(p^*(\Pi_1^\infty))) + \delta \underline{\Pi}(\delta; N). \tag{1}$$

Thus there exists a constant $\Pi_1^\infty$ satisfying (1) such that $\Pi_1^\infty \geq U^*$. In particular, $(\Pi_1^\infty, \Pi_1^\infty) \geq (U_1, U_2)$ for every balanced equilibrium-supportable payoff vector $(U_1, U_2)$.

Now, we know that every $\Pi_1^\infty$ satisfying (1) yields a symmetric stationary equilibrium with payoffs $(\Pi_1^\infty, \Pi_1^\infty)$ through the usual partially collusive construction. Let $\Pi^*$ be the maximal such $\Pi_1^\infty$ (which we know exists by continuity of $D(p^*(\cdot))$. Then it must be that $U^* = \Pi^*$, as there exists a balanced equilibrium supporting this outcome. Further, as there exists a symmetric stationary equilibrium supporting payoffs $(U^*, U^*)$, this is the unique B-optimal payoff vector. This establishes the claims of the proposition. □

□

## C.2 Basic properties of stationary equilibria

This Appendix characterizes basic properties of stationary equilibria for the duopoly setting, as well as for an extension of the model to $N+1$ firms and $N+1$ markets for any $N > 1$. (See Section C.3.1 of this Appendix for a full description of this extension.) The definition of stationarity for a duopoly setting is extended to the many-firm case in the obvious way.

**Lemma C.3.** *Let $\sigma$ be a stationary equilibrium with on-path play $\tau$. Then for each firm $i$ and market $m$, there exists a constant $\overline{\Pi}_m^i$ such that $\Pi_m^i(a_m^i, \tau_m^{-i}) = \overline{\Pi}_m^i$ with probability 1 under $\tau_m^i$.*

*Proof.* Fix a firm $i$ and a market $m$, and suppose by way of contradiction there existed a $\Pi^*$ such that $\Pi_m^i \leq \Pi^*$ and $\Pi_m^i > \Pi^*$ each occur with strictly positive probability under $\tau_m$. Then there exist actions $a^i, \widetilde{a}^i \in A^i$ such that $\Pi_m^i(a_m^i, \tau_m^{-i}) \leq \Pi^*$ and $\Pi_m^i(\widetilde{a}_m^i, \tau_m^{-i}) > \Pi^*$ and $a^i, \widetilde{a}^i$ are each profit-maximizing for firm $i$ in period 0 under $\sigma$. Further, $a^i$ and $\widetilde{a}^i$ may be chosen to lie along a compliant path for firm $i$ in period 0 under $\sigma$. But then because the set of compliant paths is rectangular, $\widehat{a}^i \equiv (\widetilde{a}_m^i, a_{-m}^i)$ lies on a compliant path as well for $i$. In a stationary equilibrium all actions lying on a compliant path yield the same expected continuation payoff. But $\widehat{a}^i$ yields a strictly higher stage-game payoff for $i$ than $a^i$ by construction, thus $a^i$ cannot be profit-maximizing for $i$. This is the desired contradiction. □

This lemma gives us a powerful accounting identity for characterizing possible equilibrium strategies: In each market, every firm must receive the same profits for all on-path actions. It is used to prove the following pair of lemmas, which establish that 1) any stationary equilibrium may be replaced by another with independent randomization across markets on-path, and 2) any SPNE featuring the same play in each period and independent randomization across markets on-path can be adapted to produce a stationary equilibrium. Therefore without loss of generality, we impose stationarity by assuming that firm use the same stage-game strategy profile in all periods and randomize independently across markets on-path.

**Lemma C.4.** *Let $\sigma$ be a stationary equilibrium with on-path play $\tau$. Then there exists another stationary equilibrium $\sigma'$ with on-path play $\tau' = \prod_{i=1}^{N+1} \prod_{m=1}^{N+1} \tau_m^i$, where $\tau_m^i$ is the marginal distribution of $\tau^i$ in market $m$. Both $\sigma$ and $\sigma'$ yield the same expected lifetime profits to all firms.*

*Proof.* Let $\overline{\Pi}_m^i$ be the constants whose existence is assured by Lemma C.3. Define $A^\dagger \equiv \{a \in A \ : \ \Pi_m^i(a_m^i, \tau_m'^{-i}) = \overline{\Pi}_m^i \ \forall i, m\}$, and let $\widetilde{\mathcal{H}} \equiv \prod_{t=0}^{\infty} A^\dagger$. Note that $A^\dagger$ is a Cartesian product of the sets $A^\dagger(i,m) \equiv \{a_m^i \in A_m^i \ : \ \Pi_m^i(a_m^i, \tau_m'^{-i}) = \overline{\Pi}_m^i\} \subset A_m^i$. Thus $\widetilde{\mathcal{H}}$ is a rectangular set of complete histories, as for each $t$ and $h \in \widetilde{\mathcal{H}}^t$ we have $A^*(h) = A^\dagger$.

Construct $\sigma'$ by setting $\sigma'(h) = \tau'$ for every $t$ and $h \in \widetilde{\mathcal{H}}^t$. Also, for $h$ in some $\widetilde{\mathcal{H}}^t$ and $a \in A$ such that $a^i \notin \prod_{m=1}^{N+1} A^\dagger(i,m)$ for at least two firms $i$, set $\sigma'|_{(h,a)} = \sigma$. Finally, consider $h$ in some $\widetilde{\mathcal{H}}^t$ and $a \in A$ such that $a^i \notin \prod_{m=1}^{N+1} A^\dagger(i,m)$ for a single firm $i$, while $a^{-i} \in \prod_{j \neq i} \prod_{m=1}^{N+1} A^\dagger(j,m)$. Let $\underline{\sigma}(i)$ be an SPNE yielding minimal lifetime profits for $i$ among all SPNEs.[2] Set $\sigma'|_{(h,a)} = \underline{\sigma}(i)$.

We claim that $\sigma'$ is the desired stationary equilibrium. We first demonstrate that $\widetilde{\mathcal{H}}$ is a set of compliant paths under $\sigma'$. Note that for all $i$ and $m$, $\tau_m'^i = \tau_m^i$ by construction, hence for all $a_m^i \in A_m^i$ we have $\Pi_m^i(a_m^i, \tau_m'^{-i}) = \Pi_m^i(a_m^i, \tau_m^{-i})$. Thus by Lemma C.3 $a_m^i \in A^\dagger(i,m)$ with probability 1 under $\tau_m'^i$. We conclude that $a \in A^\dagger$ with probability 1 under $\tau'$. It follows that $\widetilde{\mathcal{H}}$ is a set of compliant paths path, which is rectangular by construction. Obviously on-path play is $\tau'$ along any compliant path.

It remains to check that $\sigma'$ is indeed an SPNE. Following a deviation by one or more firms, continuation play is an SPNE by construction. So we need only confirm that there are no profitable unilateral deviations along any compliant path. Suppose there existed $i$ and $a^i \notin \prod_{m=1}^{N+1} A^\dagger(i,m)$ such that $(1-\delta)\Pi^i(a^i, \tau'^{-i}) + \delta U^i(\underline{\sigma}(i)) > \Pi^i(\tau')$. Because $\Pi^i(a^i, \tau'^{-i}) = \Pi^i(a^i, \tau^{-i})$ by summing the market-by-market equivalences derived earlier, it must also be the case that $(1-\delta)\Pi^i(a^i, \tau^{-i}) + \delta U^i(\underline{\sigma}(i)) > \Pi^i(\tau)$. But then $\sigma$ is not an equilibrium, as

[2] If such an SPNE does not exist because the equilibrium set is not closed, the following argument goes through by choosing an SPNE yielding profits sufficiently close to the infinum.

no matter the continuation following play of $a^i$ in period 0 by firm $i$ under $\sigma$, firm $i$ has a profitable one-shot deviation to $a^i$. So no such $a^i$ exists, ruling out profitable deviations along any compliant path.

The final claim of the lemma follows simply from noticing that lifetime profits to firm $i$ under $\sigma$ and $\sigma'$ are $\Pi^i(\tau)$ and $\Pi^i(\tau')$, respectively, and recalling that $\Pi^i(\tau) = \Pi^i(\tau')$. □

**Lemma C.5.** *Fix an SPNE $\sigma$. Suppose there exists a set of compliant paths $\mathcal{H}^*$ and mixed strategies $\tau^i_m \in \Delta(A^i_m)$ such that for all $h \in \mathcal{H}^*$ and $t$, $\sigma(h^t) = \prod_{i=1}^{N+1}\prod_{m=1}^{N+1}\tau^i_m$. Then there exists a stationary equilibrium $\sigma'$ with on-path play $\prod_{i=1}^{N+1}\prod_{m=1}^{N+1}\tau^i_m$.*

*Proof.* Let $\tau \equiv \prod_{m=1}^{N+1}\tau^i_m$. We first establish the existence of constants $\overline{\Pi}^i_m$ for each $i$ and $m$ such that $\Pi^i_m(a^i_m, \tau^{-i}_m) = \overline{\Pi}^i_m$ w.p. 1 under $\tau^i_m$. Suppose by way of contradiction that for some $i$ and $m$, there exists a profit level $\Pi^*$ such that $\Pi^i_m(a^i_m, \tau^{-i}_m) \leq \Pi^*$ occurs with probability strictly between 0 and 1 under $\tau^i_m$. Then given the independence of $i$'s actions across markets, the event $E^i = \{\Pi^i(a^i, \tau^{-i}) \leq \Pi^* + \Pi^i_{-m}(\tau_{-m})\}$ must occur with probability strictly between 0 and 1 under $\tau^i$. To see this, note first that

$$\{\Pi^i_m(a^i_m, \tau^{-i}_m) \leq \Pi^* \wedge \Pi^i_{-m}(a^i_{-m}, \tau^{-i}_{-m}) \leq \Pi^i_{-m}(\tau_{-m})\} \subset E^i,$$

and by independence the probability of the event on the lhs is equal to

$$\mathbb{P}^{\tau^i_m}\{\Pi^i_m(a^i_m, \tau^{-i}_m) \leq \Pi^*\}\mathbb{P}^{\tau^i_{-m}}\{\Pi^i_{-m}(a^i_{-m}, \tau^{-i}_{-m}) \leq \Pi^i_{-m}(\tau_{-m})\},$$

with both terms strictly positive. So $\mathbb{P}^{\tau^i}(E^i) > 0$. Similarly, letting $\overline{E^i}$ be the complementary event to $E^i$, we have

$$\{\Pi^i_m(a^i_m, \tau^{-i}_m) > \Pi^* \wedge \Pi^i_{-m}(a^i_{-m}, \tau^{-i}_{-m}) \geq \Pi^i_{-m}(\tau_{-m})\} \subset \overline{E^i},$$

and again the probability of the set on the lhs is strictly positive. So $\mathbb{P}^{\tau^i}(\overline{E^i}) > 0$, or $\mathbb{P}^{\tau^i}(E^i) < 1$.

Now, note that along any compliant path $\tau$ is played in every period, thus w.p. 1 under $\sigma$ each firm $j$'s continuation payoff after period 0 must be $\Pi^j(\tau)$. In particular, firm $i$'s expected continuation payoff given $\tau^{-i}$ must be $\Pi^i(\tau)$ w.p. 1 under $\tau^i$. But then $i$'s expected lifetime payoff from playing actions in $E^i$ is strictly lower than from playing actions in $\overline{E^i}$. This is a contradiction of the optimality of $i$'s strategy in period 0. So we conclude that the desired constants $\overline{\Pi}^i_m$ exist for all $i$ and $m$.

Now define $A^\dagger(i, m) \equiv \{a^i_m \in A^i_m : \Pi^i_m(a^i_m, \tau^{-i}_m) = \overline{\Pi}^i_m\}$ for each firm $i$ and market $m$, and let $A^\dagger \equiv \prod_{i=1}^{N+1}\prod_{m=1}^{N+1} A^\dagger(i, m)$. Consider the rectangular set of complete histories

$\widetilde{\mathcal{H}} = \prod_{t=0}^{\infty} A^{\dagger}$. Construct a repeated game strategy profile $\sigma'$ as follows. For every $t$ and $h \in \widetilde{\mathcal{H}}$, set $\sigma'(h) = \tau$. Also, for each $a \in A$ such that $a^i \notin \prod_{m=1}^{N+1} A^{\dagger}(i,m)$ for at least two firms, set $\sigma'|_{(h,a)} = \sigma$. Finally, for each $a \in A$ such that $a^i \notin \prod_{m=1}^{N+1} A^{\dagger}(i,m)$ for some firm $i$ while $a^{-i} \in \prod_{j \neq i} \prod_{m=1}^{N+1} A^{\dagger}(i,m)$, set $\sigma'|_{(h,a)} = \underline{\sigma}(i)$, where $\sigma(i)$ is an SPNE yielding minimal lifetime profits for $i$ among all SPNEs.

