Inefficient Investment Waves*

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January 2012

Abstract

We propose a dynamic model of investment and trade in a market of a specialized technology subject to two main frictions. First, agents cannot raise outside capital. Second, a random group of agents will have the opportunity to invest in new technology and these opportunities are not contractible. The first friction implies the presence of invesment cycles with abundant invesment and low returns in booms and little invesment and high returns in recessions. Only when the second friction is present invesment cycles are constrained inefficient. Often the inefficiency is two-sided with too much invesment in booms and too little in recessions from a social point of view. Interventions targetting only the underinvesment in recessions might make all agents worse off. Also, the two-sided inefficiency typically implies too volatile prices and too frequent realizations of abnormally low prices compared to fundamentals.

Key Words: Pecuniary externality, overinvestment and underinvestment, market intervention, Greenspan's put

1 Introduction

The history of modern economies is rich in boom and bust patterns. Boom periods with vast resources invested in new projects and low expected returns can abruptly turn into recessions when long-run projects are liquidated early, liquid resources are hoarded in safe short-term assets and there is little investment in new projects even if expected returns are high. Figure 1, showing the AAA corporate bond spread and the net percentage of banks tightening credit conditions and increasing spread on new loans, illustrates these investment cycles.

The financial crisis at the end of 2000s brought such investment cycles into the forefront of the academic and policy debate. Can these cycles be caused by financing frictions only? When is the investment cycle a sign of inefficiency? If it is, which part of the cycle is inefficient: is there

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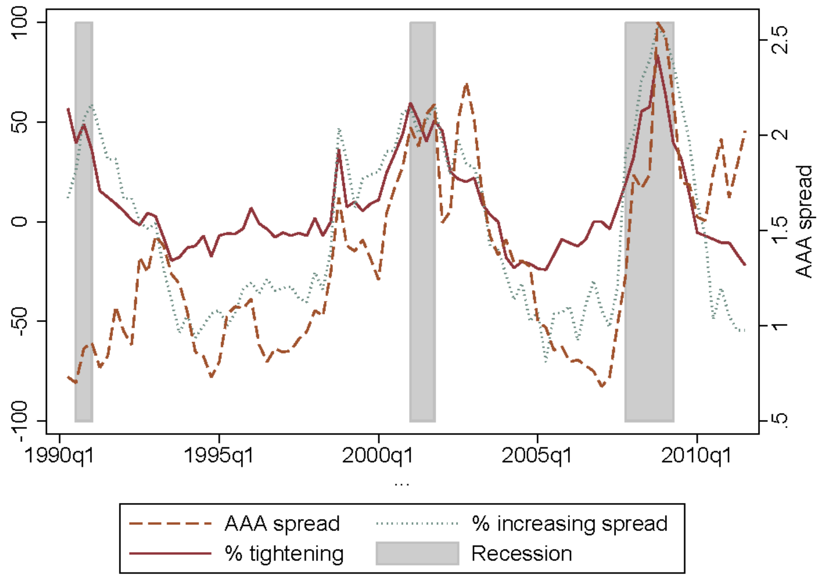

Figure 1: Credit conditions, corporate spread and recessions in the US. The solid and dashed lines show the net percentage of senior loan officers tightening lending standards and increasing spread compared to bank's cost on comercial and industrial loans to large and mid-cap firms. (Source: Survey of Senior Loan Officers, Federal Reserve.) The red curve shows Moody's AAA corporate bond spread over the 10-year treasuries. (Source: FRED.) The shaded areas are NBER recessions.

overinvestment in booms and/or underinvestment in recession? Relatedly, should the policy maker intervene in booms, in recessions or both?

In this paper, we contribute to this debate as follows. First, with the help of a parsimonious dynamic model of investment and trade, we show that in an economy where capital to finance risky projects is limited in some states, constrained efficient investment cycles arise naturally. Thus, cycles generated by financing frictions are not a sign of inefficiency per se. Second, we show that uninsurable idiosyncratic shocks regarding agents' relative valuation of available assets may induce a two-sided inefficiency. That is, this friction causes overinvestment in booms with high asset prices and underinvestment in recessions with low asset prices. As a mirror image, agents store too little liquid resources in booms and hoard too much of them in recessions. Third, we show that intervention targeted to raise prices in recessions to avoid underinvestment typically make overinvestment in booms worse. What is more, this adverse effect might be so strong that the intervention becomes Pareto inferior compared to the case of no intervention at all.

We present a simple, stochastic dynamic model of investment and trade where asset prices are endogenous. There are two assets; trees and cash. The economy is populated by specialists, who are the only agents who can operate projects in a representative technology. These projects are represented by the trees. Specialists can create new trees at a fixed cost, or liquidate the trees for a small fixed benefit both in terms of cash. Specialists can also trade trees among each other at the equilibrium price. At a random time each tree "matures", i.e., pays a one-time dividend and the world ends. Before trees mature, sometimes they generates interim cash flows, but other times they require further interim investment otherwise they have to be sold or liquidated.

The basic friction in our economy is that specialists cannot raise outside funds for the interim (or any other type of) investment. Thus, they store some cash in order to avoid inefficient liquidation of the project. The second friction in our economy is that specialists are subject to an idiosyncratic shock. Namely, at some point investment opportunity into a new technological innovation arrives, but it is available only for a subset of the specialists. These specialists sell their trees to the rest and invest all their cash into the new opportunity.

In equilibrium, the aggregate cash-to-tree ratio serves as our single state variable. It is also our proxy for the level of aggregate liquidity in our economy. When interim shocks are negative, the cash-to-tree ratio falls, and so does the equilibrium price of trees raising the expected return on buying trees. When the price drops to the level of the liquidation benefit, trees are converted back to cash keeping the cash-to-tree ratio above an endogenous lower threshold. We think of low liquidity states as a recession. Expected return is high in a recession because specialists have to be compensated for the increasing probability that they will be forced into inefficient liquidation when no cash will be available for interim investment. We refer to this as a liquidity premium. As the cash-to-tree ratio rises, this risk is reduced and the price of the trees increases and the premium decreases. When the price reaches the cost of creating new trees, specialists create new trees keeping the cash-to-tree ratio below an endogenous upper threshold. We think of the high liquidity state when new projects are created as a boom period. The focus of our analysis is whether the investment and disinvestment thresholds are at their efficient levels.

In the complete market benchmark, the market solution and the social planner's choice coincide. In this case, specialists liquidate trees only when the cash-to-tree ratio hits zero. They build trees when the cash-to-tree ratio hit a positive investment threshold. This threshold is determined by a trade-off. On one side, building trees is a positive net present value project. On the other side storing some cash is necessary to avoid costly liquidation when interim investment is required. Still, expected returns and economic activities fluctuate with the cash-to-tree ratio just as in the incomplete market equilibrium. The investment cycle is not a sign of inefficiencies per se.

However, in the market solution, the investment and disinvestment thresholds are distorted. In particular, specialists always liquidate trees at a positive cash-to-tree ratio. Also, under some conditions, they build trees at a lower threshold than the social planner would. That is, they invest too little (liquidate too much) in recessions, and overinvest in booms. As a mirror image, they hoard too much cash in a recession, and hold too little cash in a boom.

The intuition behind our mechanism is as follows. The cash-in-the-market pricing implies that the ex post trading price is high when the cash-to-tree ratio is high, i.e., when the economy is flooded with liquidity. The ex post trading price serves two roles. The first role is to move all cash (or trees) to the most efficient hands; the fact that a high price when the cash-to-tree ratio is high helps achieve this goal. However, the trading price also affects the rent distribution between tree holders and cash holders, therefore not fully reflecting the marginal rate of substitution from the social perspective. This second detrimental role drives the wedge between the private marginal rate of substitution between tree and cash and the social planner's, and distorts the individual agent's investment incentives ex ante in equilibrium. Interestingly, the direction of price distortion depends on the state of the economy. It is so, because in a boom, the aggregate cash is high, therefore the price at which specialists with the new investment opportunity can sell their trees is high. Thus, the private value of trees is higher than the social value of trees in booms. This is a pecuniary externality inducing overinvestment in trees in booms. As a symmetric argument, in recessions the price of the tree is low, inducing a negative wedge between the private and the social value of trees. This implies even lower prices and underinvestment in trees in recessions.

As we explain, our model can be applied to the boom and bust pattern in construction and house prices. Our mechanism suggests that the volume of real estate development in a boom is inefficiently high,[1] because investors build houses instead of holding liquid financial assets expecting to be able to sell the real estate for a high price in case they find a new investment opportunity. In recessions, investors hold inefficiently high level of liquid assets expecting to be able to buy real estate cheap in case a group of distressed investors have to liquidate their holdings.

The dynamic structure of our model emphasizes that there is a two-way interaction between decisions in booms and recessions. When a specialists decide to build a tree, she takes into account that this tree will have to be liquidated if the state of the economy deteriorates significantly. When she liquidates a tree in the recession, she similarly takes into account that the economy might revert to a boom. As a result of this interaction, if the policy maker taxes cash-holdings in a recession to raise the price and avoid inefficient liquidation, this one-sided intervention will typically decrease the investment threshold and make the overinvestment problem worse in the boom. Also, agents expected utility in a recession will naturally respond to the effect of the intervention in the boom. Thus, intervention in the recession, while effective in the recession, often makes all agents worse off.

Our paper also gives predictions on the different distribution of prices under complete and incomplete markets. Before the tree matures, the cash-to-tree ratio follows a uniform ergodic distribution regardless whether the constrained efficiency is achieved. When the market incompleteness implies that the disinvestment threshold is too high and the investment threshold is too low, the support of the distribution of the cash-to-tree ratio is compressed. In contrast, the asset price has a stationary distribution with a U-shaped density function regardless whether the constrained efficiency is achieved. That is, while the distribution of the cash-to-tree ratio is uniform by construction, the economy spends more time with very high and very low prices than in between. These effects is stronger with incomplete markets. We also find that the price distribution under market incompleteness is typically first-order stochastically dominated by its counterpart in economies with complete markets. These two observations imply that the consequences of our

[1] One suggestive sign of the inefficiently high level of real estate development is the frequently observed phenomenon of "overbuilding" (e.g. Wheaton and Torto, 1990; Grenadier, 1996), that is, periods of construction booms in the face of rising vacancies and plumetting demand.

two-sided inefficiency are volatile prices, more frequent high realization of the liquidity premium, and larger liquidity premium in average than under complete markets.

As a methodological contribution, we propose a novel way to analyze the effect of aggregate liquidity fluctuations on asset prices and real activity. While our model is fully dynamic, it provides analytical tractability for the full joint distribution states and equilibrium objects.

Literature. To our knowledge, our paper is the first to show that the simple friction of unverifiable idiosyncratic investment opportunities causes overinvestment in booms and underinvestment in recessions.

Our work belongs to a growing literature analyzing pecuniary externalities in incomplete markets. All this literature, including our paper, builds on the result in Geanakoplos and Polemarchakis (1985) that when markets are incomplete, the competitive equilibrium may be constrained inefficient. In this setting pecuniary externalities can have a first order effect, because prices fail to equate the marginal rate of substitution of each agent across all goods. A large stream in this literature achieves this effect by emphasizing a fire-sale feed-back loop induced, typically, by a collateral constraint (e.g. Kiyotaki and Moore, 1997; Krishnamurthy, 2003; Gromb and Vayanos, 2002; Stein, 2011; Jeanne and Korinek, 2010; Bianchi, 2010; Bianchi and Mendoza, 2011; Lorenzoni, 2008; Hart and Zingales, 2011). In these papers agents do not take into account that the more they invest ex ante, the more they have to liquidate when they hit their constraint which reduces fire-sale prices tightening the constraint and amplifying the effect. Our paper does not rely in such an amplification mechanism. Instead, our paper follows the tradition of Shleifer and Vishny (1992), Allen and Gale (1994, 2004, 2005); Caballero and Krishnamurthy (2001, 2003) and Gale and Yorulmazer (2011) where an uninsurable shock creates the dispersion in marginal rate of substitution of ex-ante identical firms.[2] Our main point of departure is that in our paper, the Geanakoplos and Polemarchakis (1985) mechanism is interacted with a theory of countercyclical liquidity premium, resulting in distortions of opposite directions in booms and recessions.[3]

A group of recent papers investigating the moral hazard problem of incentivizing banks in a macroeconomic context derive related implications to our work. In particular, our result that one-sided interventions can be inferior to no interventions is related to the debate on the pros and cons of asymmetric interest rate policy often referred to as the Greenspan's put. In their recent work, Farhi and Tirole (2011) and Diamond and Rajan (2011) argue that supporting distressed institutions by low interest rates is detrimental to ex ante incentives of financial intermediaries and encourage their excessive risk-taking ex ante. As a result, ex post intervention to save distressed institutions will be needed more often. Similar to our work, in Gersbach and Rochet (2011) banks extend too much credit in booms and too little in recessions. Their mechanism relies on the difference between the private and social solution of bank's moral hazard problem. Namely, if private benefits of banks are increasing in the size of their loans, then it is cheaper to make them exert effort by letting

[2] See Holmstrom and Tirole (2011, chap. 7.) for simplified versions and excellent discussion of Shleifer and Vishny (1992) and Caballero and Krishnamurthy (2003).

[3] See Davila (2011) for an excellent comparative analysis of the different type of fire-sale externalities explored in the literature.

them to increase their loan size in booms compared to paying them sufficient rent to avoid this. This private contract does not take into account the price effect of the resulting procyclicality in aggregate loan size. In contrast to this literature, agency frictions and related incentive problems for financial intermediaries are not central in our argument. Instead, our mechanism is based on the novel observation that incomplete might imply that the price of the productive asset is biased in the opposite direction in a boom and in a recession. Thus, whatever policy helps in a boom will typically make agents worse off in a recession and vice-versa. Ex ante welfare in any state is the weighted average of these effects.

From a methodological point of view, as a continuous time model with investment and trade, the closest paper to ours is Brunnermeier and Sannikov (2011). As their focus is balance sheet amplification, their model is more complex, but not analytically tractable.

The structure of our paper is as follows. Section 2 gives an simple static example to highlight the main intuition. In Section 3 we present our model, and analyze the market equilibrium and the constrained efficient allocations of the social planner. In Section 4 we expose the inefficiencies of the market solution, and Section 5 considers an alternative more natural shock specification. Finally, we conclude.

2 A simple example

Before we move on to set up our general model, we first illustrate the main insights of our paper by a simple example with the following 2-date-2-good economy.

Endowment and goods. At the beginning of date 0 each agent i of the unit mass of risk-neutral agents hold one unit of apple and c units of coconuts.

Transformation technology. At date 0, each agent can adjust their fruit-basket by transforming coconuts to apples or the other way around. The technology is such that each agent can convert h coconuts to an apple, or an apple to l coconuts where $h > l$. Thus, given the endowment of one unit of apple and c units of coconuts, the individual holding of $\left(A^i, C^i\right)$ at the end of date 0 must satisfy

$$\begin{aligned} hA^i + C^i &= h + c \quad \text{if } A^i > 1, \\ lA^i + C^i &= l + c \quad \text{if } A^i \leq 1. \end{aligned} \tag{1}$$

This budget constraint reflects the kinked transformation technology.

Preference and shocks. At date 0, each agent is identical in their preferences. However, at the beginning of date 1, half of the agents turn out to like only apples with a marginal utility of $R > 0$, while the rest of the agents like only coconuts with a marginal utility of $u > 0$. After these idiosyncratic shocks, agents can trade apples to coconuts with each other. As becoming clear later, we assume that $R/u \in (l, h)$.

The market solution. Recall that $\left(A^i, C^i\right)$ describe the holdings after the adjustment in date 0 but before the trade in date 1, and the aggregate counterpart $A = \int A^i di$ and $C = \int C^i di$. Given the structure of the idiosyncratic shock, it is clear that agents are happy to trade in all the fruits they dislike for the fruits they like. Thus, in a competitive market, for each unit of apple each agent will be able to obtain $p = C/A$ coconuts in date 1. For this given price, each agent solves

$$\max_{A^i, C^i} J^i\left(A^i, C^i; p\right) = \frac{1}{2}\left(A^i + \frac{C^i}{p}\right) R + \frac{1}{2}\left(A^i p + C^i\right) u$$

subject to the budget constraint in (1). Given the simple linear structure, the individual demand function is

$$\begin{array}{cc} A^i = \frac{c+h}{h}, C^i = 0 & \text{if } p > h; \\ A^i = 1, C^i = c & \text{if } l \leq p \leq h; \\ A^i = 0, C^i = c + l & \text{if } p < l. \end{array} \tag{2}$$

This is intuitive: individual agents should hold the fruit whose relative price is higher than the marginal rate of transformation, and given the kink in the transformation technology inaction may be optimal.

We can derive the unique symmetric market equilibrium by combining individual demand functions (2) with the equilibrium condition

$$\frac{C^i}{A^i} = \frac{C}{A} = p. \tag{3}$$

It is apparent that the equilibrium price p has to lie between the interval $[l, h]$. We characterize market equilibria based on the relative initial coconut endowment c.

Case 1 *Suppose $c > h$ so that the initial coconut endowment is relatively high. Then the market equilibrium has $p = h$, and individual agents invest in apples to reach the holdings of*

$$A^i = 1 + \frac{c-h}{2h} > 1, C^i = c - \frac{c-h}{2} < c.$$

Case 2 *Suppose $c < l$ so that the initial coconut endowment is relatively low. Then the market equilibrium has $p = l$, and individual agents invest in coconuts to reach the holdings of*

$$A^i = 1 - \frac{l-c}{2l} < 1, C^i = c + \frac{l-c}{2} > c.$$

Case 3 *Otherwise, when $c \in [l, h]$, the market equilibrium has $p = c$, and individual agents do not invest so that*

$$A^i = 1, C^i = c.$$

Social planner's problem and inefficiency. The planner maximizes the sum of utilities of agents. The only difference between the planner and the market is that the market takes prices as given, while the social planner takes into account how individual decisions determine prices. Thus, we can write the problem of the planner as

$$\max_{A,C} \frac{1}{2}\left(A+\frac{C}{\frac{C}{A}}\right)R+\frac{1}{2}\left(A\frac{C}{A}+C\right)u=\max_{A,C} AR+Cu$$

subject to the aggregate budget constraint similar to (1):

$$\begin{aligned} hA+C &= h+c \quad \text{if } A>C, \\ lA+C &= l+c \quad \text{if } A\leq C. \end{aligned} \tag{4}$$

Importantly, the date 1 price does not play any role in social planner's problem. Recall the parameter restriction that $\frac{R}{u}\in(l,h)$, then the optimal solution is simply the endowment allocation

$$A=1, C=c. \tag{5}$$

Intuitively, R/u gives the marginal rate of substitution for social welfare. If it lies between the two marginal rates of transformation, then it is socially wasteful to transform one fruit to the other. However, as shown in the market solution, individual agents overinvest in apples (underinvest in coconuts) when the initial endowment of coconuts is relatively high, while overinvest in coconuts (thus underinvest in apples) when the initial endowment of coconuts is relatively low.

Intuition and discussion. Let us highlight the main lessons from this example. In general inefficiency could be due to inefficient resource allocations; but it is not the case here, as the date 1 trading ensures the efficient resource allocation among ex post heterogeneous agents. In words, all agents who like apples eat all the apples and all agents who like coconuts eat all the coconuts. In fact, under both the planner's solution and the market one, given the fixed pairs of (A,C) the representative agent obtains

$$\int\left[\frac{1}{2}\left(A^i+\frac{C^i}{p}\right)R+\frac{1}{2}\left(A^ip+C^i\right)u\right]di=AR+Cu.$$

In fact in our model the inefficiency arises purely due to the divergent ex ante private incentives to transform fruits compared to the one of the social planner.

To capture divergent investment incentives, we can study the marginal rate of substitution for both the social planner and individual agents. The social planner's value, given the pair of apple-coconut holdings (A,C), is simply given by

$$J^S(A,C)=AR+uC,$$

and therefore the social planner's marginal rate of substitution (MRS^S) between apple and coconut is

$$MRS^S=\frac{J_A^S(A,C)}{J_C^S(A,C)}=\frac{R}{u}, \tag{6}$$

a constant independent of date 1 market price p. In contrast, the private value of the pair of apple-coconut holdings $\left(A^i, C^i\right)$, given the price p, is

$$J^i\left(A^i, C^i; p = \frac{C}{A}\right) = \frac{R}{2}\left(A^i + \frac{A}{C}C^i\right) + \frac{u}{2}\left(\frac{C}{A}A^i + C^i\right) = \frac{R}{2}\left(A^i + \frac{C^i}{p}\right) + \frac{u}{2}\left(pA^i + C^i\right).$$

Thus, from the perspective of individual price-taking agents the marginal rate of substitution between apple and coconut is

$$MRS^i = \frac{J^i_{A^i}\left(A^i, C^i; p = \frac{C}{A}\right)}{J^i_{C^i}\left(A^i, C^i; p = \frac{C}{A}\right)} = \frac{\frac{1}{2}\left(R + pu\right)}{\frac{1}{2}\left(\frac{1}{p}R + u\right)} = p. \tag{7}$$

This is exactly the ex post price p!

Interestingly, there is a wedge between the social planner's marginal rate of substitution R/u and that of individual agents p. The economic force behind this wedge is as follows. Although the ex post (date 1) trading guarantees the efficient resource allocation, it introduces the distribution of economic rents in a way that in general distorts the individual agent's ex ante (date 0) private marginal rate of substitution. To see this, consider the pair of trading agents so that one likes coconut but holds apple, and the other likes apple but holds coconut. The apple in the coconut agent's hand delivers him a utility of pu, while the coconut in the apple agent's hand delivers R/p (check (7)). Because the social planner's marginal rate of substitution R/u should not take the rent distribution into account, ex post trading price leads to distortion in ex ante incentives and causes a pecuniary externality. Moreover, the ex post price $p = C/A$ depends on the relative abundance of between apples and coconuts. When the private marginal rate of substitution p is higher (lower) than R/u, most of the rent from holding coconuts (apples) goes to the agents holding apples (coconuts), and thus compared to the social planner holding apples (coconuts) becomes more attractive than holding coconuts (apples).

Finally, note that in resolving the inefficiency, the social planner does need to identify (or make agents to reveal) which agent is hit by which idiosyncratic shock, as ex post trading will implement the efficient allocation of fruits. It is sufficient for the social planner to control the ex-ante investment decision.

The above intuition of welfare changing pecuniary externalities drives our main results in our general model, where we consider the choice between a more productive good called tree (instead of apples), and a less productive good called cash (instead of coconuts).[4] In our full dynamic stochastic model with cash flow shocks, instead of keeping the combination of (A, C) at a fixed ratio, both the planner and market participants will find it optimal to choose a (different) non-degenerate distribution of the A to C by transforming one good to the other only occasionally. Thus, whether the market over- or underinvests in trees depends on the realized state of the A to C ratio. This ratio is determined by the particular history of the cash-flow shock of trees, even if the efficiency of the tree remains the same.

[4] One important difference is that by replacing coconuts with cash, even the agent likes apple ex post should be able to consume cash with marginal utility of 1, which places the bound of date 1 price $p \leq R$.

3 The Model

3.1 Assets

We model a market for risky projects. These projects are represented by trees. The only other asset in the economy is fruits which we call cash; both cash and tree are perfectly storable. All trees mature at the same random time arriving with Poisson intensity ξ. A matured tree pays a single cash-dividend R or 0 as we specify below. However, before the trees mature, each tree might provide some positive cash payout, or it might require maintenance so that the cash payout is negative. This shock is common across trees and driven by σdZ_t, where $Z = \{Z_t, \mathcal{F}_t; 0 \leq t < \infty\}$ is a standard Brownian-motion on a complete probability space $(\Omega, \mathcal{F}, \mathcal{P})$. When $\sigma dZ_t > 0$ the tree provide cash; when $\sigma dZ_t < 0$, the owner of the tree has to invest this amount to the tree and otherwise the tree dies. Denote by A_t the aggregate quantity of trees. Then given the aggregate cash shock σdZ_t to each unit of tree, the evolution of aggregate cash, without investment or disinvestment (to be introduced shortly), is

$$dC_t = A_t \sigma dZ_t. \tag{8}$$

3.2 Agents and frictions

The market is populated by a unit mass of risk neutral specialists who tend the trees. We think of specialists as a pooled set of bankers-entrepreneurs representing all the agents who understand the risky projects. At each time instant specialist may decide to plant new trees, trade trees for cash at the equilibrium price p_t, or liquidate the trees. Planting a new tree costs h units of cash, while liquidating the tree (selling it for firewood) provides l units of cash where $h > l$. This scraping technology ensures the limited liability of the asset owner despite the potentially unbounded losses in (8). Specialists can also consume their cash at any moment for a constant marginal utility of 1. Because of linear technologies, in general our model features threshold strategies for (dis)investment in both market equilibria and the social planner's problem. Thus, we can simply focus on thresholds to compare different (dis)investment strategies.

There are two major frictions in this economy. First, this economy is closed in the sense that there are no outsiders who could provide cash for specialists at any point under any arrangement (with the only exception of the unlimited buying of firewood for l). That is, outsiders do not have the human capital to tend the tree and future cash flows are not pledgeable. This extreme assumption is a short-cut for frictions (e.g., informational asymmetries and agency problems) which in general prevents outside investors to finance all positive NPV projects in the economy.[5]

Second, specialists are subject to non-verifiable idiosyncratic shocks. While specialists are ex ante identical, ex post they differ in their skills. Specifically, when the tree matures, it comes with another new technology. Half of the specialists have the skill to harvest the tree, but do not have the skill to operate the new technology. Thus, in their hands each tree pays R unit of cash, but the

[5] Were we to partially relax this assumption by allowing specialists to borrow a limited amount, our results would not change.

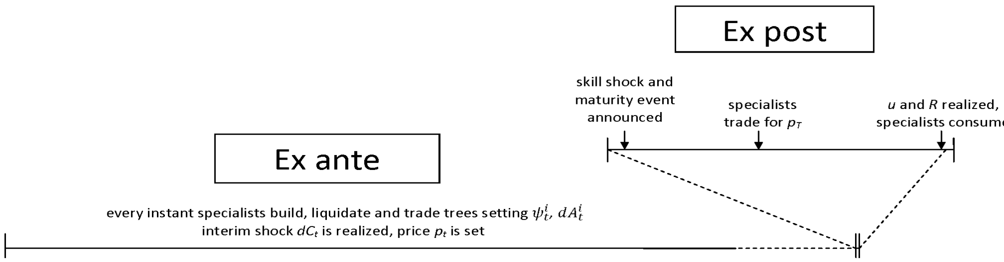

Figure 2: Time line.

new technology provides zero return. The other half of the specialists have the expertise to invest in the new technology, but do not have the skill to harvest the tree. In the hand of a specialist belonging to this group, each tree pays 0 dividend, but the new technology provides $u > 1$ cash return for every unit invested. This situation is analogous to the apple-coconut example studied in Section 2. Just as in that example, we only need that ex post there is heterogeneity among agents in their valuations of the available assets.[6] As before, we assume that the output from the new technology, i.e., u, are not pledgeable, an assumption we will discuss later in Section 4.2.2. Throughout we assume that

$$\frac{R}{h} > u, \tag{9}$$

which ensures that building trees is socially efficient when the economy has sufficient cash.

Specialists learn which group they belong to only at the moment right before the tree matures. We refer to the group with the skill to invest in the new technology as specialists hit by the idiosyncratic shock or *skill-shock*. After specialists learn whether they are hit by the shock, all specialists have a last trading opportunity to trade trees for cash. We refer to the potentially infinitely long time interval before the asset matures as ex ante, and refer to the infinitely short interval when the asset matures as ex post. We denote the ex post price by p_T which will be determined shortly.

Figure 2 summarizes the time-line of events in our model. While the dynamic structure of our model might seem unusual, we will argue that this structure unifies the advantages of two period models and infinite period models. In particular, this structure keeps the analysis analytically tractable, but still gives the opportunity for the analysis of the stationary distribution of ex ante variables. In Section 5 we consider an alternative specification where idiosyncratic skill shocks occur over time, and show that the main results still hold.

