The Growth Dynamics of Innovation, Diffusion, and the Technology Frontier

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Abstract

The recent literature on idea flows studies technology diffusion in isolation, in environments without the generation of new ideas. Without new ideas, growth cannot continue forever if there is a finite technology frontier. In an economy in which firms choose to innovate, adopt technology, or keep producing with their existing technology, we study how innovation and diffusion interact to endogenously determine the productivity distribution with a finite but expanding frontier. There is a tension in the determination of the productivity distribution—innovation tends to stretch the distribution, while diffusion compresses it. Finally, we analyze the degree to which innovation and technology diffusion at the firm level contribute to aggregate economic growth and can lead to hysteresis.

Keywords: Endogenous Growth, Technology Diffusion, Innovation, Imitation, R&D, Technology Frontier

JEL Codes: O14, O30, O31, O33, O40

1 Introduction

The productivity distribution plays a critical role in many studies in international trade (e.g., Eaton and Kortum (2002) and Melitz (2003)), macroeconomics (e.g., Hsieh and Klenow (2009)), industrial organization (e.g., Hopenhayn (1992), Foster, Haltiwanger, and Syverson (2008)), and other areas of economics. In much of this literature, the productivity of firms evolves exogenously according to some shock process, and thus key determinants of this essential object are not studied. There is a theoretical literature that does focus on productivity growth, as pioneered by Romer (1986, 1990), Segerstrom, Anant, and Dinopoulos (1990), Rivera-Batiz and Romer (1991), Grossman and

Helpman (1991, 1993), and Aghion and Howitt (1992). The key forces that generate productivity growth in these papers are innovation and imitation. Following Kortum (1997) recent papers such as Perla and Tonetti (2014) and Lucas and Moll (2014) used search theory to develop a new microfoundations for technology diffusion. In these papers all firms are alike except for their initial productivity and have no ex ante comparative advantage in innovation or imitation. However, these papers abstract away from innovation, so growth cannot continue forever if the existing technology distribution has a finite frontier, or even if it has infinite support but thin tails. Thus long-run growth in these models relies on the counter-factual assumption that at all times there are firms producing with arbitrarily large productivities, described by a distribution with infinite support.

In this paper, we build on this new microfoundation of technology diffusion by introducing endogenous innovation to explain economic growth through within-firm productivity improvements. The shape of the distribution of active technologies defines the opportunities for adoption. Innovation and adoption interact to determine the shape of the distribution of productivities, which in turn determines the incentives to adopt and to innovate. One of the aims of this paper is to model the evolution of productivity distributions with a frontier that is finite for all times, $t < \infty$. For growth to continue forever, the frontier must grow through innovation. A second important aim of this paper is to model adoption and innovation decisions and their interactions. For simplicity, we start by modeling a deterministic innovation process, and continue by introducing exogenous stochastic innovation through geometric Brownian motion or discrete-state Markov chains, and finally model stochastic innovations that are subject to firm choice.

A common interesting feature of this class of models with initial distributions that have fat tails is the existence of a continuum of stationary distributions, i.e., hysteresis. We explore the conditions under which multiple stationary distributions occur in the various models that we consider. In a number of the models considered, the shape of the stationary distribution depends on the initial distribution (e.g., Perla and Tonetti (2014)), the properties of the exogenous shock process, or both (e.g., Luttmer (2007)). However, when there is an innovation decision, the shape of the stationary distribution is endogenous and depends on the parameters of the model and the optimal adoption and innovation choices of agents.

In Section 2 we develop a model of stochastic technology adoption and deterministic innovation. Technology adoption takes the form of draws from the distribution of existing technology in use, while innovation is simply modeled as exogenous multiplicative growth for all agents. We characterize the full dynamic path of the economy starting from arbitrary initial productivity distributions. An important result is that with a finite technology frontier or a thin tailed initial productivity distribution, eventually all adoption stops and long-run growth is entirely driven by the exogenous innovation process. Furthermore, the economy exhibits hysteresis: there exists a continuum of stationary distributions parameterized by the tail index of the initial distribution.

If innovation is stochastic, incentives to adopt are renewed as successful firms pull away and unsuccessful firms fall behind. We find that there are three distinct sources of growth in these classes of models, which this paper will decompose growth into contributions from: (1) firm level research decisions, i.e., "innovation"; (2) incentives for the relatively unproductive to catchup to the aggregate distribution, i.e., "catchup diffusion"; and (3) firms receiving sequence of bad shocks relative to the growing distribution, i.e., "stochastic diffusion". We find catch-up diffusion can only occur during transition dynamics, or in the long-run with a thick-tailed productivity distribution. Stochastic diffusion can only occur in models with risky innovation.

The forces of diffusion and innovation may interact when firms endogenously choose both, as innovation investment changes with the internalized option value of future technology diffusion opportunities. A novel element of this model is that since firms internalize the value of diffusion, there are interesting trade-offs between innovation and diffusion, which can affect the optimal growth rates. This is in contrast to papers like Luttmer (2007), where diffusion effects incumbents by increasing their fixed costs relative to profits and forcing more exit–effectively, "stochastic diffusion".

Those models provide a different, and more directly Schumpeterian, mechanism.

In Section 2, to study the interaction between stochastic innovation and adoption, we first model firm innovation as an exogenous uncontrolled geometric Brownian motion. There, again, exists a continuum of stationary distributions consistent with balanced growth paths. A stationary distribution with a fatter tail induces more technology adoption, which in turn generates higher growth rates. The tractability of the Geometric Brownian Motion (GBM) growth rate, allows us to analytically decompose total growth into that from catchup vs. stochastic diffusion.

A side-effect of introducing stochastic innovation through GBM is that the support of the productivity distribution becomes infinite instantly. A desirable model property is that at any point in time the technologies in use for production and available for adoption are characterized by a distribution with a finite frontier. To achieve this, in Section 3, we depart from GBM and instead model stochastic innovation as a discrete Markov process. We begin our analysis studying an exogenous uncontrolled innovation process. If the initial distribution has infinite support, there again exist a continuum of stationary distributions, with an endogenous power law distribution, as in the GBM case. The growth rate of the economy will exceed the exogenous innovation growth rate due to growth from technology adoption, also as in the GBM case.

