Wage Dispersion in the Search and Matching Model with Intra-Firm Bargaining

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Abstract

Matched employer-employee data exhibits large and persistent wage and productivity dispersion across firms and suggests that a well defined linear relationship holds between the average wage paid and a firm productivity. The purpose of this paper is to demonstrate that these facts can be explained by a search and matching model when firms are heterogenous with respect to productivity, are composed of many workers, and face diminishing returns to labor when the wage paid to identical workers is the solution a Stole-Zwiebel bilateral bargaining problem. Helpman and Itskhoki (2008) show that a unique single wage equilibrium solution to the model exists in this environment. In this paper, I demonstrate that a unique equilibrium also exists that can be characterized by a distribution of wages in which more productive firms pay more if employed workers are able to search. Finally, employment is lower in the dispersed wage equilibrium than in the single wage equilibrium but this fact does not imply that welfare is higher in the single wage equilibrium.

1 Introduction

The simplicity of the canonical search and matching model offers many advantages for the purpose of understanding the determinants and dynamics of unemployment. However, the special assumption that a firm is composed of a single worker and employer or possess a linear technology is limiting.

Stole and Zwiebel (1996) propose a strategic bilateral wage bargaining when firms employ many workers and produce under conditions of diminishing returns but do not explicitly model labor market friction. Wolinsky (2000), Cahuc et al. (2001, 2008), and Helpman and Itskhoki (2008) incorporate Stole-Zwiebel bargaining into an extended version of the canonical search and matching model but only consider the case of no search while employed. In this paper, I extend the model by allowing for search on-the-job. The fact that the extended model can explain wage dispersion and the positive cross section relationship between the average wage paid and average labor productivity observed in micro data on firms motivates this extension.

In Stole-Zwiebel formulation, the employer bargains over wages with each worker in the firm as though he or she were the marginal employee. Because the wage decreases with the number of workers employed in the diminishing returns case, employers have an incentive to "over employ" as a means of reducing the over all wage bill. Smith (1999) reports a similar result in the context of a canonical search model with diminishing returns and rent sharing.

Wolinsky (2000) and Helpman and Itskhoki (2008) establish that Stole-Zwiebel wages paid to identical workers by different employers are equalized in a steady state search equilibrium when the production technology exhibits diminishing returns with respect to labor, even when firms are heterogeneous with respect to productivity. In this paper, I show that this result is the consequence of ignoring search while employed.

Formally, I study a particular dynamic generalization of the Stole-Zwiebel bargaining problem. In the model, long term contracts are not feasible and search, bargaining and production cannot take place simultaneous. As a consequence, not forming a producing match for a period is the only credible threat that either party has in the bargaining game. The single wage equilibrium previously derived exists as well in the model because search is not optimal when there is no wage dispersion. However, a unique equilibrium dispersed wage distribution in which more productive firms pay more is also present. Further more, I show hat employment is lower in the dispersed wage equilibrium than in the single wage equilibrium. But, because aggregate employment in the single wage equilibrium exceeds the social optimum, social welfare can be higher in the dispersed wage equilibrium.

There is a close relationship between the equilibria of the search and matching model studied in this paper and those of the dynamic monopsony models of Diamond (1971), Burdett and Judd (1983), and Burdett and

Mortensen (1998). The single wage equilibrium is the Diamond equilibrium while a unique dispersed wage equilibrium exists when employed workers search for the same reason as in the Burdett-Mortensen model. Wage dispersion arises because a higher wage increases the yield per vacancy posted and reduces the quit rate. As a consequence, there exists a distribution of vacancies over a non-degenerate interval of wages such that the expected return is equal to the common cost of posting a vacancy for every wage in the interval paid by some firm even when all firms are identical.

The analysis in this paper is also related to that of Shimer (2006). In his paper, long term contracts are allowed so that bargaining is over future income streams. Shimer starts by points out that axiomatic Nash bargaining used in the canonical model does not apply in this environment because the set of payoffs is typically not convex when employed worker search. He goes on to propose and solve a strategic formulation of the bargaining game in this environment. Shimer establishes the existence of a continuum of equilibria each characterized by a different continuous wage distribution.

2 The Model

The labor market is composed of a continuum of workers, $[0, \ell]$ and a unit continuum of employers. Both are risk neutral, live forever, and discount the future at the common rate r. Labor is the only input and production is generally subject to diminishing returns. Let $p(x)f(n)$ denote the production function of firm $x \in [0, 1]$ where the variable n is the measure of employment in the firm, and $p(x)$ is total factor productivity. The base line production function $f(n)$ is increasing and concave. Without loss of generality, firms with a higher index x are (weakly) more productive in the sense that $x > x'$ implies $p(x) \geq p(x')$. Given this convention, the firm index x is equal to the firm's percentile rank in the distribution of productivity when there are differences in productivity.

Workers are identical and can be either employed or not. While unemployed they receive an income b, which I interpreted as the flow value of home production. While employed, they earn a wage w which is determined as the outcome of a bilateral bargaining problem specified later. A worker can choose to search or not at a unit intensity in both the unemployment and employment states. Finally, all agents are risk neutral.