[6] A more intuitive assumption would be to allow for the arrival of the idiosyncratic shock in every instant. Under this alternative specification a given fraction of agents have to cash in and exit the market with marignal utility u in every instant. Then, our interpretation that the idiosyncratic shock is an idea for a new invesment opportunity is more natural. There are also other natural applications. For example, in a housing market context, the idiosyncratic shock is that an agent has to move to another city and sell his house. In that case the move provides a marginal utility of u.

Indeed, we analyze this alternative specification in Section 5.3 showing that our mechanism carries through. However,that structure is not analytically tractable. Also, our main specification makes our mechanism more transparent.

3.3 Agent's problem

Consider specialist i who holds A_t^i units of tree and C_t^i amount of cash, whose dollar value of wealth w_t^i is

$$w_t^i = p_t A_t^i + C_t^i.$$

Then the specialist i is solving the following problem:

$$\begin{aligned} & \max_{d\alpha_t^i \geq 0, A_t^i, C_t^i, dA_t^i} E\left[\int_0^\infty d\alpha_t^i\right] \qquad (10) \\ = & \max_{d\alpha_t^i, A_t^i, C_t^i, dA_t^i} E\left\{\int_0^\infty \xi e^{-\xi t}\left[\int_0^t d\alpha_u + \left[\frac{1}{2}\left(A_t^i + \frac{C_t^i}{p_T}\right)R + \frac{1}{2}\left(A_t^i p_T + C_t^i\right)u\right]\right] dt\right\}, \end{aligned}$$

where α_t^i is the specialist i's cumulative consumption (so it is non-decreasing with $d\alpha_t^i \geq 0$; later we see that it is zero in equilibrium), and dA_t^i is the amount of trees that he liquidates or build. In the first line the expectation is formed both respect to the aggregate shock Z_t and the Poisson event of the maturity of the trees. In the second line expectation is formed only with respect to the aggregate shock Z_t. In the second bracket of the second line, we also used the direct consequence of our assumptions that those hit by the skill-shock sell their trees for p_T to those who are not hit by the shock. For instance, when the skill shock hits, the specialist sells the tree to receive $A_t^i p_T$, and then invests them together with C_t^i in the new technology with productivity u.

The problem in (10) is subject to the dynamics of individual wealth,

$$dw_t^i = -d\alpha_t^i - \theta dA_t^i + A_t^i\left(dp_t + \sigma dZ_t\right),$$

where θ is the cost of changing the amount of trees so that[7]

$$\theta = \begin{cases} h & \text{if} \quad dA_t^i > 0 \\ l & \text{if} \quad dA_t^i < 0 \end{cases}.$$

Also, wealth cannot be negative at any point, i.e., $w_t^i \geq 0$ for all t.

Combining the investment/disinvestment policy dA_t, (8) implies that the dynamics of aggregate cash level in the economy is

$$dC_t = \sigma A_t dZ_t - \theta dA_t, \qquad (11)$$

where

$$A_t = \int_i A_t^i di.$$

The scale-invariance implied by the linear technology in our model suggests that it is sufficient to

[7]To simplify notation we ingore the possibility that at any given point in time some agents create trees while some agents liquidate trees. Hence, the lack of i superscript of θ. It will be easy to see that this never happens in equilibrium.

keep track of the dynamics of the cash-to-tree ratio $c_t \equiv C_t/A_t$, which follows

$$dc_t = \frac{dC_t}{A_t} - \frac{C_t}{A_t}\frac{dA_t}{A_t} = \sigma dZ_t - (\theta + c_t)\,\frac{dA_t}{A_t}. \tag{12}$$

3.4 Definition of Equilibrium

Our equilibrium concept is standard.

Definition 1 *In an incomplete market equilibrium*

1. *each specialist chooses $d\alpha_t^i, A_t^i, C_t^i, dA_t^i$ to solve (10), and*
2. *markets clear in every time instant both ex ante and ex post, i.e.,*

$$\int_i A_t^i di = A_t,\ \int_i C_t^i di = C_t,\ \text{and}\ \frac{1}{2}A_t p_T = \frac{1}{2}C_t.$$

As we will see, in our framework, the equilibrium only pins down the aggregate variables: prices, net trade, and net investment and disinvestment. Typically, any combination of individual actions consistent with the aggregate variables will be an equilibrium. Thus, often it will be convenient to pick the particular incomplete market equilibrium where all specialists follow the same action. We refer to this case as the symmetric equilibrium.

Definition 2 *A symmetric equilibrium is an incomplete market equilibrium where*

$$d\alpha_t^i = d\alpha_t,\ A_t^i = A_t,\ C_t^i = C_t,\ \text{and}\ dA_t^i = dA_t.$$

In the rest of the paper we omit the time subscript whenever it does not cause any confusion.

3.5 Incomplete Market Equilibrium

We solve for the incomplete market equilibrium in this section. It is clear that in this economy consumption before maturing event is strictly suboptimal, thus $d\alpha_t^i = d\alpha_t = 0$ always.

3.5.1 Ex post equilibrium prices

Let us start the analysis with the event when the trees mature. All specialists who are hit by the skill-shock sell their trees, because their marginal valuation of trees drop to zero. As long as the price of the tree is less than R, all cash holders who are not hit by the shock are happy to change all their cash to trees. Thus, appealing to the law of large numbers, the market clearing condition just before the asset matures is

$$\frac{1}{2}C = \frac{1}{2}Ap_T$$

implying $p_T = c$. This is an equilibrium price as long as $c < R$. As we will see, the full support of c will be endogenously determined in our model, as agent will build (dismantle) trees whenever the aggregate cash is sufficiently high (low). For simplicity, we will restrict the parameter space to ensure that the condition $c < R$ is satisfied for the full support of c that prevails in equilibrium.

3.5.2 Ex ante equilibrium values, prices, and investment polices

Before the maturity event, determining the equilibrium objects is more subtle. As we state in the next lemma, our formalization has a number of useful properties. Namely, the only relevant aggregate state variable is the cash-to-tree ratio, the value function is linear in trees and cash.

Lemma 1 *Let $J\left(C, A, A_t^i, C_t^i\right)$ the value function of specialist i. Then with aggregate cash-to-tree ratio $c = C/A$, there are functions $v(c)$ and $q(c)$ that ,*

$$J\left(C, A, A_t^i, C_t^i\right) = A_t^i v(c) + C_t^i q(c).$$

That is, regardless of the specialists portfolio, the value of every unit of tree is $v(c)$ and the value of every unit of cash is $q(c)$; both functions only depend on the aggregate cash-to-tree ratio and will be determined shortly. Because of linearity, the equilibrium price has to adjust in a way that specialists are indifferent whether to hold the tree or the cash. That is, the equilibrium price of tree $p(c)$ must satisfy that

$$p(c) = \frac{v(c)}{q(c)}.$$

Specialists build trees whenever the price of the asset p reaches the cost h, and liquidate trees whenever the price falls to the liquidation value l. Define c_h^* (c_l^*) as the endogenous threshold of the aggregate cash-to-tree ratio where specialists start to build (liquidate) trees, then we must have

$$\frac{v\left(c_h^*\right)}{q\left(c_h^*\right)} = h, \text{ and } \frac{v\left(c_l^*\right)}{q\left(c_l^*\right)} = l. \tag{13}$$

As building trees reduces the cash-to-tree ratio while liquidating trees increases it, this implies that c_h^* and c_l^* are reflective boundaries of the process c. Therefore, based on (12), the aggregate cash-to-tree ratio c must belong to the interval $[c_l^*, c_h^*]$, with a dynamics of

$$dc = \sigma dZ_t - dU_t + dL_t,$$

where $dU_t \equiv \left(h + c_h^*\right) \frac{dA_t}{A_t}$ reflects c at c_h^* from above while $dL_t \equiv \left(l + c_l^*\right) \frac{dA_t}{A_t}$ reflects c at c_l^* from below. Moreover, the standard properties of reflective boundaries imply the following smooth pasting conditions for our value functions:

$$v'\left(c_h^*\right) = q'\left(c_h^*\right) = q'\left(c_l^*\right) = v'\left(c_l^*\right) = 0. \tag{14}$$

3.5.3 Characterizing the incomplete market equilibrium

Now we turn to characterizing the value function in the range $c \in [c_l^*, c_h^*]$. We give here a draft and show the details in the Appendix. Because of Lemma 1, we can separately analyze how the value of holding a unit of the tree, $v(c)$, and the value of holding a unit of the cash $q(c)$ varies in the range $c \in [c_l^*, c_h^*]$ by the following steps. First, we write down the standard Hamiltonian for $J\left(C, A, A_t^i, C_t^i\right)$. Second, we conjecture and verify that $q(c) > 1$ holds, so that specialists do not consume before the asset matures. Finally, given the indifference among specialists in the composition of their portfolios we consider the dynamics of the value function of an agent who holds only tree and another agent with cash only. The former gives the ODE for $q(c)$:

$$0 = \frac{\sigma^2}{2} q'' + \frac{\xi}{2}(u - q(c)) + \frac{\xi}{2}\left(\frac{R}{c} - q(c)\right), \tag{15}$$

and the latter, given $q(c)$, yields the ODE for $v(c)$:

$$0 = q'(c)\sigma^2 + \frac{\sigma^2}{2} v''(c) + \frac{\xi}{2}(uc - v(c)) + \frac{\xi}{2}(R - v(c)). \tag{16}$$

One can interpret these ODEs as Euler equations. They ensure that given the dynamics of the state c, agents are indifferent whether to hold the cash (or tree). We first explain the terms without ξ in both ODEs. For the cash value q equation (15), $\frac{\sigma^2}{2} q''$ captures the impact of changing aggregate liquidity; and a similar term shows up in the asset value v equation (16). In addition, we have $q'(c)\sigma^2$ in equation (16). This is because the asset itself generates random cash flows σdZ_t that are correlated with the aggregate state $c + \sigma dZ_t$, and the expected value of these cash flows is

$$E_t\left[q(c + \sigma dZ_t)\sigma dZ_t\right] = E_t\left[q'(c)\sigma^2 (dZ_t)^2\right] = q'(c)\sigma^2 dt.$$

The terms multiplied by the intensity ξ describe the change in expected utility if the asset matures. The first of these terms in equation (15) shows that, once a specialist holding a unit of cash is hit by a skill shock, her value jumps to u from $q(c)$. If she is not hit by the shock, the second term says that she uses the unit of cash to buy $1/p_T = 1/c$ unit of tree, so her utility jumps to $\frac{R}{c}$ from $q(c)$. The interpretation in equation (16) is similar. One can solve the ODE system in (15)-(16) in closed-form, which admits the following general form:

$$q(c) = \frac{u}{2} + e^{-c\gamma} K_1 + e^{c\gamma} K_2 + R\frac{\gamma}{2} \frac{-e^{c\gamma}\,\mathrm{Ei}(-\gamma c) + e^{-c\gamma}\,\mathrm{Ei}(c\gamma)}{2}, \tag{17}$$

and

$$v(c) = R + \frac{uc}{2} + e^{c\gamma}(K_3 - cK_2) - e^{-c\gamma}(K_4 + cK_1) + cR\frac{\gamma}{2}\frac{\left(e^{\gamma c}\,\mathrm{Ei}(-\gamma c) - e^{-c\gamma}\,\mathrm{Ei}(\gamma c)\right)}{2}, \tag{18}$$

where $\gamma \equiv \frac{\sqrt{2\xi}}{\sigma}$, $\mathrm{Ei}(x)$ is the exponential integral function defined as

$$\mathrm{Ei}(x) \equiv \int_{-\infty}^{x} \frac{e^t}{t} dt,$$

and the constants K_1-K_4 are determined from boundary conditions in (14). If the resulting price $p(c) = \frac{v(c)}{q(c)}$ falls in the range of $[l, h]$ for any $c \in [c_h^*, c_l^*]$ then we have an equilibrium. The following proposition gives sufficient conditions for such an incomplete market equilibrium to exist and describe the basic properties of this equilibrium. We summarize this result below and give formal proof in the Appendix.

Proposition 1 *If the difference between the cost of liquidation, l and the cost of building a tree, h is relatively small then an incomplete market equilibrium with the following properties exist:*

1. *agents do not consume before the tree matures,*

2. *each agent in each state $c \in [c_l^*, c_h^*]$ is indifferent in the composition of her portfolio*

3. *agents do not build or liquidate trees when $c \in (c_l^*, c_h^*)$ and, in aggregate, agents spend every positive cash shock to build trees iff $c = c_h^*$ and finance the negative cash shocks by liquidating a sufficient fraction of trees iff $c = c_l^*$.*

4. *the value of holding a unit of cash and the value of holding a unit of tree are described by $v(c)$ and $q(c)$ and price ex ante is*
$$p = p(c) \equiv \frac{v(c)}{q(c)}.$$

5. *Ex post, each agents hit by the shock sells all her trees to the agents who are not hit by the shock for the price $p_T = c$, and*

6. *$q(c)$ is monotonically decreasing, $v(c)$ is monotonically increasing and $p(c)$ is monotonically increasing.*

Because all specialists are ex ante indifferent how much cash or trees to hold at the equilibrium prices, the properties of our market equilibrium leave individual portfolios undetermined. The symmetric market equilibrium picks the market equilibrium where all individual portfolios are the same.

3.5.4 Investment waves

The thick, solid lines on the three panels of Figure 3 illustrate the properties of the market equilibrium. We call the cash-to-tree ratio c the "aggregate liquidity." We think of the time with high (low) aggregate liquidity in which trees are built (liquidated) as a boom (recession). Although investment takes a simple threshold strategy so that investment (disinvestment) occurs at c_h^* (c_l^*), we believe it captures the essence of investment waves observed in the data.

The economy fluctuates between these states because the aggregate level of liquidity of specialists also fluctuate. This fluctuation is driven by the interim cash-flow shocks. When aggregate liquidity is low, the marginal value of cash increases, the price of the tree falls, and the expected return of holding trees rises. This is so, because in these states the probability that the economy slips into a recession when trees have to be liquidated is large. Thus, specialists hold a large amount of low-return cash and willing to hold trees only for a sufficiently large premium. This is consistent with the so-called slow moving capital puzzle.[8] An equivalent interpretation is that the marginal value of cash is high in these states because they can be turned into high-expected return trees. When aggregate liquidity is high, the marginal value of cash and expected returns are small, because of symmetric reasons.

The constant $\gamma \equiv \frac{\sqrt{2\xi}}{\sigma}$ parametrizing functions $v(c)$, $q(c)$ in (18) and (17) has an important role in the following analysis. Intuitively, γ drives the relative importance of the ex post payoffs for ex ante decisions. When the switching intensity ξ is high or a low σ reduces the chance of large interim shocks, ex post payoffs are important determinants of ex ante decisions. The following results on the investment and disinvestment thresholds are useful to understand the intuition behind our results.

Proposition 2 *In the incomplete market equilibrium*

1. $c_h^* > h, c_l^* < l$
2. *as* $\gamma \to \infty$, $c_h^* \to h$ *and* $c_l^* \to l$.

Consider the last result. When γ grows without bound the specialist's ex ante decisions are almost solely determined by ex post payoffs. When the specialist expects the idiosyncratic shock to be realized immediately, in expectation the specialist is better off turning h units of cash to a tree whenever

$$\left(\frac{1}{2}u + \frac{1}{2}\frac{R}{c}\right)h \leq \frac{1}{2}uc + \frac{1}{2}R$$

where the term in the bracket on the left hand side is the expected value of holding a unit of cash, while the right hand side is the expected value of holding a unit of tree when the asset matures. Clearly, this inequality holds for any $c > h$. This explains that $c_h^* \to h$ in this limit. Away from this limit, when creating trees, the specialist considers also the risk of reaching the low liquidity state when these trees are liquidated inefficiently. Thus, she decides to build trees only at a higher threshold, $c_h^* > h$.

Similarly, the specialist is better off in expectation to turn a tree into l units of cash whenever

$$\left(\frac{1}{2}u + \frac{1}{2}\frac{R}{c}\right)l \geq \frac{1}{2}uc + \frac{1}{2}R$$

or $c < l$. This gives $c_l^* \to l$. Away from this limit, when liquidating trees, the specialist considers

[8] See the presidential address of Duffie (2010).

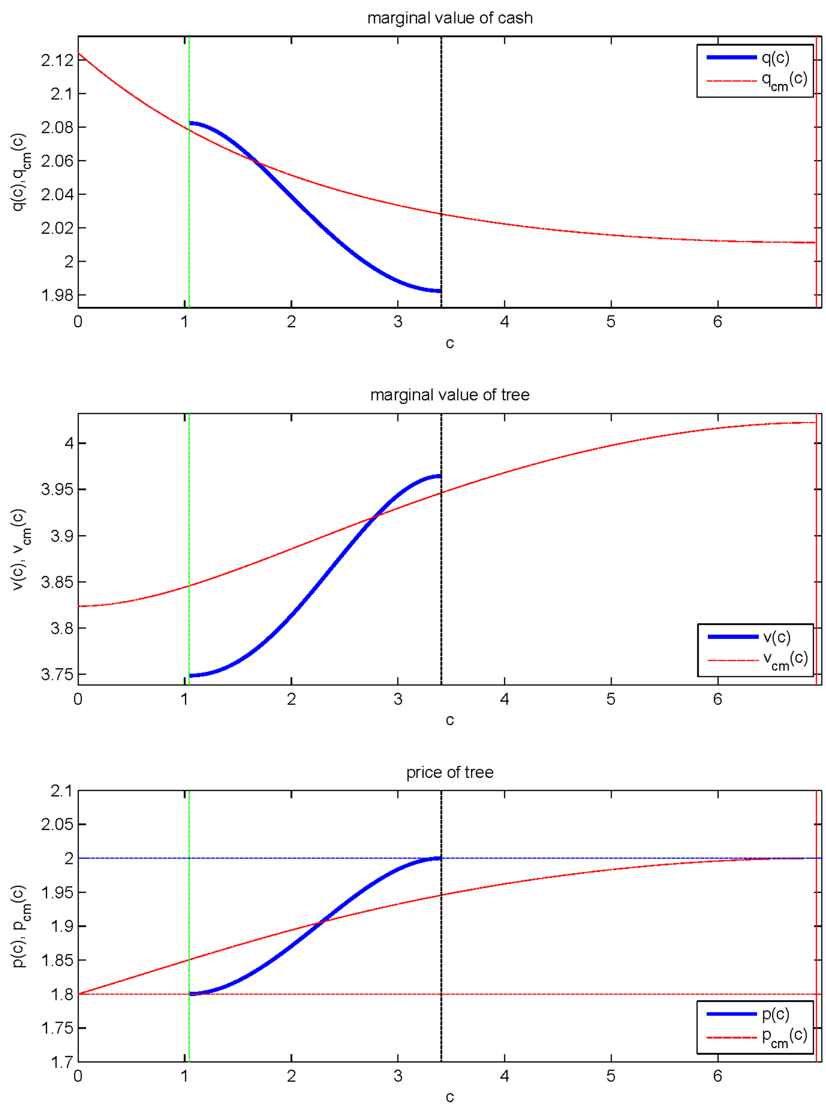

Figure 3: Panels depict the value of cash, trees and the price of trees for the baseline model with incomplete markets (thick solid curves) and the benchmark of complete markets (thin, dashed curves). The solid, vertical line on the right of each graph is at the invesment threshold in complete markets and in social planners solution, c_h^P, while the two dashed vertical lines are the disinvesent and invesment thresholds in our baseline case, c_l^*, c_h^*. The horizontal lines on the bottom panel are at the levels of l and h. Parameter values are $R = 4.1$, $\sigma^2 = 1$, $\xi = 0.1$, $u = 2$, $l = 1.8$ and $h = 2$.

also the risk of reaching the high liquidity state when these trees have to be rebuild. Thus, she decides to liquidate trees only at a lower threshold, $c_l^* < l$.

3.5.5 Price distributions

Given the function $p(c_t)$, we can find the stationary distribution of ex ante prices easily. For this, first note that the state c follows a uniform distribution on the support $[c_l^*, c_h^*]$ by standard arguments.[9] Then the pdf and the cdf of p_t is given by

$$\begin{aligned} \pi(p_t) &= \frac{1}{\left(c_h^* - c_l^*\right) p'\left(p^{-1}(p_t)\right)}, \\ \Pi(p_t) &= \frac{p^{-1}(p_t)}{\left(c_h^* - c_l^*\right)}, \end{aligned}$$

respectively.[10] As boundary conditions (14) imply $p'(h) = p'(l) = 0$, the value of the pdf is very high close to the barriers h and l. In fact, the consequence of the S-shaped $p(c)$ is that pdf is inverse U-shaped and the cdf is very steep at the barriers and flat in between. We show an example of the pdf of c and the cdf of p on Panels C and D of Figure 4. Although the state c is equally likely to be at any point of its support, the realized price is most often either very high or very low. In this sense, the volatility of the price of the tree is much higher than the volatility of the underlying cash-flows. This is a form of excess volatility consistent with the classic observation of Shiller (1981).

4 Externalities

We study pecuniary externalities in this section. As a benchmark, we first solve for constrained efficient allocation in this economy. We then show that our model features a two-sided inefficiency on investment waves, and a one-sided intervention in boosting investment in recession may lead to lower welfare everywhere including recession.

4.1 Constrained Efficient Allocations

In this part, we discuss the constrained efficient solution of our problem where the planner takes into account the technological constraint. Namely, that outside capital cannot be injected into the system. That is, the aggregate cash has to be kept non-negative by liquidating trees if necessary.[11] We will consider two benchmark economies which both produces the same constrained efficient outcome.

[9] This is so, becuase c_t is a Brownian motion with no drift regulated by reflective barriers. See Dixit (1993) pp. 59-61.

[10] Clearly, the expressions for the cdf and pdf of p_t are meaningful only if $p(c)$ is monotonically increasing. Although we do not prove this property directly, we find that it holds for every set of parameters we have experimented with.

[11] Note that without this technological constraint, condition (9) implies that the planner should convert any amount of cash to trees.

First, we discuss the social planner's problem who can dictate the investment policy in this economy. Compared to the decentralized market equilibrium, the main difference is that in the market equilibrium specialists take prices as given and this market price drives their decision to build or dismantle trees. In contrast, the social planner directly decides when to build or dismantle trees. Second, we consider a decentralized economy where markets are complete so that each individual agents have access to the same investment opportunities (through contracting), which allows us to characterize the asset prices without inefficiency.

4.1.1 Social planner's problem

Define the social planner's value function as $J_P(A, C)$. The planner can decide when to build and liquidate trees. Thus, she optimally regulates the c process subject to the constraint that the cash level C must stay non-negative.

Ex post, cash (trees) always ends up in the specialists with (without) skill shock at the market clearing price $p_T = c$. Therefore, the total value ex post is

$$AR + Cu. \tag{19}$$

Thus, given the current aggregate state pair (A, C), the problem of the social planner is

$$J_P(A, C) = \max_{dA} E\left[\int_0^\infty \xi e^{-\xi t}(A_t R + C_t u)\, dt\right] \tag{20}$$

subject to the constraint $C_t \geq 0$ and (11). The linearity implies that we can define $j_P(c)$ by rewriting (20) as

$$J_P(A, C) = A j_P(c) = \max_{dA} E\left[\int_0^\infty \xi e^{-\xi t}(AR + Cu)\, dt\right].$$

Because of the linear technology of planting and dismantling trees, regulation with reflective barriers on c is optimal.[12] That is, there exists low and high thresholds c_l^P, c_h^P, so that it is optimal to stay inactive whenever $c \in \left(c_l^P, c_h^P\right)$, and dismantle (build) just enough trees to keep $c = c_l^P$ ($c = c_h^P$) at the lower (upper) threshold. Before turning to the determination of optimal investment/disinvestment thresholds, it is useful to think of the social value given that c is regulated by any arbitrary reflecting barriers c_l, c_h. Define the corresponding (scaled) social value as $j_S(c; c_l, c_h)$ so that

$$A j_S(c; c_l, c_h) \equiv E\left[\int_0^\infty \xi e^{-\xi t}(A_t R + C_t u)\, dt | c_l, c_h\right]. \tag{21}$$

Clearly, the optimal value achieved by the social planner is

$$j_P(c) \equiv \max_{c_l, c_h} j_S(c; c_l, c_h). \tag{22}$$

[12] See Dixit (1993) for a detailed argument.

Using standard results in forming expectations on functions of regulated Brownian motions,[13] for any $c \in (c_l, c_h)$, $j_P(c)$ must satisfy

$$0 = \frac{\sigma^2}{2} j_S'' + \xi (R + uc - j_S), \tag{23}$$

where we suppressed the arguments of j_S and $j_S'' = \frac{\partial^2 j_S}{\partial^2 c}$. Also from (21), at the reflective barriers c_l, c_h the smooth pasting conditions must hold:

$$\frac{\partial [A j_S (c_l; c_l, c_h)]}{\partial A} = l \frac{\partial [A j_S (c_l; c_l, c_h)]}{\partial C}, \text{ and } \frac{\partial [A j_S (c_h; c_l, c_h)]}{\partial A} = h \frac{\partial [A j_S (c_h; c_l, c_h)]}{\partial C}. \tag{24}$$

We emphasize that these conditions are not optimality conditions. They hold for any arbitrarily chosen barriers $c_l < c_h$ as a consequence of forming expectations on a regulated Brownian motion. The ODE (23) has a closed from solution

$$j_S(c; c_l, c_h) = R + uc + D_1(c_l, c_h) e^{-\gamma c} + D_2(c_l, c_h) e^{\gamma c}. \tag{25}$$

For any fixed c_l, c_h, the constants D_1 and D_2 are solved by invoking the smooth pasting conditions in (24):

$$R + uc_h + D_1 e^{-\gamma c_h} + D_2 e^{\gamma c_h} = (h + c_h)\left(u - \gamma D_1 e^{-\gamma c_h} + \beta D_2 e^{\gamma c_h}\right), \tag{26}$$

$$R + uc_l + D_1 e^{-\gamma c_l} + D_2 e^{\gamma c_l} = (l + c_l)\left(u - \gamma D_1 e^{-\gamma c_l} + \beta D_2 e^{\gamma c_l}\right). \tag{27}$$

Following Dumas (1991), to determine the optimal barriers $\left(c_l^P, c_h^P\right)$ which solve (22), we have to add supercontact conditions. For the upper barrier, this is

$$\frac{\partial^2 A j_P \left(c_h^P; c_l, c_h\right)}{\partial A \partial C} = h \frac{\partial^2 A j_P \left(c_h^P; c_l, c_h\right)}{\partial^2 C}, \tag{28}$$

which we can rewrite as

$$0 = \frac{\partial^2 j_P (c; c_l, c_h)}{\partial c}\Big|_{c = c_h^P} = \gamma^2 \left(D_1 e^{-\gamma c_h^P} + D_2 e^{\gamma c_h^P}\right).$$

For the lower barrier, we have to take into account that at the optimal choice, the constraint $C \geq 0$ might bind. Thus, the supercontact condition is

$$\frac{\partial^2 A j_P \left(c_l^P; c_l, c_h\right)}{\partial A \partial C} \leq l \frac{\partial^2 A j_P \left(c_l^P; c_l, c_h\right)}{\partial^2 C}, \text{ for } c_l^P \geq 0$$

with complementarity. In the next proposition we state that the lower optimal threshold is always in the corner, i.e., $c_l^P = 0$, and the upper threshold is the unique solution of a simple equation.

Proposition 3 *The social planner liquidates trees when c reaches 0 and builds trees when c reaches*

[13] See Dixit (1993) for a detailed argument.

$c_h^P > 0$. *The threshold* c_h^P *is given by the unique solution of*

$$\frac{R - hu}{R - lu}\left(e^{c_h^P \gamma}\left(1 + l\gamma\right) - \left(1 - l\gamma\right)e^{-c_h^P \gamma}\right) - 2\gamma\left(c_h^P + h\right) = 0. \tag{29}$$

To understand the choice of the social planner, it is useful to consider the following comparative statics.

Proposition 4 *The socially optimal investment threshold* c_h^P

1. *is converging to 0 as* $\gamma \to \infty$, *and decreasing in* γ *given that* $\gamma > \hat{\gamma}$ *for a given* $\hat{\gamma}$,
2. *decreasing in* l *and* R *and increasing in* h.
3. *converging to* ∞ *as* $R \to uh$ *or* $u \to \frac{R}{h}$.