We claim that $\sigma'$ is a stationary equilibrium with compliant path of play $\tau$. Observe that $\widetilde{\mathcal{H}}$ is a rectangular set of compliant paths for $\sigma'$, as $a \in A^{\dagger}$ w.p. 1 under $\tau$ by definition of the $\overline{\Pi}_m^i$. And $\tau$ is the path of play under $\sigma'$ by construction. It remains only to check that $\sigma'$ is an SPNE. Off-path play follows an SPNE by construction, so we need only verify that there are no profitable deviations on-path. For each $i$, all $a^i \in \prod_{m=1}^{N+1} A^{\dagger}(i,m)$ are on-path; while all $a^i$ such that $\Pi^i(a^i, \tau^{-i}) < \Pi^i(\tau)$ yield lower immediate and continuation profits than on-path play. Finally, the unprofitability of $a^i$ such that $\Pi^i(a^i, \tau^{-i}) > \Pi^i(\tau)$ follows from the fact that $\sigma$ is an equilibrium, as $\sigma'$ provides continuation payoffs no higher than $\sigma$ following such actions. Thus $\sigma'$ is indeed an SPNE. □

## C.3 The case of many competitors

Our simple duopoly model has the implication that perfect collusion is possible even for relatively low values of $\delta$ (in particular, $\delta^M < 1/2$ from Proposition 5 in the main text). If one interprets $\delta$ literally, i.e., as reflecting discounting at the market interest rate over the intervals between competitive interactions, then one would typically expect to find $\delta > \delta^M$ in practical applications, in which case the firms would achieve perfect collusion, and the structure of optimal collusive agreements for $\delta < \delta^M$ would have little bearing on actual cartel behavior.

However, one can also interpret $\delta$ more expansively (and less literally) as a reduced-form stand-in for other factors that tend to make firms focus more on present opportunities and less on future consequences. For example, in many simple models of oligopoly, the number of competitors affects the feasibility of collusion through the same channel as discounting (because adding firms increases the potential gains from current deviations and reduces the future benefits of cooperation). Firms may also effectively discount future profits to a greater extent than market interest rates would imply because of agency problems, leadership turnover, uncertainty about future market conditions, or capital market imperfections that raise internal hurdle rates.

In this section we explore the implications of multiple competitors explicitly. We show that collusion indeed becomes more difficult to sustain as the cartel size increases, and that perfect collusion is infeasible even with a moderate number of firms and discount factors close to unity. We also generalize the results of Section 5 of the main text and, for discount factors below $\delta^M$, provide a characterization of optimal collusion that is broadly similar to the two-firm case. Our results thus provide some reassurance that our central insights concerning cartels are robust with respect to the introduction of additional factors that make collusion more difficult to sustain.

### C.3.1 Setup

We extend our model to many-firm settings while retaining symmetry across firms: there are now $N+1$ firms and $N+1$ markets, where $N \geq 2$. Firm $i$'s marginal cost is $c_H$ for units sold in market $i$ (its home market), and $c_A > c_H$ for units sold elsewhere. All other features of the model are unchanged, except for the standing assumption made in Section 5.2 of the main text, which we discard. (It will turn out to be replaced by a weaker sufficiency condition which is relaxed as $N$ grows.)

### C.3.2 Analyzing the stage game

First consider a single round of the stage game played in isolation. Because payoffs are additively separable across markets, we can focus on play in a single market. Let $\{H\} \cup \mathscr{I}$ be the set of firms, where $H$ is the home firm and $\mathscr{I} = \{1, ..., N\}$ includes the away firms. The existence of a two-firm equilibrium implies that there are many Nash equilibria when $N \geq 2$. For if $H$ and any firm $i \in \mathscr{I}$ play the two-firm equilibrium, no away firm will have an incentive to enter (as it would receive strictly less than $i$'s profits, which are zero). Hence there are at least $N$ Nash equilibria involving competition among pairs of firms.

In fact, for any non-empty subset of away firms, there is a Nash equilibria in which those firms compete with the home firm. Our main result establishes a limit on the multiplicity of equilibria: once a subset of away firms is chosen, there exists a unique Nash equilibrium involving participation by those firms. The form of this equilibrium is broadly similar to the two-firm equilibrium, with certain entry by the home firm, occasional entry by the away firms, and all firms randomly choosing prices between $\underline{p}_A$ and $p^*_H$. Further, the equilibrium is symmetric in that all away firms play identical strategies. The following result summarizes these results: (The proofs for the many-firm case rely on a number of auxiliary results developed in Appendix C.4.)

**Proposition C.1.** *For every non-empty subset $\mathscr{J} \subset \mathscr{I}$ of away firms, there exists a unique Nash equilibrium of the stage game in which every firm in $\mathscr{J}$ enters with positive probability* *and no firm in $\mathscr{I} \setminus \mathscr{J}$ ever enters. In this equilibrium:*

1. *The home firm always enters and makes profits $\Pi_H = \Delta c D(\underline{p}_A)$.*

2. *Each away firm $i \in \mathscr{J}$ enters with probability strictly less than 1 and makes profits $\Pi_i = 0$.*

3. *Each entering firm's price distribution has full support on $[\underline{p}_A, p^*_H]$.*

4. *All entering away firms play the same strategy.*

*There exist no Nash equilibria in which no away firms enter with positive probability.*

*Proof.* This is a restatement of Proposition C.10. □

### C.3.3 Asymmetric collusion with many firms

When many firms compete, the set of possible collusive arrangements is much richer than with only two firms. For the latter case, we have seen that it is always optimal to allocate production so that each firm earns all of its profits in its home market. In contrast, with three or more firms, it can be worthwhile to spread each firm's profits across several markets; this reduces the profitability of undercutting in each market and thereby relaxes incentive constraints (in some instances).

In Appendix C.3.7, we describe an equilibrium which, for particular choices of $c, \Delta c$, and $\delta$, Pareto-dominates the best equilibrium in which firms earn profits only in their home markets. Table 1 displays the division of profits for the special case of three firms. The table includes a row for each firm and a column for each market; a "+" indicates positive profits while 0 indicates zero profits.

| | M1 | M2 | M3 |
|---|---|---|---|
| F1 | + | 0 | 0 |
| F2 | + | + | 0 |
| F3 | 0 | 0 | + |

Table 1: A division of equilibrium profits that is not market-symmetric

With no further restrictions on the structure of the equilibrium, it is difficult to characterize optimal collusion. Note, however, that the equilibrium depicted in Table 1 has a feature that is arguably peculiar: within one of the markets (M1), *ex ante* identical away firms (F2 and F3) do not earn the same profits. It is reasonable to assume that symmetrically situated firms are drawn to symmetric agreements because they are easier to describe, likely simpler to negotiate, and require less coordination than asymmetric ones.

We will therefore impose symmetry going forward to provide sufficient structure for a characterization of optimal collusion with more than two firms.

### C.3.4 Optimal collusive equilibria

We begin our analysis of optimal collusive equilibria by introducing some additional notation. Recall that $\Pi^M$ is the monopoly profit of a low-cost provider in a single market. Let $\widetilde{\Pi}^M$ be the profit of a high-cost provider setting the same price $p_H^*$, which satisfies $\Pi^M - \widetilde{\Pi}^M = \Delta c D(p_H^*) > 0$. (Note that $\widetilde{\Pi}^M$ is *not* the monopoly profit of the high-cost provider, as in general $p_A^* > p_H^*$.) With this notation, define

$$\delta^M(N) \equiv 1 - \frac{1}{N+1}\left(\frac{N}{N+1}\frac{\widetilde{\Pi}^M}{\Pi^M} + \frac{1}{N+1}\right)^{-1}.$$

The notation suggests that $\delta^M(N)$ is the minimal discount factor for which perfect collusion is sustainable with $N+1$ firms, a fact we establish later, under some conditions, in Proposition C.5. Note that $\Pi^M > \widetilde{\Pi}^M$ implies that $\delta^M(N) < 1 - 1/(N+1)$.

Another useful discount factor threshold is $\overline{\delta}(N) \equiv \left(1 - \frac{1}{N}\right)\left(1 + \frac{c}{\Pi^M}\right)$. This expression plays an auxiliary role in our results and we will explain it shortly; for the moment, simply note that it is greater than $1 - 1/N$ and may be either larger or smaller than $\delta^M(N)$. Finally, let $\underline{\Pi}(\delta; N)$ be the minimum SPNE-sustainable lifetime profits with discount factor $\delta$ and $N+1$ firms.

Our first proposition is the many-firm analog of Proposition 3 from the main text:

**Proposition C.2.** *Suppose $\delta < \delta^M(N)$ and $N \geq \sqrt{1 + \Pi^M/c}$. Then the optimal symmetric stationary equilibrium payoff vector $(\Pi^*, ..., \Pi^*)$ satisfies*

$$\Pi^* = (1-\delta)((N+1)\Pi^* - \Delta c N D(p^*(\Pi^*))) + \delta \underline{\Pi}(\delta; N).$$

*Further, $\Pi^* > \Pi^C$ iff $\underline{\Pi}(\delta; N) < \Pi^C$, and $\Pi^*$ is strictly increasing in $\delta$ whenever $\underline{\Pi}(\cdot; N)$ is nonincreasing in $\delta$. Finally, $\Pi^*$ is strictly decreasing in $N$ whenever $\underline{\Pi}(\delta; \cdot)$ is nonincreasing in $N$.*

*Proof.* Note that $N \geq \sqrt{1 + \Pi^M/c}$ implies $\overline{\delta}(N) > \delta^M(N)$ and thus $\delta < \overline{\delta}(N)$ by Lemma C.15. Then this result is a consequence of Propositions C.12 through C.14. Proposition

C.12 gives a necessary condition for a profit vector to be supportable as a market-symmetric stationary equilibrium (under the conditions of the proposition), in the form of a set of inequalities. Proposition C.13 shows that these inequalities form a sufficient condition for existence of an equilibrium, while Proposition C.14 characterizes the unique symmetric optimal profit vector within the set of vectors satisfying the inequalities. □

The substance of this proposition is identical to that of Proposition 3. In essence it tells us that, provided $\delta$ is not too high, optimal collusion involves allocating markets according to cost advantages. As the proposition shows, the characterization of maximum sustainable profits then depends on the most severe punishment, $\underline{\Pi}(\delta; N)$, that firms can mete out following a deviation.

In contrast to the case of a duopoly, the optimality of allocating markets according to cost advantages is not guaranteed for all $\delta < \delta^M(N)$. We therefore also require that $N \geq \sqrt{1 + \Pi^M/c}$, which rules out the possibility of achieving higher profits by allocating business only to away firms (while respecting symmetry). In Appendix C.3.8, we show by way of example that such an arrangement can yield profits exceeding the level indicated in Proposition C.2 when this bound is violated. Table 2 depicts the division of profits for this example (in which there are three firms).

| | M1 | M2 | M3 |
|---|---|---|---|
| F1 | 0 | + | + |
| F2 | + | 0 | + |
| F3 | + | + | 0 |

Table 2: A cartel with all profits awarded to away firms

The condition $N \geq \sqrt{1 + \Pi^M/c}$ is a lower bound on the number of competitors given the ratio of monopoly profits to market-specific fixed costs. Because this bound grows slowly in $\Pi^M/c$, it may be satisfied in practice. For instance, $\Pi^M/c = 3$ implies $N \geq 2$ (which is true by assumption), while $\Pi^M/c = 15$ implies $N \geq 4$. Even if fixed costs were a trivial portion of monopoly profits, say 1%, the implied bound on $N$ would be only $N \geq 10$. Consequently, imposing $N \geq \sqrt{1 + \Pi^M/c}$ (and hence $\delta^M(N) < \overline{\delta}(N)$) strikes us as reasonably innocuous.

