Signal or noise? Uncertainty and learning about whether other traders are informed*

Snehal Banerjee
Northwestern University
Kellogg School of Management
snehal-banerjee@kellogg.northwestern.edu

Brett Green
UC Berkeley
Haas School of Business
bgreen@haas.berkeley.edu

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Abstract

We develop a model in which uninformed rational traders (*speculators*) are uncertain about whether other market participants are trading on informative signals or noise. This uncertainty generates a non-linear price that reacts more strongly to bad news than it does to good news. In fact, the price can even decrease following good news about fundamentals. We incorporate this uncertainty into a dynamic setting where speculators learn gradually about other traders, and show that this leads to a rich set of return dynamics. Expected returns and volatility are stochastic but predictable, and vary non-monotonically with disagreement across investors. The model also generates volatility clustering in which large return realizations, associated with dividend surprises, are followed by higher future volatility and expected returns. Finally, we show that the relation between returns and information quality varies endogenously and derive empirical predictions for turnover on asset prices.

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1 Introduction

As early as Keynes (1936), it has been recognized that investors face uncertainty not only about fundamentals, but also about the underlying characteristics and trading motives of other market participants.[1] Asset pricing models have focused primarily on the former, taking the latter as common knowledge. For instance, in Grossman and Stiglitz (1980), uninformed investors know the number of informed investors in the market and the precision of their signals. Similarly, each agent in Hellwig (1980) is certain about both the number other agents and the distribution of their signals. Arguably, this requires an unrealistic degree of knowledge about the economy — it seems unlikely that investors who are uncertain about fundamentals, know, *with certainty*, whether other investors are privately informed.

We develop a framework in which rational uninformed traders (*speculators*) are uncertain about whether others trade on informative signals or noise. This uncertainty generates an equilibrium price that is non-linear in the information about fundamentals, and reacts more strongly to bad news than to good news. Surprisingly, the price may even decrease following more positive news about dividends. We incorporate this uncertainty into a dynamic environment in which speculators can gradually learn whether others are informed using realized prices and dividends.

This combination of uncertainty and learning about other traders has rich implications for return dynamics. Expected returns and volatility are stochastic but predictable and vary with the disagreement across investors. The dynamic model also generates volatility clustering in which large (positive or negative), unexpected return realizations in the current period are followed by higher return volatility and higher expected returns in the next period. Further, the relation between information quality and returns moments varies endogenously over time and depends on disagreement across investors. Finally, we derive novel empirical predictions on the relation between return moments and the variation in the composition of ownership.

In order to explain the mechanism underlying these predictions, a brief overview of the model is useful. There is a risky asset in fixed supply that pays a stream of dividends, and there are two groups of investors in the market at any time. The first group consists of uninformed rational speculators (U) who have no private information and are uncertain about the type of other traders in the market. The second group of traders (θ) may be one of two types: (i) they are either informed investors (i.e., $\theta = I$), in which case they trade on a signal that is informative about next period's dividend shock, or (ii) they are noise /

[1] Keynes (1936) distinguishes between *enterprise* ("the activity of forecasting the prospective yield of assets over their whole lives") and *speculation* ("the activity of forecasting the psychology of the market").

sentiment traders (i.e., $\theta = N$) who trade on a spurious signal (noise) that they incorrectly believe to be informative.[2] Speculators are uncertain about the type of θ investors they face and, hence, whether the price is informative about dividends. All investors have mean-variance preferences and trade competitively in a centralized market by submitting limit orders.

Our benchmark model is static: speculators face uncertainty about whether other traders are informed, but there is no learning along this dimension. In equilibrium, the price and residual demand reveals the realization of the θ investors' signal to the speculators, but they are uncertain about whether it is informative. Because of this uncertainty, a surprise in the signal (in either direction) increases the speculators' posterior variance about fundamentals. As a result, the equilibrium price is (i) non-linear in the signal, and (ii) depends on the probability that speculators assign to θ investors being informed. Moreover, *disagreement* between speculators and θ investors is intimately linked to this belief. When speculators place a high probability on θ investors being informed, both groups of traders agree that the signal is informative and disagreement is low. When speculators place a low probability on θ investors being informed, disagreement is high since θ investors think the signal is informative while speculators believe it is noise.

The key additional feature of the dynamic setting is that, over time, speculators update their beliefs about whether others are informed using realized prices and dividends. When a dividend realization is in line with the information revealed through the price, speculators increase the likelihood that others are informed. The endogenous evolution of their beliefs (combined with (i) and (ii) above) implies that expected returns and volatility are stochastic, but predictable, and vary with disagreement across investors. As mentioned earlier, uncertainty and learning about other traders generates a number of additional implications, which we discuss in more detail and connect to the empirical literature below.

Asymmetric Price Reaction to News. Prices react asymmetrically to news due to the nature of the speculators' filtering problem. When there is a negative surprise, the speculators' conditional expectation is lower and their conditional variance is higher, both of which lead to a decrease in the price. However, when there is a positive surprise, the conditional expectation is higher but so is the conditional variance, and these have offsetting effects on the price. As a result, prices are more sensitive to bad news, or negative surprises, than to good news. When the overall risk concerns are sufficiently large, the effect

[2] DeLong, Shleifer, Summers, and Waldman (1990) and Mendel and Shleifer (2012) use a similar specification to model noise traders that are subject to sentiment shocks, and Hirshleifer, Subrahmanyam, and Titman (2006) use a similar specification for utility-maximizing irrational traders.

on the conditional variance dominates and the price *decreases* following additional good news about fundamentals. This occurs despite the fact that with good news, θ investors demand strictly more of the asset (at any price).

Asymmetric price reactions have been well documented in the empirical literature. Campbell and Hentschel (1992) document asymmetric price reactions to dividend shocks at the aggregate stock market level through a volatility feedback channel. At the firm level, using a sample of voluntary disclosures, Skinner (1994) documents that the price reaction to bad news is, on average, twice as large as that for good news. Skinner and Sloan (2002) document that the price response to negative earnings surprises is larger, especially for growth stocks.

Volatility Clustering. Since Mandelbrot (1963), a large number of papers have documented the phenomenon of volatility clustering for various asset classes, and at different frequencies (see Bollerslev, Chou, and Kroner, 1992 for an early survey). In our model, volatility clustering is a result of how speculators update their beliefs in response to realized dividends. Since speculators form their conditional expectations of next period's dividends based on their inference about other traders' signal, a dividend realization that is far from their conditional expectation (i.e., a large dividend surprise) leads them to revise downward the probability of others being informed. In other words, large surprises in dividend realizations, which are accompanied by large absolute return realizations, reduce the likelihood that the potentially informed investors are actually informed. In turn, this increases the speculators' uncertainty about fundamentals and, therefore, leads to higher volatility and higher expected returns in future periods.

Disagreement and Return Moments. Our model's predictions may help reconcile the mixed empirical evidence on the relation between disagreement and expected returns. For instance, while Diether, Malloy, and Scherbina (2002) and Johnson (2004) document a negative relation between analyst disagreement and expected returns, Qu, Starks, and Yan (2004) and Banerjee (2011) document a positive relation on average. In our model the relation between disagreement and return moments is not monotonic: disagreement and returns are negatively related when disagreement is high, but positively related when disagreement is low. Disagreement is high when speculators believe others are less likely to be informed — in this case, an increase in disagreement makes speculators even more certain about the type of other traders, and so leads to lower expected returns and volatility. On the other hand, disagreement is low when others are more likely to be informed. In this case, an increase in disagreement implies more uncertainty about the type of other traders, and therefore higher expected returns and higher volatility. Moreover, in our model, the level of disagreement evolves endogenously with speculators' beliefs about other traders, and as such, the relation between disagreement and return moments is time-varying.

Information Quality and Return Moments. Another implication of our model is that the relation between information quality and return moments itself depends on the disagreement across investors. When speculators believe others are more likely to be informed (and so disagreement is low), higher information quality reduces uncertainty about fundamentals for all investors, which decreases expected returns and volatility. However, if the speculators believe that other investors are not likely to be informed, the opposite relation obtains — a more informative signal for the potentially informed investors induces them to trade more aggressively. From the speculators' perspective, this introduces more noise to current and future prices leading to higher expected returns and volatility.

As such, this result may help reconcile the apparently contradictory empirical evidence on the relation between information quality and returns that has been documented in the literature. While some papers document a negative relation between information quality and expected returns (e.g., Easley, Hvidkjaer, and O'Hara, 2002; Francis, LaFond, Olsson, and Schipper, 2005; Francis, Nanda, and Olsson, 2008), others find either limited or no evidence of a relation (e.g., Core, Guay, and Verdi, 2008; Duarte and Young, 2009). Our model suggests that in order to empirically uncover the underlying relation between information quality and returns, one must *condition on the level of disagreement*.

Variation in Ownership Composition and Return Moments. Our model also predicts that expected returns and volatility can vary with the persistence of ownership composition. In the dynamic version of our model, we allow for the type of θ investors to change over time. When disagreement across investors is high, higher persistence implies that θ investors are less likely to be informed in future periods, and this translates to lower expected returns and lower volatility. On the other hand, when disagreement is low, higher persistence implies θ investors are more likely to be informed in the future, and this leads to higher expected returns and volatility. As a result, the relation between persistence in ownership composition and return moments can vary over time, and is related to disagreement across investors. To the best of our knowledge, this is a novel prediction of the model that has not been tested in the empirical literature.

The rest of the paper is organized as follows. We discuss the related literature in the next section. In Section 3, we solve the benchmark model, which allows us to highlight the intuition for many of our results transparently in a static model. Section 4 extends the analysis to a dynamic setting, which allows us to focus on the effects of learning, and discusses the implications of the model. In Section 5, we consider an alternative specification in which all investors have rational expectations and the supply of the risky asset is subject to aggregate shocks. Section 6 concludes. All proofs are in the Appendix.

2 Related Literature

A small number of recent asset-pricing models consider the effects of uncertainty about other traders. Easley, O'Hara, and Yang (2013) consider a single-period economy in which ambiguity-averse investors face uncertainty about the effective risk tolerance of other traders and show that reducing ambiguity decreases expected returns. Gao, Song, and Wang (2012) also explore a static environment where risk-averse, uninformed traders are uncertain about whether the proportion of informed traders is either low or high.[3] They show that in addition to the fully revealing equilibrium, a continuum of partially revealing rational expectations equilibria can exist. One advantage of our benchmark model relative to theirs is that we obtain a unique equilibrium, which facilitates a sharper set of predictions. More generally, we contribute to this literature by analyzing a dynamic setting, which allows us to explore the effects of *learning* about others on return dynamics.

Our paper is also related to a subset of the market microstructure literature that analyzes investors who face multiple dimensions of uncertainty. Gervais (1997) considers a static Glosten and Milgrom (1985) model in which the market maker is uncertain about the precision of informed trader's signal. Romer (1993) and Avery and Zemsky (1998) consider models in which the proportion of informed traders is uncertain (but is not learned over time). Li (2011) considers a generalization of the continuous-time, Kyle-model of Back (1992) that allows for uncertainty about whether the strategic trader is informed or not. While these papers focus on the market microstructure implications of multidimensional uncertainty (e.g., market depth, insider's profit), our focus is primarily on the asset pricing implications. Importantly, since our model considers risk-averse investors and learning over time, we are able to analyze the effects on the dynamics of risk-premia and expected returns.

While the majority of the rational expectations literature has focused on linear-normal equilibria, a number of papers, including most recently Breon-Drish (2012) and Albagli, Hellwig, and Tsyvinski (2011), have explored the effects of relaxing the assumption that fundamental shocks and signals are normally distributed.[4] Our paper contributes to this

[3] In a less closely related environment, Stein (2009) explores market efficiency in a setting where arbitrageurs are uncertainty about the total arbitrage capacity in the market.

[4] Earlier papers in this literature include Ausubel (1990), Foster and Viswanathan (1993), Rochet and Vila (1994), DeMarzo and Skiadas (1998), Barlevy and Veronesi (2000), and Spiegel and Subrahmanyam (2000).

literature by developing a model in which the non-linearity arises because of the composition of traders in the market and the information structure rather than the distribution of payoffs.[5][6]

A related non-linearity arises in the incomplete information, regime switching models of David (1997), Veronesi (1999), David and Veronesi (2008, 2009), and others, in which a representative investor updates her beliefs about which macroeconomic regime she is currently in using signals about fundamental shocks (e.g., dividends). In these models, the non-linearity in the representative investor's filtering problem leads to time-variation in uncertainty and, consequently, variation in expected returns and volatility. Stochastic volatility also arises in noisy rational expectations models, like Fos and Collin-Dufresne (2012), in which noise trader volatility is stochastic and persistent. These features arise *endogenously* in our model even though shocks to both fundamentals and news are i.i.d., and are driven by how uninformed investors learn to use the price to update their beliefs about fundamentals.

Cao, Coval, and Hirshleifer (2002) show that limited participation can also generate stochastic volatility, as well as large price movements in response to little, or no, apparent information.[7] Because of participation costs, sidelined investors update the interpretation of their private signals based on what they learn from prices, and only enter the market once they are sufficiently confident. In our model, the friction is purely informational — uninformed investors trade less aggressively because they are uncertain about the trading motives of other investors, and consequently, the informativeness of the price.

3 The Benchmark Model

This section presents the analysis for the two-date, benchmark model. This simple setting will allow us to isolate the effects of uncertainty about other traders from the effects of learning about them, which can only obtain in the dynamic setting of Section 4. The static model also allows us to solve for equilibrium prices in closed form and develop the underlying intuition more transparently.

[5]Specifically, even though shocks to fundamentals and signals are normally distributed in our model, since the uninformed investor is uncertain about whether other investors are informed, her beliefs about the price signal are given by a mixture of normals distribution.

[6]In a series of papers, Easley, O'Hara and co-authors analyze the probability of informed trading (PIN) in a sequential trade model similar to Glosten and Milgrom (1985) (e.g., Easley, Kiefer, and O'Hara, 1997a; Easley, Kiefer, and O'Hara, 1997b; Easley et al., 2002). In these papers, the risk-neutral market maker updates her valuation of the asset based on whether a specific trade is informed or not, but does not face uncertainty about the presence of informed traders in the market. In contrast, the uninformed investors in our model must update their beliefs, not only about the value of the asset, but also about the probability of other investors being informed, which leads to non-linearity in prices.

[7]Other papers that study the informational effects of limited participation include Romer (1993), Lee (1998), Hong and Stein (2003), and Alti, Kaniel, and Yoeli (2012).

Agents. There are three different groups of traders in the model. Traders within each group are identical and behave competitively.

- *Informed/Inside Traders (I).* I traders are rational agents who receive a private and informative signal about the dividend (e.g., institutional investors).
- *Uninformed/Speculative Traders (U).* U traders are rational agents who receive no private signal about fundamentals but update their beliefs by observing prices and quantities (e.g., hedge funds).
- *Noise/Sentiment Traders (N).* N traders observe and trade on a signal that they believe is informative, but is purely noise (e.g., retail investors).

The key feature we want to capture here is that speculative traders are uncertain about whether other traders in the market are trading on information or noise. To this end, we assume that either I or N traders are present in the market but not both, and further, U traders are uncertain about which type of other traders they are facing. To introduce this uncertainty, let $\theta \in \{I, N\}$ denote the random variable that represents the type of other traders that are present in the market.[8]

Securities. There are two assets: a risk-free asset and a risky asset. The gross risk-free rate is normalized to $R \equiv 1 + r > 1$. At date 1, the risky asset pays a dividend $D = \mu + d$, where $\mu > 0$ and $d \sim \mathcal{N}(0, \sigma^2)$. The aggregate supply of the risky asset is constant and equal to Z. At date 0, the risky asset is traded in a competitive market. Let P denote the market clearing price and $Q \equiv D - RP$ denote the dollar return per share of the risky asset.