Consider the case when γ is unboundedly large. The first statement shows that in this case the social planner does not store any cash, but converts it to trees immediately. The idea is as follows. In this limit either ξ is very large or σ is very small. Both imply that the social planner does not have to worry about the possible negative shocks before the tree matures. It is so, either because the asset matures very fast or because the interim shocks are small. Therefore, she decide to not to store any cash, in line with condition (9) ensuring that at maturity the payoff of a tree is larger than the marginally utility weighted cost of creating a tree. When γ is not so large the social planner worries about negative shocks. As reinvesting from cash reserves is cheaper than liquidating trees for reinvestment, she stores some cash and builds the trees only when the cash-to-tree ratio reaches the positive threshold c_h^P. At that point the marginal utility of cash is sufficiently small that turning cash to trees is optimal. The second statement is also intuitive. When planting trees is expensive, liquidating them is very inefficient or their return is low than reinvesting by liquidating trees as opposed to by using cash is more painful. Thus, the social planner holds on to the cash until a higher threshold. Finally, when $R - uh$ is sufficiently small, the social planner would only slightly prefer to turn cash to trees even absent of the possibility that a series of bad interim shocks induces inefficient liquidation of trees. Thus, the threshold to turn cash to trees increases without bound.

4.1.2 Market prices under constrained efficient allocations with complete market

Consider the variant of our decentralized model where markets are complete so that constrained efficient solution is achieved. There are many different ways to model complete markets. In the context of our model with investment, we simply assume that the proceeds R and u are fully pledgeable so that individual agents can enjoy the investment opportunities of others.[14] Thus, all agents know that they invest their cash-holdings to the new technology and none of them loses their expertise to tend the trees. The critical point is that the ex post heterogeneity among agents

[14] In the context of ex post perference as in the simple example in Section 2, complete market requires Arrow-Debreu securities written on the idiosyncratic preference shocks and thus verifiable idiosyncratic preference shocks.

effectively disappear. We refer to this variant as the complete market economy or the subscript cm. By following the same derivation, in this case $q_{cm}(c)$ and $v_{cm}(c)$ solve the ODE system

$$0 = \frac{\sigma^2}{2} q''_{cm}(c) + \xi(u - q_{cm}(c)), \tag{30}$$

$$0 = q'_{cm}(c)\sigma^2 + \frac{\sigma^2}{2} v''_{cm}(c) + \xi(R - v_{cm}(c)). \tag{31}$$

The following statement characterizes the equilibrium in this variant of our model.

Proposition 5 *In the complete market economy, there is an equilibrium for any set of parameters where*

1. *agents do not consume before the tree matures,*
2. *each agent in each state $c \in [0, c_h^P]$ is indifferent in the composition of her portfolio*
3. *each agent holding trees use every positive cash shock to build trees iff $c = c_h^P$ and finance the negative cash shocks by liquidating the tree iff $c = 0$.*
4. *the value of holding a unit of cash, the value of holding a unit of tree and the price of the tree is described by*

$$q_{cm}(c) = \frac{u}{2} + e^{-c\gamma} L_1 + e^{c\gamma} L_2, \tag{32}$$

$$v_{cm}(c) = R + \frac{uc}{2} + e^{c\gamma}(L_3 - cL_2) + e^{-c\gamma}(L_4 - cL_1). \tag{33}$$

   where L_1, L_2, L_3, L_4 and c_h^P is given by boundary conditions

$$\frac{v_{cm}(c_h^P)}{q_{cm}(c_h^P)} = h,\ \frac{v_{cm}(0)}{q_{cm}(0)} = l,\ v'_{cm}(c_h^P) = q'_{cm}(c_h^P) = v'_{cm}(0) = 0 \tag{34}$$

   and

$$j_P(c) = v_{cm}(c) + cq_{cm}(c)$$

   for all c_t.

5. *$v_{cm}(c)$ is increasing in c, $q_{cm}(c)$ is decreasing in c and $p_{cm}(c) \equiv \frac{v_{cm}(c)}{q_{cm}(c)}$ is increasing in c.*

The Proposition states that in this economy, the market implements the social planner's solution. This economy is constrained efficient, as individual agents have the same objective as the social planner. It is also important to note that the qualitative properties of the constrained-efficient economy is quite similar to the market solution of our baseline economy. In particular, as the cash-to-tree ratio decreases, the price fall and the tree trades with a significant liquidity premium. We illustrate this equilibrium by the thin, dashed curves on Figure 3. The intuition for all these results are the same as in the economy with an idiosyncratic skill-shock. Thus, underpriced assets, large liquidity premium and slow moving capital is not inconsistent with a constrained efficient economy.

The state c follows a uniform distribution on the support $\left[0, c_h^P\right]$. Just as in the baseline model, we can determine the pdf and cdf of prices,

$$\begin{aligned} \pi_{cm}\left(p_t\right) &= \frac{1}{c_h^P p'_{cm}\left(p_{cm}^{-1}\left(p_t\right)\right)} \\ \Pi\left(p_t\right) &= \frac{p_{cm}^{-1}\left(p_t\right)}{c_h^P}. \end{aligned}$$

The S-shape of $p_{cm}(c)$ implies an inverse U-shaped pdf, just in the baseline case. Thus, as before, although the economy is in any state c with equal probability, most of the time the price is either very high or very low. However, with complete markets the boundary conditions (34) imply only $p'_{cm}(h) = 0$, but does not imply $p'_{cm}(l) = 0$. As a consequence, the pdf of the price has a very high value only close to the upper barrier. The thin, dashed curves on Panel C and D of Figure 4 show the pdf of c and $\Pi_{cm}(p_t)$. It is apparent that $\Pi_{cm}(p_t)$ is very steep only around h, but not around l. We will return this issue in the next section, where we discuss how our Second Best benchmarks compare to our baseline, decentralized model with incomplete markets.

4.2 Two-sided inefficiency

In this part, we argue that there is a large subset of parameters where the externality imposed by the idiosyncratic shock imply both overinvestment in productive assets in a boom and underinvestment in productive assets in a recession. We refer to this case as *two-sided inefficiency*. In particular, we show that unlike the social planner, in the market equilibrium specialists dismantle trees when still some cash is around, $c_l^* > 0$. Also, specialists create new trees at a lower liquidity level than the social planner would do, $c_h^* < c_h^P$. We show that in this case any policy which raises the upper threshold keeping the lower one constant, or decrease the lower threshold keeping the upper one constant would unambiguously increase total welfare in our economy. Constraining ourselves to the symmetric market equilibrium, this is equivalent to a Pareto improvement both in the ex ante sense, while ex post welfare is not effected for any given realized state. Our case is illustrated on Figures 3 and 4, where the dashed vertical lines show the thresholds, c_l^*, c_h^* of the baseline market equilibrium, and the solid vertical line shows the investment thresholds in the constrained-efficient economy with complete market.

4.2.1 The existence of two-sided inefficiency and intuition

The following proposition states that while the social planner would dismantle trees only when all cash in the economy is gone, the market solution described by Proposition 1 implies $c_l^* > 0$ for any parameters. That is, in the market equilibrium agents dismantle trees when the social planner would still avoid it. In this sense there is underinvestment in productive assets or, equivalently,

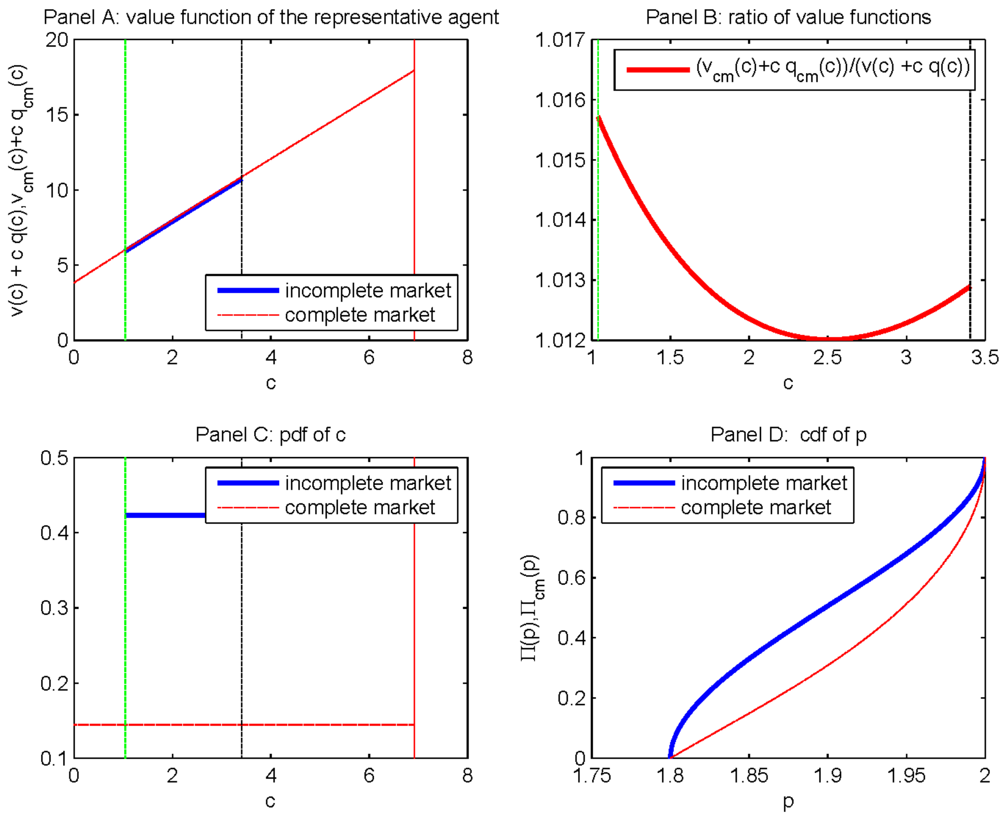

Figure 4: The total value of the representative agent, the ratio of value functions, the probability density of the state c and the cumulative density of the price of the tree, p, for our baseline model with incomplete markets (thick solid curves) and the benchmark of complete markets (thin, dashed curves). On Panels A and C, the solid, vertical line on the right is at the invesment threshold in complete markets and in social planners solution, c_P^*, while the two dashed vertical lines are the disinvesent and invesment thresholds in our baseline case, c_l^*, c_h^*. Parameter values are $R = 4.1$, $\sigma^2 = 1$, $\xi = 0.1$, $u = 2$, $l = 1.8$ and $h = 2$.

over hoarding of liquidity in a recession. As we explain below, the market equilibrium imply over or underinvestment in productive assets in booms, i.e., $c_h^* \gtrless c_h^P$ depending on the parameter values.

Proposition 6 *1. For any parameters, $c_l^* > 0$, so the market solution implies underinvestment in trees and over hoarding of liquidity in recessions.*

2. Keeping u, l, h, R fixed, there is a threshold $\hat{\gamma}$ that if $\gamma > \hat{\gamma}$, $c_h^* > c_h^P$, so the market solution implies underinvestment in trees and over hoarding of liquidity in booms as well.

3. Keeping u, l, h fixed, there are threshold $\hat{\gamma}$, and function $R(\gamma)$ that if $\gamma > \hat{\gamma}$ and $R = R(\gamma)$ then $c_h^* < c_h^P$, so the market solution implies overinvestment in trees and underinvestment in liquidity in booms. Also, $\hat{R}(\gamma)$ is decreasing in γ for $\gamma > \hat{\gamma}$ and as $\gamma \to \infty$, $\hat{R}(\gamma) \to uh$.

The general intuition behind our mechanism is essentially given in the simple example in Section 2. The ex post market clearing price not only moves resources to the most efficient hands but also allocates the rent among different agents, and this distorted price changes the investment and disinvestment thresholds. Importantly, the direction of price distortion depends on the state of the economy as illustrated in the bottom panel of Figure 3. Because the representative specialist will sell the tree in the market given a skill-shock, the private (expected) ex post value of a tree s

$$\frac{1}{2}up_T + \frac{1}{2}R,$$

while from the social perspective, the ex post value of a tree is always R. Therefore, whether the representative agent overvalues the trees compared to the planner crucially depends on whether

$$p_T > \frac{R}{u}.$$

That is, whether the private marginal rate of substitution is larger than the social marginal rate of substitution as calculated in Section 2. Given that $p_T = c$ fluctuates in the interval $[c_l^*, c_h^*]$ we should expect overinvestment in booms and underinvestment in recessions whenever

$$uc_l^* < R < uc_h^*. \tag{35}$$

Consistent with Proposition 6, the first inequality in (35) is always satisfied because $uc_l^* < ul < uh < R$, whereas the second inequality might or might not hold, depending on whether $c_h^* u > R$.[15]

An intuitive application of our model is the boom and bust pattern in real estate development and house prices.[16] Our mechanism suggests that the volume of construction in a boom is inefficiently high, because banks and investors invest in real estate developments instead of holding liquid financial assets expecting to be able to sell the real estate for a high price in case they find a new investment opportunity.[17] One suggestive sign of this inefficiency is the frequently observed

[15] There is an analogous argument by comparing of the social and private values of a unit of cash. This arguement leads to the same inequalities.

[16] Shiller (2007) illustrates this pattern by the cyclicality of the residential investment to GDP ratio. He points out that cycles in this ratio correspond closely to the recessions after 1950, typically peaking few quarters before the start of the recession. This pattern was not observed before the 2000-01 recession but was observed again before the 2007-2009 recession.

[17] Related arguments were made in connection to the development of Japan.

> It took most Japanese banks years to whittle down the tens of billions of dollars in unrecoverable loans left on their books after the collapse of a real estate bubble in Japan's overheated 1980's. They finally succeeded in the last two or three years [...]But analysts criticize most banks for failing to find new, more profitable – and less risky – ways of doing business. Instead, analysts say many have gone back to lending heavily to real estate development companies and investment funds, as the rebounding economy has touched off a construction boom in Tokyo. "If the economy stalled, Japanese banks would have a bad loan problem all over again," said Naoko Nemoto, an analyst for Standard & Poor's in Tokyo. Ms. Nemoto estimates that banks loaned 1.6 trillion yen ($14 billion) to real estate developers in the six months that ended last September – half of all new bank lending in that period." (The New York Times, January 17, 2006, pg.4)

phenomenon of "overbuilding," that is, periods of construction booms in the face of rising vacancies and plummeting demand.[18] On the other hand, in recessions, our model suggests that banks and investors hold inefficiently high level of liquid assets expecting to be able to buy real estate cheap in case a group of distressed investors have to liquidate their holdings.

Proposition 6 translates the above intuition in terms of the deep parameters of the model. We construct these restrictions by combining results in Proposition 4 and Proposition 2. Vaguely speaking, Proposition 6 suggests that there is overinvestment in booms and underinvestment in recessions, if $\frac{R}{h} - u$ is small, i.e., the profitability of the existing tree technology is close to that of the new investment opportunity. As pointing out the two-sided inefficiency is the major novelty in our paper, we focus mostly on this case in the rest of the paper.

4.2.2 What market failures drive the inefficiency?

To be completed.

First, the pledgeability of future project payoffs in ex post period definitely help resolve the distorted ex ante investment incentives. Here, the pledgeability means that we can write contract on the proceeds from both projects (both u and R). Thus, the individual agent with skill-shock can hire the agents without skill shock to harvest the tree on behalf of the agents with skill shock, and still value the full marginal return of R from tree. Similarly, the agent do not have investment opportunity u can lend their cash to the agent with skill shock and receive the investment benefit u. This way, all agents are essentially facing the same investment opportunities, and thus inefficiency disappears.

Second, the pledgeability does not apply in our example where R and u is the agent's preference. There, completing the market by introducing the Arrow-Debreu securities help. We can show that if individual states are contractible, then date zero trading of these securities will restore the investment incentives.

4.3 Social welfare

Now we turn into the explicit comparison of welfare under the second-best achieved by the social planner and the market equilibrium with incomplete markets. For simplicity, we focus on the symmetric equilibrium where all specialists hold the same portfolio ex ante.

As emphasized earlier in Section 2, in our model trading leads to ex post efficient resource allocation. Under both the social planner's solution and in our incomplete market solution, the representative specialist hit by the skill-shock gets

$$u\left(p_T A + C\right) = u\left(cA + C\right) = 2Cu,$$

[18] See Wheaton and Torto (1990) and Grenadier (1996) for alternative explanations of overbuilding. Overbuidling was also observed before the 2007-2009 recession in the sense that rental vacancies peaked in 2004, before the peak of the contstruction boom. (See http://www.census.gov/hhes/www/housing/hvs/historic/index.html.)

while, if she is not affected, she gets

$$RA + \frac{R}{p_T}C = RA + \frac{R}{c}C = 2AR.$$

That is, given the state (A, C) the social planner does not change the welfare of the representative agent ex post. This is intuitive, as trading after the shock moves the assets (cash) to the hands with the highest profitability.

Instead, by changing the thresholds c_h and c_l, the social planner can influence the future distribution of c (or, equivalently, the joint distribution of (A, C)), which affects the representative agents ex ante welfare. To be more specific, following the argument in Section 4.1, given any thresholds (c_l, c_h) which regulate the process c, total welfare is given by

$$Aj_S\left(c; c_l, c_h\right)$$

with an explicit solution determined by (25)-(27). Note that in a symmetric equilibrium, $Aj_S\left(c; c_l, c_h\right)$ is also the ex ante value of the representative agent. If a policy increase total welfare with respect to the market equilibrium, it is an ex ante Pareto improvement with respect to the symmetric market equilibrium. Given that the total welfare is state dependent, we can make a distinction between policies which improve total welfare at some states, e.g. in recessions only, and policies which improve welfare everywhere. [19]

In the next proposition we show that increasing the lower threshold or decreasing the upper threshold compared to the social planners' solution unambiguously decreases welfare everywhere.

Proposition 7 *For any $c_h < c_h^P$ and $c_l > 0$ and c*

$$\begin{aligned}\frac{\partial j_S\left(c; c_l, c_h\right)}{\partial c_l} &< 0 \\ \frac{\partial j_S\left(c; c_l, c_h\right)}{\partial c_h} &< 0.\end{aligned}$$

The proposition ensures that whenever $c_h^* < c_h^P$, specialists underinvest in recessions and overinvest in booms in the productive asset in our market equilibrium in a well defined sense. Indeed, a lower disinvestment threshold or a higher investment threshold would increase total welfare in a market equilibrium and would lead to a Pareto improvement in the symmetric market equilibrium. The comparison of the solid curves and dashed curves on Panel A and B on Figure 4 illustrate this point. It is apparent that in an economy with two-sided externalities, the social planner raises ex ante welfare at every state.

At a basic level, our mechanism is in line with the welfare effects of pecuniary externalities identified by Geanakoplos and Polemarchakis (1985). That seminal paper shows that when markets

[19] Note that in our structure for any current c_t, the total welfare function factors in the effect of the policy in each other state. The idea behind a policy that improves welfare, e.g., only in a recession is that the probability of arriving in a given state depends on the current c_t, i.e.,in a recession, a boom looks less likely then a continuing recession.

are incomplete and, consequently, prices do not equate marginal rate of substitution of agents, then pecuniary externalities might have first-order effects on welfare. Our mechanism works by the same logic. The ex post price p_T cannot equate marginal rates of substitutions because agents who are not hit by the shock cannot pay more than c for the asset, even if their valuation is R, because this is all the cash they have. In a market equilibrium, for a given p_T agents behave optimally when they build trees at c_h^* and liquidate their trees at c_l^*. However, they fail to take into account that, because of the missing market, price p_T does not serve its Wallrasian function of signalling relative social value of different goods. This makes specialists' decisions socially inefficient. Our main contribution relative to Geanakoplos and Polemarchakis (1985) and the subsequent literature is to point out that the distortion implied by the pecuniary externality is likely to change sign with the state of the economy, because the distortion in the price changes sign.

Our complete market benchmark helps to see how the two-sided inefficiency changes the ex ante distribution of prices. Panel D in Figure 4 compares the cdfs in the complete market case and the incomplete market case. The first thing to note that the price distribution in the complete market case first-order stochastically dominates the incomplete market case. That is, two-sided inefficiency implies that lower liquidity premium states happen with higher probability compared to the second-best. We also find that the unconditional volatility of prices is higher in our baseline model and the distribution is more positively skewed.[20] All these are consistent with our previous observation that while in our baseline model the pdf of p_t approaches infinity both at the extremes h and l, it is only the case at the high-end h in the complete market case. This asymmetry is implied by the difference in boundary conditions (14) and (34).

An important advantage of the dynamic structure of our model is that in any ex ante state c agents' decisions are affected by their expectation of economic conditions in all other states. Suppose that the economy is in a recession and a policy is introduced with the promise that it will be abandoned as soon as the economy recovers. This policy will necessarily influence agents' choices in the boom, which agents foresee. Thus, in turn, this effect will influence their current reaction to the policy. This feedback effect is in the centre of our analysis of possible government interventions in the next section.

4.4 One-sided interventions

In this part, we will analyze suboptimal policies of a class we call one-sided interventions. The idea is that at a state close c_l^* the policy maker might realize that the price falls dangerously close to the disinvestment threshold l and might decide to intervene to raise prices and to avoid inefficient liquidation of productive assets. We do not allow the policy maker to regulate prices directly. Instead, the tool we give to the policy maker is any combination of ex ante taxes and subsidies to the cash holders and asset holders subject to a balanced-budget condition. The policy maker can effect the equilibrium prices and the equilibrium investment/disinvestment thresholds through

[20] We do not have analytical proofs for these statements, due to the lack of a closed form expression for functions $p^{-1}(p_t)$ and $p_{cm}^{-1}(p_t)$. Still, we find these results robust across all set of parameters we experimented with.

these taxes and subsidies. Since the policy maker might realize that raising prices by taxes and subsidies is unnecessary in a boom, she might make the policy conditional on being in a sufficiently low c state.

4.4.1 Tax-subsidy scheme

A one sided intervention lowers the disinvestment threshold by definition. We know from Proposition (7) that if the investment threshold, c_h^*, remained constant, this policy would improve welfare. While we show that typically a one-sided intervention reduces the investment threshold, c_h^*, implying a negative effect of welfare, this is not our main result. The main result of this section is that this negative effect can be so strong that it might imply that the policy reduces welfare everywhere! That is, even if the policy reduces inefficient liquidation in the recession, agents' welfare might reduce even in the recession, because they expect that the current policy will make overinvestment in a future boom much worse. Before stating these results formally, we define one-sided intervention and the corresponding intervention equilibrium. We distinguish equilibrium objects under intervention from their counterpart under no intervention with the index τ as in $\{c_l^\tau, c_h^\tau, p_\tau(c), v_\tau(c), q_\tau(c)\}$.

Definition 3 *A one-sided intervention is a tax-subsidy scheme $\tau(c)$ and an intervention-threshold c_0 such that*

1. *for each unit of cash held in state c specialists pay $\tau(c)$,*
2. *for each unit of tree held in state c specialists receive $c\tau(c)$,*
3. *$\tau(c) \equiv 0$ for any $c > c_0$,*
4. *the disinvestment threshold is reduced by the intervention, $c_l^\tau < c_l^*$*
5. *the equilibrium price is increased at the intervention threshold, $p_\tau(c_0) > p(c_0)$.*

Definition 4 *An intervention equilibrium is an incomplete market equilibrium under the a tax-subsidy scheme defined by a one-sided intervention.*

Note that what makes one-sided interventions truly asymmetric is not the arbitrarily large intervention-threshold c_0 , but the requirement that the price at that threshold must be raised by the intervention. Intuitively, we think of one-sided interventions as policies which raise prices for every $c \in [c_l^*, c_0]$ compared to the market equilibrium by increasing the marginal value of the asset $v_\tau(c)$ and/or decreasing the marginal value of cash $q_\tau(c)$ for every $c \in [c_l^*, c_0]$. However, for our results we need less. Thus, we do not restrict the sign of $\tau(c)$ and impose only the weaker requirement on the effect on prices in part 5 of the definition.

In the next proposition, we show that a one-sided intervention typically decreases the investment threshold, $c_h^\tau < c_h^*$, which in this sense makes overinvestment in the boom worse. The condition of the proposition is weak in the sense that we would expect a one sided intervention which is designed to raise prices up to the point c_0 to decrease the marginal value of cash at that point.

Proposition 8 *Any one-sided intervention* $(\tau(c), c_0)$ *for which the value of cash decreases at* c_0, $q_\tau(c_0) \leq q(c_0)$, *reduces the investment threshold,* $c_h^\tau < c_h^*$.

Proposition 8 is quite intuitive. After all, the value of the tree in one state is naturally positively related to its value in every other state. Thus, when intervention raises the price of the tree in low states, its price tend to increase also in high states. However, this implies that c_h^τ has to decrease to make sure that the condition $p_\tau(c_h^\tau) = h$ is not violated. Thus, an intervention focusing on improving underinvestment in the recession will typically make overinvestment worse in the boom.

4.4.2 An example with a Pareto-dominated one-sided intervention

The intuitive result in Proposition 8 opens an interesting question. The price-boosting one-sided intervention in the recession alleviates the underinvestment problem in recession; however, because it leads to higher prices in the boom, this one-sided intervention necessarily results in more severe overinvestment in the boom. Is it possible that the latter negative equilibrium effect dominates the earlier positive effect in every state, even in the recession where the one-sided intervention is designed for? We provide an affirmative answer to this question, by constructing an example where a one-sided intervention is Pareto inferior to no-intervention at all in every state.

