Competing Contests with Reimbursement

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Abstract

This paper explores the impact of reimbursement as a policy tool when two contests compete with one another for contestant effort. Whether or not winners and/or losers are reimbursed by either contest in equilibrium depends on the number of contestants, the amount of effort they have available to allocate, the curvature of the contest success function (sensitivity to effort), and the size of the two prizes being offered. Depending on all those factors together, equilibria featuring no reimbursement at all, winner-only reimbursement, and the reimbursement of both winners and losers are all possible. Losers, however, are not reimbursed at all unless winners are fully reimbursed. It is also possible for only one contest to reimburse its participants while the other does not, in which case the contest that does reimburse is always the one that offers the smaller prize value. Finally, the competition between the two contests is often a Prisoner's Dilemma – both would be better off in a state of zero reimbursement, but nevertheless do reimburse in Nash equilibrium.

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1 Introduction

Contests are useful models of competition via effort provision. Political lobbying, litigation, R&D races, and employee promotion schemes are all examples of such scenarios. More generally, contests embody any setting in which scarce prizes are awarded to contenders based, at least in part, on their expenditure of resources.

Much of the theoretical literature on contests rightly focuses on how much effort contestants provide in equilibrium. Critical to that determination are basic structural aspects such as the size of the prize(s) involved, the number of contestants, and the curvature of the contest success function (see Corchón (2007) for a survey of results). Knowledge of how these and other features affect equilibrium effort is obviously of great importance to those who wish to implement a contest environment for various goals.

Recent work has revealed that *reimbursements*, payments made to compensate contestants for their efforts, can be an especially useful policy tool for contest designers. In particular, Matros and Armanios (2009) show that, depending on the designer's goals, it may be best to fully reimburse the expenditures of either the winner or the losers of a contest. Another work, by Cohen and Sela (2005), shows that if contestants have asymmetric valuations, fully reimbursing the winner can lead to equilibria in which a weaker contestant is more likely to win.

This paper is concerned with contests that compete with one another. As chronicled by the *Economist* magazine (2010), cash prizes are becoming more and more popular, among both public and private organizations, as incentive devices to motivate researchers toward the pursuit of solutions to specific and often very difficult problems. The increased use of the contest format therefore offers potential problem-solvers a greater freedom of choice regarding where they will direct their efforts. This in turn begs the question of how the design of a contest will affect the amount of effort directed at it when other alternatives are available.

Azmat and Möller (2009) study a model of competing contests, focusing on how sensitivity to effort (the curvature of the contest success function) affects the optimal prize structure when two contests compete for participation. They find that more sensitive contests are better suited to flatter multi-prize structures, while less sensitive contests are better suited to steeper (for example, winner-take-all) prize structures. Here we are similarly concerned with two competing contests, but we take the prize structure as fixed and instead focus on reimbursements as contest designers' key strategic variables.

We model a pool of contestants, possibly interpreted as competing researchers, each of whom must allocate a fixed amount of effort between two contests, possibly interpreted as alternative research projects. One project yields a bigger reward, perhaps in terms of prestige, so in the absence of any reimbursement, researchers will spend more time pursuing that goal. However, if the other project partially or fully reimburses expenditures, researchers will change their division of effort accordingly. Each contest designer must therefore consider the other contest's reimbursement policies in addition to contestants' equilibrium behavior when making their own decision of how much to reimburse. This concurrent reimbursement decision by the contest designers is the (Nash) equilibrium we are most interested in in this paper.

Whether or not winners and/or losers are reimbursed by either contest in equilibrium depends on the number of contestants, the amount of effort they have available to allocate, the curvature of the contest success function (sensitivity to effort), and the size of the two prizes being offered. Intuitively, a greater number of contestants with more effort available to allocate leads to higher levels of reimbursement. Equilibrium levels of reimbursement also decline with sensitivity, which is in line with the results of Azmat and Möller (2009), who find that the size of the winner's prize should decline with sensitivity to effort. Reimbursements are also greater the greater the discrepancy between the two prize values.

Equilibria featuring no reimbursement at all, winner-only reimbursement, and the reimbursement of both winners and losers are all possible. The reimbursement of losers only never occurs in equilibrium, however. In fact, losers are not reimbursed at all unless winners are fully reimbursed. This in turn means that losers are never fully reimbursed. In any equilibrium, the weaker contest (the one offering the smaller prize value) always reimburses more than the stronger contest, due to the fact that contestants otherwise direct the majority of their effort to the contest with the larger prize value. The weaker contest therefore always finds it worthwhile to win back any extra effort the stronger contest attempts to procure with its own reimbursement, since it can do so at a lower cost. If only one of two contests reimburses positively in equilibrium, then it is always the one with the smaller prize value.

Finally, the ultimate consequences of reimbursement on the contest designers' payoffs (total expenditures of effort by contestants net of reimbursement) are worth noting. The stronger of the two contests is always worse off in any equilibrium featuring positive levels of reimbursement, relative to the state in which neither contest reimburses. The weaker contest can be better off, for example if it does reimburse effort while the stronger does not. In many circumstances, then, the competition between the two contest designers resembles a Prisoner's Dilemma; though both end up reimbursing to some degree in equilibrium, both would have higher payoffs if neither reimbursed at all.

2 Model

Consider a group of n symmetric contestants competing in two asymmetric contests, contest 1 and contest 2. The prizes for winning each contest are V_1 and V_2, respectively, with $V_1 > V_2 > 0$, and are allocated according to the well-known Tullock (1980) contest success function. In addition to those prizes, each contest may reimburse the efforts of participants to some extent. Following Matros and Armanios (2009) and Baye et al. (2005) we assume that reimbursements are linear functions of individual effort. Winners of contest i are reimbursed α_i of their effort, while losers of contest i are reimbursed β_i of their effort, $i = 1, 2$. Also following Matros and Armanios (2009), we assume that individual reimbursements do not exceed individual effort ($0 \leq \alpha_i \leq 1$ and $0 \leq \beta_i \leq 1$, $i = 1, 2$).