If instead the initial distribution has finite support, it will maintain finite support for all finite times. However, even though the support remains finite, the normalized stationary distribution will be unbounded. That is, the ratio of the frontier productivity to the mean productivity, $\bar{z}$, will grow towards infinity as time progresses. The distinction between finite and bounded stationary distributions is an important distinction, as we do not observe a perpetual spreading of the productivity distribution in data. As in previous cases with finite support, with unbounded support there exists a unique stationary distribution with long-run adoption. Since the frontier remains finite $\forall\, t\ < \infty$, the long-run growth rate must equal the exogenous innovation growth rate.

To achieve a stationary distribution that is both finite and bounded, in that $\bar{z}$ is a finite constant, we modify the innovation process to allow for a positive fraction of some agents to leap-frog to the frontier. This is a continuous time analog of the traditional quality-ladder model, in which successful innovators jump to the technology frontier. In this case, we again have a unique stationary distribution, where the long-run growth rate is equal to the exogenous innovation rate, even though firms continue to adopt.

Thus far, we have considered innovation processes that were exogenous. In Section 4, we generalize the model to allow for agents to choose their innovation growth rate at some cost. Agents optimally choose to either invest in adoption or innovation, with the result that innovation growth rates are increasing with an agent's productivity. Starting with a finite initial distribution, if agents cannot leap-frog to the frontier, the unique stationary distribution is unbounded. In the long run, the growth rate of the economy is equal to the innovation growth rate chosen by the frontier agent.

However, if we also allow some agents to leap-frog to the frontier, there now exists a continuum of stationary distributions that are bounded in relative terms (i.e., the ratio of most to least productive does not diverge). The growth rate of the economy is endogenous and equal to the innovation growth rate chosen by the firm at the frontier. However, in contrast to all previous cases, even with finite initial distributions, there exists a continuum of stationary distributions. This is because there is an important interaction between the incentives to adopt and to innovate that generates a self-sustaining feedback. We can index the stationary distributions by the relative frontier, $\bar{z}$. Even with a finite frontier, a distribution that has more weight in higher productivities produces stronger incentives to adopt. The optimal innovation policy is increasing in productivity, as opposed to the exogenous innovation policy that was flat. This innovation policy generates more mass in the right tail, and thus generates more incentives to adopt, as adopters internalize the option value of innovation.

1.1 Recent Literature

Related papers in this class of models include Lucas and Moll (2014), Kortum (1997), Luttmer (2007, 2014) and Sampson (2014) which emphasize selection from optimal entry/exit, and Alvarez, Buera, and Lucas (2008, 2013), which emphasizes diffusion as an arrival rate of ideas from the productivity distribution. Buera and Oberfield (2014) is a related semi-endogenous growth model of international diffusion of technology and its connection to trade. Another approach, taken in Jovanovic and Rob (1989), is to add positive spillovers from the diffusion process itself, which can create balanced growth.

While Perla and Tonetti (2014) isolated the role of growth through "catch-up diffusion", in some sense the role of "stochastic diffusion" is isolated in Luttmer (2007). The "catchup diffusion" effect is not present in Luttmer (2007) in the same sense, as the incumbent firms lower in the productivity distribution gain no benefit from growth. However, in our model, "stochastic diffusion" is different from Luttmer (2007), as firms internalize the value of an upgrade rather than being driven into un-profitability and exit from GE effects.

In this paper we are considering process innovation rather than new product innovation. Smaller firms may be especially innovative in coming up with new products as in Klette and Kortum (2004) or Acemoglu, Akcigit, Bloom, and Kerr (2013), but this is not considered in our innovation technology, as all firms have one product and the number of products in equilibrium is kept fixed for simplicity. Other papers emphasizing the role of an endogenous innovation choice include Atkeson and Burstein (2010) and Stokey (2014).

Acemoglu, Aghion, and Zilibotti (2006), König, Lorenz, and Zilibotti (2012), Chu, Cozzi, and Galli (2014), Stokey (2014), and Benhabib, Perla, and Tonetti (2014) also explore the relationship between innovation and diffusion from different perspectives. The crucial element that enables the interesting trade-off between innovation and technology diffusion in our model is that the incumbents internalize some of the value from the evolving distribution of technologies, distorting their innovation choices. We describe this as an "option value of diffusion", where incumbents take into account the possibility of future improvements in their productivity through jumps from technology diffusion. The lower the relative productivity of a firm, the higher the expected benefit of adoption via a jump to a superior technology, and the sooner the expected time to execute the adoption option. Therefore, low productivity firms have high option values of diffusion, while very high productivity firms may have an asymptotically irrelevant contribution from technology diffusion.

This tension between innovation and technology diffusion explored here has a different emphasis in Luttmer (2007, 2012, 2014), where the generator of diffusion is entry/exit in equilibrium, and only new entrants can internalize the benefits of technology diffusion. The main similarity is that in both papers, some firms sample from the existing distribution of productivity. In particular, Luttmer (2007) is interested in the role of technology diffusion through entry, so it is the entrants who gain the benefits of a growing economy. As incumbents pay a fixed cost that grows with the scale of the economy, entry can spur more exit. Therefore negative profits that results in exit leads to entry and to technology diffusion. In our model incumbents, or operating firms, choose when to exploit the incentives to adopt a new technology.[1] The difference between whether incumbents or entrants internalize the value of a growing economy leads to very different implications for technology diffusion.

[1]Mechanically, these differences manifest themselves in the option value of the Bellman equation. In Luttmer (2007), incumbents are only affected negatively by growth and have a zero option value of technology diffusion, whereas in our model incumbents have a positive option value of diffusion as they can always adopt from by taking a draw from the existing distribution.

