The Common Factor in Idiosyncratic Volatility

Bernard Herskovic
NYU

Bryan Kelly
Chicago Booth

Hanno Lustig
UCLA Anderson

Stijn Van Nieuwerburgh
NYU

First Draft: November 1, 2012, This Draft: December 2013

Abstract

We show that firms' idiosyncratic volatility in returns and cash flows obeys a strong factor structure. We find that the stocks of firms with large, negative common idiosyncratic volatility (CIV) factor betas earn high average returns. The CIV beta quintile spread is 5.6% per year. To explain this spread, we develop a heterogeneous investor model with incomplete markets in which the idiosyncratic volatility of investor consumption growth inherits the factor structure of firm cash flow growth. In our model, the CIV factor is a priced state variable, because an increase in volatility represents a worsening of the investment opportunity set for the average investor. The calibrated model is able to match the high degree of comovement in idiosyncratic volatilities, the CIV beta spread, along with a host of asset price moments.

JEL: E3, E20, G1, L14, L25
Keywords: Firm volatility, Idiosyncratic risk, Cross-section of stock returns

1 Introduction

This paper presents three central findings. First, we document a strong factor structure in firm-level volatility, even after removing common variation in returns via factor models. Second, stocks that tend to lose value when common idiosyncratic volatility (CIV) rises earn comparatively high average returns. Third, to account for these findings, we develop a heterogeneous agent model with common idiosyncratic volatility in investors' consumption as well as the firms' cash flow processes, both measured by the CIV factor in stock returns. In our model, CIV is a priced state variable with a negative risk price. Our model generates comovement in cash flow and return volatility and a spread in stock returns based on CIV exposure that are quantitatively consistent with the data.

In our first analysis, we estimate monthly realized return volatilities for over 20,000 CRSP stocks firms over the 1926-2010 sample. The first principal component explains 39% of the variation in this panel. At first glance this may not appear surprising. A wide range of finance theories model returns as linear functions of common factors[1] – if the factors themselves have time-varying volatility, then firm-level volatility will naturally inherit a factor structure as well.

More surprising is that the firm-level volatility factor structure is effectively unchanged after accounting for common factors in returns. We examine residuals from factor models that include the Fama-French (1993) three factor, as well as statistical factor decompositions using as many as 10 principal components. Stock return residuals from these models possess an extremely high degree of common variation in their second moments. Residual volatility accounts for the vast majority (over 90%) of the variation in a typical stock's volatility, thus there is little distinction between total and idiosyncratic volatility at the firm level. Total and idiosyncratic volatility possess effectively the same volatility factor structure.

To emphasize that volatility comovement does not arise from omitted common factors,

[1] Prominent examples include the CAPM (Sharpe (1964)), ICAPM (Merton (1973)), APT (Ross (1976)) and the Fama and French (1993) model.

we show that return factor model residuals are virtually uncorrelated. Consider returns on the 100 Fama-French size and value portfolios. The average pairwise correlation between returns on these portfolios is 64%. But their residuals from a Fama-French three factor model or a five factor principal components model have pairwise correlations below 1% on average. However, correlations among the monthly volatilities of the 100 portfolios is 75% on average, and volatility comovement remains extremely high after removing common factors from returns. The average correlation among the idiosyncratic volatilities of the 100 portfolios is 54% based on the Fama-French factors and 59% based on a five factor principal components model. Therefore, omitted factors are not an viable explanation for comovement in volatilities.

Comovement in volatilities is not only a feature of returns, but also of the volatility of fundamentals. We estimate volatilities of firm-level sales growth using quarterly Compustat data. Despite the fact that these volatility estimates are far noisier than the return data, we again find a strong factor structure among total fundamental volatilities as well as among volatilities of cash flow factor model residuals. We thus argue that volatility patterns identified in this paper are not wholly (or even primarily) explicable with investor preferences or other pure discount rate variation. We know of no extant model in the firm growth or asset pricing literature that generates a factor structure in both fundamental and return volatilities through an economic mechanism.

To explore the asset pricing implications of the factor structure in firm-level volatility in returns and cash flows, we develop a heterogeneous investor model in which the idiosyncratic volatility of investor consumption growth inherits the same factor structure as firm cash flow growth. In this model, an increase in idiosyncratic volatility represents a deterioration of the investment opportunity set for the average investor, whose individual consumption growth is exposed to the idiosyncratic volatility factor.

In a large class of Breeden-Lucas-Rubinstein representative agent models, aggregate volatility, albeit of aggregate consumption growth or the market return, can be a priced state variable provided that she has a preference for early or late resolution of uncertainty, as pointed by Campbell (1993, 1996).[2] An increase in aggregate volatility raises the marginal utility of wealth for the stand-in investor if she has a preference for early resolution of uncertainty. The representative agent is willing to sacrifice some portion of her expected returns in exchange for insurance against a rise in volatility. However, she does not seek to hedge against innovations in idiosyncratic volatility (even if idiosyncratic volatility has common factor) because the idiosyncratic risk can be diversified.

We create a role for the idiosyncratic volatility factor by shutting down some markets. This breaks the aggregation result that is the foundation for all representative agent models. Instead, we propose an incomplete markets model in which investors' post-trade consumption is exposed to idiosyncratic risk. Our model is motivated by the incompleteness of household consumption insurance (see, e.g. Cochrane (1991), Attanasio and Davis (1996)). This market incompleteness implies that an increase in idiosyncratic risk at the firm level will carry over to the cross-sectional distribution of household consumption growth, via increased labor income risk, job loss risk and housing price risk that cannot be insured away. The local bias in households' financial portfolios (Coval and Moskowitz (1999)) exacerbates the exposure of their consumption to local, firm-driven shocks.

The main source of common variation in idiosyncratic shocks experienced by households and investors has to be the the employer, the firm, and the labor income, broadly defined, that these investors derive from the firm. While there many other sources of idiosyncratic risk (e.g., illness, divorce), these types of risks are less likely to have a factor structure in the volatility. A large literature documents an idiosyncratic volatility factor in labor income.[3]

Motivated by this fact, we exogenously impose the same common factor structure on the idiosyncratic volatility of consumption growth and on firm dividend growth. Indeed,

[2]Campbell, Giglio, and Polk (2012) extend the closed-form solutions to handle stochastic volatility.

[3]Storesletten, Telmer, and Yaron (2004) document evidence of counter-cyclical variation idiosyncratic labor income variance, while Guvenen, Ozkan, and Song (2012) conclude that the left-skewness is counter-cyclical.

Cogley (2002) and Brav, Constantinides, and Geczy (2002) document a counter-cyclical factor structure in the cross-sectional variance of U.S. household consumption growth.[4]

In our model, the average investor wants to hedge against an increase in idiosyncratic volatility, even if she is indifferent about the timing of uncertainty resolution. As a result, the common component in idiosyncratic volatility is a priced state variable, as in Constantinides and Duffie (1996).[5] Stocks with positive loadings on CIV shocks provide a hedge and earn a lower risk premium in equilibrium.

In general, exposure to constant uninsurable idiosyncratic risk will not affect equilibrium risk premia, but it will merely lower the risk-free rate and increase all securities prices in a large class of incomplete market models (see. e.g. Grossman and Shiller (1982) and Krueger and Lustig (2010)). Building on Mankiw (1986)'s insight, Constantinides and Duffie (1996) showed that counter-cyclical variation in idiosyncratic volatility can increase the equity risk premium in an equilibrium model with heterogeneous agents. It is exposure to the idiosyncratic volatility factor that is priced, not idiosyncratic volatility itself.

Measuring the cross-sectional dispersion in investors' consumption growth is hard (see Vissing-Jorgensen (2002) and Brav, Constantinides, and Geczy (2002) for recent examples), but our model gives us a license to use the CIV factor in stock returns as the priced factor, because investor consumption growth inherits its factor structure from the firms' cash flows. Only the common variation in the dispersion of investors' consumption growth matters for cross-sectional asset pricing. We provide empirical evidence that the common factor in idiosyncratic firm volatility is a priced state variable in the cross-section of U.S. stocks with a negative risk price, as predicted by the model. This cross-sectional evidence directly lends support to models with investor heterogeneity, and it cannot be reconciled with standard, representative agent models.

[4]Advances in the risk sharing technology could disentangle consumption from labor income risk, a theme explored by Krueger and Perri (2006), but we abstract from this.

[5]In a standard representative agent model, aggregate consumption growth volatility is only a priced factor if the agent has a preference for early or late resolution of uncertainty (Campbell (1993)). Bansal and Yaron (2004) exploit this mechanism for generating an aggregate volatility risk premium in an endowment economy.

Our paper complements the evidence in Ang, Hodrick, Xing, and Zhang (2006, 2009) who find that exposure to market volatility is priced in the cross section of stocks. They also document that that high idiosyncratic volatility stocks also earn low average returns. Our finding is distinct from the cross-sectional association between stock returns and exposure to market volatility or the level of idiosyncratic firm volatility. Instead, we establish that stocks with positive exposures to CIV earn lower returns, because, according to our model, the common idiosyncratic volatility in stock returns proxies for uninsurable investor consumption risk. Our calibrated model matches moments of the investor consumption and firm dividend distribution, the commonality in idiosyncratic volatility, and the spread in returns associated with exposures to the idiosyncratic volatility factor.

Other Related Literature Idiosyncratic volatility has been studied in several asset pricing contexts. Campbell, Lettau, Malkiel, and Xu (2001a) examine secular variation in average idiosyncratic volatility over time, though do not study the cross section properties of idiosyncratic volatility. This gave rise to several papers that explore this fact in more detail, such as Bennett, Sias, and Starks (2003), Irvine and Pontiff (2009), and Brandt, Brav, Graham, and Kumar (2010), including analyzing which firm characteristics correlate with its idiosyncratic volatility. Wei and Zhang (2006) study aggregate time series variation in fundamental volatility. Bekaert, Hodrick, and Zhang (2010) find comovement in average idiosyncratic volatility across countries. We analyze comovement among volatilities at the firm-level for both returns and fundamentals. Our focus is on the joint dynamics of the entire panel of firm-level volatilities, which we document is a prominent empirical feature of returns and growth rates that is new to the literature.

Gilchrist and Zakrajsek (2010) also study the time series behavior of the average firm-level volatility of the idiosyncratic component of returns. They explore the impact of uncertainty on corporate bond prices in a structural model. Recently, Atkeson, Eisfeldt, and Weill (2013) study the distribution of volatility across financial and non-financial firms to make inference about their financial soundness, and confirm our findings. In closely related work, Engle and Figlewski (2012) document a common factor in option-implied volatilities since 1996, and Barigozzi, Brownlees, Gallo, and Veredas (2010) and Veredas and Luciani (2012) examine the factor structure in realized volatilities of intra-daily returns since 2001. Bloom, Floetotto, Jaimovich, and Terry (2012) shows that firm-specific sales growth and productivity show cross-sectional dispersion that fluctuates with the macroeconomy.

As long as cash flow volatility is idiosyncratic, it is valued by stock market investors (see, e.g., Pastor and Veronesi (2003, 2009)), because of the convex relation between cash flow growth variance and terminal value. In our model, the stocks of firms which experience high idiosyncratic cash flow volatility will endogenously inherit a larger exposure to the CIV factor, because a positive volatility innovation increases the value of the stock more than that of other stocks. This in turn will lower the stock's equilibrium risk premium and increase its valuation, potentially helping to resolve the Ang, Hodrick, Xing, and Zhang (2006, 2009) idiosyncratic risk puzzle.

In related work, Constantinides and Ghosh (2013) explore the asset pricing implications of a factor structure not only in the cross-sectional volatility of household consumption growth, but also in the higher-order moments, but they do not explore the cross-sectional asset implications, which are the focus of our work, and they do not connect the factor in the cross-sectional moments of consumption growth to the factor structure in firm-level cash flow growth.

To account for the factor structure in idiosyncratic volatility, Kelly, Lustig, and Van-Nieuwerburgh (2013) propose a simple model in which firms are connected to other firms in a customer-supplier network. Firms' idiosyncratic growth rate shocks, which are homoskedastic, are transmitted in part to their trading partners. Differences in firms' network connections, and evolution of the network over time, impart total firm volatility with cross section and time series heteroskedasticity. Firm-level volatilities exhibit a common factor structure where the factor is firm size dispersion in the economy. In the model, each supplier's network is a random draw from the entire population of firms, so that any firm's customer network inherits similar dispersion to that of the entire size distribution. An increase in dispersion slows down every firm's shock diversification and increases their volatility. In the data, we find that firm volatilities possess a strong factor structure, and we show that size dispersion explains 25% of the variation in (realized) firm volatilities, as much as is explained by average volatility, a natural benchmark.[6]

The rest of the paper is organized as follows. Section 2 describes the data, section 3 describes the CIV factor in U.S. stock returns and firm-level cash flows, while section 4 establishes that CIV is a priced factor. Section 5 describes the heterogeneous agent model with CIV as priced state variable. Finally, section 6 calibrates a version of this model.

2 Data

2.1 Data Construction

To document these facts, we present evidence in the form of firm-year volatility panels. Return volatility is estimated each year for each CRSP stock as the standard deviation of the roughly 250 daily returns within the year. Fundamental volatility is estimated each year for all Compustat firms using the four quarterly year-on-year sales growth observations within the year. We also show that our return volatility results are robust to using twelve monthly returns within each year rather than daily returns to calculate volatility. Similarly, we show that our fundamental volatility results are robust to estimating volatility with a

[6]The factor structure implies strong time series correlations between moments of the size and volatility distributions. An increase in the size dispersion translates into higher average volatility among firms. It also raises the cross section dispersion in volatilities. In the time series, size dispersion has a 72% correlation with mean firm volatility and 79% with the dispersion of firm volatility. Our paper is the first to provide an economic explanation for the factor structure in firm-level volatility by connecting it to firm concentration. A persistent widening in the firm size dispersion should lead to a persistent rise in mean firm volatility. We observe such a widening (increase in firm concentration) between the early 1960s and the late 1990s, providing a new explanation for the trend in mean firm volatility studied by Campbell et al. (2001)Campbell, Lettau, Malkiel, and Xu (2001b).

five-year rolling window of quarterly observations (rather than one year of quarterly data), which reduces estimation noise.

The focus of our analysis is on idiosyncratic volatility. Idiosyncratic returns are constructed within each calendar year τ by estimating a factor model using all observation within that year (we estimate it for all firms with no missing observations during the year). Our factor models are of the form

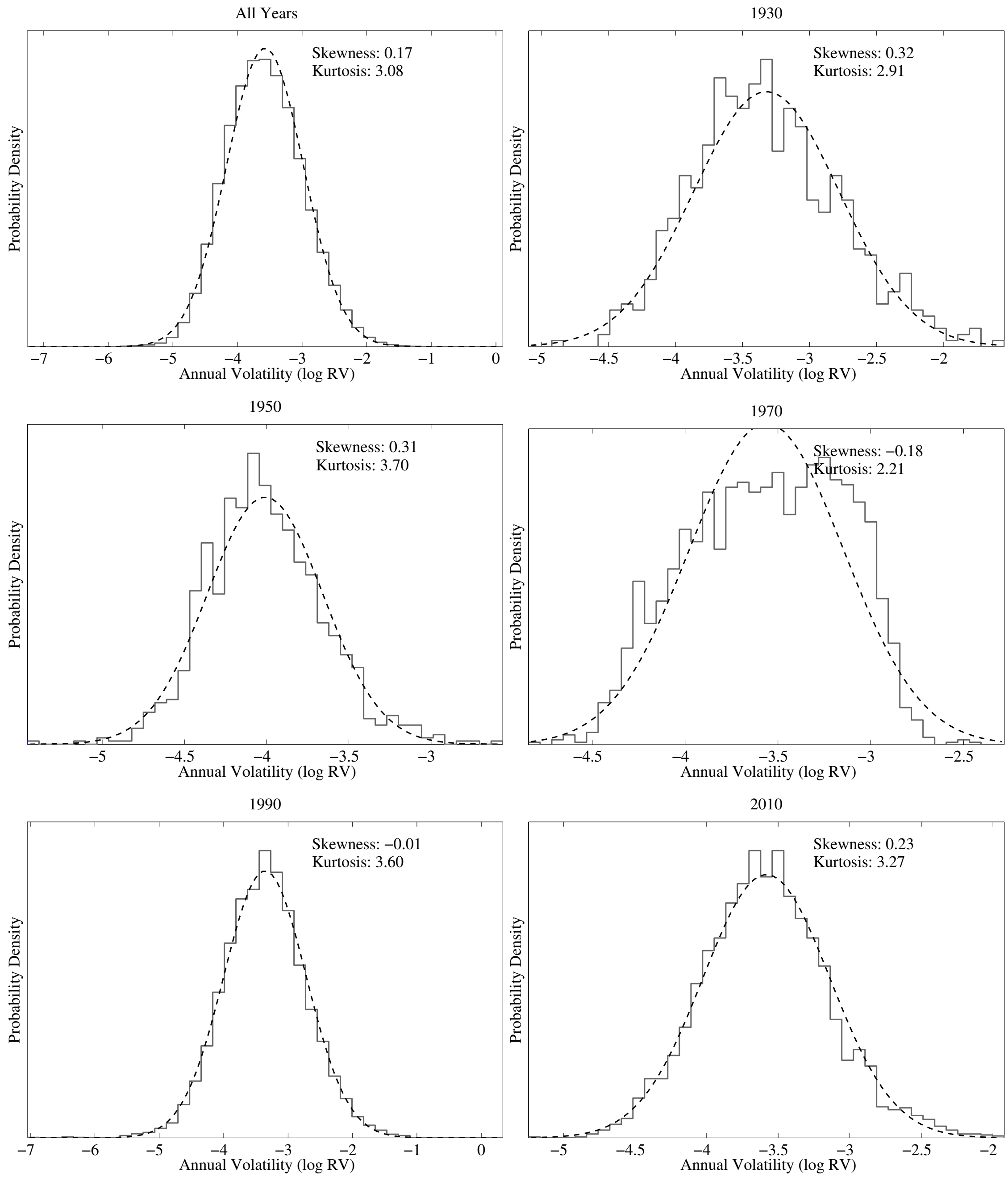
$$r_{i,t} = \gamma_{0,i} + \boldsymbol{\gamma}_i' \boldsymbol{F}_t + \varepsilon_{i,t} \tag{1}$$

and use all date t return observations in the year (where the frequency of t is either daily or monthly). A firm's idiosyncratic volatility is then calculated as the standard deviation of residuals $\varepsilon_{i,t}$ within the calendar year. The result of this procedure is a panel of firm-year idiosyncratic volatility estimates. The first return factor model that we consider specifies $\boldsymbol{F}_t$ as the 3×1 vector of Fama-French (1993) factors.[7] The second return factor model we use is purely statistical. In this case, $\boldsymbol{F}_t$ contains the first $K \times 1$ principal components of returns within the year, where we allow K to range between one and ten.