Each contestant has a fixed amount of effort, $\bar{x} > 0$, to allocate between the two contests. This is an important assumption, since if contestants were entirely unconstrained, the two contests would not really have to compete with one another, and the analysis and conclusions would be identical to those of Matros and Armanios (2009) for optimal reimbursement in a single, stand-alone contest. Hence, the objective of each contestant is to exhaust $\bar{x}$ to maximize their payoffs from expected prize winnings and reimbursement payments. In the context of the competing researchers interpretation mentioned in the introduction, this is akin to the assumption that all researchers will always try their hardest in terms of total effort because it has no use outside of the competing activities. The question is simply how they will divide those total efforts among the two possible projects. To ensure that contestants are at least compensated for their efforts in (symmetric) equilibrium, we assume $(V_1 + V_2)/n \geq \bar{x}$.[1]

To influence the decision of the contestants, contest designers are in charge of setting reimbursement parameters. Prize valuations are assumed fixed, perhaps because of perceptions of prestige, and the designers themselves value the prizes at zero. Our primary assumption regarding the designers' objective is that they seek to maximize the amount of effort directed toward their contest net of reimbursement payments, though we also briefly consider the possibility that they maximize total effort irrespective of reimbursement.

The model is thus a two-stage game with the two contest designers choosing their reimbursement levels simultaneously in the first stage. In the second stage, the n contestants simultaneously choose how to divide their effort between the two contests, and prizes are awarded according to the (Tullock) success function. Solving backward, we first consider the contestant's behavior given the reimbursement levels of the two contests.

[1] This assumption also helps to satisfy second-order conditions for the contestants' decision problem, though it is actually much more strict than we actually require.

3 Equilibrium

3.1 Contestant Behavior

Each contestant $i = 1, 2, \ldots, n$ is concerned with the following objective function, where x_i is the amount of effort expended in contest 1, $(\bar{x} - x_i)$ is the amount of effort expended in contest 2, and $0 < r \leq 1$ is the curvature of the contest success function.

$$\max_{x_i} \frac{x_i^r}{\sum_{j=1}^n x_j^r}(V_1 + \alpha_1 x_i) + \Big(1 - \frac{x_i^r}{\sum_{j=1}^n x_j^r}\Big)\beta_1 x_i \quad + \\ \frac{(\bar{x} - x_i)^r}{\sum_{j=1}^n (\bar{x} - x_j)^r}(V_2 + \alpha_2(\bar{x} - x_i)) \quad + \quad \Big(1 - \frac{(\bar{x} - x_i)^r}{\sum_{j=1}^n (\bar{x} - x_j)^r}\Big)\beta_2(\bar{x} - x_i)$$

The first-order condition for this problem is

$$\frac{r x_i^{r-1}(\sum_{j \neq i} x_j^r)}{(\sum_{j=1}^n x_j^r)^2}(V_1 + (\alpha_1 - \beta_1)x_i) + \frac{x_i^r}{\sum_{j=1}^n x_j^r}(\alpha_1 - \beta_1) + \beta_1 \\ - \frac{r(\bar{x} - x_i)^{r-1}(\sum_{j \neq i}(\bar{x} - x_j)^r)}{(\sum_{j=1}^n (\bar{x} - x_j)^r)^2}(V_2 + (\alpha_2 - \beta_2)(\bar{x} - x_i)) - \frac{(\bar{x} - x_i)^r}{\sum_{j=1}^n (\bar{x} - x_j)^r}(\alpha_2 - \beta_2) - \beta_2 \\ = 0. \quad (1)$$

In a symmetric equilibrium $x_i = x^*$ for all i, so (1) simplifies to

$$\frac{r(n-1)}{n^2 x^*}V_1 + \frac{r(n-1)+n}{n^2}(\alpha_1 - \beta_1) + \beta_1 = \frac{r(n-1)}{n^2(\bar{x} - x^*)}V_2 + \frac{r(n-1)+n}{n^2}(\alpha_2 - \beta_2) + \beta_2$$

or

$$\frac{(\bar{x} - x^*)r(n-1)}{n^2}V_1 + (\bar{x} - x^*)x^*\Big(\frac{r(n-1)+n}{n^2}(\alpha_1 - \beta_1 - \alpha_2 + \beta_2) + \beta_1 - \beta_2\Big) - \frac{x^* r(n-1)}{n^2}V_2 = 0. \quad (2)$$

If the middle term in (2) is zero, the solution is $x^* = \bar{x}\frac{V_1}{V_1+V_2}$. Otherwise, (2) is quadratic, and there are two potential solutions given by the classic

$$x^* = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a},$$

where

$$a = \frac{r(n-1)+n}{n^2}(\alpha_2 - \beta_2 - \alpha_1 + \beta_1) + \beta_2 - \beta_1, \quad (3)$$

$$b = \bar{x}(-a) - \frac{r(n-1)}{n^2}(V_1 + V_2), \quad (4)$$

and

$$c = \frac{r(n-1)}{n^2} V_1 \bar{x}. \tag{5}$$

Somewhat surprisingly, in spite of the above quadratic form, it turns out that the contestants' equilibrium does not suffer from problems of non-existence (due to complex components) or multiplicity.

Proposition 1. *The symmetric equilibrium effort level for contest 1 (and thus also for contest 2) exists and is unique regardless of parameter specifications. Specifically,*

$$x^*(\alpha_1, \beta_1, \alpha_2, \beta_2) = \begin{cases} \frac{V_1}{V_1+V_2}\bar{x} & \text{if } a = 0 \\ \frac{-b-\sqrt{b^2-4ac}}{2a} & \text{otherwise.} \end{cases} \tag{6}$$

Proof. See Appendix A. □

Appendix B verifies that these equilibrium effort levels respond in expected ways to changes in the reimbursement parameters. That is, any increase in α_1 or β_1 increases x^*, while an increase in α_2 or β_2 has the opposite effect. More reimbursement by one contest elicits more effort directed at that contest, and thus less directed at the other.

Important to recognize, however, is the role of the term a in contestants' equilibrium effort division. Rather than simply the absolute values of the reimbursement parameters, it is the way those parameters combine in a determines x^*. For $a < 0$ (which happens, for example, if $\alpha_1 > \alpha_2$ and $\beta_1 > \beta_2$), equilibrium efforts directed toward contest 1 are greater than they would be if both contests reimbursed at the same levels (which would make $a = 0$ and $x^* = \frac{V_1}{V_1+V_2}\bar{x}$). For $a > 0$, the opposite is true.

In fact, as depicted in Figure 1, it can be verified that regardless of model specifications, x^* follows a backwards-S shape when graphed against a. Most important for the analysis in the next section, however, is simply the monotonic nature of the relationship between the two.