2 Baseline Model with Exogenous Innovation

2.1 Model Summary

We begin by describing features of the baseline model of catchup and stochastic diffusion in the absence of a finite technology frontier, similar to Perla, Tonetti, and Waugh (2014) and Perla and Tonetti (2014). To investigate the role of stochastic innovation, this model nests a stochastic, exogenous innovation process modelled as Geometric Brownian Motion (GBM).[2,3]

For notational simplicity, define the differential operator $\boldsymbol{\partial}$ such that $\boldsymbol{\partial}_z \equiv \frac{\partial}{\partial z}$ and $\boldsymbol{\partial}_{zz} \equiv \frac{\partial^2}{\partial z^2}$. When a function is univariate, derivatives will be denoted as $v'(z) \equiv \frac{\mathrm{d}v(z)}{\mathrm{d}z}$.

Firm Heterogeneity and Choices Assume firms producing a homogeneous product are heterogeneous over their productivity, Z. The distribution of productivity at time t is $\Phi(t, Z)$. Define the technology frontier as the maximum productivity, $B(t) \equiv \sup\{\text{support}\{\Phi(t, \cdot)\}\} \leq \infty$, and normalize the mass of firms to 1 so that $\Phi(t, B(t)) = 1$. At any point in time, the minimum of the support of the distribution will be an endogenously determined $M(t)$, so that $\Phi(t, M(t)) = 0$.

A firm with productivity Z can choose to continue producing with its existing technology, in which case it would grow stochastically, or it can choose to adopt a new technology instantaneously. The cost of adoption scales with the economy, and for simplicity is proportional to the endogenous scale of the economy, $M(t)$.[4]

Diffusion and Evolution of the Distribution If a firm adopts a new technology, then it immediately changes its productivity to a draw from the distribution $\Phi(t, Z)$, potentially distorted.[5] The degree of imperfect mobility is indexed by $\kappa > 0$ where the agent draws its Z from the cdf $\Phi(t, Z)^\kappa$. Note that for higher κ, the probability of a better draw increases. As $\Phi(t, B(t))^\kappa = 1$ and $\Phi(t, M(t))^\kappa = 0$, for all $\kappa > 0$, this is a valid probability distribution.

In equilibrium, all firms choose an identical threshold, $M(t)$, above which they will continue operating with their existing technology. A firm with $Z \leq M(t)$ chooses to adopt a new technology. This endogenous $M(t)$ is the evolving minimum of the $\Phi(t, Z)$ distribution. To show that the minimum of support is the endogenous threshold, assume a Poisson arrival rate of draw opportunities approaching infinity. In any infinitesimal time interval firms would gain an acceptable draw with probability 1, so that $Z > M(t)$ almost surely.[6] Hence the equilibrium does not depend on whether draws are from the unconditional distribution or are from the distribution conditional on being above the current adoption threshold.

A flow $S(t) \geq 0$ of firms cross into the adoption region at time t and choose to adopt a new technology. For the case in which innovation is driven by GBM, with a drift of γ and variance σ,

[2] Luttmer (2014) emphasizes the role of a stochastic shock as "experimentation" as distinct from deterministic innovation, and important in the generation of endogenous tail parameters. Staley (2011) adds exogenous geometric Brownian motion to an economy with a Lucas (2009) technology diffusion model, and investigates the evolution of the productivity distribution and growth rates.

[3] For a version of the model using monopolistic competition and associated equilibrium conditions, see Section 2.7 and Appendix C.

[4] On a BGP, this is equivalent to having the firm pay an upgrade cost as a fraction of the Z they draw. In Section 2.7 and Appendix C, a more elaborate version of this is derived in general equilibrium where ζ is the quantity of labor required for adoption, but it ends up being qualitatively equivalent. An alternative is to have the cost scale with the firm's Z, which introduces a less convenient smooth pasting condition, but remains otherwise tractable.

[5] See Appendix A.2 for a proof that the ability for a firm to recall its last productivity doesn't change the equilibrium conditions, and Appendix A.1 for a derivation of this where adoption is not instantaneous.

[6] The derivation of the cost function as the limit of the arrival rate of unconditional draws is in Appendix A.1.

the Kolmogorov Forward Equation (KFE) below (in cdfs) is

$$\boldsymbol{\partial}_t \Phi(t,Z) = \underbrace{-(\gamma+\sigma^2/2)Z\,\boldsymbol{\partial}_Z \Phi(t,Z)}_{\text{Deterministic Drift}} + \underbrace{\frac{\sigma^2}{2}Z^2\,\boldsymbol{\partial}_{ZZ}\Phi(t,Z)}_{\text{Brownian Motion}} + \underbrace{S(t)\Phi(t,Z)^\kappa - S(t)}_{\text{Firm draws - Adopters}}, \quad \text{for } M(t) \le Z \le B(t) \tag{1}$$

If $\sigma > 0$, then $B(t) = \infty$ immediately. Otherwise, if $\sigma = 0$ and $B(t) < \infty$, then the frontier grows at rate $B'(t)/B(t) = \gamma$.

2.2 Firm's Problem

The firm maximizes the present discounted value of profits, discounting at rate $r > 0$, where Z evolves following a GBM. The firm chooses the productivity threshold $M(t)$, below which they choose to adopt a new technology. A firm's productivity may hit $M(t)$ due to a sequence of bad relative shocks or because the $M(t)$ barrier is overtaking their Z. This problem can be cast as an optimal stopping problem where $V(t,Z)$ is the continuation value.[7]