2.1 Worker Search

If the environment is stationary, the workers' common value of unemployment solves the Bellman equation

$$rU = b + \max_{\phi \in \{0,1\}} \left\{ \lambda\phi \int \max \langle W(w) - U, 0 \rangle \, dF(w) - \varepsilon\phi \right\} \tag{1}$$

given random search where ϕ is an indicator that takes the value 1 if the worker searches and 0 otherwise, ε is a small fixed cost of search, b represents the value of home production, r is the interest rate, λ is the job finding rate, δ is the job destruction rate, $W(w)$ is the value of employment at wage w, and $F(w)$ is the distribution of wage offers. The value of employment is the expected present value of the worker future income when employed. For a firm paying wage w, the state contingent value of employment satisfies the Bellman equation

$$rW(w) = \begin{array}{c} w + \delta(U - W(w)) \\ + \max_{\phi \in \{0,1\}} \left\{ \lambda\phi \int \max \langle W(z) - W(w), 0 \rangle \, dF(z) - \varepsilon\phi \right\}. \end{array} \tag{2}$$

Of course, the worker can choose whether or not to search when employed as well as when unemployed and does so to maximize expected future income. Although the "search technology" as represented by the offer arrival rate may be different in the two states, we assume they are same in the paper for simplicity. Hence, in the limit as $\varepsilon \to 0$, the worker's optimal search strategy is to search when unemployed if and only $1 - F(b) > 0$ and search when employed at wage w if and only if $1 - F(w) > 0$. Hence, all employed workers except those employed by the firm paying the highest wage search if wages are disperse and no employed worker search if there is no dispersion. Let $\phi_0 \in \{0, 1\}$ represent the optimal choice if unemployed and $\phi_1 \in \{0, 1\}$ if employed. Below I only consider the case in which trade takes place in equilibrium so that $\phi_0 = 1$ and assume that $\phi_1 = 1$ if and only if $1-F(w) > 0$ for $w \le \overline{w}$ where $\overline{w}$ is the upper support of the wage offer distribution.

Given that employed workers do search, they move if and only if offered a strictly higher employment value. Hence, for a firm that pays w, the separation rate is

$$s(w) = \delta + \lambda\phi_1(1 - F(w)). \tag{3}$$

Because an unemployed worker accepts all offers in equilibrium and an employed worker accepts only if offered a higher wage, the expected yield per vacancy advertised is

$$h(w) = \eta \left[\frac{u + (1-u)\phi_1 G^-(w)}{u + (1-u)\phi_1} \right] \tag{4}$$

where u is the unemployment rate, $G^-(w) = lim_{x \uparrow w} G(w)$ is the fraction of workers earning strictly less than W, $G(w)$ is wage cdf, and η is the rate at which employers contact searching workers per vacancy posted. Because $s(w)$ is decreasing and $h(w)$ is increasing, an increase in the a firm's relative wage reduces turnover if and only if employed workers search.

Note that $W'(w) = 1/(r + s(w))$. Hence, by rearranging terms and integrating by parts, equations (1) and (2) imply

$$\begin{aligned}
(r+\delta)\left(W(w) - U\right) &= w - rU + \lambda\phi_1 \int_w^{\overline{w}} \left[W(z) - W(w)\right] dF(z) \\
&= w - b + \lambda\left(\phi_1 - \phi(b)\right) \int_w^{\overline{w}} \left(\frac{1 - F(z)}{r + s(z)} \right) dz.
\end{aligned}$$

in the limit as the fixed cost of search vanishes. Therefore, the definition of the reservation wage $U = W(\widehat{w})$ imply that an unemployed worker accepts a job offer if and only if the wage exceeds the solution to

$$\widehat{w} = b + \lambda[\phi(b) - \phi(\widehat{w})] \int_{\widehat{w}}^{\overline{w}} \left(\frac{1 - F(z)}{r + s(z)} \right) dz. \tag{5}$$

In other words, if employed workers search when employed at the reservation wage, then the reservation wage is the value of home production. Otherwise, the reservation wage is equal to the value of home production plus the option value of continued search while unemployed.

2.2 Wage Bargaining

Wages are determined by bilateral bargaining between employer and each worker separately in the spirit of Stole and Zwiebel (1996). Specifically, renegotiation is costless, long term contracts are not binding, and employer and potential employees all are free to terminate the relationship "at will." Wolinsky (2000) suggests and studies a dynamic generalization of the original model in the case of no on-the-job search which is adopted by Cahuc et al. (2001,2007) and Helpman and Itskhoki (2008). Although the bargaining game assumed in this paper differs from the Wolinsky formulation when search on-the-job is possible, I contend that it is consistent with the spirit of the Stole-Zwiebel formulation.

Think of each "period" as divided into three subintervals. In the first, search and recruiting takes place. In the second, the employer bargains individually with each the workers available. Finally, production takes place using the subset of worker who agree to the bargaining outcome in the last subperiod. Note that search and recruiting determines the allocation of job applicants among firms and hence cannot occur during the bargaining phase. As a consequence, the only default option for either side in the bargaining game is to refrain from production as in Hall and Milgrom (2008). Finally, employers commit not to respond to any alternative employment opportunity available to the worker. Hence, those employed workers with alternative offers must choose one or the other prior to entering into bargaining.

Information is complete and symmetric in the sense that both sides know the values of their match and all options available at each node of the bargaining game. Bargaining between each worker-employer pair takes place in a sequence of rounds as in Rubinstein (1982). Worker can engage in home production during a delay in negotiation and employer forgo marginal profit. As in any of the Rubinstein type bargaining games with complete information, agreement is immediate because any offer made by either party is equal to the value of continuing the bargaining to the other. Hence, if J and W represent the values of continuing to the production stage respectively for the employer and the marginal worker, then the value of delaying a sub-interval of length Δ is $e^{-\Delta}J$ for the employer but is $b\Delta + e^{-\Delta}W$ for the worker.