Preferences. Traders have mean-variance preferences over terminal wealth, and trade competitively (i.e., are price takers). In particular, trader i submits a limit order, x_i, such that

$$x_i = \arg\max_x \mathbb{E}_i[W_i R + xQ] - \tfrac{\alpha}{2}\text{var}_i[W_i R + xQ], \tag{1}$$

and where $\mathbb{E}_i[\cdot]$ and $\text{var}_i[\cdot]$ denote her conditional expectation and variance given her information set, W_i denotes her wealth, and α denotes her risk-aversion. Given these preferences, investor i's optimal demand for the risky asset is given by

$$x_i = \frac{\mathbb{E}_i[Q]}{\alpha \text{var}_i[Q]} = \frac{\mathbb{E}_i[D] - RP}{\alpha \text{var}_i[D]}. \tag{2}$$

[8]One could also consider a setting in which all three types of traders are present in the market, and speculators are uncertain about the proportion of informed traders vs. noise traders they face. We discuss such a setting in Section 5.

Information and Beliefs. The θ traders are either informed traders (i.e., $\theta = I$) or noise traders (i.e., $\theta = N$), where the prior probability of being informed is $\pi_0 \equiv \Pr(\theta = I)$. Prior to submitting their order, investor θ receives a signal S_θ of the form:

$$S_\theta = \begin{cases} d + \varepsilon & \text{if} \quad \theta = I \\ u + \varepsilon & \text{if} \quad \theta = N, \end{cases}$$

where $\varepsilon \sim \mathcal{N}(0, \sigma_\varepsilon^2)$ and u is distributed identically to d and where (ε, u, d) are mutually independent. It is convenient to parametrize the information quality of the informed investors' signal (i.e., S_I) by the Kalman gain, λ, where

$$\lambda \equiv \frac{\text{cov}[S_I, d]}{\text{var}[S_I]} = \frac{\sigma^2}{\sigma^2 + \sigma_\varepsilon^2}.$$

Note that λ is decreasing in the noise of the signal (i.e., σ_ε^2) and takes values between zero and one. When $\lambda = 0$, S_I is completely uninformative; investors learn nothing about future dividends by observing it. Conversely, when $\lambda = 1$, S_I perfectly reveals the realization of next period's dividend. Unless otherwise noted, we assume $\lambda > 0$.

3.1 Remarks on Noise Traders

Our specification for noise trading is different from the standard approach in noisy rational expectations models (e.g., aggregate supply shocks or private stochastic investment opportunities). Yet, it is in line with models of utility-maximizing traders that are subject to sentiment shocks (e.g., DeLong et al., 1990; Hirshleifer et al., 2006; Mendel and Shleifer, 2012). More generally, investors in our model lack a common prior and may be interpreted as having a difference of opinions. We view this specification of noise trading as an appealing feature, both for its tractability and its empirical relevance.[9]

Nevertheless, our results do not rely on this particular specification of noise trading. In Section 5, we show how a rational expectations model with noisy aggregate supply that generalizes Grossman and Stiglitz (1980) generates qualitatively similar results to our benchmark model.[10] Thus, our key departure from prior work is that the type of investors trading in the market is neither deterministic nor common knowledge. U traders do not know whether

[9]See Shleifer and Summers (1990) and Hirshleifer (2001) for discussions of the implications of noise / sentiment traders, and Morris (1995) for a discussion of the implications of relaxing the common prior assumption.

[10]In this setting, the two types of θ investors submit orders with different distributions, which allows the U investor to learn about θ solely from prices and quantities. Hence, learning is relevant even in a static environment.

informed investors are actively trading and so are unsure whether the prices conveys relevant information about the asset. It is this uncertainty that underlies the novel predictions of the benchmark model.

One could also interpret the behavior of noise traders in our model as a form of over-confidence (e.g., Daniel, Hirshleifer, and Subrahmanyam, 1998; Odean, 1998).[11] The implications of the model remain qualitatively the same if, instead, we assume that noise traders receive an informative signal about the asset, but over-estimate the informativeness of the signal. In particular, suppose the $\theta = N$ investor receives a signal

$$S_N = \psi d + \sqrt{(1-\psi^2)}\, u + \varepsilon, \tag{3}$$

for some $\psi \in (0,1)$, but believes she observes a signal $S_N = d + \varepsilon$. In this case, the true informativeness of her signal is given by

$$\lambda_N = \frac{\text{cov}\left[S_N, d\right]}{\text{var}\left[d\right]} = \frac{\psi\sigma^2}{\sigma^2 + \sigma_e^2} = \psi\lambda, \tag{4}$$

while she believes the informativeness of the signal is λ. We consider the extreme case for which $\psi = 0$ for ease of exposition.

3.2 Equilibrium Characterization

An equilibrium consists of a price for the risky asset, P, and investor demands, x_i, such that: (i) investor i's demand are optimal, given their beliefs and information, (i.e., satisfy (2)) and (ii) the market for the risky asset clears i.e.,

$$x_U + x_\theta = Z. \tag{5}$$

Since there are no additional sources of noise, one expects that in equilibrium, speculators will be able to infer S_θ from the price and the aggregate residual supply and use this to update their beliefs about the dividend.[12] We say that an equilibrium is *signal-revealing*, if the equilibrium price and allocations reveal S_θ, but not θ, to the U investors.[13] Below we show that the unique equilibrium is signal-revealing.

[11] Overconfidence has been shown to have important implications for trading behavior in financial markets (e.g., Odean, 1999; Barber and Odean, 2000; and Grinblatt and Keloharju, 2000).

[12] As we will see, because the equilibrium price is non-monotonic in S_θ, observing the price alone does not necessarily reveal the signal S_θ. We follow Kreps (1977) and allow the U investors to condition their order on both price and quantity.

[13] Formally, that S_θ is measurable with respect to U's information set. Conceptually, when U is uncertain about θ, a signal-revealing equilibrium differs from a *fully-revealing* equilibrium in which *both* S_θ and θ are revealed.

Regardless of type, θ believes that her signal is informative with probability one. This implies that the conditional beliefs of the θ investor are symmetric across $\theta \in \{I, N\}$. Investor θ's conditional beliefs about the value d next period are given by

$$\mathbb{E}_\theta[d] = \lambda S_\theta, \quad \text{and} \quad \text{var}_\theta[d] = \sigma^2(1-\lambda). \tag{6}$$

However, since U investor faces uncertainty about θ, her beliefs about d conditional on inferring the signal, S_θ, are given by

$$\mathbb{E}_U[d] = \pi_0 \mathbb{E}_U[d|\theta = I] + (1-\pi_0)\mathbb{E}_U[d|\theta = N] = \pi_0 \lambda S_\theta, \text{ and} \tag{7}$$

$$\text{var}_U[d] = \underbrace{\pi_0\sigma^2(1-\lambda) + (1-\pi_0)\sigma^2}_{\text{expected conditional variance}} + \underbrace{\pi_0(1-\pi_0)(\lambda S_\theta)^2}_{\text{variance of conditional expectation}}. \tag{8}$$

Equation (8) highlights the key feature of the benchmark model: U's conditional variance depends on the realization of the signal S_θ. Note that if U is certain that θ is informed (i.e., $\pi_0 = 1$), then her conditional expectation of d depends on S_θ. On the other hand, if U were certain that θ is *not* informed (i.e., $\pi_0 = 0$), then her conditional expectation is identical to her prior and is unaffected by S_θ. In either case, since U is certain about θ, the conditional variance is independent of S_θ. When U is uncertain about θ, the variance of her conditional expectation is (generically) not zero, and this leads to additional uncertainty about dividends. Furthermore, this additional uncertainty is increasing in the magnitude of the signal—larger realizations of $|S_\theta|$ are further from the unconditional expectation (recall $\mathbb{E}[d] = 0$) and this increases disparity between the expected dividend conditional on $\theta = I$ and the expected dividend conditional on $\theta = N$. As we shall see, the dependence of the U investor's conditional variance on the realization of the signal plays an important role in our results.

Since there is only one trading period, U investors do not trade subsequent to updating their beliefs about whether others are informed. The prior belief, π_0, is a key parameter of interest; comparative statics with respect to π_0 will help develop the intuition for the dynamic model, in which beliefs evolve over time. Moreover, note that π_0 is intimately tied to *disagreement* across investors. Specifically, the disagreement about dividend forecasts between the U and θ investors is given by

$$|\mathbb{E}_U[D] - \mathbb{E}_\theta[D]| = (1-\pi_0)\lambda|S_\theta|.$$

The following result characterizes the unique equilibrium of our benchmark model.

Proposition 1. *In the benchmark model, there exists a unique equilibrium. This equilibrium* *is signal-revealing and the price is given by*

$$P = \tfrac{1}{R}\left(\mu + \left(\kappa + (1-\kappa)\pi_0\right)\lambda S_\theta - \kappa\alpha\sigma^2(1-\lambda)Z\right), \tag{9}$$

where the weight κ is given by

$$\kappa \equiv \frac{var_U\left[d\right]}{var_U\left[d\right] + var_\theta\left[d\right]} = \frac{\sigma^2(1-\pi_0\lambda) + \pi_0(1-\pi_0)(\lambda S_\theta)^2}{\sigma^2(1-\lambda) + \sigma^2(1-\pi_0\lambda) + \pi_0(1-\pi_0)(\lambda S_\theta)^2} \in [0,1]. \tag{10}$$

The equilibrium price can be decomposed into a market expectations component and a risk-premium component, since

$$P = \tfrac{1}{R}\left(\underbrace{\mu + \kappa\mathbb{E}_\theta\left[d\right] + (1-\kappa)\mathbb{E}_U\left[d\right]}_{\text{expectations}} - \underbrace{\kappa\alpha\sigma^2(1-\lambda)Z}_{\text{risk premium}}\right). \tag{11}$$

The risk-aversion coefficient, α, and the aggregate supply of the asset, Z, scale the risk-premium component, but not the expectations component. Thus, the product, αZ, determines the relative role of each component in the price. When risk aversion is low or the aggregate supply of the asset is small, the price is primarily driven by the expectations component. On the other hand, when risk aversion is high, or the aggregate supply of the asset is large, the risk-premium component drives the price. As such, it will be useful to characterize separately how each component of the price behaves. The following corollary presents these results.

Corollary 1.

(i) The expectations component of the price is increasing in S_θ, increasing in both λ and π_0 for $S_\theta > 0$, and decreasing both in λ and π_0 for $S_\theta < 0$.

(ii) The risk-premium component of the price is hump-shaped in S_θ around zero, U-shaped in π_0 around $\frac{1}{2}\left(1 - \frac{\sigma^2}{\lambda S_\theta^2}\right)$, increasing in λ for small and large $|S_\theta|$, but decreasing in λ for intermediate $|S_\theta|$.

Intuitively, the comparative statics for the expectations component follow because it is a weighted average of investors' conditional expectations, which are increasing in S_θ. The risk-premium component of prices depends on the uncertainty that investors face. In particular, note that the risk-premium component can be rewritten as

$$-\kappa\alpha\sigma^2(1-\lambda)Z = -\alpha\left(\tfrac{1}{\mathrm{var}_\theta[d]} + \tfrac{1}{\mathrm{var}_U[d]}\right)^{-1} Z, \tag{12}$$

which is linear in the harmonic mean of the conditional variance of both U and θ investors. Unlike standard rational expectations models with linear equilibria, because the conditional variance of the uninformed investors depends on the signal realization, so too does the risk-premium component. The conditional variance $\text{var}_U[d]$ (see (8)) increases in $|S_\theta|$: larger realizations of $|S_\theta|$ increase the uninformed investors' uncertainty about fundamentals, since they are unsure about whether the signal is informative. Finally, note that $\text{var}_\theta[d]$ always decreases in λ, while $\text{var}_U[d]$ decreases in λ for small realizations of $|S_\theta|$, but increases in λ for large realizations of $|S_\theta|$. Moreover, the larger the conditional variance of the U investors, the smaller its contribution to the risk-premium term. As a result, the risk-premium component increases in λ for small and large realizations of $|S_\theta|$, but decreases in λ for intermediate values (when the increase in $\text{var}_U[d]$ dominates the decrease in $\text{var}_\theta[d]$).

3.3 Asymmetric Price Reaction to News

The overall effect of S_θ on the price in our model distinguishes it from linear models that are standard in the literature. While the expectations component of price is monotonic in S_θ, the risk-premium component is hump-shaped in S_θ around zero. This implies that the two components reinforce each other when bad news arrives ($S_\theta < 0$), but offset each other when good news arrives ($S_\theta > 0$).

Proposition 2. *The equilibrium price reacts asymmetrically to news about fundamentals. Specifically, it decreases more with bad news than it increases with good news. For any $s > 0$,*

$$\frac{d}{dS_\theta}P(s) < \frac{d}{dS_\theta}P(-s).$$

Since the risk-premium component is bounded, the expectations component dominates when $|S_\theta|$ is large enough. However, for S_θ small enough, the risk-premium component dominates. This means that the price can actually *decrease* with the signal.

Proposition 3. *For any two signal realizations s_1, s_2 such that $0 < s_1 < s_2$, there exists a $\gamma > 0$ such that if $\alpha Z > \gamma$, the equilibrium price is strictly greater when s_1 is realized than it is when s_2 is realized.*

Intuitively, if the overall risk concerns in the market (as measured by αZ) are large enough, more positive news about fundamentals can have a bigger impact on prices through the uncertainty it generates for uninformed investors than through its effect on the market's expectations about future dividends.

The mechanism through which the asymmetry in prices arises in our model differs from those in the regime-switching models of Veronesi (1999) and others. Specifically, in Veronesi

(1999), the asymmetry in price reaction is driven by uncertainty about whether the underlying state of the economy is good or bad. The representative investor "over-reacts" to bad news only if he believes with sufficiently high probability that the current state is good, and "under-reacts" to good news only if he believes that the current state is bad, because these are the instances in which the realization of the news increases uncertainty about the underlying state. In our model, the asymmetry is not state-dependent: the price is more sensitive to bad news for *any* $\pi_0 \in (0,1)$, even in the absence of any learning about θ. This is because the asymmetry is driven by uncertainty about the informativeness of the price signal, not the underlying fundamentals.[14]

3.4 Expected returns and volatility

Given the results from Proposition 1, we now turn to investigating the moments of returns. The decomposition in (11) implies that dollar returns can be expressed as

$$Q = d - (\kappa \mathbb{E}_\theta[d] + (1-\kappa)\mathbb{E}_U[d]) + \kappa\alpha\sigma^2(1-\lambda)Z. \tag{13}$$

Return moments are computed based on the information set of the U investor, since she has rational expectations.[15] We refer to *conditional* expected returns as the expected returns conditional on all information up to and including the current period (i.e., the price and residual demand, and consequently, S_θ). *Unconditional* returns are computed based on all information prior to the current period.