3.2 Equilibrium Reimbursement

With the knowledge of contestants' (symmetric) equilibrium behavior, the contest designers each simultaneously choose the degree of reimbursement their contest will provide for winners and losers. Assuming the designers are interested in maximizing total expenditures net of

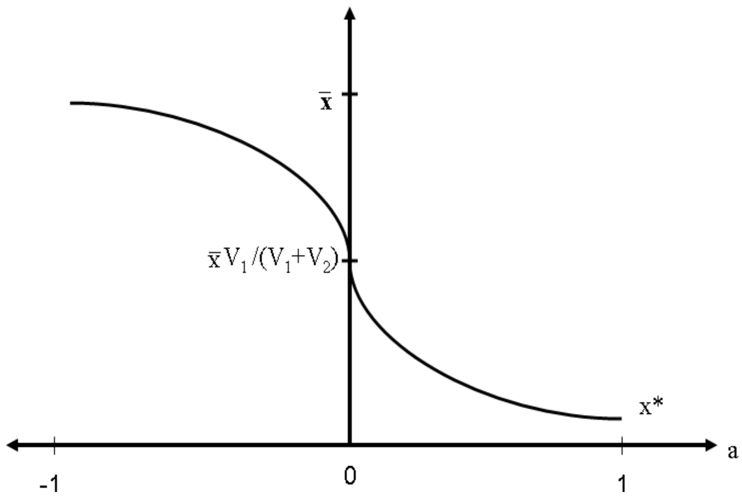

Figure 1: Contestants' symmetric equilibrium effort level directed at contest 1.

reimbursement payments (NT), their payoff functions are

$$NT_1(\alpha_1, \beta_1, \alpha_2, \beta_2) = (n - \alpha_1 - (n-1)\beta_1)x^*$$

and

$$NT_2(\alpha_1, \beta_1, \alpha_2, \beta_2) = (n - \alpha_2 - (n-1)\beta_2)(\bar{x} - x^*)$$

for contests 1 and 2 respectively, where the arguments of x^* are suppressed.

The (potential) discontinuity in contestants' effort levels makes finding analytical expressions for the contests' equilibrium reimbursement levels extremely difficult. It is not possible to simply solve four first-order conditions for four reaction functions. Nevertheless, we can be sure that a Nash equilibrium for the contests (and thus a subgame perfect Nash equilibrium for the entire game) does always exist in pure strategies.

Proposition 2. *A pure-strategy Nash equilibrium in reimbursement levels always exists.*

Proof. It is straightforward to verify that NT_1 and NT_2 are always either non-increasing or non-decreasing and then non-increasing in their own levels of reimbursement, and therefore quasiconcave. They also satisfy a condition known as *pseudocontinuity*, described by

Morgan and Scalzo (2007).[2] In turn, by Proposition 4.1 of Morgan and Scalzo (2007, p. 181), the game played by the two contests is *better-reply secure*, a condition defined by Reny (1999). Along with the compactness of the strategy space and quasiconcavity, this guarantees the existence of a pure-strategy Nash equilibrium (Reny, 1999, Theorem 3.1, p. 1033). □

Despite lacking closed-form solution, we can establish several key properties of the contests' reimbursement levels in any pure-strategy equilibrium. Specifically, we can be sure of how reimbursement by the stronger contest will compare to that of the weaker, and also how the reimbursement of winners will compare to that of losers. To those ends, the following result is crucial.

Proposition 3. *In any pure-strategy Nash equilibrium, $a \geq 0$, so that $x^* \leq \frac{V_1}{V_1+V_2}\bar{x}$.*

Proof. To prove that $a \geq 0$ in any pure-strategy Nash equilibrium, we show that any state of $a < 0$ can *not* be a Nash equilibrium, because it is always profitable for the second contest to at least match the first contest's reimbursement levels, thereby making $a \geq 0$.

Thus, suppose that $\{\alpha_1, \beta_1, \alpha_2, \beta_2\}$ such that $a < 0$ did constitute a Nash equilibrium. To be a Nash equilibrium, it would have to be the case that

$$NT_1(\alpha_1, \beta_1, \alpha_2, \beta_2) \geq NT_1(\alpha_2, \beta_2, \alpha_2, \beta_2). \tag{7}$$

Otherwise, it would be profitable for contest 1 to deviate to match the reimbursement levels of contest 2, thus making $a = 0$ and $x^* = \frac{V_1}{V_1+V_2}\bar{x}$. We now exploit the inequality in (7).

Let x' denote $x^*(\alpha_2, \beta_2, \alpha_2, \beta_2) = \bar{x}\frac{V_1}{V_1+V_2}$, contestants' symmetric equilibrium effort levels when both contest designers choose reimbursements $\{\alpha_2, \beta_2\}$. Also, let x denote $x^*(\alpha_1, \beta_1, \alpha_2, \beta_2) > x'$, where $\alpha_1 = \alpha_2 + \epsilon$ and $\beta_1 = \beta_2 + \gamma$ with ϵ and γ such that $a < 0$. $NT_1(\alpha_1, \beta_1, \alpha_2, \beta_2) \geq NT_1(\alpha_2, \beta_2, \alpha_2, \beta_2)$ implies

$$(n - (\alpha_2 + \epsilon) - (n-1)(\beta_2 + \gamma))\, x \geq (n - \alpha_2 - (n-1)\beta_2)\, x'.$$

[2] A real valued function f defined on a topological space Z is upper pseudocontinuous at $z_o \in Z$ if for all $z \in Z$ such that $f(z_o) < f(z)$, $\limsup_{y \to z_o} f(y) < f(z)$, lower pseudocontinuous at z_o if $-f$ is upper pseudocontinuous at z_0, and pseudocontinuous if it is both upper and lower pseudocontinuous for all $z_o \in Z$ (Morgan and Scalzo, 2007, Definition 2.1, p. 175). NT_1 and NT_2 satisfy this condition since x^* is increasing in α_1 and β_1 and decreasing in α_2 and β_2.

Rearranging, we can obtain

$$(n - \alpha_2 - (n-1)\beta_2)\,(x - x') \geq (\epsilon + (n-1)\gamma)x. \tag{8}$$

Comparing the states of $a < 0$ and $a = 0$, then, (8) shows the gain to the designer of contest 1 from additional contestant effort on the left-hand side and the loss from additional reimbursement payments on the right-hand side. But note that if the designer of contest 2 deviates from the supposed equilibrium to reimburse at the same levels as contest 1, its gain would would in fact be the same as the left-hand side of (8) since its share of contestants' effort would increase from $(\bar{x} - x)$ to $(\bar{x} - x') = \bar{x}\frac{V_2}{V_1+V_2}$. Its loss, however, would be strictly less than the right-hand side of (8) since $(\bar{x} - x') < x'$, since $V_2 < V_1$. The designer of contest 2 would therefore be made strictly better off by matching contest 1's reimbursement levels, contradicting the supposition of Nash equilibrium. □

In an important sense then, in equilibrium the designer of contest 2 always reimburses "more" than contest 1. Since the stronger contest receives a larger share of contestant effort when both contests reimburse at exactly the same level, its cost of any additional reimbursement will be greater than that of the weaker contest. Thus, if increasing effort share (x^*) is worthwhile for contest 1, it must also be worthwhile for contest 2 to win that share back, since it would mean the same gain but a lower added cost. This yields the following corollary, which itself is another important feature of equilibrium reimbursement.