We estimate idiosyncratic volatility of firm fundamentals analogously. Since there is no single predominant factor model for sales growth in the literature, we only consider principal components as factors. The approach is the same as in Equation 1, with the exception that the left hand side variable is sales growth, and the frequency of t is quarterly. $\boldsymbol{F}_t$ contains the first $K \times 1$ principal components of growth rates within a five-year window ending in year τ, and residual volatility in year τ is estimated from the four model residuals within that year. Again, the number of principal components K ranges from one to ten.

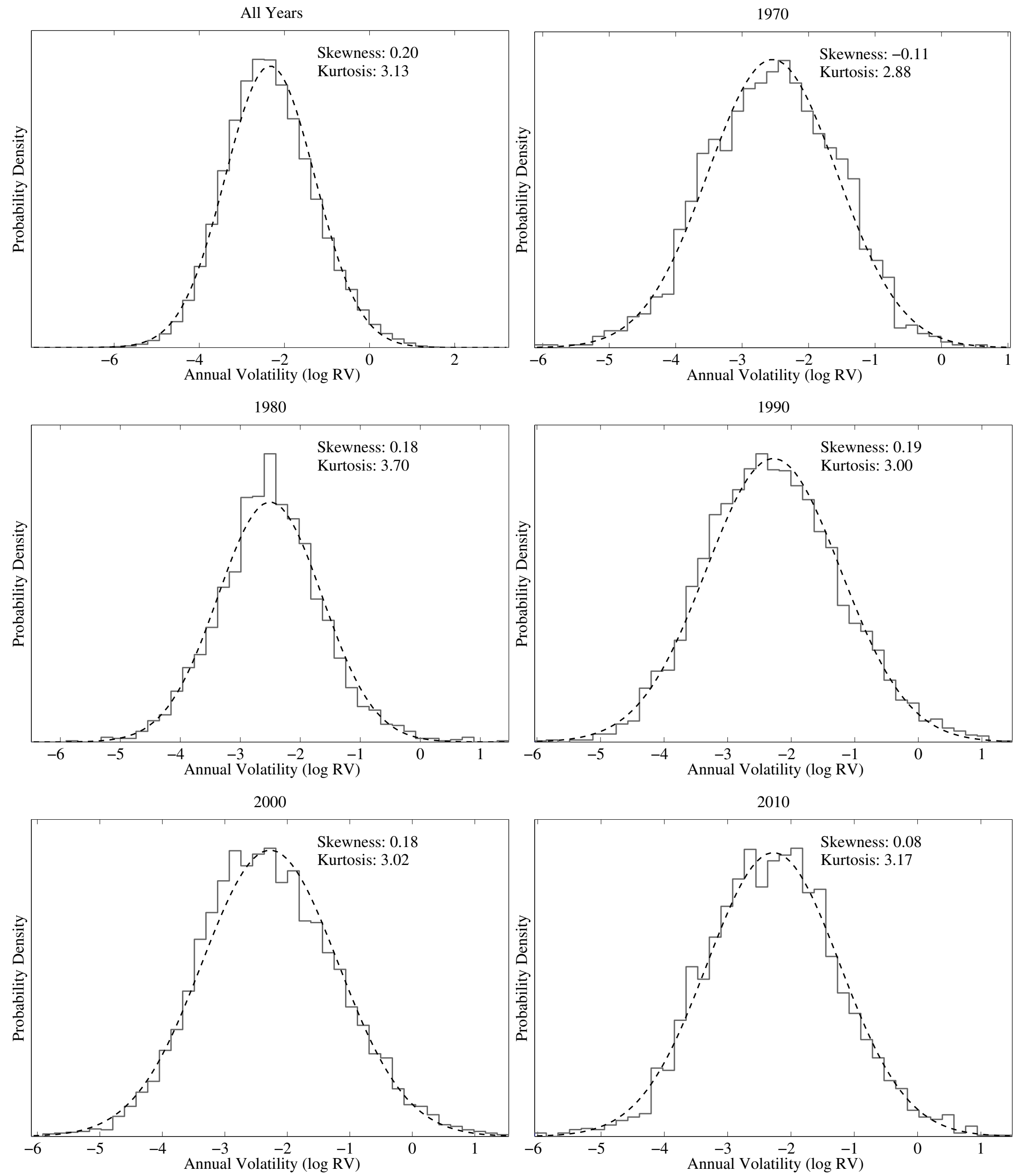
[7] Our estimates diverge slightly from the standard Fama-French model in which returns in excess of the risk free rate are the left-hand side variables, and the excess market return is the first factor. We use gross returns on the left-hand side, and the gross market return as the first factor.

Figure 1: Total Return Log Volatility: Empirical Density Versus Normal Density



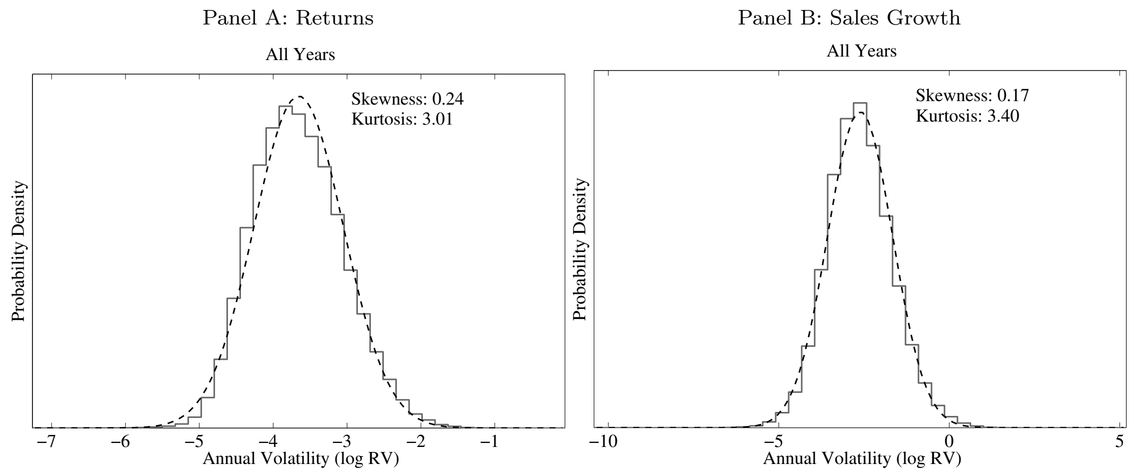
Notes: The figure plots histograms of the empirical cross section distribution of annual firm-level volatility (in logs). Within each calendar year, we calculate the standard deviation of daily returns for each stock. The upper left-hand corner histogram pools all years (1926-2010). Selected one-year snapshots of the firm volatility cross section distribution are also show (for years 1930, 1950, 1970, 1990 and 2010). Overlaid on these histograms is the exact normal density with mean and variance set equal to that of the empirical distribution. Each figure reports the skewness and kurtosis of the data in the histogram.

Figure 2: Log Sales Volatility: Empirical Density Versus Normal Density



Notes: The figure plots histograms of the empirical cross section distribution of annual firm-level volatility (in logs). Within each calendar year, we calculate the standard deviation of four quarterly observations of year-on-year sales growth for each stock. The upper left-hand corner histogram pools all years (1926-2010). Selected one-year snapshots of the firm volatility cross section distribution are also show (for years 1970, 1980, 1990, 2000 and 2010). Overlaid on these histograms is the exact normal density with mean and variance set equal to that of the empirical distribution. Each figure reports the skewness and kurtosis of the data in the histogram.

Figure 3: Log Idiosyncratic Volatility: Empirical Density Versus Normal Density



Notes: The figure plots histograms of the empirical cross section distribution of annual idiosyncratic firm-level volatility (in logs). Within each calendar year, we calculate the standard deviation of residuals from a factor model for daily returns for each stock (left panel) or quarterly sales growth (right panel). In both cases, residuals are constructed from a five factor principal components model. For returns, principal components are estimated from daily data within the year, while for sales growth a five year rolling window of quarterly data is used. The histograms pool all years (1926-2010 for return volatility, 1975-2010 for sales growth volatility). Overlaid on these histograms is the exact normal density with mean and variance set equal to that of the empirical distribution. Each figure reports the skewness and kurtosis of the data in the histogram.

3 The Factor Structure in Volatility

3.1 The Cross Section Distribution of Volatility

We begin by noting that the cross-sectional distributions of return volatility and fundamental volatility are lognormal to a close approximation, which motivates us to estimate our factor models using volatility in logs rather than levels.

We plot histograms of the empirical cross section distribution of firm-level volatility (in logs). The upper left-hand corner of Figure 1 shows the distribution of log realized volatility pooling all firm-years from 1926-2010. The figure also shows empirical distributions for selected one-year snapshots throughout the sample (years 1930, 1950, 1970, 1990 and 2010). Overlaid on these histograms is the exact normal density with mean and variance set equal to that of the empirical distribution, and each figure reports the skewness and kurtosis of the data in the histogram.

The pooled histograms and each of the snapshots (with the exception of 1970) look nearly normally distributed. They demonstrate only slight skewness (typically less than 0.3 in absolute value) and do not appear to be substantively leptokurtotic.

Figure 2 reports cross section distributions of yearly sales growth volatility (in logs) for all CRSP/Compustat firms. Fundamental volatility also appears to closely fit a lognormal distribution, with skewness no larger than 0.3 and kurtosis never exceeding 3.5.

While Figures 1 and 2 demonstrate near lognormality of total return and growth rate volatility, the same feature holds for residual volatility. Figure 3 shows the distributions of log idiosyncratic return volatility (Panel A) and log idiosyncratic fundamental volatility (Panel B) pooling all firm-years. For both returns and sales growth, residuals are constructed from the five factor principal components model. The distributions are qualitatively identical to the empirical histograms for total volatility. They are nearly normal, with a slight amount of right skewness and mild excess kurtosis.

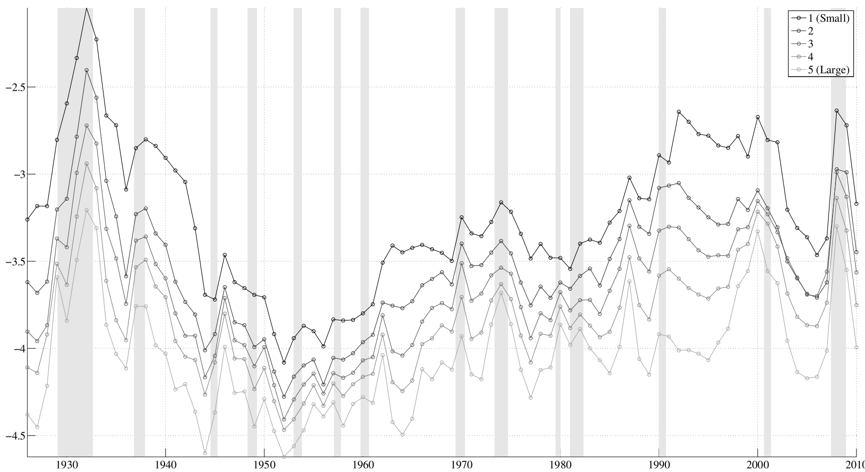
3.2 Common Secular Patterns in Firm-Level Volatility

3.2.1 Return Volatility

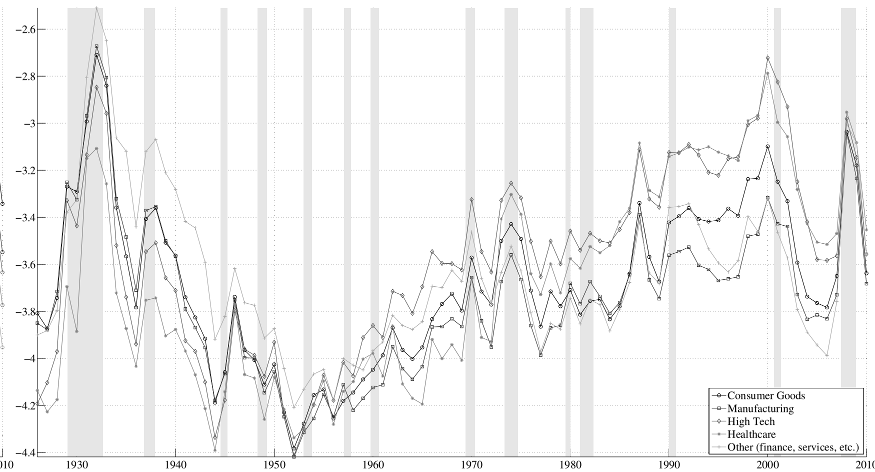
Next, we document common time variation in volatility across stocks. Panel A of Figure 4 plots firm-level log total return volatility, averaged within size quintiles. There is a striking degree of common variation in the volatilities of the largest quintile and smallest quintile of stocks. The same is true of industry groups. Panel B reports average return volatilities among the stocks in the five-industry categorization of SIC codes provided on Ken French's website. This is perhaps unsurprising given that firm-level returns are believed to have a substantial degree of common return variation, as evidenced by the predominance of factor-based models of individual stock returns. If returns have common factors and the volatility of those factors varies over time, then firm-level variances will also inherit a factor structure.

Figure 4: Log Total and Idiosyncratic Return Volatility by Size and Industry Group

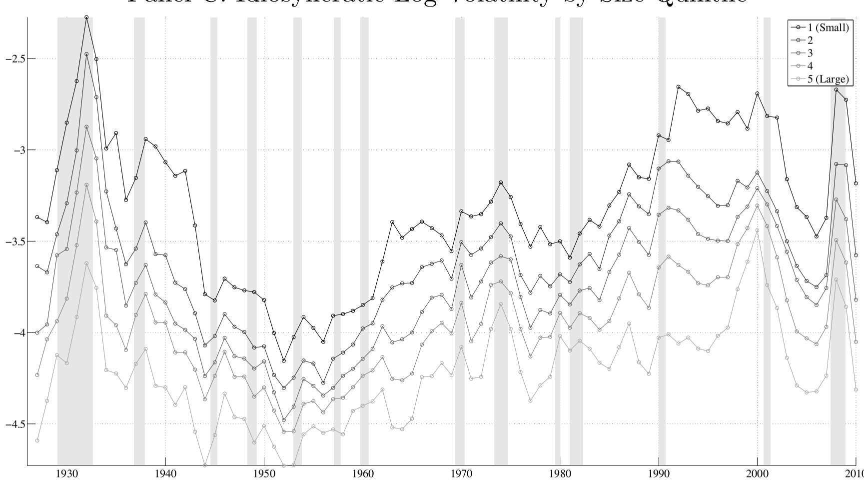
Panel A: Total Log Volatility by Size Quintile



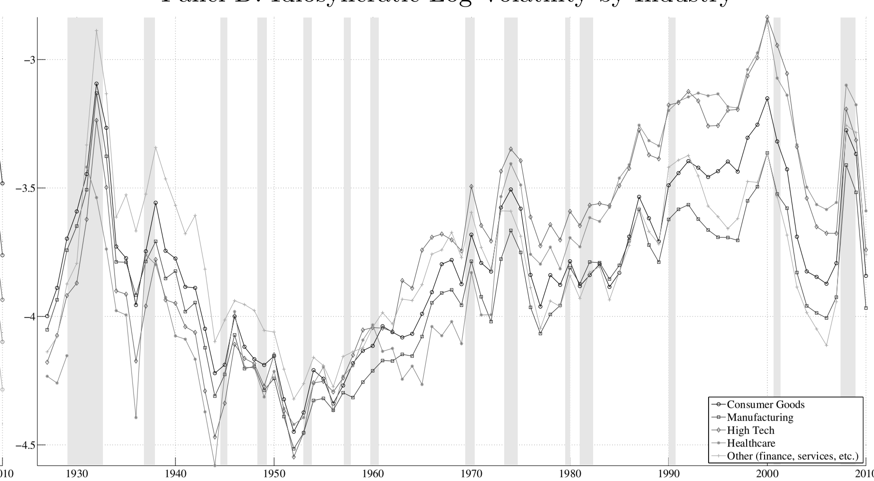
Panel B: Total Log Volatility by Industry



Panel C: Idiosyncratic Log Volatility by Size Quintile



Panel D: Idiosyncratic Log Volatility by Industry



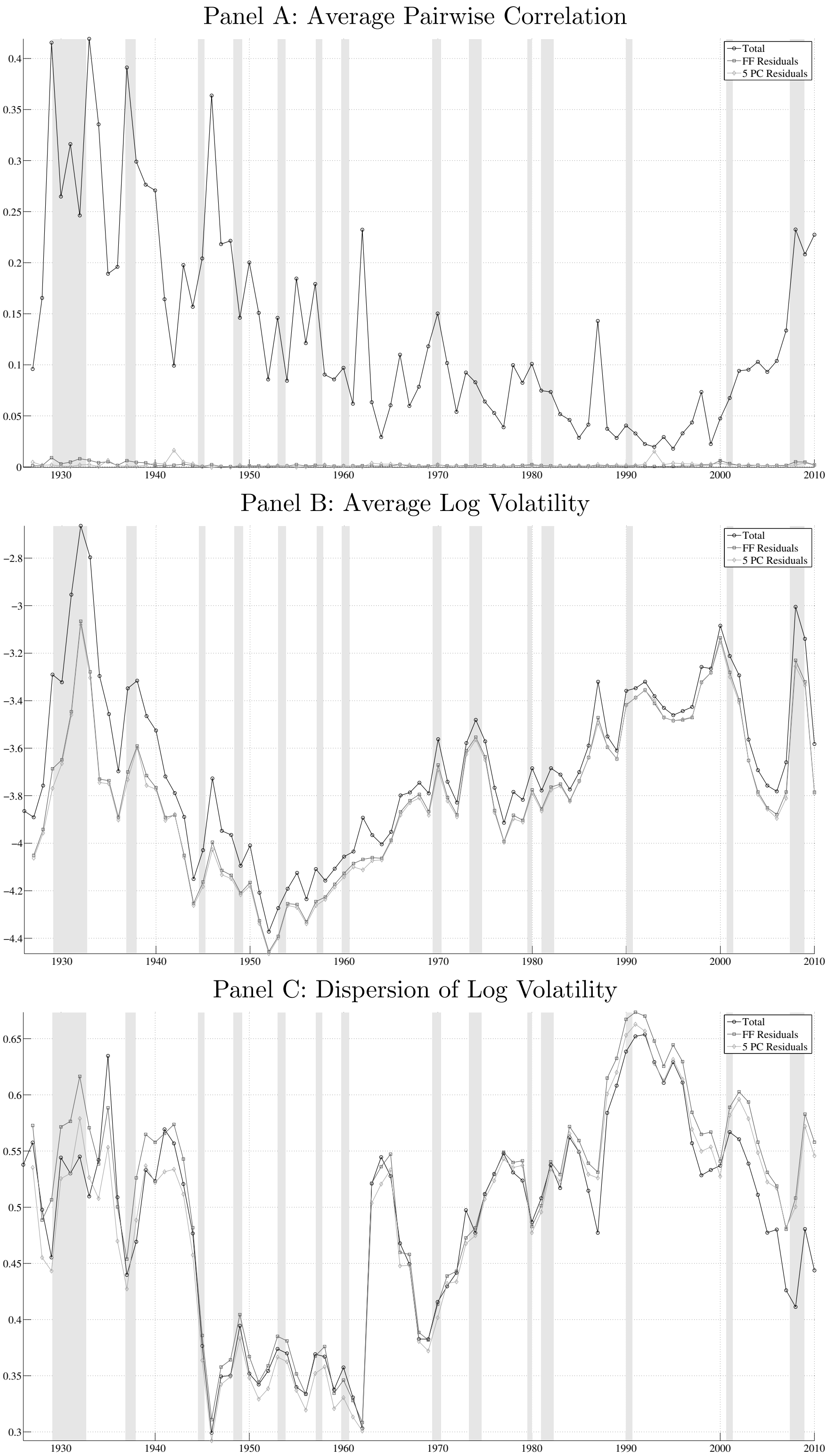
Notes: The figures plot firm-level log volatility averaged within size and industry groups. Within each calendar year, volatilities are estimated as the standard deviation of daily returns for each stock. Panel A shows firm-level log total return volatility averaged within market equity quintiles. Panel B shows log total return volatility averaged within the five-industry categorization of SIC codes provided on Ken French's website. Panels C and D report the same within-group averages of firm-level log idiosyncrantic volatility. Idiosyncratic volatility is the standard deviation of residuals from a five factor principal components model for daily returns.