Proposition 4. *In the static model, the conditional expected return and volatility are given by*

$$\mathbb{E}[Q|P, x_\theta] = -(1-\pi_0)\lambda\kappa S_\theta + \kappa\alpha\sigma^2(1-\lambda)Z, \;\; \textit{and} \tag{14}$$

$$\mathrm{var}[Q|P, x_\theta] = \sigma^2(1-\pi_0\lambda) + \pi_0(1-\pi_0)(\lambda S_\theta)^2. \tag{15}$$

The unconditional expected return and volatility are given by

$$\mathbb{E}[Q] = \mathbb{E}[\kappa]\alpha\sigma^2(1-\lambda)Z, \;\; \textit{and} \tag{16}$$

$$\mathrm{var}[Q] = \sigma^2(1-\pi_0^2\lambda) + (1-\pi_0)^2\lambda^2\mathrm{var}[\kappa S_\theta] + (\sigma^2(1-\lambda)\alpha Z)^2\mathrm{var}[\kappa] \tag{17}$$

To gain some intuition for the expressions in Proposition 4, we note that the expectation

[14] Note that in the dynamic version of our model, the uninformed investor updates her beliefs based on realizations of fundamentals, but this is not what drives the asymmetric reaction of prices to signals.

[15] This corresponds to the information set of an econometrician who observes the price and quantity of executed trades as well as dividends.

of (13) with respect to an arbitrary information set $\mathcal{I}$ can be decomposed into the following two components:

$$\mathbb{E}[Q|\mathcal{I}] = \underbrace{\mathbb{E}\left[\kappa(\mathbb{E}_U[d] - \mathbb{E}_\theta[d])|\mathcal{I}\right]}_{\text{expectations}} + \underbrace{\mathbb{E}[\kappa\alpha\sigma^2(1-\lambda)Z|\mathcal{I}]}_{\text{risk premium}}. \tag{18}$$

As noted earlier, because the U investor is uncertain about the interpretation of S_θ, her conditional variance about d depends on both π_0 and S_θ. This means that κ and, as a result, the risk premium component of expected returns also depend on both π_0 and S_θ. As we will see in the next section, in the dynamic setting when θ is persistent, this dependence on π_0 and S_θ gives rise to expected returns that are stochastic, predictable and vary with disagreement across investors.

The expression for the unconditional volatility of returns given in equation (17) can be decomposed into three terms, each of which captures a different source of risk,

$$\text{var}[Q] = \underbrace{\sigma^2(1-\pi_0^2\lambda)}_{\text{fundamental}} + \underbrace{(1-\pi_0)^2\lambda^2\text{var}[\kappa S_\theta]}_{\text{expectations}} + \underbrace{(\sigma^2(1-\lambda)\alpha Z)^2\text{var}[\kappa]}_{\text{risk premium}}. \tag{19}$$

The first term is the expectation of the conditional variance in returns and so captures the volatility in returns due to uncertainty about next period's fundamental dividend shock d. The second term in (19) reflects the volatility in returns due to variation in the expectations component of conditional expected returns. Finally, the third term is volatility due to variation in the risk-premium component of conditional expected returns. As in the case of expected returns, each of these components depends on π_0. Consequently, in a dynamic setting with persistence in θ, the model generates stochastic, predictable return volatility. For the interested reader, we explore these components in greater detail and discuss comparative statics on return moments in the benchmark model in Appendix B.

4 The Dynamic Model

This section extends our analysis to a dynamic setting by developing an overlapping generations (OLG) model. There are two key additional considerations in the dynamic setting that are not present in the static one. First, the price is affected not only by investors' beliefs about fundamentals and other traders, but also their beliefs about future prices. Second, uninformed investors' beliefs about other traders evolve stochastically over time as they learn from realized prices and dividends.

We retain the key features of the benchmark model as described in Section 3 with the following additions.

Agents. As before, there are three types of traders: uninformed speculative traders (U), informed traders (I), and noise/sentiment traders (N). In each generation, U traders are uncertain about which type of other traders they face, and $\theta_t \in \{I, N\}$ denotes the random variable that represents the type of these other traders at date t. We allow θ to vary over time and consider two specifications: (i) θ_t is i.i.d. over time and (ii) θ_t exhibits persistence according to a Markov switching process. These cases together allow us to model the composition of traders in the market quite generally.

Securities. In date t, the risky asset pays a dividend D_t, which evolves according to an $AR(1)$ process:

$$D_{t+1} = (1-\rho)\mu + \rho D_t + d_{t+1}, \tag{20}$$

where $d_{t+1} \sim \mathcal{N}(0, \sigma^2)$, and $\rho < 1$. The dollar return at time t on a share of the risky asset is given by $Q_t \equiv P_t + D_t - RP_{t-1}$.

Preferences. Each generation of investor lives for two dates, and has mean-variance preferences over terminal wealth. An investor i, who is born in date t and consumes in date $t+1$, has optimal demand for the risky asset given by:

$$x_{i,t} = \frac{\mathbb{E}_{i,t}\left[P_{t+1} + D_{t+1}\right] - RP_t}{\alpha \mathrm{var}_{i,t}\left[P_{t+1} + D_{t+1}\right]}. \tag{21}$$

Information and Beliefs. In addition to the information structure defined in Section 3, each generation of investor can observe the history of dividend realizations, prices and trades. Since the type of other investors can change over time, we denote the date t beliefs of U traders about whether others are informed by $\pi_t = \Pr_t(\theta_t = I)$.

4.1 When the distribution of types is i.i.d.

When the distribution of types of other traders (i.e., the distribution of θ_t) is independent and identical across generations, there exists an equilibrium of the dynamic model which resembles the equilibrium of the benchmark model. Moreover, as we show below, this stationary equilibrium is the limit of the unique equilibrium of the finite horizon version of the model.

Proposition 5. *Suppose $R - \rho - 2\alpha\sigma Z > 0$ and the distribution of θ_t is i.i.d. with $\pi = \Pr(\theta_t = I)$. Then, there exists a stationary equilibrium. This equilibrium is signal-revealing* *and the price is given by* $P_t = A\mu + BD_t + p(S_{\theta,t})$*, where* $A = \frac{R(1-\rho)}{(R-1)(R-\rho)}$, $B = \frac{\rho}{R-\rho}$,

$$p(S_{\theta,t}) = \tfrac{1}{R}\left((1+B)(\kappa_t + (1-\kappa_t)\pi)\lambda S_{\theta,t} + m - \alpha\kappa_t((1+B)^2\sigma^2(1-\lambda)+v)Z\right), \tag{22}$$

$$\kappa_t = \tfrac{(1+B)^2(\sigma^2(1-\pi\lambda)+\pi(1-\pi)(\lambda S_{\theta,t})^2)+v}{(1+B)^2(\sigma^2(1-\pi\lambda)+\pi(1-\pi)(\lambda S_{\theta,t})^2)+(1+B)^2\sigma^2(1-\lambda)+2v}, \tag{23}$$

$$m = \mathbb{E}\left[p(S_{\theta,t})\right] = -\tfrac{1}{R-1}((1+B)^2\sigma^2(1-\lambda)+v)\alpha Z\mathbb{E}\left[\kappa_t\right], \tag{24}$$

and v *is implicitly characterized by*

$$v = \mathrm{var}[p(S_{\theta,t}] \tag{25}$$

Moreover, the equilibrium price is the limit of the price of the unique equilibrium of the finite horizon (T *period) model, as the horizon increases (i.e., as* $T \to \infty$*).*

As in other OLG models (e.g., Spiegel, 1998), the sufficient condition for existence $R - \rho - 2\alpha\sigma Z > 0$ ensures that the aggregate risk in holding the risky asset is not too large. Intuitively, when the aggregate amount of risk (i.e., $\alpha\sigma Z$) increases, the risk-premium component of the current price is large and more sensitive to shocks (in κ_t), which in turn increases the risk-premium in the previous period. As a result, if $\alpha\sigma Z$ is too large, the risk-premium terms explode and a stationary equilibria does not exist.

Though we do not have an analytical proof that the equilibrium is unique, we have been unable to find parameters in which there are multiple solutions to (25). This is in contrast to standard (linear) OLG models (e.g., Spiegel, 1998; Banerjee, 2011) which generally exhibit two equilibria. A likely explanation for this difference is due to our specification of noise traders. In standard OLG models, the price is exposed to two types of shocks: fundamental shocks and noise shocks. The multiplicity in equilibria arise due to multiplicity in self-fulfilling beliefs about the price sensitivity to noise shocks.[16] In our model, the price is a non-linear function of a single shock ($S_{\theta,t}$) and the price sensitivity of this shock is pinned down by the traders' beliefs of its informativeness.

Note that the equilibrium price in Proposition 5 depends not only on beliefs about current dividends and whether others are informed, it also depends on investors' beliefs about the price next period. However, because the distribution of the type of other traders (i.e., θ) is i.i.d., U investors do not learn about θ. The effect of learning about θ on the equilibrium price is the focus of the following subsections.

[16] Generically, there is a low volatility equilibrium (when investors believe prices are not very sensitive to noise) and a high volatility equilibrium (when investors believe prices are sensitive to noise). See Spiegel (1998) for a discussion.

4.2 When the distribution of types is persistent

Suppose that instead of being distributed i.i.d. across generations, θ_t follows a Markov switching process with persistence q i.e., $\Pr(\theta_{t+1} = I|\theta_t = I) = \Pr(\theta_{t+1} = N|\theta_t = N) = q$. In this case, prices and dividends in period t are informative about the composition of traders in the market at $t+1$.

Recall that $\pi_t = \Pr_t(\theta_t = I)$ with respect to the U investors' information set at date t. Because of the symmetry in the equilibrium trades of the I and N investors, observing date-t prices and quantities alone are not informative about θ_t — the U investor must also observe dividends. Following the realization of D_{t+1}, the posterior probability that U assigns to $\theta_t = I$, is given by

$$\Pr(\theta_t = I|S_{\theta,t}, d_{t+1}) = \frac{\pi_t \Pr(S_{\theta,t}|\theta_t = I, d_{t+1})}{\pi_t \Pr(S_{\theta,t}|\theta_t = I, d_{t+1}) + (1-\pi_t)\Pr(S_{\theta,t}|\theta_t = N, d_{t+1})} \tag{26}$$

$$= \frac{\frac{\pi_t}{\sigma_\varepsilon}\phi\left(\frac{S_{\theta,t}-d_{t+1}}{\sigma_\varepsilon}\right)}{\frac{\pi_t}{\sigma_\varepsilon}\phi\left(\frac{S_{\theta,t}-d_{t+1}}{\sigma_\varepsilon}\right) + \frac{1-\pi_t}{\sqrt{\sigma^2+\sigma_\varepsilon^2}}\phi\left(\frac{S_{\theta,t}-0}{\sqrt{\sigma^2+\sigma_\varepsilon^2}}\right)}, \tag{27}$$

where $\phi(\cdot)$ is the probability distribution function for a standard normal random variable. As a result, the probability U assigns to $\theta_{t+1} = I$ is given by

$$\pi_{t+1} = q\Pr(\theta_t = I|S_{\theta,t}, d_{t+1}) + (1-q)(1-\Pr(\theta_t = I|S_{\theta,t}, d_{t+1})) \tag{28}$$

$$= (1-q) + (2q-1)\frac{\frac{\pi_t}{\sigma_\varepsilon}\phi\left(\frac{S_{\theta,t}-d_{t+1}}{\sigma_\varepsilon}\right)}{\frac{\pi_t}{\sigma_\varepsilon}\phi\left(\frac{S_{\theta,t}-d_{t+1}}{\sigma_\varepsilon}\right) + \frac{1-\pi_t}{\sqrt{\sigma^2+\sigma_\varepsilon^2}}\phi\left(\frac{S_{\theta,t}-0}{\sqrt{\sigma^2+\sigma_\varepsilon^2}}\right)}. \tag{29}$$

Equation (29) illustrates an important feature of the model; uninformed investors' belief about other traders is persistent and evolves stochastically. As we will see, this leads to time-variation and predictability in return moments, even though shocks to fundamentals and information are i.i.d. The next result provides a characterization of signal revealing equilibria in the general dynamic setting.

Proposition 6. *In any signal-revealing equilibrium, investor i's optimal demand is given by expression* (21)*, investor beliefs are given by*

$$\mathbb{E}_{U,t}[D_{t+1}] = (1-\rho)\mu + \rho D_t + \pi_t\lambda S_{\theta,t}, \quad \mathbb{E}_{\theta,t}[D_{t+1}] = (1-\rho)\mu + \rho D_t + \lambda S_{\theta,t},$$
$$\mathrm{var}_{U,t}[D_{t+1}] = \sigma^2(1-\pi_t\lambda) + \pi_t(1-\pi_t)(\lambda S_{\theta,t})^2, \ \textit{and}\ \mathrm{var}_{\theta,t}[D_{t+1}] = \sigma^2(1-\lambda),$$

and the price of the risky asset is given by

$$P_t = \tfrac{1}{R}\left(\underbrace{\bar{\mathbb{E}}_t[P_{t+1}+D_{t+1}]}_{expectations} - \underbrace{\alpha\kappa_t \mathrm{var}_{\theta,t}[P_{t+1}+D_{t+1}]Z}_{risk\ premium}\right), \tag{30}$$

where $\bar{\mathbb{E}}_t[\cdot] \equiv \kappa_t \mathbb{E}_{\theta,t}[\cdot] + (1-\kappa_t)\mathbb{E}_{U,t}[\cdot]$*, and* κ_t *is given by*

$$\kappa_t = \frac{\mathrm{var}_{U,t}[P_{t+1}+D_{t+1}]}{\mathrm{var}_{U,t}[P_{t+1}+D_{t+1}] + \mathrm{var}_{\theta,t}[P_{t+1}+D_{t+1}]}. \tag{31}$$

The characterization of the price has a familiar form; it is a weighted average of investors' conditional expectations about future payoffs, adjusted for a risk-premium.[17] The weight of each investor's expectation in $\bar{\mathbb{E}}_t[\cdot]$ depends on the conditional variance of her beliefs relative to those of the others.

The equilibrium price cannot be characterized in closed form. We solve for it numerically, by using an iterative procedure to compute the equilibrium price function as the limit of the (unique) signal-revealing equilibrium of the finite horizon model, as characterized by the following result.