For the sake of introducing asymmetry in the surplus shares, I suppose that nature determines the agent who make the offer in every bargaining round. Specifically, the worker makes the offer with exogenous probability $\beta \in (0,1)$ and the employer with complementary probability $1-\beta$. Therefore, the expected values of the agreement net of the defaults solve

$$(1-\beta)(\pi_n\Delta + e^{-\Delta}J - e^{-\Delta}J) = \beta(w\Delta + e^{-\Delta}W - b\Delta - e^{-\Delta}W).$$

Hence,

$$(1-\beta)\pi_n = (1-\beta)\left(pf'(n) + w\frac{\partial w}{\partial n}\right) = \beta(w-b) \tag{6}$$

characterizes the wage agreement where $\pi_n = \partial\pi/\partial n$ represents the profit attributable to the marginal worker and

$$\pi(n,p) = pf(n) - wn. \tag{7}$$

is gross profit from production during the period.

The solution to the differential equation (6) is

$$w(n,p) = (1-\beta)b + p\int_0^1 z^{\frac{1-\beta}{\beta}} f'(zn)dz. \tag{8}$$

which I will refer to as the wage function. For example, if the production function is Cobb-Douglas of the form $f(n) = n^{\alpha}$,$\alpha \in (0,1)$ then the wage function,

$$w(n,p) = (1-\beta)b + \left(\frac{\beta\alpha}{1-\beta+\beta\alpha}\right)\frac{pf(n)}{n}, \tag{9}$$

is linear in average product per worker. However, this equation need not imply that the equilibrium wage varies with firm productivity p because employment is endogenous.[1]

2.3 Job Vacancies

A firm posts v vacancies, which is chosen to maximize the expected present value of the firm's future profit. The hire yield per vacancy is $h(w)$ and the separate rate is $s(w)$ as defined respectively by equations (3) and (4). Hence, the law of motion for a firm's labor force is

$$\dot{n} = vh(w(n,p)) - s(w(n,p))n \tag{10}$$

where v represents the number of vacancies posted by the firm. The value of the firm satisfies the following continuous time Bellman equation

$$rV(n,p) = \max_{v\geq 0}\left\{\begin{array}{c} \pi(n,p) - cv \\ +J(n,p)\left[h(w(n,p)v - s(w(n,p))n\right] \end{array}\right\} \tag{11}$$

where c is the cost of posting a vacancy and $J(n,p) = \partial V/\partial N$ is the value of the marginal worker to the firm. Hence,

$$v(n,p) = \arg\max_{v\geq 0}\left\{h(w(n,p))vJ(n,p) - cv\right\} \tag{12}$$

[1]The wage $w(n,p)$ will be the outcome of bargaining with any worker who is employed in a steady state only if it exceeds the flow value of their outside option of unemployed search. This condition always holds in the equilibria described below.

is the optimal number of vacancies posted.

Because the vacancy posting cost, c, is constant by assumption, the firm posts no vacancies if the vacancy posting cost exceeds the expected return, $h(w(n_0, p)J(n_0, p)$, evaluated at its initial labor force size n_0. In this case, the firm either allows its labor force to fall through attrition until equality holds or it immediately lays off the redundant workers if doing so is costless. Once its steady state size is achieved, the firm posts the vacancies needed to replace those that quit. On the other hand, if the return exceeds the cost, the firm instantaneously hires the workers required to achieve equality of the marginal cost to the expected return from posting a vacancy. In sum, the firm's labor force size quickly adjusts to the desired level, which is defined by the requirement that the cost of posting a vacancy is equal to its expected return

$$c = h(w(n,p))J(n,p), \tag{13}$$

where the number of vacancies posted,

$$v(n,p) = \frac{s(w(n,p))n}{h(w(n,p))}, \tag{14}$$

is that required to keep the labor force at the desired size. In the job search literature, equation (13) is referred to as the "free entry" condition for vacancy creation.

As an increase in employment require a decrease in the wage, the value of the marginal worker satisfies

$$\begin{aligned} rJ(n,p) &= \pi_n(n,p) - s(w(n,p))J(n,p) \\ &\quad + J_n(n,p)\left[h(w(n,p))v(n,p) - s(w(n,p))n\right] \\ &\quad + J(n,p) \lim_{w \downarrow w(n,p)} \left\{ \frac{\begin{array}{c}[h(w(n,p)) - h(w)]\, v(n,p) \\ -\, [s(w(p,n)) - s(w)]\, n\end{array}}{w(p,n)) - w} \right\} w_n(n,p) \end{aligned} \tag{15}$$

where $w_n = \partial w(n,p)/\partial n$ by the envelope theorem. The first term on the RHS of (15) is the profit earned on the marginal worker as defined in equation (7), the second term is the cost of separations per worker, and the last two terms account for the total effect of an additional worker on the capital gain or loss associated with any rate of change in the size of the labor force. There are two effects of adding a worker to the firm's labor force that are not present in the canonical search and matching model. The first of the two, the impact of a change in employment on the value of the marginal match, vanished in steady state. The second effect, represented by the last term on the RHS of (15), is more novel. It represents the fact than an increase in the number of employees drives down the wage thereby decreasing the yield on vacancies and increasing the separation rate. *Of course, equations (3) and (4) imply that this effect is present if and only if employed worker search or the wage paid by the firm is not the lowest in the market.* These facts provide the rational for all of the original results found in the paper.

2.4 Labor Market Matching

The aggregate flow of matches that form per period is determined by an increasing, concave, and homogenous of degree one matching function of the aggregate number of vacancies and searching worker, denoted $M(v, z)$. The rates at which workers are randomly matched with vacant jobs and vacancies with workers are respectively

$$\lambda = M(v, z)/z = m(\theta) \tag{16}$$

and

$$\eta = M(v, z)/v = m(\theta)/\theta \tag{17}$$

where $m(\theta) = M(\theta, \ell)$,

$$z = [u + (1 - u)\phi_1]\, \ell \tag{18}$$

is the measure of searching workers, ℓ is the size of the aggregate labor force, and $\theta = v/z$ is market tightness.