In fact we can do better. Arrangements in which profits are allocated against cost advantage be shown to be suboptimal whenever the regularity condition $\delta < \overline{\delta}(N)$ holds. (The condition $N \geq \sqrt{1 + \Pi^M/c}$ is merely a sufficient condition for regularity to hold.) And the irregular case $\overline{\delta}(N) \leq \delta < \delta^M(N)$, when the discount factor might be low enough to require partially collusive arrangements but high enough to violate regularity is demonstrably unimportant. In particular:

**Proposition C.3.** $[\overline{\delta}(N), \delta^M(N)] \subset (1-1/N, 1-1/(N+1))$. *Therefore if* $\delta \in [\overline{\delta}(N), \delta^M(N))$, *then* $\delta < \overline{\delta}(N+1)$ *and* $\delta > \delta^M(N-1)$.

*Proof.* The set inclusion follows from the fact that $\delta^M(N) < 1-1/(N+1)$ while $\overline{\delta}(N) > 1-1/N$, inequalities which are obvious by inspection of the relevant definitions. The remaining inequalities are immediate corollaries of the fact that the collection of intervals $(1-1/N, 1-1/(N+1))$ are pairwise disjoint. □

This result tells us that $\delta \in [\overline{\delta}(N), \delta^M(N))$ is a knife-edge case: add one more firm and we will have $\delta < \overline{\delta}(N+1)$, which means we can focus on cartel structures that allocate markets according to cost. Subtract one firm, and the resulting cartel can sustain perfect collusion. The size of the problematic interval $[\overline{\delta}(N), \delta^M(N))$ is also at most $1/(N(N+1))$, and so collapses rapidly with $N$. We therefore consider the possibility of alternative collusive structures (ones that allocate profits to away firms) a minor issue that we can safely ignore.

The next result generalizes Proposition 4 of the main text:

**Proposition C.4.** *Whenever*

$$\delta \geq \underline{\delta}(N) \equiv \left(1 - \frac{1}{N+1}\right)\left(1 + \frac{N}{N+1}\frac{\Pi^M - \widetilde{\Pi}^M}{c}\right)^{-1},$$

*there exists an SPNE supporting lifetime profits of 0 for each firm, so that* $\underline{\Pi}(\delta; N) = 0$.

*Proof.* This is a restatement of Proposition C.15. □

The structure of the punishment equilibrium resembles the one used for duopolies, but it has an asymmetric element: the punishment for firm $i$ consists of a price war between $i$ and another firm, say $i+1$, in their respective markets. All other firms stay out of those markets and play the stage-game Nash equilibrium in the remaining markets. Firms revert to cooperation after one round of a successful price war. Note that $\underline{\delta}(N)$ is increasing in $N$, but is strictly bounded away from 1 given $\Pi^M > \widetilde{\Pi}^M$. Thus even with a large number of somewhat impatient competitors, minmax punishments are feasible.

Next we generalize Proposition 5 from the main text, and fully characterize optimal collusive payoffs for a range of discount factors below $\delta^M(N)$ under mildly restrictive conditions:

**Proposition C.5.** *Suppose* $\Pi^M > \Delta c D(p_H^*) + \frac{c}{N}$ *and* $N \geq \sqrt{1 + \Pi^M/c}$. *Then* $\underline{\delta}(N) < \delta^M(N)$, *and for all* $\delta \in [\underline{\delta}(N), \delta^M(N)]$ *the optimal symmetric stationary equilibrium profit* *vector* $(\Pi^*, ..., \Pi^*)$ *satisfies*

$$\Pi^* = (1-\delta)((N+1)\Pi^* - \Delta c N D(p^*(\Pi^*))).$$

*Further,* $\Pi^*$ *is continuous, strictly greater than* $\Pi^C$*, and strictly increasing in* $\delta$*. Finally,* $\delta^M(N)$ *is the minimal discount factor at which perfect collusion is sustainable.*

*Proof.* This result follows from Propositions C.2 and C.4, once we have established $\underline{\delta}(N) < \delta^M(N)$. Write $\delta^M(N)$ as

$$\delta^M(N) = \frac{1}{1 + \frac{1}{N}\Pi^M/\widetilde{\Pi}^M}$$

and $\underline{\delta}(N)$ as

$$\underline{\delta}(N) = \frac{N}{N+1}\frac{1}{1 + \frac{N}{N+1}\frac{\Pi^M - \widetilde{\Pi}^M}{c}}.$$

Then re-arrangement of the inequality yields

$$1 + \frac{N}{N+1}\frac{\Pi^M - \widetilde{\Pi}^M}{c} > \frac{N}{N+1}\left(1 + \frac{1}{N}\frac{\Pi^M}{\widetilde{\Pi}^M}\right).$$

Multiplying through by $N+1$ and cancelling terms leaves

$$1 + N\frac{\Pi^M - \widetilde{\Pi}^M}{c} > \frac{\Pi^M}{\widetilde{\Pi}^M}.$$

Subtracting both sides by 1 and combining terms on the rhs allows us to cancel a common factor of $\Pi^M - \widetilde{\Pi}^M$. Finally, we are left with $\widetilde{\Pi}^M > c/N$, which is equivalent to the condition $\Pi^M > \Delta c D(p_H^*) + c/N$ in the proposition statement. □

As in the duopoly case, we impose a mild sufficiency condition on $\Pi^M$ to ensure $\underline{\delta}(N) < \delta^M(N)$. This condition is weaker than the one imposed for duopoly, grows weaker as $N$ increases, and is always satisfied for sufficiently large $N$. Note that Proposition C.5 does not directly speak to the form of $\Pi^*$ for any discount factor when $\overline{\delta}(N) < \delta^M(N)$. To address this deficiency, in Proposition C.16 we derive a very mild lower bound on $N$ that ensures $\underline{\delta}(N) < \overline{\delta}(N)$, in which case Proposition C.5 continues to characterize optimal profits for discount factors in the range $[\underline{\delta}(N), \overline{\delta}(N))$.

Finally, we characterize an equilibrium supporting profits $\Pi^*$ for each firm. As in the case of a duopoly, this construction holds regardless of the value of $\underline{\Pi}(\delta; N)$.

**Proposition C.6.** *Suppose $\delta < \delta^M(N)$. Then lifetime profits $(\Pi^*, ..., \Pi^*)$ are supported by a symmetric stationary equilibrium with the following properties:*

1. *The home firm's strategy is the same in all markets, and all away firms play the same strategy in all markets.*

2. *The home firm enters with probability 1, while all away firms enter with a probability that is strictly between zero and 1 and decreasing in $\Pi^*$.*

3. *The home firm earns profits $\Pi^*$, while all away firms make zero profits.*

4. *Each firm posts prices only in $[p^*(\Pi^*), p^*_H]$, and firms' price distributions have full support on $(p^*(\Pi^*), p^*_H)$.*

5. *If $\Pi^* > \Delta c D(\underline{p}_A)$, the home firm plays $p^*(\Pi^*)$ with some strictly positive probability, which is increasing in $\Pi^*$.*

6. *Each market is captured by an away firm with some strictly positive probability, which is strictly decreasing in $\Pi^*$ when $\Pi^* \geq \frac{1}{2}\Pi^M$.*

7. *Any unilateral deviation by an away firm to a price at or below $p^*(\Pi^*)$ is punished by a continuation payoff of $\underline{\Pi}(\delta; N)$ to that firm.*

*Proof.* This is a special case of Proposition C.13, with the inequality of property 6 weakened to provide a simpler expression. □

This result mirrors our conclusions concerning the optimal collusive structure for a duopoly, and features business-stealing for essentially the same reason.

### C.3.5 Imperfect collusion in large cartels

The following result explores how the range of discount factors for which we have characterized optimal collusion changes with cartel size.

**Proposition C.7.** *$\underline{\delta}(N)$ and $\delta^M(N)$ are strictly increasing in $N$, and $\lim_{N\to\infty} \underline{\delta}(N) < 1$ while $\lim_{N\to\infty} \delta^M(N) = 1$. Further, $\delta^M(N) - \underline{\delta}(N)$ is strictly increasing in $N$ whenever $\delta^M(N) \geq \underline{\delta}(N)$.*

*Proof.* Writing $\delta^M(N)$ as

$$\delta^M(N) = 1 - \frac{1}{1 + N\widetilde{\Pi}^M/\Pi^M}$$

proves that it is strictly increasing in $N$ and approaches 1 as $N \to \infty$. Similarly, writing $\underline{\delta}(N)$ as

$$\underline{\delta}(N) = \left(1 + \frac{1}{N} + \frac{\Pi^M - \widetilde{\Pi}^M}{c}\right)^{-1}$$

shows that $\underline{\delta}(N)$ is strictly increasing but bounded below 1.

To finish the proof, we must show that $\Delta(N) \equiv \delta^M(N) - \underline{\delta}(N)$ is increasing whenever $\Delta(N) \geq 0$. We showed in the proof of Proposition C.5 that the latter inequality holds iff $\widetilde{\Pi}^M \geq c/N$. It is then sufficient to verify that $\Delta'(N) > 0$ whenever $N \geq c/\widetilde{\Pi}^M$.

Computing the derivative of $\Delta(N)$ yields

$$\Delta'(N) = \frac{\widetilde{\Pi}^M/\Pi^M}{\left(1 + N\frac{\widetilde{\Pi}^M}{\Pi^M}\right)^2} - \frac{1/N^2}{\left(1 + \frac{1}{N} + \frac{\Pi^M - \widetilde{\Pi}^M}{c}\right)^2}.$$

Some re-arrangement shows that $\Delta'(N) > 0$ iff

$$1 + \frac{\Pi^M - \widetilde{\Pi}^M}{c} > \sqrt{\frac{\widetilde{\Pi}^M}{\Pi^M}} + \frac{1}{N}\left(\sqrt{\frac{\Pi^M}{\widetilde{\Pi}^M}} - 1\right).$$

Because $\Pi^M > \widetilde{\Pi}^M$, the rhs is largest when $N$ is smallest, i.e. at $N = c/\widetilde{\Pi}^M$. It is therefore sufficient to show that

$$1 + \frac{\Pi^M}{c} > \sqrt{\frac{\widetilde{\Pi}^M}{\Pi^M}} + \frac{\sqrt{\Pi^M \widetilde{\Pi}^M}}{c}.$$

But the first term on the lhs is strictly greater than the first term on the rhs, with a similar comparison holding for the second terms. So indeed $\Delta'(N) > 0$ whenever $N \geq c/\widetilde{\Pi}^M$, completing the proof. □

Because $\delta^M(N)$ goes to 1 as $N$ grows large, large cartels can aspire only to imperfect collusion even when their members are extremely patient. The minimal discount factor required to sustain a price war yielding zero profits also grows with $N$, but more slowly. Thus the range of discount factors for which we completely characterize optimal collusion expands with $N$. Accordingly, this proposition establishes the robustness of our results with respect to cartel size. It also illustrates the point that our analysis of imperfect collusion applies in settings where firms' discount rates are in line with market interest rates.

### C.3.6 Comparative statics

The results of this section establish straightforward generalizations of the comparative statics results of Section 5.4 in the main text for the many-firm case. $p^{**}$ is defined analogously to the two-firm case.

**Proposition C.8.** *Fix $\delta \in (0,1)$. If $\delta < N/(N+1)$, then $\Pi^* \to \Pi^C$ as $c_A \downarrow c_H$. If $\delta \geq N/(N+1)$ then $\Pi^* = \Pi^M$ for all $c_A > c_H$.*

*Proof.* The $N = 1$ case is Proposition 9 in the main text. The $N \geq 2$ case is a direct consequence of Proposition C.17 when combined with Proposition C.14, which implies that $\Pi^\dagger = \Pi^*$. □

**Proposition C.9.** *Fix $\delta \in (0,1)$. Let $(F_H(\cdot), F_A(\cdot), \pi_A)$ be the home and away firms' price distributions and the away firm's entry probability, respectively, for the equilibrium characterized in Proposition C.6. As $c \to 0$, $F_H(\cdot)$ converges uniformly to $\mathbf{1}\{p \leq p^{**}\}$ while $\pi_A F_A(p^{**}) \to 0$. The probability of business stealing therefore falls to zero as c vanishes, and in the limit the home firm wins the market at price $p^{**}$ with probability 1.*

*Proof.* The $N = 1$ case is Proposition 10 in the main text. The $N \geq 2$ case is a direct consequence of Proposition C.18 when combined with Proposition C.14, which implies that $\Pi^\dagger = \Pi^*$. □

### C.3.7 An equilibrium in asymmetric strategies

In this subsection we demonstrate parameters under which collusion in symmetric strategies is Pareto-dominated by collusion in more general strategies. Fix $D(p) = \mathbf{1}\{p \leq v\}$, $N = 2$, $\delta = 0.6$, $\Delta c = 0.2$, and $c = 0.1$. $v$ will be assumed to be sufficiently large. The largest symmetric profits which can be supported in this environment by a stationary equilibrium satisfy

$$\Pi^* = (1-\delta)(N\Pi^* - (N-1)\Delta c),$$

yielding $\Pi^* = 0.8$.