The simplest example of a one sided intervention is when the tax-subsidy is constant, i.e., $\tau(c) \equiv \tau$ for every $c < c_0$. Following our derivation of the market equilibrium, value functions in the intervention equilibrium are defined by the ODEs

$$0 = \frac{\sigma^2}{2} q_\tau'' - \mathbf{1}_{c<c_0}\tau + \xi\left(\frac{u + R/c}{2} - q_\tau\right) \tag{36}$$

$$0 = \frac{\sigma^2}{2} v_\tau'' + q_\tau'\sigma^2 + \mathbf{1}_{c<c_0}\tau c + \xi\left(\frac{uc + R}{2} - v_\tau\right) \tag{37}$$

where $\mathbf{1}$ is the indicator function, subject to the boundary conditions

$$\frac{v_\tau(c_h^\tau)}{q_\tau(c_h^\tau)} = h, \frac{v_\tau(c_l^\tau)}{q_\tau(c_l^\tau)} = l \tag{38}$$

$$v_\tau'(c_h^\tau) = q_\tau'(c_h^\tau) = q_\tau'(c_l^\tau) = v_\tau'(c_l^\tau) = 0. \tag{39}$$

We also have to make sure that each function is smooth at c_0. It is simple to check that the following general solution satisfies the system

$$q(c) = -1_{c<c_0}\frac{\tau}{\xi} + \frac{u}{2} + e^{-c\gamma}\left(1_{c>c_0}M_1 + 1_{c<c_0}M_5\right) + e^{c\gamma}\left(1_{c>c_0}M_2 + 1_{c<c_0}M_6\right) + \\ + R\frac{\gamma}{2}\frac{-e^{c\gamma}\,\mathrm{Ei}(-\gamma c) + e^{-c\gamma}\,\mathrm{Ei}(c\gamma)}{2}$$

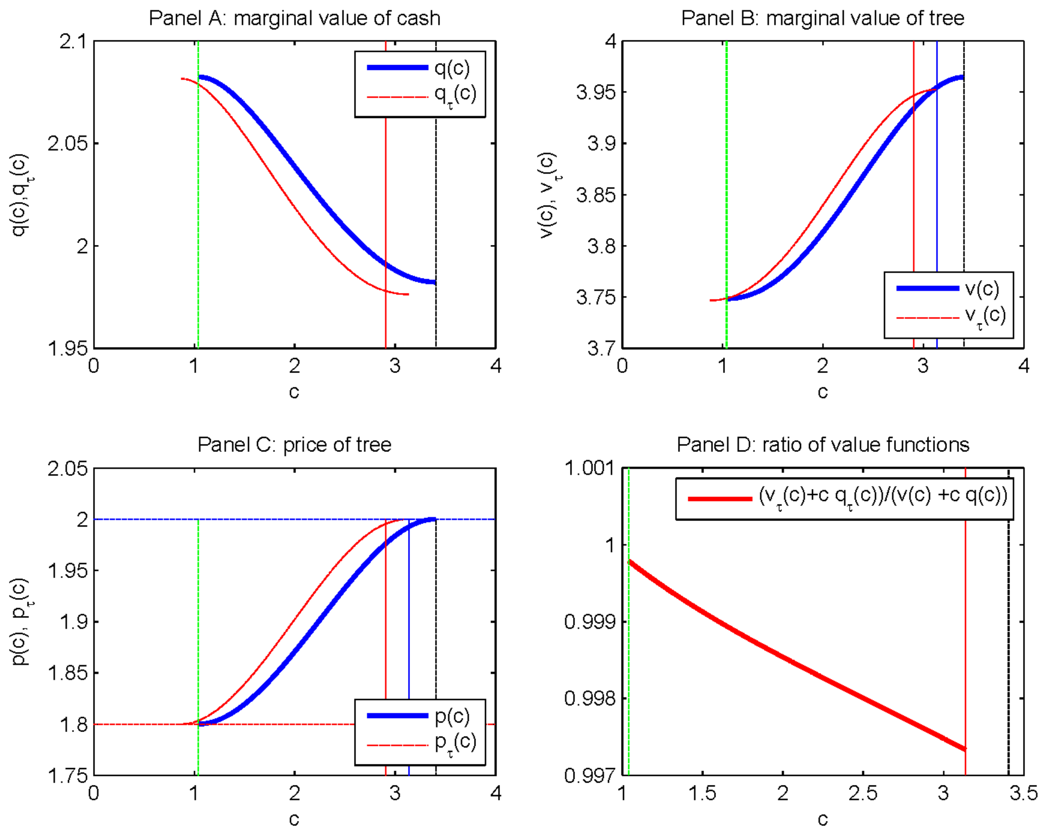

Figure 5: The marginal value of cash, the marginal value of a tree, the price of the tree and the ratio of value functions for our baseline model with incomplete markets (thick solid curves) and a particular one-sided intervention (thin, dashed curves). On Panels A, B and C, the solid, vertical lines in the midle are the thresholds for intervention, c_0, and the post-intervention investment threshold, c_h^τ.while the two dashed vertical lines close to the edges of each Panel are the disinvesent and invesment thresholds in our baseline case, c_l^*, c_h^*. Parameter values are $R = 4.1$, $\sigma^2 = 1$, $\xi = 0.1$, $u = 2$, $l = 1.8$ and $h = 2$ and $c_0 = c_h^* - 0.5$ and $\tau = 0.015$.

$$v(c) = 1_{c<c_0}\frac{\tau}{\xi}c + R + \frac{uc}{2} + e^{c\gamma}\left((1_{c>c_0}M_3 + 1_{c<c_0}M_7) - c(1_{c>c_0}M_1 + 1_{c<c_0}M_5)\right)$$
$$- e^{-c\gamma}\left((1_{c>c_0}M_4 + 1_{c<c_0}M_8) + c(1_{c>c_0}M_2 + 1_{c<c_0}M_6)\right) + cR\frac{\gamma}{2}\frac{\left(e^{\gamma c}\,\mathrm{Ei}(-\gamma c) - e^{-c\gamma}\,\mathrm{Ei}(\gamma c)\right)}{2} \quad (40)$$

where the constants $M_1, ..., M_8$ are given by (38)-(39) and the smooth-pasting conditions at c_0 for $v(c)$ and $q(c)$.

We plot one particular example on Figure 5. It is apparent that while the policy raises prices, decreases both the investment and disinvestment thresholds, $c_l^\tau < c_l^*, c_h^\tau < c_h^*$, it also reduces welfare at every point. Thus, the depicted one-sided intervention is Pareto inferior compared to the symmetric market equilibrium with no-intervention.

It is instructive to connect this result to the current debate on "Greenspan's put", i.e., the doctrine that it is sufficient if monetary policy intervenes in a recession, but stays inactive when the economy is recovered. In our abstract model we can interpret our taxes-and-subsidies schemes as vague representations of an expansionary monetary policy. An interest rate cut decreases incentives to save cash and increases incentives to invest in productive assets, just as our simple one-sided intervention does. Our result shows that such interventions might be harmful even at the recession.

Recently, several papers proposed arguments against the Greenspan's put including Farhi and Tirole (2011) and Diamond and Rajan (2011). However, their argument is different In Diamond and Rajan (2011) the main friction is that banks can provide only non-state contingent demand deposit contract to households. In such a world, ex post inefficient bank-runs serve as a disciplining device for banks to honour these contracts. Anticipated interest rate cuts in bad times helps insolvent banks ex post, but weakens this disciplining device ex ante. As result, banks take on too much leverage ex ante, and subject to runs ex post too often. Farhi and Tirole (2011) makes a similar argument showing that there is strategic complementarity in the choice of increasing leverage ex ante, and, consequently, needing a more frequent non-directed bail-out in the form of low interest rates ex post. In both of these papers incentive problems related to the agency friction inherent in financial intermediation play the central role. In contrast, the interaction of a pecuniary externality and incomplete markets is in the center of our mechanism.

Until now we have put little emphasis on the parameters which imply a one-sided inefficiency. That is, on the case implying underinvestment in trees both in the boom and in the recession. It is useful to note, however, that in this case, a price increasing one-sided intervention (at least if it is sufficiently small) improves welfare by pushing the economy closer to the second-best both in a recession and in a boom. Thus, an alternative reading of our results is that the pros and cons of an asymmetric interest rate policy depend on the nature of the externality we face. In our case, it depends on whether the technology represented by trees are much more productive than the idiosyncratic investment opportunity or not. Only in the latter case a one-sided intervention tends to be harmful.

5 Robustness: an alternative specification

In this section, we argue that in an abstract level, our mechanism is based on three main ingredients:

1. Two assets of which relative supply is affected by a stochastic process.

2. A group of agents who can transform each asset to the other one by a linear technology, but with some loss in the process.

3. An idiosyncratic shock which changes some agents' relative valuation of the assets compared to other agents.

In particular, we present an alternative specification which also incorporate this three ingredients but a different dynamic structure. This variant generates the same main results. We emphasize that the specific structure in our baseline model that the idiosyncratic shock is connected to the maturity event is immaterial for our results.

5.1 The Setting

In this variant, we make two major changes. First, the Poisson event that the shock matures and the idiosyncratic shock is separated. In particular, the asset matures with Poisson intensity ϕ. When it does, all agents who are still in the market and hold the tree harvest the R fruit or cash dividend. However, in each point of time ξdt fraction of the agents are hit by the skill shock. That is, they do not have the skill to harvest the tree, but have the skill to invest in a new opportunity with a marginal return of $u > 1$. As a result, in each instant, a group of agents with measure ξdt sell all their trees to the rest of the specialists and exit the market.

The second change is about timing of (dis)investment opportunities. Instead of letting the agents to invest and disinvest at any point, we assume that they can do so only irregularly. In particular, with intensity η, a Poisson event realizes when all specialists on the market are allowed to build trees at cost h or liquidate trees at cost l in any amount they wish. Given that in this variant there is no guarantee that the aggregate cash level is kept away from zero by disinvesting if necessary, we also assume an infinite pool of outside capital which can inject 1 unit of cash to this market for a total cost of $\lambda > 1$. That is, outside investors can acquire the knowledge of specialists, but it is costly. We think of λ to be sufficiently high.

In a certain sense, in this variant, the dynamic structure is the mirror image of that of our baseline model. While in our baseline model specialists can invest and disinvest in any instant before the tree matures, they learn the realization of the idiosyncratic shock only at maturity at which they cannot invest or disinvest further. In this variant, in each instant a group of agents learn the realization of the idiosyncratic shock, specialists can invest and disinvest only in discrete time periods.

5.2 HJB Equations

Following our logic before, the value of tree $v(c)$ and the value of cash $q(c)$ in the market solution have to satisfy the following ODEs as HJB equations:

$$\begin{aligned} 0 &= -v'(c+p)\xi + q'(c_t)\sigma^2 + \frac{\sigma^2}{2}v'' + \xi(pu - v) + \phi(R - v) + \\ & \quad \eta\left(\left(\mathbf{1}_{c>c_h}v(c_h) + \mathbf{1}_{c<c_l}v(c_l) + \mathbf{1}_{c_h>c>c_l}v(c)\right) - v(c)\right), \quad (41) \\ 0 &= -q'(c+p)\xi + \frac{\sigma^2}{2}q'' + \xi(u - q) + \phi(1 - q) \\ & \quad +\eta\left(\left(\mathbf{1}_{c>c_h}q(c_h) + \mathbf{1}_{c<c_l}q(c_l) + \mathbf{1}_{c_h>c>c_l}q(c)\right) - q(c)\right). \quad (42) \end{aligned}$$

with

$$p(c) = \frac{v(c)}{q(c)},$$

and the following six boundary conditions

$$
\begin{aligned}
v'(0) &= 0 && (43)\\
q(0) &= \lambda && (44)\\
p(c_l) &= \frac{v(c_l)}{q(c_l)} = l && (45)\\
p(c_h) &= \frac{v(c_h)}{q(c_h)} = h && (46)\\
\lim_{c\to\infty} v'(c) &= 0 && (47)\\
\lim_{c\to\infty} q'(c) &= 0. && (48)
\end{aligned}
$$

In the ODEs, there are two main changes compared to our baseline model. First, the first term in each equation is due to the fact that this variant introduces a drift term into the dynamics of c, as agents with idiosyncratic shocks are leaving with cash at any instant of time. In particular, whenever there is no investment opportunities, we have

$$dc = -\xi(c+p)\,dt + \sigma dZ_t.$$

The drift is because ξdt fraction of agents leave with the cash with the normalized value of their portfolio $c+p$. More specifically, they leave with their cash holding c, and also sell their tree holdings at a price of p and leave the market with these proceeds.

Second, the last bracketed term in each equation, i.e., (41) and (42), is due to the fact that when specialists have the opportunity to invest or disinvest, they will do so, only if the price level makes the decision optimal. In particular, if $p > h$, they build new trees until the point where the cash level falls to c_h where $p(c_h) = h$. Similarly, if $p < l$, they liquidate trees until the point where the cash level is raised to c_l where $p(c_l) = l$. Due to this adjustment, whenever the opportunity to change the measure of trees arrives, the value of a tree, $v(c)$, and the value of cash, $q(c)$, jumps to $v(c_h)$ and $q(c_h)$ if $c > c_h$, and to $v(c_l)$ and $q(c_l)$ if $c < c_l$, and remains unchanged in every other case.

Turning to the boundary conditions, condition (43) holds because $c = 0$ is a reflective barrier. Condition (44) has to hold, because outside capital is injected whenever the value of cash is larger than λ. Conditions (45)-(46) are determined by the adjustment of the measure of trees explained above. The last two conditions have to hold to ensure that the value of cash and trees does not increase or decrease without bound even when the cash level is very high.

Unlike in the case of the baseline model, no analytical solution of this system is available. Thus, we have to rely on numerical solutions. Panels A,B and C on Figure 6 show the functions $q(c)$, $v(c)$ and $p(c)$ in a particular example. It is apparent, that just as before, $q(c)$ is a decreasing function while $p(c)$ is an increasing function. Intuitively, this variant also generates investment waves where the economy fluctuates between boom periods characterized by investment and low expected returns and bust periods characterized by disinvestment and high expected returns. The difference is that in our baseline model whenever c hit the lower or upper threshold disinvestment or investment began and continued as long as a contrarian shock does not hit the system. Under our alternative specification, investment and disinvestment is lumpy. Whenever c is below or above the corresponding thresholds and the Poisson even hits, a large amount of investment or disinvestment occurs in one instant. When the Poisson event does not hit, there is no change in the number of trees regardless of the value of c.

5.3 Two-Sided Inefficiency and Government Intervention

Does this economy constrained efficient? Just as before, we asses whether a social planner could improve welfare by only changing the investment/disinvestment thresholds c_h, c_l instead of leaving their determination to the market. We characterizes the ODE for the equilibrium value functions v and q in the Appendix, and solve them numerically.

On Panel D of Figure 6 we compare the market equilibrium with three economies with suboptimal policies. In the first one, we marginally decrease c_l compared to the market solution c_l^*. In the second one, we marginally increase c_h compared to c_h^*, in the third one we do both.[21] Under each scenarios, the value increases compared to the decentralized outcome in every state. This illustrates that just as in our baseline model, there is a two-sided inefficiency in this variant as well.

The intuition is similar to what we have illustrated in the baseline model. As in the baseline model, agents might suffer idiosyncratic skill shocks which force them to sell the tree. And, when the aggregate cash level is low (high), the equilibrium price is low (high), exactly because of the cash-in-the-market setting that we are considering. Therefore, as in our baseline model, these prices should affect the agents' ex ante investment incentives when the investment/disinvestment opportunities arrive occasionally. Take the example when the current aggregate cash is low and agents now can disinvest to convert trees to cash. Because agents worry that before the next opportunity comes they might be hit by a skill shock and therefore sell their trees to other agents, while the social planner should ignore this transfer issues, agents will disinvest excessively so that they hold more cash more than the social planner wants. The same idea applies to the state with abundant aggregate cash. There, agents will invest more than the social planner wants, because they will take into account the fact that in the near future, once hit by liquidity shocks, the tree can be sold at a high equilibrium price. As a result, they will invest cash to obtain the tree, although the social value of the cash, i.e., u, can be higher than $v(c_h)/h = q(c_h)$. These distorted ex ante investment/disinvestment incentives are no longer there if the investment technologies are always available.

[21] In the first case $c_h = c_h^* + 0.1$, in the second case $c_l = c_l^* - 0.01$, while in the third case $c_h = c_h^* + 0.05$ and $c_l = c_l^* - 0.02$. Were we to set $c_h = c_h^* + 0.1$ and $c_l = c_l^* - 0.01$ in the third case, the dotted curve would be indistinguishable from the upper envelope of the solid and dashed curves.

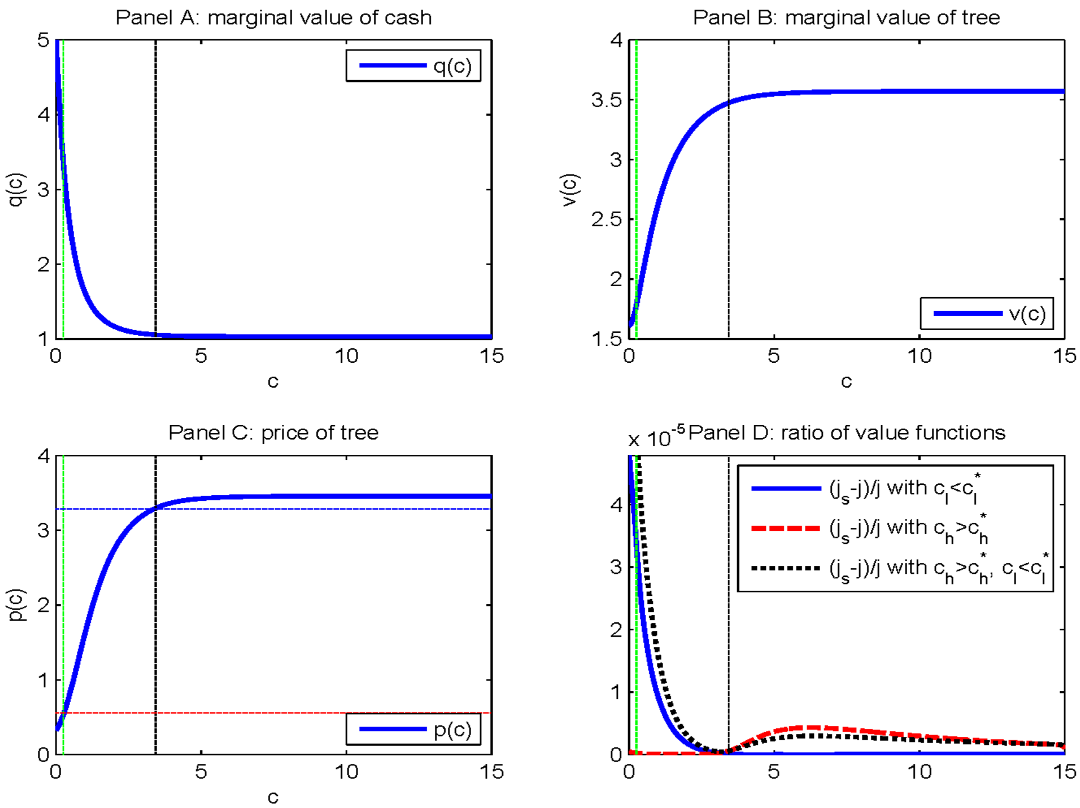

Figure 6: Panel A, B and C shows the value of a unit of cash, $q(c)$, the value of a unit of asset, $v(c)$, and the price of the asset, $p(c)$, in the decentralized equilibrium under our alternative specification. Panel D shows the relative change in the value when the lower and upper thresholds are changed as follows. The dashed curve corresponds to $c_h = c_h^* + 0.1$, the solid curve corresponds to $c_l = c_l^* - 0.01$, while the dotted curve corresponds to $c_h = c_h^* + 0.05$ and $c_l = c_l^* - 0.02$. Figures of functions $v(c), q(c)$ and $p(c)$ under these scenarios are indistinguishable from the baseline case. Vertical lines in all panels correspond to c_l^*, c_h^*, while the horizontal lines on Panel C correspond to h and l. Parameter values are $R = 3.5$, $\sigma^2 = 0.3$, $\xi = 0.1$, $u = 1.25$, $l = 0.55$, $h = 3.28$, $\phi = 0.8$, $\lambda = 5$, $\eta = 0.2$.

6 Conclusion

In this paper, we built an analytically tractable, dynamic stochastic model of investment and trade of a specialized asset. We argue that if specialized technologies are a determining part of the economy, investment cycles arise as a dominant pattern. That is, boom periods with abundant investment and low returns on this technology will interchange with bust periods with low investment and high returns. We showed that these cycles might or might not be constrained efficient. In particular, while under complete markets, there are constrained efficient investment cycles, in the presence of unverifiable idiosyncratic investment opportunities a two-sided inefficiency can arise. That is, there are two much investment in the technology and too low buffer in cash in booms, and there are too little investment and too much cash holdings in recessions. We show that in this case a one-sided policy targeting only the underinvestment in the recession period might be ex ante Pareto inferior to no intervention in every state.

Apart from analyzing two-sided inefficiencies, we also presented a novel way of modelling problems of investment and trade. This method provides analytical tractability in a dynamic stochastic framework for the full joint distribution of states and equilibrium objects. To explore its potential, we use this framework to analyze the role of sovereign wealth funds in financial crises by introducing groups of specialists with different level of skills in a parallel project.

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A Appendix

A.1 Proof of Lemma 1 and Proposition 1

We construct the proof in steps. In particular, we separate Proposition 1 into the following four Lemmas.

Lemma A.1 *If the equation system (17)-(18), (13)-(14) has a solution where* $c_h^* < R$ *and* $v(c)$ *is increasing and* $q(c)$ *is increasing in the range* $c \in [c_h^*, c_l^*]$ *then Proposition 1 hold.*

Lemma A.2 *The system (17)-(18), (13)-(14) always have at least one solution.*

Lemma A.3 *If* $h - l$ *is sufficiently small,* $c_h^* < R$.

Lemma A.4 *If* $h-l$ *is sufficiently small* $q(c)$ *is monotonically decreasing and* $v(c)$ *is monotonically increasing in the range* $c \in [c_h^*, c_l^*]$.

Clearly, the four lemmas are sufficient to prove Proposition 1.

A.1.1 Step 1: Proof of Lemma 1 and Lemma A.1

Denote the dollar share of tree in the specialist's portfolio by ψ_t^i, so that $\psi_t^i = A_t^i p_t / w_t^i$. According to our conjecture, the value function can be written as (recall the aggregate cash-to-tree ratio $c = C/A$)

$$J\left(A_t, C_t, A_t^i, C_t^i\right) = w_t^i \left[\left(1 - \psi_t^i\right) q\left(c_t\right) + \frac{\psi_t^i}{p_t} v\left(c\right)\right] = J\left(A_t, C_t, w_t^i\right),$$

is linear in w_t. Note that this is equivalent to the statement

$$J\left(C, A, A_t^i, C_t^i\right) = A_t^i v\left(c\right) + C_t^i q\left(c\right)$$

stated in the Lemma. Also, we have the wealth dynamics, expressed in terms of portfolio choice ψ_t^i, as

$$dw_t^i = -d\alpha_t^i - \theta dA_t^i + \psi_t^i w_t^i \frac{1}{p_t}\left(dp_t + \sigma dZ_t\right).$$

The HJB of problem (10) can be equivalently written as the agent is choosing consumption $d\alpha_t^i$ and portfolio share ψ_t^i, and the tree to build or liquidate dA_t^i

$$0 = \max_{d\alpha_t^i, \psi_t^i, dA_t^i} d\alpha_t^i + J_C E_t\left(dC_t\right) + \frac{1}{2} J_{CC} dC_t^2 + J_w E_t\left(dw_t\right) + J_A' dA_t^i + J_{w,C} E_t\left(dw_t dC_t\right).$$

Let also conjecture that $q\left(c_t\right) > 1$ for every c_t, thus specialists do not consume before the tree matures. Denote the endogenous drift and volatility of prices as

$$dp_t = \frac{1}{2}\sigma^2 p''\left(c\right) dt + \sigma p'\left(c\right) dZ_t + dL_t^p - dU_t^p$$

where dL_t^p (dU_t^p) reflects p at $p(c_l^*) = l$ $(p(c_h^*) = h)$. This is because as we explained in the main text, in any market equilibrium specialists will create trees if $p_t = h$ and destroy trees if $p_t = l$ otherwise they would use the market to adjust the number of their trees. We derived the boundary conditions in the main text. Also, by risk neutrality and ex ante homogeneity of agents, before the tree matures the price of the tree has to make specialists indifferent whether to hold trees or cash. Otherwise markets could not clear. We also explained that $p_T = c_t$.

Thus, for the range of $p \in (l, h)$ we can rewrite the Hamiltonian as (we drop i from now on)

$$0 = \max_{\psi_t, \psi_t^B} \left\{ \begin{array}{c} \frac{\sigma^2}{2} w_t q_c'' + q(c_t)\psi_t w_t \frac{\frac{1}{2}\sigma^2 p''(c)}{p_t} dt + q'(c_t)\left(\left(\psi_t w_t \frac{1}{p_t}(\sigma + p'\sigma)\right)\sigma\right) dt \\ +\xi w_t \left(\frac{1}{2}\left(\frac{\psi_t R}{p_t} + (1-\psi_t)\frac{R}{c_t}\right) + \frac{1}{2}\left(\frac{\psi_t}{p_t} u c_t + (1-\psi_t) u\right) - q(c)\right) \end{array} \right\}.$$

Note that this is indeed linear in w_t. It is also linear in ψ_t, so specialists are indifferent in their choice of ψ_t. Thus, we can separate the problem for calculating the dynamics of the cash value by choosing $\psi_t = 0$ and dynamics of tree value by choosing $\psi = 1$. The first choice directly implies (15). As long as $u, \frac{R}{c} > 1$, our conjecture that $q(c) > 1$ must also hold as specialists holding only cash get either u or $\frac{R}{c}$ at maturity and they do not discount the future. Choosing $\psi_t = 1$ gives

$$0 = \frac{\sigma^2}{2} q''(c) + q(c)\frac{\frac{1}{2}\sigma^2 p''(c)}{p_t} + q'(c)\left(\frac{1}{p_t}(\sigma + p'\sigma)\sigma\right) + \frac{1}{p_t}\left(\frac{\xi}{2}(R + uc) - q(c)p_t\right).$$

Rearrange, we have

$$0 = \frac{\sigma^2}{2} p(c) q'' + q(c)\left(\frac{1}{2}\sigma^2 p''(c) + R\right) + q'(c_t)\left((\sigma + p'\sigma)\sigma\right) + \left(\frac{\xi}{2}(uc - q(c)p_t) + \frac{\xi}{2}(R - q(c)p_t)\right)$$

Since $v(c) = p(c)q(c)$ and $v' = q'p + p'q$, we can rewrite this

$$0 = \frac{\sigma^2}{2} p(c) q''(c) + q(c)\left(\frac{1}{2}\sigma^2 p''(c) + R\right) + q'(c_t)\left((\sigma + p'\sigma)\sigma\right) + \left(\frac{\xi}{2}(uc - v) + \frac{\xi}{2}(R - v)\right)$$

using that $v'' = q''p + 2p'q' + p''q$ gives (16). Given that the ODEs for $v(c)$ and $q(c)$ were derived by substituting in $\psi_t = 1$ and $\psi_t = 0$, it is easy to see that these functions can be interpreted as the value of a tree and that of a unit of cash. This implies that

$$J\left(C, A, w_t^i\right) = \left(w_t^i\left(1 - \psi_t^i\right) q(c) + \frac{\psi_t^i}{p_t} w_t^i v(c)\right) = q(c) w_t$$

verifying both Lemma 1 and our conjecture on the form of $J\left(C, A, w_t^i\right)$. Also, the definition of $v(c)$ is equivalent to a no-arbitrage condition ensuring that specialists are indifferent whether to hold the tree or cash of equivalent value. One can check that (18) and (17) are indeed solutions of our system of ODEs.