3 Steady State Equilibria

3.1 Steady State Conditions

In a market steady state, the unemployment rate and the distribution of employment over firms are stationary by assumption. As the steady state unemployment rate equates the flows in and out of employment, it satisfies

$$\frac{u}{1 - u} = \frac{\delta}{m(\theta)}. \tag{19}$$

The analogous requirement that the flow into employment with a firm that offers value w or less is equal to the flow out determines the employment weighted distribution of wages paid by employers, given any distribution of values over vacancies offered. Formally,

$$\lambda F(w)u\ell = (\delta + m(\theta)\phi_1[1 - F(w)])\, G(w)(1-u)\ell$$

where the left side is the flow of workers into employment paying wage w or less and the right side is the flow of job-worker matches that are destroyed plus the flow of quits to jobs paying more than w. Hence, the steady state relationship between the distribution of wages paid employed workers and the distribution of wages offered over vacancies is given by

$$G(w) = \frac{\delta F(w)}{\delta + m(\theta)\phi_1[1 - F(w)]}. \tag{20}$$

These steady state conditions together with equations (4) and (17) imply that vacancy yield is

$$h(w) = \frac{m(\theta)}{\theta}\left[\frac{u + (1-u)\phi_1 G^{-}(w)}{u + (1-u)\phi_1}\right] = \frac{\delta m(\theta)}{\theta\,(\delta + m(\theta)\phi_1[1 - F^{-}(w)])} \tag{21}$$

where $F^{-}(w) = \lim_{x \uparrow w} F(w)$. Furthermore,

$$s(w) = \delta + m(\theta)\phi_1[1 - F(w)] \tag{22}$$

is the separation rate from (3) and (16).

Because both the yield per vacancy and the separation rate are discontinuous at any mass point in the distribution of wages in the Burdett-Mortensen model, an equilibrium wage distribution must be atomless and characterized by a continuous density function. The same is true for this model for the same reasons.

Lemma 1: If employed workers search, then an equilibrium offer distribution has a convex support and no mass points.

Proof. If $w(n,p)$ is a mass point in the support of $F(\cdot)$ then the vacancy $h(w)$ jumps down in the limit as the w approaches $w(n,p)$ from above by equations (21) and the separation jumps up as w approaches $w(n,p)$ from below by (22). As a consequence, equation (15) implies that the value of the marginal worker converges to zero as the mass point is approach from above or from below. Hence, the FONC for an optimal vacancy choice requires that no employer offering such a wage posts vacancies. However,, since $F(w)$ is the fraction of all vacancies in the market that offer wage w or less this fact contradicts the supposition that $w(n,p)$ is a mass point of an equilibrium offer distribution.

Now suppose that there is a gap in the support of the offer distribution, i.e., $F(w_1) = F(w_2)$ for some $w_2 > w_1$ both of which are in the support of $F(w)$. As there are no mass point, it follows that $h(w_1) = h(w_2)$ and $s(w_1) = s(w_2)$ so that in steady state

$$\begin{aligned} h(w_1)J(n(w_2,p_1,p_1) &= \frac{(1-\beta)(w_2-b)/\beta}{r+s(w')} > \frac{(1-\beta)(w_1-b)/\beta}{r+s(w_1)} \\ &\geq h(w_1)J(n(w_1,p_1,p_1) = c \end{aligned}$$

from equations (6), (15) and the fact that the last term on the right hand side of (15) is negative where p_1 is the productivity of any firm offering wage w_1. In other words, the firm offering wage w_1 can do better by hiring $n(w_2,p_1)$ workers and paying wage w_2. ■

At this point, I remind the reader that a reduction in the wage has no effect on either the vacancy yield or the separation rate if the wage paid is the lowest in the market. Namely, because the sharing rule implies $\beta\pi_n = (1-\beta)(w-b)$ from (6), equation (15) can be rewritten as

$$J(n(w,p),p) = \begin{cases} \frac{(1-\beta)(\underline{w}-b)/\beta}{r+s(\underline{w})} \text{ if } w = \underline{w} \\ \\ \frac{(1-\beta)(w-b)/\beta}{(r+s(w)+2m(\theta)g(w,p)\phi_1 F'(w)} \text{ if } w > \underline{w} \end{cases} \tag{23}$$

where $\underline{w}$ is the lower support of $F(w)$ as a consequence of Lemma 1. This derivation uses that fact that an employer with n employees must post $v(n,p) = s(w(n,p))n/h(w(n,p))$ vacancies. Hence, the steady state relationship between the vacancy yield and the separation rate function given by (21), (22), imply that the effect of adding a worker to future turnover costs can be expressed as

$$(h'(w(n,p)) - s'(w(n,p))\,n)w_n(n,p) = -2m(\theta)\phi_1 F'(w)g(w,p) \tag{24}$$

where from equation (8)

$$g(w,p) \equiv -w_n(w,p)n(w,p) = -n(w,p)p\int_0^1 z^{\frac{1}{\beta}} f''(zn(w,p))dz > 0 \tag{25}$$

given that $n(w,p)$ is the inverse of the Stole-Zwiebel wage function defined in equation (9).