Corollary. *In any pure-strategy Nash equilibrium, if only one contest reimburses positively, it will be the weaker contest 2.*

Proof. Follows directly from the fact that $a \geq 0$ in any equilibrium of the first stage. □

Beyond just being sure that $a \geq 0$ in equilibrium, we can actually be sure that contest 2 will reimburse all of its participants at least as much as the stronger contest 1. To show that, first we show that for either contest $i = 1, 2$, $\beta_i > 0$ only if $\alpha_i = 1$, again another important feature of equilibrium reimbursement.

Proposition 4. *In any pure-strategy Nash equilibrium, losers are reimbursed by a contest only if winners are fully reimbursed by that contest.*

Proof. The benefit from a contest reimbursing its participants comes from the increased effort they direct toward it. But, as previously emphasized, participants' effort depends on the relative reimbursement values as measured by a in (3). Either contest partially reimbursing its winners $\alpha_i < 1$ and its losers $\beta_i > 0$ could therefore always increase its net payoff by decreasing β_i and increasing α_i, keeping a exactly the same and thereby not changing the effort directed at it, but decreasing its total reimbursement payments since there are $n-1$ losers and only one winner. □

Combining Propositions 3 and 4, we have the following.

Proposition 5. *In any pure-strategy Nash equilibrium,* $\alpha_2 \geq \alpha_1$ *and* $\beta_2 \geq \beta_1$.

Proof. Since $\beta_i > 0$ only if $\alpha_i = 1$ for $i = 1, 2$, there are four relevant cases to consider (all other cases trivially imply $\alpha_2 \geq \alpha_1$ and $\beta_2 \geq \beta_1$).
(i) $\alpha_1 > 0$, $\alpha_2 = 0$, $\beta_1 = 0$, $\beta_2 = 0$.
(ii) $\alpha_1 > 0$, $\alpha_2 > 0$, $\beta_1 = 0$, $\beta_2 = 0$.
(iii) $\alpha_1 = 1$, $\alpha_2 \geq 0$, $\beta_1 > 0$, $\beta_2 = 0$.
(iv) $\alpha_1 = 1$, $\alpha_2 = 1$, $\beta_1 > 0$, $\beta_2 > 0$.
Cases (i) and (iii) cannot be Nash equilibria since $a \geq 0$ by Proposition 3. Similarly, in case (ii), it must be that $\alpha_2 > \alpha_1$, and in case (iv), it must be that $\beta_2 \geq \beta_1$, again because $a \geq 0$. □

With these properties established, it is now prudent to make a few comments regarding the welfare of the two contest designers. First, since contestants' total effort is fixed, clearly the combined payoff of the two contests designers decreases with increased reimbursement by either side. Second, since the designer of contest 2 always reimburses participants at least as much as contest 1, the amount of effort directed at contest 1 in equilibrium is never more than it is when neither contest reimburses at all. Thus, the designer of contest 1 is always worse off in any equilibrium with positive levels of reimbursement relative to a state of zero reimbursement by both. The designer of contest 2, on the other hand, may be better off. In particular, contest 2 is always better off (relative to a state of zero reimbursement) in an equilibrium in which it reimburses positively but contest 1 does not. Finally, it is also possible that both contest designers are made worse off by reimbursing in equilibrium. For example, in any equilibrium in which the two designers reimburse contestants positively but equally, both are clearly worse off since they would receive the same amount of effort if neither reimbursed.

This leads us to conclude the section by acknowledging the possibility of contest designers having an alternative motivation. It is possible that contest designers may be interested in maximizing total effort directed toward their contest rather than effort net of reimbursement payments. Since increased reimbursement always affects the amount of effort directed towards a contest positively, the only equilibrium outcome of that scenario would be both contests fully reimbursing all participants. Welfare in that case would be the same as if neither contest reimbursed, since total efforts would be the same. Though not entirely uninteresting, we continue to focus for the remainder of the paper on the former motivation, since it admits a wider variety of outcomes and is potentially more realistic.

4 Determinants of Equilibrium Reimbursement

In this section we examine how the game's underlying structural environment (the size of the two prizes, the amount of contestant effort available, the number or contestants and the curvature of the contest success function) affects equilibrium reimbursements. Though we are unable to provide analytical solutions for the contests' reimbursement levels, we are able to establish necessary and sufficient conditions for positive levels of reimbursement to occur in equilibrium. These conditions then guide us in in exploring parameterized scenarios, computing and comparing equilibria.

4.1 Positive Reimbursement in Equilibrium

For $\alpha_1 = \beta_1 = \alpha_2 = \beta_2 = 0$ to *not* be part of an equilibrium, it must be the case that $NT_1(\alpha_1, \beta_1, 0, 0) \geq NT_1(0, 0, 0, 0)$ for some $\alpha_1 + \beta_1 > 0$, or $NT_2(0, 0, \alpha_2, \beta_2) \geq NT_2(0, 0, 0, 0)$ for some $\alpha_2 + \beta_2 > 0$, or both. Though we know from Proposition 3 that if any deviation from zero reimbursement is profitable for the stronger contest, it must also be so for the weaker one, it is instructive to consider first what it would require to make a deviation profitable for the designer of contest 1.