A.1.2 Step 2: Proof of Lemma A.2

First, note that for any arbitrary c_h and c_l from (14), we can express K_1, K_2, K_3 and K_4 in (17)-(18) as a function of c_h and c_l only. Substituting back to (17)-(18) we get our functions parameterized by c_h and c_l which we denote as $v(c; c_l, c_h)$ and $q(c; c_l, c_h)$. Evaluating these functions at $c = c_l$ and $c = c_h$, we get the following expressions. These notations help us rewrite our critical equations as

$$
\begin{aligned}
q(c_l; c_l, c_h) &= \frac{u}{2} + \frac{R\xi}{\gamma\sigma^2} \frac{e^{-\gamma c_h}(\mathrm{Ei}[c_h\gamma] - \mathrm{Ei}[c_l\gamma]) + e^{\gamma c_h}(\mathrm{Ei}[-c_h\gamma] - \mathrm{Ei}[-c_l\gamma])}{e^{\gamma(c_h-c_l)} - e^{-\gamma(c_h-c_l)}} = \\
&= \frac{u}{2} + \frac{R\xi}{\gamma\sigma^2} f_l(c_l, c_h) \\
v(c_l; c_l, c_h) &= R + \frac{c_l u}{2} + \frac{u}{2\gamma} m(c_l, c_h) + \frac{R\xi}{\gamma\sigma^2}\left(\frac{g_l(c_l, c_h)}{\gamma} - c_l f_l(c_l, c_h)\right)
\end{aligned}
$$

$$
\begin{aligned}
q(c_h; c_l, c_h) &= \frac{u}{2} + e^{-c_h\gamma}K_1 + e^{\gamma c_h}K_2 + \frac{R\xi}{2}\frac{\left(-e^{\gamma c_h}\,\mathrm{Ei}[-\gamma c_h] + e^{-\gamma c_h}\,\mathrm{Ei}[c_h\gamma]\right)}{\sqrt{2\xi\sigma^2}} = \\
&= \frac{u}{2} + \frac{R\xi}{\gamma\sigma^2}\frac{e^{-\gamma c_l}(\mathrm{Ei}[c_h\gamma] - \mathrm{Ei}[c_l\gamma]) + e^{\gamma c_l}(\mathrm{Ei}[-\gamma c_h] - \mathrm{Ei}[-\gamma c_l])}{e^{\gamma(c_h-c_l)} - e^{-\gamma(c_h-c_l)}} = \\
&= \frac{u}{2} + \frac{R\xi}{\gamma\sigma^2} f_h(c_l, c_h)
\end{aligned}
$$

$$
\begin{aligned}
v(c_h) = R + \frac{c_h u}{2} + e^{c_h\gamma}K_3 - e^{-c_h\gamma}K_4 - c_h\left(q(c_h) - \frac{u}{2}\right) &= \\
= R + \frac{c_h u}{2} + \frac{u\left(2 - \left(e^{\gamma(c_h-c_l)} + e^{-\gamma(c_h-c_l)}\right)\right)}{2\gamma\left(e^{\gamma(c_h-c_l)} - e^{-\gamma(c_h-c_l)}\right)} &+ \\
\frac{R\xi}{\gamma\sigma^2}\frac{e^{-\gamma c_l}(\mathrm{Ei}[c_h\gamma] - \mathrm{Ei}[c_l\gamma]) + e^{\gamma c_l}(\mathrm{Ei}[-\gamma c_l] - \mathrm{Ei}[-\gamma c_h])}{\gamma\left(e^{\gamma(c_h-c_l)} - e^{-\gamma(c_h-c_l)}\right)} - c_h\left(q(c_h) - \frac{u}{2}\right) & \\
= R + \frac{c_h u}{2} - \frac{u}{2\gamma} m(c_l, c_h) + \frac{R\xi}{\gamma\sigma^2}\left(\frac{g_h(c_l, c_h)}{\gamma} - c_h f_h(c_l, c_h)\right) &
\end{aligned}
$$

where we used the definitions

$$
\begin{aligned}
f_l(c_l, c_h) &\equiv \frac{e^{-\gamma c_h}(\mathrm{Ei}[c_h\gamma] - \mathrm{Ei}[c_l\gamma]) + e^{\gamma c_h}(\mathrm{Ei}[-c_h\gamma] - \mathrm{Ei}[-c_l\gamma])}{e^{\gamma(c_h-c_l)} - e^{-\gamma(c_h-c_l)}} \\
g_l(c_l, c_h) &\equiv \frac{e^{-\gamma c_h}(\mathrm{Ei}[c_h\gamma] - \mathrm{Ei}[c_l\gamma]) + e^{\gamma c_h}(\mathrm{Ei}[-\gamma c_l] - \mathrm{Ei}[-\gamma c_h])}{e^{\gamma(c_h-c_l)} - e^{-\gamma(c_h-c_l)}} \\
f_h(c_l, c_h) &\equiv \frac{e^{-\gamma c_l}(\mathrm{Ei}[c_h\gamma] - \mathrm{Ei}[c_l\gamma]) + e^{\gamma c_l}(\mathrm{Ei}[-\gamma c_h] - \mathrm{Ei}[-\gamma c_l])}{e^{\gamma(c_h-c_l)} - e^{-\gamma(c_h-c_l)}} \\
g_h(c_l, c_h) &\equiv \frac{e^{-\gamma c_l}(\mathrm{Ei}[c_h\gamma] - \mathrm{Ei}[c_l\gamma]) + e^{\gamma c_l}(\mathrm{Ei}[-\gamma c_l] - \mathrm{Ei}[-\gamma c_h])}{e^{\gamma(c_h-c_l)} - e^{-\gamma(c_h-c_l)}} \\
m(c_l, c_h) &\equiv \frac{e^{\gamma(c_h-c_l)} - 1}{1 + e^{\gamma(c_h-c_l)}} \in (0, 1)
\end{aligned}
$$

Define the function $c_l = H(c_h)$ by

$$p(c_h; c_l, c_h) \equiv \frac{v(c_h; c_l, c_h)}{q(c_h; c_l, c_h)} = h$$

while $c_l = L(c_h)$ is defined implicitly by

$$p(c_l; c_l, c_h) \equiv \frac{v(c_l; c_l, c_h)}{q(c_l; c_l, c_h)} = l.$$

It is easy to see that if there is a c_h that $H(c_h) = L(c_h)$ then this particular c_h and the corresponding $c_l = H(c_h)$ is a solution of (13)-(14), (17)-(18). To show that this intercept exists, we first establish properties of $L(c_h)$ then we proceed to the properties of $H(c_h)$.

Properties of $L(c_h)$ It is useful to observe that

$$\begin{aligned}
\frac{\partial f_l}{\partial c_l} &= \frac{\left(e^{2\gamma c_h} + e^{2\gamma c_l}\right)}{\left(e^{2\gamma c_h} - e^{2\gamma c_l}\right)}\left(\gamma f_l - \frac{1}{c_l}\right) \\
\frac{\partial f_l}{\partial c_h} &= 2\frac{\frac{1}{c_h} - \gamma f_h}{e^{\gamma(c_h - c_l)} - e^{\gamma(c_l - c_h)}} \\
\frac{\partial g_l}{\partial c_l} &= \frac{1}{c_l} + \frac{\left(e^{2\gamma c_h} + e^{2\gamma c_l}\right)}{\left(e^{2\gamma c_h} - e^{2\gamma c_l}\right)}\gamma g_l \\
\frac{\partial g_l}{\partial c_h} &= -\frac{2\gamma g_h}{e^{\gamma(c_h - c_l)} - e^{\gamma(c_l - c_h)}}
\end{aligned}$$

$$\lim_{c_l \to c_h} f_l = \frac{1}{\gamma c_h}, \quad \lim_{c_l \to c_h} g_l = 0, \quad \lim_{c_l \to c_h} m = 0.$$

1. We show that $f_l(c_h, c_l)$ is monotonically decreasing in c_l.Its slope in c_l is

$$\frac{\partial f_l}{\partial c_l} = \frac{\left(e^{2\gamma c_h} + e^{2\gamma c_l}\right)}{\left(e^{2\gamma c_h} - e^{2\gamma c_l}\right)}\left(\gamma f_l(c_h, c_l) - \frac{1}{c_l}\right) < 0$$

second derivative is

$$\begin{aligned}
\frac{\partial^2 f_l}{\partial^2 c_l} &= -\left(4\gamma e^{2\gamma c_h}\frac{e^{2\gamma c_l}}{\left(e^{2\gamma c_h} - e^{2\gamma c_l}\right)^2} - \frac{\left(e^{2\gamma c_h} + e^{2\gamma c_l}\right)}{\left(e^{2\gamma c_h} - e^{2\gamma c_l}\right)}\gamma\frac{\left(e^{2\gamma c_h} + e^{2\gamma c_l}\right)}{\left(e^{2\gamma c_h} - e^{2\gamma c_l}\right)}\right)\left(\frac{1}{c_l} - \gamma f_l(c_h, c\right. \\
-\frac{\left(e^{2\gamma c_h} + e^{2\gamma c_l}\right)}{\left(e^{2\gamma c_h} - e^{2\gamma c_l}\right)}\left(-\frac{1}{c_l^2}\right) &= \gamma\left(\frac{1}{c_l} - \gamma f_l(c_h, c_l)\right) + \frac{\left(e^{2\gamma c_h} + e^{2\gamma c_l}\right)}{\left(e^{2\gamma c_h} - e^{2\gamma c_l}\right)}\frac{1}{c_l^2}
\end{aligned}$$

Thus, this function can have only minima, but no maxima. At the limit

$$\lim_{c_l \to c_h}\left(\frac{1}{c_l}\frac{\left(e^{2\gamma c_h} + e^{2\gamma c_l}\right)}{\left(e^{2\gamma c_h} - e^{2\gamma c_l}\right)}\left(\gamma c_l f_l(c_h, c_l) - 1\right)\right) = \frac{1}{c_h}\left(-\frac{1}{2\gamma c_h}\right) < 0$$

Thus, $f_l(c_h, c_l)$ is decreasing at $c_h = c_l$. this implies that it has to be monotonically decreasing over the whole range of $c_l < c_h$. (suppose that there is a $\hat{c}_l < c_h$ where the first derivative is zero, so f is minimal. as we decrease c_l from c_h f is increasing as long as $\hat{c}_l < c_l < c_h$, but it is a contradiction with $\hat{c}_l$ being a minimum point)

2. We show that the function $\frac{g_l(c_h,c_l)}{\gamma} - c_l f_l(c_h, c_l)$ is monotonically increasing in c_l.(all the derivatives in this part are with respect to c_l)

$$\begin{aligned}\left(\frac{g_l}{\gamma} - c_l f_l\right)' &= \frac{1}{\gamma c_l} + \frac{\left(e^{2\gamma c_h} + e^{2\gamma c_l}\right)}{\left(e^{2\gamma c_h} - e^{2\gamma c_l}\right)} g_l(c_l, c_h) - \left(\frac{\left(e^{2\gamma c_h} + e^{2\gamma c_l}\right)}{\left(e^{2\gamma c_h} - e^{2\gamma c_l}\right)} \left(c_l \gamma f_l(c_h, c_l) - 1\right) + f_l(c_l, c_h)\right) \\ &= \left(\frac{1}{\gamma c_l} + \gamma \frac{\left(e^{2\gamma c_h} + e^{2\gamma c_l}\right)}{\left(e^{2\gamma c_h} - e^{2\gamma c_l}\right)} \left(\frac{g_l(c_l, c_h)}{\gamma} - c_l f_l(c_h, c_l)\right) + \frac{\left(e^{2\gamma c_h} + e^{2\gamma c_l}\right)}{\left(e^{2\gamma c_h} - e^{2\gamma c_l}\right)} - f_l(c_l, c_h)\right)\end{aligned}$$

if the first derivative is zero than at that point

$$\left(\frac{g_l}{\gamma} - c_l f_l\right) = \frac{\left(f_l(c_l, c_h) - \frac{1}{\gamma c_l}\right)}{\gamma \frac{\left(e^{2\gamma c_h} + e^{2\gamma c_l}\right)}{\left(e^{2\gamma c_h} - e^{2\gamma c_l}\right)}} - \frac{1}{\gamma}$$

we also know that

$$\begin{aligned}\lim_{c_l \to c_h} \left(\frac{g_l}{\gamma} - c_l f_l\right)' &= 0 \\ \lim_{c_l \to c_h} \left(\frac{g_l}{\gamma} - cf\right)'' &= -\frac{1}{3\gamma c_h^2} < 0\end{aligned}$$

so for any fixed c_h, $c_l = c_h$ is a local maximum. In general

$$\begin{aligned}\left(\frac{g_l}{\gamma} - c_l f_l\right)'' &= -\frac{1}{\gamma c_l^2} + \gamma \frac{\left(e^{2\gamma c_h} + e^{2\gamma c_l}\right)}{\left(e^{2\gamma c_h} - e^{2\gamma c_l}\right)} \left(\frac{g_l}{\gamma} - c_l f_l\right)' - \frac{\partial f_l}{\partial c_l} + \\ &\quad 4e^{2\gamma c_h} \frac{e^{2\gamma c_l}}{\left(e^{2\gamma c_h} - e^{2\gamma c_l}\right)^2} \gamma^2 \left(\left(\frac{g_l}{\gamma} - c_l f_l\right) + \frac{1}{\gamma}\right)\end{aligned}$$

Thus, if there were a $\hat{c}_l$ that the first derivative were zero and $c_h > \hat{c}_l$ then the second derivative were

$$\begin{aligned}-\frac{1}{\gamma c_l^2} - f'(c_l, c_h) + 4e^{2\gamma c_h} \frac{e^{2\gamma c_l}}{\left(e^{2\gamma c_h} - e^{2\gamma c_l}\right)^2} \gamma^2 \left(\frac{\left(f_l - \frac{1}{\gamma c_l}\right)}{\gamma \frac{\left(e^{2\gamma c_h} + e^{2\gamma c_l}\right)}{\left(e^{2\gamma c_h} - e^{2\gamma c_l}\right)}} - \frac{1}{\gamma} + \frac{1}{\gamma}\right) &= \\ = -\frac{1}{\gamma c_l^2} - \gamma \frac{\left(e^{2\gamma c_h} - e^{2\gamma c_l}\right)}{e^{2\gamma c_h} + e^{2\gamma c_l}} \left(f_l - \frac{1}{\gamma c_l}\right)\end{aligned}$$

can this be non-negative? for that we would need f_l

$$(c_l, c_h) < -\frac{\frac{1}{\gamma c_l}\left(\frac{1}{c_l} - \gamma\frac{\left(e^{2\gamma c_h} - e^{2\gamma c_l}\right)}{e^{2\gamma c_h} + e^{2\gamma c_l}}\right)}{\gamma\frac{\left(e^{2\gamma c_h} - e^{2\gamma c_l}\right)}{e^{2\gamma c_h} + e^{2\gamma c_l}}} = \frac{1}{\gamma c_l}\frac{\left(\gamma\frac{\left(e^{2\gamma c_h} - e^{2\gamma c_l}\right)}{e^{2\gamma c_h} + e^{2\gamma c_l}} - \frac{1}{c_l}\right)}{\gamma\frac{\left(e^{2\gamma c_h} - e^{2\gamma c_l}\right)}{e^{2\gamma c_h} + e^{2\gamma c_l}}} < \frac{1}{\gamma c_l}$$

as $f(c_l, c_h)$ is decreasing in c_l and at

$$\lim_{c_l \to c_h} f_l(c_l, c_h) = \frac{1}{\gamma c_l}$$

this could be non-negative only at a $c_l > c_h$. So the second derivative is always negative at a point where the first derivative is zero, which implies it does not have another extreme point so $\left(\frac{g}{\gamma} - cf\right)' > 0$ for any $c_l < c_h$.

3. Given that f_l is decreasing c_l, $q(c_l; c_l, c_h)$ is also decreasing in c_l for any $c_l < c_h$. Given that $\left(\frac{g_l}{\gamma} - c_l f_l\right)' > 0$ and

$$\frac{\partial\left(\frac{c_l u}{2} + \frac{u\left(e^{-\gamma(c_h - c_l)} + e^{\gamma(c_h - c_l)} - 2\right)}{2\gamma\left(e^{\gamma(c_h - c_l)} - e^{-\gamma(c_h - c_l)}\right)}\right)}{\partial c_l} = \frac{1}{2}u\frac{\exp(-2\gamma c_h + 2\gamma c_l) + 1}{\left(e^{-\gamma c_h + \gamma c_l} + 1\right)^2} > 0$$

$v(c_l; c_l, c_h)$ is increasing in c_l. Thus, $p(c_l; c_l, c_h)$ is increasing in c_l for any $c_l < c_h$. Also

$$\lim_{c_l \to 0} = p(c_l; c_l, c_h) = -\frac{\tanh(\gamma c_h)}{\gamma} < 0$$

and

$$\lim_{c_l \to c_h} p(c_l; c_l, c_h) = \frac{R + c_h\frac{u}{2} + \frac{R\xi}{\gamma\sigma^2}\left(-c_h\frac{1}{\gamma c_h}\right)}{\frac{u}{2} + \frac{R\xi}{\gamma\sigma^2}\frac{1}{\gamma c_h}} = \frac{R + c_h\frac{u}{2} - \frac{R\xi}{\gamma^2\sigma^2}}{\frac{u}{2} + \frac{R\xi}{\gamma^2\sigma^2}\frac{1}{c_h}}.$$

Thus, as long as

$$\frac{R + c_h\frac{u}{2} - \frac{R\xi}{\gamma^2\sigma^2}}{\frac{u}{2} + \frac{R\xi}{\gamma^2\sigma^2}\frac{1}{c_h}} \geq l$$

there is a unique solution c_l for any c_h of

$$p(c_l; c_l, c_h) = l.$$

Thus, $L(c_h)$ exist. From the monotonicity in c_l, and continuity of $p(c_l; c_l, c_h)$ we also know that $L(c_h)$ is continuous.

Properties of $H(c_h)$ first we show that for any fixed $c_h \in [l, R]$, $H(c_h)$ is a continuous function and $H(c_h) \in [0, c_h]$.

It will be useful to know that

$$\begin{aligned}
\frac{\partial f_h}{\partial c_l} &= \frac{2\left(\gamma f_l\left(c_h, c_l\right) - \frac{1}{c_l}\right)}{\left(e^{\gamma(c_h - c_l)} - e^{-\gamma(c_h - c_l)}\right)} \\
\frac{\partial g_h}{\partial c_l} &= \frac{2\left(\gamma g_l\left(c_h, c_l\right)\right)}{\left(e^{\gamma(c_h - c_l)} - e^{-\gamma(c_h - c_l)}\right)} \\
\frac{\partial f_h}{\partial c_h} &= \frac{\left(e^{2\gamma c_h} + e^{2\gamma c_l}\right)}{\left(e^{2\gamma c_h} - e^{2\gamma c_l}\right)}\left(\frac{1}{c_h} - \gamma f_h\left(c_h, c_l\right)\right) \\
\frac{\partial g_h}{\partial c_h} &= \frac{1}{c_h} - \frac{\left(e^{2\gamma c_h} + e^{2\gamma c_l}\right)}{\left(e^{2\gamma c_h} - e^{2\gamma c_l}\right)}\gamma g_h\left(c_l, c_h\right)
\end{aligned}$$

and in the limit

$$\lim_{c_l \to c_h} f_h = \frac{1}{\gamma c_h}, \quad \lim_{c_l \to c_h} g_h = 0$$

1. $\frac{\partial f_h}{\partial c_l} = \frac{2\left(\gamma f_l(c_h, c_l) - \frac{1}{c_l}\right)}{\left(e^{\gamma(c_h - c_l)} - e^{-\gamma(c_h - c_l)}\right)} < 0$ from previous results.

2. we have

$$\frac{\partial\left(\frac{g_h}{\gamma} - f_h c_h\right)}{\partial c_l} = 2\frac{g_h\left(c_h, c_l\right) - c_h \gamma f_l\left(c_h, c_l\right) + c_h \frac{1}{c_l}}{\left(e^{\gamma(c_h - c_l)} - e^{-\gamma(c_h - c_l)}\right)}$$

and

$$\begin{aligned}
\frac{\partial^2\left(\frac{g_h}{\gamma} - f_h c_h\right)}{\partial^2 c_l} &= \frac{2g_h'\left(c_h, c_l\right) - c_h 2\gamma f_h'\left(c_h, c_l\right) - c_h 2\frac{1}{c_l^2}}{\left(e^{\gamma(c_h - c_l)} - e^{-\gamma(c_h - c_l)}\right)} + \\
&\quad + \gamma e^{-\gamma(c_h - c_l)} \frac{e^{2(-\gamma(c_h - c_l))} + 1}{\left(e^{-2\gamma(c_h - c_l)} - 1\right)^2}\left(2g_l\left(c_h, c_l\right) - c_h 2\gamma f_l\left(c_h, c_l\right) + c_h 2\frac{1}{c_l}\right)
\end{aligned}$$

if the first derivative is zero at a point $c_h > c_l$,then the second derivative is

$$\frac{2\frac{1}{c_l} + 2\gamma\frac{\left(e^{2\gamma c_h} + e^{2\gamma c_l}\right)}{\left(e^{2\gamma c_h} - e^{2\gamma c_l}\right)}\left(g_l\left(c_l, c_h\right) - c_h \gamma f_l\left(c_h, c_l\right) + \frac{c_h}{c_l}\right) - c_h 2\frac{1}{c_l^2}}{\left(e^{\gamma(c_h - c_l)} - e^{-\gamma(c_h - c_l)}\right)} = \frac{-2\frac{c_h - c_l}{c_l^2}}{\left(e^{\gamma(c_h - c_l)} - e^{-\gamma(c_h - c_l)}\right)} < 0.$$

for any $c_h > c_l$, which implies that it can have no minimum in that range. Also

$$\begin{aligned}
\lim_{c_l \to c_h} \frac{\partial\left(\frac{g_h}{\gamma} - f_h c_h\right)}{\partial c_l} &= 0 \\
\lim_{c_l \to c_h} \frac{\partial^2\left(\frac{g_h}{\gamma} - f_h c_h\right)}{\partial^2 c_l} &= -\frac{1}{3\gamma c_h^2}
\end{aligned}$$

so $c_l = c_h$ must be the unique maximum in the range $c_h \geq c_l$. Thus,

$$\frac{\partial \left(\frac{g_h}{\gamma} - f_h c_h\right)}{\partial c_l} > 0$$

for $c_l < c_h$.

3. Consequently, $q(c_h; c_h, c_l)$ is monotonically decreasing and $v(c_h; c_h, c_l)$ is monotonically increasing in c_l. Thus, $p(c_h; c_h, c_l)$ is monotonically increasing in c_l.

4. Also

$$\begin{aligned}
\lim_{c_l \to c_h} \frac{v_h(c_l, c_h)}{q_h(c_l, c_h)} &= \frac{R + c_h \frac{u}{2} + \frac{R\xi}{\gamma\sigma^2}\left(-c_h \frac{1}{\gamma c_h}\right)}{\frac{u}{2} + \frac{R\xi}{\gamma\sigma^2}\frac{1}{\gamma c_h}} = \frac{Rc_h + c_h^2 \frac{u}{2} - \frac{R\xi}{\gamma^2\sigma^2}}{\frac{u}{2}c_h + \frac{R\xi}{\gamma^2\sigma^2}} > h \\
Rc_h + c_h^2 \frac{u}{2} - \frac{R\xi}{\gamma^2\sigma^2} c_h - h\left(\frac{u}{2}c_h + \frac{R\xi}{\gamma^2\sigma^2}\right) &> 0 \\
\frac{1}{2}uc_h^2 + \frac{1}{2}(R - hu)\, c_h - R\frac{h}{2} &> 0 \\
\frac{1}{2}uc_h^2 + \frac{1}{2}(R - hu)\, c_h - R\frac{h}{2} &> 0
\end{aligned}$$

holds if $c_h > h$.

while

$$\lim_{c_l \to 0} = p(c_h; c_l, c_h) = -c_h$$

Thus, for any $c_h \geq h$ there is a unique $c_l \in [0, c_h]$ which solves $p(c_h; c_l, c_h) = h$. From the monotonicity of $p(c_h; c_h, c_l)$ in c_l and the continuity in c_h, the resulting function $H(c_h)$ is continuous in $c_{h.}$

Intercept of $H(c_h)$ and $L(c_h)$

1. Here we show that $H(h) > L(h)$ ..We know that $H(h) = h$ as

$$\lim_{c_l \to h} = \frac{v_h(c_l, h)}{q_h(c_l, h)} = \frac{R + h\frac{u}{2} + \frac{R\xi}{\gamma\sigma^2}\left(-h\frac{1}{\gamma h}\right)}{\frac{u}{2} + \frac{R\xi}{\gamma\sigma^2}\frac{1}{\gamma h}} = \frac{R + h\frac{u}{2} + \frac{R}{2}\gamma\left(-h\frac{1}{\gamma h}\right)}{\frac{u}{2} + \frac{R}{2}\gamma\frac{1}{\gamma h}} = h$$

however as

$$\lim_{c_l \to h} \frac{v_l(c_l, h)}{q_l(c_l, h)} = \frac{R + h\frac{u}{2} + \frac{R\xi}{\gamma\sigma^2}\left(-h\frac{1}{\gamma h}\right)}{\frac{u}{2} + \frac{R\xi}{\gamma\sigma^2}\frac{1}{\gamma h}} = \frac{R + h\frac{u}{2} + \frac{R}{2}\gamma\left(-h\frac{1}{\gamma h}\right)}{\frac{u}{2} + \frac{R}{2}\gamma\frac{1}{\gamma h}} = h$$

and $\frac{v_l(c_l,h)}{q_l(c_l,h)}$ is increasing in c_l, and $L(h)$ is defined by $\frac{v_l(L(h),h)}{q_l(L(h),h)} = l < h$,

$$L(h) < h$$

must hold.

2. Here we show that $\lim_{c_h\to\infty} H(c_h) = 0 < \lim_{c_h\to\infty} L(c_h)$. Observe, that

$$\begin{aligned}
\lim_{c_h\to\infty} f_l &= \lim_{c_h\to\infty} \frac{e^{-\gamma 2c_h}\left(\mathrm{Ei}[c_h\gamma]-\mathrm{Ei}[c_l\gamma]\right)+\left(\mathrm{Ei}[-c_h\gamma]-\mathrm{Ei}[-c_l\gamma]\right)}{e^{\gamma(-c_l)}-e^{-\gamma(2c_h-c_l)}} = \frac{-\mathrm{Ei}[-c_l\gamma]}{e^{\gamma(-c_l)}}\\
\lim_{c_h\to\infty} g_l &= \lim_{c_h\to\infty} \frac{e^{\gamma c_h}\left(\mathrm{Ei}\left[-\gamma c_l\right]-\mathrm{Ei}\left[-\gamma c_h\right]\right)+e^{-\gamma c_h}\left(\mathrm{Ei}[c_h\gamma]-\mathrm{Ei}[c_l\gamma]\right)}{\left(e^{\gamma(c_h-c_l)}-e^{-\gamma(c_h-c_l)}\right)} = \frac{\mathrm{Ei}\left[-\gamma c_l\right]}{e^{\gamma(-c_l)}}\\
\lim_{c_h\to\infty} f_h &= \lim_{c_h\to\infty} \frac{e^{-\gamma c_l}\left(\mathrm{Ei}[c_h\gamma]-\mathrm{Ei}[c_l\gamma]\right)+e^{\gamma c_l}\left(\mathrm{Ei}\left[-\gamma c_h\right]-\mathrm{Ei}\left[-\gamma c_l\right]\right)}{e^{\gamma(c_h-c_l)}-e^{-\gamma(c_h-c_l)}} = \frac{0}{e^{\gamma(-c_l)}-e^{-\gamma(2c_h-c_l)}} = 0\\
\lim_{c_h\to\infty} g_h &= \frac{e^{-\gamma c_l}\left(\mathrm{Ei}[c_h\gamma]-\mathrm{Ei}[c_l\gamma]\right)+e^{\gamma c_l}\left(\mathrm{Ei}\left[-\gamma c_l\right]-\mathrm{Ei}\left[-\gamma c_h\right]\right)}{\gamma\left(e^{\gamma(c_h-c_l)}-e^{-\gamma(c_h-c_l)}\right)} =\\
&= \frac{e^{-\gamma c_l}e^{-\gamma c_h}\left(\mathrm{Ei}[c_h\gamma]-\mathrm{Ei}[c_l\gamma]\right)+e^{\gamma c_l}e^{-\gamma c_h}\left(\mathrm{Ei}\left[-\gamma c_l\right]-\mathrm{Ei}\left[-\gamma c_h\right]\right)}{\gamma\left(e^{\gamma(-c_l)}-e^{-\gamma(2c_h-c_l)}\right)} = 0
\end{aligned}$$

Thus,

$$\begin{aligned}
\lim_{c_h\to\infty}\frac{v(c_l;c_l,c_h)}{q(c_l;c_l,c_h)} &= \lim_{c_h\to\infty}\frac{R+\frac{c_lu}{2}+\frac{u}{2\gamma}m(c_l,c_h)+\frac{R\xi}{\gamma\sigma^2}\left(\frac{g_l(c_l,c_h)}{\gamma}-c_lf_l(c_l,c_h)\right)}{\frac{u}{2}+\frac{R\xi}{\gamma\sigma^2}f_l(c_l,c_h)} =\\
&= \frac{R+\frac{c_lu}{2}+\frac{u}{2\gamma}+\frac{R\xi}{\gamma\sigma^2}\left(\frac{\mathrm{Ei}[-\gamma c_l]}{\gamma e^{\gamma(-c_l)}}-c_l\frac{-\mathrm{Ei}[-c_l\gamma]}{e^{\gamma(-c_l)}}\right)}{\frac{u}{2}-\frac{\mathrm{Ei}[-c_l\gamma]}{e^{\gamma(-c_l)}}}
\end{aligned}$$

and $\lim_{c_h\to\infty} L(c_h)$ is the finite positive solution of

$$\frac{R+\frac{c_lu}{2}+\frac{u}{2\gamma}+\frac{R\xi}{\gamma\sigma^2}\left(\frac{\mathrm{Ei}[-\gamma c_l]}{\gamma e^{\gamma(-c_l)}}-c_l\frac{-\mathrm{Ei}[-c_l\gamma]}{e^{\gamma(-c_l)}}\right)}{\frac{u}{2}-\frac{\mathrm{Ei}[-c_l\gamma]}{e^{\gamma(-c_l)}}} = l.$$

In contrast,

$$\begin{aligned}
\lim_{c_h\to\infty}\frac{v(c_h;c_l,c_h)}{q(c_h;c_l,c_h)} &= \lim_{c_h\to\infty}\frac{R+\frac{c_hu}{2}-\frac{u}{2\gamma}m(c_l,c_h)+\frac{R\xi}{\gamma\sigma^2}\left(\frac{g_h(c_l,c_h)}{\gamma}-c_hf_h(c_l,c_h)\right)}{\frac{u}{2}+\frac{R\xi}{\gamma\sigma^2}f_h(c_l,c_h)} =\\
&= \lim_{c_h\to\infty}\frac{\frac{R}{c_h}+\frac{u}{2}-\frac{u}{c_h2\gamma}+\frac{R\xi}{\gamma\sigma^2}\left(\frac{g_h(c_l,c_h)}{c_h\gamma}-f_h(c_l,c_h)\right)}{\frac{u}{2c_h}+\frac{R\xi}{\gamma\sigma^2}\frac{f_h(c_l,c_h)}{c_h}} =\\
&= \lim_{c_h\to\infty}\frac{\frac{u}{2}+\frac{R\xi}{\gamma\sigma^2}\left(\frac{g_h(c_l,c_h)}{c_h\gamma}\right)}{\frac{R\xi}{\gamma\sigma^2}\frac{f_h(c_l,c_h)}{c_h}} = \infty,
\end{aligned}$$

As $\frac{v(c_h;c_l,c_h)}{q(c_h;c_l,c_h)}$ grows without bound for any fixed c_l and $\frac{v(c_h;c_l,c_h)}{q(c_h;c_l,c_h)}$ is monotonically increasing in c_l, In order to have a solution of $\lim_{c_h\to\infty}\frac{v(c_h;c_l,c_h)}{q(c_h;c_l,c_h)} = l$, c_l has to go to zero, implying $\lim_{c_h\to\infty} H(c_h) = 0$.