Proposition 7. *In the finite horizon (T-period) model, there exists a unique, signal-revealing equilibrium. The equilibrium price is of the form* $P_t = A_t\mu + B_t D_t + p_t(S_{\theta,t}, \pi_t)$*, where* $p_T = 0$*,* $A_T = 0$*,* $B_T = 0$*,* $A_t = \frac{1}{R}(A_{t+1} + (1-\rho)(1+B_{t+1}))$*,* $B_t = \frac{\rho}{R}(1+B_t)$*,*

$$p_t(S_{\theta,t},\pi_t) = \frac{1}{R}\left\{\begin{array}{c} \bar{\mathbb{E}}_t\left[(1+B_{t+1})\,d_{t+1} + p_{t+1}(S_{\theta,t+1},\pi_{t+1})\right] \\ -\alpha\kappa_t \mathrm{var}_{\theta,t}\left[(1+B_{t+1})\,d_{t+1} + p_{t+1}(S_{\theta,t+1},\pi_t)\right]Z \end{array}\right\},$$

$$\kappa_t = \frac{\mathrm{var}_{U,t}\left[(1+B_{t+1})d_{t+1}+p_{t+1}(S_{\theta,t+1},\pi_{t+1})\right]}{\mathrm{var}_{U,t}\left[(1+B_{t+1})d_{t+1}+p_{t+1}(S_{\theta,t+1},\pi_{t+1})\right]+\mathrm{var}_{\theta,t}\left[(1+B_{t+1})d_{t+1}+p_{t+1}(S_{\theta,t+1},\pi_{t+1})\right]},$$

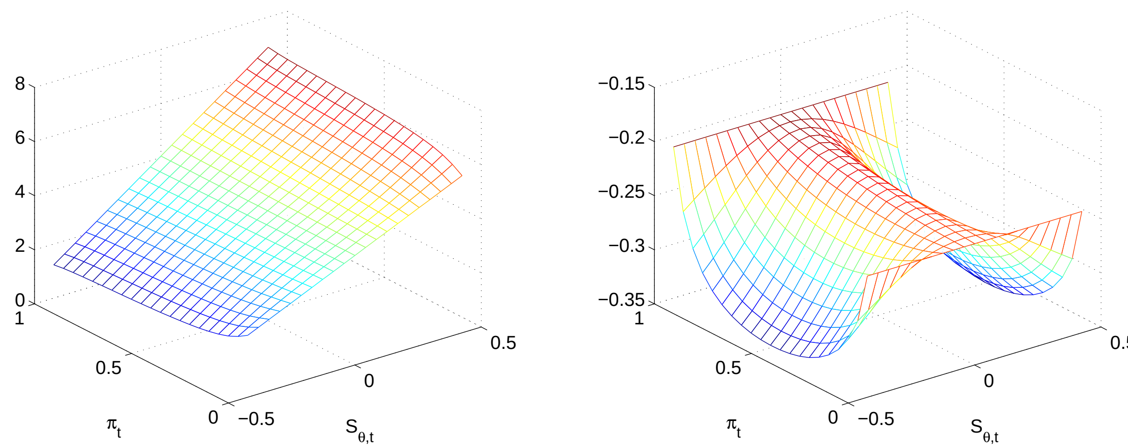
and $\bar{\mathbb{E}}_{i,t}[\cdot] = \kappa_t\mathbb{E}_{\theta,t}[\cdot] + (1-\kappa_t)\mathbb{E}_{U,t}[\cdot]$*.*

Specifically, we first solve for the unique equilibrium price function of the static model on a grid over the space of signals and beliefs. Then, we iterate backwards, using the current price function (evaluated on the grid) as next period's price for the next iteration, until the mean squared difference in the price functions across iterations is negligible. Under the

[17] To this point, we do not have a proof of existence for the general case as doing so requires solving a non-standard fixed point problem and then verifying that the fixed point retains certain properties. However, we have established existence and uniqueness in the finite horizon model, existence in the dynamic model with i.i.d. θ, and existence and uniqueness in the limiting cases where $\pi_t \in \{0,1\}$, as discussed in Appendix B. We have also verified existence numerically for a wide range of parameters in the general case.

sufficient condition given in Proposition 5 (i.e., $R - \rho - 2\alpha\sigma Z > 0$), we find that the price function converges for a wide range of parameters.

As in the static setting, the price is more sensitive to bad news (i.e., negative $S_{\theta,t}$) than it is to good news (i.e., positive $S_{\theta,t}$). All else equal, investors' expectation of dividends next period, and hence the expectations component of the price, increases in $S_{\theta,t}$ as in Figure 1(a). However, a surprise in $S_{\theta,t}$ in either direction also leads to an increase in uncertainty for the U investor, and so the risk-premium component is hump-shaped in $S_{\theta,t}$ as in Figure 1(b). For negative $S_{\theta,t}$ these two effects reinforce each other, while for positive $S_{\theta,t}$, the effects offset each other, and this leads to the asymmetric reaction of the price to $S_{\theta,t}$.[18]



(a) Expectations Component of P_{t+1} (b) Risk Premium Component of P_{t+1}

FIGURE 1: The two components of the equilibrium price function as they depend on the underlying state variable and the realization of information.

In Appendix B, we present two natural limiting cases in which U investors are *not* uncertain about whether θ investors are informed: (i) $\pi_t = 1$ and $\theta_t = I$, and (ii) $\pi_t = 0$ and $\theta_t = N$. In both cases, without uncertainty about other traders, the model's predictions are more standard — the equilibrium price is linear, return volatility is constant, and expected returns are either constant or i.i.d. This exercise illustrates our main results are driven by U's uncertainty about θ and subsequent learning, neither of which are present in these limiting cases.

[18]The comparative statics with respect to π_t are familiar from the static case — the sensitivity of the expectations component to $S_{\theta,t}$ increases in π_t and the risk-premium component is U shaped in π_t for any $S_{\theta,t}$.

4.3 Implications of the Model

Since the solution to the dynamic model with persistent types is not analytically tractable, we explore the implications by analyzing the model numerically. While the analysis in the earlier sections has focused on characterizing properties of dollar returns per share, $Q_{t+1} = P_{t+1} + D_{t+1} - RP_t$, we will now characterize properties of the excess rate of return, Q_{t+1}/P_t, in order to highlight the robustness of the results and to facilitate comparisons to the broader literature. The parameters are set to the following baseline values unless otherwise specified: $r = 2.5\%$, $\mu = 3\%$, $\rho = 0.95$, $\sigma = 6\%$, $Z = 1$, $\alpha = 0.75$, and $q = 0.65$.[19] For these baseline parameter values, the expected excess return on the risky asset is 5.7% and the volatility is 20.4%, when evaluated at $\pi_t = 1$ and $\lambda = 0.5$.

4.3.1 Disagreement and predictability in return moments

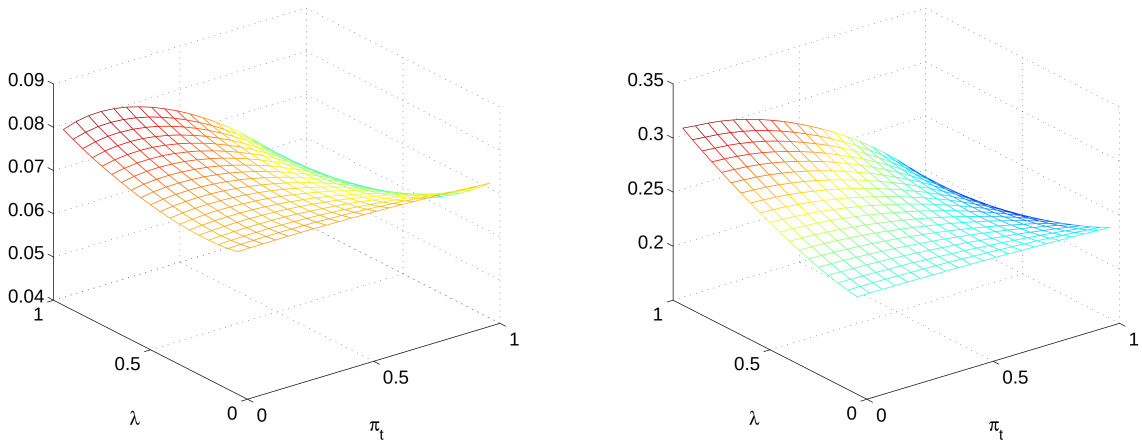
When speculators are uncertain about θ, the effect of π_t on the price generates novel empirical predictions that distinguish our model form standard, dynamic rational expectations models. In particular, the belief π_t is an endogenous state variable of the model, which evolves stochastically and is persistent. As a result, in addition to generating *stochastic* expected returns and volatility, the model predicts that these moments are *persistent*, despite the fact that shocks to fundamentals and signals are i.i.d.

As Figure 2 shows, excess returns are first increasing in π_t but decreasing for larger π_t. Similarly, expected returns and volatility are decreasing in λ for high π_t, since better quality information reduces uncertainty when traders are sufficiently confident of its source. However, returns and volatility are *increasing* in λ for low π_t; higher quality information leads to more volatile asset prices when traders are skeptical of the information source.

Figure 2 also suggests that the magnitude of the comparative statics results is economically meaningful. For instance, an increase in λ from 0.3 to 0.7 implies an increase in expected returns from 7% to 7.2% and an increase in volatility from 24% to 26% percent (for $\pi_t = 0.5$); an increase in π_t from 0.3 to 0.7 implies a decrease in expected returns from 7.3% to 6.7% percent and a decrease in volatility from 26% to 23% percent (for $\lambda = 0.5$).

Finally, recall that a natural proxy for π_t in our model is disagreement across investors. As such, the model predicts that time-variation in expected returns and volatility are closely related to disagreement across investors. However, the relation between return moments

[19] Because we are using normally distributed random variables, the population moments of Q_{t+1}/P_t are not well defined due to prices arbitrarily close to zero — see Campbell, Grossman, and Wang (1993) and Llorente, Michaely, Saar, and Wang (2002) for a discussion. We adopt the conventional approach and choose D_t large enough relative to the volatility of dividend shocks (setting $D_t = 1$ and $\sigma = 0.06$) such that the numerical estimation of these moments is well behaved.



(a) Expected Excess Rate of Return

(b) Volatility of Rate of Return

FIGURE 2: This figure plots the expected excess rate of return and volatility as a function of π_t and λ. The parameters are set to the following baseline values unless otherwise specified: $r = 2.5\%$, $\mu = 3\%$, $\rho = 0.95$, $\sigma = 6\%$, $Z = 1$, $\alpha = 0.75$, and $q = 0.65$

and disagreement is not monotonic: an increase in disagreement (decrease in π) leads to higher expected returns and volatility when initial disagreement is low, but lower expected returns and volatility when initial disagreement is high. Intuitively, when disagreement is high, speculators assign a low likelihood to other investors being informed (i.e., π_t is low). An increase in disagreement reduces π_t further, which reduces the uncertainty that speculators have about others, and so leads to lower expected returns and volatility. In contrast, when disagreement is low (i.e., π_t is high), an increase in disagreement increases uncertainty about whether θ investors are informed, and this leads to higher expected returns and volatility.[20] As discussed in the introduction, these results may also help reconcile the mixed empirical evidence on the relation between disagreement and returns documented in the literature.

4.3.2 Information quality and return moments

Figure 2 suggests that the relation between return moments and information quality (i.e., λ) depends on the state π_t, and, therefore, can vary over time. Specifically, when investors agree on the informativeness of $S_{\theta,t}$ (i.e., π_t is close to one), higher information quality leads to lower uncertainty and therefore lower expected returns. However, if investors disagree on the interpretation of the signal (i.e., π_t is close to zero), a signal distribution with a higher

[20]The intuition is consistent with the predictions of Banerjee (2011) which derives a negative relation between disagreement and return moments for models in which investors exhibit differences of opinion, but a positive relation for models in which investors exhibit rational expectations and condition on prices. In the current model, when π_t is low, investors behave as if they "agree to disagree," since U traders do not believe $S_{\theta,t}$ is informative, but θ traders do. On the other hand, when π_t is high, both groups of investors agree on the informativeness of $S_{\theta,t}$, as they would in a rational expectations model.

λ generates more uncertainty for the U investor since it makes the θ investor trade more aggressively on his information. All else equal, this leads to higher volatility of prices and higher expected returns.

The above results may be useful in reconciling the mixed empirical evidence on the relation between information quality and returns discussed in the introduction. In particular, the model suggests that conditioning on an empirical proxy of π_t (e.g., institutional ownership or disagreement) may be useful in uncovering the underlying relation.

4.3.3 Volatility clustering

The dynamic model generates time-series predictability in both expected returns and volatility. In particular, for π_t close to one and persistent θ_t (i.e., q close to one), the model predicts *volatility clustering* — return surprises in either direction are followed by an increase in both volatility and expected returns. The intuition for these results follows from how U updates her beliefs about whether θ_t is informed. An unanticipated realization of D_{t+1} leads the U investor to revise her beliefs about θ being informed downwards (i.e., $\pi_{t+1} < \pi_t$).[21] This revision in beliefs generates additional uncertainty for U investors, and as a result, leads to higher future volatility and higher expected returns going forward. Figure 3 illustrates this clustering effect. Specifically, the figure plots expected returns and volatility in period $t+1$ as a function of the current realization of D_{t+1} (scaled by its standard error) starting from π_t close to one. Starting from zero on the x-axis, increasing the dividend surprise in either direction implies π_{t+1} is closer to $\frac{1}{2}$ and therefore U is more uncertain about θ, which leads to higher expected returns and volatility.[22]

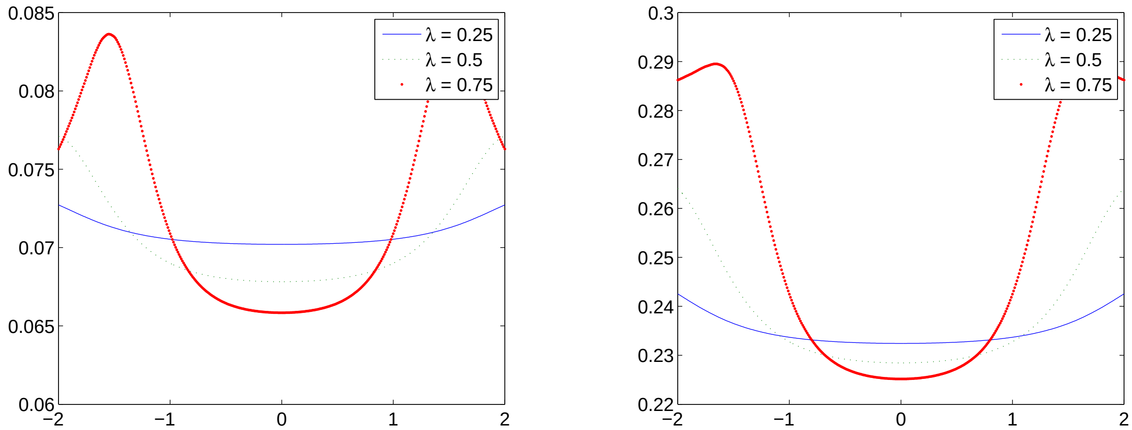
For π_t close to zero, the opposite relationship can obtain; returns in line with expectations cause the U investor to revise her belief upwards, which again increases the uncertainty about other traders and hence volatility and expected returns. In this sense, no *news* (i.e., little to no surprise in returns) can either be good news (when π_t is close to one) or bad news (when π_t is close to zero). More generally, our model highlights a channel through which cash-flow news (i.e., dividend surprises) can affect discount rates (i.e., expected returns) in the future through its effect on uncertainty about other investors.[23]

For the baseline parameters, Figure 3 provides magnitudes for the volatility clustering effect. For $\lambda = 0.75$, a one-standard deviation surprise in dividend realizations predicts an increase in future expected excess returns of roughly 50 b.p. (from 6.5% to 7%) and an

[21] This follows from the evolution of π_t in (29), q being close to one, and π_t being large initially.

[22] Note that for sufficiently large (and unlikely) surprises, the posterior, π_{t+1}, tends to zero and expected returns and volatility may decrease.

[23] We thank Karl Diether for this observation.



(a) Future Excess Returns

(b) Future Volatility

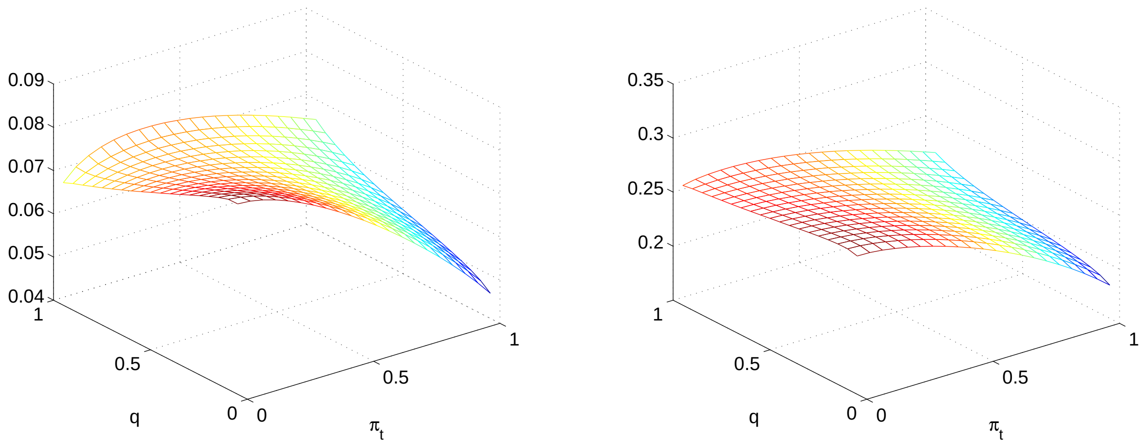
FIGURE 3: The figure plots expected returns and volatility in period $t+1$ as they depend on dividend surprises in period t and the quality of the information (in both figures $\pi_t = 0.95$). The parameters are set to the following baseline values unless otherwise specified: $r = 2.5\%$, $\mu = 3\%$, $\rho = 0.95$, $\sigma = 6\%$, $Z = 1$, $\alpha = 0.75$, and $q = 0.975$.

increase in future volatility of 1.75% (from 22.5% to 24.25%), while a two-deviation surprise generates increases of 1% in expected returns and about 6% in volatility. Finally, note that in the limiting cases without uncertainty about others (i.e., where $\pi_t \in \{0, 1\}$), these plots are perfectly flat. Thus, even for small deviations from the standard model ($\pi_t = 0.95$ instead of $\pi_t = 1$), the clustering effect can be quite economically significant.