The function $g(w,p)$ plays a crucial role in the analysis in the paper. It represents the absolute amount by which the wage bill paid to existing employees falls when a new worker is hired or rises when an existing employee leaves the firm. In general, the function is not monotone in productivity. Indeed, in the log linear production function case, that defined in equation (9)

$$g(w,p) = -np\int_0^1 z^{\frac{1}{\beta}} f''(zn)dz = \alpha p n(w,p)^{\alpha-1} = \left(\frac{1-\beta+\alpha\beta}{\alpha\beta}\right)(w-(1-\beta)b) \tag{26}$$

is independent of p. One can also show that the Cobb-Douglas case is the only one for $g(w,p)$ is independent of p everywhere.

3.2 Degenerate Wage Equilibrium

I begin by verifying that the "law of one price" can hold in the sense that an equilibrium solution exists with this property. Since employed workers do not search in such an equilibrium ($\phi_1 \equiv 0$), the FONC for an optimal choice of vacancies, $c = h(\underline{w})J(n(w,p))$ and equation (23) imply that the common wage paid is equal to the value of home production plus the amortized cost of replacing a worker. Formally,

$$w = b + \left(\frac{\beta}{1-\beta}\right)\frac{(r+\delta)c\theta}{m(\theta)}\ . \tag{27}$$

provided that the participation constraint, $w \geq \widehat{w}$ is satisfies. As equation (5) and the fact $w > b$ imply

$$\widehat{w} = b + m(\theta)(w-\widehat{w}) = b + \frac{m(\theta)(w-b)}{1+m(\theta)} < w,$$

the workers' participation constraint is slack. Finally, the steady state value of labor market tightness satisfies the employment identify

$$\frac{m(\theta)\ell}{\delta+m(\theta)} = (1-u)\,\ell = \int_0^1 n(w,p(x))dx. \tag{28}$$

Definition 1 *A degenerate steady state equilibrium is a wage w and a market tightness parameter θ that satisfy (27) and (28).*

The following conditions on the matching and production technologies are standard:

Assumption 1 *The job finding rate $m(\theta)$ is increasing and strictly concave function of θ and $m(0) = 0$, and $\lim_{\theta \to 0}\{\theta/m(\theta)\} = 0$.*

Assumption 2 *The baseline production function $f(n)$ is increasing, strictly concave and twice differentiable, and satisfies the Inada conditions, $\lim_{n\to 0} f'(n) = \infty$ and $\lim_{n\to\infty} f'(n) = 0$.*

Proposition 1 *A unique degenerate steady state equilibrium exists.*

Proof. Because the solution to (27) for the wage implicitly defines a strictly increasing and continuous functional relationship between w and θ, the right side of (28) can be represented as a positive continuous decreasing function of θ. As the LHS is continuous and increasing in θ and is equal to zero at $\theta = 0$, there is at most one positive solution for θ. ■

3.3 Equilibrium Wage Dispersion

Next consider the implications of search on-the job. If wages are disperse, then employed workers search when employed. As a consequently, the reservation wage is $\widehat{w} = b$ by equation (5) so the FONC for vacancy creation (13) and equation (23) imply that the lowest wage paid contingent on market tightness is

$$\begin{aligned} \underline{w} &= b + \left(\frac{\beta}{1-\beta}\right)(r + s(\underline{w}))\frac{c}{h(\underline{w})} \qquad (29) \\ &= b + \left(\frac{\beta}{1-\beta}\right)\left(\frac{(r+\delta+m(\theta))(\delta+m(\theta))}{\delta m(\theta)}\right)c\theta \end{aligned}$$

where the second equality follows from (21). The FONC for vacancy choice, equation (23), also implies that any equilibrium wage offer density must satisfy the differential equation

$$F'(w) = \frac{h(w)(1-\beta)(w-b)/\beta - c(r+s(w))}{2cm(\theta)g(w,p)}, w \in (\underline{w}, \overline{w}). \qquad (30)$$

Of course, the fact that that $h(w)$, $s(w)$, and $g(w,p)$ are continuous in w, implies that

$$F'(\underline{w}) = 0. \qquad (31)$$

Note that in case of homogenous firms, where p is a common parameter for all the firm, equation (30) is an single ordinary differential equation. As equation (31) provides an initial condition, a unique solution exists. Conditional on market tightness, θ, it is the unique candidate equilibrium offer distribution. Indeed, that solution together with a value of tightness that satisfies the employment identity form an dispersed wage equilibrium.

In the general case, the denominator of (30), varies with productivity. Consequently, another differential equation is needed to fully describe an equilibrium wage offer distribution when firms are heterogenous in productivity. Let $\omega(x)$, $x \in [0,1]$, represent a function that assigns a steady state wage rate to firm x and let $\digamma(x)$ denote the fraction of vacancies posted by firms with productivity rank x or less. We seek a steady state equilibrium in which more productive employers pay more, at least weakly. Given $\omega'(x) \geq 0$, $\digamma(x) = F(\omega(x))$ and, consequently, $\digamma'(x) = F'(\omega(x))\omega'(x)$. Hence,

$$\omega'(x) = \frac{2cm(\theta)g(n(\omega(x), p(x)))\digamma'(x)}{h(\omega(x))\left(1-\beta)\right)(\omega(x)-b)/\beta-(r+s(\omega(x)))c}, x \in [0,1) \tag{32}$$

which provides one of the two differential equations for the ODE system. The other is supplied by the employment identity for all firms of productivity x or less

$$\begin{aligned}(1-u)\ell G(\omega(x)) &= \frac{m(\theta)\ell}{\delta+m(\theta)}\left(\frac{\delta\digamma(x)}{\delta+m(\theta)[1-\digamma(x)]}\right) \\ &= \int_0^x n(\omega(z), p(z))dz, , x \in [0,1].\end{aligned}$$