In a state of zero reimbursement, $a = 0$. A deviation to $\alpha_1 > 0$ (analogous reasoning holds if $\alpha_1 = 1$ and the deviation is to $\beta_1 > 0$) makes $a < 0$, thereby changing the symmetric equilibrium effort level directed at contest 1 according to (6). Specifically, x^* changes from $\bar{x}\frac{V_1}{V_1+V_2}$ to

$$\frac{-b - \sqrt{b^2 - 4ac}}{2a} = \frac{\bar{x}}{2} + \frac{(V_1 + V_2)}{2a}\frac{r(n-1)}{n^2} - \frac{\sqrt{\Delta}}{2a},$$

where a from now on indicates its new, non-zero value, and Δ is the discriminant, equal to $b^2 - 4ac = \bar{x}^2 a^2 + 2\bar{x}a(V_2 - V_1)\frac{r(n-1)}{n^2} + \left((V_1 + V_2)\frac{r(n-1)}{n^2}\right)^2$. For the deviation to be profitable for the designer of contest 1, it must be that

$$n\bar{x}\frac{V_1}{V_1+V_2} < (n-\alpha_1)\left(\frac{\bar{x}}{2}+\frac{(V_1+V_2)}{2a}\frac{r(n-1)}{n^2}-\frac{\sqrt{\Delta}}{2a}\right) \tag{9}$$

for some $\alpha_1 > 0$. Since $V_1 > V_2$ and $a < 0$, a necessary condition for (9) to hold is

$$\sqrt{\Delta} > (V_1+V_2)\frac{r(n-1)}{n^2}. \tag{10}$$

In turn, this requires that

$$\bar{x}^2a^2 + 2\bar{x}a(V_2-V_1)\frac{r(n-1)}{n^2} > 0 \tag{11}$$

for some a resulting from increased reimbursement by contest 1.

On its own, (11) is not sufficient to ensure that (9) is satisfied. For sufficiency, the left-hand side of (10) must be large enough that the increased effort from contestants outweighs the loss from reimbursement. Thus, although the inequality in (11) is always satisfied when $a < 0$ (when the designer of contest 1 is the deviator), that does not mean it is always profitable for contest 1 to reimburse if contest 2 does not. Rather, it requires the right combination of parameters. For example, (11) suggests that the more effort contestants have available to expend or the greater the number of contestants, the more likely it is that increased reimbursement (from a state of zero reimbursement) will be profitable for the designer of contest 1. Also, the left side of (9) suggests that a higher prize valuation for contest 1 makes it less likely that reimbursement will be profitable, *ceteris paribus*.

Similarly, for the designer of contest 2 deviating to $\alpha_2 > 0$ from an initial state of zero reimbursement,

$$n\bar{x}\frac{V_2}{V_1+V_2} < (n-\alpha_2)\left(\frac{\bar{x}}{2}-\frac{(V_1+V_2)}{2a}\frac{r(n-1)}{n^2}+\frac{\sqrt{\Delta}}{2a}\right) \tag{12}$$

must be satisfied for the deviation to be profitable. Since $V_2 < V_1$, although $a > 0$ the inequality in (10) is no longer necessary for (12) to hold. Nevertheless, the larger the left-hand side of (10), the more likely it is that contest 2 will find it worthwhile to positively reimburse, so the intuition from (11) also applies to contest 2. In fact, we can be a bit more specific by substituting for a

$$\bar{x}\alpha_2 > 2(V_1-V_2)\frac{r(n-1)}{r(n-1)+n}, \tag{13}$$

again assuming that the deviation is $\alpha_2 > 0$ while all other reimbursement parameters are zero.

Just as the left-hand side of (9) suggests that a higher valuation for contest 1's prize makes it less worthwhile to reimburse from a state of zero reimbursement, the left-hand side (12) suggests the same relationship holds for the size of contest 2's prize and its incentive to reimburse, *ceteris paribus*. And as does (11), (13) suggests that the more effort contestants have available to expend or the greater the number of contestants, the more likely it is that increased reimbursement (from a state of zero reimbursement) will be profitable for contest 2. But (13) also suggests that reimbursement is less likely to be profitable when the r parameter is higher, that is, when contestants' winning probabilities are more sensitive to individual effort. This is an interesting aspect to the model which we will elaborate upon in just a moment, in the next subsection.

As expected from Proposition 3, of course, (12) is a weaker condition than (9). The designer of the stronger contest will never have an incentive to deviate from zero reimbursement if the weaker contest does not, and so (12) is the necessary and sufficient condition for at least some reimbursement to occur in equilibrium. Though the same parameter relations and combinations enhance the incentive to reimburse from a state of zero reimbursement for both contest designers, the threshold is much lower for the designer of the weaker contest.

More generally, since $a = 0$ whenever the two contests have identical reimbursement policies, the deviation conditions (9) and (12), and the parameter relations implied by (11) and (13) apply more broadly. The same factors that make deviation from a zero-reimbursement state more profitable also make changes from any state of $a = 0$ more profitable, though when both contests are already reimbursing positively the maximum possible change in a becomes smaller. Thus, in general, a greater amount of effort available for contestants to allocate, a greater the number of contestants, a lower the sensitivity to effort, and prizes that are closer in valuation are all important factors leading to higher levels of reimbursement in equilibrium.

4.2 Parameterized Examples

To support the implications of the previous subsection, here we compute Nash equilibrium reimbursement levels for parameterized scenarios. Though we can not obtain closed form expressions that would enable comparative-statics analysis, comparing equilibrium reimbursement levels as the parameters change relative to one another helps to illustrate the comparative-statics properties suggested by (13).

To compute equilibria for a given parameter setting we discretize the possible reim-