4.3.4 Variation in the composition of ownership and return moments

The analysis also suggests that the effect of the persistence in types of other traders (i.e., q) on return moments also depends on π_t, and therefore is time-varying. As an instance, Figure 4 plots the expected excess rate of return and volatility as a function of π_t and q for the baseline parameters. The plots suggest that except near the boundaries of $q = 0$ and $q = 1$, expected returns and volatility are increasing in q for large π_t, but decreasing in q or low π_t.

Intuitively, changing q does not change beliefs about next period's dividends, but it does affect beliefs about future prices. Specifically, when the likelihood of other traders being informed is low in the current period (i.e., π_t is low), and increase in persistence of θ implies that the likelihood of other traders being informed is lower in future periods. On the other hand, an increase in q when π_t is high implies that the likelihood of other traders being informed is higher in future periods. Since future prices are more sensitive to signals, and therefore riskier, when the likelihood of $\theta = I$ is higher, expected returns and volatility is



(a) Expected Excess Rate of Return

(b) Volatility of Rate of Return

FIGURE 4: This figure plots the expected excess rate of return and volatility as a function of π_t and q. The parameters are set to the following baseline values unless otherwise specified: $r = 2.5\%$, $\mu = 3\%$, $\rho = 0.95$, $\sigma = 6\%$, $Z = 1$, $\alpha = 0.75$, and $\lambda = 0.5$.

increasing in q for high π_t but decreasing in q for low π_t.

These results suggest a novel prediction of the model, which has not been tested in the literature (to the best of our knowledge). As discussed above, in our model, higher disagreement across investors corresponds to lower π_t. One could also argue that higher *variation in ownership composition* of the risky asset reflects lower persistence (i.e., lower q) in the type of θ traders. As such, the model predicts that higher variation in ownership composition (lower q) should be associated with higher expected returns and higher volatility when disagreement across investors is high, and lower expected returns and volatility when disagreement is low.

5 Robustness: Common priors and noisy aggregate supply

In this section, we consider an alternative specification to our benchmark model. We focus on the two-date version (and normalize $R = 1$, $\mu = 0$) using the same setup is as in Section 3, with the following exceptions: investors share a common prior, N investors are fully rational, and the aggregate supply of the risky asset is stochastic. This specification is useful in highlighting the main effects of uncertainty about others in a more familiar setting — the model reduces to the noisy rational expectations model of Grossman and Stiglitz (1980) when $\pi = 1$. This specification also helps to illustrate the robustness of our qualitative implications though it is less analytically tractable than the benchmark model.

Conditional on $\theta = N$, investors agree that the signal is uninformative.[24] As a result,

[24]Or, equivalently, N investors do not observe a signal prior to submitting orders.

the optimal demand for a θ investor is given by:

$$x_\theta = \begin{cases} \frac{\lambda S_\theta - P}{\alpha\sigma^2(1-\lambda)} & \text{if} \quad \theta = I \\ \frac{0-P}{\alpha\sigma^2} & \text{if} \quad \theta = N, \end{cases} \tag{32}$$

Note that without an additional source of noise, observing price and quantities perfectly reveals both S_θ and θ. Thus, we follow the noisy rational expectations literature and introduce aggregate supply shocks (e.g., noise traders). In particular, the aggregate supply of the risky asset is $Z + z$, where $z \sim N(0, \sigma_z^2)$.[25] The market clearing condition is given by:

$$x_\theta + x_U = Z + z. \tag{33}$$

Finally, as in the benchmark model, we assume U investors can condition on the equilibrium price and residual supply when determining their optimal demand. In this setup, we show that there exists a rational expectations equilibrium which is characterized by the following proposition.

Proposition 8. *There exists a rational expectations equilibrium in which the price is given by the solution to:*

$$P = \underbrace{\left(\kappa^* + \left(1-\kappa^*\right)\pi^*\lambda_y\right) y}_{\text{expectations}} - \underbrace{\kappa^* \alpha\sigma^2 \left(1-\lambda\right) Z}_{\text{risk premium}}.$$

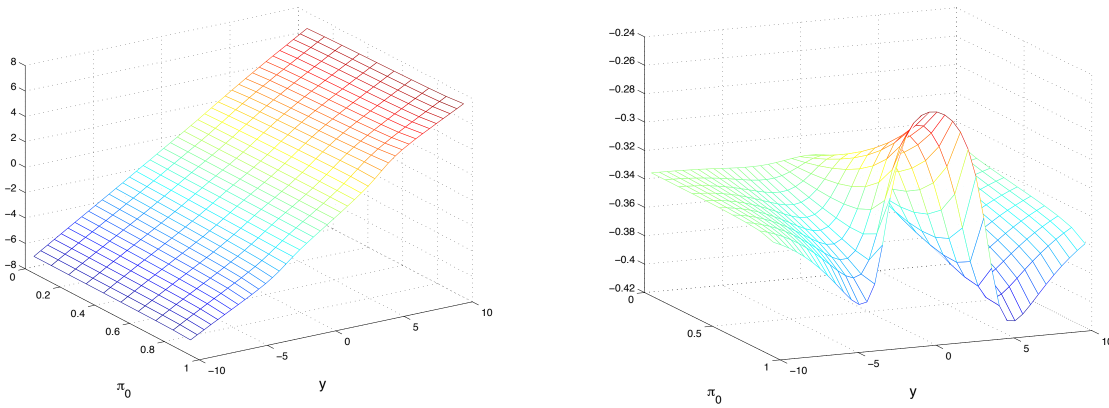
where $y = \alpha\sigma^2(1-\lambda)(x_\theta - z) + P$*,* $\pi^* = \Pr(\theta = I | y, P)$*, and*

$$\kappa^* = \frac{\sigma^2(1-\pi^*\lambda_y) + \pi^*(1-\pi^*)(\lambda_y y)^2}{\sigma^2(1-\lambda) + \sigma^2(1-\pi^*\lambda_y) + \pi^*(1-\pi^*)(\lambda_y y)^2}.$$

Analogous to the decomposition in equation (11), the price can be decomposed into the expectations and risk-premium components. Figure 5 illustrates these two components and suggests that uncertainty about whether others are informed has qualitatively similar implications in this setup. As in the static model of Section 3, the expectations component is monotonic in the price signal. Moreover, U investors are unsure about the informativeness of y, which implies that their posterior variance, and therefore, the risk-premium component of price depends on the realization of y. Thus, as in the benchmark model, the price reacts asymmetrically to good news versus bad news.

In contrast to the benchmark model, U investors learn directly about θ from the signal

[25] Alternatively, one could assume that in addition to being potentially informed, θ investors anticipate an endowment to their wealth of zd in the next period, where z is known to θ investors but not to U investors, and $z \sim N(0, \sigma_z^2)$.



(a) Expectations Component of Price

(b) Risk-premium Component of Price

FIGURE 5: Using the alternative specification given in Section 5, this figure illustrates the two components of the equilibrium price function as they depend on the price signal y and the prior beliefs π_0.

y; large realizations of y lead to large updates in π^* (either towards zero or one). That is, both uncertainty and learning are present in the two date version of this specification. As a result, the risk-premium component is dampened for large realizations of y, since for these realizations π^* is closer to zero or one. Though we do not numerically solve the dynamic version of this specification, clearly both uncertainty and learning will play a role in such a setting. Based on the analysis here, we expect that similar results to those in Section 4 would obtain.

Note that in all specification thus far, we assume that either I or N traders are present in the market, but not both. One could instead consider a setting in which both I and N investors are present simultaneously, but the U investor is uncertain about the *proportion* of each type of investor. The analysis of such a setting would not be dissimilar to the specification above. Specifically, by conditioning on the information in the residual demand and the price, speculators will be able to update their beliefs about the proportion of informed investors even in a static setting. Conditional on these beliefs, the residual demand provides a noisy signal about the dividend next period, which speculators can use to update their beliefs about fundamentals. A complete analysis of such a model is left for future work.

6 Final Remarks

Asset pricing models have focused primarily on uncertainty about the underlying fundamentals and assume that the characteristics of other traders in the market is common knowledge. We consider a framework in which investors are uncertain about whether others are informed and gradually learn about them by observing prices and dividends. We show that these additional channels of uncertainty and learning can have important implications for return dynamics. Specifically, the model generates non-linear prices, which are more sensitive to bad news than good news; stochastic, predictable expected returns and volatility, that vary with disagreement; volatility clustering (i.e., big return realizations of either sign are followed by higher volatility and higher expected returns); time-variation in the relation between information quality and return moments; and time-variation in the relation between return moments and variation in ownership composition.

We have focused on a setting in which investors are uncertain about whether other traders are informed. However, one could also consider alternative settings in which uninformed investors are uncertain about other characteristics of other traders such as their risk aversion or hedging demands (Section 2 discusses some recent advances along similar lines). The predictions of such a model will depend on the exact source of uncertainty, yet a number of similarities should arise: the multi-dimensional uncertainty will generally lead to a non-linearity in prices, and learning about others should generate rich return dynamics.

We have kept the model as parsimonious as possible to highlight the key mechanism behind our results. Although beyond the scope of the current paper, we believe that an enriched version of the model may be suitable for calibration. Moreover, our theoretical framework could potentially be extended to study information acquisition in a dynamic setting, thereby endogenizing the information structure.

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Appendix A - Proofs

Proof of Proposition 1. First, note that in the static model the optimal demand given in (2) reduces to

$$x_i = \frac{\mu + \mathbb{E}_i[d] - RP}{\alpha \mathrm{var}_i[d]}. \tag{34}$$

For θ investors, this can be expressed as

$$x_\theta = \frac{\mu + \lambda S_\theta - RP}{\alpha(1-\lambda)\sigma^2}. \tag{35}$$

We argue that any equilibrium must be signal revealing. If the equilibrium is not signal revealing, then there must exist two signal realizations, $s_1 > s_2$, for which P is the same. But in this case, from (35), the θ investor would demand strictly more after observing s_1, which implies that the U investor can distinguish between s_1 and s_2 using residual demand (i.e., $Z - x_\theta$). Next, since I and N have symmetric optimal strategies, prices and quantities cannot reveal information about θ. Hence, the equilibrium cannot be fully revealing and therefore, U's beliefs about the dividend must be given by (7) and (8). Existence and uniqueness follow by plugging the formulas for the optimal demand of U and θ investors given by (34) into the market clearing condition and solving for P as given by (9). □

Proof of Corollary 1. To demonstrate the results, it will be useful to establish the following properties of κ:

$$\begin{aligned}
\frac{\partial}{\partial\lambda}\kappa &= \frac{\sigma^2(1-\pi_0)(\pi_0 S_\theta^2(2-\lambda)\lambda+\sigma^2)}{(\sigma^2(1-\lambda)+\sigma^2(1-\pi_0\lambda)+\pi_0(1-\pi_0)(\lambda S_\theta)^2)^2} = \frac{(1-\pi_0)(\pi_0 S_\theta^2(2-\lambda)\lambda+\sigma^2)}{\sigma^2(1-\lambda)^2}(1-\kappa)^2 \geq 0 \\
\frac{\partial}{\partial\pi_0}\kappa &= -\frac{\left(\sigma^2-(1-2\pi_0)\lambda S_\theta^2\right)\lambda(1-\lambda)\sigma^2}{(\sigma^2(1-\lambda)+\sigma^2(1-\pi_0\lambda)+\pi_0(1-\pi_0)(\lambda S_\theta)^2)^2} = -\frac{\left(\sigma^2-(1-2\pi_0)\lambda S_\theta^2\right)\lambda}{\sigma^2(1-\lambda)}(1-\kappa)^2 \\
\frac{\partial}{\partial S_\theta}\kappa &= \frac{2\pi_0(1-\pi_0)(1-\lambda)\lambda^2\sigma^2 S_\theta}{(\sigma^2(1-\lambda)+\sigma^2(1-\pi_0\lambda)+\pi_0(1-\pi_0)(\lambda S_\theta)^2)^2} = \frac{2\pi_0(1-\pi_0)\lambda^2 S_\theta}{\sigma^2(1-\lambda)}(1-\kappa)^2
\end{aligned} \tag{36}$$

which imply κ is (i) increasing in λ, (ii) hump shaped in π_0 around $\frac{1}{2}\left(1-\frac{\sigma^2}{\lambda S_\theta^2}\right)$, and (iii) U-shaped in S_θ around 0.

<u>Effect of λ:</u> The derivative of the expectations component of P with respect to λ is given by

$$\frac{\partial}{\partial\lambda}\left((\kappa+(1-\kappa)\pi_0))\lambda S_\theta\right) = \left(\pi_0 + (1-\pi_0)\left(\kappa + \lambda\frac{\partial}{\partial\lambda}\kappa\right)\right) S_\theta.$$

From (i) above, this component increases with λ for $S_\theta > 0$ and decreases in λ otherwise. The derivative of risk-premium component is given by

$$\frac{\partial}{\partial\lambda}(-\alpha\sigma^2(1-\lambda)\kappa Z) = \frac{\alpha Z\sigma^2\left((-1+\pi_0)^2{\pi_0}^2 S_\theta^4\lambda^4+2\pi_0\sigma^2 S_\theta^2\lambda\left(-1+\pi_0+3\lambda-3\pi_0\lambda-\lambda^2+{\pi_0}^2\lambda^2\right)+\sigma^4\left(1+{\pi_0}^2\lambda^2+\pi_0\left(1-4\lambda+\lambda^2\right)\right)\right)}{\left((-1+\pi_0)\pi_0 S_\theta^2\lambda^2+\sigma^2(-2+\lambda+\pi_0\lambda)\right)^2}$$

The expression can be positive or negative depending on $|S_\theta|$. For $S_\theta = 0$ and as $|S_\theta| \to \infty$, the derivative is strictly positive and so the risk-premium component of price increases in λ at these extremes. However, for intermediate values of $|S_\theta|$, the derivative is negative.