Assuming continuity of $\Phi(0,Z)$, then the necessary conditions for an equilibrium, $\Phi(t,Z)$ and $M(t)$, are,

$$rV(t,Z) = Z + (\gamma+\sigma^2/2)Z\,\boldsymbol{\partial}_Z V(t,Z) + \frac{\sigma^2}{2}Z^2\,\boldsymbol{\partial}_{ZZ}V(t,Z) + \boldsymbol{\partial}_t V(t,Z) \tag{2}$$

$$V(t,M(t)) = \int_{M(t)}^{B(t)} V(t,\tilde{Z})\mathrm{d}\Phi(t,\tilde{Z})^\kappa - \zeta M(t) \tag{3}$$

$$\boldsymbol{\partial}_Z V(t,M(t)) = 0, \quad \text{if } S(t) > 0 \tag{4}$$

$$\boldsymbol{\partial}_t \Phi(t,Z) = -(\gamma+\sigma^2/2)Z\,\boldsymbol{\partial}_Z \Phi(t,Z) + \frac{\sigma^2}{2}Z^2\,\boldsymbol{\partial}_{ZZ}\Phi(t,Z) + S(t)\Phi(t,Z) - S(t) \tag{5}$$

$$\Phi(t,M(t)) = 0 \tag{6}$$

$$\Phi(t,B(t)) = 1 \tag{7}$$

$$B'(t)/B(t) = \gamma, \text{ if } \sigma > 0 \tag{8}$$

$$S(t) = \big(M'(t) - \gamma M(t)\big)\,\boldsymbol{\partial}_Z \Phi(t,M(t)) + \frac{\sigma^2}{2}M(t)^2\boldsymbol{\partial}_{ZZ}\Phi(t,M(t)) \tag{9}$$

where equation (2) is the Bellman Equation in the continuation region, and equations (3) and (4) are the value matching and smooth pasting conditions. While the value matching condition always holds, the smooth pasting condition is only neccesary if there is negative drift relative to the boundary $M(t)$. Equations (5) to (7) are the Kolmogorov forward equation with the appropriate boundary conditions. Equation (8) is the deterministic growth of the boundary, which is simply the growth rate of frontier agents as $M(t) < B(t)$ in equilibrium. Finally, equation (9) is the flow of adopters as a function of the pdf at the optimal boundary and the drift of firms relative to the moving boundary.

2.3 Normalization and Stationarity

To find a balanced growth path (BGP), it is convenient to transform this system to a stationary set of equations by normalizing variables relative to the endogenous boundary $M(t)$. Define the

[7]The sequential formulation and connection to a recursive optimal stopping of a deterministic process is given on page 110-112 of Stokey (2009).

change of variables, normalized distribution, and normalized value functions as,

$$z \equiv \log(Z/M(t)) \tag{10}$$
$$F(t,z) = F(t, \log(Z/M(t))) \equiv \Phi(t,Z) \tag{11}$$
$$v(t,z) = v(t, \log(Z/M(t))) \equiv \frac{V(t,Z)}{M(t)} \tag{12}$$

The relative technology frontier is then,

$$\bar{z}(t) \equiv \log(B(t)/M(t)) \leq \infty \tag{13}$$

The adoption threshold was chosen to be normalized to $z = \log(M(t)/(M(t))) = 0$. See Figure 1 for a comparison of the normalized and unnormalized distributions.

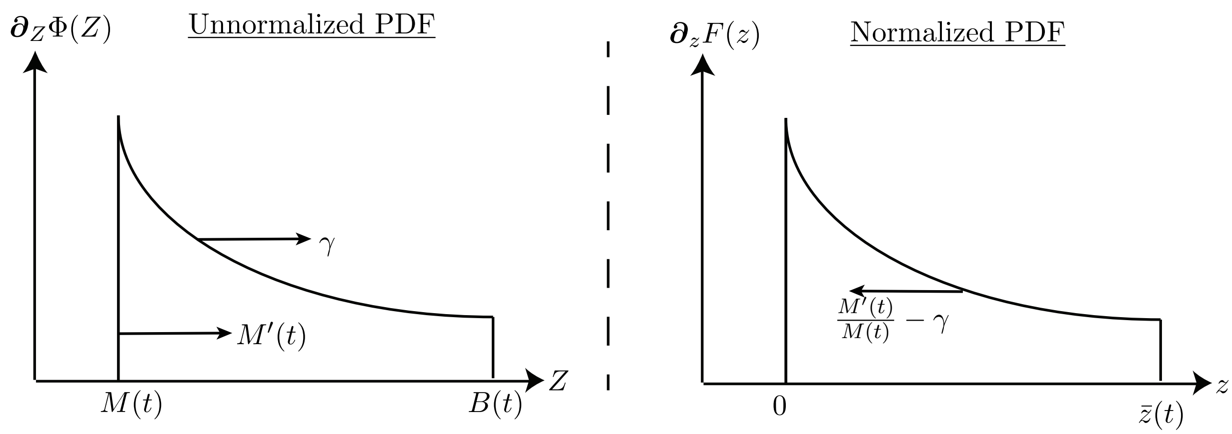

Figure 1: Normalized vs. Unnormalized Distributions

With the above normalizations, the value function, productivity distribution, and growth rates can be stationary and independent of time. An important example is when $\Phi(t,Z)$ is Pareto with minimum of support $M(t)$ and tail parameter α:

$$\Phi(t,Z) = 1 - \left(\frac{M(t)}{Z}\right)^{\alpha}, \text{ for } M(t) \leq Z \tag{14}$$

Then $F(t,z)$ is independent of M and t:

$$F(t,z) = 1 - e^{-\alpha z}, \text{ for } 0 \leq z. \tag{15}$$

This is the cdf of an exponential distribution, with parameter $\alpha > 1$. From a change of variables, if $X \sim Exp(\alpha)$, then $e^X \sim Pareto(1,\alpha)$. Hence, α is the tail index of the unnormalized Pareto distribution for Z. When the distribution is not time varying, let $F'(z)$ denote the probability density function.