The two results imply that there is always an intercept $c_h \in (h, \infty)$ that $H(c_h) = L(c_h)$. This concludes the step proving that (13)-(14), (17)-(18) has a solution

A.1.3 Step 3: Proof of Lemma A.3

We have shown that $H(h) = h$. Note also that if $c_h = c_l$ then

$$\frac{v_h}{q_h} = \frac{v_l}{q_l}.$$

This, and the continuity of $H(\cdot)$ and $L(\cdot)$ in l, implies that at the limit $l \to h$, there is a solution of the system (13)-(14), (17)-(18) that $c_l^* - c_h^* \to 0$ and $c_h^*, c_l^* \to h$. Then, the statement comes from $h < hu < R$,

A.1.4 Step 3: Proof of Lemma A.4

First we show that $q(c)$ is always deceasing, and there exists a critical value $\widehat{c} \in (c_l, c_h)$ so that $q''(c) < 0$ for $c \in (c_l, \widehat{c})$ and $q''(c) > 0$ for $c \in (\widehat{c}, c_h)$. Moreover, for $c \in (c_l, \widehat{c})$ where $q''(c) < 0$, we have that $q'''(c) > 0$.

We first show that $q' < 0$. From the ODE $0 = \frac{\sigma^2}{2} q'' + \frac{\xi}{2}\left(u + \frac{R}{c}\right) - \xi q$ satisfied by q, we have

$$0 = \frac{\sigma^2}{2} q''' - \frac{\xi}{2} \frac{R}{c^2} - \xi q'. \tag{A.1}$$

Due to boundary conditions, we have $\frac{\sigma^2}{2} q''' = \frac{\xi}{2} \frac{R}{c^2} > 0$ at both ends c_l^* and c_h^*. Define $F(c) \equiv q'(c)$; then $F(c_l) = F(c_h) = 0$ and $F''(c_l) = F''(c_h) > 0$. Suppose $F(\widehat{c}) > 0$ for some points; then there must exist a point $\widehat{c}$ so that $F(\widehat{c}) > 0$ and $F''(\widehat{c}) = 0$ (otherwise $F(c)$ is convex always and never comes back to zero). But because

$$\frac{\sigma^2}{2} F''(\widehat{c}) = \frac{\xi}{2} \frac{R}{\widehat{c}^2} + \xi F(\widehat{c}) > 0,$$

contradiction. This proves that $q' < 0$. We know that $q''(c_l) < 0$ and $q''(c_h) > 0$, therefore there exists $\widehat{c}$ so that

$$q''(\widehat{c}) = 0.$$

We show this point is unique. Because $0 = \frac{\sigma^2}{2} q'' + \frac{\xi}{2}\left(u + \frac{R}{c}\right) - \xi q$, we have

$$0 = \frac{\sigma^2}{2} q''' - \frac{\xi}{2} \frac{R}{c^2} - \xi q' \tag{A.2}$$

$$0 = \frac{\sigma^2}{2} q'''' + \frac{\xi R}{c^3} - \xi q'' \tag{A.3}$$

Suppose we have multiple solutions for $q''(\widehat{c}) = 0$. Clearly, it is impossible to have the possibility that $q''(\widehat{c}) = 0$ but $q''(\widehat{c}-) > 0$ and $q''(\widehat{c}+) > 0$; otherwise $q''''(\widehat{c}) > 0$ which contradicts with (A.3).

Then there must exist two points $c_1 > \widehat{c}$ and $c_2 > c_1 > \widehat{c}$ that

$$q''(c_1) = 0,\ q''(c_2) < 0 \text{ and } q''''(c_2) > 0,$$

and

$$q''(c) < 0 \text{ for } c \in (c_1, c_2)$$

Therefore

$$\frac{\sigma^2}{2} q''''(c_1) = -\frac{\xi R}{c_1^3} + \xi q''(c_1) < 0.$$

As a result, there exists another point $c_3 \in (c_1, c_2)$ so that $q''''(c_3) = 0$ with $q''(c_3) < 0$. But this contradicts with (A.3). Now we show that for $c \in (c_l, \widehat{c})$ so that $q''(c) < 0$, we have $q'''(c) > 0$, i.e., $q''(c)$ is increasing. Suppose not. Since $q'''(c_l) > 0$ so that $q''(c)$ is increasing at the beginning, there must exist some reflecting point c_4 so that $q''''(c_4) = 0$. But because $q''(c_4) < 0$, it contradicts with (A.3).

Now we show that if $v''(c_l) > 0$, then $v(c)$ is increasing. Let $F(c) = v'(c)$, so that

$$0 = q''\sigma^2 + \frac{\sigma^2}{2} F'' + \frac{\xi}{2} u - \xi F$$

with boundary conditions that $F(c_l) = F(c_h) = 0$. Since $F'(c_l) > 0$ we know that if there are some points with $F'(c) < 0$ then it must exists two points c_1 and c_2 (a maximum and a minimum) so that $c_1 < c_2$ but $F''(c_1) < 0$ $F''(c_2) > 0$, $F'(c_1) = F'(c_2) = 0$ and $F(c_1) > 0 > F(c_2)$. From ODE

$$\begin{aligned}
0 &= q''(c_1)\sigma^2 + \frac{\sigma^2}{2} F''(c_1) + \frac{\xi}{2} u - \xi F(c_1) \\
0 &= q''(c_2)\sigma^2 + \frac{\sigma^2}{2} F''(c_2) + \frac{\xi}{2} u - \xi F(c_2)
\end{aligned}$$

It is easy to show that $q''(c_2) < 0$, which implies that $c_1 < c_2 < \widehat{c}$. However, the above two equations also imply that

$$q''(c_1)\sigma^2 > \frac{\xi}{2} u > q''(c_2)\sigma^2$$

contradiction with the previous lemma which shows that q'' is increasing over $[c_l, \widehat{c}]$.

Finally we show that if $h - l$ is sufficiently close to 0, then $v''(c_l) > 0$.

From our ODE,

$$\begin{aligned}
v''(c_l) &= -\frac{\xi}{\sigma^2} 2 \left(\frac{(uc_l + R)}{2} - v(c_l) \right) = \\
&= \frac{\xi}{\sigma^2} 2 \left(\frac{R}{2} + \frac{u}{2\gamma} h(c_l, c_h) + \frac{R\xi}{\gamma\sigma^2} \left(\frac{g_l(c_l, c_h)}{\gamma} - c_l f_l(c_l, c_h) \right) \right).
\end{aligned}$$

We know that as $h - l \to 0$, $c_h - c_l \to 0$. We will prove the statement by showing that (1)

$\lim_{c_l \to c_h} \left(\frac{(uc_l+R)}{2} - v(c_l) \right) = 0$ and (2) $\lim_{c_l \to c_h} \frac{\partial \left(\frac{(uc_l+R)}{2} - v(c_l) \right)}{\partial c_l} < 0$. These two statements imply that when $c_h - c_l$ is sufficiently small then

$$v''(c_l) > \lim_{c_l \to c_h} v''(c_l) = 0.$$

Thus, the chain of equalities

$$\begin{aligned} \lim_{c_l \to c_h} \left(\frac{(uc_l + R)}{2} - v(c_l) \right) &= \\ &= \lim_{c_l \to c_h} \left(\frac{R}{2} + \frac{u}{2\gamma} h(c_l, c_h) + \frac{R\xi}{\gamma\sigma^2} \left(\frac{g_l(c_l, c_h)}{\gamma} - c_l f_l(c_l, c_h) \right) \right) = \\ &= \frac{R}{2} + 0 + \frac{R\xi}{\gamma\sigma^2} \left(0 - \frac{1}{\gamma} \right) = 0 \end{aligned}$$

Proof. and

$$\begin{aligned} \lim_{c_l \to c_h} \frac{\partial \left(\frac{(uc_l+R)}{2} - v(c_l) \right)}{\partial c_l} &= \\ &= \lim_{c_l \to c_h} \frac{\partial \left(\frac{u}{2\gamma} h(c_l, c_h) + \frac{R\xi}{\gamma\sigma^2} \left(\frac{g_l(c_l, c_h)}{\gamma} - c_l f_l(c_l, c_h) \right) \right)}{\partial c_l} = \\ &= \lim_{c_l \to c_h} \left(-u \frac{e^{\gamma(c_h - c_l)}}{\left(e^{\gamma(c_h - c_l)} + 1 \right)^2} + \frac{R\xi}{\gamma\sigma^2} \left(\frac{1}{\gamma c_l} + \frac{\left(e^{2\gamma c_h} + e^{2\gamma c_l} \right)}{\left(e^{2\gamma c_h} - e^{2\gamma c_l} \right)} g_l - \frac{\left(e^{2\gamma c_h} + e^{2\gamma c_l} \right)}{\left(e^{2\gamma c_h} - e^{2\gamma c_l} \right)} (c_l \gamma f_l + \right. \right. \\ &= -u \frac{1}{(1+1)^2} + \frac{R\xi}{\gamma\sigma^2} \left(\frac{1}{\gamma c_h} - \frac{1}{2\gamma c_h} - \frac{1}{2\gamma c_h} \right) = -\frac{u}{4} < 0 \end{aligned}$$

prove the statement. ■

A.2 Proof of Proposition 2

The result $c_h^* > h$ is a consequence of the fact that we defined $H(c_h)$ as the unique c_l solving $\frac{v_h(c_l, c_h)}{q_h(c_l, c_h)} = h$ when $c_h > h$. (see part 4 in section A.1.2.)

For the result $c_l^* \leq l$, consider the possibility that $c_l^* > l$. The following Lemma states that in this case $p''(c_l^*) < 0$. This implies that this is not an equilibrium as $p'(c_l^*) = 0$ by the boundary conditions $v'(c_l^*) = q'(c_l^*) = 0$, thus $p''(c_l^*) < 0$ would imply that $p(c) < l$ for a c sufficiently close to c_l^*.

Lemma A.5 *The sign of $p''(c_l^*)$ is the same as the sign of $(l - c_l^*)$.*

Proof.

$$\begin{aligned}
p''\left(c_l^*\right) &= \left(\frac{v'q-q'v}{q^2}\right)' = \frac{\left(v''q+v'q'-\left(q''v+v'q'\right)\right)}{q^2} - 2q^{-3}\left(v'q-q'v\right) = \\
&= \frac{v''q-q''v}{q^2} = \frac{\left(-\frac{\xi}{2}\left(uc_l^*+R\right)+\xi lq\left(c_l^*\right)\right)\frac{2}{\sigma^2}q - \left(-\frac{\xi}{2}\left(uc_l^*+R\right)+\xi c_l^* q\left(c_l^*\right)\right)\frac{2}{\sigma^2 c_l^*}v}{q^2} = \\
&= \frac{\left(-\frac{\xi}{2}\left(uc_l^*+R\right)+\xi lq\left(c_l^*\right)+\xi c_l^* q\left(c_l^*\right)-\xi c_l^* q\left(c_l^*\right)\right)\frac{2}{\sigma^2}q - \left(-\frac{\xi}{2}\left(uc_l^*+R\right)+\xi c_l^* q\left(c_l^*\right)\right)\frac{2}{\sigma^2 c_l^*}v}{q^2} = \\
&= \frac{\left(-\frac{\xi}{2}\left(uc_l^*+R\right)+\xi c_l^* q\left(c_l^*\right)\right)\frac{2}{\sigma^2}\left(q-\frac{v}{c_l^*}\right)+\left(l-c_l^*\right)\xi q\left(c_l^*\right)\frac{2}{\sigma^2}q}{q^2} \\
&= \left(l-c_l^*\right)\frac{\frac{1}{c_l^*}\left(\frac{\xi}{2}\left(uc_l^*+R\right)-\xi c_l^* q\left(c_l^*\right)\right)\frac{2}{\sigma^2}+\xi q\left(c_l^*\right)\frac{2}{\sigma^2}}{q}
\end{aligned}$$

which gives the Lemma by realizing that our observation that q is decreasing in c and the boundary $q'\left(c_l^*\right)=0$ implies that

$$-\frac{\xi}{2}\left(uc_l^*+R\right)+\xi c_l^* q\left(c_l^*\right) \propto q'' < 0$$

and $q>0$. ■

The third statement is a consequence of the following Lemma.

Lemma A.6 *We have the following limiting results:*

$$\begin{aligned}
\lim_{\gamma\to\infty}\gamma f_l &= \frac{1}{c_l},\ \lim_{\gamma\to\infty}\gamma f_h = \frac{1}{c_h},\ \lim_{\gamma\to\infty} g_h = 0, \\
\lim_{\gamma\to\infty} g_l &= 0,\ \lim_{\gamma\to\infty} c_h^* = h,\ \lim_{\gamma\to\infty} c_l^* = l.
\end{aligned}$$

Proof. Using L'Hopital rule repeatedly, we have

$$\begin{aligned}
\lim_{\gamma\to\infty}\gamma f_l &= \lim_{\gamma\to\infty}\frac{\gamma\left(\mathrm{Ei}[-c_h\gamma]-\mathrm{Ei}[-c_l\gamma]\right)}{e^{\gamma(-c_l)}} = \lim_{\gamma\to\infty}\frac{\mathrm{Ei}[-c_h\gamma]-\mathrm{Ei}[-c_l\gamma]}{\frac{1}{\gamma}e^{\gamma(-c_l)}} \\
&= \lim_{\gamma\to\infty}\frac{\frac{e^{-c_h\gamma}}{\gamma}-\frac{e^{-c_l\gamma}}{\gamma}}{-\frac{1}{\gamma^2}e^{\gamma(-c_l)}+\frac{(-c_l)}{\gamma}e^{\gamma(-c_l)}} = \lim_{\gamma\to\infty}\frac{-e^{-c_l\gamma}/\gamma}{\frac{(-c_l)}{\gamma}e^{\gamma(-c_l)}} = \frac{1}{c_l}
\end{aligned}$$

$$\begin{aligned}
\lim_{\gamma\to\infty}\gamma f_h &= \lim_{\gamma\to\infty}\gamma\frac{e^{-\gamma c_l}\left(\mathrm{Ei}[c_h\gamma]-\mathrm{Ei}[c_l\gamma]\right)+e^{\gamma c_l}\left(\mathrm{Ei}\left[-\gamma c_h\right]-\mathrm{Ei}\left[-\gamma c_l\right]\right)}{e^{\gamma(c_h-c_l)}-e^{-\gamma(c_h-c_l)}} = \\
&= \lim_{\gamma\to\infty}\gamma\frac{e^{-\gamma c_l}\left(\mathrm{Ei}[c_h\gamma]-\mathrm{Ei}[c_l\gamma]\right)}{e^{\gamma(c_h-c_l)}} = \lim_{\gamma\to\infty}\frac{\left(\mathrm{Ei}[c_h\gamma]-\mathrm{Ei}[c_l\gamma]\right)}{\frac{1}{\gamma}e^{\gamma c_h}} = \\
&= \lim_{\gamma\to\infty}\frac{\frac{e^{c_h\gamma}}{\gamma}-\frac{e^{c_l\gamma}}{\gamma}}{-\frac{1}{\gamma^2}e^{\gamma(c_h)}+\frac{c_h}{\gamma}e^{\gamma c_h}} = \lim_{\gamma\to\infty}\frac{e^{c_h\gamma}}{c_h e^{\gamma c_h}} = \frac{1}{c_h}
\end{aligned}$$

where we used that

$$\lim_{\gamma\to\infty} e^{\gamma c_l}\left(\mathrm{Ei}\left[-\gamma c_h\right]-\mathrm{Ei}\left[-\gamma c_l\right]\right)=\lim_{\gamma\to\infty}\frac{\left(\mathrm{Ei}\left[-\gamma c_h\right]-\mathrm{Ei}\left[-\gamma c_l\right]\right)}{e^{-\gamma c_l}}=\lim_{\gamma\to\infty}\frac{\frac{e^{-c_h\gamma}-e^{-c_l\gamma}}{\gamma}}{\left(-c_l\right)e^{\gamma(-c_l)}}=\lim_{\gamma\to\infty}\frac{\frac{-1}{\gamma}}{\left(-c_l\right)}=0.$$

Also,

$$\begin{aligned}\lim_{\gamma\to\infty} g_h &= \lim_{\gamma\to\infty}\frac{e^{-\gamma c_l}\left(\mathrm{Ei}[c_h\gamma]-\mathrm{Ei}[c_l\gamma]\right)+e^{\gamma c_l}\left(\mathrm{Ei}\left[-\gamma c_l\right]-\mathrm{Ei}\left[-\gamma c_h\right]\right)}{e^{\gamma(c_h-c_l)}-e^{-\gamma(c_h-c_l)}}= \\ &= \lim_{\gamma\to\infty}\frac{\left(\mathrm{Ei}[c_h\gamma]-\mathrm{Ei}[c_l\gamma]\right)}{e^{\gamma c_h}}=\lim_{\gamma\to\infty}\frac{\frac{e^{c_h\gamma}}{\gamma}-\frac{e^{c_l\gamma}}{\gamma}}{c_h e^{\gamma c_h}}=\lim_{\gamma\to\infty}\frac{\frac{1}{\gamma}}{c_h}=0\end{aligned}$$

and

$$\begin{aligned}\lim_{\gamma\to\infty} g_l &= \lim_{\gamma\to\infty}\frac{e^{-\gamma c_h}\left(\mathrm{Ei}[c_h\gamma]-\mathrm{Ei}[c_l\gamma]\right)+e^{\gamma c_h}\left(\mathrm{Ei}\left[-\gamma c_l\right]-\mathrm{Ei}\left[-\gamma c_h\right]\right)}{e^{\gamma(c_h-c_l)}-e^{-\gamma(c_h-c_l)}}= \\ &= \lim_{\gamma\to\infty}\frac{\gamma\left(\mathrm{Ei}[-c_h\gamma]-\mathrm{Ei}[-c_l\gamma]\right)}{e^{\gamma(-c_l)}}=\lim_{\gamma\to\infty}\frac{\frac{e^{-c_h\gamma}}{\gamma}-\frac{e^{-c_l\gamma}}{\gamma}}{\left(-c_l\right)e^{\gamma(-c_l)}}=\lim_{\gamma\to\infty}\frac{-e^{-c_l\gamma}/\gamma}{\left(-c_l\right)e^{\gamma(-c_l)}}=0.\end{aligned}$$

This implies that

$$\begin{aligned}\lim_{\gamma\to\infty}\frac{v_h}{q_h} &= \lim_{\gamma\to\infty}\frac{R+\frac{c_h u}{2}-\frac{u}{2\gamma}m\left(c_l,c_h\right)+R\frac{\gamma}{2}\left(\frac{g_h(c_l,c_h)}{\gamma}-c_h f_h\left(c_l,c_h\right)\right)}{\frac{u}{2}+R\frac{\gamma}{2}f_h\left(c_l,c_h\right)}= \\ &= \frac{R+\frac{c_h u}{2}-R\frac{1}{2}}{\frac{u}{2}+R\frac{1}{2c_h}}\end{aligned}$$

Thus, in the limit the solution of $\frac{v_h}{q_h}=h$ is the solution of

$$\frac{R+\frac{c_h u}{2}-R\frac{1}{2}}{\frac{u}{2}+R\frac{1}{2c_h}}=h$$

which gives $\lim_{\gamma\to\infty} c_h^*=h$. Similarly,

$$\begin{aligned}\lim_{\gamma\to\infty}\frac{v_l}{q_h} &= \lim_{\gamma\to\infty}\frac{R+\frac{c_l u}{2}+\frac{u}{2\gamma}m\left(c_l,c_h\right)+R\frac{\gamma}{2}\left(\frac{g_l(c_l,c_h)}{\gamma}-c_l f_l\left(c_l,c_h\right)\right)}{\frac{u}{2}+R\frac{\gamma}{2}f_l\left(c_l,c_h\right)}= \\ &= \frac{R+\frac{c_l u}{2}+R\frac{1}{2}}{\frac{u}{2}+R\frac{1}{2c_l}}\end{aligned}$$

implying that the solution of $\frac{v_l}{q_l}$ in this limit has to solve

$$\frac{R+\frac{c_l u}{2}-R\frac{1}{2}}{\frac{u}{2}+R\frac{1}{2c_l}}=l$$

which gives $\lim_{\gamma\to\infty} c_l^*=l$. ■

A.3 Proof of Proposition 3

From (26)-(28) and our conjecture that $c_l^P = 0$, we have

$$
\begin{aligned}
R + D_1 + D_2 &= l\left(u - \gamma D_1 + \gamma D_2\right) && \text{(A.4)}\\
R + uc_h^P + D_1 e^{-\gamma c_h^P} + D_2 e^{\gamma c_h^P} &= \left(h + c_h^P\right)\left(u - \gamma D_1 e^{-\gamma c_h^P} + \gamma D_2 e^{\gamma c_h^P}\right) && \text{(A.5)}\\
D_1 e^{-\gamma c_h^P} + D_2 e^{\gamma c_h^P} &= 0 && \text{(A.6)}
\end{aligned}
$$

The first and the last equation give

$$
D_1 = -\frac{(R - lu)\, e^{2\gamma c_h^P}}{1 - l\gamma + (1 + l\gamma)\, e^{2\gamma c_h^P}},\ D_2 = \frac{R - lu}{1 - l\gamma + (1 + l\gamma)\, e^{2\gamma c_h^P}}
$$

which if we substitute in to the second equation we get (29). To validate that $c_l^P = 0$, we have to check that $j_P''(0) < 0$. As

$$
j_P''(0) = D_1 + D_2 = -(R - lu)\, \frac{e^{2\gamma c_h^P} - 1}{e^{2\gamma c_h^P} + l\gamma\left(e^{2\gamma c_h^P} - 1\right) + 1} < 0
$$

this is indeed the case.

Now we show that under certain conditions the solution exists and unique. Let

$$
G(c) = \frac{R - hu}{R - lu}\left(e^{c\gamma}(1 + l\gamma) - (1 - l\gamma)\, e^{-c\gamma}\right) - 2\gamma(c + h)
$$

with

$$
G(0) = 2R\gamma\frac{l - h}{R - lu} < 0, \text{ and } G(\infty) = \infty.
$$

Now define

$$
g(c) \equiv G'(c) = \gamma\left(\frac{(R - hu)}{(R - lu)}\left((l\gamma + 1)\, e^{c\gamma} + e^{-c\gamma}(1 - l\gamma)\right) - 2\right).
$$

Since

$$
g(0) = 2R\gamma\frac{l - h}{R - lu} < 0.
$$

and $g(c)$ changes sign only once. Consequently, there is a unique $\hat{c}$ that $g(\hat{c}) = 0$. This implies that $G(c)$ is decreasing for any $c < \hat{c}$ and increasing for any $c > \hat{c}$. As $G(0) < 0$ and $G(\infty) = \infty$, there must be a unique c_h^P that $G\left(c_h^P\right) = 0$.

The following series of results characterize the property of social planner's value function $j_P(c)$, which satisfies

$$
0 = \frac{\sigma^2}{2} j_P''(c) + \xi\left(R + uc - j_P(c)\right) \tag{A.7}
$$

with boundary conditions

$$
j_P(0) = l j_P'(0),\ j_P\left(c_h^P\right) = \left(t + c_h^P\right) j_P'\left(c_h^P\right),\ \text{and } j_P''\left(c_h^P\right) = 0.
$$

Note that the boundary conditions imply that $j_P\left(c_h^P\right) = R + uc_h^P$.

Lemma A.7 *The social value function $j_P(c)$ is concave and increasing over $[0, c^*]$, and $j_P(c) < R + uc$.*

Proof. *First of all, from smooth pasting condition we have*

$$u - j_P'\left(c_h^P\right) = u - \frac{j_P\left(c_h^P\right)}{t + c_h^P} = u - \frac{R + uc_h^P}{t + c_h^P} = \frac{ut - R}{t + c_h^P} < 0.$$

Second, taking derivative again on (A.7) and evaluate at the optimal policy point c_h^P, we have

$$0 = \frac{\sigma^2}{2} j_P'''\left(c_h^P\right) + \xi\left(u - j_P'\left(c_h^P\right)\right).$$

Combining both results, we have

$$j_P'''\left(c_h^P\right) = \frac{2\xi}{\sigma^2}\frac{R - ut}{t + c_h^P} > 0,$$

and as a result $j_P''\left(c_h^P - \epsilon\right) < 0$. Suppose that j_P fails to be globally concave over $[0, c^*]$. Then there exists some point $j_P'' > 0$, and pick the largest one $\widehat{c}$ so that j_P'' is concave over $[\widehat{c}, c^*]$ with

$$j_P''(\widehat{c}) = 0 \text{ and } j_P'''(\widehat{c}) < 0$$

but since j_P'' is concave over $[\widehat{c}, c^*]$, $j_P'(\widehat{c}) > j_P'(c^*) > u$, therefore

$$\frac{\sigma^2}{2} j_P'''\left(c_h^P\right) = \xi\left(j_P'\left(c_h^P\right) - u\right) > 0,$$

contradiction. Therefore j_P is globally concave over $\left[0, c_h^P\right]$. Finally, since $j_P'\left(c_h^P\right) > u$, we must have $j_P'(c) > u > 0$ always. Combining with the fact that $j_P\left(c_h^P\right) = R + uc_h^P$, we have $j_P(c) < R + uc$. QED. ■

We can further extend $j_P(c)$ outside c_h^P by recognizing the optimal policy is investing. Suppose that $C > Ac^*$; then immediately the economy should build x trees so that

$$\frac{C - xt}{A + x} = c^* \Rightarrow x = \frac{C - c^* A}{h + c^*} = A\frac{c - c^*}{h + c^*},$$

and the total value is

$$Aj(c) = (A + x)\, j_P\left(c_h^P\right) = A\left(1 + \frac{c - c_h^P}{h + c_h^P}\right) j\left(c_h^P\right) = A\left(\frac{h + c}{h + c_h^P}\right) j\left(c_h^P\right)$$

Therefore the envelope of value is

$$j_P(c) = \begin{cases} j_P(c) & \left[0, c_h^P\right] \\ \frac{h+c}{h+c_h^P} j_P\left(c_h^P\right) & c > c_h^P \end{cases}.$$

A.4 Proof of Proposition 4

Let

$$G(c) = \frac{R-hu}{R-lu}\left(e^{c\gamma}(1+l\gamma) - (1-l\gamma)e^{-c\gamma}\right) - 2\gamma(c+h).$$

Note that limit of c_h^P as $\gamma \to \infty$ has to satisfy

$$\lim_{\gamma\to\infty} \frac{\frac{R-hu}{R-lu}\left(e^{c\gamma}(1+l\gamma) - (1-l\gamma)e^{-c\gamma}\right)}{2\gamma c} - \left(1+\frac{h}{c}\right)|_{c=c_h^p} = 0$$

as

$$\lim_{\gamma\to\infty} \frac{\frac{R-hu}{R-lu}\left(e^{c\gamma}(1+l\gamma) - (1-l\gamma)e^{-c\gamma}\right)}{\gamma c} = \infty$$

,c has to converge to 0. This is the first part of the first statement. Also,

$$\frac{\partial G(c)}{\partial \gamma} = \frac{R-hu}{R-lu}\left(ce^{c\gamma}(1+l\gamma) + le^{c\gamma} - c(l\gamma-1)e^{-c\gamma} + le^{-c\gamma}\right) - 2(c+h)$$

which is clearly positive for sufficiently large γ. Finally, from the proof of Proposition 3 we know that

$$\frac{\partial G(c)}{\partial c}|_{c=c_h^P} > 0.$$

Thus, for sufficiently large γ,

$$\frac{\partial c_h^p}{\partial \gamma} = -\frac{\frac{R-hu}{R-lu}\left(ce^{c\gamma}(1+l\gamma) + le^{c\gamma} - c(l\gamma-1)e^{-c\gamma} + le^{-c\gamma}\right) - 2(c+h)}{\frac{\partial G(c)}{\partial c}}|_{c=c_h^p} < 0.$$

This concludes the first part. Also,

$$\begin{aligned}
\frac{\partial G(c)}{\partial h} &= \frac{-u}{R-lu}\left(e^{c\gamma}(1+l\gamma) - (1-l\gamma)e^{-c\gamma}\right) - 2\gamma < 0 \\
\frac{\partial G(c)}{\partial l} &= (R-hu)\frac{u\left(1-e^{-2c\gamma}\right) + R\gamma + R\gamma e^{2(-c\gamma)}}{e^{-c\gamma}(R-lu)^2} > 0 \\
\frac{\partial\left(\frac{R-hu}{R-lu}\right)}{\partial R} &= u\frac{h-l}{(R-lu)^2} > 0
\end{aligned}$$

which conclude the second part. Finally, $\lim_{R\to hu} G(c) = \lim_{u\to\frac{R}{h}} G(c) = -2\gamma(c+h)$, so there is no finite solution in this limit. Instead,

$$\lim_{R\to hu} \frac{\frac{R-hu}{R-lu}\left(e^{c\gamma}(1+l\gamma) - (1-l\gamma)e^{-c\gamma}\right)}{c} - 2\gamma\left(1+\frac{h}{c}\right)|_{c=c_h^p} = 0$$

has to hold, which implies that $\lim_{R\to hu} \frac{\frac{R-hu}{R-lu}\left(e^{c\gamma}(1+l\gamma)-(1-l\gamma)e^{-c\gamma}\right)}{c}$ has to converge to a constant. This is possible only if $c_h^p \to \infty$. This concludes the proposition.