What is surprising is that volatilities of residuals display the same degree of common variation despite the fact that common return factors have been removed. Instead of averaging total volatility within size and industry groups, Panels C and D plot average residual volatility from a factor model that uses the first five principal components of returns as factors. The plots show that the same dynamics appear for all groups of firms when considering idiosyncratic rather than total volatility. The correlation between average log idiosyncratic volatility within size quintiles one and five is 83%. The lowest correlation among the five industry groups is 67%, which is for idiosyncratic volatilities of firms in the healthcare industry versus those in the "other" category (including construction, transportation, services, and finance).

This common variation is idiosyncratic and cannot be explained by excess comovement among factor model residuals, for instance due to omitted common factors. Figure 5 shows how firms' idiosyncratic daily return volatility estimates are affected by using different factor models for returns. It compares raw returns to residuals from the Fama-French three factor model, as well as to residuals from a five factor principal components model. Panel A shows that raw returns share substantial common variation, with an average pairwise correlation of 13% over the 1926-2010 sample. However, the Fama-French model captures effectively all of this common variation at the daily frequency, as correlations among its residuals are less than 0.2% on average, and are never above 0.9% in a year. The same is true for the principal components model, whose residual correlation is also below 0.2% on average.

The interesting fact is that the common variation in returns, which is well accounted for by these factor models, is responsible for very little of the total variation in returns. Panel B of Figure 5 shows that the average log idiosyncratic volatility from the factor models is virtually the same as average volatility of total returns. In the typical year, only 4% of average log total volatility is accounted for by the five principal components factor model, while average log idiosyncratic volatility inherits 96% of the average total volatility level (the same is true for the Fama-French model).

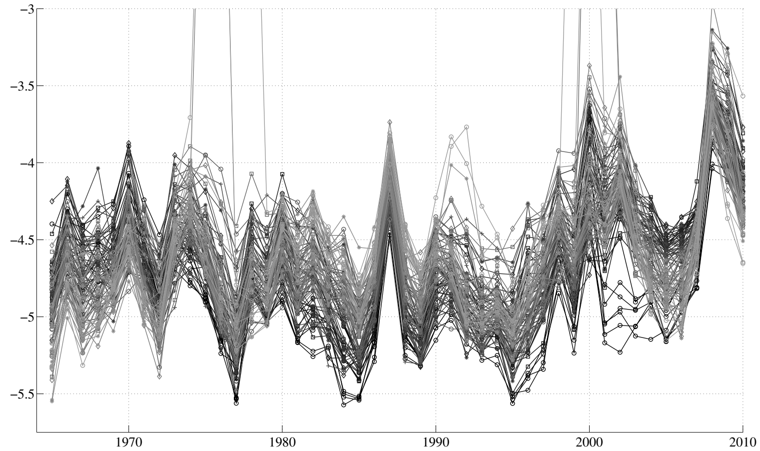
Figure 5: Volatility and Correlation of Daily Returns



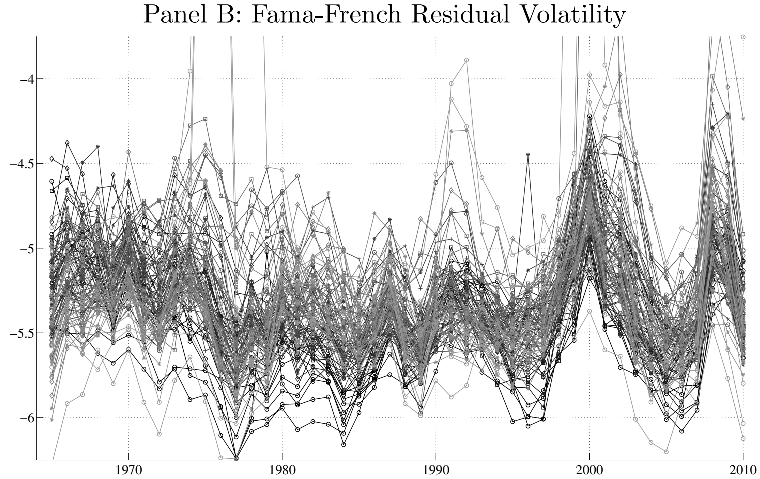
Notes: Panel A shows cross section average firm-level log volatility each year for total and idiosyncratic returns. Panel B shows the cross section standard deviation of firm-level log volatility and Panel C shows the average pairwise correlation for total and idiosyncratic returns within each calendar year. Idiosyncratic volatility is the standard deviation of residuals from the three factor Fama-French model or a five factor principal components model for daily returns (factor models also estimated within each calendar year).

Figure 6: Volatility of 100 Size and Value Portfolios

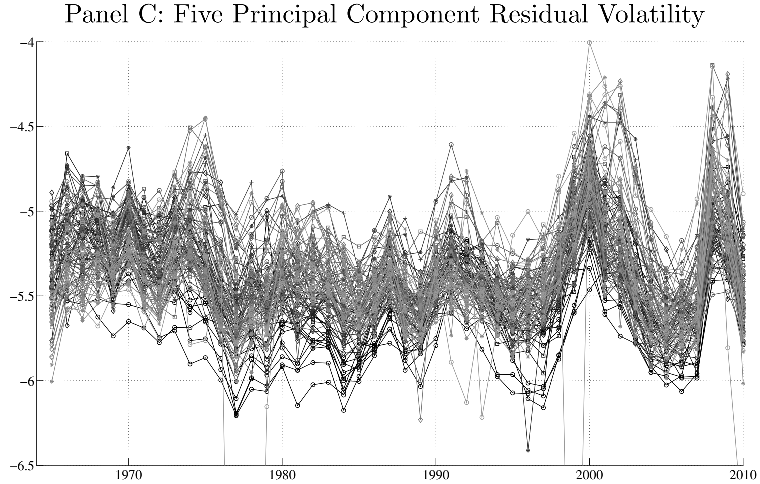
Panel A: Total Volatility



Panel B: Fama-French Residual Volatility

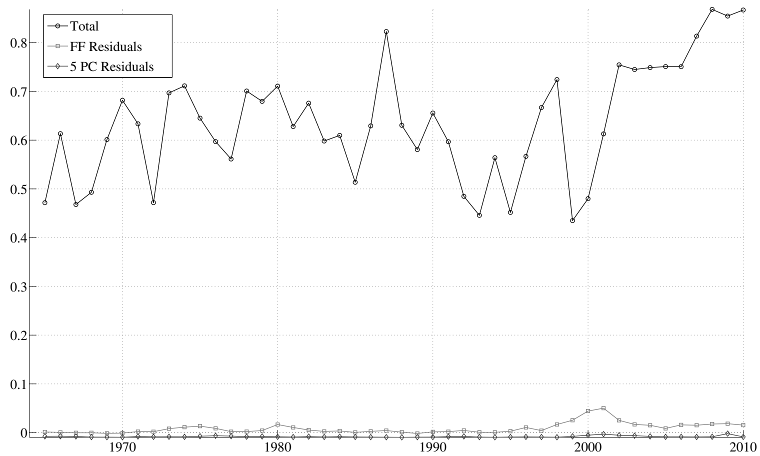


Panel C: Five Principal Component Residual Volatility



Notes: The figures plot log volatility of total and idiosyncratic returns on 100 size and value portfolios. Within each calendar year, total return volatilities are estimated from daily returns for each portfolio (Panel A), while idiosyncratic return volatility is the standard deviation of residuals from the three factor Fama-French model (Panel B) or a five factor principal components model (Panel C) for daily returns (factor models also estimated within each calendar year).

Figure 7: Average Pairwise Correlation of 100 Size and Value Portfolios



Notes: The figure shows average pairwise correlation for total and idiosyncratic returns on 100 size and value portfolios within each calendar year (refer to Figure 5 for details).

Similarly, the dispersion in firms' log volatility is more or less unaffected by removing common factors, as shown in Panel C. Since log volatilities are approximately normally distributed, Panels B and C contain most of the relevant information about the cross section of firm volatilities over time. In short, commonalities among returns have very little influence on the commonalities in return volatilities. The cross section distribution of total volatility and idiosyncratic volatility are qualitatively identical.

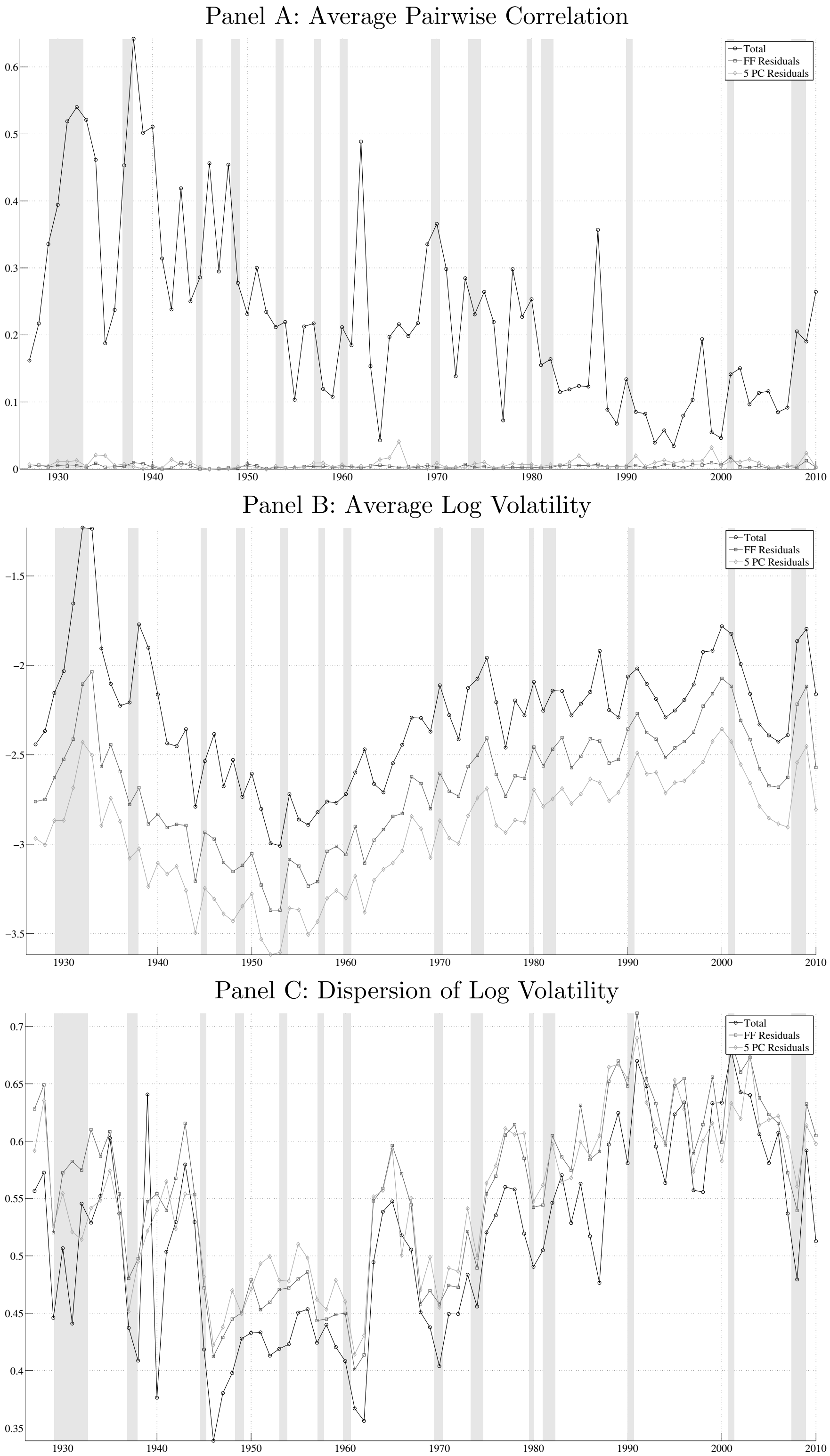
The strong comovement of return volatility is similarly discernible in portfolio returns. Figure 6 reports annual volatilities of the 100 Fama-French size and value portfolios. These are also calculated from daily returns within the year over the 1964-2010 sample (the first available full year of Ken French's data is 1964). Panel A shows log total volatility, Panel B shows log idiosyncratic volatility using the Fama-French three factor model, and Panel C shows idiosyncratic volatilities for a five factor principal components model. Portfolio volatilities show a strikingly similar degree of comovement across the size and book-to-market spectrum, even after accounting for common factors. Like the individual stock results above, factor models remove the vast majority of common variation in returns, thus common volatility patterns are unlikely to be driven by omitted common return factors. This can be seen clearly in Figure 7. Raw portfolio returns have an average pairwise correlation of 64% between 1964 and 2010, while the correlation of factor model residuals is below 1% on average for both models. However, the average pairwise correlation between portfolio volatilities remain high whether total or idiosyncratic volatilities are analyzed. The average pairwise correlation of the volatility series in Figure 6 Panel A is 77%, falling only to 54% and 59% in Panels B and C, respectively.

One may be concerned that daily factor models miss some portion of common variation among returns due to non-synchronicity in when aggregate information is incorporated into individual stock prices. To address this, we re-estimate factor models and firm-level idiosyncratic volatilities using data at the monthly frequency, and re-plot average correlations, average volatilities, and the dispersion of volatilities in Figure 8. Panel A shows that, indeed, there is a higher correlation among monthly raw returns relative to daily, with an average pairwise correlation of 23% over the 1926-2010. At the monthly frequency the Fama-French model continues to captures nearly all common variation, with correlations below 0.4% on average. The five factor principal components model has monthly residual correlation below 0.8% on average. At the monthly frequency, 22% of average log total volatility is accounted for by the principal components factor model, while average log idiosyncratic volatility inherits 78% of the average total volatility level. Thus, monthly return factor models do explain a larger fraction of the total return variation, but return volatility continues to be dominated by idiosyncratic rather than common variation. It is also worth noting that the bulk of the idiosyncratic volatility literature estimates volatility from daily data (e.g. Ang et al. (2006)).

3.2.2 Fundamental Volatility

Strong comovement among volatilities is not distinct to return volatilities, but is also true for fundamental volatility. Figure 9 reports average yearly sales growth volatility (in logs) by size quintile and French's five-industry categories (Panels A and B). Despite the fact that yearly sale growth volatilities are estimated from only four observations per year, the data

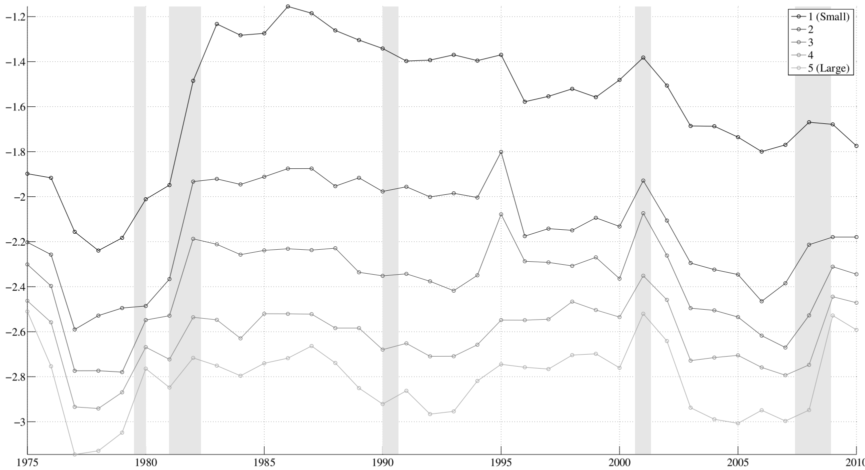
Figure 8: Volatility and Correlation of Monthly Returns



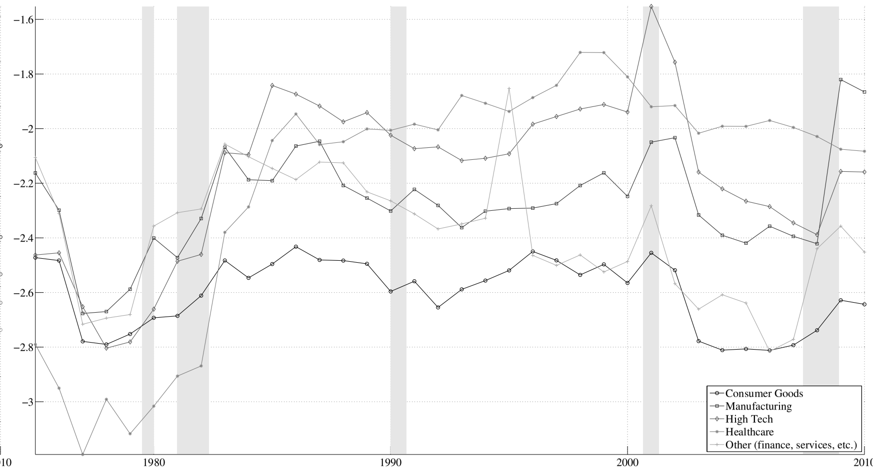
Notes: The figure repeats the analysis of Figure 5 using monthly return observations within each calendar year, rather than daily.