A full derivation of the normalization is done in Appendix B.1, which leads to the following normalized set of equations. Given an initial condition $F(0,z)$, the dynamics of $v(t,z)$, $F(t,z)$,

$g(t) \geq 0$, and $S(t) \geq 0$, must satisfy

$$(r - g(t))v(t,z) = e^z + (\gamma - g(t))\boldsymbol{\partial}_z v(t,z) + \tfrac{\sigma^2}{2}\boldsymbol{\partial}_{zz} v(t,z) + \boldsymbol{\partial}_t v(t,z) \tag{16}$$

$$v(t,0) = \int_0^\infty v(t,z)\mathrm{d}F(t,z)^\kappa - \zeta \tag{17}$$

$$\boldsymbol{\partial}_z v(t,0) = 0 \tag{18}$$

$$0 = (g(t) - \gamma)\boldsymbol{\partial}_z F(t,z) + \tfrac{\sigma^2}{2}\boldsymbol{\partial}_{zz} F(t,z) + S(t)F(t,z)^\kappa - S(t) \tag{19}$$

$$F(t,0) = 0 \tag{20}$$

$$F(t,B(t)) = 1 \tag{21}$$

$$S(t) = (g(t) - \gamma)\boldsymbol{\partial}_z F(t,0) + \tfrac{\sigma^2}{2}\boldsymbol{\partial}_{zz} F(t,0) \tag{22}$$

In stationary form, these become $v(z)$, $F(z)$, $g \geq 0$, $S > 0$, and $0 < \bar{z} \leq \infty$ such that,

$$(r-g)v(z) = e^z + (\gamma - g)v'(z) + \tfrac{\sigma^2}{2}v''(z) \tag{23}$$

$$v(0) = \int_0^\infty v(z)\mathrm{d}F(z)^\kappa - \zeta \tag{24}$$

$$v'(0) = 0 \tag{25}$$

$$0 = (g - \gamma)F'(z) + \tfrac{\sigma^2}{2}F''(z) + SF(z) - S \tag{26}$$

$$F(0) = 0 \tag{27}$$

$$F(\infty) = 1 \tag{28}$$

$$S = (g - \gamma)F'(0) + \tfrac{\sigma^2}{2}F''(0) \tag{29}$$

The value matching condition in (24) can also be written, using (D.124), as

$$\zeta = \int_0^\infty v'(z)\left(1 - F(z)^\kappa\right)\mathrm{d}z \tag{30}$$

To interpret (30), in equilibrium, as the firm is already about to gain $v(0)$ costlessly, it is indifferent between production and adoption only if the sum of all marginal values over the counter-cdf of draws is identical to the cost of adoption. (29) can be understood as the flux crossing the endogenous barrier, where in normalized terms the barrier is moving at rate $g - \gamma$ and collecting the infinitesimal mass at the boundary, i.e. the pdf F'(0). Additionally, there is a Brownian diffusion term where a σ dependent flow of agents are moving back purely randomly.

2.4 Deterministic Balanced Growth Path

We begin by analyzing the deterministic balanced growth path, in which $\sigma = 0$, and innovation is common and constant for all firms. First we assume that firms are adopting technologies from the unconditional distribution by setting $\kappa = 1$. Proposition 1 characterizes the balanced growth path equilibrium.

Proposition 1 (Deterministic Equilibrium with Pareto Initial Condition and $\kappa = 1$). *If* $\Phi(0, Z) = 1 - \left(\frac{M_0}{Z}\right)^\alpha$, $\alpha > 1$, *and* $r > \gamma + (\zeta\alpha(\alpha - 1))^{-1} > 0$, *then*

$$g = \frac{1 - \zeta(\alpha - 1)(r - \alpha\gamma)}{\zeta(\alpha - 1)^2}, \tag{31}$$

$$v(z) = \frac{1}{r - \gamma}e^z + \frac{1}{\nu(r - \gamma)}e^{-\nu z}, \tag{32}$$

where,

$$\nu \equiv \frac{r-g}{g-\gamma} > 0, \tag{33}$$

and the stationary distribution in logs is

$$F(z) = 1 - e^{-\alpha z}. \tag{34}$$

Proof. See Appendix B.2. □

The first term of equation (32) is the value of production in perpetuity. This would be the value of the firm if it did not have the option of adopting a better technology. The second term of equation (32) is the *option value of technology diffusion*. It is decreasing in z since the optimal time to adopt is increasing with better relative technologies. The exponent, ν in (33) determines the rate at which the option value is discounted. More discounting of the future, or slower growth rates, lead to a more rapid drop-off of this option value.

For $\kappa > 0$, the draws are distorted and the stationary distribution is a non-Pareto power-law.

Definition 1 (Power Law Distribution)**.** *A distribution $\Phi(Z)$ is defined as a power-law, or equivalently is fat-tailed, if there exists an $\alpha > 0$ such that for large Z, the counter-cdf is asymptotically Pareto $1 - \Phi(Z) \approx Z^{-\alpha}$. Under the change of variables $z \equiv \log(Z)$ with $F(z) \equiv \Phi(e^z)$, the distribution $\Phi(Z)$ is a power-law if the counter-cdf is asymptotically exponential $1 - F(z) \approx e^{-\alpha z}$.*

Pareto distributions trivially fulfill the requirements of a Power Law. See Appendix B.5.1 for more formal definition based on the theory of regularly varying functions, and Appendix B.3 for more on the tail index α. We say a distribution is thin-tailed if there does not exist any $\alpha > 0$ such that the definition can hold. The following proposition generalizes Proposition 1 for $\kappa > 0$.

Proposition 2 (Deterministic Equilibrium with Pareto Initial Condition and $\kappa > 0$)**.** *For a given g, the value function is independent of κ. For $\nu \equiv \frac{r-g}{g-\gamma}$,*

$$v(z) = \frac{1}{r-\gamma} e^z + \frac{1}{\nu(r-\gamma)} e^{-\nu z} \tag{35}$$

Given $F'(0)$ parameterizing the set of stationary equilibria, the quantile of $F(z)$ is defined as $Q(q) = F^{-1}(q)$, as a form based on the Hypergeometric function, ${}_1\mathbb{F}_2(\cdot)$.[8]

$$Q(q) = \frac{q}{F'(0)}\ {}_1\mathbb{F}_2\left(1, 1/\kappa, 1 + 1/\kappa, q^\kappa\right) \tag{39}$$