Now, consider a stationary equilibrium with profits taking the signs indicated in Table 1. Markets 2 and 3 take the standard structure of Proposition C.13.

Fix $\Pi_1^1$, and in market 1, let $p^L \equiv \Pi_1^1 + c + c_H$, $p_1^U \equiv \frac{1}{2}(v + c_H)$, and $p_2^U \equiv \frac{1}{2}(v + c_A)$. Firm

3 does not enter. Firms 1 and 2 always enter and play

$$F_1^1(p) = \begin{cases} 0, & p < p^L, \\ 1 - \frac{p^L - c_A}{p - c_A}, & p \in [p^L, p_1^U), \\ 1 - \frac{p^L - c_A}{p_2^U - c_A}, & p \in [p_1^U, v), \\ 1, & p \geq v \end{cases}$$

and

$$F_1^2(p) = \begin{cases} 0, & p < p^L, \\ 1 - \frac{p^L - c_H}{p - c_H}, & p \in [p^L, p_1^U), \\ 1 - \frac{p^L - c_H}{p_1^U - c_H}, & p \in [p_1^U, v), \\ 1, & p \geq v. \end{cases}$$

These two distributions are continuous with full support on $[p^L, p_1^U)$ and then have a gap on $(p_1^U, v)$. Firm 1 also places an atom at $p_1^U$. Finally, both firms place an atom at $v$, of sizes

$$\Delta F_1^1 = 2\left(\frac{\Pi_1^2 + c}{\Pi^M + c - \Delta c}\right), \quad \Delta F_1^2 = 2\left(\frac{\Pi_1^1 + c}{\Pi^M + c}\right),$$

where $\Pi_1^2 = \Pi_1^1 - \Delta c$. In order for this construction to be well-defined, we need $p^L < p_1^U$, or equivalently $\Pi_1^1 < \frac{1}{2}(\Pi^M - c)$, which is satisfied for $v$ sufficiently large.

The best deviation by each of firms 1 and 2 in market 1 is to undercut the atom at $p = v$, which yields them profits $2\Pi_1^i + c$. Meanwhile firm 3 has two possible maximally profitable deviations, one at $p = p^L$ and another by undercutting $p = v$. (It can't be more profitable to price in the firms' price support, as this will make firm 3 strictly less than firm 2 would by playing there, and thus strictly less than he would make by playing $p = p^L$.) His profits at $p^L$ are $\Pi_1^2$, while his profits undercutting $v$ are

$$(\Pi^M + c - \Delta c)\Delta F_1^1 \Delta F_1^2 - c = 4(\Pi_1^2 + c)\frac{\Pi_1^1 + c}{\Pi^M + c} - c.$$

For sufficiently large $v$, these are lower than his profits at $p^L$.

Thus, for sufficiently large $v$ the incentive constraints which need to be satisfied are

$$\Pi_1^1 \geq (1 - \delta)(2\Pi_1^1 + c + \Pi_2^2 + \Pi_3^3 - 2\Delta c)$$

for firm 1,

$$\Pi_1^2 + \Pi_2^2 \geq (1-\delta)(2\Pi_1^2 + c + \Pi_2^2 + \Pi_3^3 - \Delta c)$$

for firm 2, and

$$\Pi_3^3 \geq (1-\delta)(\Pi_1^2 + \Pi_2^2 + \Pi_3^3 - \Delta c)$$

for firm 3. It is easily checked that all firms' profits are simultaneously maximized subject to the IC constraints when $\Pi_1^1 = 1.4$, $\Pi_2^2 = 0.2$, and $\Pi_3^3 = 0.8$. This equilibrium is actually a Pareto-improvement on the best partially collusive one!

### C.3.8 An equilibrium with no home market profits

In this subsection we demonstrate parameters under which (symmetric) collusion in which profits are won in each firm's home market is Pareto-dominated by collusion in which profits are won only in firms' away markets. Fix $D(p) = \mathbf{1}\{p \leq v\}$, $N = 2$, $\delta = 0.62$, $\Delta c = 0.2$, and $c = 0.1$. $v$ will be assumed to be sufficiently large. The maximal profits supportable by an equilibrium of the type characterized in Proposition C.13 (which is the best that can be done by a symmetric stationary equilibrium when firms earn profits only in their home market) satisfy

$$\Pi^* = (1-\delta)(N\Pi^* - (N-1)\Delta c),$$

or $\Pi^* \simeq 1.09$.

Now consider a symmetric equilibrium in which all away firms make positive profits $\Pi/N$ in each market, while the home firm makes no profits. Thus each firm makes total equilibrium profits $\Pi$.

The home firm refrains from entering, while the away firms always enter. Let $p^L \equiv \Pi/N + c + c_A$ and $p^U \equiv c_A + \frac{1}{N}(v - c_A)$. Each away firm plays

$$F^A(p) = \begin{cases} 0, & p < \Pi + c + c_A, \\ 1 - \left(\frac{p^L - c_A}{p - c_A}\right)^{1/(N-1)}, & p \in [p^L, p^U), \\ 1 - \left(\frac{p^L - c_A}{p^U - c_A}\right)^{1/(N-1)}, & p \in [p^U, v), \\ 1, & p \geq v. \end{cases}$$

Each firm's price distribution is continuous with full support on $[p^L, p^U]$, has a gap on $[p^U, v)$, and places an atom at $p^U$ of strength

$$\Delta F^A = \left[ N \left( \frac{p^L - c_A}{v - c_A} \right) \right]^{1/(N-1)}.$$

For the construction to be well-defined, we need $p^U > p^L$, i.e. $\Pi < v - c_A - Nc$, which is possible for $v$ sufficiently large.

Now, each away firm has a deviation to undercutting $p = v$, yielding profits $\Pi + (N - 1)c$. Meanwhile the home firm has two candidate deviations. Setting $p = p^L$ yields profits $\Pi/N + \Delta c$, while undercutting $p = v$ yields profits

$$(v - c_H)(\Delta F^A)^N - c = N^{N/(N-1)} \frac{v - c_H}{(v - c_A)^{N/(N-1)}} - c.$$

For $v$ sufficiently large the home firm's most profitable deviation is to $p^L$.

The IC constraint required to support this equilibrium is then

$$\Pi \geq (1 - \delta) \left[ \left( N + \frac{1}{N} \right) \Pi + N(N-1)c + \Delta c \right],$$

with the additional constraint that $\Pi < v - c_A - Nc$. Given that $(1-\delta)(N+1/N) = 0.95 < 1$, any $\Pi \geq 3.04$ will satisfy the IC constraint. So for $v$ sufficiently high, there exist $\Pi > \Pi^*$ supportable in equilibrium.

## C.4 Auxiliary results for the many-firm case

### C.4.1 The stage game

The following result characterizes the set of Nash equilibria of the stage game.

**Proposition C.10.** *For every non-empty subset $\mathscr{J} \subset \mathscr{I}$ of away firms, there exists a unique Nash equilibrium of the stage game in which every firm in $\mathscr{J}$ enters with positive probability and no firm in $\mathscr{I} \setminus \mathscr{J}$ ever enters. In this equilibrium:*

1. *The home firm always enters and makes profits $\Pi_H = \Delta c D(\underline{p}_A)$.*

2. *Each away firm $i \in \mathscr{J}$ enters with probability strictly less than 1 and makes profits $\Pi_i = 0$.*

3. *Each entering firm's price distribution has full support on $[\underline{p}_A, p_H^*]$.*

*4. All entering away firms play the same strategy.*

*There exist no Nash equilibria in which no away firms enter with positive probability.*

*Proof.* Let $(\pi_H, F_H, \{\pi_i, F_i\}_{i \in \mathscr{I}})$ be a Nash equilibrium of the stage game. Define $\mathscr{J} \equiv \{i \in \mathscr{I} : \pi_i > 0\}$. We first establish that at least one away firm must occasionally enter in equilibrium

**Lemma C.6.** *$\mathscr{J}$ is non-empty.*

*Proof.* If no away firm entered, then each away firm makes zero profits in equilibrium. Meanwhile, the unique profit-maximizing strategy of the home firm is to post price $p_H^*$. But then each away firm can make strictly positive profits by pricing just under $p_H^*$, a contradiction of equilibrium. □

Define $\Pi_i(p)$ to be the expected profits of firm $i \in \{H\} \cup \mathscr{J}$ upon entering and setting price $p$ given the equilibrium strategies of all other firms. We will often overload notation by letting $\Pi_i$ (with no argument) represent the equilibrium profits of firm $i$.

The next lemma establishes that $\Pi_i(p)$ is continuous at $p$ iff no other firm places an atom at $p$, and that when an atom exists the profit function is discontinuous from both directions.

**Lemma C.7.** $\Pi_i(p-) \geq \Pi_i(p) \geq \Pi_i(p+)$ *for all* $i \in \{H\} \cup \mathscr{J}$ *and* $p \in [\underline{p}_A, p_H^*]$, *with equality for given firm* $i$ *iff no other firm places an atom at* $p$.

*Proof.* Obvious. □

The next lemma establishes that firms set prices only in the interval $[\underline{p}_A, p_H^*]$, that the home firm always enters the market, and that the away firm occasionally enteres the market.

**Lemma C.8.** $F_H([\underline{p}_A, p_H^*]) = F_i([\underline{p}_A, p_H^*]) = 1$ *for all* $i \in \mathscr{J}$ *and* $\pi_H = 1$.

*Proof.* Each $i \in \mathscr{J}$ receives strictly negative profits below $\underline{p}_A$ no matter the other firms' strategies. So $F_i(\underline{p}_A-) = 0$ in equilibrium. Then the home firm is never profit-maximizing below $\underline{p}_A$ given that his profits are non-positive below $\underline{p}_H$, zero at $\underline{p}_H < \underline{p}_A$, and strictly increasing on $[\underline{p}_H, \underline{p}_A]$. Hence $F_H(\underline{p}_A-) = 0$ as well. Additionally, the home firm achieves strictly positive profits by setting a price just below $\underline{p}_A$, so his equilibrium profits must be strictly positive and therefore $\pi_H = 1$.

At the other end of the price support, the home firm always makes strictly lower profits setting a price above $p_H^*$ than by pricing at $p_H^*$, no matter the away firms' strategies. Then $F_H(p_H^*) = 1$. This result, combined with the fact that the home firm always enters the market, means that any away firm pricing above $p_H^*$ will make no sale and achieve negative profits. This is less profitable than not entering the market, so $F_i(p_H^*) = 1$ for each $i \in \mathscr{J}$ in equilibrium. $\square$

We next establish a "no overlapping atoms" result:

**Lemma C.9.** *For each $p \in [\underline{p}_A, p_H^*]$, there exists at most one firm in $\{H\} \cup \mathscr{J}$ whose price distribution is not continuous at $p$.*

*Proof.* Suppose some firm $i \in \{H\} \cup \mathscr{J}$ places an atom at $p \in (\underline{p}_A, p_H^*]$. Then $i$ must be profit-maximizing at $p$. If some other firm also placed an atom at $p$, then the limit of $i$'s profits for prices just below $p$ would be strictly higher than his profits at $p$, by the previous lemma. This contradicts the optimality of $p$ for $i$, so no other firm can have an atom at $p$.

Finally, consider placement of an atom at $\underline{p}_A$ by two firms. At least one of these firms must be an away firm; but as the away firm loses the market with positive probability at that price, he makes strictly negative profits given the definition of $\underline{p}_A$. This means $\underline{p}_A$ cannot be optimal for that firm, ruling out the placement of an atom there. Reaching a contradiction, we conclude that at most one firm can place an atom at $\underline{p}_A$. $\square$

Let $p_i^U \equiv \sup\{p \ : \ F_i(p) < 1\}$ be the supremum of firm $i$'s price support for $i \in \{H\} \cup \mathscr{J}$, and similarly let $p_i^L \equiv \inf\{p \ : \ F_i(p) > 0\}$ be the infimum.