Figure 9: Log Total and Idiosyncratic Sales Growth Volatility by Size and Industry Group

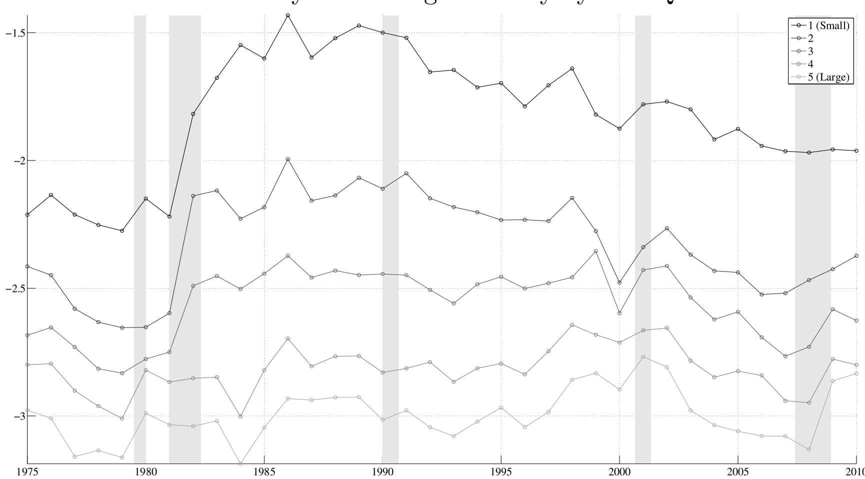
Panel A: Total Log Volatility by Size Quintile



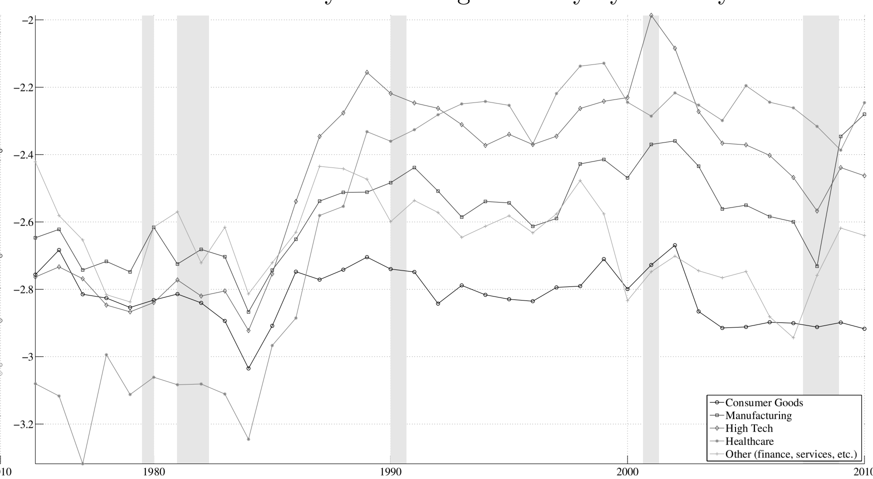
Panel B: Total Log Volatility by Industry



Panel C: Idiosyncratic Log Volatility by Size Quintile

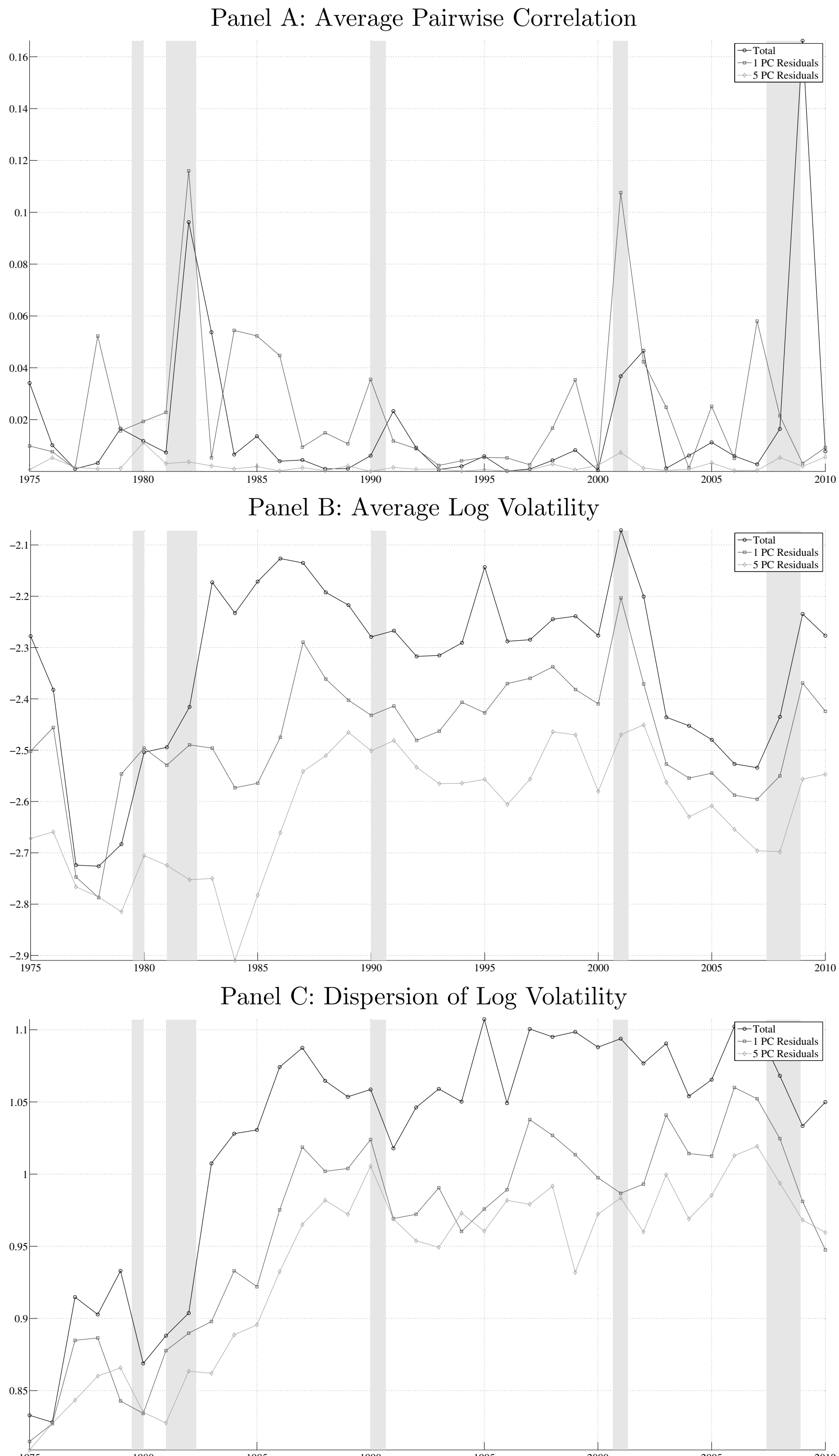


Panel D: Idiosyncratic Log Volatility by Industry



Notes: The figures plot firm-level log volatility averaged within size and industry groups. Within each calendar year, volatilities are estimated as the standard deviation of four quarterly year-on-year sales growth observations for each stock. Panel A shows firm-level log total volatility averaged within market equity quintiles. Panel B shows log total return volatility averaged within the five-industry categorization of SIC codes provided on Ken French's website. Panels C and D report the same within-group averages of firm-level log idiosyncrantic volatility. Idiosyncratic volatility is the standard deviation of residuals (four observation within each year) from a five factor principal components model for quarterly sales growth. The components are estimated in a five year rolling window ending in the year that the residual volatility is calculated.

Figure 10: Volatility and Correlation of Total and Idiosyncratic Sales Growth



Notes: Panel A shows cross section average firm-level log volatility each year for total and idiosyncratic sales growth. Panel B shows the cross section standard deviation of firm-level log volatility and Panel C shows the average pairwise correlation for total and idiosyncratic returns within each calendar year. Idiosyncratic volatility is the standard deviation of residuals from a one or five factor principal components model for quarterly sales growth. The components are estimated in a five year rolling window ending in the year that the residual volatility is calculated.

continues to display a high degree of volatility commonality.

This is a feature of both total and residual volatility of fundamentals. Panels C and D show within-group average log idiosyncratic volatility estimated from a five factor principal components model for sales growth. These panels display the same volatility patterns as those in the top two panels.

A common factor model for firms' sales growth is perhaps less relevant than that for returns, as shown in Panel A of Figure 10. The average pairwise sales growth correlation in the 1975-2010 sample is only 2%, though it reaches as high as 17% in 2009. Accounting for common factors with a five principal component factor model lowers these correlations to below 0.3% on average, with correlations reaching a high of only 1% in 1980.

Panel B shows that, like returns, average idiosyncratic volatility of fundamentals shares the same broad pattern as total volatility (correlation of 59%), and inherits 89% of the average total volatility (11% is accounted for by the factor model). Given the near lognormality of sales growth volatility in the cross section, along with the same overall patterns between the cross section mean and standard deviation of for the total volatility and idiosyncratic volatility distribution, we conclude that idiosyncratic volatility rather than common variation drives the entire panel of firm-level fundamental volatilities.

3.3 Volatility Factor Model Estimates

Next, we estimate a one-factor model for volatility. We consider total volatility, as well as idiosyncratic volatility estimated from a Fama-French three factor model or a K factor principal component model ($K = 5$ or 10). In all cases, time series regressions are run firm-by-firm, use log volatility as the left-hand side variable, and the right-hand side factor is an equally weighted average of the left-hand size volatility measure across all firms.

Our first set of results, shown in Panel A of Table 1, reports factor model results for daily return volatilities. Columns correspond to the factor model used to construct return

Table 1: Log Volatility Factor Model Estimates

Panel A: Daily Returns	Total	FF	5 PCs	10 PCs
Loading (average)	0.925	0.920	0.924	0.925
Loading (median)	0.911	0.884	0.886	0.888
Loading (25th %ile)	0.446	0.362	0.363	0.355
Loading (75th %ile)	1.387	1.400	1.405	1.412
Intercept (average)	-0.181	-0.201	-0.191	-0.190
Intercept (median)	-0.272	-0.354	-0.351	-0.344
Intercept (25th %ile)	-1.946	-2.324	-2.312	-2.361
Intercept (75th %ile)	1.491	1.584	1.585	1.617
R^2 (average univariate)	0.363	0.349	0.352	0.351
R^2 (pooled)	0.385	0.346	0.356	0.357
Panel B: Portfolio Returns	Total	FF	5 PCs	10 PCs
Loading (average)	1.000	1.000	1.000	1.000
Loading (median)	0.982	0.907	0.989	0.985
Loading (25th %ile)	0.914	0.789	0.869	0.868
Loading (75th %ile)	1.049	1.046	1.090	1.116
Intercept (average)	0.000	0.000	0.000	0.000
Intercept (median)	-0.122	-0.485	-0.055	-0.097
Intercept (25th %ile)	-0.368	-1.172	-0.777	-0.750
Intercept (75th %ile)	0.313	0.224	0.562	0.642
R^2 (average univariate)	0.760	0.547	0.597	0.606
R^2 (pooled)	0.628	0.441	0.497	0.474

Table 1: Log Volatility Factor Model Estimates, Continued

Panel C: Monthly Returns	Total	FF	5 PCs	10 PCs
Loading (average)	0.918	0.888	0.872	0.926
Loading (median)	0.918	0.847	0.841	0.848
Loading (25th %ile)	0.416	0.193	0.050	-0.935
Loading (75th %ile)	1.417	1.524	1.657	2.746
Intercept (average)	-0.099	-0.195	-0.276	-0.251
Intercept (median)	-0.152	-0.348	-0.408	-0.603
Intercept (25th %ile)	-1.258	-1.998	-2.558	-8.191
Intercept (75th %ile)	1.005	1.405	1.875	7.574
R^2 (average univariate)	0.266	0.214	0.186	0.126
R^2 (pooled)	0.288	0.207	0.180	0.087
Panel D: Sales Growth	Total (1yr)	1 PC	10 PCs	Total (5yr)
Loading (average)	0.876	0.849	0.938	0.897
Loading (median)	0.777	0.776	0.889	0.864
Loading (25th %ile)	-0.711	-0.852	-0.876	-0.489
Loading (75th %ile)	2.401	2.523	2.652	2.207
Intercept (average)	-0.231	-0.211	-0.096	-0.262
Intercept (median)	-0.514	-0.432	-0.271	-0.476
Intercept (25th %ile)	-3.983	-3.379	-4.647	-4.530
Intercept (75th %ile)	3.327	2.767	4.210	3.820
R^2 (average univariate)	0.140	0.229	0.127	0.144
R^2 (pooled)	0.174	0.168	0.167	0.283

Notes: The table reports estimates for one factor regression models of yearly log volatility. In each panel, the single volatility factor is the equal weighted average of all firms' log volatilities within that year. Thus all estimated volatility factor models take the form: $\log \sigma_{i,t} = \text{intercept}_i + \text{loading}_i \cdot \overline{\log \sigma_{\cdot,t}} + e_{i,t}$. Columns represent different volatility measures. For returns (Panels A through C), the first column represents estimates for a factor model of log total return volatility, the second column for idiosyncratic volatility based on Fama-French model residuals, and the third and fourth columns to idiosyncratic volatility from one and five factor principal component models. For sales growth volatility (Panel D), the last column reports model estimates for yearly volatilities estimated in a rolling 20 quarter window to reduce estimation noise. We report means and quantiles of the empirical distribution of firm-level intercepts and volatility factor loadings, as well as time series regression R^2 average over all firms. We also report a pooled factor model R^2, which compares the estimated factor model to a model with only a firm-specific constant.

residuals. The mean loading of an individual return volatility on the volatility factor is 0.925 for total return volatility, and is 0.920, 0.924 and 0.925 for idiosyncratic volatility based on the Fama-French, five PC and ten PC models, respectively. Median loadings are similar. The inter-quartile ranges for loadings span the interval from 0.355 to 1.412. The average firm's intercept is between −0.181 and −0.201, with slightly higher median intercept and inter-quartile ranges covering −2.361 to 1.617. The average univariate time series R^2 is 38.5% for the total volatility model, and around 35% for idiosyncratic volatility models. Pooling all volatilities, we find a pooled R^2 34.9% and 36.3% (relative to a volatility model with only a firm-specific constant).

In Panel B we re-estimate the volatility factor model using daily return volatility of 100 size and value portfolios. The interquartile range of loadings on the volatility factor go from 0.789 to 1.116, and are between −1.172 and 0.642 for intercepts. The common variation among idiosyncratic portfolio volatility is exceptional, with average time series R^2 between 54.7% and 64.2%, and pooled R^2 between 44.1% and 49.7%.

Panel C shows volatility factor model estimates for monthly (rather than daily) return volatilities. The picture is broadly similar to daily results. The average (median) firm has a loading between 0.826 and 0.926 (0.776 to 0.918) and an intercept of −0.099 to −0.276 (−0.152 to −0.603). The average time series R^2 is between 12.6% for ten factor model residuals and 26.6% for raw returns.

In Panel D we show volatility factor model estimates for sales growth volatility. The first three columns report total volatility, and idiosyncratic volatility from one and five principal component models in which volatility is estimated from four quarterly observations within each year. The last column reports model estimates for an annual volatility panel that uses a rolling 20 quarter window to estimate each firm-year's volatility.

Due to the excessively small number of observations used to construct volatility, we might expect poorer fit in these regressions. Yet the results are closely in line with those for return volatility. The average firm has a volatility factor loading of between 0.849 and 0.938, with an intercept between -0.096 and -0.262. The time series R^2 for raw and idiosyncratic growth rate volatility ranges between 12.7% and 22.9% on average. The pooled R^2 reaches as high as 28.3% when volatilities are estimated in a 20 quarter window.

4 Cross Section of Stock Returns

In this section we investigate whether pervasive fluctuations in firms' idiosyncratic volatility are associated with differences in average returns across stocks. We proceed by constructing a common idiosyncratic variance factor, CIV, as the equal weighted average of CAPM residual variances computed each month.[8] We then calculate innovations to CIV based on a monthly AR(1) model. Finally, we orthogonalize these innovations with respect to innovations in monthly market variance. This orthogonalization disentangles our CIV exposures from the market variance exposures studied by Ang et al. (2006).

Each month from 1926 until 2010, we regress monthly individual firm stock returns on orthogonalized CIV innovations using a trailing 60-month window. These CIV betas are used to sort stocks into CIV quintile portfolios, for which we construct value-weighted returns over the subsequent month. Stocks in the lowest quintile (Q1) have low/negative CIV betas and tend to lose value when CIV rises. In contrast, stocks in Q5 tend to hedge CIV rises, paying off in high volatility states.

Average returns on CIV beta-sorted portfolios are reported in Table 3. Panel A shows that average returns are decreasing in CIV beta. The spread between highest and lowest quintiles is -5.6% per year with a t-statistic of -2.7.

In Panel B we report portfolios sorted simultaneously on CIV beta and market variance beta, where rows correspond the the market variance beta dimension. This provides a

[8]We estimate the CAPM each month using all daily returns within the month. The CAPM residual variance is then estimated from the output of this regression. Results are qualitatively identical if residuals from alternative factor models, such as Fama-French or principal components, are used.

Table 3: Portfolios Formed on Common Idiosyncratic Variance Beta

Average returns on monthly CIV beta-sorted portfolios in percent per year. Panel A shows portfolios of all stocks. Panel B we report portfolios sorted simultaneously on CIV beta and market variance beta, where rows correspond the the market variance beta dimension. Panel C reports bivariate sorts based on CIV beta and stock-level idiosyncratic volatility.