Effect of π_0: The derivative of the expectations component of P with respect to π_0 is given by

$$\tfrac{\partial}{\partial \pi_0}\left((\kappa+(1-\kappa)\pi_0)\right)\lambda S_\theta) = \left((1-\kappa)+(1-\pi_0)\tfrac{\partial}{\partial \pi_0}\kappa\right)\lambda S_\theta$$

Inserting the expression from (36) for $\frac{\partial}{\partial \pi_0}\kappa$ gives:

$$\left((1-\kappa)+(1-\pi_0)\tfrac{\partial}{\partial \pi_0}\kappa\right)\lambda S_\theta = \frac{\sigma^2(1-\lambda)\left(((1-\pi_0)S_\theta\lambda)^2+2(1-\lambda)\sigma^2\right)}{(1-\lambda)\sigma^2+(1-\pi_0\lambda)\sigma^2+(\lambda S_\theta)^2(1-\pi_0)}\lambda S_\theta$$

Therefore, the derivative of the expectations component of prices with respect to π_0 has the same sign as S_θ. The risk premium component of the price is U-shaped in π_0 around $\frac{1}{2}\left(1-\frac{\sigma^2}{\lambda S_\theta^2}\right)$. This can be seen by using (ii) above and

$$\tfrac{\partial}{\partial \pi_0}(-\alpha\sigma^2(1-\lambda)\kappa Z) = -\alpha\sigma^2(1-\lambda)Z\tfrac{\partial}{\partial \pi_0}\kappa.$$

Effect of S_θ: The expectations component of P is increasing in S_θ. This can be seen by using (iii) above and

$$\tfrac{\partial}{\partial S_\theta}\left((\kappa+(1-\kappa)\pi_0)\right)\lambda S_\theta) = (\kappa+(1-\kappa)\pi_0))\lambda+(1-\pi_0)\lambda S_\theta\tfrac{\partial}{\partial S_\theta}\kappa > 0.$$

The risk-premium component of the price is hump-shaped in S_θ around zero. This can also be seen by using (iii) above and

$$\tfrac{\partial}{\partial S_\theta}(-\alpha\sigma^2(1-\lambda)\kappa Z) = -\alpha\sigma^2(1-\lambda)Z\tfrac{\partial}{\partial S_\theta}\kappa.$$

This completes the proof of the comparative static results. $\square$

Proof of Proposition 2. To be completed. $\square$

Proof of Proposition 3. Let $P(s)$ denote the equilibrium price (as given by (11)) for an arbitrary signal realization s, and similarly for $\kappa(s)$ (which is given by (10)). Note that for $0<s_1<s_2$, the difference in the price is given by

$$P(s_2)-P(s_1) = \tfrac{1}{R}\begin{pmatrix} (\kappa(s_2)+(1-\kappa(s_2))\pi_0)\lambda s_2-(\kappa(s_1)+(1-\kappa(s_1))\pi_0)\lambda s_1 \\ -\alpha\sigma^2(1-\lambda)Z(\kappa(s_2)-\kappa(s_1)) \end{pmatrix}$$

Since $\kappa(s_2)>\kappa(s_1)$ (see proof of Corollary 1), setting

$$\gamma = \frac{(\kappa(s_2)+(1-\kappa(s_2))\pi_0)\lambda s_2-(\kappa(s_1)+(1-\kappa(s_1))\pi_0)\lambda s_1}{\sigma^2(1-\lambda)(\kappa(s_2)-\kappa(s_1))}$$

gives the result. $\square$

Proof of Proposition 4. The expressions for the conditional expected return and volatility follow from the observation that the only source of randomness in returns, conditional on P and x_θ, is the realization of the dividend d. In particular, this implies that $\mathrm{var}[Q|P,x_\theta] =$

$\text{var}_U[d|P, x_\theta]$. To derive the expression for unconditional expected return, take the expectation of the right-hand side (RHS) of (13) and using that $\mathbb{E}_U[d] = \pi_0\lambda S_\theta$, we have

$$\mathbb{E}[Q] = \mathbb{E}[\mathbb{E}[Q|P, x_\theta]] = \alpha\sigma^2(1-\lambda)Z\mathbb{E}[\kappa] - (1-\pi_0)\lambda\mathbb{E}[\kappa S_\theta]$$

Thus, it suffices to show that $\mathbb{E}[\kappa S_\theta] = 0$. For this, note that $\kappa \cdot S_\theta$ is an odd-function (of S_θ) and the distribution of S_θ is symmetric around zero. Thus, $\mathbb{E}[\kappa S_\theta | S_\theta > 0] = -\mathbb{E}[\kappa S_\theta | S_\theta < 0]$, which implies $\mathbb{E}[\kappa S_\theta] = 0$.

For unconditional volatility of returns, we have that

$$\begin{aligned}
\text{var}[Q] &= \mathbb{E}[\text{var}[Q|P, x_\theta]] + \text{var}[\mathbb{E}[Q|P, x_\theta]] \\
&= \mathbb{E}[\sigma^2(1-\pi_0\lambda) + \pi_0(1-\pi_0)(\lambda S_\theta)^2] + \text{var}\left[\alpha\sigma^2(1-\lambda)\kappa Z - (1-\pi_0)\kappa\lambda S_\theta\right] \\
&= \sigma^2(1-\pi_0^2\lambda) + (\alpha\sigma^2(1-\lambda)Z)^2\text{var}[\kappa] + (1-\pi_0)^2\lambda^2\text{var}[\kappa S_\theta] \\
&\quad - 2\alpha\sigma^2(1-\lambda)\lambda Z(1-\pi_0)\text{cov}(\kappa, \kappa S_\theta).
\end{aligned}$$

Stein's Lemma implies that for $Y \sim \mathcal{N}(0, \sigma_Y^2)$, and $g(Y)$ such that $\mathbb{E}[g(Y)Y] < \infty$ and $\sigma_Y^2\mathbb{E}[g'(Y)] < \infty$, we have $\text{cov}(g(Y), X) = \mathbb{E}[g'(Y)]\text{cov}(Y, X)$. Therefore

$$\begin{aligned}
\text{cov}(\kappa, S_\theta) &= \mathbb{E}\left[\tfrac{\partial}{\partial S_\theta}\kappa\right]\text{var}(S_\theta) \\
\text{var}[\kappa S_\theta] &= \mathbb{E}[\kappa^2 S_\theta^2] - (\mathbb{E}[\kappa S_\theta])^2 = \text{cov}(\kappa^2 S_\theta, S_\theta) - \text{cov}(\kappa, S_\theta) \\
&= \left(\mathbb{E}\left[\kappa^2 + 2\kappa S_\theta\tfrac{\partial}{\partial S_\theta}\kappa\right] - \mathbb{E}\left[\tfrac{\partial}{\partial S_\theta}\kappa\right]\right)\text{var}(S_\theta) \\
\text{cov}(\kappa, \kappa S_\theta) &= \mathbb{E}[\kappa^2 S_\theta] - \mathbb{E}[\kappa]\mathbb{E}[\kappa S_\theta] = \text{cov}(\kappa^2, S_\theta) - \mathbb{E}[\kappa]\text{cov}(\kappa, S_\theta) \\
&= \left(\mathbb{E}\left[2\kappa\tfrac{\partial}{\partial S_\theta}\kappa\right] - \mathbb{E}[\kappa]\mathbb{E}\left[\tfrac{\partial}{\partial S_\theta}\kappa\right]\right)\text{var}(S_\theta).
\end{aligned}$$

Since $\frac{\partial}{\partial S_\theta}\kappa(S_\theta) = -\frac{\partial}{\partial S_\theta}\kappa(-S_\theta)$, we have that $\mathbb{E}\left[\frac{\partial}{\partial S_\theta}\kappa\right] = 0$, and $\mathbb{E}\left[\kappa\frac{\partial}{\partial S_\theta}\kappa\right] = 0$. This implies that volatility can be expressed as:

$$\text{var}[Q] = \sigma^2(1-\pi_0^2\lambda) + (\alpha\sigma^2(1-\lambda)Z)^2\text{var}[\kappa] + (1-\pi_0)^2\lambda^2\text{var}[\kappa S_\theta]$$

since $\lambda = \sigma^2/\text{var}(S_\theta)$. □

Proof of Proposition 5. We first establish that there is a unique equilibrium in the finite horizon model.

Lemma 1. *In the finite horizon model, the unique equilibrium price is of the form*

$$P_t = A_t\mu + B_t D_t + p_t(S_{\theta,t}), \tag{37}$$

where $p_T = 0$, $A_T = 0$, $B_t = \frac{\rho}{R}(1+B_t)$, $A_t = \frac{1}{R}(A_{t+1} + (1-\rho)(1+B_{t+1}))$, *and*

$$p_t(S_{\theta,t}) = \tfrac{1}{R}\left\{\bar{\mathbb{E}}_t\left[(1+B_{t+1})\,d_{t+1} + p_{t+1}(S_{\theta,t+1})\right] - \alpha\kappa_t var_{\theta,t}\left[(1+B_{t+1})\,d_{t+1} + p_{t+1}(S_{\theta,t+1})\right]Z\right\}, \tag{38}$$

where
$$\kappa_t = \frac{var_{U,t}\left[(1+B_{t+1})d_{t+1}+p_{t+1}(S_{\theta,t+1})\right]}{var_{U,t}\left[(1+B_{t+1})d_{t+1}+p_{t+1}(S_{\theta,t+1})\right]+var_{\theta,t}\left[(1+B_{t+1})d_{t+1}+p_{t+1}(S_{\theta,t+1})\right]}$$
and $\bar{\mathbb{E}}_{i,t}[\cdot] = \kappa_t \mathbb{E}_{\theta,t}[\cdot] + (1-\kappa_t)\mathbb{E}_{U,t}[\cdot]$.

Proof of Lemma 1. We shall establish the claim by verifying the recursion.
Base Step. The terminal date is T (i.e., $P_T = 0$) and so P_{T-1} is given by:

$$\begin{aligned} P_{T-1} &= \tfrac{1}{R}\left((1-\rho)\mu + \rho D_{T-1} + \bar{E}_{i,T-1}[d_T] - \alpha\kappa_{T-1}\mathrm{var}_{\theta,t}[d_T]\, Z\right) && (39)\\ &= A_{T-1}\mu + B_{T-1}D_{T-1} + p_{T-1}(S_{\theta,T-1}), && (40)\end{aligned}$$

since $A_T = B_T = p_T = 0$. Also, note that at date $T-1$, both investors' beliefs about p_T has finite first and second moments (degenerately).
Recursive Step. Suppose the price in the next period satisfies $P_{t+1} = A_{t+1}\mu + B_{t+1}D_{t+1} + p_{t+1}(S_{\theta,t+1})$, and both investors' date t beliefs about p_{t+1} has finite first and second moments. Then,

$$\begin{aligned} x_{i,t} &= \frac{\mathbb{E}_{i,t}[P_{t+1}+D_{t+1}] - RP_t}{\alpha \mathrm{var}_{i,t}[P_{t+1}+D_{t+1}]} && (41)\\ &= \frac{\mathbb{E}_{i,t}[A_{t+1}\mu + (1+B_{t+1})D_{t+1} + p_{t+1}(S_{\theta,t+1})] - RP_t}{\alpha\mathrm{var}_{i,t}[(1+B_{t+1})D_{t+1} + p_{t+1}(S_{\theta,t+1})]} && (42)\\ &= \frac{(A_{t+1}+(1+B_{t+1})(1-\rho))\mu + (1+B_{t+1})\rho D_t + \mathbb{E}_{i,t}\left[(1+B_{t+1})d_{t+1}+p_{t+1}(S_{\theta,t+1})\right] - RP_t}{\alpha\mathrm{var}_{i,t}\left[(1+B_{t+1})d_{t+1}+p_{t+1}(S_{\theta,t+1})\right]} && (43)\end{aligned}$$

Market clearing implies $\sum_i x_{i,t} = Z$, or equivalently,

$$\begin{aligned} P_t &= \frac{1}{R}\begin{pmatrix} (A_{t+1}+(1+B_{t+1})(1-\rho))\mu + (1+B_{t+1})\rho D_t \\ +\bar{\mathbb{E}}_{i,t}[(1+B_{t+1})d_{t+1} + p_{t+1}(S_{\theta,t+1})] \\ -\alpha\kappa_t \mathrm{var}_{\theta,t}[(1+B_{t+1})d_{t+1}+p_{t+1}(S_{\theta,t+1})]\, Z \end{pmatrix} && (44)\\ &\equiv A_t\mu + B_t D_t + p_t(S_{\theta,t}), && (45)\end{aligned}$$

which verifies our conjecture. Finally, since $S_{\theta,t+1}$ is independent of d_{t+1} and both investors' date t beliefs about p_{t+1} has finite first and second moments, the above implies both investors' date $t-1$ beliefs about p_t has finite first and second moments. □

Stationary solution to OLG IID θ model. Now we turn to the proof of the main result. Note that in a stationary equilibrium of the infinite horizon OLG model in which θ is IID, we should have $A_t = A$ and $B_t = B$, which implies $A = \frac{R(1-\rho)}{(R-1)(R-\rho)}$, and $B = \frac{\rho}{R-\rho}$. Denote $m = \mathbb{E}_{i,t}[p_{t+1}] = \mathbb{E}_i[p_{t+1}]$ and $v = \mathrm{var}_{i,t+1}[p_{t+1}]$. Note that $\mathbb{E}[\kappa_t S_{\theta,t}] = 0$ and $\mathrm{cov}[\kappa_t S_{\theta,t}, \kappa_t] = 0$, since κ_t is even in $S_{\theta,t}$, and $\mathrm{var}[\kappa_t S_{\theta,t}] \le \sigma^2 + \sigma_\varepsilon^2$ and $\mathrm{var}[\kappa_t] \le 1$, since

$\kappa_t \leq 1$. Finally, note that d_{t+1} and p_{t+1} are uncorrelated. Then,

$$m = \mathbb{E}_i\left[p_t\right] = \mathbb{E}_i\left[\tfrac{1}{R}\left\{\bar{\mathbb{E}}_t\left[(1+B)\,d_{t+1} + p_{t+1}\right] - \alpha\kappa_t \mathrm{var}_{\theta,t}\left[(1+B)\,d_{t+1} + p_{t+1}\right] Z\right\}\right] \tag{46}$$

$$= \tfrac{1}{R}\left(\mathbb{E}\left[(1+B)(\kappa_t + (1-\kappa_t)\pi)\lambda S_{\theta,t} + m - \alpha\kappa_t((1+B)^2\sigma^2(1-\lambda) + v)Z\right]\right) \tag{47}$$

$$\Rightarrow m = -\tfrac{1}{R-1}((1+B)^2\sigma^2(1-\lambda) + v)\alpha Z \mathbb{E}\left[\kappa_t\right] \tag{48}$$

Let $\sigma_E^2 \equiv \mathrm{var}\left[(\kappa_t + (1-\kappa_t)\pi)S_{\theta,t}\right]$ and $\sigma_\kappa^2 = \mathrm{var}\left[\kappa\right]$. Then, we have

$$v = \mathrm{var}_i\left[\tfrac{1}{R}\left((1+B)(\kappa_t + (1-\kappa_t)\pi)\lambda S_{\theta,t} + m - \alpha\kappa_t((1+B)^2\sigma^2(1-\lambda) + v)Z\right)\right] \tag{49}$$

$$= \tfrac{1}{R^2}\left((1+B)^2\lambda^2\sigma_E^2 + \alpha^2((1+B)^2\sigma^2(1-\lambda) + v)^2 Z^2\sigma_\kappa^2\right) \tag{50}$$

$$\Rightarrow v = J(v) \tag{51}$$

where $J(v) = \frac{1}{R^2}\left((1+B)^2\lambda^2\sigma_E^2 + \alpha^2((1+B)^2\sigma^2(1-\lambda) + v)^2 Z^2\sigma_\kappa^2\right)$. Note that $J(0) > 0$ and since

$$J(v) \leq \tfrac{1}{R^2}\left((1+B)^2\lambda^2(\sigma^2 + \sigma_\varepsilon^2) + \alpha^2((1+B)^2\sigma^2(1-\lambda) + v)^2 Z^2\right) \tag{52}$$

$$\equiv F + G(v+H)^2 \tag{53}$$

where $F = \frac{1}{R^2}(1+B)^2\lambda^2(\sigma^2 + \sigma_\varepsilon^2)$, $G = \frac{1}{R^2}\alpha^2 Z^2$, and $H = (1+B)^2\sigma^2(1-\lambda)$. The solution to the quadratic equation $v = F + G(v+H)^2$ is given by:

$$v^* = \tfrac{1 - 2GH \pm \sqrt{1 - 4G(F+H)}}{2G} \tag{54}$$

and a sufficient condition for existence is

$$1 - 4G(F+H) \geq 1 - \tfrac{4\alpha^2 Z^2\sigma^2}{(R-\rho)^2} > 0, \tag{55}$$

since $\sigma_E^2 \leq \sigma^2 + \sigma_\varepsilon^2$, and $\sigma_\kappa^2 \leq 1$. This implies that $J(v^*) \leq v^*$ which implies there exists a solution to $J(v) = v$, and consequently, an equilibrium.