A.5 Proof of Proposition 5

The argument that we can separate the value of cash and that of a unit of asset in the value function of agents and the derivation of (30)-(31) are analogous to the incomplete market case, so it is omitted. It is easy to check that the general forms (32)-(33) are indeed solutions of (30)-(31).

Now we show that equations (34) have a solution $L_1, L_2, L_3, L_4, c_h^{cm}$ where $c_h^{cm} = c_h^P$, and that

$$j_P(c) = v_{cm}(c) + cq_{cm}(c). \tag{A.8}$$

For this, first observe that first that

$$v_{cm}(c) + cq_{cm}(c) = R + uc + e^{-c\gamma}L_3 + e^{c\gamma}L_4. \tag{A.9}$$

Also, (34) can be written as

$$\begin{aligned}
\frac{R+L_3+L_4}{\frac{u}{2}+L_1+L_2} &= l && \text{(A.10)}\\
\frac{u}{2} + \gamma L_3 - L_2 - \gamma L_4 - L_1 &= 0 && \text{(A.11)}\\
-\gamma e^{-c_h^{cm}\gamma}L_1 + \gamma e^{c_h^{cm}\gamma}L_2 &= 0 && \text{(A.12)}\\
\frac{u}{2} + \gamma e^{c_h^{cm}\gamma}\left(L_3 - c_h^{cm}L_2\right) - L_2 e^{c_h^{cm}\gamma} - \gamma e^{-c_h^{cm}\gamma}\left(L_4 - c_h^{cm}L_1\right) - L_1 e^{-c_h^{cm}\gamma} &= 0 && \text{(A.13)}\\
\frac{R + uc_h^{cm} + e^{c_h^{cm}\gamma}L_3 + e^{-c_h^{cm}\gamma}L_4}{\frac{u}{2} + e^{-c_h^{cm}\gamma}L_1 + e^{c_h^{cm}\gamma}L_2} &= h + c_h^{cm}. && \text{(A.14)}
\end{aligned}$$

Adding c_h^{cm} times (A.12) to (A.13) gives

$$\frac{u}{2} + e^{c_h^{cm}\gamma}(\gamma L_3 - L_2) - e^{-c_h^{cm}\gamma}(\gamma L_4 + L_1) = 0. \tag{A.15}$$

Together with (A.11) this implies

$$\gamma L_3 = L_2, \quad -L_1 = \gamma L_4. \tag{A.16}$$

Substituting this into (A.12) gives

$$e^{-c_h^{cm}\gamma}L_4+e^{c_h^{cm}\gamma}L_3=0. \tag{A.17}$$

Also,expressing (L_1+L_2) from (A.11) and plugging into (A.10) gives

$$R+L_3+L_4=l\left(u+\gamma L_3-\gamma L_4\right) \tag{A.18}$$

and by (A.16), (A.14) is equivalent to

$$R+uc_h^{cm}+e^{c_h^{cm}\gamma}L_3+e^{-c_h^{cm}\gamma}L_4=\left(h+c_h^{cm}\right)\left(u-\gamma L_4e^{-c_h^{cm}\gamma}+\gamma L_3e^{c_h^{cm}\gamma}\right). \tag{A.19}$$

Observe that the system (A.17)-(A.19) is equivalent with the system (A.4)-(A.6), thus $L_3=D_2$, $L_4=D_1$ and $c_h^{cm}=c_h^P$. Given (A.9) and the fact that (A.16), we proved the statement.

Finally, we show that $v_{cm}'(c)>0$ and $q_{cm}'(c)<0$ for every $c\in(0,c_h^{cm})$ which proves that the price is monotonically increasing. For $q_{cm}'(c)<0$ observe that

$$\begin{aligned}
q_{cm}'(c) &= -\gamma e^{-c\gamma}L_1+\gamma e^{c\gamma}L_2=\gamma e^{-c\gamma}\gamma L_4+\gamma e^{c\gamma}\gamma L_3= \\
&= \gamma e^{-c\gamma}\gamma D_1+\gamma e^{c\gamma}\gamma D_2= \\
&= \gamma^2\left(R-lu\right)e^{c\gamma}\frac{1-e^{2\gamma\left(c_h^P-c\right)}}{e^{2\gamma c_h^P}+l\gamma\left(e^{2\gamma c_h^P}-1\right)+1}<0.
\end{aligned}$$

For $v_{cm}'(c)>0$ observe that

$$\begin{aligned}
v_{cm}'(c) &= \frac{u}{2}+\gamma e^{c\gamma}\left(L_3-cL_2\right)-L_2e^{c\gamma}-\gamma e^{-c\gamma}\left(L_4-cL_1\right)-L_1e^{-c\gamma}= \\
&= \frac{u}{2}-c\gamma^2D_2e^{c\gamma}-c\gamma^2D_1e^{-c\gamma}= \\
&= \frac{u}{2}+c\gamma^2\left(R-lu\right)e^{c\gamma}\frac{e^{2\gamma\left(c_h^P-c\right)}-1}{e^{2\gamma c_h^P}+l\gamma\left(e^{2\gamma c_h^P}-1\right)+1}>0.
\end{aligned}$$

A.6 Proof of Proposition 6

The first statement comes from the construction of the Proof of Proposition **??**. In particular, from the fact that c_h^* and c_l^* are constructed as the intercept of continuous functions $H(c_h), L(c_h)$ which map $[h,\infty)\to R^{++}$ and $H(h)=h>L(h)>0$ and $0<\lim_{c_h\to\infty}L(c_h)$ $\lim_{c_h\to\infty}H(c_h)=0<\lim_{c_h\to\infty}L(c_h)<\infty$. Thus, both $c_h^*\in(h,\infty)$ and $c_l^*\in(0,c_h^*)$.

The second statement is the consequence of Lemma A.6 and the first result in Proposition 4.

For the last statement, we provide a constructive proof. By the third statement in Proposition 4 for any parameters, we can pick an R sufficiently close to uh that $c_h^P>h$. Call this $\hat{c}_h^P$. By the fact that c_h^P solves

$$\frac{R-hu}{R-lu}\left(e^{c_h^P\gamma}(1+l\gamma)-(1-l\gamma)e^{-c_h^P\gamma}\right)-2\gamma\left(c_h^P+h\right)=0$$

for any γ there is an $R(\gamma)$ defined as

$$R(\gamma)\equiv u\frac{h\left(e^{\hat{c}_h^P\gamma}(1+l\gamma)-e^{-\hat{c}_h^P\gamma^i}(1-l\gamma)\right)-2l\gamma\left(\hat{c}_h^P+h\right)}{\left(e^{\hat{c}_h^P\gamma}(1+l\gamma)-(1-l\gamma^i)e^{-\hat{c}_h^P\gamma}\right)-2\gamma\left(\hat{c}_h^P+h\right)}$$

than for the pair γ, $R(\gamma)$ $c_h^P=\hat{c}_h^P$. Also, for sufficiently large γ, $c_h^*(\gamma,R(\gamma))\to h, c_l^*(\gamma,R(\gamma))\to l$ by an analogous argument to Lemma A.6. Thus, there is a sufficiently large $\hat{\gamma}_1$ that for any $\gamma>\hat{\gamma}_1$ and $R(\gamma)$ we have $c_h^*<c_h^P$. Also

$$\frac{\partial(R(\gamma))}{\partial\gamma}=2ue^{-\hat{c}_h^P\gamma}(h-l)\left(\hat{c}_h^P+h\right)\frac{\left(\hat{c}_h^P l\gamma^2-1\right)\left(e^{-2\hat{c}_h^P\gamma}-1\right)-c\gamma\left(e^{-2\hat{c}_h^P\gamma}+1\right)}{\left(-l\gamma+e^{2(-\hat{c}_h^P\gamma)}-l\gamma e^{2(-\hat{c}_h^P\gamma)}+2\hat{c}_h^P\gamma e^{-\hat{c}_h^P\gamma}+2h\gamma e^{-\hat{c}_h^P\gamma}-1\right)^2}$$

which is negative if $\gamma>\frac{1}{2l}\left(\sqrt{\frac{1}{\hat{c}_h^P}\left(\hat{c}_h^P+4l\right)}-1\right)$. Thus, to complete the proof, let us pick

$$\hat{\gamma}=\max\left(\gamma_1,\frac{1}{2l}\left(\sqrt{\frac{1}{\hat{c}_h^P}\left(\hat{c}_h^P+4l\right)}-1\right)\right).$$

A.7 Proof of Proposition 7

Suppose that we are given the policy (c_l,c_h) pair, so that $c_l>0$ and $c_h<c_h^P$ where c_h^P satisfies the super-contact condition $j_P''\left(c_h^P;0,c_h^P\right)=0$. For simplicity, we denote the social value (which is $j_S(c;c_l,c_h)$ in the main text) as $j(c;c_l,c_h)$, and we need to show that

$$\frac{\partial j(c;c_l,c_h)}{\partial c_l}<0 \text{ and } \frac{\partial j(c;c_h,c_l)}{\partial c_h}>0.$$

If this holds for any pair of (c_l,c_h) that lies in the interval of $\left[0,c_h^P\right]$, then we know that for $0<c_l^2<c_l^1<c_h^1<c_h^2<c_h^P$, we have

$$j\left(c;c_l^1,c_h^1\right)<j\left(c;c_l^2,c_h^2\right).$$

Before proceed, let us show that given (c_l,c_h), we have the failure of super contact condition on both ends c_h and c_l:

$$j''(c_h;c_l,c_h)<0 \text{ and } j''(c_l;c_l,c_h)<0.$$

To see this, notice that given (c_l, c_h) as non-optimal policies, it must be that

$$j\left(c; c_l, c_h\right) < j_P\left(c\right) \leq R + uc$$

where the last inequality comes from Lemma A.7. Then $0 = \frac{\sigma^2}{2} j''\left(c\right) + \xi\left(R + uc - j\left(c\right)\right)$ implies that the value function is strictly concave at both ends. And, for all c_l and c_h we must have

$$\begin{aligned} j\left(c_h; c_l, c_h\right) - \left(c_h + h\right) j'\left(c_h; c_l, c_h\right) &= 0, & \text{(A.20)} \\ j\left(c_l; c_l, c_h\right) - \left(c_l + l\right) j'\left(c_l; c_l, c_h\right) &= 0. & \text{(A.21)} \end{aligned}$$

These two conditions hold for any policy (including the market solution, the social planner's solution, or any solution.)

First, we focus on the top policy c_h. Define

$$F_h\left(c; c_l, c_h\right) \equiv \frac{\partial}{\partial c_h} j\left(c; c_l, c_h\right)$$

which is the impact of policy on value. Then since

$$0 = \frac{\sigma^2}{2} j''\left(c; c_l, c_h\right) + \xi\left(R + uc - j\left(c; c_l, c_h\right)\right), \tag{A.22}$$

we have $\frac{\sigma^2}{2} \frac{\partial}{\partial c_h} j''\left(c; c_l, c_h\right) - \xi \frac{\partial}{\partial c_h} j\left(c; c_l, c_h\right) = 0$, or

$$\frac{\sigma^2}{2} F_h''\left(c; c_l, c_h\right) - \xi F_h\left(c; c_l, c_h\right) = 0. \tag{A.23}$$

Moreover, take the total derivative with respect to c_h on the equality (A.20), i.e., take derivative that affects both the policy c_h and the state c_h, we have

$$\begin{aligned} & \frac{\partial}{\partial c_h} j\left(c_h; c_l, c_h\right) + j'\left(c_h; c_l, c_h\right) = j'\left(c_h; c_l, c_h\right) + \left(c_h + h\right)\left(\frac{\partial}{\partial c_h} j'\left(c_h; c_l, c_h\right) + j''\left(c_h; c_l, c_h\right)\right) \\ \Rightarrow \quad & \frac{\partial}{\partial c_h} j\left(c_h; c_l, c_h\right) - \left(c_h + h\right) \frac{\partial}{\partial c_h} j'\left(c_h; c_l, c_h\right) = \left(c_h + h\right) j''\left(c_h; c_l, c_h\right) < 0. \end{aligned}$$

Since $j''\left(c_h; c_l, c_h\right) < 0$, we have

$$F_h\left(c_h; c_l, c_h\right) - \left(c_h + h\right) F_h'\left(c_h; c_l, c_h\right) < 0, \tag{A.24}$$

which gives the boundary condition of F at c_h. At c_l we can take total derivative with respect to c_h on the equality (A.21), we have

$$\frac{\partial}{\partial c_h} j\left(c_l; c_l, c_h\right) = \left(c_l + l\right) \frac{\partial}{\partial c_h} j'\left(c_l; c_l, c_h\right) \Rightarrow F_h\left(c_l; c_l, c_h\right) - \left(c_l + l\right) F_h'\left(c_l; c_l, c_h\right) = 0 \tag{A.25}$$

Based on (A.23) we can show that these two boundary conditions imply F_h has to be positive always. As a result, $F_h(c; c_l, c_h) \equiv \frac{\partial}{\partial c_h} j(c; c_l, c_h)$ as the impact of raising c_h for any state and any lower policy c_l must be positive.

Lemma A.8 *We have $F_h(c) > 0$ for $c \in [c_l, c_h]$.*

Proof. *First we show that $F(c)$ cannot change sign over $[c_l, c_h]$. Suppose that $F_h(c_l) > 0$; then from (A.25) we know that $F'(c_l) > 0$. The ODE (A.23) implies that $F_h'' > 0$ always, so it is convex and always positive. If $F_h(c_l) < 0$ then the similar argument implies that F is concave and negative always. Now suppose that $F_h(c_l) = 0$ but F_h changes sign at some point. Without loss of generality, there must exist some point $\widehat{c}$ so that*

$$F_h'(\widehat{c}) = 0,\ F_h(\widehat{c}) > 0 \text{ and } F_h''(\widehat{c}) < 0;$$

but this contradicts with the ODE (A.23).

Now let $G_h(c) \equiv F_h(c) - (l+c) F_h'(c)$ so that $G_h'(c) = -(l+c) F_h''(c) = -\frac{2\xi(l+c)}{\sigma^2} F_h(c)$. As a result, $G_h'(c)$ cannot change sign. Since further that $G_h(c_l) = 0$, $G_h(c) = 0$ cannot change sign either. Suppose counterfactually that $F_h(c) < 0$ so that $G_h'(c) > 0$ and $G_h(c) > 0$. But we then have

$$\begin{aligned} G_h(c_h) &= F_h(c_h) - (l+c) F_h'(c_h) = F_h(c_h) - (h+c) F_h'(c_h) + (h-l) F_h'(c_h) \\ &= F_h(c_h) - (h+c) F_h'(c_h) + \frac{h-l}{l+c}(F_h - G_h) < 0 \end{aligned}$$

where we have used (A.24). Therefore $F_h(c) < 0$. ■

The argument for the effect of c_l is similar. Define $F(c; c_l, c_h) \equiv \frac{\partial}{\partial c_l} j(c; c_l, c_h)$ which is the impact of lower-end policy. Then since

$$0 = \frac{\sigma^2}{2} j''(c; c_l, c_h) + \xi (R + uc - j(c; c_l, c_h)),$$

which implies that

$$\frac{\sigma^2}{2} F''(c; c_l, c_h) - \xi F(c; c_l, c_h) = 0$$

Using (A.20) and (A.21), and take total derivatives, we can show that

$$\begin{aligned} F(c_h; c_l, c_h) - (c_h + h) F'(c_h; c_l, c_h) &= 0 \\ F(c_l; c_l, c_h) - (c_l + l) F'(c_l; c_l, c_h) &< 0 \end{aligned}$$

Based on (A.23) we can show that these two boundary conditions imply F has to be negative always.

A.8 Proof of Proposition 8

Consider the functions $\tilde{q}(c; q_0, v_0, c_h)$ and $\tilde{v}(c; q_0, v_0, c_h)$ of c parameterized by q_0, v_0, and c_h defined by the system

$$0 = \frac{\sigma^2}{2}\tilde{q}''(c) + \frac{\xi}{2}(u - \tilde{q}(c)) + \frac{\xi}{2}\left(\frac{R}{c} - \tilde{q}(c)\right) \tag{A.26}$$

$$0 = \tilde{q}'(c)\sigma^2 + \frac{\sigma^2}{2}\tilde{v}''(c) + \frac{\xi}{2}(uc - \tilde{v}(c)) + \frac{\xi}{2}(R - \tilde{v}(c)). \tag{A.27}$$

and the boundary conditions

$$\tilde{v}'(c_h) = \tilde{q}'(c_h) = 0 \tag{A.28}$$

$$\tilde{q}(c_0) = q_0, \tilde{v}(c_0) = v_0. \tag{A.29}$$

Define the function $c_h(q_0, v_0)$ implicitly by

$$\tilde{v}(c_h; q_0, v_0, c_h) - h\tilde{q}(c_h; q_0, v_0, c_h) \equiv 0.$$

We are interested in the derivatives

$$\frac{\partial c_h}{\partial q_0} = -\frac{\tilde{v}'_{q_0} - h\tilde{q}'_{q_0}}{\tilde{v}'_{c_h} - h\tilde{q}'_{c_h}}$$
$$\frac{\partial c_h}{\partial v_0} = -\frac{\tilde{v}'_{v_0} - h\tilde{q}'_{v_0}}{\tilde{v}'_{c_h} - h\tilde{q}'_{c_h}}$$

For this, consider the following Lemmas.

Lemma A.9 $\frac{\partial \tilde{q}(c_h; q_0, v_0, c_h)}{\partial q_0} = \frac{2}{e^{c_h\gamma}e^{-c_0\gamma} + e^{-c_h\gamma}e^{\gamma c_0}} > 0$

Proof. we know that $q(c_0) = q_0$ and $q'(c_h) = 0$. We can rewrite the earlier as

$$e^{-c_0\gamma}K_1 + e^{\gamma c_0}K_2 + l_q = q_0$$

which implies

$$\frac{-l_q - e^{\gamma c_0}K_2 + q_0}{e^{-c_0\gamma}} = K_1.$$

Rewrite the latter as

$$-e^{-c_h\gamma}\gamma K_1 + e^{c_h\gamma}\gamma K_2 + s_q = 0$$

which implies

$$K_2 = \frac{e^{-c_h\gamma}\gamma K_1 - s_q}{e^{c_h\gamma}\gamma} = \frac{e^{-c_h\gamma}\gamma\frac{-l_q - e^{\gamma c_0}K_2 + q_0}{e^{-c_0\gamma}} - s_q}{e^{c_h\gamma}\gamma}$$
$$\frac{-s_q e^{-c_0\gamma} + e^{-c_h\gamma}\gamma(-l_q + q_0)}{e^{-c_0\gamma} + e^{\gamma c_0}}.$$

or

$$K_2 = \frac{e^{-c_h\gamma}\gamma\frac{-l_q+q_0}{e^{-c_0\gamma}} - s_q}{\left(1+e^{-2c_h\gamma}e^{\gamma 2c_0}\right)e^{c_h\gamma}\gamma}$$

Thus,

$$\frac{\partial K_2}{\partial q_0} = \frac{e^{-c_h\gamma}}{e^{c_h\gamma}e^{-c_0\gamma}+e^{-c_h\gamma}e^{\gamma c_0}}$$

and

$$\frac{\partial K_1}{\partial q_0} = \frac{1}{e^{-c_0\gamma}} - e^{\gamma 2c_0}\frac{e^{-c_h\gamma}}{e^{c_h\gamma}e^{-c_0\gamma}+e^{-c_h\gamma}e^{\gamma c_0}} = \frac{e^{c_h\gamma}}{e^{c_h\gamma}e^{-c_0\gamma}+e^{-c_h\gamma}e^{\gamma c_0}}$$

implying that

$$\frac{\partial q\left(c_h\right)}{\partial q_0} = \frac{1}{e^{c_h\gamma}e^{-c_0\gamma}+e^{-c_h\gamma}e^{\gamma c_0}} + \frac{1}{e^{c_h\gamma}e^{-c_0\gamma}+e^{-c_h\gamma}e^{\gamma c_0}} = \frac{2}{e^{c_h\gamma}e^{-c_0\gamma}+e^{-c_h\gamma}e^{\gamma c_0}} > 0$$

■

Lemma A.10 $\frac{\partial \tilde{v}(c_h;q_0,v_0,c_h)}{\partial v_0} = \frac{2}{e^{-\gamma(c_h-c_0)}+e^{\gamma(c_h-c_0)}} > 0$

Proof. The general solution is

$$\begin{aligned}
v\left(c\right) &= R + \frac{cu}{2} + e^{c\gamma}\left(K_3 - cK_2\right) - e^{-c\gamma}\left(K_4 + cK_1\right) + \frac{cR\xi}{2\sqrt{2\xi\sigma^2}}\left(e^{\gamma c}\,\mathrm{Ei}\left(-\gamma c\right) - e^{-c\gamma}\,\mathrm{Ei}\left(\gamma c\right)\right) \\
v'\left(c\right) &= \frac{u}{2} + \frac{R\xi\left(-e^{-c\gamma}\,\mathrm{Ei}[c\gamma] + e^{c\gamma}\,\mathrm{Ei}[-c\gamma]\right)}{2\sqrt{2\xi\sigma^2}} + \frac{Rc\xi\gamma\left(e^{-c\gamma}\,\mathrm{Ei}[c\gamma] + e^{c\gamma}\,\mathrm{Ei}[-c\gamma]\right)}{2\sqrt{2\xi\sigma^2}} \\
&\quad + + e^{c\gamma}\left(\left(-\gamma c - 1\right)K_2 + \gamma K_3\right) + e^{-c\gamma}\left(\left(\gamma c - 1\right)K_1 + \gamma K_4\right)
\end{aligned}$$

our boundaries are $v\left(c_0\right) = v_0$ and $v'\left(c_h\right) = 0$ from the first

$$e^{c_0\gamma}K_3 - e^{-c_0\gamma}K_4 + l_v = v_0$$

where l_v does not depend on K_3 andK_4 or v_0

$$\begin{aligned}
e^{c_0\gamma}K_3 &= v_0 + e^{-c_0\gamma}K_4 - l_v \\
K_3 &= e^{-c_0\gamma}v_0 + e^{-2c_0\gamma}K_4 - e^{-c_0\gamma}l_v
\end{aligned}$$

and we rewrite the second boundary as

$$\begin{aligned}
s_v + e^{c_h\gamma}\gamma K_3 + e^{-c_h\gamma}\gamma K_4 &= 0 \\
s_v + e^{c_h\gamma}\gamma\left(e^{-c_0\gamma}v_0 + e^{-2c_0\gamma}K_4 - e^{-c_0\gamma}l_v\right) + e^{-c_h\gamma}\gamma K_4 &= 0 \\
K_4 &= \frac{-s_v - e^{c_h\gamma}\gamma\left(e^{-c_0\gamma}v_0 - e^{-c_0\gamma}l_v\right)}{\left(e^{c_h\gamma}e^{-2c_0\gamma} + e^{-c_h\gamma}\right)\gamma}
\end{aligned}$$

$$
\begin{aligned}
\frac{\partial K_4}{\partial v_0} &= \frac{-e^{c_h\gamma}e^{-c_0\gamma}}{e^{c_h\gamma}e^{-2c_0\gamma}+e^{-c_h\gamma}} \\
\frac{\partial K_3}{\partial v_0} &= e^{-c_0\gamma}+e^{-2c_0\gamma}\frac{-e^{c_h\gamma}e^{-c_0\gamma}}{e^{c_h\gamma}e^{-2c_0\gamma}+e^{-c_h\gamma}}
\end{aligned}
$$

$$
\frac{\partial v\left(c_h\right)}{\partial v_0} = e^{c_h\gamma}\left(e^{-c_0\gamma}+e^{-2c_0\gamma}\frac{-e^{c_h\gamma}e^{-c_0\gamma}}{e^{c_h\gamma}e^{-2c_0\gamma}+e^{-c_h\gamma}}\right) - e^{-c_h\gamma}\frac{-e^{c_h\gamma}e^{-c_0\gamma}}{e^{c_h\gamma}e^{-2c_0\gamma}+e^{-c_h\gamma}} = \frac{2}{e^{-\gamma(c_h-c_0)}+e^{\gamma(c_h-c_0)}} > 0
$$

■

Lemma A.11 $\frac{\partial\tilde{v}(c_h;q_0,v_0,c_h)}{\partial q_0} = 2\frac{e^{\gamma(c_h-c_0)}-e^{-\gamma(c_h-c_0)}-\gamma(c_h-c_0)\left(e^{-\gamma(c_h-c_0)}+e^{\gamma(c_h-c_0)}\right)}{\gamma\left(e^{\gamma c_0}e^{-\gamma c_h}+e^{-\gamma c_0}e^{\gamma c_h}\right)^2} < 0$ *and* $\frac{\partial\tilde{v}(c_h;q_0,v_0,c_h)}{\partial q_0} - h\frac{\partial q(c_h)}{\partial q_0} = 2\frac{e^{\gamma(c_h-c_0)}-e^{-\gamma(c_h-c_0)}-\gamma(c_h+h-c_0)\left(e^{-\gamma(c_h-c_0)}+e^{\gamma(c_h-c_0)}\right)}{\gamma\left(e^{\gamma c_0}e^{-\gamma c_h}+e^{-\gamma c_0}e^{\gamma c_h}\right)^2} < 0.$