**Lemma C.10.** *$p_i^U = p_H^*$ for all $i \in \{H\} \cup \mathscr{J}$.*

*Proof.* We first show that $p_i^U = p_j^U$ for all $i, j \in \{H\} \cup \mathscr{J}$. Suppose $p_i^U > \max_{j \neq i} p_j^U \equiv p_{-i}^U$ for some $i \in \{H\} \cup \mathscr{J}$. As $i$ makes non-negative equilibrium profits, he must win with strictly positive probability when playing any price (strictly) above $p_{-i}^U$. Thus his profits are strictly increasing on $(p_{-i}^U, p_H^*]$, meaning $i$ places an atom at $p_H^*$ and does not set prices in $(p_{-i}^U, p_H^*)$.

Now, if no atom existed at $p_{-i}^U$, then some firm $j$ whose support supremum lies at $p_{-i}^U$ has continuous profits there. Hence $p_{-i}^U$ must be profit-maximizing for $j$. In particular, as $j$ makes non-negative profits, he wins with positive probability by pricing at $p_{-i}^U$, so also by setting prices in $[p_{-i}^U, p_H^*)$. But then $j$'s profits are also strictly increasing on $[p_{-i}^U, p_H^*)$, a contradiction.

Then it must be that some firm, say $j$ again, places an atom at $p_{-i}^U$. But by the overlapping atoms result no other firm can place an atom there. Then $j$'s profits are strictly increasing on $[p_{-i}^U, p_H^*)$, contradicting the optimality of $p_{-i}^U$ for $j$ implied by his placement of an atom there. We conclude that every firm's price ceiling is the same, say $p^U$.

Suppose $p^U < p_H^*$. If no firm places an atom at $p^U$, then each firm's profits are continuous at $p^U$ and hence this price is profit-maximizing for all firms. For equilibrium profits to be non-negative, each firm must win the market with positive probability at $p^U$, meaning profits are strictly increasing on $[p^U, p_H^*]$, a contradiction of optimality. So some firm $i$ must place an atom at $p^U$, which is then profit-maximizing for $i$. Since there can be no overlapping atoms, $i$'s profits are continuous at $p^U$, meaning they are strictly increasing on $[p^U, p_H^*]$, another contradiction. Hence $p^U = p_H^*$. □

**Lemma C.11.** *$p_i^L = \underline{p}_A$ and $\Pi_i = D(\underline{p}_A)(\underline{p}_A - c_i) - c$ for all $i \in \{H\} \cup \mathscr{J}$.*

*Proof.* Suppose $p_i^L < \min_{j \neq i} p_j^L \equiv p_{-i}^L$ for some $i$. Then $i$ wins w.p. 1 on $[p_i^L, p_{-i}^L]$, meaning his profits are strictly increasing on this interval. This contradicts the optimality of prices strictly less than $p_{-i}^L$. Hence $p_i^L = p^L$ for some $p^L \in [\underline{p}_A, p_H^*]$ and all $i \in \{H\} \cup \mathscr{J}$.

Suppose some firm placed an atom at $p^L$. Then every other firm's profits at $p^L$ are strictly higher than at prices just above $p^L$, meaning no other firm's price distribution assigns positive measure to $(p^L, p^L + \varepsilon)$ for $\varepsilon > 0$ sufficiently small. But given the definition of $p^L$, this means every other firm must place an atom at $p^L$, contradicting the overlapping atom result.

So there exist no atoms at $p^L$, meaning by continuity of the profit function there that each firm's profits are maximized at $p^L$. Hence $\Pi_i = D(p^L)(p^L - c_i) - c$ for all $i$. Suppose $p^L > \underline{p}_A$. Then $\Pi_i > 0$ for all $i$, meaning $\pi_i = 1$ for each $i$. Now consider the possible existence of an atom at $p_H^*$. In light of the previous lemma, there must be profit-maximizing prices for each firm arbitrarily close to $p_H^*$. But then at most one firm can place an atom there, say firm $i$. In this case given sure entry by all other participating firms below this price, $i$ makes $-c < 0$ at $p_H^*$, a contradiction of $\Pi_i > 0$. So no firm places an atom at $p_H^*$. But then each firm's profits are continuous at $p_H^*$, meaning that each firm's profits are maximized there. But their profits are again $-c < 0$ there, another contradiction. So we must have $p^L = \underline{p}_A$. □

**Lemma C.12.** *There can exist at most one atom in equilibrium, by the home firm at $p_H^*$.*

*Proof.* Suppose some firm $i$ places an atom at $p \in [\underline{p}_A, p_H^*)$. Then there exists an $\varepsilon > 0$ such that each firm $j \neq i$ places no support on $[p, p+\varepsilon)$. Now, we know that $i$ has profit-maximizing prices arbitrarily close to $p_H^*$ given that this is the supremum of his price support. Then $i$ wins with positive probability arbitrarily close to $p_H^*$, meaning he must win with (constant) positive probability on $[p, p+\varepsilon)$ (assuming $p + \varepsilon < p_H^*$, which we can assume wlog by taking $\varepsilon$ small). But then $i$'s profits are strictly increasing on $[p, p+\varepsilon)$. This contradicts the fact that $p$ is profit-maximizing for $i$ implied by his placement of an atom there. So no such atom exists.

The existence of at most a single atom at $p_H^*$ follows from the overlapping atoms result. If this atom were placed by an away firm, the fact that $\pi_H = 1$ implies that the away firm never wins the market at $p_H^*$ and thus that his profits are $-c < 0$ there. This contradicts the optimality of $p_H^*$ implied by placement of an atom there. Hence only the home firm can place an atom at $p_H^*$. □

**Lemma C.13.** *$F_H$ has full support on $[\underline{p}_A, p_H^*]$.*

*Proof.* Suppose not. Then there exists a non-degenerate open interval $S \subset (\underline{p}_A, p_H^*)$ assigned zero measure under $F_H$. Let $\widehat{F} \equiv F_H(p)$ for any $p \in S$. Because the infimum and supremum of the support of $F_H$ are $\underline{p}_A$ and $p_H^*$, we must have $\widehat{F} \in (0,1)$. Expand $S$ so that $S = (p_L, p_H)$, where $p_L \equiv \inf\{p \ : \ F_H(p) = \widehat{F}\}$ and $p_H \equiv \sup\{p \ : \ F_H(p) = \widehat{F}\}$. Because $\widehat{F} \in (0,1)$, each of $p_L$ and $p_H$ is finite and lies in $[\underline{p}_A, p_H^*]$. Also, by assumption $p_L < p_H$.

Because no other firm can place an atom at $p_L$ or $p_H$, the home firm's profits are continuous at these prices, and therefore both prices are profit-maximizing for him given $\widehat{F} \in (0,1)$. Then no price in $S$ can provide higher profits than at one of the endpoints. This implies the inequalities

$$D(p)(p - c_H) \prod_{j \in \mathscr{J}} (1 - \pi_j F_j(p)) \leq D(p_H)(p_H - c_H) \prod_{j \in \mathscr{J}} (1 - \pi_j F_j(p_H)) \ \forall p \in S,$$

with the lhs and rhs being $H$'s profits at $p$ and $p_H$ respectively. (Recall that no away firm places an atom in $S$.) Now multiply each side by $\frac{p-c_A}{p-c_H} \frac{1-\pi_H F_H(p)}{1-\pi_i F_i(p)}$ for $i \in \mathscr{J}$. Then we obtain

$$\begin{aligned}
& D(p)(p - c_A) \prod_{j \neq i} (1 - \pi_j F_j(p)) \\
\leq\ & D(p_H)(p_H - c_H) \frac{p - c_A}{p - c_H} \frac{1 - \pi_H F_H(p)}{1 - \pi_i F_i(p)} \prod_{j \in \mathscr{J}} (1 - \pi_j F_j(p_H)) \\
<\ & D(p_H)(p_H - c_H) \frac{p_H - c_A}{p_H - c_H} \frac{1 - \pi_H \widehat{F}}{1 - \pi_i F_i(p_H)} \prod_{j \in \mathscr{J}} (1 - \pi_j F_j(p_H)) \\
=\ & D(p_H)(p_H - c_A)(1 - \pi_H \widehat{F}) \prod_{j \in \mathscr{J} \setminus \{i\}} (1 - \pi_j F_j(p_H)).
\end{aligned}$$

The second inequality follows from the fact that $(p - c_A)/(p - c_H)$ is strictly increasing in $p$, $1 - \pi_H F_H(p)$ is constant, and $1 - \pi_i F_i(p)$ is (weakly) decreasing in $p$ on $S$. We arrive at the inequalities

$$D(p)(p - c_A) \prod_{j \neq i} (1 - \pi_j F_j(p)) < D(p_H)(p_H - c_A)(1 - \pi_H \widehat{F}) \prod_{j \in \mathscr{J} \setminus \{i\}} (1 - \pi_j F_j(p_H)) \ \forall p \in S.$$

The lhs are $i$'s profits for $p \in S$ (recall that no firm places on atom in $S$). Meanwhile the rhs are $i$'s profits in the limit for prices just below $p_H$. (No away firm places an atom at $p_H$.) Thus no price in $S$ can be profit-maximizing for $i$, as there is always some price very close to $p_H$ which will do better. So $i$'s price distribution assigns zero measure to $S$ as well.

This reasoning holds for all $i$, so we conclude that no firm plays in $S$. As no firm places an atom at $p_L$, the home firm's profits are then strictly increasing on $[p_L, p_H)$, contradicting the optimality of $p_L$. Hence no such interval $S$ can exist. □

**Lemma C.14.** *$F_i$ has full support on $[\underline{p}_A, p_H^*]$ for each $i \in \mathscr{J}$.*

*Proof.* Suppose not. Then for some $i \in \mathscr{J}$ we can construct an $S = (p_L, p_H)$ as in the previous lemma. We know that the home firm's price distribution has full support in this interval; if no other away firm played in $S$, then the home firm's profits would be strictly increasing in $S$, a contradiction. We will show, however, that no other away firm will play in $S$, proving the result. If $|\mathscr{J}| = 1$, then the result is trivial, so assume $|\mathscr{J}| \geq 2$.

It can't be the case that the home firm places an atom at $p_H$, for then neither this price nor any price just above it would be profit-maximizing for $i$. (Recall $\widehat{F} < 1$, so $i$ must have profit-maximizing profit-maximizing prices arbitrarily close to $p_H$ from above.) Because no other firm places an atom at $p_H$, firm $i$'s profits are continuous there and hence $p_H$ is profit-maximizing for $i$. Then

$$D(p)(p - c_A)\prod_{j \neq i}(1 - \pi_j F_j(p)) \leq D(p_H)(p_H - c_A)\prod_{j \neq i}(1 - \pi_j F_j(p_H)) \;\forall p \in S.$$

Choose $k \in \mathscr{J} \setminus \{i\}$. Multiplying both sides by $\frac{1-\pi_i F_i(p)}{1-\pi_k F_k(p)}$ yields

$$\begin{aligned}
&D(p)(p - c_A)\prod_{j \neq k}(1 - \pi_j F_j(p)) \\
&\leq D(p_H)(p_H - c_A)\frac{1 - \pi_i F_i(p)}{1 - \pi_k F_k(p)}\prod_{j \neq i}(1 - \pi_j F_j(p_H)) \\
&\leq D(p_H)(p_H - c_A)\frac{1 - \pi_i F_i(p_H)}{1 - \pi_k F_k(p_H)}\prod_{j \neq i}(1 - \pi_j F_j(p_H)) \\
&= D(p_H)(p_H - c_A)\prod_{j \neq k}(1 - \pi_j F_j(p_H)).
\end{aligned}$$

The second inequality is strict whenever $F_k(p_H) > F_k(p)$, in which case the inequality implies $k$'s profits at $p_H$ are strictly higher than at $p$. (Recall no firm places an atom at $p_H$.)

Suppose by way of contradiction that $F_k$ assigns positive measure to some subset of $S$. Then given continuity of $F_k$ there exists a $p \in S$ such that $p$ is profit-maximizing for $k$ and $F_k(p_H) > F_k(p)$. But the latter inequality implies that $k$'s profits at $p$ are strictly lower than at $p_H$, a contradiction of profit-maximization. So $k$ assigns zero measure to $S$. This yields the desired contradiction. □

It is an immediate consequence of the previous lemma that each firm's profits are equal to $\Pi_i = D(\underline{p}_A)(\underline{p}_A - c_i) - c$ for all $p \in [\underline{p}_A, p_H^*)$. For the lack of atoms implies continuity of profits at all such $p$, and the full support result implies a sequence of profit-maximizing prices converging to $p$.