	CIV Beta					
	Low				High	
	1	2	3	4	5	5-1
	Panel A: All Stocks					
VW Avg. Ret.	15.57	13.21	13.24	10.61	9.98	-5.59
t	5.10	5.33	5.84	5.26	4.38	-3.08
EW Avg. Ret.	20.57	17.76	17.10	15.81	14.55	-6.03
t	5.10	5.33	5.84	5.26	4.38	-3.08
VW CAPM Alpha	1.69	0.88	1.54	-0.18	-1.50	-3.19
t	1.37	1.22	2.84	-0.37	-1.91	-1.88
	Panel B: Market Variance Beta					
	VW Avg. Ret.					
Low 1	14.76	18.20	15.17	14.97	11.31	-2.66
2	16.13	14.53	13.33	12.28	13.29	-3.13
3	15.41	12.81	13.67	11.56	11.50	-3.91
4	17.90	12.91	12.79	11.77	11.15	-6.72
High 5	14.95	12.01	12.68	10.29	8.75	-6.29
5-1	1.00	-6.08	-2.49	-4.87	-2.11	
	t-stat					
Low 1	3.80	5.24	4.64	4.81	3.48	-1.09
2	4.68	5.09	4.76	4.89	4.66	-1.34
3	4.46	4.67	5.96	5.10	4.65	-1.58
4	5.64	4.83	5.82	5.53	4.59	-2.80
High 5	4.56	4.60	5.33	4.62	3.37	-2.17
5-1	0.36	-2.49	-1.03	-2.02	-0.80	
	Panel C: Idiosyncratic Stock Variance					
	VW Avg. Ret.					
Low 1	14.84	12.99	13.74	11.16	10.08	-5.10
2	15.64	13.19	13.83	11.02	11.86	-3.78
3	16.88	14.04	13.22	15.06	11.07	-5.81
4	15.33	13.87	12.05	12.24	11.79	-3.54
High 5	7.40	7.78	8.53	7.24	7.70	0.30
5-1	-8.52	-5.21	-5.21	-3.91	-2.38	
	t-stat					
Low 1	5.37	5.60	6.61	5.85	4.87	-2.57
2	5.01	4.84	5.28	4.67	4.84	-1.96
3	4.85	4.61	4.57	5.21	4.05	-2.64
4	4.33	4.11	3.62	3.89	3.54	-1.74
High 5	1.77	1.93	2.02	1.83	1.98	0.10
5-1	-2.82	-1.59	-1.61	-1.22	-0.77	

comparison with the results of Ang et al. (2006). We see that high CIV beta stocks continue to earn substantially lower average returns within each market beta quintile. While the 5-1 CIV beta spread is significant only in market beta Q4 and Q5, all quintiles show a spread of at least -2.7% per year.

Panel C reports bivariate sorts based on CIV beta and stock-level idiosyncratic volatility. Again, the CIV beta spread remains strong except for within the most extreme idiosyncratic volatility quintile (Q5). It is significant in 3 of the remaining 4 quintiles with a spread of at least -3.5% per year.

5 Model

This section studies an equilibrium asset pricing model with heterogeneous agents in the spirit of Constantinides and Duffie (1996) and Constantinides and Ghosh (2013). The idiosyncratic volatility factor, denoted σ_{gt}^2, is the key state variable which drives the residual return volatility in stocks, as well the the cross-sectional volatility of investor consumption growth. Innovations to this factor are priced, with a negative price of risk. Stocks with more negative exposure with respect to this innovation (a more negative "dispersion beta") carry a higher risk premium.

We start with a simple model that makes the point that idiosyncratic consumption risk can help explain the idiosyncratic volatility puzzle. In that model, the sole source of difference in dispersion betas arises from heterogeneity in the exposure of idiosyncratic cash-flow risk to the cross-sectional dispersion. That model provides most of the intuition. The next section presents a richer model that adds additional links between cross-sectional dispersion and cash flow growth, and that generates quantitatively larger risk premium differences.

5.1 Preferences

There is a unit mass of atomless agents. Each one of them has Epstein-Zin preferences. Let $U_t(C_t)$ denote the utility derived from consuming C_t. The value function of each agent takes the following recursive form:

$$U_t(C_t) = \left[(1-\delta)C_t^{\frac{1-\gamma}{\theta}} + \delta \left(E_t U_{t+1}^{1-\gamma}\right)^{\frac{1}{\theta}}\right]^{\frac{\theta}{1-\gamma}}.$$

The time discount factor is δ, the risk aversion parameter is $\gamma \geq 0$, and the inter-temporal elasticity of substitution (IES) is $\psi \geq 0$. The parameter θ is defined by $\theta \equiv (1-\gamma)/(1-\frac{1}{\psi})$. When $\psi > 1$ and $\gamma > 1$, then $\theta < 0$ and agents prefer early resolution of uncertainty.

Aggregate labor income is defined as I_t. There is a large number of securities in zero or positive net supply. There combined total (and per capita) dividends are D_t. Aggregate dividend income plus aggregate labor income equals aggregate consumption: $C_t = I_t + D_t$. Individual labor income is defined by

$$I_{j,t} = S_t^j C_t - D_t$$

All agents can trade in all securities at all times and are endowed with an equal number of all securities at time zero. As in Constantinides and Ghosh (2013), given the symmetric and homogenous preferences, households choose not to trade away from their initial endowments. That is, autarky is an equilibrium and individual j' equilibrium consumption is $C_{j,t} = I_{j,t} + D_t = S_t^j C_t$.

5.2 Technology

On the technology side, we impose the same idiosyncratic volatility factor structure on investor consumption growth and firm diviidend growth by adopting the following specification for aggregate consumption growth, consumption growth of agent j, and dividend growth of firm i:

$$\begin{aligned}
\Delta c_{t+1}^{a} &= \mu_g + \sigma_c \eta_{t+1} && (2)\\
\Delta s_{t+1}^{j} &= \sigma_{g,t+1} v_{t+1}^{j} - \frac{1}{2}\sigma_{g,t+1}^{2} && (3)\\
\Delta d_{t+1}^{i} &= \mu_i + \chi_i \left(\sigma_{gt}^{2} - \sigma_g^{2}\right) + \varphi_i \sigma_c \eta_{t+1} + \kappa_i \sigma_{gt} e_{t+1}^{i} + \zeta_i \sigma_{it} \varepsilon_{t+1}^{i} && (4)\\
\sigma_{g,t+1}^{2} &= \sigma_g^{2} + \nu_g \left(\sigma_{gt}^{2} - \sigma_g^{2}\right) + \sigma_{gw} w_{g,t+1} && (5)\\
\sigma_{i,t+1}^{2} &= \sigma_i^{2} + \nu_i \left(\sigma_{it}^{2} - \sigma_i^{2}\right) + \sigma_{iw} w_{i,t+1} && (6)
\end{aligned}$$

All shocks are i.i.d standard normal and mutually uncorrelated. Lowercase letters denote logs. The cross-sectional mean and variance of the consumption share process are:

$$\begin{aligned}
\mathbb{E}_j \left[\Delta s_{t+1}^{j}\right] &= -\frac{1}{2}\sigma_{g,t+1}^{2}\\
\mathbb{V}_j \left[\Delta s_{t+1}^{j}\right] &= \sigma_{g,t+1}^{2}
\end{aligned}$$

Thus, the idiosyncratic volatility factor, $\sigma_{g,t+1}$ is the cross-sectional standard deviation of consumption share growth. Individual consumption growth is $\Delta c_{t+1}^{j} = \Delta c_{t+1}^{a} + \Delta s_{t+1}^{j}$. The mean consumption share in levels is one: $\mathbb{E}_j \left[S_t^j\right] = 1$.

The conditional variance of aggregate consumption growth is constant. However, the conditional variance of dividend growth for an individual stock i is time varying and depends on the idiosyncratic volatility factor. The common factor in residual dividend growth volatility is the idiosyncratic volatility factor. We now show that positive innovations in the idiosyncratic volatility factor ($w_{g,t+1} > 0$) are associated with bad times and carry a negative price of risk. Thus, bad times are times with little risk sharing (high dispersion in equilibrium consumption share growth). Assets whose returns are low exactly when risk sharing is impaired must pay higher risk premia.

5.3 Claim to Individual Consumption Stream

We start by pricing a claim to individual consumption growth, using the individual's own intertemporal marginal rate of substitution (IMRS). We conjecture that the log wealth-consumption ratio of agent j is linear in the state variable σ^2_{gt}, and does not depend on any agent-specific characteristics:

$$wc^j_t = \mu_{wc} + W_{gs}\left(\sigma^2_{gt} - \sigma^2_g\right)$$

We verify this conjecture by plugging in this guess into the Euler equation for the consumption claim of agent j: $E_t[SDF^j_{t+1}R^j_{t+1}] = 1$. Under symmetric preferences, this conjecture implies that the individual wealth-consumption ratio does not depend on agent-specific attributes, only on aggregate objects.

The return to agent j's consumption claim r^j_{t+1} equals:

$$r^j_{t+1} = r^c_0 + \left[W_{gs}\left(\nu_g - \kappa^c_1\right) - \frac{1}{2}\nu_g\right]\left(\sigma^2_{gt} - \sigma^2_g\right) + \sigma_c\eta_{t+1} + \left(W_{gs} - \frac{1}{2}\right)\sigma_{gw}w_{g,t+1} + \sigma_{g,t+1}v^j_{t+1}$$

where r^c_0 in the unconditional mean and κ^c_1 is a linearization constant slightly exceeding 1. The intermediate steps are provided in the appendix, along with all pother derivations.

Epstein and Zin (1989) show that the log real stochastic discount factor is a function of consumption growth and the return to the consumption claim:

$$\begin{aligned} sdf^j_{t+1} &= \theta\log\delta - \frac{\theta}{\psi}\Delta c^j_{t+1} + (\theta-1)\, r^j_{t+1} \\ &= \mu_s + \left[(\theta-1)\, W_{gs}\left(\nu_g - \kappa^c_1\right) + \gamma\frac{1}{2}\nu_g\right]\left(\sigma^2_{gt} - \sigma^2_g\right) \\ &\quad -\gamma\sigma_c\eta_{t+1} - \gamma\sigma_{g,t+1}v^j_{t+1} + \left[(\theta-1)\, W_{gs} + \frac{1}{2}\gamma\right]\sigma_{gw}w_{g,t+1} \end{aligned}$$

where μ_s is the unconditional mean SDF.[9]

5.4 Aggregate SDF

Since all agents can invest in all risky assets, the Euler equation has to be satisfied for any two agents j and j' and for every stock i (with returns orthogonal to the agents' idiosyncratic income shocks v^j and $v^{j'}$). This also implies that the average SDF must also price all financial assets if all the individual SDFs price the return R^i_{t+1}:

$$\begin{aligned} 1 &= \mathbb{E}_t\left[SDF^j_{t+1}R^i_{t+1}\right] = \mathbb{E}_t\left[\mathbb{E}_j\left(SDF^j_{t+1}R^i_{t+1}\right)\right] = \mathbb{E}_t\left[\mathbb{E}_j\left(SDF^j_{t+1}\right)R^i_{t+1}\right] \\ &= \mathbb{E}_t\left[SDF^a_{t+1}R^i_{t+1}\right]. \end{aligned}$$

We can rewrite the expression for the average log real stochastic discount factor:

$$\begin{aligned} sdf^a_{t+1} &= \mathbb{E}_j\left[sdf^j_{t+1}\right] + \frac{1}{2}\mathbb{V}_j\left[sdf^j_{t+1}\right] \\ &= \mu_s + \frac{1}{2}\gamma^2\sigma^2_g + s_{gs}\left(\sigma^2_{gt} - \sigma^2_g\right) - \lambda_\eta\sigma_c\eta_{t+1} - \lambda_w\sigma_{gw}w_{g,t+1} \end{aligned}$$

where the loadings are given by:

$$\begin{aligned} s_{gs} &\equiv \frac{1}{2}\gamma\nu_g\left(\frac{1}{\psi}+1\right), \\ \lambda_\eta &\equiv \gamma, \\ \lambda_w &\equiv \frac{\gamma\nu_g\left(\frac{1}{\psi}-\gamma\right)}{2\left(\kappa^c_1-\nu_g\right)} - \frac{1}{2}\gamma(1+\gamma), \end{aligned}$$

[9] The appendix shows that the coefficient W_{gs} is given by:

$$W_{gs} = \frac{\nu_g\gamma(\gamma-1)}{2\theta(\kappa^c_1-\nu_g)} = -\frac{\gamma\nu_g\left(1-\frac{1}{\psi}\right)}{2\left(\kappa^c_1-\nu_g\right)} \tag{7}$$

If the IES exceeds 1, then $W_{gs} < 0$. Less risk sharing, or higher consumption share dispersion, leads to a lower wealth-consumption ratio.

Hence, there are two priced sources of aggregate risk in our model: shocks to aggregate consumption growth, which carry a positive price of risk λ_η, equal to the coefficient of relative risk aversion, and shocks to the idiosyncratic volatility factor. The latter carry a negative price of risk λ_w, indicating that deterioration in risk sharing is bad news for the stand-in agent, provided that the agent has a preference for early resolution of uncertainty. In the case of time-additive utility, the standard Mankiw (1986) result obtains and only the current volatility matters, but, in general, the average investor also cares about the future dispersion of consumption growth. The size of this effect is governed by the persistence (ν_g) of the idiosyncratic volatility factor. This result will allow us to directly use the volatility factor constructed from stock returns to test our asset pricing mechanism empirically rather than measure the cross-sectional dispersion of investor consumption growth directly.

The maximum Sharpe ratio in the economy is larger, the bigger these prices of risk and the more volatile the shocks:

$$\max SR_t = Std_t[sdf^a_{t+1}] = \sqrt{\lambda_\eta^2 \sigma_c^2 + \lambda_w^2 \sigma_{gw}^2}$$

It follows that the risk-free interest rate is:

$$\begin{aligned} r_t^f &= -\mathbb{E}_t[sdf^a_{t+1}] - \frac{1}{2}\mathbb{V}_t[sdf^a_{t+1}] \\ &= -\mu_s - \frac{1}{2}\gamma^2\sigma_g^2 - \frac{1}{2}\lambda_\eta^2\sigma_c^2 - \frac{1}{2}\lambda_w^2\sigma_{gw}^2 - s_{gs}\left(\sigma_{gt}^2 - \sigma_g^2\right) \end{aligned}$$

Interest rates contain the usual impatience and intertemporal substitution terms, included in μ_s. The next three terms capture the precautionary savings motive: when idiosyncratic risk is high, agents increase savings, lowering interest rates. Interest rates move negatively with the state variable because $s_{gs} > 0$. The higher consumption share dispersion, the lower rates.

5.5 Firm Return

Turning to the pricing of the dividend claim defined by equation (4), we guess and verify that its log price-dividend ratio is affine in the common and idiosyncratic variance terms:

$$pd_t^i = \mu_{pdi} + A_{gs}^i \left(\sigma_{gt}^2 - \sigma_g^2\right) + A_{is}^i \left(\sigma_{it}^2 - \sigma_i^2\right)$$

As usual, log returns are approximated as:

$$r_{t+1}^i = \Delta d_{t+1}^i + \kappa_0^i + \kappa_1^i pd_{t+1}^i - pd_t^i$$

Innovations in individual stock returns and the return variance reflect the additional sources of idiosyncratic risk:

$$\begin{aligned} r_{t+1}^i - \mathbb{E}_t\left[r_{t+1}^i\right] &= \beta_{\eta,i}\sigma_c\eta_{t+1} + \beta_{gs,i}\sigma_{gw}w_{g,t+1} + \kappa_i\sigma_{gt}e_{t+1}^i + \zeta_i\sigma_{it}\varepsilon_{t+1}^i + \kappa_1^i A_{is}^i\sigma_{iw}w_{i,t+1} \\ \mathbb{V}_t\left[r_{t+1}^i\right] &= \beta_{\eta,i}^2\sigma_c^2 + \beta_{gs,i}^2\sigma_{gw}^2 + \left(\kappa_1^i A_{is}^i\right)^2\sigma_{iw}^2 + \kappa_i^2\sigma_{gt}^2 + \zeta_i^2\sigma_{it}^2 \end{aligned}$$

where

$$\begin{aligned} \beta_{\eta,i} &\equiv \varphi_i, \\ \beta_{gs,i} &\equiv \kappa_1^i A_{gs}^i \end{aligned}$$

Innovations in stock returns contain two sources of aggregate risk and three sources of idiosyncratic risk (equation 8). The variance of individual stock returns are driven by the common σ_{gt} and idiosyncratic σ_{it} processes (equation 8). In the empirical section, we demonstrated the presence of a large first principal component in both total and residual stock returns, and showed that it was the same component in both. This model generates that feature and associates the common component in residual variance with changes in the cross-sectional dispersion of consumption growth across agents. Times of low risk sharing are times of high idiosyncratic (and total) stock return variance.

The coefficients of the price-dividend equation are obtained from the Euler equation:

$$A^i_{gs} = \frac{2s_{gs} + 2\chi_i + \kappa_i^2}{2\left(1-\kappa^i_1\nu_g\right)} = \frac{\left(1+\frac{1}{\psi}\right)\gamma\nu_g + 2\chi_i + \kappa_i^2}{2\left(1-\kappa^i_1\nu_g\right)} \tag{8}$$

$$A^i_{is} = \frac{\zeta_i^2}{2(1-\kappa^i_1\nu_i)} \tag{9}$$

and the constant μ_{pdi} is the mean log pd ratio which solves the following non-linear equation:

$$\begin{aligned} 0 &= r^i_0 + \mu_s + \frac{1}{2}\gamma^2\sigma_g^2 + \frac{1}{2}(\beta_{gs,i}-\lambda_w)^2\sigma^2_{gw} + \frac{1}{2}(\beta_{\eta,i}-\lambda_\eta)^2\sigma^2_c \\ &\quad + \frac{1}{2}\kappa_i^2\sigma_g^2 + \frac{1}{2}\zeta_i^2\sigma_i^2 + \frac{1}{2}\left(\kappa^i_1 A^i_{is}\right)^2\sigma^2_{iw} \end{aligned}$$

where

$$\begin{aligned} r^i_0 &= \mu_i + \kappa^i_0 + (\kappa^i_1 - 1)\mu_{pdi}, \\ \kappa^i_1 &= \frac{\exp(\mu_{pdi})}{1+\exp(\mu_{pdi})}, \qquad \kappa^i_0 = \log(1+\exp(\mu_{pdi})) - \kappa^i_1\mu_{pdi} \end{aligned}$$

The expression for the equity risk premium on an individual stock is:

$$\mathbb{E}_t\left[r^i_{t+1} - r^f_t\right] + .5\mathbb{V}_t[r^i_{t+1}] = \beta_{\eta,i}\lambda_\eta\sigma^2_c + \beta_{gs,i}\lambda_w\sigma^2_{gw}. \tag{10}$$

The first term is the standard consumption CAPM term. It is typically small because consumption growth is not very volatile. The second term is a new term which compensates investors for movements in the cross-sectional (income and) consumption distribution, today and in the future. Stocks that have low returns exactly when risk sharing deteriorates ($\beta_{gs,i} < 0$) are risky and carry high expected returns because $\lambda_w < 0$. If χ^i is sufficiently negative, $A^i_{gs} < 0$ and $\beta_{gs,i} < 0$. A negative χ_i is natural because empirically, poor risk sharing are bad economic times that are associated with lower future dividend growth.