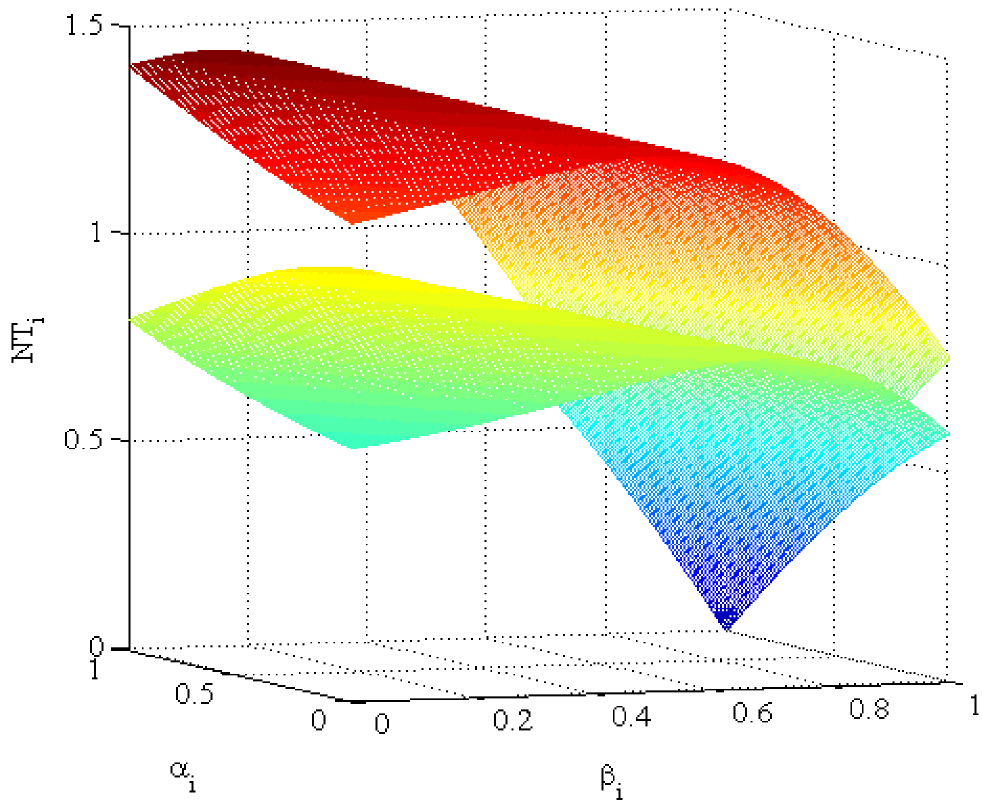

Figure 2: Contests' equilibrium payoff functions.

bursement levels to create an $\alpha \times \beta$ strategy space grid. We then find each contest's optimal choice of reimbursement parameters (those that maximize the contest's net total expenditures) while holding the other's choice constant, iterating back and forth until convergence. To check for multiple equilibria, we repeat the process for each specification of parameters several times with different initializations of reimbursement parameters, but multiplicity does not appear to be an issue.

To begin, we illustrate a case in which winners and losers are reimbursed by both contests in equilibrium: $V_1 = 4$, $V_2 = 2$, $\bar{x} = 1$, $n = 4$, and $r = 0.25$. The equilibrium reimbursements for that scenario are $\alpha_1 = 1$, $\beta_1 = 0.14$, $\alpha_2 = 1$, and $\beta_2 = 0.32$ (note that winners are fully reimbursed since losers receive some reimbursement). Figure 2 illustrates each contest's payoff as function of own reimbursement, holding the other's constant at the equilibrium values. The higher of the two is $NT_1(\alpha_1, \beta_1, 1, 0.32)$, while the lower is $NT_2(1, 0.14, \alpha_2, \beta_2)$.

Now we vary the parameter values to see how each alters equilibrium reimbursement. These comparisons are summarized by Table 1.

Table 1: Parameterizations and Corresponding Equilibrium Reimbursements.

$\{V_1,$	$V_2,$	$\bar{x},$	$n,$	$r\}$	$\{\alpha_1,$	$\beta_1,$	$\alpha_2,$	$\beta_2\}$
4	2	1	4	0.25	1	0.14	1	0.32
4	2	1	4	0.33	1	0	1	0.15
4	2	1	4	0.5	0.38	0	1	0
4	2	1	4	0.67	0	0	0.73	0
4	2	1	4	1	0	0	0.14	0
4	2	1.25	4	0.25	1	0.31	1	0.45
4	2	1.5	4	0.25	1	0.42	1	0.54
20	2	1.5	4	0.25	0	0	1	0.28
6	2	1.5	4	0.25	1	0.24	1	0.47
6	1	1.5	4	0.25	1	0.36	1	0.65
6	5	1.5	4	0.25	1	0	1	0.06
6	5	1.5	2	0.25	0.32	0	0.42	0
6	5	1.5	3	0.25	0.85	0	0.98	0
6	5	1.5	6	0.25	1	0.34	1	0.38
6	5	1.5	6	1	0	0	0	0

The first row of Table 1 lists the baseline case, also illustrated in Figure 2. The next four rows list the effects of increased sensitivity to effort, which exhibits a negative relationship with equilibrium reimbursement. Such an effect is predicted by (13) and the intuition for it is as follows. Increased r in the contest success function increases the productivity of individual effort in affecting the probability of winning a contest. With an increase in r, then, each contestant has stronger incentives to allocate effort toward affecting their chance of winning a contest, *ceteris paribus*, so each contest designer can relax their reimbursements.

The second group of (two) rows in Table 1 demonstrate the effect of an increase in the contestants' effort budget. As expected, reimbursements are greater in equilibrium when there is more total effort to compete for. And the fourth group of (three) rows in Table 1 similarly demonstrates that equilibrium reimbursements are greater with a larger number of contestants.

The third group of (four) rows in Table 1 demonstrates changes in the contest prize valuations. Again, as expected from (9) and (12), an increase in the value of a contest's prize diminishes its own incentive to reimburse. Not entirely expected, though (13) does hint at it, is that an increase in either prize's value leads to lower equilibrium reimbursement for both contests. The effect seems larger for the contest whose valuation actually changes, especially in the case of V_1, but both exhibit different levels of equilibrium reimbursement with a change in just one of the prize values.

The last row in Table 1 simply exhibits a case in which there is no equilibrium reimbursement at all.

5 Discussion

Our results should be of interest to all those looking to incentivize efforts by awarding prizes, but particularly those who recognize that the presence of other prizes diverts the attention of contestants. As Matros and Armanios (2009) show, reimbursement payments can be an important policy tool for any contest. But our work emphasizes that this is especially true when multiple contests compete, albeit with somewhat different outcomes.

Whereas Matros and Armanios (2009) find a continuum of equilibria for a stand-alone contest looking to maximize net total spending (featuring full reimbursement for winners and an arbitrary level of reimbursement for losers), this is not the case when contests compete with one another and account for each other's reimbursements as well as their own. We find that although there are scenarios in which reimbursing winners fully and losers partially can be optimal for a contest, such equilibria tend to be unique. And positive levels of reimbursement, even for just winners, are not always in a contest's best interests. There are instances in which neither contest will reimburse at all. While it is true that reimbursing losers is never optimal unless winners are fully reimbursed, it is also possible that the weaker contest may reimburse both winners and losers while the stronger one does not.

Accordingly, designers of contests that offer relatively small prizes amidst a field of others benefit most from the use of reimbursements as a policy tool. Designers of such contests may be able to use reimbursements to increase the amount of net spending directed their way, although not always. If the stronger contest also reimburses, both contest designers may ultimately end up worse off. It is crucial for both to be aware of the presence of reimbursement payments, however; even though the stronger contest is always worse off in equilibria featuring reimbursement, reimbursing its own participants can serve an important retaliatory purpose.