[8] One definition for the Hypergeometric function is derived by simplifying Euler's representation in its integral form, is,

$$\int_0^y \frac{1}{a + b\tilde{y}^\kappa}\, d\tilde{y} = \frac{y}{a}\ {}_1\mathbb{F}_2(1, 1/\kappa, 1 + 1/\kappa, -\tfrac{b}{a} y^\kappa) \tag{36}$$

Derivatives of Hypergeometric functions can use the following result:

$$\tfrac{\mathrm{d}}{\mathrm{d}p}\ {}_1\mathbb{F}_2(a, b, c; p) = \frac{ab}{c}\ {}_1\mathbb{F}_2(a+1, b+1, c+1; p) \tag{37}$$

In our specific case, other transformations can be made,

$$\tfrac{\mathrm{d}}{\mathrm{d}p}\ {}_1\mathbb{F}_2(1, 1/\kappa, 1 + 1/\kappa; p) = \frac{\frac{1}{1-p} - {}_1\mathbb{F}_2(1, 1/\kappa, 1 + 1/\kappa; p)}{\kappa p} \tag{38}$$

The power law tail index of $F(z)$ is

$$\alpha = \kappa F'(0) \tag{40}$$

Given a $F'(0)$, the equilibrium growth rate fulfills the following equation

$$\frac{1}{r-g} + \zeta = \int_0^1 \left(\frac{1}{r-\gamma} e^{Q(q^{1/k})} + \frac{1}{\nu(r-\gamma)} e^{-\frac{r-g}{g-\gamma} Q(q^{1/k})} \right) \mathrm{d}q \tag{41}$$

Proof. See Appendix B.3. □

The definition in (39) provides an expression for the quantile of the distribution, and (41) can be solved numerically for g. In the simple case of a Pareto distribution and $\kappa = 1$, $\alpha = F'(0)$, so (40) nests the Pareto case of $\kappa = 1$. Intuitively, as κ grows, the draws are more skewed towards the lower tail and hence the thinner tailed distribution. As the $\kappa > 0$ case features the same qualitative behavior as the $\kappa = 1$ case, but is less analytically tractable, we will concentrate on $\kappa = 1$ for most of our analysis.

2.5 Dynamic Deterministic Equilibrium

In this section, we characterize the transition dynamics of an economy that features deterministic innovation, i.e., $\sigma = 0$. For the dynamics, it will sometimes be more convenient to calculate the growth rate of M at the state M. This can be translated to t from the law of motion if $g(t) > 0$ on some interval $t \in [0, T]$ for $T \leq \infty$.[9] Denote $\hat{g}(M) = M'(M)/M$. On a balanced growth path, $g = M'(t)/M$, which also is the growth rate of output. For notational simplicity, define a truncation of the initial condition, $\Phi(0, Z)$, at M as,

$$\Phi_M(Z) \equiv \frac{\Phi(0,Z) - \Phi(0,M)}{1 - \Phi(0,M)}, \quad Z \in [M, \infty) \tag{42}$$

with pdf

$$\Phi'_M(Z) \equiv \frac{\Phi'(0,Z)}{1 - \Phi(0,M)}, \quad Z \in [M, \infty). \tag{43}$$

For simplicity, set $\kappa = 1$ and $\gamma = 0$ to analyze the de-trended version of the model with unconditional draws. To simplify algebra, let the cost of adoption be ζZ rather than $\zeta M(t)$.[10] Under these conditions, Proposition 3 describes the growth rate along the transition path and its dependence on initial conditions.

Proposition 3 (Dynamic Solution for an Arbitrary Initial Condition)**.** *The growth rate of M given an initial condition $\Phi(0, Z)$ is determined by is given by*

$$\hat{g}(M) = \frac{\frac{1}{M}\int_M^\infty Z\Phi'_M(Z)\mathrm{d}Z - (1 + \zeta r)}{\zeta M \Phi'_M(M)}. \tag{44}$$

Additionally,

[9] To see this, note that if $g(t) > 0$ on some interval, then $M(t)$ is strictly increasing. Therefore, due to the lack of aggregate shocks, $M(t)$ is bijective on this interval.

[10] In the recursive formulation with a ζZ cost, the only change to the set of equations is that the smooth pasting condition becomes $\boldsymbol{\partial}_Z V(t, Z) = -\zeta$. This ends up simplifying the algebra for solving for the transition dynamics. In the rest of the paper, we use the cost function $\zeta M(t)$. Recall that the gross value of search is independent of Z. Hence, an economic reason to use a non-Z dependent cost in the other sections is if we believe the cost should be symmetrically independent of Z. See Appendix B.5.1 for details.

1. *On a BGP, if* $\phi(0, Z)$ *thin-tailed then* $\lim_{M\to\infty} \hat{g}(M) = \gamma$*. The asymptotic distribution does not converge to the frontier, and is not degenerate, so* $\lim_{t\to\infty} M(t)/B(t) < 1$*.*

2. *Otherwise, on a BGP with* $g > \gamma$*, then then* $\Phi(0, Z)$ *has power law tails and the asymptotic growth rate is determined by the tail parameter* α *of its distribution, where*[11]

$$g = \gamma + \frac{1 - \zeta(r-\gamma)(\alpha - 1)}{\zeta\alpha(\alpha - 1)} \tag{45}$$

Proof. See Appendix B.5.1. (44) can be extended to the $\gamma > 0$ case by simply adding the γ trend to the growth of M. □

The key result is that growth driven through technology diffusion can only continue forever if the distribution has power law tails. Figure 2 shows an example of a Frechet initial condition converging towards the growth rate determined from its power law tail. The initial growth rate is higher since the non-monotonicity of the Frechet density provides strong incentives for catchup diffusion, and the tail becomes relatively thinner as z increases.