By differentiating both side and solving the result for $\digamma'(x)$, one obtains

$$\digamma'(x) = \left(\frac{(\delta+m(\theta))\left(\delta+m(\theta)[1-\digamma(x)]\right)^2}{\delta m(\theta)\ell}\right)n(\omega(x), p(x)), x \in (0,1). \tag{33}$$

Of course, market tightness must satisfy the aggregate employment identity

$$\frac{m(\theta)\ell}{\delta+m(\theta)} = \int_0^1 n(\omega(z), p(z))dz. \tag{34}$$

The phase diagram for the planar system composed of equations (32) and (33) is illustrated as Figure 1. Although the phase space generally has three dimensions with each point representing a particular $(\digamma, \omega, x)$ combination, a slice of the space at any given value of x has the qualitative properties illustrated in Figure 1. The curve in the space with end points labeled w_0 and w_1 is the locus along which the denominator of the expression on the right side of equation (32) is zero. In other words, the curve is defined by

$$\begin{aligned} & \left(\frac{m(\theta)\delta}{\theta(\delta+m(\theta)[1-\digamma(x)])}\right)(1-\beta)(\omega-b)/\beta \\ = \quad & (r+\delta+m(\theta)[1-\digamma(x)])c \end{aligned} \tag{35}$$

where w_0 is the solution for ω when $\digamma(0) = 0$ and w_1 is the solution when $\digamma(1) = 1$. Because the curve is independent of x, it is in the same position in Figure 1 for all x.

Definition 2 *A monotone increasing steady state dispersed wage equilibrium is an increasing wage assignment function $\omega : [0,1] \rightarrow [\omega(0), \omega(1)]$, a vacancy c.d.f. $\digamma : [0,1] \rightarrow [0,1]$, and a market tightness parameter θ that satisfy equations (32), (33), and (34).*

Proposition 2 *If $p(x)$ is continuous, a unique monotone dispersed wage steady state equilibrium exists.*

Proof. For a given value of the tightness parameter θ, one can use the phase diagram in Figure 1 to characterize the set of solutions to the ODE system that could serve as possible wage offer distributions when $p(x)$ is continuous. The curves in the phase diagram represented by arrows are the solution trajectories of the system where the points of each indicates the "direction of motion" along the curve as x increases. All of these are potential offer distributions.

Note that there are two solution trajectories that initiate from any point on the curve defined by (35). Local multiplicity arises because $\omega'(0)$ converges to infinity as any such point is approached. In other words, the standard sufficient condition for a unique local solution to the ODE system, Lipschitz continuity, does not hold along the curve. However, because the RHS of (33) is always strictly positive, all solution trajectory that initiate above and to the right of the curve are unique and tend toward the north east in the diagram while only those to the left of it move northwest. Obviously, only the former can represent a possible offer distribution density function when more productive employers pay more since $F'(w) = \digamma'(x)/\omega'(x)$ and

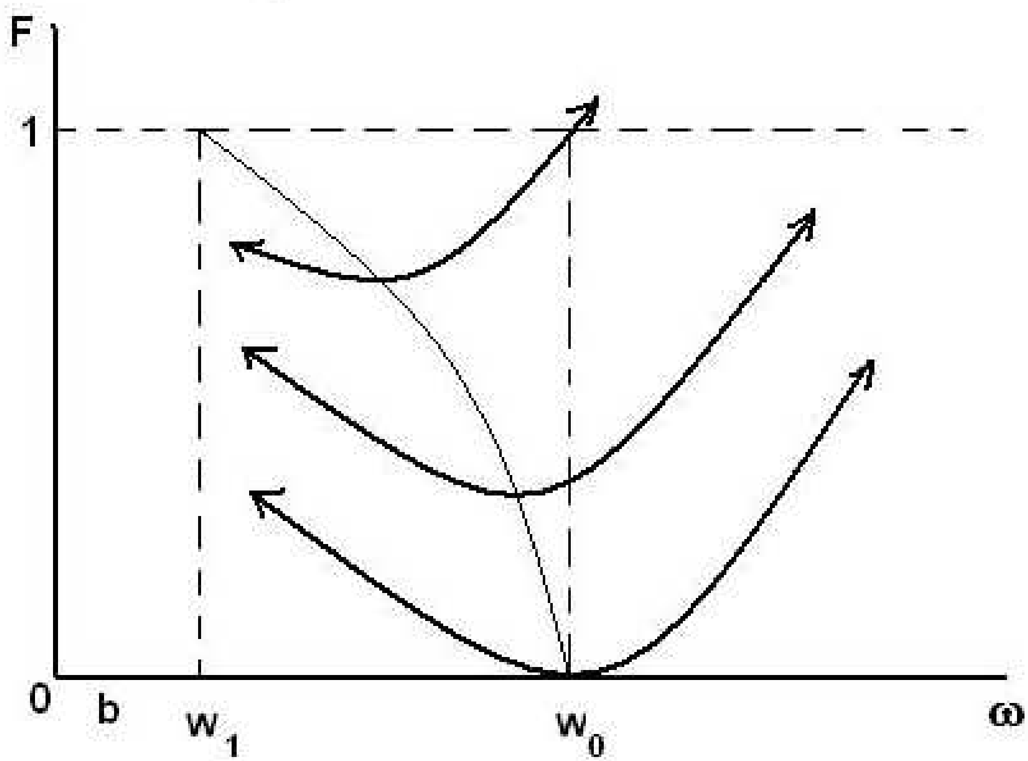

Figure 1: Phase Diagram

$\mathcal{F}'(x)$ must both be p.d.f.s. Finally, for each candidate, the upper support of the distribution, $\overline{w}$, is the value of $\omega(x)$ at the point where the trajectory intersects the line $\mathcal{F} = 1$.