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Appendix A: Proof of proposition 1

To begin the proof of proposition 1, recall the definitions $a = \frac{r(n-1)+n}{n^2}(\alpha_2 - \beta_2 - \alpha_1 + \beta_1) + \beta_2 - \beta_1$, $b = \bar{x}(-a) - \frac{r(n-1)}{n^2}(V_1 + V_2)$, and $c = \frac{r(n-1)}{n^2}V_1\bar{x}$. Then note that if $a = 0$ (which happens, for example, if $\alpha_1 = \alpha_2$ and $\beta_1 = \beta_2$), the first order condition in (4) simplifies to

$$\frac{(\bar{x}-x)r(n-1)}{n^2}V_1 - \frac{xr(n-1)}{n^2}V_2 = 0,$$

which, has a unique solution of $x^* = \frac{V_1}{V_1+V_2}\bar{x}$.

If $a \neq 0$, the relevant solution candidates are given by (5). To establish that an equilibrium will always exist in these cases, it is prudent to show that the discriminant (Δ) is always positive so that the potential solutions are always real.

Lemma A1. $\Delta = b^2 - 4ac > 0$.

Proof. Since $b^2 > 0$ and $c > 0$, if $a < 0$ then clearly $\Delta > 0$. Assume then that $a > 0$, so that

$$\Delta = \bar{x}^2a^2 + 2\bar{x}a(V_2 - V_1)\frac{r(n-1)}{n^2} + [(V_1 + V_2)\frac{r(n-1)}{n^2}]^2$$

$\Delta > 0$ can therefore be rearranged as

$$n^2\bar{x}^2a^2 + 2\bar{x}aV_2r(n-1) + \frac{[(V_1+V_2)r(n-1)]^2}{n^2} > 2\bar{x}aV_1r(n-1)$$

$$\Leftrightarrow \frac{n^2\bar{x}a}{V_1r(n-1)} + \frac{V_2}{V_1} + \frac{(V_1+V_2)^2r(n-1)}{2\bar{x}an^2V_1} > 1$$

$$\Leftrightarrow \frac{n^2\bar{x}a}{V_1r(n-1)} + \frac{V_2}{V_1} + \frac{V_1r(n-1)}{2\bar{x}an^2} + \frac{V_2r(n-1)}{\bar{x}an^2} + \frac{V_2^2r(n-1)}{2\bar{x}an^2V_1} > 1 \tag{14}$$

To see that (14) must hold, it is sufficient to consider three cases, noting that all terms on the left side of (14) are positive. First, if $n^2\bar{x}a \geq V_1r(n-1)$, then the first term in (14) alone is greater than one. Second, if $2n^2\bar{x}a \leq V_1r(n-1)$, then the third term alone is greater than one. Finally, if $n^2\bar{x}a < V_1r(n-1) < 2n^2\bar{x}a$, then the first and third terms together are greater than one.$\square$

Next it is shown that one of the solution candidates, $\frac{-b+\sqrt{b^2-4ac}}{2a}$, is never feasible. That is, it is always either less than zero or greater than the budget of $\bar{x}$.

Lemma A2. $x^+ = \frac{-b+\sqrt{b^2-4ac}}{2a}$ is not feasible. If $a < 0$, $x^+ < 0$, and if $a > 0$, $x^+ > \bar{x}$.

Proof. If $a < 0$, $\sqrt{\Delta} > |b|$. Thus, whether $-b$ is positive or not, $x^+ < 0$. If $a > 0$, it is helpful to substitute in for $-b$ in the first part of x^+ to obtain

$$x^+ = \frac{\bar{x}}{2} + (V_1 + V_2)\frac{r(n-1)}{2an^2} + \frac{\sqrt{\Delta}}{2a}.$$

The last part of A2 is then evident since $x^+ > \bar{x}$

$$\Leftrightarrow \frac{\sqrt{\Delta}}{2a} > \frac{\bar{x}}{2} - (V_1 + V_2)\frac{r(n-1)}{2an^2}$$

$$\Leftrightarrow \sqrt{\Delta} > \bar{x}a - (V_1 + V_2)\frac{r(n-1)}{n^2}$$

$$\Leftrightarrow \Delta > \bar{x}^2 a^2 - 2\bar{x}a(V_1 + V_2)\frac{r(n-1)}{n^2} + \left[(V_1 + V_2)\frac{r(n-1)}{n^2}\right]^2,$$

which holds since

$$\Delta = \bar{x}^2 a^2 + 2\bar{x}a(V_2 - V_1)\frac{r(n-1)}{n^2} + \left[(V_1 + V_2)\frac{r(n-1)}{n^2}\right]^2.$$

□

Finally, it is shown that the remaining candidate, x^-, is always feasible. That is, regardless of parameter specifications, $0 \leq x^- \leq \bar{x}$.

Lemma A3. $x^- = \frac{-b-\sqrt{b^2-4ac}}{2a}$ is always feasible.

Proof. If $a < 0$, clearly $x^- > 0$ since $\sqrt{\Delta} > |b|$. If $a > 0$, on the other hand, $\sqrt{\Delta} < |b|$, but x^- is still greater than zero because $\frac{-b}{2a} > 0$.

To show that $x^- < \bar{x}$, again substitute for $-b$ to obtain $x^- < \bar{x}$

$$\Leftrightarrow -\frac{\sqrt{\Delta}}{2a} < \frac{\bar{x}}{2} - (V_1 + V_2)\frac{r(n-1)}{2an^2},$$

which simplifies as in the proof of lemma A2. Substituting for the terms in Δ, $x^- < \bar{x} \Leftrightarrow$

$$-(\bar{x}^2a^2+2\bar{x}a(V_2-V_1)\frac{r(n-1)}{n^2}+\left[(V_1+V_2)\frac{r(n-1)}{n^2}\right]^2) < \bar{x}^2a^2-2\bar{x}a(V_2+V_1)\frac{r(n-1)}{n^2}+\left[(V_1+V_2)\frac{r(n-1)}{n^2}\right]^2$$

$$\Leftrightarrow -2\Big(\bar{x}^2a^2 + \left[(V_1 + V_2)\frac{r(n-1)}{n^2}\right]^2\Big) < -4\bar{x}aV_1\frac{r(n-1)}{n^2}. \qquad (15)$$