Proof. We rewrite $v\left(c_0\right)$ and $v'\left(c_h\right)$ as

$$
\begin{aligned}
v\left(c_0\right) &= e^{c_0\gamma}\left(K_3-c_0K_2\right)-e^{-c_0\gamma}\left(K_4+c_0K_1\right)+l_{vq} \\
v'\left(c_h\right) &= s_{vq}+e^{c_h\gamma}\left(\left(-\gamma c_h-1\right)K_2+\gamma K_3\right)+e^{-c_h\gamma}\left(\left(\gamma c_h-1\right)K_1+\gamma K_4\right)
\end{aligned}
$$

Thus, the boundaries $v\left(c_0\right)=v_0$ imply

$$
K_3 = c_0K_2+e^{-c_0\gamma}v_0-e^{-c_0\gamma}l_{vq}+e^{-2c_0\gamma}\left(K_4+c_0K_1\right)
$$

and $v'\left(c_h\right)=0$

$$
K_4 = -\frac{\begin{array}{c}\left(-e^{\gamma c_h}\left(\gamma c_h-\gamma c_0+1\right)\right)K_2+\left(e^{-\gamma c_h}\left(\gamma c_h-1\right)+\gamma c_0e^{-2\gamma c_0}e^{\gamma c_h}\right)K_1 \\ +\left(\gamma e^{-\gamma c_0}e^{\gamma c_h}\right)v_0+\left(s_{qv}-\gamma e^{-\gamma c_0}e^{\gamma c_h}l_{qv}\right)\end{array}}{\gamma e^{-\gamma c_h}+\gamma e^{-2\gamma c_0}e^{\gamma c_h}}
$$

Thus,

$$
\begin{aligned}
\frac{\partial K_4}{\partial q_0} = \frac{e^{c_h\gamma}}{e^{c_h\gamma}e^{-c_0\gamma}+e^{-c_h\gamma}e^{\gamma c_0}}&\left(-\frac{\left(e^{\gamma c_0}e^{-\gamma c_h}\left(\gamma c_h-1\right)+\gamma c_0e^{-\gamma c_0}e^{\gamma c_h}\right)}{\gamma e^{\gamma c_0}e^{-\gamma c_h}+\gamma e^{-\gamma c_0}e^{\gamma c_h}}\right) \\
&-\frac{\left(-e^{\gamma c_0}e^{\gamma c_h}\left(\gamma c_h-\gamma c_0+1\right)\right)}{\gamma e^{\gamma c_0}e^{-\gamma c_h}+\gamma e^{-\gamma c_0}e^{\gamma c_h}}\frac{e^{-c_h\gamma}}{e^{c_h\gamma}e^{-c_0\gamma}+e^{-c_h\gamma}e^{\gamma c_0}} = \\
&= e^{\gamma c_h}\frac{2e^{\gamma c_0}e^{-\gamma c_h}-\gamma c_0\left(e^{\gamma c_0}e^{-\gamma c_h}+e^{-\gamma c_0}e^{\gamma c_h}\right)}{\gamma\left(e^{\gamma c_0}e^{-\gamma c_h}+e^{-\gamma c_0}e^{\gamma c_h}\right)^2}
\end{aligned}
$$

where we used the previous results on $\frac{\partial K_1}{\partial q_0}$, $\frac{\partial K_2}{\partial q_0}$. Also

$$
\begin{aligned}
\frac{\partial K_3}{q_0} &= \frac{\partial K_1}{q_0}e^{-2c_0\gamma}c_0+\frac{\partial K_2}{q_0}c_0+\frac{\partial K_4}{q_0}e^{-2c_0\gamma} \\
&= \frac{e^{c_h\gamma}}{e^{c_h\gamma}e^{-c_0\gamma}+e^{-c_h\gamma}e^{\gamma c_0}}e^{-2c_0\gamma}c_0+\frac{e^{-c_h\gamma}}{e^{c_h\gamma}e^{-c_0\gamma}+e^{-c_h\gamma}e^{\gamma c_0}}c_0+e^{\gamma c_h}\frac{2e^{\gamma c_0}e^{-\gamma c_h}-\gamma c_0\left(e^{\gamma c_0}e^{-\gamma c_h}+e^{-\gamma c_0}e^{\gamma c_h}\right)}{\gamma\left(e^{\gamma c_0}e^{-\gamma c_h}+e^{-\gamma c_0}e^{\gamma c_h}\right)^2} \\
&= e^{-\gamma c_h}\frac{2e^{-\gamma c_0}e^{\gamma c_h}+\gamma c_0\left(e^{\gamma c_0}e^{-\gamma c_h}+e^{-\gamma c_0}e^{\gamma c_h}\right)}{\gamma\left(e^{\gamma c_0}e^{-\gamma c_h}+e^{-\gamma c_0}e^{\gamma c_h}\right)^2}
\end{aligned}
$$

Consequently,

$$\frac{\partial v\left(c_h\right)}{\partial q_0}=e^{c_h\gamma}\frac{\partial K_3}{q_0}-e^{-c_h\gamma}\frac{\partial K_4}{q_0}-c_h\frac{\partial q\left(c_h\right)}{\partial q_0}=$$
$$=\frac{2e^{-\gamma c_0}e^{\gamma c_h}+\gamma c_0\left(e^{\gamma c_0}e^{-\gamma c_h}+e^{-\gamma c_0}e^{\gamma c_h}\right)}{\gamma\left(e^{\gamma c_0}e^{-\gamma c_h}+e^{-\gamma c_0}e^{\gamma c_h}\right)^2}-\frac{2e^{\gamma c_0}e^{-\gamma c_h}-\gamma c_0\left(e^{\gamma c_0}e^{-\gamma c_h}+e^{-\gamma c_0}e^{\gamma c_h}\right)}{\gamma\left(e^{\gamma c_0}e^{-\gamma c_h}+e^{-\gamma c_0}e^{\gamma c_h}\right)^2}-c_h\frac{\partial q\left(c_h\right)}{\partial q_0}=$$
$$=2\frac{e^{\gamma(c_h-c_0)}-e^{-\gamma(c_h-c_0)}+\gamma c_0\left(e^{\gamma c_0}e^{-\gamma c_h}+e^{-\gamma c_0}e^{\gamma c_h}\right)}{\gamma\left(e^{\gamma c_0}e^{-\gamma c_h}+e^{-\gamma c_0}e^{\gamma c_h}\right)^2}-c_h\frac{\partial q\left(c_h\right)}{\partial q_0}=$$
$$=2\frac{e^{\gamma(c_h-c_0)}-e^{-\gamma(c_h-c_0)}-\gamma\left(c_h-c_0\right)\left(e^{-\gamma(c_h-c_0)}+e^{\gamma(c_h-c_0)}\right)}{\gamma\left(e^{\gamma c_0}e^{-\gamma c_h}+e^{-\gamma c_0}e^{\gamma c_h}\right)^2}<0.$$

the inequality comes from the fact that the function

$$e^x-e^{-x}-x\left(e^{-x}+e^x\right)$$

is negative and monotonically decreasing for all $x>0$. The second statement comes directly from the expression for $\frac{\partial q(c_h)}{\partial q_0}$. ■

Lemma A.12 *if* $\frac{v_0}{q_0}<h$, *then* $\tilde{v}_{c_h}-h\tilde{q}_{c_h}>0$.

Proof. Using ... we write

$$\begin{aligned} q\left(c\right) &= \frac{u}{2}+e^{-c\gamma}K_1+e^{\gamma c}K_2+\frac{R\xi}{2}\frac{F\left(c\right)}{\gamma\sigma^2} \\ v\left(c\right) &= R+\frac{cu}{2}+e^{c\gamma}\left(K_3-cK_2\right)-e^{-c\gamma}\left(K_4+cK_1\right)-\frac{cR\xi}{2\gamma\sigma^2}F\left(c\right) \end{aligned}$$

and $F'\left(c\right)=\gamma G\left(c\right)$ where

$$\begin{aligned} F\left(c\right) &= -e^{\gamma c}\,\mathrm{Ei}\left[-\gamma c\right]+e^{-\gamma c}\,\mathrm{Ei}\left[c\gamma\right] \\ G\left(c\right) &= -e^{-c\gamma}\,\mathrm{Ei}[c\gamma]-e^{c\gamma}\,\mathrm{Ei}[-c\gamma]. \end{aligned}$$

We want to show that the function $\tilde{v}\left(c;q_0,v_0,c\right)-h\tilde{q}\left(c;q_0,v_0,c\right)$ is negative at $c=c_0$ and positive at $c\to\infty$. Therefore, there is a $c=c_h$ where this function is zero (satisfying the definition of c) and where the slope of this function is positive.

To get $\tilde{v}\left(c;q_0,v_0,c\right)-h\tilde{q}\left(c;q_0,v_0,c\right)$ we have to solve (A.26)-(A.27) with the boundary conditions

$$\tilde{v}'\left(c\right) = \tilde{q}'\left(c\right)=0 \tag{A.30}$$
$$\tilde{q}\left(c_0\right) = q_0,\tilde{v}\left(c_0\right)=v_0. \tag{A.31}$$

Thus, indeed,

$$\tilde{v}\left(c_0;q_0,v_0,c_0\right)-h\tilde{q}\left(c_0;q_0,v_0,c_0\right)=v_0-hq_0<0$$

by the condition of the proposition.

Now we show that $\lim_{c\to\infty}\tilde{q}\left(c;q_0,v_0,c\right)=\frac{u}{2}$. For this note that solving for K_1 and K_2 from (A.26)-(A.27), (A.30)-(A.31) we get

$$\begin{aligned} e^{c\gamma}\frac{\frac{u}{2}+e^{(c-c_0)\gamma}\frac{R\xi}{2}\frac{G(c)}{\gamma\sigma^2}+\frac{R\xi}{2}\frac{F(c_0)}{\gamma\sigma^2}-q_0}{\left(e^{(2c-c_0)\gamma}+e^{\gamma c_0}\right)} &= e^{c\gamma}K_2 \\ e^{c\gamma}K_2+\frac{R\xi}{2}\frac{G\left(c\right)}{\gamma\sigma^2} &= e^{-c\gamma}K_1. \end{aligned}$$

Using $\lim_{c\to\infty}F\left(c\right)=\lim_{c\to\infty}G\left(c\right)=0$ gives

$$\lim_{c\to\infty}e^{c\gamma}K_2=\lim_{c\to\infty}e^{-c\gamma}K_1=0.$$

implying the result.

Now we show that $\lim_{c\to\infty}\tilde{v}\left(c;q_0,v_0,c\right)=\infty$, (A.26)-(A.27), (A.30)-(A.31) gives

$$\begin{aligned} e^{-c\gamma}K_4 &= \frac{e^{(c_0-c)\gamma}\left(R+c_0u-v_0-c_0q\left(c_0\right)\right)+e^{(2c_0-2c)\gamma}\left(-\frac{u}{\gamma}+\frac{Rc\xi G(c)}{\gamma 2\sigma^2}+\frac{1}{\gamma}q\left(c\right)\right)}{\left(1+e^{(-2c+2c_0)\gamma}\right)} \\ e^{c\gamma}K_3 &= -\frac{u}{\gamma}+\frac{Rc\xi G\left(c\right)}{\gamma 2\sigma^2}+\frac{q\left(c\right)}{\gamma}-e^{-c\gamma}K_4 \end{aligned}$$

using that $\lim_{c\to\infty}\frac{Rc\xi G(c)}{\gamma 2\sigma^2}=0$ gives

$$\lim_{c\to\infty}e^{-c\gamma}K_4=0,\ \lim_{c\to\infty}e^{c\gamma}K_3=-\frac{u}{\gamma}$$

implying the result.

Thus, $\lim_{c\to\infty}\tilde{v}\left(c;q_0,v_0,c\right)-h\tilde{q}\left(c;q_0,v_0,c\right)=\infty$ which proves our claim. ■

Putting together the three Lemma gives

$$\begin{aligned} \frac{\partial c_h}{\partial q_0} &= -\frac{\tilde{v}'_{q_0}-h\tilde{q}'_{q_0}}{\tilde{v}'_{c_h}-h\tilde{q}'_{c_h}}<0 \\ \frac{\partial c_h}{\partial v_0} &= -\frac{\tilde{v}'_{v_0}-h\tilde{q}'_{v_0}}{\tilde{v}'_{c_h}-h\tilde{q}'_{c_h}}>0 \end{aligned}$$

This implies that $c_h^\tau<c_h^*$ whenever $q_\tau\left(c_0\right)\leq q\left(c_0\right)$ and $v_\tau\left(c_0\right)\geq v\left(c_0\right)$. The only remaining part of the Lemma is to show that even if v_0 and q_0 decrease proportionally so $\frac{v_0}{q_0}$ remains constant then $\frac{v(c_h)}{q(c_h)}$ would decrease. We increase q_0 to $\bar{q}_0=q_0+\varepsilon$ where ε is very small. To make sure that $\frac{\bar{v}_0}{\bar{q}_0}=\frac{v_0}{q_0}$, we need that $\bar{v}_0=v_0+a\varepsilon$ where $a=\frac{v_0}{q_0}$. Let us refer to all the objects after the change with the bar. Using the first two Lemmas above, we have

$$\begin{aligned} \bar{q}\left(c_h\right) &= q\left(c_h\right)+\varepsilon\frac{2}{e^{c_h\gamma}e^{-c_0\gamma}+e^{-c_h\gamma}e^{\gamma c_0}} \\ \bar{v}\left(c_h\right) &= v\left(c_h\right)+\varepsilon 2\frac{e^{\gamma(c_h-c_0)}-e^{-\gamma(c_h-c_0)}-\gamma\left(c_h-c_0\right)\left(e^{-\gamma(c_h-c_0)}+e^{\gamma(c_h-c_0)}\right)}{\gamma\left(e^{\gamma c_0}e^{-\gamma c_h}+e^{-\gamma c_0}e^{\gamma c_h}\right)^2}+\frac{v_0}{q_0}\varepsilon\frac{2}{e^{c_h\gamma}e^{-c_0\gamma}+e^{-c_h\gamma}e^{\gamma c_0}} \end{aligned}$$

we show we want to show that

$$\frac{v\left(c_{h}\right)+\varepsilon 2\frac{e^{\gamma\left(c_{h}-c_{0}\right)}-e^{-\gamma\left(c_{h}-c_{0}\right)}-\gamma\left(c_{h}-c_{0}\right)\left(e^{-\gamma\left(c_{h}-c_{0}\right)}+e^{\gamma\left(c_{h}-c_{0}\right)}\right)}{\gamma\left(e^{\gamma c_{0}}e^{-\gamma c_{h}}+e^{-\gamma c_{0}}e^{\gamma c_{h}}\right)^{2}}+\frac{v_{0}}{q_{0}}\varepsilon\frac{2}{e^{c_{h}\gamma}e^{-c_{0}\gamma}+e^{-c_{h}\gamma}e^{\gamma c_{0}}}}{q\left(c_{h}\right)+\varepsilon\frac{2}{e^{c_{h}\gamma}e^{-c_{0}\gamma}+e^{-c_{h}\gamma}e^{\gamma c_{0}}}}<\frac{v\left(c_{h}\right)}{q\left(c_{h}\right)}$$

which is equivalent to

$$\frac{e^{\gamma\left(c_{h}-c_{0}\right)}-e^{-\gamma\left(c_{h}-c_{0}\right)}-\gamma\left(c_{h}-c_{0}\right)\left(e^{-\gamma\left(c_{h}-c_{0}\right)}+e^{\gamma\left(c_{h}-c_{0}\right)}\right)}{\gamma\left(e^{\gamma c_{0}}e^{-\gamma c_{h}}+e^{-\gamma c_{0}}e^{\gamma c_{h}}\right)}<\left(\frac{v\left(c_{h}\right)}{q\left(c_{h}\right)}-\frac{v_{0}}{q_{0}}\right)$$

which holds, because the left hand side is negative as $e^{x}-e^{-x}-x\left(e^{-x}+e^{x}\right)<0$ for every x.

A.9 Appendix for Section 5.3

The following Proposition gives the system of ODEs determining the social welfare for an arbitrarily given c_h and c_l in our alternative setting.

Proposition 9 *The total welfare in the alternative specification for arbitrarily given thresholds $c_l < c_h$ is*

$$Aj_{S}\left(c;c_{l},c_{h}\right)=A\left(v_{S}\left(c\right)+q_{S}\left(c\right)c\right)$$

where v_S and q_S is given by the system

$$\begin{aligned}
0 &= -q_{S}^{\prime}\left(c+p_{S}\right)\xi+\frac{\sigma^{2}}{2}q_{S}^{\prime\prime}+\xi\left(u-q_{S}\right)+\phi\left(1-q_{S}\right)\\
&\quad+\eta\left(1_{c>c_{h}}\left(-\frac{c-c_{h}}{ch+cc_{h}}\left(hq_{S}\left(c_{h}\right)-v_{S}\left(c_{h}\right)\right)+q_{S}\left(c_{h}\right)-q_{S}\left(c\right)\right)+1_{c<c_{l}}\left(q_{S}\left(c_{l}\right)-q_{S}\left(c\right)\right)\right),\\
0 &= q_{S}^{\prime}\left(c_{t}\right)\sigma^{2}-v_{S}^{\prime}\left(c+p_{S}\right)\xi+\frac{\sigma^{2}}{2}v^{\prime\prime}+\xi\left(p_{S}u-v_{S}\right)+\phi\left(R-v\right)\\
&\quad+\eta\left(1_{c>c_{h}}\left(v_{S}\left(c_{h}\right)-v_{S}\left(c\right)\right)+1_{c<c_{l}}\left(-\frac{c_{l}-c}{l+c_{l}}\left(v_{S}\left(c_{l}\right)-lq_{S}\left(c_{l}\right)\right)+v_{S}\left(c_{l}\right)-v_{S}\left(c\right)\right)\right).
\end{aligned}$$

with the boundary conditions (43)-(44) and (47)-(48).

Proof. *For social welfare $j(c)$ so that $c > c_h^g$, once investment opportunity arrives, immediately the economy should build x trees so that*

$$\frac{C-xh}{A+x}=c_{h}^{g}\Rightarrow x=\frac{C-c_{h}^{g}A}{h+c_{h}^{g}}=A\frac{c-c_{h}^{g}}{h+c_{h}^{g}},$$

and the total value is

$$Aj\left(c\right)=\left(A+x\right)j\left(c_{h}^{g}\right)=A\left(1+\frac{c-c_{h}^{g}}{h+c_{h}^{g}}\right)j\left(c_{h}^{g}\right)=A\left(\frac{h+c}{h+c_{h}^{g}}\right)j\left(c_{h}^{g}\right)$$

If instead that $C < Ac_l^g$, then the economy should dismantle x trees so that

$$\frac{C+xl}{A-x} = c_l^g \Rightarrow x = \frac{c_l^g A - C}{l + c_l^g} = A\frac{c_l^g - c}{l + c_l^g},$$

and the total value is

$$Aj\left(c\right) = \left(A-x\right) j\left(c_l^g\right) = A\left(1 - \frac{c_l^g - c}{l + c_l^g}\right) j\left(c_l^g\right) = A\left(\frac{l+c}{l+c_l^g}\right) j\left(c_l^g\right)$$

So essentially we have to evaluate

$$0 = j''\frac{\sigma^2}{2} - j'\left(c+p\right)\xi + \xi\left(p+c\right)u + \phi\left(R+c-j\right) + \eta\left(\left(\frac{\theta+c}{\theta+B^*\left(c\right)}\right) j\left(B^*\left(c\right)\right) - j\left(c\right)\right)$$

However we need to know p and therefore we need to solve for v and q. Take the upper as example. Suppose that social planner build trees through taxing cash and do not touch tree. we need to build

$$A\frac{c - c_h^g}{t + c_h^g}$$

amount of trees, which need

$$A\frac{c - c_h^g}{h + c_h^g} h$$

amount of cash. Since existing cash is Ac, per unit of cash the taxation is

$$\frac{c - c_h^g}{h + c_h^g}\frac{h}{c}$$

in the meantime each gets $A\frac{c-c_h^g}{h+c_h^g}$ trees that they can easily sell to the market to get

$$A\frac{c - c_h^g}{h + c_h^g}\frac{v\left(c_h\right)}{q\left(c_h\right)}$$

As a result, the net taxation per unit of cash is

$$\frac{c - c_h^g}{h + c_h^g}\frac{h}{c} - \frac{c - c_h^g}{h + c_h^g}\frac{v\left(c_h\right)}{q\left(c_h\right)c} = \frac{c - c_h^g}{ch + cc_h^g}\left(t - \frac{v\left(c_h\right)}{q\left(c_h\right)}\right) = \frac{c - c_h^g}{ch + cc_h^g}\left(h - p\left(c_h\right)\right)$$

If $p\left(c_h\right) > h$ then cash is getting positive taxes. Therefore for q equation, we have (we need to multiply above the by $q\left(c_h^g\right)$ to get back to utilities)

$$0 = -q'\left(c+p\right)\xi + \frac{\sigma^2}{2}q'' + \xi\left(u-q\right) + \phi\left(1-q\right) + \eta\left(-\frac{c - c_h^g}{ch + cc_h^g}\left(hq\left(c_h^g\right) - v\left(c_h^g\right)\right) + q\left(c_h^g\right) - q\left(c\right)\right) \text{ for } c > c_h^g.$$

Similarly, when c is low, social planner wants to dismantle amount of trees

$$x = A\frac{c_l^g - c}{l + c_l^g}$$

which brings cash of

$$A\frac{c_l^g - c}{l + c_l^g}l$$

So for each tree, investor has to send to the social planner of $\frac{c_l^g - c}{l + c_l^g}$ *amount of tree, and getting back* $\frac{c_l^g - c}{l + c_l^g}l$ *amount of cash. Because they can immediately sell these trees to the market, effectively each tree is taxed at*

$$\frac{c_l^g - c}{l + c_l^g}\left(v\left(c_l^g\right) - lq\left(c_l^g\right)\right).$$

When $p\left(c_l^g\right) < l$ *then trees are getting tax subsidy, and we have*

$$0 = q'\left(c_t\right)\sigma^2 - v'\left(c + p\right)\xi + \frac{\sigma^2}{2}v'' + \xi\left(pu - v\right) + \phi\left(R - v\right) + \eta\left(-\frac{c_l^g - c}{l + c_l^g}\left(p\left(c_l^g\right) - l\right) + v\left(c_l^g\right) - v\left(c\right)\right) \text{ for } c < c_l^g$$

■

B Appendix (online only): An alternative equilibrium

In the main text, we showed that an equilibrium exist when $h - l$ is sufficiently small. While our condition is only sufficient, and not necessary, it is possible that the type of equilibrium we present does not exist. In this Appendix, we provide some insights on the type of equilibrium which arises instead. We argue that the main properties of this alternative equilibrium are very similar to the one presented.

While the equation system (17)-(18), (13)-(14) always have a solution, for some parameters this solution implies that for a c sufficiently close to c_l^*, the price is below the threshold l. This obviously cannot be an equilibrium. This is so, because agents would liquidate the first instant when the price drops below the liquidation value. For this set of parameters, the equilibrium is changed. The main difference is that there is a $c_x \in (c_l^*, c_h^*)$ that for every $c \in [c_l^*, c_x]$

$$p(c) = \frac{v(c)}{q(c)} = l$$

and an endogenous fraction of trees are liquidated at every instant. That is, in this range the price is constant in c and specialists liquidate an increasing fraction of their trees as c drops further from c_x. The following Proposition describes this equilibrium.

Proposition B.1 *Suppose that there is a $c_h^* < R$, $c_x \in (l, c_h^*)\,, q_0, K_1, K_2, K_3, K_4$ solving (17)-(18), (13)*

$$\begin{aligned}
\frac{\xi}{2\sigma^2}\left(u + \frac{R}{c_x}\right)(l - c_x) &= q'(c_x) \\
l\frac{\xi}{2\sigma^2}\left(u + \frac{R}{c_x}\right)(l - c_x) &= v'(c_x) \\
\frac{v(c_x)}{q(c_x)} &= l, \\
\frac{v(c_h^*)}{q(c_h^*)} &= h \\
v'(c_h) &= q'(c_h) = 0.
\end{aligned}$$

Then there is an incomplete market equilibrium with partial liquidation where

1. *agents do not consume before the tree matures,*
2. *each agent in each state $c \in [l, c_h^*]$ is indifferent in the composition of her portfolio*
3. *agents do not build or liquidate trees when $c \in (c_x, c_h^*)$ and, in aggregate, agents spend every positive cash shock to build trees iff $c = c_h^*$ and finance an endogenous fraction of the negative cash shocks by liquidating a fraction of trees iff $c \in [l, c_x]$. When $c = l$, agents finance the every negative cash shock by liquidating trees.*

4. the value of holding a unit of cash and the value of holding a unit of tree are described by $q(c)$ *and* $v(c)$ *and the ex ante price is* $p = \frac{v(c)}{q(c)}$ *when* $c \in [c_x, c_h^*]$ *and by*

$$\begin{aligned} q_m(c) &= q_0 + \frac{\xi}{2\sigma^2}\left[(ul - R)(c - l) - \frac{u}{2}\left(c^2 - l^2\right) + lR(\ln c - \ln l)\right] \\ v_m(c) &= lq_m(c) \end{aligned}$$

and the ex ante price is $p = l$ *when* $c \in [l, c_x]$.

5. Ex post, each agents hit by the shock sells all her trees to the agents who are not hit by the shock for the price $p_T = c$.

Proof. Under the conditions of the Proposition, agents start to disinvest whenever

$$p(c) = l$$

and along the way

$$dc = x(c)\,dt + \sigma dZ_t.$$

Here, if the disinvestment rate is $y = -\frac{dA}{A}$, then

$$x(c) = \frac{dC}{A} - \frac{C}{A}\frac{dA}{A} = -\frac{ldA}{A} - \frac{C}{A}\frac{dA}{A} = (l + c)\,y.$$

The following must hold as agents are always indifferent.

$$\begin{aligned} 0 &= x(c)\,q'(c) + \frac{\sigma^2}{2}q''(c) + \frac{\xi}{2}\left(u + \frac{R}{c}\right) - \xi q(c) \\ 0 &= x(c)\,v'(c) + q'(c)\,\sigma^2 + \frac{\sigma^2}{2}v''(c) + \frac{\xi}{2}(uc + R) - \xi v(c) \\ v(c) &= lq(c) \end{aligned}$$

Then we must have

$$\begin{aligned} 0 &= x(c)\,lq'(c) + q'(c)\,\sigma^2 + \frac{\sigma^2}{2}lq''(c) + \frac{\xi}{2}(uc + R) - \xi lq(c) \\ 0 &= x(c)\,lq'(c) + \frac{\sigma^2}{2}lq''(c) + \frac{\xi l}{2}\left(u + \frac{R}{c}\right) - \xi lq(c) \end{aligned}$$

so that

$$q'(c)\,\sigma^2 + \frac{\xi}{2}(uc + R) - \frac{\xi l}{2}\left(u + \frac{R}{c}\right) = 0$$

or,

$$q'(c) = \frac{\xi}{2\sigma^2}\left(u + \frac{R}{c}\right)(l - c) = 0.$$

As $q'(c_l) = 0$ has to hold, $c_l = l$. The closed-form solution is

$$q(c) = q_0 + \frac{\xi}{2\sigma^2}\left[(ul - R)(c - l) - \frac{u}{2}\left(c^2 - l^2\right) + lR(\ln c - \ln l)\right]$$

And,

$$q''(c) = -\frac{\xi}{2\sigma^2}\left(u + \frac{lR}{c^2}\right) < 0.$$

We know that for $c \in [l, c_x]$ we have $v(c) = lq(c)$ which gives

$$x(c) = \frac{-\frac{\sigma^2}{2}q''(c) - \frac{\xi}{2}\left(u + \frac{R}{c}\right) + \xi q(c)}{q'(c)}$$

For $c > c_x$ we have the ODE as usual. We then search for the c_x, c_h pair that satisfies the conditions of the proposition. ■

Plotting v, q and p give very similar graphs to Figure 3 with the main difference that at the range $c \in [l, c_x]$ the price is flat at the level l. In the same range $q(c)$ is decreasing implying that $v(c) = lq(c)$ is also decreasing.