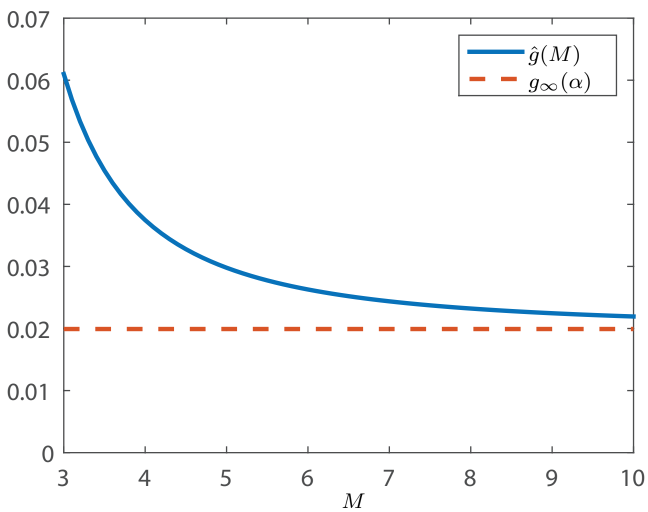

Figure 2: Dynamics from a Frechet Initial Condition

Without an initial power-law tail, the growth rate is asymptotically determined by the innovation rate. In that case, growth from catchup diffusion stops when the productivity distribution compresses enough to balance the returns from innovation with the returns from adoption at the margin: the adoption threshold grows at the same rate as that of innovation. Hence, no firms will hit the diffusion threshold as all are growing with deterministic innovation.[12]

[11] The difference between (45) and the $\sigma = 0$ case of (48) come from the alternative cost function.

[12] In order to see if an endogenous choice of R&D intensity changes these results within the deterministic version of the model, Technical Appendix B adds an innovation decision for operating firms within this simple framework. As it can be shown that the innovation rate is weakly increasing in z, it is conceivable that in a stationary equilibrium, agents near the adoption barrier may choose not to innovation, which could lead to equilibrium technology diffusion. However, as in the exogenous innovation case, with a finite productivity distribution, the only stationary equilibrium is one in which every firm does (or does not) conduct innovation and there is no equilibrium technology diffusion. Endogenous innovation within the stochastic model is discussed in Section 4.

2.6 Stochastic Exogenous Innovation

If innovation is stochastic and driven by GBM, even with a finite $F(0, z)$ initial condition, the support of a stationary $F(z)$ must be $[0, \infty)$. With a continuum of agents, Brownian motion instantaneously increases the support of the distribution.

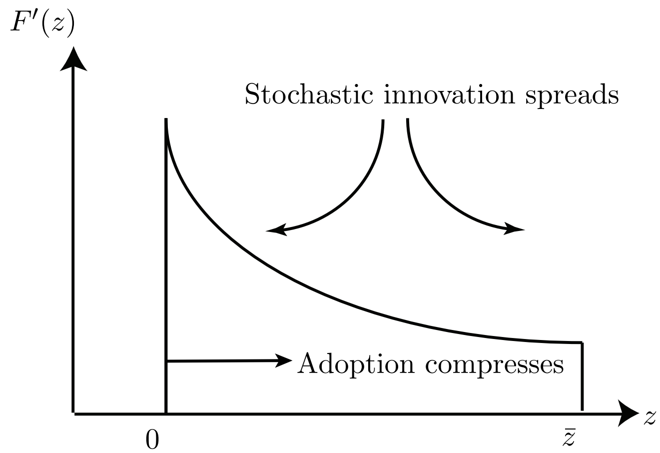

Figure 3: Tension between Stochastic Innovation and Adoption

When geometric random shocks are added, the stationary solutions will endogenously become power-law distributions, as discussed with generality in Gabaix (2009). Figure 3 provides some intuition on how these forces can create a stationary distribution with technology diffusion. Stochastic innovation spreads out the distribution and in the absence of endogenous adoption this would prevent the existence of a stationary distribution.[13] However, as the distribution spreads, the incentives to adopt a new technology increase, and this in turn acts to compress the distribution. In equilibrium, technology diffusion occurs with certainty because otherwise the returns to adopt a new technology become infinite in relative terms.

To see this intuitively, consider the alternative where there are geometric stochastic shocks for operating firms, but no firm chooses to adopt new technologies. A distribution generated by a random walk has a growing variance, and its support is unbounded unless there is adoption or death, even if the drift is zero. But when firms are choosing whether to adopt a new technology or not, the increasing variance of the distribution implies the returns to adoption go to infinity, overcoming any finite adoption cost. The firms at the lower end of the productivity distribution would choose to adopt, and the spread of the distribution would be contained.

Proposition 4 (Equilibrium with Geometric Brownian Motion Innovations)**.** *A continuum of equilibria parameterized by α exist satisfying*

$$\alpha > \frac{1}{2}\left(1 + \sqrt{\frac{4 + \zeta(r - \gamma - \sigma^2/2)}{\zeta(r - \gamma - \sigma^2/2)}}\right) \tag{46}$$

and

$$0 > (\alpha - 1)^2\alpha\zeta^2\sigma^4 - 2(\alpha - 1)\zeta\sigma^2\left((\alpha - 3)\alpha + \left(\alpha^2 - 1\right)\zeta(r - \gamma)\right) + 4((\alpha - 1)\zeta(r - \gamma) - 1)^2. \tag{47}$$

[13] Without endogenous adoption there is no "absorbing" or "reflecting" barrier and geometric random shocks lead to a diverging variance in the KFE.