When $a < 0$ the inequality in (15) clearly holds because the term on the left side is negative while the term on the right is positive. When $a > 0$, (15) still holds, though it is not as obvious. One more step is helpful to turn (15) into

$$\bar{x}^2 a^2 + \left[(V_1 + V_2)\frac{r(n-1)}{n^2}\right]^2 > 2\bar{x}aV_1\frac{r(n-1)}{n^2},$$

which can be shown to hold in the same way that (14) holds. □

To complete the proof, we now verify that the second-order conditions are satisfied. The second-order condition at the symmetric equilibrium x is

$$\frac{(r-1)n^2 - 2nr}{n^2x^2}(V_1+(\alpha_1-\beta_1)x)+\frac{2}{x}(\alpha_1-\beta_1)+\frac{(r-1)n^2-2nr}{n^2(\bar{x}-x)^2}(V_2+(\alpha_2-\beta_2)(\bar{x}-x))+\frac{2}{(\bar{x}-x)}(\alpha_2-\beta_2) \leq 0$$

or

$$\begin{aligned} &2(\alpha_1-\beta_1)(\bar{x}-x) + 2(\alpha_2-\beta_2)x \leq \\ &\frac{(1-r)n^2+2nr}{n^2x}(V_1+(\alpha_1-\beta_1)x)(\bar{x}-x) + \frac{(1-r)n^2+2nr}{n^2(\bar{x}-x)}(V_2+(\alpha_2-\beta_2)(\bar{x}-x))x \end{aligned} \quad (16)$$

Since reimbursements can not exceed individual effort, the left side of (16) is at most $2\bar{x}$. Similarly, the right side of (16) is at least

$$((1-r)n^2+2nr)\left(V_1\frac{(\bar{x}-x)}{n^2x} + V_2\frac{x}{n^2(\bar{x}-x)}\right),$$

making (16) more than satisfied as long as

$$2\bar{x} \leq ((1-r)n+2r)\left(V_1\frac{(\bar{x}-x)}{nx} + V_2\frac{x}{n(\bar{x}-x)}\right) \quad (17)$$

Given that $0 < r \leq 1$, $n \geq 2$ and $V_1 > V_2$, and that $x \geq \frac{V_1}{V_1+V_2}$ in equilibrium, (17) is satisfied so long as $\bar{x} \leq (V_1+V_2)/n$, as was assumed at the beginning of section 2.
This concludes the proof of proposition 1.

Appendix B: The effect of increased reimbursement on equilibrium effort levels

Here we establish that a contest increasing the amount of reimbursement it offers leads to increased effort levels directed at that contest. Though we only show the case of changes in α_1 or β_1 explicitly, the analysis for the second contest's variables is straightforwardly analogous.

The effect of a change in reimbursement variables depends on the variables' initial values. Specifically, whether or not $a = 0$ to begin with. If $a \neq 0$, a differential approach is possible. For $\gamma \in \{\alpha_1, \beta_1\}$, and letting a' stand for $\partial a/\partial \gamma$,

$$\frac{\partial x^*}{\partial \gamma} = \frac{-2a'(V_1+V_2)\frac{r(n-1)}{n^2} - 2a\frac{1}{2}\Delta^{-1/2}\left(2\bar{x}^2 aa' + 2\bar{x}a'(V_2-V_1)\frac{r(n-1)}{n^2}\right) + 2a'\Delta^{1/2}}{4a^2} > 0$$

$$\Leftrightarrow a'\left[-(V_1+V_2)\frac{r(n-1)}{n^2} - a^2\Delta^{-1/2}\bar{x}^2 - a\Delta^{-1/2}\bar{x}(V_2-V_1)\frac{r(n-1)}{n^2} + \Delta^{1/2}\right] > 0$$

$$\Leftrightarrow -\Delta^{1/2}(V_1+V_2)\frac{r(n-1)}{n^2} + \bar{x}a(V_2-V_1)\frac{r(n-1)}{n^2} + \left[(V_1+V_2)\frac{r(n-1)}{n^2}\right]^2 < 0$$

where the last step comes from dividing by a' and multiplying by $\Delta^{1/2}$, and the inequality flip occurs because $a' < 0$.

Now just rearrange so that

$$\bar{x}a(V_2-V_1)\frac{r(n-1)}{n^2} + \left[(V_1+V_2)\frac{r(n-1)}{n^2}\right]^2 < \Delta^{1/2}(V_1+V_2)\frac{r(n-1)}{n^2}.$$

Squaring both sides and substituting for Δ then simplifies to

$$\bar{x}^2a^2\left[(V_2-V_1)\frac{r(n-1)}{n^2}\right]^2 < \bar{x}^2a^2\left[(V_1+V_2)\frac{r(n-1)}{n^2}\right]^2$$

or

$$(V_2-V_1) < (V_1+V_2).$$

If a change in reimbursement occurs from an initial scenario with $a = 0$, the initial x^* is $\bar{x}\frac{V_1}{V_1+V_2}$. An increase in α_1 or β_1 then changes the equilibrium effort level to

$$\frac{-b-\sqrt{b^2-4ac}}{2a} = \frac{\bar{x}}{2} + \frac{(V_1+V_2)}{2a}\frac{r(n-1)}{n^2} - \frac{\sqrt{\Delta}}{2a}$$

, where a from now on indicates the new, non-zero value, which is negative. A bit of algebra shows that this new x^* is indeed greater than the original.

$$\frac{\bar{x}}{2} + \frac{(V_1+V_2)}{2a}\frac{r(n-1)}{n^2} - \frac{\sqrt{\Delta}}{2a} > \bar{x}\frac{V_1}{V_1+V_2}$$

$$\Leftrightarrow -2a\bar{x}\frac{V_1}{V_1+V_2} + (V_1+V_2)\frac{r(n-1)}{n^2} + a\bar{x} < \sqrt{\Delta}$$

$$\Leftrightarrow (2a\bar{x}\frac{V_1}{V_1+V_2})^2 - 4a\bar{x}V_1\frac{r(n-1)}{n^2} - 4a^2\bar{x}^2\frac{V_1}{V_1+V_2} + ((V_1+V_2)\frac{r(n-1)}{n^2})^2 + 2a\bar{x}(V_1+V_2)\frac{r(n-1)}{n^2} + \bar{x}^2a^2 < \Delta$$

Substituting for Δ,

$$(2a\bar{x}\frac{V_1}{V_1+V_2})^2 - 4a\bar{x}V_1\frac{r(n-1)}{n^2} - 4a^2\bar{x}^2\frac{V_1}{V_1+V_2} < -4a\bar{x}V_1\frac{r(n-1)}{n^2}$$

$$\Leftrightarrow (2a\bar{x}\frac{V_1}{V_1+V_2})^2 < 4a^2\bar{x}^2\frac{V_1}{V_1+V_2}$$

$$\Leftrightarrow (\frac{V_1}{V_1+V_2})^2 < \frac{V_1}{V_1+V_2}.$$