In the cross-section, a stock with higher exposure to the common idiosyncratic risk term κ_i will have a higher (less negative) beta and therefore carry a lower expected return, all else equal. Since stocks with high idiosyncratic volatility have high exposures κ_i empirically, the model generates a lower expected return for high idiosyncratic risk stocks. That is, the model generates the idiosyncratic risk anomaly. Intuitively, when there is a positive shock to the volatility factor, the quantity of idiosyncratic risk goes up more so for stocks with greater exposure κ_i. As a result of the convexity in the relation between growth and terminal value, also explored by Pastor and Veronesi (2003, 2009)), this in turn raises the price more of those high volatility stocks, thus increasing their beta to the vol factor and lowering their equilibrium expected return. The next section discusses a calibrated version of this model to see how close it comes to explaining the idiosyncratic risk anomaly quantitatively.

6 Calibration

The model in the previous section cleanly illustrates how increases in idiosyncratic risk, resulting in less risk sharing among agents, affect interest rates and risk premia in the cross-section of stocks. We now add a few additional model ingredients while preserving the main intuition given in the simple model. The full model allows for an additional correlation between aggregate consumption growth and dividend growth with innovations in the risk sharing process $w_{g,t+1}$. Second, it allows for the volatility of these innovations to be time-varying rather than constant. The latter change makes risk premia time-varying. It also makes the volatility of the return on the market portfolio, defined as an asset that pays aggregate dividend growth (one without any idiosyncratic risk), time varying, which is a desirable feature. In the current version, the variability of the market return is driven solely by the idiosyncratic volatility factor. In reality, the two are positively but not perfectly correlated. In the next version of this paper, we plan to introduce a separate aggregate volatility factor, σ_{ct}. In sum, the following technology processes are modified:

$$\begin{aligned}
\Delta c^{a}_{t+1} &= \mu_g + \sigma_c \eta_{t+1} + \phi_c \sigma_{gt} w_{g,t+1} \\
\Delta d^{i}_{t+1} &= \mu_i + \chi^i \left(\sigma^2_{gt} - \sigma^2_g\right) + \varphi_i \sigma_c \eta_{t+1} + \phi_i \sigma_{gt} w_{g,t+1} + \kappa_i \sigma_{gt} e^i_{t+1} + \zeta_i \sigma_{it} \varepsilon^i_{t+1} \\
\sigma^2_{g,t+1} &= \sigma^2_g + \nu_g \left(\sigma^2_{gt} - \sigma^2_g\right) + \sigma_w \sigma_{gt} w_{g,t+1}
\end{aligned}$$

The market portfolio is modeled as a claim to aggregate dividend growth:

$$\Delta d^{a}_{t+1} = \mu_m + \chi^m \left(\sigma^2_{gt} - \sigma^2_g\right) + \varphi_m \sigma_c \eta_{t+1} + \phi_m \sigma_{gt} w_{g,t+1}$$

Appendix B works out the details of this model extension.

The variance of an individual stock return now takes the following form:

$$\mathbb{V}_t \left[r^i_{t+1}\right] = \beta^2_{\eta,i} \sigma^2_c + \beta^2_{gs,i} \sigma^2_{gt} + \left(\kappa^i_1 A^i_{is}\right)^2 \sigma^2_{iw} + \kappa^2_i \sigma^2_{gt} + \zeta^2_i \sigma^2_{it} \tag{11}$$

where

$$\begin{aligned}
\beta_{\eta,i} &\equiv \varphi_i, \\
\beta_{gs,i} &\equiv \kappa^i_1 A^i_{gs} \sigma_w + \phi_i,
\end{aligned}$$

Appendix B provides the expressions for A^i_{gs}, which measures the sensitivity of the *pd* ratio to changes in the cross-sectional variance of consumption growth σ^2_{gt}, and A^i_{is}, which captures the sensitivity to changes in stock-specific variance σ^2_{it}. The dependence of A^i_{gs} on the parameters of the model is similar to that in equation (8) while the expression for A^i_{is} is identical as in (9). The former is usually negative while the latter is always positive. The equity risk premium on any stock i is:

$$\mathbb{E}_t \left[r^i_{t+1} - r^f_t\right] + .5\mathbb{V}_t[r^i_{t+1}] = \beta_{\eta,i} \lambda_\eta \sigma^2_c + \beta_{gs,i} \lambda_w \sigma^2_{gt}. \tag{12}$$

Similarly, the variance of the market portfolio is:

$$\mathbb{V}_t\left[r^m_{t+1}\right] \;=\; \beta^2_{\eta,m}\sigma^2_c + \beta^2_{gs,m}\sigma^2_{gt} \tag{13}$$

The model contains three new features. First, equations (11) and (13) show that the variance of stock returns moves around over time not only via the idiosyncratic sources of return volatility -as in the previous model- but also via the priced source of risk associated with $w_{g,t+1}$ shocks. Market variance is high when risk sharing is poor. Second, the equity risk premium in equation (12) is now time-varying rather than constant. This variation arises because of the stochastic volatility in the σ^2_{gt} process. Third, we built in an additional cash-flow channel that relates innovations in cross-sectional income dispersion $w_{g,t+1}$ to changes in dividend growth and aggregate consumption growth. The former changes the beta of a stock. If contemporaneous deteriorations in risk sharing coincide with low dividend growth realizations ($\phi_i < 0$) for a stock, then that stock has a lower (more negative) beta. Given the negative price of risk, it will carry a higher expected return. The latter effect ($\phi_c < 0$) changes the price of risk λ_w and makes it more negative. It is natural to associate recessions (negative aggregate consumption growth episodes) with periods where risk sharing deteriorates. Hence, the contemporaneous cash-flow effects ($\phi_c < 0$ and $\phi_i < 0$) increase the equity risk premium, all else equal.

The following objects are useful in what follows. The idiosyncratic stock return variance is the variance of the idiosyncratic return components:

$$\mathbb{V}_t\left[r^{idio,i}_{t+1}\right] = \left(\kappa^i_1 A^i_{is}\right)^2\sigma^2_{iw} + \kappa^2_i\sigma^2_{gt} + \zeta^2_i\sigma^2_{it}$$

Define the idiosyncratic variance (IV) factor as the first principal component of the idiosyncratic return variance, as in section 2:

$$IV_t \equiv \mathbb{V}_t\left[r^{IV}_{t+1}\right] = \bar{\kappa}^2\sigma^2_{gt} + \bar{\zeta}\sigma^2_i,$$

where the last term is constant by virtue of the iid nature of the σ_{it} processes in the cross-section of stocks.

Two key parameters are κ_i and ζ_i. The following relationships identify these parameters. First, we can infer κ_i from a regression of idiosyncratic stock return variance on the idiosyncratic variance factor:

$$\frac{Cov\left(\mathbb{V}_t\left[r_{t+1}^{idio,i}\right], IV_t\right)}{Var[IV_t]} = \frac{\kappa_i^2}{\bar{\kappa}^2} \tag{14}$$

This regression slope is informative about κ_i, holding all other parameters fixed. Second, the R-squared of that regression is informative about ζ_i, all else equal:

$$R^2 = 1 - \frac{\zeta_i^4 \frac{\sigma_{iw}^2}{1-\nu_i^2}}{\kappa_i^4 \frac{\sigma_w^2 \sigma_g^2}{1-\nu_g^2} + \zeta_i^4 \frac{\sigma_{iw}^2}{1-\nu_i^2}} \tag{15}$$

Table 5 shows our parameter choices; the model is calibrated and simulated at annual frequency. Risk aversion γ is set to 10 and the inter-temporal elasticity of substitution ψ is set to 2, both common values in the consumption-based asset pricing literature. The time discount factor δ is set to 0.91, which produces a mean risk-free rate of 0.96% per year, given all other parameters. The model produces a risk-free rate with low volatility of 0.91% per year. Mean consumption growth μ_g is 2% per year. We set σ_c to 1.5%. We set ϕ_c equal to -0.22 to capture the negative correlation between aggregate consumption growth and the degree of risk sharing. Aggregate consumption growth volatility is modest at 2.67% per year.

We set the mean of the cross-sectional dispersion in consumption growth, σ_g, to 10%. This value is a compromise between the data, which -while noisy- indicate a higher value of dispersion and our ability to solve the model.[10] The persistence of the cross-sectional dispersion process, ν_g, is set to an intermediate value of 0.77 per year. This choice will produce annual persistence in idiosyncratic return volatility around 0.77, close to the 0.77 in

[10] For high values of σ_g, the equation for the mean price-dividend ratio no longer has a solution.

the data. This choice implies that our main state variable moves at business cycle rather than at much lower frequencies. We set σ_w to 1.5%. This ensures that σ_{gt}^2 remains positive, while creating substantial action in the extent of risk sharing over time. The time series standard deviation of σ_{gt} is 1.18%. The model results in a negative market price of "dispersion risk" $\lambda_w = -5.24$ and a substantial maximum conditional Sharpe ratio of 0.55.

Section 2 documented facts about firm-level volatilities. Our calibrated model aims to match both firm volatility behavior as well as understand the asset pricing implications of this behavior. The model suggests a sort of firms in terms of their exposure to the idiosyncratic volatility factor IV_t. Each month from 1926 until 2010, we regress monthly individual firm stock returns on market returns to form residual returns using 60-month windows. We calculate residual variance in month t as the time series standard deviation of the 60 months of returns ending in month t. We define the IV factor in month t as the equally-weighted mean of all stocks' residual variances in month t. We then regress each stock's residual variance on the IV factor, controlling for the lagged IV factor and for the contemporaneous market return variance, using a 60 month window, and sort stocks into five quintile portfolios from lowest IV factor exposure (Q1) to highest (Q5). We then hold the portfolio for one month until $t+1$ and calculate portfolio returns over that month. They are reported in row 1 of Table 4. We observe a declining pattern in return from Q1 to Q5. The return spread between portfolio 5 and portfolio 1 is 5.59% per year and has a t-statistic of 3.08. This shows that stocks with higher exposure to the IV factor have lower returns. The pattern cannot be explained by exposure to the market variance factor since market variance was controlled for in the portfolio formation step.

We also calculate post-formation betas $\beta_{gs,i}$ with respect to the IV factor. These are reported in row 5 of Table 4. The betas are very negative for Q1 (-1.77) and less negative for Q5 (-0.97). The spread in betas is -0.80.

We calculate total and residual return variances as the average total (idiosyncratic) return variances of the stocks in that portfolio in that month. These are individual stock variances,

Table 4: Main Results

This tables reports moments from the model and compares them to the data. The first two rows report the average excess return in model and data. The next two rows split out the equity risk premium into a contribution representing compensation for η risk and a compensation for w_g risk. Rows 5 and 6 report return volatilities in data and model, followed by a breakdown of volatility into its five components in rows 7-11 (see equation 11). Since the variance but not the volatility components are additive, we calculate the square root of each variance component, and then rescale all components so they sum tot total volatility. Rows 12 and 13 report the empirical object in equation (14), $\frac{Cov\left(\mathbb{V}_t\left[r^{idio,i}_{t+1}\right],\mathbb{V}_t\left[r^{IV}_{t+1}\right]\right)}{Var[\mathbb{V}_t\left[r^{IV}_{t+1}\right]]}$, in data and in model, multiplied by 100. Rows 14 and 15 report the R-squared in equation (15) in data and in model, multiplied by 100. The model is simulated at annual frequency for 60,000 periods. All moments in the data are expressed as annual quantities and computed from the July 1926 to December 2010 sample.

	Moment		Q1	Q2	Q3	Q4	Q5	M
1	Excess Ret	Data	12.00	9.64	9.67	7.04	6.41	7.47
2		Model	9.96	9.23	5.97	5.55	5.76	7.02
3		η risk	0.68	0.68	0.68	0.68	0.68	0.68
4		w_g risk	9.28	8.55	5.30	4.88	5.09	6.35
5	Beta $\beta_{gs,i}$	Data	−1.77	−1.63	−1.01	−0.93	−0.97	−1.21
6		Model	−1.77	−1.63	−1.01	−0.93	−0.97	−1.21
7	Return Vol.	Data	81.20	60.08	53.74	52.38	66.66	16.69
8		Model	83.27	53.61	49.20	44.58	67.05	12.91
9		η risk	2.72	2.61	2.66	2.65	2.75	3.50
10		w_g risk	10.70	9.46	5.97	5.48	5.92	9.41
11		e_i risk	40.47	24.57	21.86	19.95	30.67	×
12		ε_i risk	27.78	16.39	18.02	15.95	26.29	×
13		w_i risk	1.60	0.58	0.69	0.54	1.41	×
14	Eq. (14)	Data	1.83	0.89	0.68	0.57	1.03	×
15		Model	1.98	0.79	0.60	0.51	1.12	×
16	Eq. (15)	Data	92.27	93.05	85.18	86.65	83.08	×
17		Model	92.27	93.05	85.18	86.65	83.08	×

not portfolio return variances. We annualize the series. Annual return volatilities (standard deviations) range from 48% to 77%. They are highest for portfolios Q1 and Q5. The market portfolio has a volatility of 16.7% in the data. Row 7 of Table 4 shows the return volatilities.

We set μ_i equal to the values observed in the data. We set $\varphi_{di} = \varphi_{dm} = 3$. This is a standard leverage parameter. By setting this parameter equal for all portfolios, we ensures that all differences in risk premia across portfolios arise from differences in exposure to the $w_{g,t+1}$ shocks. That contribution to the risk premium from this standard CCAPM (η-risk) is

0.68% per year. We set ϕ_{dm} equal to -0.65 and χ^m equal to -7.98 in order to exactly match the observed market beta $\beta_{gs,m}$ of -1.21.[11] Our parameter choices imply a return volatility of 12.91% for the market portfolio, somewhat below the data. The model-implied equity risk premium is 7.02% per annum, close to the historical average for the 1926-2010 sample of 7.47%.

The idiosyncratic volatility process parameters are common to all firms. We set σ_i to 0.4%, ν_i to 0.40, and σ_{iw} to 2e-6. The persistence of σ_{it} is set lower to that of σ_{gt} to generate the fact that the persistence of the return volatility of the quintile portfolios is lower than that of the market. σ_{iw} is chosen as large as possible while preventing σ_{it} to go negative. Finally, the value for σ_i is a normalization because every portfolio's variance is pre-multiplied by ζ_i.

The individual dividend growth parameters, ϕ_i, χ_i, κ_i, and ζ_i are chosen as follows. First, to add further parsimony, we also set all parameters $\phi_i = \phi_{dm} = -0.65, \forall i$. Second, we chose ζ_i to match the R^2 of equations (15) exactly for each stock, given our choice for κ_i. See rows 16 and 17 of Table 4. Third, for κ_i we start from the values implied by the slope in equations (14) for each stock and multiply by $\bar{\kappa}^2$ before taking the square root. The value for $\bar{\kappa}^2$ is chosen to match the observed unconditional average of the IV factor of 35.45%. From this base level, we adjust up the κ_i values for Q1 and Q5 up and adjust the ones for Q2, Q3, and Q4 down so as to keep the average $\bar{\kappa}^2$ unchanged.[12] This helps us to match the higher volatilities of stocks in portfolios Q1 and Q5, at the expense of missing our target for the slopes of equations (14) somewhat. Rows 14 and 15 of Table 4 shows that the fit is still good. Fourth, we set the parameters χ_i to exactly match the observed betas; see rows 5 and 6 of Table 4. Table 5 shows that this implies a strongly increasing pattern for χ_i, given all other parameters.

[11] There are multiple combinations of ϕ_{dm} and χ_m that deliver the same $\beta_{gs,m}$. These combinations have roughly the same volatility implications and identical risk premium implications. We select a value for ϕ_{dm} equal to -0.65 which is 3 times the value for ϕ_c, the same leverage effect of 3 as we assume with respect to the η shock ($\varphi_{dm} = 3$).

[12] The adjustment factors are 1.12, 0.92, 0.92, 0.92 and 1.12.

Table 5: CALIBRATION

This table lists the parameters of the model. The last panel discusses the calibration of five stock portfolios, sorted from lowest volatility (Q1) to highest volatility (Q5). The market portfolios is indicated by the letter M.

			Preferences			
δ	0.91	γ	10	ψ	2	
			Aggregate Consumption Growth Process			
μ_g	0.02	σ_c	.015	ϕ_c	-0.22	
			Consumption Share Process			
σ_g	0.10	ν_g	0.77	σ_w	.015	
			Dividend Growth Process			
σ_i	0.004	ν_i	0.50	σ_{iw}	2e-6	
Parameter	Q1	Q2	Q3	Q4	Q5	M
μ_i	4.86%	3.84%	4.46%	1.88%	-5.57%	4.14%
φ_{di}	3	3	3	3	3	3
ϕ_i	-0.65	-0.65	-0.65	-0.65	-0.65	-0.65
χ^i	-35.72	-20.62	-11.43	-9.44	-16.76	-7.98
κ_i	6.69	4.23	3.70	3.39	5.02	$\times$
ζ_i	114.90	70.57	76.20	67.68	107.63	$\times$

Table 4 shows the results. The model does a good job at replicating the overall return volatility of the stocks in each of the five portfolios. the same is true for the idiosyncratic return volatilities (not reported). The main finding is that the model generates a decreasing pattern in equity risk premia. Hence, the model generates the fact that stocks with large negative return exposure to the idiosyncratic volatility factor carry higher expected returns. The spread in betas of -0.80, combined with a market price of risk of $\lambda_w = -5.24$ generates a spread in expected returns of 4.2%, three quarters of the observed spread. The stocks in portfolio Q5 have high exposure to the IV factor. Their returns fall the least when risk sharing opportunities deteriorate in so they are the best hedge against such deteriorations. As a result, they carry the lowest risk premia.

The model generates the observed pattern in betas mainly via the increasing pattern in the predictive coefficient χ_i which go from more negative to less negative. A low degree of risk sharing (high σ_{gt}^2) signals lower future cash-flow growth persistently into the future, and more so for the Q1 portfolio than for the Q5 portfolio. That makes the Q1 portfolio riskier than the Q5 portfolio, all else equal. There also is the Jensen effect highlighted in the previous section arising from differences in κ_i. [13] Hence the cash flow predictability effect must offset the Jensen effect for the betas to line up with the observed ones.

7 Conclusion

We document strong comovement of individual stock return volatilities. Removing common variation in returns has little effect on volatility comovement, as the volatility of residual returns demonstrates effectively the same factor structure as total returns, despite the fact that these residuals are uncorrelated. The distinction between stocks' total volatility and idiosyncratic volatility is tiny – almost all return variation at the stock level is idiosyncratic. Volatility comovement is not only a feature of returns, but also for volatility of firms' fundamentals. Like returns, we find a strong factor structure among sales growth volatilities, both for total growth rates as well as among idiosyncratic, uncorrelated factor model residual growth rates.