The equilibrium price is then given by

$$P_t = A\mu + BD_t + \tfrac{1}{R}\left((1+B)(\kappa_t + (1-\kappa_t)\pi)\lambda S_{\theta,t} + m - \alpha\kappa_t((1+B)^2\sigma^2(1-\lambda) + v)Z\right),$$

where A, B, m and v are characterized above.

□

Proof of Proposition 6. Optimality of $x_{i,t}$ follows from (2), the expressions for beliefs are given by (7)–(6), and the expression for the price follows from the market clearing condition. □

Proof of Proposition 7. We shall establish the claim by verifying the recursion.
Base Step. The terminal date is T (i.e., $P_T = 0$) and so P_{T-1} is given by:

$$P_{T-1} = \tfrac{1}{R}\left((1-\rho)\mu + \rho D_{T-1} + \bar{E}_{i,T-1}[d_T] - \alpha\kappa_{T-1}\mathrm{var}_{\theta,t}\left[d_T\right] Z\right) \tag{56}$$

$$= A_{T-1}\mu + B_{T-1}D_{T-1} + p_{T-1}(S_{\theta,T-1}), \tag{57}$$

since $A_T = B_T = p_T = 0$. Also, note that at date $T-1$, both investors' beliefs about p_T has finite first and second moments (degenerately).
Recursive Step. Suppose the price in the next period satisfies $P_{t+1} = A_{t+1}\mu + B_{t+1}D_{t+1} + p_{t+1}(S_{\theta,t+1}, \pi_{t+1})$, and both investors' date t beliefs (at any $(S_{\theta,t}, \pi_t)$) about p_{t+1} has finite first and second moments. Then,

$$x_{i,t} = \frac{\mathbb{E}_{i,t}\left[P_{t+1} + D_{t+1}\right] - RP_t}{\alpha \text{var}_{i,t}\left[P_{t+1} + D_{t+1}\right]} \tag{58}$$
$$= \frac{\mathbb{E}_{i,t}\left[A_{t+1}\mu + (1+B_{t+1})D_{t+1} + p_{t+1}(S_{\theta,t+1}, \pi_{t+1})\right] - RP_t}{\alpha \text{var}_{i,t}\left[(1+B_{t+1})\, D_{t+1} + p_{t+1}(S_{\theta,t+1}, \pi_{t+1}\right]} \tag{59}$$
$$= \frac{(A_{t+1}+(1+B_{t+1})(1-\rho))\mu + (1+B_{t+1})\rho D_t + \mathbb{E}_{i,t}\left[(1+B_{t+1})d_{t+1} + p_{t+1}(S_{\theta,t+1},\pi_{t+1})\right] - RP_t}{\alpha \text{var}_{i,t}\left[(1+B_{t+1})d_{t+1} + p_{t+1}(S_{\theta,t+1},\pi_{t+1})\right]} \tag{60}$$

Market clearing implies $\sum_i x_{i,t} = Z$, or equivalently,

$$P_t = \frac{1}{R}\begin{pmatrix} (A_{t+1} + (1+B_{t+1})(1-\rho))\mu + (1+B_{t+1})\rho D_t \\ +\bar{\mathbb{E}}_{i,t}\left[(1+B_{t+1})d_{t+1} + p_{t+1}(S_{\theta,t+1}, \pi_{t+1})\right] \\ -\alpha\kappa_t \text{var}_{\theta,t}\left[(1+B_{t+1})d_{t+1} + p_{t+1}(S_{\theta,t+1}, \pi_{t+1})\right] Z \end{pmatrix} \tag{61}$$
$$\equiv A_t\mu + B_t D_t + p_t(S_{\theta,t}, \pi_t), \tag{62}$$

which verifies our conjectured form. To verify this is an equilibrium, we need to confirm that the conditional expectation and variance of $p_t(\cdot)$ is bounded.

Suppose $|\mathbb{E}_{i,t}\left[p_{t+1}(S_{\theta,t+1}, \pi_{t+1})|\pi_{t+1}\right]| \le m_t$ and $\text{var}_{i,t}\left[p_{t+1}(S_{\theta,t+1}, \pi_{t+1})\right]| \le v_t$. Note that the first constraint implies that $|\mathbb{E}_{i,t}\left[p_{t+1}(S_{\theta,t+1}, \pi_{t+1})\right]| \le m_t$. Also, since $p_T = 0$, these conditions hold degenerately for T. We want to show that the same moments of $p_t(S_{\theta,t}, \pi_t)$ are also bounded. Note that since the equilibrium is signal-revealing, we have:

$$p_t = \tfrac{1}{R}\left(\bar{\mathbb{E}}_t\left[(1+B_{t+1})d_{t+1} + p_{t+1}\right] - \alpha\kappa_t \text{var}_{\theta,t}\left[(1+B_{t+1})d_{t+1} + p_{t+1}\right] Z\right) \tag{63}$$
$$= \tfrac{1}{R}\left((1+B_{t+1})(\kappa_t + (1-\kappa_t)\pi_t)\lambda S_{\theta,t} + \bar{\mathbb{E}}_t\left[p_{t+1}\right] - \alpha\kappa_t \text{var}_{\theta,t}\left[(1+B_{t+1})d_{t+1} + p_{t+1}\right] Z\right) \tag{64}$$

Given our conjecture,

$$\text{var}_{\theta,t}\left[(1+B_{t+1})d_{t+1} + p_{t+1}\right] \tag{65}$$
$$= (1+B_{t+1})^2 \text{var}_{\theta,t}\left[d_{t+1}\right] + \text{var}_{\theta,t}\left[p_{t+1}\right] + 2(1+B_{t+1})\text{cov}_{\theta,t}\left[d_{t+1}, p_{t+1}\right] \tag{66}$$
$$\le (1+B_{t+1})^2\sigma^2 + v_t + 2(1+B)\sigma\sqrt{v_t} \equiv V_t, \tag{67}$$

But this implies that:

$$\mathbb{E}_{i,t-1}\left[\kappa_t \text{var}_{\theta,t}\left[(1+B_{t+1})d_{t+1} + p_{t+1}\right]|\pi_t\right] \le V_t + \sqrt{V_t}, \tag{68}$$

since $\kappa_t \in [0,1]$. Finally, since κ_t is even in $S_{\theta,t}$, and $S_{\theta,t}$ has zero mean, we have

$$\mathbb{E}_{i,t-1}\left[(\kappa_t + (1-\kappa_t)\pi_t)\lambda S_{\theta,t}|\pi_t\right] = 0, \tag{69}$$

and so,

$$|\mathbb{E}_{i,t-1}\left[p_t(S_{\theta,t},\pi_t)|\pi_t\right]| \leq \tfrac{1}{R}\left(m_t + \alpha Z(V_t + \sqrt{V_t})\right) \equiv m_{t-1}, \tag{70}$$

which verifies the conjecture for the conditional mean of the price.

Next, note that

$$\mathrm{var}_{i,t-1}\left[p_t|\pi_t\right] \leq \tfrac{1}{R^2}\left(\begin{array}{c}(1+B_{t+1})^2\lambda^2(\sigma^2+\sigma_\varepsilon^2) + m_t^2 + \alpha^2 Z^2 V_t^2 \\ +2\left((1+B_{t+1})\lambda\sqrt{\sigma^2+\sigma_\varepsilon^2}(m_t+\alpha Z V_t) + m_t \alpha Z V_t\right)\end{array}\right) \equiv W_t. \tag{71}$$

But this implies that by applying the law of total variance, we have

$$\begin{aligned}\mathrm{var}_{i,t-1}\left[p_t(S_{\theta,t},\pi_t)\right] &= \mathbb{E}_{i,t-1}\left[\mathrm{var}_{i,t-1}\left[p_t|\pi_t\right]\right] + \mathrm{var}_{i,t-1}\left[\mathbb{E}_{i,t-1}\left[p_t|\pi_t\right]\right] && (72)\\ &\leq W_t + m_{t-1}^2 \equiv v_{t-1}, && (73)\end{aligned}$$

which verifies the conjecture for the conditional variance in the price. □

Proof of Proposition 8. Define the random variable

$$y \equiv \alpha\sigma^2(1-\lambda)(x_\theta - z) + P$$

Since U investors can observe the equilibrium price and the residual supply, y is measurable w.r.t. their information. Given the optimal demand of the θ investors, y takes the form

$$y = \begin{cases}\lambda S_\theta - \alpha\sigma^2(1-\lambda)z & \text{if} \quad \theta = I\\ \lambda P - \alpha\sigma^2(1-\lambda)z & \text{if} \quad \theta = N,\end{cases}$$

Note that this implies that, unlike the static version of the base model, the U investor can use the "signal", y, to learn about θ. In particular, her updated belief conditional on (y, P) is given by:

$$\pi^*(y,P) = \frac{\frac{\pi_0}{\sqrt{\sigma_{Y,I}^2}}\phi\left(\frac{y}{\sqrt{\sigma_{Y,I}^2}}\right)}{\frac{\pi_0}{\sqrt{\sigma_{Y,I}^2}}\phi\left(\frac{y}{\sqrt{\sigma_{Y,I}^2}}\right) + \frac{1-\pi_0}{\sqrt{\sigma_{Y,NI}^2}}\phi\left(\frac{y-\lambda P}{\sqrt{\sigma_{Y,NI}^2}}\right)}$$

where

$$\sigma_{Y,I}^2 = \lambda^2\left(\sigma^2+\sigma_\varepsilon^2\right) + \alpha^2\sigma^4(1-\lambda)^2\sigma_z^2, \qquad \text{and} \quad \sigma_{Y,NI}^2 = \alpha^2\sigma^4(1-\lambda)^2\sigma_z^2.$$

Conditional on $\theta = I$, U's belief about d is given by

$$\mathbb{E}_U\left[d|y,\theta=I\right] = \lambda_y y, \qquad \mathrm{var}_U\left[d|y,\theta=I\right] = \sigma^2\left(1-\lambda_y\right),$$

where

$$\lambda_y = \frac{\mathrm{cov}(y,d)}{\mathrm{var}(y)} = \frac{\lambda\sigma^2}{\sigma_{Y,I}^2} = \frac{\lambda\sigma^2}{\lambda\sigma^2 + \lambda^2\alpha^2\sigma_\epsilon^4\sigma_z^2}.$$

Optimal demand for the U investor is then given by

$$x_U(x_\theta - z, P) = \frac{1}{\alpha}\frac{\pi^*\lambda_y y - P}{\pi^*\sigma^2(1-\lambda_y)+(1-\pi^*)\sigma^2+\pi^*(1-\pi^*)(\lambda_y y)^2}.$$

The market clearing condition (33) implies that the equilibrium price, P, can be implicitly characterized as the solution to the following equation:

$$x_U(x_\theta - z, P) = Z - (x_\theta - z). \tag{74}$$

Note that since

$$\frac{\partial\pi^*}{\partial P} = \frac{\pi_0(1-\pi_0)}{\sigma^2_{Y,I}\sigma^2_{Y,NI}}\left(\alpha\sigma^2(1-\lambda)(x_\theta - z)\left(\sigma^2_{Y,I}(1-\lambda) - \sigma^2_{Y,NI}\right) + P\left(\left(\sigma^2_{Y,I}(1-\lambda)^2 - \sigma^2_{Y,NI}\right)\right)\right),$$

for any realization of $x_\theta - z$, we have:

- If $\sigma^2_{Y,I}(1-\lambda)^2 - \sigma^2_{Y,NI} > 0$, the derivative is increasing in P and, for large enough values of P, it is positive, which implies $\lim_{|P|\to\infty}\pi^* = 1$, and consequently

$$\lim_{|P|\to\infty} x_U = \frac{\lambda_y y - P}{\alpha\sigma^2(1-\lambda_y)}.$$

- If $\sigma^2_{Y,I}(1-\lambda)^2 - \sigma^2_{Y,NI} < 0$, the derivative is decreasing in P_t and, for large enough P_t, it is negative. This implies that $\lim_{|P|\to\infty}\pi^* = 0$, and consequently

$$\lim_{|P|\to\infty} x_U = \frac{0-P}{\alpha\sigma^2}.$$

Since x_U is continuous in P and π^*, this implies that in either case, there exists a P that satisfies equation (74). Rearranging equation (74) gives the expression for the price in the proposition. □

Appendix B - Supplementary Analysis

Comparative statics on return moments

To investigate comparative statics, we start by presenting the following result.

Proposition 9. *In the static model,*

(i) The unconditional expected return is homogeneous of degree 1 (HD1) in σ^2 and αZ.

(ii) The unconditional volatility component due to fundamental shocks is HD1 in σ^2 and HD0 in αZ.

(iii) The unconditional volatility component due to the expectations component of returns is HD1 in σ^2 and HD0 in αZ.

(iv) The unconditional volatility component due to the risk premium component of returns is HD2 in σ^2 and αZ.

Proof of Proposition 9. It suffices to show that $\mathbb{E}[\kappa]$ and $\text{var}[\kappa]$ are HD0 in σ^2, while $\text{var}[\kappa S_\theta]$ is HD1 in σ^2. Recall that

$$\kappa = \frac{\sigma^2(1-\pi_0\lambda)+\pi_0(1-\pi_0)\lambda^2 S_\theta^2}{\sigma^2(1-\lambda)+\sigma^2(1-\pi_0\lambda)+\pi_0(1-\pi_0)\lambda^2 S_\theta^2}$$

and by definition, $\lambda = \frac{\sigma^2}{\sigma^2+\sigma_\epsilon^2}$ and $S_\theta \sim N(0,\sigma^2+\sigma_\epsilon^2) = N(0,\sigma^2/\lambda)$, we have

$$\mathbb{E}[\kappa] = \frac{1}{\sqrt{2\pi\sigma^2/\lambda}}\int_{-\infty}^{\infty}\frac{\sigma^2(1-\pi_0\lambda)+\pi_0(1-\pi_0)\lambda^2 s^2}{\sigma^2(1-\lambda)+\sigma^2(1-\pi_0\lambda)+\pi_0(1-\pi_0)\lambda^2 s^2}\exp\left(\frac{-s^2}{2\sigma^2/\lambda}\right)ds$$

Using a change of variables, by letting $x \equiv \frac{\sqrt{\lambda}}{\sigma}s$, we get that

$$\begin{aligned}
E[\kappa] &= \frac{1}{\sqrt{2\pi\sigma^2/\lambda}}\frac{\sigma}{\sqrt{\lambda}}\int\frac{\sigma^2(1-\pi_0\lambda)+\pi_0(1-\pi_0)\lambda^2\frac{\sigma^2}{\lambda}x}{\sigma^2(1-\lambda)+\sigma^2(1-\pi_0\lambda)+\pi_0(1-\pi_0)\lambda^2\frac{\sigma^2}{\lambda}x}\exp\left(\frac{-x^2}{2}\right)dx \\
&= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\frac{(1-\pi_0\lambda)+\pi_0(1-\pi_0)\lambda x}{(1-\lambda)+(1-\pi_0\lambda)+\pi_0(1-\pi_0)\lambda x}\exp\left(\frac{-x^2}{2}\right)dx \qquad (75)
\end{aligned}$$

And clearly (75) is independent of σ. To see that $\text{var}[\kappa]$ is also independent of σ, note that same proof as above applies to $E[\kappa^2]$.