We now construct the unique equilibrium for a given (arbitrary) non-empty subset $\mathscr{J} \subset \mathscr{I}$ of entering away firms. In light of continuity of each $F_i$ below $p_H^*$, the profit-maximization condition on $[\underline{p}_A, p_h^*)$ is

$$D(\underline{p}_A)(\underline{p}_A - c_i) - c = D(p)(p - c_i) \prod_{j \neq i} (1 - \pi_j F_j(p)) - c$$

for all $i \in \{H\} \cup \mathscr{J}$ and $p \in [\underline{p}_A, p_H^*)$. For $i = H$ this becomes

$$\prod_{j \in \mathscr{J}} (1 - \pi_j F_j(p)) = \frac{D(\underline{p}_A)(\underline{p}_A - c_H)}{D(p)(p - c_H)}.$$

Inserting into the condition for $i \in \mathscr{J}$ yields

$$1 - \pi_i F_i(p) = \frac{D(\underline{p}_A)(\underline{p}_A - c_H)}{c} \frac{p - c_A}{p - c_H} (1 - \pi_H F_H(p)).$$

Hence, inserting back into the $i = H$ condition,

$$\pi_H F_H(p) = 1 - \frac{c}{D(\underline{p}_A)(\underline{p}_A - c_H)} \frac{p - c_H}{p - c_A} \left( \frac{D(\underline{p}_A)(\underline{p}_A - c_H)}{D(p)(p - c_H)} \right)^{1/|\mathscr{J}|}.$$

As $\pi_H = 1$, this pins down $F_H(p)$ for $p < p_H^*$. Note that $F_H(\cdot)$ is strictly increasing in $p$, as required. Further, $F_H(p_H^*-) < 1$, so $H$ must place an atom at $p_H^*$.

We re-write $H$'s mixing distribution in final form as

$$F_H(p) = 1 - \left( \frac{c}{D(p)(p - c_A)} \right) \left( \frac{D(p)(p - c_H)}{D(\underline{p}_A)(\underline{p}_A - c_H)} \right)^{1 - 1/|\mathscr{J}|}$$

for $p < p_H^*$, with $F_H(\underline{p}_A) = 0$ and $F_H(p_H^*) = 1$.

Next, by inserting $F_H(\cdot)$ into the relationship between $F_i$ and $F_H$, we find that each away firm's mixing distribution satisfies

$$\pi_i F_i(p) = 1 - \left( \frac{D(\underline{p}_A)(\underline{p}_A - c_H)}{D(p)(p - c_H)} \right)^{1/|\mathscr{J}|}.$$

Because $F_i$ must be continuous and equal to 1 at $p_H^*$, this pins down $\pi_i$ as

$$\pi_i = 1 - \left( \frac{D(\underline{p}_A)(\underline{p}_A - c_H)}{D(p_H^*)(p_H^* - c_H)} \right)^{1/|\mathscr{J}|}.$$

Solving for $F_i(\cdot)$ yields each away firm's mixing distribution. Note that all participating away firms play an identical strategy.

Finally, we must check that it is optimal for each non-participating away firm not to enter. But each away firm $i \in \mathscr{J}$ makes zero profits when playing any $p \in (\underline{p}_A, p_H^*)$ when faced with $|\mathscr{J}| - 1$ other away firms. Then at each price $p > \underline{p}_A$, all non-participating firms make strictly lower profits than $i$ because they will occasionally lose the sale to $i$. Then no non-participating firm wants to enter above $\underline{p}_A$. And entering at $\underline{p}_A$ yields zero profits. So indeed it is optimal for all firms in $\mathscr{I} \setminus \mathscr{J}$ to refrain from entering. This completely characterizes all Nash equilibria of the stage game. □

### C.4.2 Optimal collusion in stationary equilibria

**Lemma C.15.** $N \geq \sqrt{1 + \Pi^M/c}$ *implies* $\overline{\delta}(N) > \delta^M(N)$.

*Proof.* $\overline{\delta}(N) \geq \delta^M(N)$ is equivalent to

$$1 - \frac{(N-1)c}{\Pi^M} \leq \frac{1}{\frac{N+1}{N} - \frac{\Delta c D(p_H^*)}{\Pi^M}}.$$

As the rhs of this inequality is always greater than $N/(N+1)$, a sufficient condition is

$$1 - \frac{(N-1)c}{\Pi^M} \leq 1 - \frac{1}{N+1} \Rightarrow N \geq \sqrt{1 + \Pi^M/c}.$$

□

**Definition C.2.** *Fix a stationary equilibrium $\sigma$ with associated on-path stage-game strategy profile $\tau$. Then $\sigma$ is* market-symmetric *if, for each market $m$ and all away firms $i$ and $j$, $\tau_m^i = \tau_m^j$.*

This class is more general than the stationary equilibria studied in Appendix C.3, as every symmetric equilibrium is also market-symmetric, but not vice versa. Most of the results in this appendix go through in the broader class. In addition, in the two-firm case every stationary equilibrium is automatically market-symmetric. Thus any restriction to market-symmetric equilibrium has no consequences in the two-firm case. Hence focusing on market-symmetric equilibria simplifies the proof structure, as results may be proven all at once for arbitrary $N$.

**Proposition C.11.** *Suppose that $\delta < \delta^M(N)$. Then if $N = 1$ or $\delta < \overline{\delta}(N)$, at most one firm earns positive intra-period profits in each market in every market-symmetric stationary equilibrium.*

*Proof.* We first show that the existence of multiple firms making positive profits in a given market implies a minimally profitable deviation for each such firm.

**Lemma C.16.** *Fix a stationary equilibrium yielding strictly positive intra-period profits to $2 \leq K \leq N+1$ firms $i_1, i_2, ..., i_K \in \mathscr{I}$ in market $m$. Then for each $i = i_1, ..., i_K$, stage profits are bounded above as*

$$\Pi^i_m \leq \frac{1}{K}(\Pi^M + c - \Delta c D(p^*_A)\mathbf{1}\{i \neq m\}) - c$$

*and there exists a deviation in market $m$ yielding intra-period profits of at least $K\Pi^i_m + (K-1)c$.*

*Proof.* Wlog fix $m = 1$. (It will not matter whether the home firm is one of the firms receiving positive profits.) As a first observation, we must have $\pi^{i_k}_1 = 1$ for all $k = 1, ..., K$. For strictly positive intra-period profits in market 1 imply existence of an on-path action $a$ involving entry which gives strictly positive profits. Failing to enter yields lower intra-period profits and a lower continuation than playing $a$, so each firm must choose to enter w.p. 1.

Define $p^i_U \equiv \sup\{p \ : \ F^i_1(p) < 1\}$ for $i \in \mathscr{I}$. Then $p^i_U$ is the supremum of the support of firm $i$'s price distribution in market 1. $p^i_U < \infty$ for $i = i_1, ..., i_k$ given (A1) and the fact that intra-period on-path profits are strictly positive for each such firm in market 1.

We claim that $p^{i_k}_U = p^{i_{k'}}_U$ for all $k, k' \in \{1, ..., K\}$. Suppose not, say $p^{i_1}_U > \max_{k=2,...,K} p^{i_k}_U \equiv p^{i-1}_U$. Then there exists an interval $[p_A, p_B] \subset (p^{i-1}_U, p^{i_1}_U]$ such that $F^{i_1}_1([p_A, p_B]) > 0$. In particular, there exists a $p_C \in [p_A, p_B]$ such that $\Pi^{i_1}_1(p_C) = \Pi^{i_1}_1$. But given $\pi^{i_k}_1 = 1$ and $F^{i_k}_1(p_C) = 1$ for all $k \geq 2$, we have $\Pi^{i_1}_1(p_C) = -c < 0$, contradicting the assumption that $\Pi^{i_1}_1 > 0$.

Let $p_U$ be the mutual supremum of the price supports for firms $i_1$ through $i_K$. We next argue that each of these firms places an atom at $p_U$. Suppose that some firm, say $i_1$, places no atom at $p_U$. Then each of $i_2$ through $i_K$ receive intra-period profits of $-c < 0$ at $p_U$ given that $\pi_1^{i_1} = 1$. So none of these firms places an atom at $p_U$ either. But then $\Pi_1^{i_1}(p_U-) = -c$, since for prices approaching $p_U$ firm $i_1$ will be underbid by one of $i_2$ through $i_k$ with probability approaching 1. Then $i_1$'s profits are non-positive for prices sufficiently close to $p_U$, meaning they cannot be profit-maximizing. But then for some $\varepsilon > 0$ the interval $[p_U - \varepsilon, p_U]$ is assigned measure zero by $F_1^{i_1}$, a contradiction of the definition of $p_U$. So $i_1$ must place an atom at $p_U$.

Now, the existence of overlapping atoms generates a profitable intra-period deviation for each firm. Consider the case of firm $i_1$. His equilibrium intra-period profits in market 1 are equal to his profits at $p^U$, which are bounded above as

$$\Pi_1^{i_1} \leq \left[\prod_{j \notin \{i_1,\ldots,i_K\}} (1 - \pi_j F_1^j(p_U-))\right] \left[\prod_{k=2,..,K} \Delta F_1^{i_k}(p_U)\right] \frac{1}{K} D(p_U)(p_U - c_{i_1}) - c.$$

(The inequality will be strict if some other firm also places an atom at $p^U$.) This bound is loosest if no firms enter below $p_U$ and the price ceiling is set to the profit-maximizing value for a given firm. This yields the upper bound in the lemma statement.

Meanwhile, by deviating to just under $p_U$ he can obtain profits of at least

$$\widetilde{\Pi}_1^{i_1} = \left[\prod_{j \notin \{i_1,\ldots,i_K\}} (1 - \pi_j F_1^j(p_U-))\right] \left[\prod_{k=2,..,K} \Delta F_1^{i_k}(p_U)\right] D(p_U)(p_U - c_{i_1}) - c \geq K\Pi_1^{i_1} + (K-1)c.$$

This is the deviation claimed in the lemma statement. □

Now, consider an arbitrary market-symmetric stationary equilibrium. Assume that in some market, at least two firms earn positive profits.

*The $N = 1$ case:* Wlog suppose both firms make positive profits in market 1. There are two possibilities: either both firms earn positive profits in market 2 as well, or some firm $i$ earns non-positive profits. In the former case, firm $i$ has a deviation worth $2\Pi_m^i + c$ in each market, which when summed imply the IC constraint

$$\Pi^i \geq (1-\delta)(2\Pi^i + 2c) + \delta\underline{\Pi}(\delta; N) \geq (1-\delta)(2\Pi^i + 2c).$$

In the latter case, firm $i$ has a deviation worth at least $2\Pi_1^i + c$ in market 1 and $2\Pi_2^i$ in market 2 (the latter following trivially from $\Pi^i_2 \leq 0$), hence the IC constraint

$$\Pi^i \geq (1-\delta)(2\Pi^i + c) + \delta\underline{\Pi}(\delta; N) \geq (1-\delta)(2\Pi^i + c)$$

holds. In either case incentive-compatibility demands $\Pi^i \geq (1-\delta)2\Pi^i$. Now, $\Pi^i > 0$, else $i$ could deviate and obtain positive profits by exiting market 2 given $\Pi^i_1 > 0$. So this inequality implies $\delta \geq 1/2 > \delta^M(1)$.

*The $N \geq 2$ case:*

**Lemma C.17.** *Suppose $N \geq 2$. Fix a stationary equilibrium in which for some market, at least two firms earn positive intra-period profits in each period on-path. Then there exists a firm $i \in \mathscr{I}$ and an integer $n \in \{1, ..., N, N+1\}$ for which the IC constraint*

$$\Pi^i \geq (1-\delta)(N\Pi^i + n(N-1)c)$$

*holds, and* $0 < \Pi^i \leq \frac{n}{N}(\Pi^M - (N-1)c)$.

*Proof.* Suppose first that the home firm makes strictly positive profits in every market. Assume multiple firms make positive profits in market 1. Then by market symmetry all $N+1$ firms make positive profits in that market, and firm 1 therefore has a deviation worth $(N+1)\Pi^1_1 + Nc$ in that market by Lemma C.16. Additionally, in every other market $m \geq 2$ he either makes non-positive profits, and so trivially has a deviation worth $(N+1)\Pi^1_1$; or else he makes positive profits and has a deviation worth $(N+1)\Pi^1_m + Nc$ in that market as well.