We explore the asset pricing implicates of these findings in a model with heterogeneous investors whose consumption growth is subject to some of this variation in idiosyncratic risk. In this model, CIV is a priced state variable. Increases in CIV lead to a deterioration in risk sharing and are are associated with high marginal utility for the average investor. Stocks with less negative exposure to positive innovations in risk sharing opportunities are less risky and carry lower returns. Sorting stocks into portfolios based on their exposure to the CIV, we find that stocks with more negative betas carry higher average returns. The model explores various channels that can generate the observed pattern in betas. It generates three quarters of the observed return spread for plausible parameters.

[13] For the portfolios under investigation in this section, that effect leads to a 4.8% spread in the wrong direction if we set $\chi^i = \chi^5 \; \forall i$.

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A Appendix Section 3

Starting from the budget constraint for agent j:

$$W^j_{t+1} = R^j_{t+1}(W^j_t - C^j_t). \tag{16}$$

The beginning-of-period (or cum-dividend) total wealth W^j_t that is not spent on consumption C^j_t earns a gross return R^j_{t+1} and leads to beginning-of-next-period total wealth W^j_{t+1}. The return on a claim to consumption, the *total wealth return*, can be written as

$$R^j_{t+1} = \frac{W^j_{t+1}}{W^j_t - C^j_t} = \frac{C^j_{t+1}}{C^j_t}\frac{WC^j_{t+1}}{WC^j_t - 1}.$$

We use the Campbell (1991) approximation of the log total wealth return $r^j_t = \log(R^j_t)$ around the long-run average log wealth-consumption ratio $\mu_{wc} \equiv E[w^j_t - c^j_t]$:

$$r^j_{t+1} = \kappa^c_0 + \Delta c^j_{t+1} + wc^j_{t+1} - \kappa^c_1 wc^j_t,$$

where the linearization constants κ^c_0 and κ^c_1 are non-linear functions of the unconditional mean log wealth-consumption ratio μ_{wc}:

$$\kappa^c_1 = \frac{e^{\mu_{wc}}}{e^{\mu_{wc}} - 1} > 1 \text{ and } \kappa^c_0 = -\log\left(e^{\mu_{wc}} - 1\right) + \frac{e^{\mu_{wc}}}{e^{\mu_{wc}} - 1}\mu_{wc}.$$

The return on a claim to the consumption stream of agent j, R^j, evaluated at her intertemporal marginal rate of substitution SDF^j satisfies the Euler equation:

$$\begin{aligned}
1 &= \mathbb{E}_t\left[SDF^j_{t+1}R^j_{t+1}\right] \\
1 &= \mathbb{E}_t\left[\mathbb{E}_j\left[SDF^j_{t+1}R^j_{t+1}\right]\right] \\
1 &= \mathbb{E}_t\left[\mathbb{E}_j\left[\exp\{sdf^j_{t+1} + r^j_{t+1}\}\right]\right] \\
1 &= \mathbb{E}_t\left[\exp\{\mathbb{E}_j\left(sdf^j_{t+1} + r^j_{t+1}\right) + \frac{1}{2}\mathbb{V}_j\left(sdf^j_{t+1} + r^j_{t+1}\right)\}\right]
\end{aligned} \tag{17}$$

where the second equality applies the law of iterated expectations, and the last equality applies the cross-sectional normality of consumption share growth.

We combine the approximation of the log total wealth return with our conjecture for the wealth-consumption ratio of agent j:

$$wc^j_t = \mu_{wc} + W_{gs}\left(\sigma^2_{gt} - \sigma^2_g\right)$$

We solve for the coefficients μ_{wc} and $W_g s$ by imposing the Euler equation for the consumption claim.

First, we compute r^j_{t+1}:

$$
\begin{aligned}
r^j_{t+1} &= \kappa^c_0 + \mu_g + (1-\kappa^c_1)\,\mu_{wc} + W_{gs}\,(\nu_g - \kappa^c_1)\left(\sigma^2_{gt} - \sigma^2_g\right) + \sigma_c\eta_{t+1} + W_{gs}\sigma_{gw}w_{g,t+1} + \Delta s^j_{t+1} \\
&= \kappa^c_0 + \mu_g + (1-\kappa^c_1)\,\mu_{wc} + W_{gs}\,(\nu_g - \kappa^c_1)\left(\sigma^2_{gt} - \sigma^2_g\right) + \sigma_c\eta_{t+1} + W_{gs}\sigma_{gw}w_{g,t+1} + \sigma_{g,t+1}v^j_{t+1} - \frac{1}{2}\sigma^2_{g,t+1} \\
&= r^c_0 + \left[W_{gs}\,(\nu_g - \kappa^c_1) - \frac{1}{2}\nu_g\right]\left(\sigma^2_{gt} - \sigma^2_g\right) + \sigma_c\eta_{t+1} + (W_{gs} - \frac{1}{2})\sigma_{gw}w_{g,t+1} + \sigma_{g,t+1}v^j_{t+1}
\end{aligned}
$$

where $r^c_0 = \kappa^c_0 + \mu_g + (1-\kappa^c_1)\,\mu_{wc} - \frac{1}{2}\sigma^2_g$ is the unconditional mean log return.

Second, Epstein and Zin (1989) show that the log real stochastic discount factor is

$$
\begin{aligned}
sdf^j_{t+1} &= \theta\log\delta - \frac{\theta}{\psi}\Delta c^j_{t+1} + (\theta-1)\,r^j_{t+1} \\
&= \theta\log\delta - \gamma\Delta c^j_{t+1} + (\theta-1)\left(\kappa^c_0 + wc^j_{t+1} - \kappa^c_1 wc^j_t\right) \\
&= \theta\log\delta - \gamma\left(\mu_g + \sigma_c\eta_{t+1}\right) - \gamma\Delta s^j_{t+1} + (\theta-1)\left(\kappa^c_0 + wc^j_{t+1} - \kappa^c_1 wc^j_t\right) \\
&= \theta\log\delta - \gamma\mu_g + (\theta-1)[\kappa^c_0 + (1-\kappa^c_1)\mu_{wc}] + (\theta-1)\,W_{gs}\,(\nu_g - \kappa^c_1)\left(\sigma^2_{gt} - \sigma^2_g\right) \\
&\quad -\gamma\sigma_c\eta_{t+1} - \gamma\Delta s^j_{t+1} + (\theta-1)\,W_{gs}\sigma_{gw}w_{g,t+1} \\
&= \mu_s + \left[(\theta-1)\,W_{gs}\,(\nu_g - \kappa^c_1) + \gamma\frac{1}{2}\nu_g\right]\left(\sigma^2_{gt} - \sigma^2_g\right) \\
&\quad -\gamma\sigma_c\eta_{t+1} - \gamma\sigma_{g,t+1}v^j_{t+1} + \left[(\theta-1)\,W_{gs} + \frac{1}{2}\gamma\right]\sigma_{gw}w_{g,t+1}
\end{aligned}
$$

where $\mu_s = \theta\log\delta - \gamma\mu_g + (\theta-1)[\kappa^c_0 + (1-\kappa^c_1)\mu_{wc}] + \gamma\frac{1}{2}\sigma^2_g$ in the unconditional mean log SDF.

We have that:

$$
\begin{aligned}
\mathbb{E}_j\left(sdf^j_{t+1}\right) &= \mu_s + \left[(\theta-1)\,W_{gs}\,(\nu_g - \kappa^c_1) + \gamma\frac{1}{2}\nu_g\right]\left(\sigma^2_{gt} - \sigma^2_g\right) \\
&\quad -\gamma\sigma_c\eta_{t+1} + \left[(\theta-1)\,W_{gs} + \frac{1}{2}\gamma\right]\sigma_{gw}w_{g,t+1} \\
\mathbb{E}_j\left(r^j_{t+1}\right) &= r^c_0 + \left[W_{gs}\,(\nu_g - \kappa^c_1) - \frac{1}{2}\nu_g\right]\left(\sigma^2_{gt} - \sigma^2_g\right) \\
&\quad +\sigma_c\eta_{t+1} + \left(W_{gs} - \frac{1}{2}\right)\sigma_{gw}w_{g,t+1} \\
\mathbb{V}_j\left[sdf^j_{t+1} + r^j_{t+1}\right] &= (1-\gamma)^2\,\sigma^2_{g,t+1} = (1-\gamma)^2\left(\sigma^2_g + \nu_g\left(\sigma^2_{gt} - \sigma^2_g\right) + \sigma_{gw}w_{g,t+1}\right)
\end{aligned}
$$

All the equations above imply that:

$$\begin{aligned}
\mathbb{E}_j\left(sdf_{t+1}^j + r_{t+1}^j\right) + \frac{1}{2}\mathbb{V}_j\left[sdf_{t+1}^j + r_{t+1}^j\right] &= \mu_s + \left[(\theta-1)W_{gs}\left(\nu_g - \kappa_1^c\right) + \gamma\frac{1}{2}\nu_g\right]\left(\sigma_{gt}^2 - \sigma_g^2\right) \\
&\quad -\gamma\sigma_c\eta_{t+1} + \left[(\theta-1)W_{gs} + \frac{1}{2}\gamma\right]\sigma_{gw}w_{g,t+1} \\
&\quad +r_0^c + \left[W_{gs}\left(\nu_g - \kappa_1^c\right) - \frac{1}{2}\nu_g\right]\left(\sigma_{gt}^2 - \sigma_g^2\right) \\
&\quad +\sigma_c\eta_{t+1} + (W_{gs} - \frac{1}{2})\sigma_{gw}w_{g,t+1} \\
&\quad +\frac{1}{2}(1-\gamma)^2\left(\sigma_g^2 + \nu_g\left(\sigma_{gt}^2 - \sigma_g^2\right) + \sigma_{gw}w_{g,t+1}\right) \\
&= \mu_s + r_0^c + \frac{1}{2}(1-\gamma)^2\sigma_g^2 \\
&\quad + \left[\theta W_{gs}\left(\nu_g - \kappa_1^c\right) + \frac{1}{2}\gamma(\gamma-1)\nu_g\right]\left(\sigma_{gt}^2 - \sigma_g^2\right) \\
&\quad +(1-\gamma)\sigma_c\eta_{t+1} + \left[\theta W_{gs} + \frac{1}{2}\gamma(\gamma-1)\right]\sigma_{gw}w_{g,t+1}
\end{aligned}$$

Now, we can take expected value and variance conditioning on t:

$$\begin{aligned}
\mathbb{E}_t\left[\mathbb{E}_j\left(sdf_{t+1}^j + r_{t+1}^j\right) + \frac{1}{2}\mathbb{V}_j\left[sdf_{t+1}^j + r_{t+1}^j\right]\right] &= \mu_s + r_0^c + \frac{1}{2}(1-\gamma)^2\sigma_g^2 \\
&\quad + \left[\theta W_{gs}\left(\nu_g - \kappa_1^c\right) + \frac{1}{2}\gamma(\gamma-1)\nu_g\right]\left(\sigma_{gt}^2 - \sigma_g^2\right) \\
\mathbb{V}_t\left[\mathbb{E}_j\left(sdf_{t+1}^j + r_{t+1}^j\right) + \frac{1}{2}\mathbb{V}_j\left[sdf_{t+1}^j + r_{t+1}^j\right]\right] &= (1-\gamma)^2\sigma_c^2 + \left[\theta W_{gs} + \frac{1}{2}\gamma(\gamma-1)\right]^2\sigma_{gw}^2
\end{aligned}$$

Plugging these different components into equation (17), and setting all the constant terms to zero yields:

$$0 = \mu_s + r_0^c + \frac{1}{2}(1-\gamma)\sigma_g^2 + \frac{1}{2}(1-\gamma)^2\sigma_c^2 + \frac{1}{2}\left[\theta W_{gs} + \frac{1}{2}\gamma(\gamma-1)\right]^2\sigma_{gw}^2 \tag{18}$$

Then setting all coefficients in $\left(\sigma_{gt}^2 - \sigma_g^2\right)$ equal to zero we obtain:

$$W_{gs} = \frac{\nu_g\gamma(\gamma-1)}{2\theta(\kappa_1^c - \nu_g)} = -\frac{\gamma\nu_g\left(1 - \frac{1}{\psi}\right)}{2\left(\kappa_1^c - \nu_g\right)} \tag{19}$$

If the IES exceeds 1, then $W_{gs} < 0$.

Plugging these coefficients back into equation (18) implicitly defines a nonlinear equation in one unknown (μ_{wc}), which can be solved for numerically, characterizing the average wealth-consumption ratio.

We can derive an expression for the common log real stochastic discount factor:

$$
\begin{aligned}
sdf^a_{t+1} &= \mathbb{E}_j\left[sdf^j_{t+1}\right] + \frac{1}{2}\mathbb{V}_j\left[sdf^j_{t+1}\right] \\
&= \mu_s + \left[(\theta-1)\,W_{gs}\left(\nu_g-\kappa^c_1\right) + \gamma\frac{1}{2}\nu_g\right]\left(\sigma^2_{gt}-\sigma^2_g\right) \\
&\quad -\gamma\sigma_c\eta_{t+1} + \left[(\theta-1)\,W_{gs} + \frac{1}{2}\gamma\right]\sigma_{gw}w_{g,t+1} + \frac{1}{2}\gamma^2\sigma^2_{g,t+1} \\
&= \mu_s + \left[(\theta-1)\,W_{gs}\left(\nu_g-\kappa^c_1\right) + \gamma\frac{1}{2}\nu_g\right]\left(\sigma^2_{gt}-\sigma^2_g\right) \\
&\quad -\gamma\sigma_c\eta_{t+1} + \left[(\theta-1)\,W_{gs} + \frac{1}{2}\gamma\right]\sigma_{gw}w_{g,t+1} + \frac{1}{2}\gamma^2\left(\sigma^2_g + \nu_g\left(\sigma^2_{gt}-\sigma^2_g\right) + \sigma_{gw}w_{g,t+1}\right) \\
&= \mu_s + \frac{1}{2}\gamma^2\sigma^2_g + \left[(\theta-1)\,W_{gs}\left(\nu_g-\kappa^c_1\right) + \frac{1}{2}\gamma(1+\gamma)\nu_g\right]\left(\sigma^2_{gt}-\sigma^2_g\right) \\
&\quad -\gamma\sigma_c\eta_{t+1} + \left[(\theta-1)\,W_{gs} + \frac{1}{2}\gamma(1+\gamma)\right]\sigma_{gw}w_{g,t+1} \\
&= \mu_s + \frac{1}{2}\gamma^2\sigma^2_g + s_{gs}\left(\sigma^2_{gt}-\sigma^2_g\right) - \lambda_\eta\sigma_c\eta_{t+1} - \lambda_w\sigma_{gw}w_{g,t+1}
\end{aligned}
$$

where

$$
\begin{aligned}
s_{gs} &\equiv (\theta-1)\,W_{gs}\left(\nu_g-\kappa^c_1\right) + \frac{1}{2}\gamma(1+\gamma)\nu_g = \frac{1}{2}\gamma\nu_g\left(\frac{1}{\psi}+1\right), \\
\lambda_\eta &\equiv \gamma, \\
\lambda_w &\equiv (1-\theta)\,W_{gs} - \frac{1}{2}\gamma(1+\gamma) = \frac{\gamma\nu_g\left(\frac{1}{\psi}-\gamma\right)}{2\left(\kappa^c_1-\nu_g\right)} - \frac{1}{2}\gamma(1+\gamma),
\end{aligned}
$$

The risk-free rate is:

$$
\begin{aligned}
r^f_t &= -\log\left(\mathbb{E}_t[SDF^j_{t+1}]\right) = -\log\left(\mathbb{E}_t[\mathbb{E}_j[SDF^j_{t+1}]]\right) \\
&= = -\mathbb{E}_t[sdf^a_{t+1}] - \frac{1}{2}\mathbb{V}_t[sdf^a_{t+1}] \\
&= -\mu_s - \frac{1}{2}\gamma^2\sigma^2_g - \frac{1}{2}\lambda^2_\eta\sigma^2_c - \frac{1}{2}\lambda^2_w\sigma^2_{gw} - s_{gs}\left(\sigma^2_{gt}-\sigma^2_g\right)
\end{aligned}
$$

B Appendix Section 4

The derivations follow the same steps as in appendix A. We first compute r^j_{t+1}:

$$
\begin{aligned}
r^j_{t+1} &= \kappa^c_0 + \mu_g + \left(1-\kappa^c_1\right)\mu_{wc} + W_{gs}\left(\nu_g-\kappa^c_1\right)\left(\sigma^2_{gt}-\sigma^2_g\right) + \sigma_c\eta_{t+1} + (\phi_c + W_{gs}\sigma_w)\sigma_{gt}w_{g,t+1} + \Delta s^j_{t+1} \\
&= r^c_0 + \left[W_{gs}\left(\nu_g-\kappa^c_1\right) - \frac{1}{2}\nu_g\right]\left(\sigma^2_{gt}-\sigma^2_g\right) + \sigma_c\eta_{t+1} + (\phi_c + W_{gs}\sigma_w - \frac{1}{2}\sigma_w)\sigma_{gt}w_{g,t+1} + \sigma_{g,t+1}v^j_{t+1}
\end{aligned}
$$

where $r^c_0 = \kappa^c_0 + \mu_g + \left(1-\kappa^c_1\right)\mu_{wc} - \frac{1}{2}\sigma^2_g$.