We need two initial conditions to determine a unique particular solution to the ODE system. Of course, equation (29) provides one; namely, $\omega(0) = \underline{w} = w_0$ from (29) and (35) and the other is $F'(\underline{w}) = 0$. Hence, the only candidate equilibrium is the increasing solution trajectory initiating at the point $(\omega, \mathcal{F}'(0)) = (\underline{w}, 0)$.

An equilibrium value of market tightness solves equation (34) when its RHS is evaluated using the unique candidate offer distribution. Obviously, equation (29) implies that the lowest wage of the candidate distribution increases with market tightness. Since $s(w)$ increases and $h(w)$ decreases with θ from (21) and (22) and $F'(w) = \mathcal{F}'(\omega(x))/\omega'(x) > 0$ imply that the slopes of the all the solution trajectories decrease continuously with θ. As a consequence, the wage offer distribution is stochastically increasing in θ which implies that the RHS of (34) is a positive, continuous and decreasing function of θ. Finally, the fact that the LHS is increasing in θ and zero when $\theta = 0$ implies that there is a unique solution for market tightness. ■

Corollary: If all firms are equally productive ($p(x) = p \forall x \in [0, 1]$), a unique dispersed wage steady state equilibrium exists.

Specifically, the case of identical firms is simply the limit of a sequence of equilibria generated as the productivity distribution, the inverse of $p(x)$, tends to a single mass point.

4 Comparing the Equilibria

As $h(w) < m(\theta)/\theta$ for all $w < \overline{w}$ from (21) when employed workers search, the lower support of the wage distribution in the disperse equilibrium is greater than the equilibrium single wage when evaluated at the same level of market tightness by equations (27) and (29). Workers employed at the lowest wage receive more rent when they search because the cost of replacing the marginal worker is higher given the same arrival rates. Because higher wages lower the incentive to post vacancies, the level of market tightness will be lower and unemployment will be higher in the disperse wage equilibrium than in the degenerate equilibrium. Indeed, an application of the argument used to prove existence of the dispersed wage equilibrium also implies that the following relationship between wages and employment in the two equilibria.

Proposition 3 *Market tightness is lower in a dispersed wage equilibrium than in the degenerate wage equilibrium and the equilibrium single wage is an element in the interior of the equilibrium wage offer distribution support.*

Proof. Suppose that employed worker search and for the moment consider the case in which value of market tightness in the degenerate equilibrium; call it θ^*. When $\theta = \theta^*$, $w_0 > w^*$ in Figure 1 which implies that all the wages in the support of the candidate distribution exceed w^*. As a consequence, the LHS of (34) is strictly greater than the RHS when $\theta = \theta^*$ which in turn implies that the unique value of θ in the disperse wage equilibrium is strictly less than θ^*. Obviously, an analogous argument implies that the value of θ for which the upper support of the wage distribution is equal to w^* is such that the LHS of (34) is strictly greater than the RHS. Hence, the support of the equilibrium wage distribution includes the single wage equilibrium in its interior. ■

Employment is lower in the disperse wage equilibrium because the cost of turnover per worker to the firm is higher. In addition, the act of hiring another worker reduces the wage paid which adversely affects the net change in employment attributable to posting a vacancy. Both effects reduce the incentive to post vacancies which results in lower market tightness and employment in steady state.

Does lower employment imply less welfare in the dispersed wage equilibrium? Not necessarily. As Stole and Zwiebel (1996) point out in a particular equilibrium setting, their bilateral intra-firm bargaining solution provides an incentive for employers to "over employ" as a means of driving down wage costs. Cahuc, Marque, and Wasmer (2007) suggest that this conclusion continues to hold in search equilibrium when firms are composed of many workers and firms are equally productive. Below I extend the inefficiency result to include the case of heterogenous firms. Hence, wage dispersion and the additional search it induces off sets to some extent the incentive to over employ.

Suppose that the planner chooses vacancies for every firm to maximize the expected present value of aggregate income including home production. As is well known, search externalities generally exist in a matching model. However, as both Pissarides (1984) and Hosios (1990) have shown, these are internalized in the canonical search and matching model if the employers' share of match rent is equal to the elasticity of the matching function with respect to vacancies. In this section, I demonstrate that employment in the single wage equilibrium exceeds the solution to the planner's problem when the Hosios condition holds and production exhibits diminishing returns. In addition to excessive employment, workers are generally misallocated across firms as well when firm productivities differ.

The law of motion for employment in firm x is

$$\dot{n}(x) = v(x)q(\theta) - \delta n(x), x \in [0,1]$$

where $v(x)$ represents vacancies posted and θ is market tightness, the ratio of aggregate vacancies to the number of unemployed workers. Letting Ψ represent the maximal expected value of future aggregate income, the continuous time Bellman equation is

$$r\Psi = \max_{v(x),n(x),\theta} \left\{ \begin{array}{c} \int_0^1 p(x)f(n(x))dx \\ +b\left(1-\int_0^1 n(x)dx\right) - c\int_0^1 v(x)dx \\ +\lambda\left[\theta\left(1-\int_0^1 n(x)dx\right) - \int_0^1 v(x)dx\right] \\ +\int_0^1 \Psi_{n(x)}\left[v(x)q\left(\theta\right) - \delta n(x)\right]dx \end{array} \right\} \tag{36}$$

where Ψ is the value of the optimal program, $\Psi_{n(x)}$ is its partial derivative with respect to $n(x)$, and λ is the shadow price of market tightness. The FONCs for interior vacancy choices are

$$\begin{aligned} \Psi_{n(x)}q(\theta) - \lambda - c &= 0 \ \forall x \in [0,1] \\ \lambda\left(1-\int_0^1 n(x)dx\right) + \int_0^1 \Psi_{n(x)}v(x)q'\left(\theta\right)dx &= 0 \\ \theta\left(1-\int_0^1 n(x)dx\right) - \int_0^1 v(x)dx &= 0 \end{aligned}$$

where

$$(r+\delta)\,\Psi_{n(x)} = p(x)f'(n(x)) - b - \lambda\theta + \dot{\Psi}_{n(x)}$$

by the envelope theorem.