Summing the profits for firm 1 from deviating across all markets, assuming that 1 makes positive profits in $n-1$ markets other than his own, we obtain the IC constraint

$$\Pi^1 \geq (1-\delta)((N+1)\Pi^1 + nNc).$$

As $\Pi^1 > 0$ (else he could deviate by withdrawing from all markets $i \geq 2$ to make positive profits), this IC constraint implies the desired one.

Further, the previous lemma implies that his profits in each positive-profit market $m$ are at most

$$\frac{1}{N+1}(\Pi^M + c - \Delta c D(p^*_A)\mathbf{1}\{m \geq 2\}) - c < \frac{\Pi^M}{N+1} - \frac{N}{N+1}c.$$

Hence by making positive profits in $n$ markets, he can make no more than $\frac{n}{N+1}(\Pi^M - Nc)$, implying the desired bound given that $N/(N+1) > (N-1)/N$.

Now suppose that the home firm makes non-positive profits in some market, say market 1. Then firm 1 must make positive profits in some other market $m \geq 2$, else he could deviate to achieve positive profits by undercutting in whatever market yields positive profits to multiple firms. In market $m$ all other away firms also make positive profits by market-symmetry, hence 1 has a deviation worth $N\Pi^1_m + (N-1)c$. The same reasoning holds in all other markets in which he makes positive profits. And when he makes non-positive profits, he trivially has a deviation worth $N$ times his profits. Summing these deviations, assuming 1 makes positive profits in $n$ markets, we obtain the IC constraint

$$\Pi^1 \geq (1-\delta)(N\Pi^1 + n(N-1)c),$$

which is the desired one. Reasoning as in the previous case shows that he can make no more than $\frac{n}{N}(\Pi^M - (N-1)c)$. $\square$

By the lemma just proven, for two or more firms to make positive profits in some market there must be some form $i$ and integer $n$ between 1 and $N+1$ such that

$$\delta \geq 1 - \frac{\Pi^i}{N\Pi^i + n(N-1)c}.$$

Further $\Pi^i \leq \frac{n}{N}(\Pi^M - (N-1)c)$, so that

$$\delta \geq 1 - \frac{\frac{n}{N}(\Pi^M - (N-1)c)}{n(\Pi^M - (N-1)c) + n(N-1)c} = \overline{\delta}(N).$$

$\square$

**Proposition C.12.** *Suppose $\delta < \delta^M(N)$ and $N = 1$ or $\delta < \overline{\delta}(N)$. Then payoffs $(\Pi^1, ..., \Pi^{N+1})$ of any market-symmetric stationary equilibrium must satisfy $\Pi^i \in [0, \Pi^M]$ and*

$$\Pi^i \geq (1-\delta)\left(\sum_{j=1}^{N+1} \Pi^j - \Delta c \sum_{j \neq i} D(p^*(\Pi^j))\right) + \delta\, \underline{\Pi}(\delta; N),$$

*for all $i \in \mathscr{I}$, where $p^*(\Pi)$ is the unique solution in $[\underline{p}_H, p^*_H]$ to $D(p)(p - c_H) - c = \Pi$ for $\Pi \in [0, \Pi^M]$.*

*Further, any such equilibrium yields strictly positive profits to each firm in at most one market. When $N \geq 2$, this market is the firm's home market.*

*Proof.* Assume the conditions of the proposition statement, and fix a market-symmetric stationary equilibrium with profits $(\Pi^1, ..., \Pi^{N+1})$. We must have $\Pi^i \geq 0$ for all $i$, else any firm earning negative profits could withdraw from all markets permanently as a profitable deviation.

**Lemma C.18.** *Each firm $i \in \mathscr{I}$ can receive positive profits in at most one market.*

*Proof.* Suppose some firm, say firm 1, received positive profits in two or more markets. Then by the pigeonhole principle, along with the fact (Proposition C.11) that at most one firm can earn positive profits in any one market, there must be some firm $i \geq 2$ which receives non-positive profits in all markets and hence non-positive lifetime profits. But firm $i$ has a deviation yielding positive lifetime profits by undercutting the infimum of firm 1's price support in some market in which 1 is an away firm and makes positive profits. (Such a market exists by assumption.) So no firm can receive positive profits in multiple markets. □

The lemma implies that each $\Pi^i \leq \Pi^M$, as this is the most any firm could make in a single market. So $p^*(\cdot)$ is well-defined over the range of profits allowable in equilibrium. Also, when $N \geq 2$ market symmetry implies that firm $i$'s positive-profit market must be his home market, else all other away firms would also make positive profits in that market.

Let $m(i)$ be the (unique) market in which firm $i \in \mathscr{I}$ receives positive profits. If $i$ receives positive profits in no market, choose $m(i)$ to be some market in which he receives zero profits. $m : \mathscr{I} \to \mathscr{I}$ can always be constructed so that it is a bijection, which we will assume in what follows.

Now consider the following deviation by firm $i$ : he follows his equilibrium strategy in market $i$, while for each $j \neq i$ firm $i$ just undercuts the infimum of firm $j$'s price support in market $m(j)$. Because firm $j$ makes his entire profits $\Pi^j$ in market $m(j)$, his price support infimum must be at least $p^*(\Pi^j)$ there. Hence, since $D(p^*(\cdot))$ is a decreasing function, $i$ makes at least $\Pi^j - \Delta c D(p^*(\Pi^j))$ through undercutting in market $m(j)$. (He makes more from undercutting if the infimum of $j$'s price support is larger than $p^*(\Pi^j)$, or if $m(j)$ is not $j$'s home market, but we will not need to make use of this fact.)

Summing the intra-period profits from this deviation yields

$$\sum_{j=1}^{N+1} \Pi^j - \Delta c \sum_{j \neq i} D(p^*(\Pi^j)).$$

Thus, to deter a profitable deviation the IC constraint stated in the problem statement must hold. □

**Proposition C.13.** *Fix a profit vector* $(\Pi^1, ..., \Pi^{N+1}) \in [\Delta c D(\underline{p}_A), \Pi^M]^{N+1}$ *satisfying*

$$\Pi^i \geq (1-\delta)\left(\sum_{j=1}^{N+1} \Pi^j - \Delta c \sum_{j \neq i} D(p^*(\Pi^j))\right) + \delta\, \underline{\Pi}(\delta; N)$$

*for all* $i \in \mathscr{I}$*. Then there exists a market-symmetric stationary equilibrium supporting this profit vector with the following properties:*

1. *If* $\Pi^m = \Pi^{m'}$ *for markets* $m$ *and* $m'$*, then the home and away firms' strategies are the same in both markets.*
2. *The home firm in market* $m$ *enters with probability 1, while all away firms enter with a probability that is strictly between zero and 1 and decreasing in* $\Pi^m$*.*
3. *The home firm earns profits* $\Pi^m$ *in market* $m$*, while all away firms make zero profits.*
4. *Each firm posts prices only in* $[p^*(\Pi^m), p_H^*]$ *in market* $m$*, and firms' price distributions have full support on* $(p^*(\Pi^m), p_H^*)$*.*
5. *If* $\Pi^m > \Delta c D(\underline{p}_A)$*, the home firm in market* $m$ *plays* $p^*(\Pi^m)$ *with some strictly positive probability, which is increasing in* $\Pi^m$*.*
6. *Each market where* $\Pi^m < \Pi^M$ *is captured by an away firm with some strictly positive probability, which is strictly decreasing in* $\Pi^m$ *when* $\Pi^m \geq \frac{N-1}{2N-1}\Pi^M - \frac{N}{2N-1}c$*.*
7. *Any unilateral deviation by an away firm to a price at or below* $p^*(\Pi^m)$ *in any market* $m$ *is punished by a continuation payoff of* $\underline{\Pi}(\delta; N)$ *to that firm.*

*Proof.* We construct an equilibrium analogous to the stage-game, with a higher price floor. Fix a market $m$. If $\Pi^m = \Pi^M$, then the home firm plays $p_H^*$ w.p. 1 while the away firms do not enter. So assume $\Pi^m < \Pi^M$. The home firm mixes so as to make each away firm indifferent over prices in $(p^*(\Pi^m), p_H^*)$. The indifference condition, assuming symmetric play by all away firms, is

$$0 = D(p)(p - c_A)(1 - \pi_A F_A(p))^{N-1}(1 - F_H(p)) - c.$$

Meanwhile the away firms play to make the home firm indifferent over all prices in the same interval, yielding

$$\Pi^m = D(p)(p - c_H)(1 - \pi_A F_A(p))^N - c.$$

Hence

$$\pi_A F_A(p) = 1 - \left( \frac{\Pi^m + c}{D(p)(p - c_H)} \right)^{1/N},$$

which when substituted into the away firm's indifference condition allows us to solve for $F_H(p)$ on $(p^*(\Pi^m), p_H^*)$ :

$$F_H(p) = 1 - \frac{c}{D(p)(p - c_A)} \left( \frac{\Pi^m + c}{D(p)(p - c_H)} \right)^{1-1/N}.$$

In the limit as $p \searrow p^*(\Pi^m)$, we have

$$F_H(p^*(\Pi^m)+) = 1 - \frac{c}{D(p^*(\Pi^m))(p^*(\Pi^m) - c_A)} = 1 - \frac{c}{\Pi^m + c - \Delta c D(p^*(\Pi^m))}.$$

Then whenever $\Pi^m > \Delta c D(\underline{p}_A) \geq \Delta c D(p^*(\Pi^m))$, this expression is strictly positive. So the home firm places an atom at $p^*(\Pi^m)$ of size

$$\Delta F_H(p^*(\Pi^m)) = 1 - \frac{c}{\Pi^m + c - \Delta c D(p^*(\Pi^m))}.$$

Note that $p^*(\cdot)$ is increasing while $D(\cdot)$ is decreasing, and hence $\Pi^m - \Delta c D(p^*(\Pi^m))$ is strictly increasing in $\Pi^m$. Thus $\Delta F_H(p^*(\Pi^m))$ is strictly increasing in $\Pi^m$.

Further, as $p \nearrow p_h^*$ we have

$$F_H(p_H^*-) = 1 - \frac{c}{\Pi^M + c - \Delta c D(p_H^*)} \left( \frac{\Pi^m + c}{\Pi^M + c} \right)^{1-1/N},$$

which is strictly less than 1. So the home firm places another atom at $p_H^*$ of size

$$\Delta F_H(p_H^*) = \frac{c}{\Pi^M + c - \Delta c D(p_H^*)} \left( \frac{\Pi^m + c}{\Pi^M + c} \right)^{1-1/N}.$$

Finally, we complete our characterization of the strategy played by all away firms. Note that $\pi_A F_A(p^*(\Pi^m)+) = 0$, so the away firm's price distribution is continuous at its lower end. On the other hand,

$$\pi_A F_A(p_H^*) = 1 - \left( \frac{\Pi^m + c}{\Pi^M + c} \right)^{1/N},$$

which is strictly positive. In order to avoid the away firm placing an atom at $p_H^*$ (which would not be optimal given the atom placed there by the home firm), we set

$$\pi_A = 1 - \left(\frac{\Pi^m + c}{\Pi^M + c}\right)^{1/N},$$

which lies strictly between zero and 1 and is decreasing in $\Pi^m$.

Having exhibited a strategy profile in each market, we must show that it is supportable in equilibrium. The home firm has no profitable deviation in each market, while the away firms' most profitable deviation is to just below $p^*(\Pi^m)$, yielding profits $\Pi^m - \Delta c D(p^*(\Pi^m))$. So each firm has $N$ profitable deviations in each market other than his own, and our strategies are an equilibrium iff the IC constraint stated in the proposition holds.

The fact that business-stealing occurs is immediate from the fact that the home and away firms play price distributions with overlapping support. To evaluate when it is decreasing in $\Pi^m$, we first compute the probability of business-stealing. The probability that some firm undercuts a given price $p$ is the complement of the probability that no firm does, which is $(1 - \pi_A F_A(p))^N$. Then, taking account of the atom by the home firm at $p_H^*$, we have

$$\mathbb{P}\{\text{Business stealing}\} = (1 - (1 - \pi_A)^N)\Delta F_H(p_H^*) + \int_{(p^*(\Pi^m), p_H^*)} (1 - (1 - \pi_A F_A(p))^N)\, dF_H(p).$$

(The open-set notation for the limits of integration indicates that the atoms at the top and bottom of the support of $F_H(\cdot)$ are excluded from the integral.)