The log real stochastic discount factor for agent j is

$$
\begin{aligned}
sdf^j_{t+1} &= \theta \log\delta - \gamma\left(\mu_g + \sigma_c\eta_{t+1} + \phi_c\sigma_{gt}w_{g,t+1}\right) - \gamma\Delta s^j_{t+1} + (\theta-1)\left(\kappa^c_0 + wc^j_{t+1} - \kappa^c_1 wc^j_t\right) \\
&= \mu_s + \left[(\theta-1)\,W_{gs}\left(\nu_g - \kappa^c_1\right) + \gamma\frac{1}{2}\nu_g\right]\left(\sigma^2_{gt} - \sigma^2_g\right) \\
&\quad -\gamma\sigma_c\eta_{t+1} - \gamma\sigma_{g,t+1}v^j_{t+1} + \left[(\theta-1)\,W_{gs}\sigma_w - \gamma\phi_c + \frac{1}{2}\gamma\sigma_w\right]\sigma_{gt}w_{g,t+1}
\end{aligned}
$$

where $\mu_s = \theta\log\delta - \gamma\mu_g + (\theta-1)[\kappa^c_0 + (1-\kappa^c_1)\mu_{wc}] + \gamma\frac{1}{2}\sigma^2_g$.

$$
\begin{aligned}
\mathbb{E}_j\left(sdf^j_{t+1}\right) &= \mu_s + \left[(\theta-1)\,W_{gs}\left(\nu_g - \kappa^c_1\right) + \gamma\frac{1}{2}\nu_g\right]\left(\sigma^2_{gt} - \sigma^2_g\right) \\
&\quad -\gamma\sigma_c\eta_{t+1} + \left[(\theta-1)\,W_{gs}\sigma_w - \gamma\phi_c + \frac{1}{2}\gamma\sigma_w\right]\sigma_{gt}w_{g,t+1} \\
\mathbb{E}_j\left(r^j_{t+1}\right) &= r^c_0 + \left[W_{gs}\left(\nu_g - \kappa^c_1\right) - \frac{1}{2}\nu_g\right]\left(\sigma^2_{gt} - \sigma^2_g\right) \\
&\quad +\sigma_c\eta_{t+1} + (\phi_c + W_{gs}\sigma_w - \frac{1}{2}\sigma_w)\sigma_{gt}w_{g,t+1} \\
\mathbb{V}_j\left[sdf^j_{t+1} + r^j_{t+1}\right] &= (1-\gamma)^2\left(\sigma^2_g + \nu_g\left(\sigma^2_{gt} - \sigma^2_g\right) + \sigma_w\sigma_{gt}w_{g,t+1}\right)
\end{aligned}
$$

All the equations above imply that:

$$
\begin{aligned}
\mathbb{E}_j\left(sdf^j_{t+1} + r^j_{t+1}\right) + \frac{1}{2}\mathbb{V}_j\left[sdf^j_{t+1} + r^j_{t+1}\right] &= \mu_s + \left[(\theta-1)\,W_{gs}\left(\nu_g - \kappa^c_1\right) + \gamma\frac{1}{2}\nu_g\right]\left(\sigma^2_{gt} - \sigma^2_g\right) \\
&\quad -\gamma\sigma_c\eta_{t+1} + \left[(\theta-1)\,W_{gs}\sigma_w - \gamma\phi_c + \frac{1}{2}\gamma\sigma_w\right]\sigma_{gt}w_{g,t+1} \\
&\quad +r^c_0 + \left[W_{gs}\left(\nu_g - \kappa^c_1\right) - \frac{1}{2}\nu_g\right]\left(\sigma^2_{gt} - \sigma^2_g\right) \\
&\quad +\sigma_c\eta_{t+1} + (\phi_c + W_{gs}\sigma_w - \frac{1}{2}\sigma_w)\sigma_{gt}w_{g,t+1} \\
&\quad +\frac{1}{2}(1-\gamma)^2\left(\sigma^2_g + \nu_g\left(\sigma^2_{gt} - \sigma^2_g\right) + \sigma_w\sigma_{gt}w_{g,t+1}\right) \\
&= \mu_s + r^c_0 + \frac{1}{2}(1-\gamma)^2\sigma^2_g \\
&\quad + \left[\theta W_{gs}\left(\nu_g - \kappa^c_1\right) + \frac{1}{2}\gamma(\gamma-1)\nu_g\right]\left(\sigma^2_{gt} - \sigma^2_g\right) \\
&\quad +(1-\gamma)\sigma_c\eta_{t+1} + \left[\theta W_{gs}\sigma_w + (1-\gamma)\phi_c + \frac{1}{2}\gamma(\gamma-1)\sigma_w\right]\sigma_{gt}w_{g,t+1}
\end{aligned}
$$

Now, we can take expected value and variance conditioning on t:

$$\begin{aligned}
\mathbb{E}_t\left[\mathbb{E}_j\left(sdf^j_{t+1}+r^j_{t+1}\right)+\frac{1}{2}\mathbb{V}_j\left[sdf^j_{t+1}+r^j_{t+1}\right]\right] &= \mu_s+r^c_0+\frac{1}{2}(1-\gamma)^2\sigma^2_g \\
&\quad+\left[\theta W_{gs}\left(\nu_g-\kappa^c_1\right)+\frac{1}{2}\gamma(\gamma-1)\nu_g\right]\left(\sigma^2_{gt}-\sigma^2_g\right) \\
\mathbb{V}_t\left[\mathbb{E}_j\left(sdf^j_{t+1}+r^j_{t+1}\right)+\frac{1}{2}\mathbb{V}_j\left[sdf^j_{t+1}+r^j_{t+1}\right]\right] &= (1-\gamma)^2\sigma^2_c+\left[\theta W_{gs}\sigma_w+(1-\gamma)\phi_c+\frac{1}{2}\gamma(\gamma-1)\sigma_w\right]^2\sigma^2_{gt}
\end{aligned}$$

Plugging these different components into equation (17) yields:

$$\begin{aligned}
0 &= \mathbb{E}_t\left[\mathbb{E}_j\left(sdf^j_{t+1}+r^j_{t+1}\right)+\frac{1}{2}\mathbb{V}_j\left[sdf^j_{t+1}+r^j_{t+1}\right]\right]+\frac{1}{2}\mathbb{V}_t\left[\mathbb{E}_j\left(sdf^j_{t+1}+r^j_{t+1}\right)+\frac{1}{2}\mathbb{V}_j\left[sdf^j_{t+1}+r^j_{t+1}\right]\right] \\
&= \mu_s+r^c_0+\frac{1}{2}(1-\gamma)^2\sigma^2_g+\left[\theta W_{gs}\left(\nu_g-\kappa^c_1\right)+\frac{1}{2}\gamma(\gamma-1)\nu_g\right]\left(\sigma^2_{gt}-\sigma^2_g\right) \\
&\quad+\frac{1}{2}\left\{(1-\gamma)^2\sigma^2_c+\left[\theta W_{gs}\sigma_w+(1-\gamma)\phi_c+\frac{1}{2}\gamma(\gamma-1)\sigma_w\right]^2\sigma^2_{gt}\right\} \\
&= \mu_s+r^c_0+\frac{1}{2}\left(1-\gamma\right)^2\sigma^2_g+\frac{1}{2}\left(1-\gamma\right)^2\sigma^2_c+\frac{1}{2}\left[\theta W_{gs}\sigma_w+(1-\gamma)\phi_c+\frac{1}{2}\gamma(\gamma-1)\sigma_w\right]^2\sigma^2_g \\
&\quad+\left\{\theta W_{gs}\left(\nu_g-\kappa^c_1\right)+\frac{1}{2}\gamma(\gamma-1)\nu_g+\frac{1}{2}\left[\theta W_{gs}\sigma_w+(1-\gamma)\phi_c+\frac{1}{2}\gamma(\gamma-1)\sigma_w\right]^2\right\}\left(\sigma^2_{gt}-\sigma^2_g\right)
\end{aligned}$$

Using method of undetermined coefficients, μ_{wc} solves:

$$0 = \mu_s+r^c_0+\frac{1}{2}\left(1-\gamma\right)^2\sigma^2_g+\frac{1}{2}\left(1-\gamma\right)^2\sigma^2_c+\frac{1}{2}\left[\theta W_{gs}\sigma_w+(1-\gamma)\phi_c+\frac{1}{2}\gamma(\gamma-1)\sigma_w\right]^2\sigma^2_g$$

and W_{gs} solves the following quadratic equation:

$$\begin{aligned}
0 &= \theta W_{gs}\left(\nu_g-\kappa^c_1\right)+\frac{1}{2}\gamma(\gamma-1)\nu_g+\frac{1}{2}\left[\theta W_{gs}\sigma_w+(1-\gamma)\phi_c+\frac{1}{2}\gamma(\gamma-1)\sigma_w\right]^2 \\
&= \theta W_{gs}\left(\nu_g-\kappa^c_1\right)+\frac{1}{2}\gamma(\gamma-1)\nu_g \\
&\quad+\frac{1}{2}\sigma^2_w\theta^2W^2_{gs}+\frac{1}{2}\left[(1-\gamma)\phi_c+\frac{1}{2}\gamma(\gamma-1)\sigma_w\right]^2+\theta\left[(1-\gamma)\phi_c+\frac{1}{2}\gamma(\gamma-1)\sigma_w\right]\sigma_wW_{gs} \\
&= \underbrace{\frac{1}{2}\sigma^2_w\theta^2}_{>0}W^2_{gs}+\left(\underbrace{\nu_g-\kappa^c_1}_{<0}+\underbrace{\sigma_w\left[(1-\gamma)\phi_c+\frac{1}{2}\gamma(\gamma-1)\sigma_w\right]}_{>0}\right)\underbrace{\theta}_{<0}W_{gs} \\
&\quad+\underbrace{\frac{1}{2}\gamma(\gamma-1)\nu_g+\frac{1}{2}\left[(1-\gamma)\phi_c+\frac{1}{2}\gamma(\gamma-1)\sigma_w\right]^2}_{>0}
\end{aligned}$$

If $\sigma_w > 0$ is sufficiently small, we have

$$\nu_g-\kappa^c_1+\sigma_w\left[(1-\gamma)\phi_c+\frac{1}{2}\gamma(\gamma-1)\sigma_w\right]<0$$

which makes both roots of the quadratic equation strictly negative if not complex, i.e. $W_{gs} < 0$.

We can rewrite the expression for the common log real stochastic discount factor:

$$\begin{aligned}
sdf^a_{t+1} &= \mathbb{E}_j\left[sdf^j_{t+1}\right] + \frac{1}{2}\mathbb{V}_j\left[sdf^j_{t+1}\right] \\
&= \mu_s + \left[(\theta-1)\,W_{gs}\left(\nu_g - \kappa^c_1\right) + \gamma\frac{1}{2}\nu_g\right]\left(\sigma^2_{gt} - \sigma^2_g\right) \\
&\quad -\gamma\sigma_c\eta_{t+1} + \left[(\theta-1)\,W_{gs}\sigma_w - \gamma\phi_c + \frac{1}{2}\gamma\sigma_w\right]\sigma_{gt}w_{g,t+1} + \frac{1}{2}\gamma^2\left(\sigma^2_g + \nu_g\left(\sigma^2_{gt} - \sigma^2_g\right) + \sigma_w\sigma_{gt}w_{g,t+1}\right) \\
&= \mu_s + \frac{1}{2}\gamma^2\sigma^2_g + \left[(\theta-1)\,W_{gs}\left(\nu_g - \kappa^c_1\right) + \frac{1}{2}\gamma(1+\gamma)\nu_g\right]\left(\sigma^2_{gt} - \sigma^2_g\right) \\
&\quad -\gamma\sigma_c\eta_{t+1} + \left[(\theta-1)\,W_{gs}\sigma_w - \gamma\phi_c + \frac{1}{2}\gamma(1+\gamma)\sigma_w\right]\sigma_{gt}w_{g,t+1} \\
&= \mu_s + \frac{1}{2}\gamma^2\sigma^2_g + s_{gs}\left(\sigma^2_{gt} - \sigma^2_g\right) - \lambda_\eta\sigma_c\eta_{t+1} - \lambda_w\sigma_{gt}w_{g,t+1}
\end{aligned}$$

where

$$\begin{aligned}
s_{gs} &\equiv (\theta-1)\,W_{gs}\left(\nu_g - \kappa^c_1\right) + \frac{1}{2}\gamma(1+\gamma)\nu_g, \\
\lambda_\eta &\equiv \gamma, \\
\lambda_w &\equiv (1-\theta)\,W_{gs}\sigma_w + \gamma\phi_c - \frac{1}{2}\gamma(1+\gamma)\sigma_w,
\end{aligned}$$

The risk-free rate is:

$$r^f_t = -\mu_s - \frac{1}{2}\gamma^2\sigma^2_g - s_{gs}\left(\sigma^2_{gt} - \sigma^2_g\right) - \frac{1}{2}\lambda^2_\eta\sigma^2_c - \frac{1}{2}\lambda^2_w\sigma^2_{gt} \tag{20}$$

For individual firm's stock returns, we guess and verify that

$$pd^i_t = \mu_{pdi} + A^i_{gs}\left(\sigma^2_{gt} - \sigma^2_g\right) + A^i_{is}\left(\sigma^2_{it} - \sigma^2_i\right)$$

As usual, returns are approximated as:

$$\begin{aligned}
r^i_{t+1} &= \Delta d^i_{t+1} + \kappa^i_0 + \kappa^i_1 pd^i_{t+1} - pd^i_t \\
&= \Delta d^i_{t+1} + \kappa^i_0 + \mu_{pdi}\left(\kappa^i_1 - 1\right) + \left(\sigma^2_{gt} - \sigma^2_g\right)A^i_{gs}\left(\kappa^i_1\nu_g - 1\right) + \left(\sigma^2_{it} - \sigma^2_i\right)A^i_{is}\left(\kappa^i_1\nu_i - 1\right) \\
&\quad + A^i_{gs}\kappa^i_1\sigma_w\sigma_{gt}w_{g,t+1} + A^i_{is}\kappa^i_1\sigma_{iw}w_{i,t+1} \\
&= \mu_i + \chi_i\left(\sigma^2_{gt} - \sigma^2_g\right) + \varphi_i\sigma_c\eta_{t+1} + \phi_i\sigma_{gt}w_{g,t+1} + \kappa_i\sigma_{gt}e^i_{t+1} + \zeta_i\sigma_{it}\varepsilon^i_{t+1} \\
&\quad + \kappa^i_0 + \mu_{pdi}\left(\kappa^i_1 - 1\right) + \left(\sigma^2_{gt} - \sigma^2_g\right)A^i_{gs}\left(\kappa^i_1\nu_g - 1\right) + \left(\sigma^2_{it} - \sigma^2_i\right)A^i_{is}\left(\kappa^i_1\nu_i - 1\right) \\
&\quad + A^i_{gs}\kappa^i_1\sigma_w\sigma_{gt}w_{g,t+1} + A^i_{is}\kappa^i_1\sigma_{iw}w_{i,t+1} \\
&= r^i_0 + \left[-A^i_{gs}\left(1 - \kappa^i_1\nu_g\right) + \chi_i\right]\left(\sigma^2_{gt} - \sigma^2_g\right) - A^i_{is}\left(1 - \kappa^i_1\nu_i\right)\left(\sigma^2_{it} - \sigma^2_i\right) \\
&\quad + \varphi_i\sigma_c\eta_{t+1} + \left(\phi_i + A^i_{gs}\kappa^i_1\sigma_w\right)\sigma_{gt}w_{g,t+1} + \kappa_i\sigma_{gt}e^i_{t+1} + \zeta_i\sigma_{it}\varepsilon^i_{t+1} + A^i_{is}\kappa^i_1\sigma_{iw}w_{i,t+1}
\end{aligned}$$

where $r^i_0 = \mu_i + \kappa^i_0 + (\kappa^i_1 - 1)\mu_{pdi}$

Innovations in individual stock market return and individual return variance reflect the additional sources of idiosyncratic risk:

$$\begin{aligned} r^i_{t+1} - \mathbb{E}_t\left[r^i_{t+1}\right] &= \beta_{\eta,i}\sigma_c\eta_{t+1} + \beta_{gs,i}\sigma_{gt}w_{g,t+1} + \kappa_i\sigma_{gt}e^i_{t+1} + \zeta_i\sigma_{it}\varepsilon^i_{t+1} + \kappa^i_1 A^i_{is}\sigma_{iw}w_{i,t+1} \\ \mathbb{V}_t\left[r^i_{t+1}\right] &= \beta^2_{\eta,i}\sigma^2_c + \beta^2_{gs,i}\sigma^2_{gt} + \left(\kappa^i_1 A^i_{is}\right)^2\sigma^2_{iw} + \kappa^2_i\sigma^2_{gt} + \zeta^2_i\sigma^2_{it} \end{aligned}$$

where

$$\begin{aligned} \beta_{\eta,i} &\equiv \varphi_i, \\ \beta_{gs,i} &\equiv \kappa^i_1 A^i_{gs}\sigma_w + \phi_i, \end{aligned}$$

The expression for the equity risk premium on an individual stock is:

$$\mathbb{E}_t\left[r^i_{t+1} - r^f_t\right] + .5\mathbb{V}_t[r^i_{t+1}] = \beta_{\eta,i}\lambda_\eta\sigma^2_c + \beta_{gs,i}\lambda_w\sigma^2_{gt}.$$

The coefficients of the price-dividend equation are obtained from the Euler equation. A^i_{gs} solves the following quadratic equation:

$$\begin{aligned} 0 &= s_{gs} + A^i_{gs}(\kappa^i_1\nu_g - 1) + \chi_i + \frac{1}{2}\kappa^2_i + \frac{1}{2}\left(\kappa^i_1 A^i_{gs}\sigma_w + \phi_i - \lambda_w\right)^2 \\ &= s_{gs} + A^i_{gs}(\kappa^i_1\nu_g - 1) + \chi_i + \frac{1}{2}\kappa^2_i + \frac{1}{2}{\kappa^i_1}^2 {A^i_{gs}}^2\sigma^2_w + \frac{1}{2}(\phi_i - \lambda_w)^2 + (\phi_i - \lambda_w)\kappa^i_1 A^i_{gs}\sigma_w \\ &= \frac{1}{2}\sigma^2_w{\kappa^i_1}^2 {A^i_{gs}}^2 + A^i_{gs}\left(\kappa^i_1\nu_g - 1 + (\phi_i - \lambda_w)\kappa^i_1\sigma_w\right) + s_{gs} + \chi_i + \frac{1}{2}\kappa^2_i + \frac{1}{2}(\phi_i - \lambda_w)^2, \end{aligned}$$

A^i_{is} is given by:

$$A^i_{is} = \frac{\zeta^2_i}{2(1 - \kappa^i_1\nu_i)},$$

and the constant μ_{pdi} is the mean log pd ratio which solves the following non-linear equation:

$$\begin{aligned} 0 = \; & r^i_0 + \mu_s + \frac{1}{2}\gamma^2\sigma^2_g + \frac{1}{2}(\beta_{gs,i} - \lambda_w)^2\sigma^2_g + \frac{1}{2}(\beta_{\eta,i} - \lambda_\eta)^2\sigma^2_c \\ & + \frac{1}{2}\kappa^2_i\sigma^2_g + \frac{1}{2}\zeta^2_i\sigma^2_i + \frac{1}{2}\left(\kappa^i_1 A^i_{is}\right)^2\sigma^2_{iw} \end{aligned}$$

where r^i_0 in the unconditional mean stock return

$$\begin{aligned} r^i_0 &= \mu_i + \kappa^i_0 + (\kappa^i_1 - 1)\mu_{pdi}, \\ \kappa^i_1 &= \frac{\exp(\mu_{pdi})}{1 + \exp(\mu_{pdi})}, \\ \kappa^i_0 &= \log(1 + \exp(\mu_{pdi})) - \kappa^i_1\mu_{pdi} \end{aligned}$$

Given our parameters, only one of the four roots (two for $W_g s$ and two for A^i_{gs}) solves the system of equations for the mean wealth-consumption and mean price-dividend ratio. That root delivers negative values for both coefficients. this is consistent with the previous model.