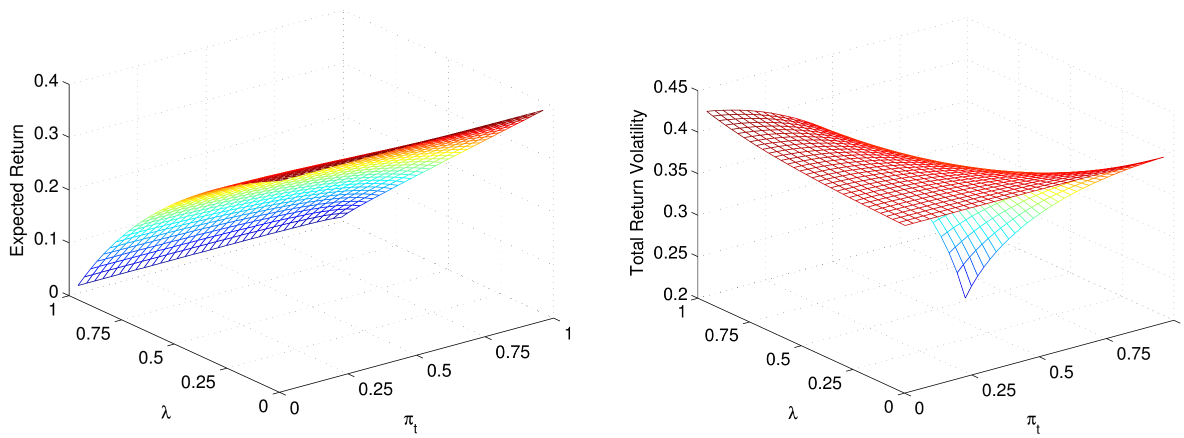
For $\text{var}\left[\kappa S_\theta\right]$, again using the same change of variables, we have that

$$\begin{aligned}
\mathbb{E}[\kappa S_\theta] &= \frac{1}{\sqrt{2\pi\sigma^2/\lambda}}\int_{-\infty}^{\infty}\frac{\sigma^2(1-\pi_0\lambda)+\pi_0(1-\pi_0)\lambda^2 s^2}{\sigma^2(1-\lambda)+\sigma^2(1-\pi_0\lambda)+\pi_0(1-\pi_0)\lambda^2 s^2}s\exp\left(\frac{-s^2}{2\sigma^2/\lambda}\right)ds \\
&= \frac{\sigma}{\sqrt{2\pi}}\int\frac{(1-\pi_0\lambda)+\pi_0(1-\pi_0)\lambda x}{(1-\lambda)+(1-\pi_0\lambda)+\pi_0(1-\pi_0)\lambda x}\frac{x}{\sqrt{\lambda}}\exp\left(\frac{-x^2}{2}\right)dx
\end{aligned}$$

which clearly scales with σ and hence $(E[\kappa S_\theta])^2$ scales with σ^2. The same change of variables can be used to show that the same is true of $E[(\kappa S_\theta)^2]$, which completes the proof. □

As expected, (i) implies that unconditional expected returns are increasing in the fundamental volatility and the overall risk concerns in the market as captured by αZ. Results (ii) through (iv) are also fairly intuitive, but they have important implications for which component drives overall volatility. In particular, when overall concerns about risk in the market are relatively high, the risk premium component of expression (19) is the key driver of overall return volatility. When αZ and σ^2 are relatively small, the first and second components of expression (19) drive overall volatility.

Proposition 9 is also useful for exploring comparative static results with respect to λ and π_0. For example, (i) implies that when exploring how expected returns change with λ and π_0, it is without loss to normalize σ^2 and αZ. By doing so, we are left with a two-dimensional parameter space (i.e., $(\pi_0,\lambda)\in[0,1]^2$), over which the expected return can be plotted to obtain comparative-static results that obtain *for any parameter specification* of the model. Figure 6(a) illustrates the result; both higher quality information and greater likelihood of an informed trader decrease the expected return. This is because both higher quality information and a higher likelihood of an informed trader imply that the price is more informative about the fundamentals in expectation, and the uncertainty faced by the uninformed investor is lower.



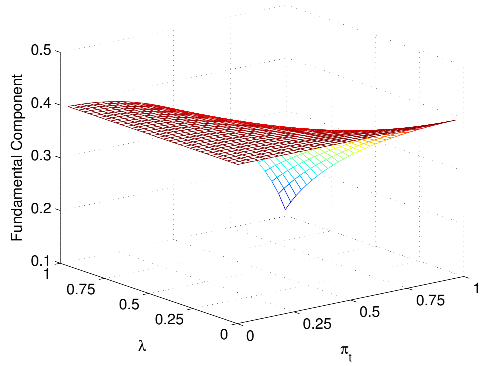
(a) Expected Return (b) Volatility

FIGURE 6: Illustration of expected returns (a) and total volatility (b) as they depend on the quality of information λ and the probability of a θ being informed, i.e., π_0. The other parameters are $\sigma = 40\%$, and $\alpha Z = 3$.

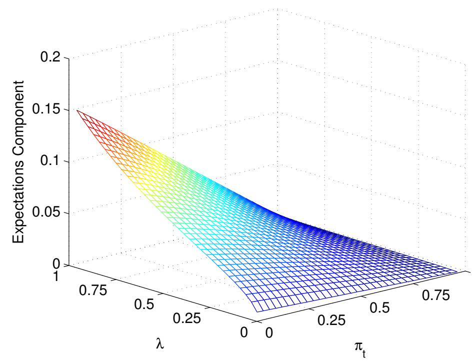
Using (ii) through (iv), we can conduct a similar exercise to characterize the comparative static effects of each of the individual components of volatility. Figure 7(a) shows the volatility in returns due to fundamental dividend shocks is decreasing in π_0 and λ, since an increase in either parameter reduces the uncertainty that investors face about next period's dividend. Figure 7(b) shows that the variance in the expectations component of conditional expected returns is decreasing in π_0 but increasing in λ. Recall that the expectations component of the conditional expected returns is non-zero because investors exhibit differences of opinion, and in particular, because uninformed θ investors believe they are informed. This effect is larger when π_0 is smaller (since θ investors are less likely to actually be informed) and when λ is larger (since uninformed θ investors put more weight on their signals), which leads to the effect on volatility. Figure 7(c) shows the risk-premium component of volatility is non-monotonic in both π_0 and λ. This is because the risk-premium component of returns is stochastic only when both λ and π_0 are strictly between zero and one.[26]

Of course, comparative statics on the total return volatility depend on the relative magnitudes of σ^2 and αZ, which determine the relative weight on each component. For instance, Figure 6(b) presents the effect of π_0 and λ on overall volatility for a given set of parameters, for which the fundamental and expectations components dominate the risk-premium component.

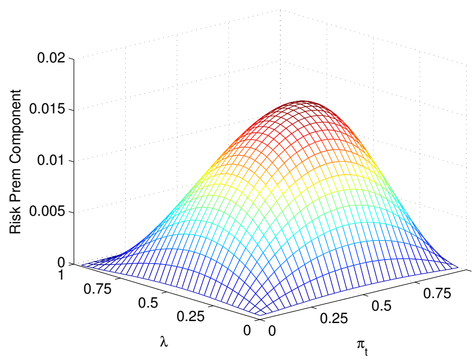
[26]If $\pi_0 \in \{0, 1\}$, the conditional variance of U investors does not depend on S_θ. Consequently, κ and the risk-premium component of expected returns are constant. Similarly, when $\lambda = 1$, the risk-premium is zero, while when $\lambda = 0$, all θ investors are effectively uninformed, and so the risk-premium is, again, constant.



(a) Fundamental Component



(b) Expectations Component



(c) Risk-Premium Component

FIGURE 7: The three components of volatility as they depend on the quality of information, λ, and the probability of a θ being informed, π_0. Panel (a) plots the fundamental component of volatility (i.e., $\sigma^2(1-\pi_0^2\lambda)$), panel (b) plots the expectations component (i.e., $(1-\pi_0)^2\lambda^2\text{var}[\kappa S_\theta]$), and panel (c) plots the risk-premium component (i.e., $(\sigma^2(1-\lambda)\alpha Z)^2\text{var}[\kappa]$). The other parameters are set as in Figure 6.

Limiting Cases: No uncertainty (or learning) about other traders

In this subsection, we present two natural limiting cases of the general model. First, we characterize the equilibrium for the case in which $\pi_t = 1$. This setting is analogous to a standard rational expectations environment. Next, we consider the other extreme, when $\pi_t = 0$. In this case, U investors do not condition on the price when updating their beliefs about the fundamental value of the asset and thus it is analogous to a standard difference of opinions (or Walrasian) setting. The analysis implies that our main results are driven by speculators' uncertainty, and learning, about θ investors — neither feature is present in these limiting cases.

Proposition 10. *If $\pi_0 = 1$ and $\theta = I$, there exists a unique linear, stationary equilibrium. The equilibrium is signal-revealing, and the price of the risky asset is given by $P_t = A\mu + BD_t + CS_{\theta,t} + F$, where $A = \frac{R(1-\rho)}{(R-1)(R-\rho)}$, $B = \frac{\rho}{R-\rho}$, $C = \frac{\lambda}{R-\rho}$, and $F = -\frac{1}{2(R-1)}\left(\frac{C^2}{\lambda} + (1+B)^2(1-\lambda)\right)\sigma^2\alpha Z$. Conditional on $S_{\theta,t}$, expected returns and volatility* *are given by*

$$\mathbb{E}_t[Q_{t+1}|S_{\theta,t}] = \tfrac{1}{2}\alpha \mathrm{var}_t[Q_{t+1}|S_{\theta,t}]Z, \qquad \textit{and} \tag{76}$$

$$\mathrm{var}_t[Q_{t+1}|S_{\theta,t}] = \sigma^2 \left(\tfrac{C^2}{\lambda} + (1+B)^2(1-\lambda)\right) \tag{77}$$

When θ investors are informed and U investors are certain about this, the price is linear in the signal $S_{\theta,t}$ and informationally efficient. The expected return is constant, and reflects only the risk-premium that investors require for holding the risky asset. The conditional volatility of returns is also constant since the equilibrium price is linear in $S_{\theta,t}$.

Proposition 11. *If $\pi_0 = 0$ and $\theta = NI$, there exists a unique linear, stationary equilibrium. The equilibrium is signal-revealing, and the price of the risky asset is given by $P_t = A\mu + BD_t + CS_{\theta,t} + F$, where $A = \frac{R(1-\rho)}{(R-1)(R-\rho)}$, $B = \frac{\rho}{R-\rho}$, C is the unique solution to*

$$C = \frac{\lambda\left(R^2\lambda + C^2(R-\rho)^2\right)}{\left(R^2(2-\lambda)\lambda + 2C^2(R-\rho)^2\right)(R-\rho)}, \tag{78}$$

and $F = -\frac{1}{R-1}\frac{(C^2/\lambda+(1+B)^2)(C^2/\lambda+(1+B)^2(1-\lambda))}{(C^2/\lambda+(1+B)^2+C^2/\lambda+(1+B)^2(1-\lambda))}\sigma^2\alpha Z$. Conditional on $S_{\theta,t}$, expected returns and volatility are given by

$$\mathbb{E}_t[Q_{t+1}|S_{\theta,t}] = \tfrac{(C^2/\lambda+(1+B)^2)(C^2/\lambda+(1+B)^2(1-\lambda))}{(C^2/\lambda+(1+B)^2+C^2/\lambda+(1+B)^2(1-\lambda))}\sigma^2\alpha Z - RCS_{\theta,t}, \qquad \textit{and} \tag{79}$$

$$\mathrm{var}_t[Q_{t+1}|S_{\theta,t}] = \sigma^2\left(\tfrac{C^2}{\lambda} + (1+B)^2\right) \tag{80}$$

Again, without uncertainty about θ, the price is linear in $S_{\theta,t}$ and return volatility is constant. Though θ investors are not informed, they believe they have payoff relevant information — as a result, the price responds to realizations of $S_{\theta,t}$. However, since the signals are spurious, the price is expected to mean-revert in the next period, and this induces predictability in expected returns through the $-RCS_{\theta,t}$ term in expression (79).

Proofs of Propositions 10 and 11. One can conjecture and verify the specified price function in each case. In particular, suppose $P_{t+1} = A\mu + BD_{t+1} + CS_{\theta,t+1} + F$. Since $S_{\theta,t+1}$ is uncorrelated with d_{t+1}, we have that optimal demand for investor i is given by:

$$x_{i,t} = \frac{\mathbb{E}_{i,t}\left[P_{t+1} + D_{t+1}\right] - RP_t}{\mathrm{var}_{i,t}\left[P_{t+1} + D_{t+1}\right]} \tag{81}$$

$$= \frac{A\mu + (B+1)((1-\rho)\mu + \rho D_t + \mathbb{E}_{i,t}\left[d_{t+1}\right]) + F - RP_t}{(B+1)^2\mathrm{var}_{i,t}\left[d_{t+1}\right] + C^2(\sigma^2 + \sigma_\varepsilon^2)}. \tag{82}$$

This implies that the equilibrium is signal-revealing, since the optimal demand for θ investors is linear in $S_{\theta,t}$. Moreover, note that for $\pi_t = 1$ and $\theta = I$, we have

$$\mathbb{E}_{i,t}\left[d_{t+1}\right] = \lambda S_{\theta,t}, \text{ and } \mathrm{var}_{i,t}\left[d_{t+1}\right] = \sigma^2(1-\lambda), \tag{83}$$

for $i \in \{U, \theta\}$, while for $\pi_t = 0$ and $\theta = N$, we have

$$\mathbb{E}_{\theta,t}\left[d_{t+1}\right] = \lambda S_{\theta,t}, \text{ and } \mathrm{var}_{\theta,t}\left[d_{t+1}\right] = \sigma^2(1-\lambda), \tag{84}$$

$$\mathbb{E}_{U,t}\left[d_{t+1}\right] = 0, \text{ and } \mathrm{var}_{U,t}\left[d_{t+1}\right] = \sigma^2. \tag{85}$$

Plugging in these beliefs into the optimal demand for each type of investor, and imposing the market clearing condition (i.e., $x_{U,t} + x_{\theta,t} = Z$) verifies the conjectured linear form. Matching coefficients implies uniqueness of the equilibrium. In particular, for $\pi_t = 1$ and $\theta = I$, we have

$$C = \frac{\lambda}{R-\rho}, \tag{86}$$

while for $\pi_t = 0$ and $\theta = N$, we can show that C is the solution to the cubic equation:

$$C = \frac{\lambda\left(R^2\lambda + C^2(R-\rho)^2\right)}{\left(R^2(2-\lambda)\lambda + 2C^2(R-\rho)^2\right)(R-\rho)}. \tag{87}$$

Since the discriminant of the above equation is less than zero, there is one real solution, which pins down the unique linear equilibrium in this case.

The expressions for expected returns and volatility in returns follow from plugging in the expression for price and computing the moments. □