Let $\widetilde{F}_H(p) \equiv \frac{F_H(p)}{(\Pi^m + c)^{1-1/N}}$. Note that $\widetilde{F}_H(p)$ has the property that its increments are independent of $\Pi^m$. Now, inserting the expression for $\pi_A F_A(p)$ derived earlier, we obtain

$$\begin{aligned} &\mathbb{P}\{\text{Business stealing}\} \\ &= (\Pi^m + c)^{1-1/N}\left(1 - \frac{\Pi^m + c}{\Pi^M + c}\right)\Delta \widetilde{F}_H(p_H^*) \\ &\quad + \int_{(p^*(\Pi^m), p_H^*)} (\Pi^m + c)^{1-1/N}\left(1 - \frac{\Pi^m + c}{\Pi^M + c}\right) d\widetilde{F}_H(p). \end{aligned}$$

Now we differentiate wrt $\Pi^m$, assuming that both $\widetilde{F}_H$ and $p^*$ are differentiable. (We will show later that this assumption is innocuous.) Differentiating wrt $\Pi^m$ in the limit of integration yields no contribution, as the integrand is zero at the lower limit. The remaining terms are then

$$\begin{aligned}
&\frac{d}{d\Pi^m}\mathbb{P}\{\text{Business stealing}\}\\
&= (\Pi^m + c)^{-1/N}\left[\left((1-1/N) - (2-1/N)\frac{\Pi^m + c}{\Pi^M + c}\right)\Delta\widetilde{F}_H(p_H^*)\right.\\
&\quad \left. + \int_{(p^*(\Pi^m),p_H^*)}\left((1-1/N) - (2-1/N)\frac{\Pi^m + c}{D(p)(p-c_H)}\right) d\widetilde{F}_H(p)\right].
\end{aligned}$$

(Recall that the increments of $\widetilde{F}_H$ are not a function of $\Pi^m$.) Both the leading term and the integrand are negative over the entire integration range when

$$\left((1-1/N) - (2-1/N)\frac{\Pi^m + c}{\Pi^M + c}\right) \leq 0,$$

i.e. when

$$\Pi^m \geq \frac{1-1/N}{2-1/N}(\Pi^M + c) - c = \frac{N-1}{2N-1}\Pi^M - \frac{N}{2N-1}c.$$

Recall that we had assumed sufficient regularity to be able to differentiate wrt the lower limit of integration. In the general case, the increase in business stealing for small increase in $\Pi^m$ will be less than if we had assumed $p^*(\Pi^m)$ to be fixed, as the integrand is non-negative and the integrator is increasing. Thus, an upper bound on the first-order change in business-stealing is the one just derived. Whenever that bound is negative, the change in business-stealing must also be negative for sufficiently small changes in $\Pi^m$. □

**Proposition C.14.** *Suppose $N \geq 2$ and $\delta < \delta^M(N)$. Among symmetric stationary equilibrium payoff profiles $(\Pi, ..., \Pi)$ satisfying*

$$\Pi \geq (1-\delta)\left((N+1)\Pi - \Delta c D(p^*(\Pi))\right) + \delta\,\underline{\Pi}(\delta; N), \quad i \in \mathscr{I}$$

*there is a unique profile $(\Pi^*, ..., \Pi^*)$ simultaneously maximizing both firms' payoffs, where $\Pi^*$ is the unique solution to*

$$\Pi = (1-\delta)((N+1)\Pi - \Delta c N D(p^*(\Pi))) + \delta\,\underline{\Pi}(\delta; N).$$

*Further, $\Pi^* > \Pi^C$ iff $\underline{\Pi}(\delta; N) < \Pi^C$, and $\Pi^*$ is strictly increasing in $\delta$ whenever $\underline{\Pi}(\cdot; N)$ is non-increasing in $\delta$.*

*Proof.* Any symmetric stationary equilibrium payoff profile satisfying the IC constraints is subject to the single IC constraint

$$\Pi \geq (1-\delta)((N+1)\Pi - \Delta c N D(p^*(\Pi))) + \delta\, \underline{\Pi}(\delta; N).$$

As $\delta < \delta^M(N) < 1 - 1/(N+1)$, raising $\Pi$ increases the rhs more quickly than the lhs. And at $\Pi^C$ the rhs is at most $\Pi^C$ given $\underline{\Pi}(\delta; 1) \leq \Pi^C$, while at $\Pi^M$ the rhs is at least

$$(1-\delta)((N+1)\Pi^M - \Delta c N D(p_H^*)) = (1-\delta)(\Pi^M + N\widetilde{\Pi}^M) > \Pi^M$$

given $\delta < \delta^M(N)$. Hence, given continuity of the rhs, there exists a unique $\Pi \in [\Pi^C, \Pi^M]$ for which the IC constraint just binds. This is the largest possible payoff satisfying the IC constraint.

Finally, note that

$$(N+1)\Pi - \Delta c N D(p^*(\Pi)) \geq (N+1)\Pi^C - \Delta c N D(p^*(\Pi^C)) = \Pi^C,$$

with equality when $\Pi = \Pi^C$. Thus $\Pi^* > \Pi^C$ iff $\underline{\Pi}(\delta; N) < \Pi^C$. And since raising $\delta$ lowers the rhs of the IC constraint whenever $\Pi > \Pi^C$ and $\underline{\Pi}(\delta; N)$ is decreasing in $\delta$, we conclude that $\Pi^*$ is increasing in $\delta$ whenever the latter condition holds. □

**Proposition C.15.** *Suppose that*

$$\delta \geq \left(1 + \frac{1}{N} + \frac{\Pi^M - \widetilde{\Pi}^M}{c}\right)^{-1}.$$

*Then for each firm $m$ there exists an SPNE supporting lifetime profits of 0 for firm $m$, so that $\underline{\Pi}(\delta; N) = 0$.*

*Proof.* We explicitly construct such an equilibrium strategy profile. Wlog let $m = 1$. The equilibrium consists of two phases: "punishment" and "cooperation." In the punishment phase, firm 1 enters his home market and prices at some $p^{PW} \leq \underline{p}_H$, while in market 2 he enters and mixes with any distribution assigning measure 1 to $(p^{PW}, \underline{p}_H]$. Firm 2 plays symmetrically in the two markets. All other firms stay out of markets 1 and 2, and in all remaining markets all firms play the stage-game NE. In the cooperative phase firms play an SPNE yielding profits $(\Pi^*, ..., \Pi^*)$ characterized in Proposition C.14. Players transit from the punishment to the cooperative phase after a single stage, and stay in the cooperative phase forever. All deviations result in a reversion to the punishment phase.

We choose $p^{PW}$ so that lifetime profits for firms 1 and 2 are zero, implying

$$(1-\delta)(D(p^{PW})(p^{PW}-c_H)-2c)+\delta\Pi^*=0.$$

In order that $p^{PW} \leq \underline{p}_H$ (crucial to preventing a profitable deviation yielding positive lifteime profits to firms 1 or 2), we therefore require

$$\Pi^* \geq \frac{1-\delta}{\delta}c.$$

Now, $\Pi^*$ is characterized by

$$\Pi^* = (1-\delta)((N+1)\Pi^* - \Delta c N D(p^*(\Pi^*)))$$

whenever the strategy profile outlined is indeed an equilibrium. Re-arranging yields

$$\Pi^* = (1-\delta)((N+1)\Pi^* - \Delta c N D(p^*(\Pi^*))).$$

Combining this expression with the lower bound on $\Pi^*$ derived earlier yields the bound

$$\delta \geq \left(1+\frac{1}{N}+\frac{\Delta c D(p^*(\Pi^*))}{c}\right)^{-1}.$$

Our strategy profile satisfies $p^{PW} \leq \underline{p}_H$ iff this inequality holds. Now, note that $\Delta c D(p^*(\Pi^*)) \geq \Delta c D(p^*_H) = \Pi^M - \widetilde{\Pi}^M$, so the inequality in the problem statement implies this one.

We complete the proof by checking that no firm has any profitable deviations. By construction no such deviations exist in the cooperation phase. As for the punishment phase, no firm has any profitable deviation in markets 3 through $N+1$ because the stage-game NE is played there. As for markets 1 and 2, firms 3 through $N+1$ could never win the market at a profitable price by entering, so they have no incentive to enter. As for firms 1 and 2, given $p^{PW} \leq \underline{p}_H$ their most profitable deviation is to exit both markets at once, yielding 0 in the current stage and a punishment continuation of 0. This is precisely the same as their lifetime payoffs from playing their equilibrium strategies, so they have no profitable deviations.

We conclude that our proposed strategy profile is indeed an equilibrium whenever $p^{PW} \leq \underline{p}_H$, and in particular whenever the inequality in the proposition statement holds. □

**Proposition C.16.** *Suppose* $N \geq 1 + \sqrt{\frac{\Pi^M - \widetilde{\Pi}^M}{c}}$. *Then* $\overline{\delta}(N) > \underline{\delta}(N)$.

*Proof.* As $\overline{\delta}(N) > 1 - 1/N$, a sufficient condition for $\overline{\delta}(N) > \underline{\delta}(N)$ is

$$1 - \frac{1}{N} \geq \left(1 + \frac{1}{N} + \frac{\Pi^M - \widetilde{\Pi}^M}{c}\right)^{-1}.$$

Some rearrangement yields the equivalent inequality

$$N(N-1) \geq \frac{\Pi^M - \widetilde{\Pi}^M}{c},$$

which is in turn implied by

$$(N-1)^2 \geq \frac{\Pi^M - \widetilde{\Pi}^M}{c}.$$

Solving for $N$ yields the inequality in the proposition statement. □

For the following pair of propositions, let $\Pi^\dagger$ be defined as the unique solution in $[\Pi^C, \Pi^M]$ to

$$\Pi^\dagger = (1-\delta)(N\Pi^\dagger - (N-1)\Delta c D(p^*(\Pi^\dagger))) + \delta \underline{\Pi}(\delta; N)$$

whenever $\delta < \delta^M(N)$, with $\Pi^\dagger = \Pi^M$ whenever $\delta \geq \delta^M(N)$. (See the proof of Proposition C.14 for existence and uniqueness of this $\Pi^\dagger$.)

**Proposition C.17.** *Fix $\delta \in (0,1)$. If $\delta < N/(N+1)$, then $\Pi^\dagger \to 0$ as $c_A \downarrow c_H$. If $\delta \geq N/(N+1)$, then $\Pi^\dagger = \Pi^M$ for all $c_A > c_H$.*

*Proof.* Fix $N$, $\delta$, $c$, and $c_H$. If there exists $c_A > c_H$ such that assumptions (A1) through (A3) hold, then these assumptions continue to hold for all smaller $c_A$. We will assume that $c_A$ is sufficiently close to $c_H$ that these assumptions hold everywhere.

Note that $\delta^M(N) < N/(N+1)$. Hence if $\delta \geq N/(N+1)$, $\delta > \delta^M(N)$ and so $\Pi^\dagger = \Pi^*$. So assume $\delta < N/(N+1)$. As $\delta^M(N) \to N/(N+1)$ as $c_A \downarrow c_H$, for $c_A$ sufficiently close to $c_H$ we have $\delta < \delta^M(N)$. Also observe that $\Pi^M$ is independent of $c_A$, while $\Pi^C = \Delta c D(p_A^*) \to 0$ as $c_A \downarrow c_H$ given $D(p) \in [0,1]$. Then as $c_A \downarrow c_H$, $\Delta c D(p^*(\Pi^*)) \to 0$ and $\underline{\Pi}(\delta; N) \to 0$, the latter because $\underline{\Pi}(\delta; N) \in [0, \Pi^C]$. Hence $\Pi^* \to 0$ as well. □

In the following proposition, we define $p^{**}$ to be the limiting value of the price floor $p^*(\Pi^\dagger)$ as fixed costs fall to zero.

**Proposition C.18.** *Fix $\delta \in (0,1)$. Let $(F_H(\cdot), F_A(\cdot), \pi_A)$ be the home and away firms' price distributions and the away firm's entry probability, respectively, for the equilibrium characterized in Proposition C.13 with $\Pi^i = \Pi^\dagger$ for all $i$. As $c \to 0$, $F_H(\cdot)$ converges uniformly to*

$\mathbf{1}\{p \leq p^{**}\}$ *while* $\pi_A F_A(p^{**}) \to 0$. *The probability of business stealing therefore falls to zero as* $c$ *vanishes, and in the limit the home firm wins the market at price* $p^{**}$ *with probability 1.*

*Proof.* Follows from reasoning closely analogous to the proof of Proposition 9 in the main text. □