In steady state, the marginal value of a worker is the same across firms. Letting $\Psi_{n(x)} = \Psi_n$ for all x, market tightness satisfies

$$\Psi_n\left[q(\theta) + \theta q'(\theta)\right] - c = 0 \tag{37}$$

where the common marginal value of a worker solves

$$(r+\delta)\,\Psi_n = p(x)f'(n(x)) - b - \left(\frac{\theta q'(\theta)}{q(\theta)+\theta q'(\theta)}\right)c\theta, x \in [0,1] \tag{38}$$

and total employment satisfies

$$\frac{m(\theta)}{\delta + m(\theta)} = \frac{\theta q(\theta)\ell}{\delta + \theta q(\theta)} = \int_0^1 n(w, p(x))dx. \tag{39}$$

However, in a degenerate wage market equilibrium, equations (6) and (27) require

$$\beta\left(p(x)f'(n(x)) + g(p(x), w) - w\right) = (1-\beta)\left(w - b - \frac{c\beta\theta}{1-\beta}\right)$$

where

$$pf'(n(w, p(x))) + g(w, p(x)) - w = \frac{(r+\delta)c}{q(\theta)}$$

which together imply

$$w = b + \frac{\beta}{1-\beta}\left[\frac{(r+\delta)\,c}{q(\theta)} + c\theta\right]$$

In combination, these equations imply

$$p(x)f'(n(x))) + g(w, p(x)) - b - \frac{\beta}{1-\beta}c\theta = \frac{(r+\delta)c}{(1-\beta)q(\theta)}, x \in [0,1] \tag{40}$$

while equation (37) and (38) require

$$p(x)f'(n(x)) - b - \left(\frac{\theta q'(\theta)}{q(\theta)+\theta q'(\theta)}\right)c\theta = \left(\frac{q(\theta)}{q(\theta)+\theta q'(\theta)}\right)\frac{(r+\delta)c}{q(\theta)}, x \in [0,1] \tag{41}$$

Given the Hosios condition

$$1 - \beta = 1 + \frac{\theta q'(\theta)}{q(\theta)} = \frac{\theta m'(\theta)}{m(\theta)}, \tag{42}$$

intra-firm bargaining generates two distortions. First, because the marginal contribution to profit, $p(x)f'(n(x)) + g(w, p(x))$, is equalized across firms in the degenerate wage equilibrium rather than the marginal products, the allocation of workers across firms is not output maximizing except in the Cobb-Douglas production function case where $g(w, p(x))$ is independent of $p(x)$. Second, because $g(w, p(x)) > 0$, employment in every firm is too high given tightness.

Proposition 4 *If the Hosios condition holds, then the value of market tightness in the degenerate wage equilibrium exceeds that in the planner's solution.*

Proof. By comparison of (40) and (41), the solution for $n(x)$ using the first equation is larger than that for the second for all w and $p(x)$ given that $g(w, p(x)) > 0$ contingent on the same value of θ. Hence, if the value of θ is chosen to be the solution to the planner's problem, the RHS of (39) exceeds the left in the single wage equilibrium. Hence, the equilibrium value of θ, that which equates to side, must be larger. ■

Although I did not allow for search on-the-job in formulating the social planner's problem, this restriction is not binding when the vacancy posting cost is linear. In steady state at least, the planner has no incentive to reallocated workers across firms because marginal products are equal. However, because employment is too high in the single wage equilibrium and the allocation of employment is sub-optimal except in the case of a Cobb-Douglas production function, it is possible that the dispersed equilibrium with search-on-the job yields higher welfare than the single wage equilibrium.

5 Concluding Remarks

The purpose of this paper is to show that a dispersed wage steady state equilibrium exists in a version of the search and matching model in which firms have many employees, face diminishing returns in production, and wages are the outcome of intra-firm bargaining as modeled by Stole and Zwiebel (1996). Helpman and Itskhoki (2008) establish that a unique single wage equilibrium exists in this environment. In this paper, I prove that a unique dispersed wage equilibrium also exists with the property that more productive firms pay higher wages because employed workers will search. I also find that, employment is lower in the dispersed wage equilibrium but welfare need not be because employment exceeds the social optimum. Excessive employment arises in the single wage equilibrium because employing another worker reduces the wage paid to all employees as originally pointed out by Stole and Zwiebel. Employment is higher in the dispersed wage equilibrium because search by employed workers reduces each employer's incentive to create jobs.

The obvious unanswered question has to do with stability of equilibrium. Will the market converge to the single wage or to the disperse wage steady state equilibrium? When the market is not in steady state, the nature of the wage equation implies that wages will differ and, consequently, employed worker will search. Is this fact sufficient to guarantee continued search in the transition and hence convergence of the market distribution to the non-degenerate steady state? Answering this question is a subject for future theoretical research.

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