For a given α, the growth rate is

$$g = \underbrace{\frac{1-(\alpha-1)\zeta\left(r-\alpha\gamma\right)}{(\alpha-1)^2\zeta}}_{\text{Catch-up Diffusion}} + \underbrace{\frac{\sigma^2}{2}\frac{\alpha\left(\alpha(\alpha-1)\left(r-\gamma-\frac{\sigma^2}{2}\right)\zeta-2\right)+1}{(\alpha-1)\left((\alpha-1)\left(r-\gamma-\frac{\sigma^2}{2}\right)\zeta-1\right)}}_{\text{Stochastic Diffusion}}. \tag{48}$$

Furthermore the stationary distribution and value function are

$$F(z) = 1 - e^{-\alpha z} \tag{49}$$

$$v(z) = \frac{1}{r-\gamma-\sigma^2/2}\left(e^z + \frac{1}{\nu}e^{-\nu z}\right), \tag{50}$$

where,

$$\nu = -\frac{\gamma - g}{\sigma^2} + \sqrt{\left(\frac{g-\gamma}{\sigma^2}\right)^2 + \frac{r-g}{\sigma^2/2}}. \tag{51}$$

Proof. See Appendix B.4. □

A lower bound on α is defined by (46), while the upper bound on α is a root of the cubic (47) at equality. The continuum of equilibria in this case is similar to that discussed in Luttmer (2007, 2012, 2014). While there exist multiple stationary equilibria, the uniqueness of a stationary equilibrium given a particular initial condition in a similar model is discussed in Luttmer (2014). This corresponds to hysteresis (i.e., dependence on initial condition $\Phi(0, Z)$): a unique path exists given a particular set of parameters and initial conditions. Furthermore, given α, the flow of adopters, S, satisfies

$$\alpha = \frac{g-\gamma}{\sigma^2} - \sqrt{\left(\frac{g-\gamma}{\sigma^2}\right)^2 - \frac{S}{\sigma^2/2}}, \tag{52}$$

that is,

$$S = \frac{\sigma^2}{2}\left(\alpha - \frac{g-\gamma}{\sigma^2}\right)^2 + \left(\frac{g-\gamma}{\sigma^2}\right)^2. \tag{53}$$

The tail index in (52) is of a very similar form to that in Luttmer (2014) Proposition 1, where our endogenous flow of adopters, S, is related to Luttmer's exogenous arrival rate of learning opportunities.

The growth rate and value function as $\sigma \to 0$ is identical to that in Proposition 1. Hence, in the decomposition of growth rates of (48), the first term is the "catch-up diffusion" caused by the same incentives as in the deterministic case, while the second is the "stochastic diffusion" caused by negative (unlucky) shocks to firms close to the adoption threshold. While some firms will receive positive shocks and be lucky when near the threshold, half of the drift-adjusted Brownian motion ends up crossing the threshold.

Another place where the Brownian motion can be seen to influence the equilibrium is in (51). This is a stochastic version of the ν in (33). Higher variance decreases the expected hitting time at which productivity reaches the normalized zero threshold, and hence increases the value of technology diffusion. Note that, due to risk neutrality, the variance is constructed to have no direct effect on the expected continuation profits, just on the expected length of time the firm will operate its existing technology.

To decompose the contributions to growth, define the growth rate with no stochastic shocks to productivity as $g_c(\alpha)$ as in (54). Since the contributions to growth rates from stochastic diffusion (i.e., unlucky experimentation pushing firms below the boundary in relative terms) have been removed, this can be interpreted as "catchup diffusion".

$$g_c(\alpha) \equiv \lim_{\sigma \to 0} g(\alpha; \sigma) \tag{54}$$

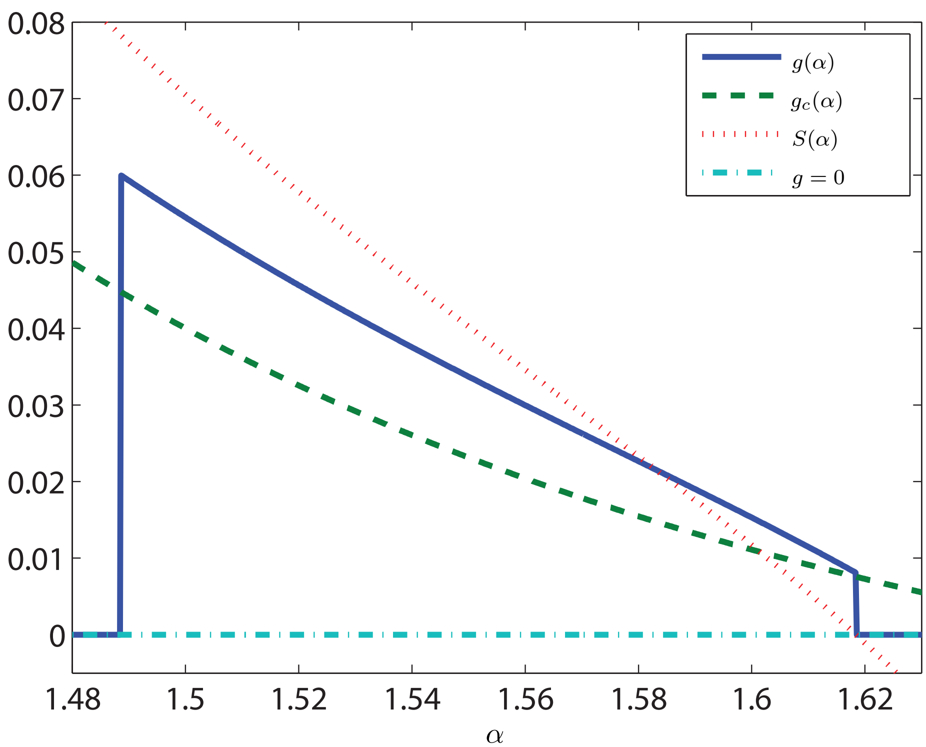

Figure 4: Growth rate and Growth from Catchup Diffusion as a Function of α

Figure 4 shows a plot of $g(\alpha)$ from equation (48) and $g_c(\alpha)$ from (54), for parameter values: $r = 0.06, \zeta = 25, \sigma = .1$, and $\gamma = 0$. The range of admissible α from the intersection of the sets in (46) and (47) is relatively tight, between about 1.49 and 1.62. As discussed before, a stationary equilibrium with strictly positive Brownian term and no equilibrium technology diffusion is not possible, so the $\alpha \simeq 1.62$ is likely the stationary distribution for a large number of initial conditions with relatively thin tails. Equilibria where $\alpha < 1.49$ here are simply not defined as the option value explodes.[14]

The minimum growth rate of around 1% occurs at the maximum α, and is strictly positive. This occurs at the point where the contribution of "stochastic diffusion", $g - g_c$, is zero. Otherwise, the contribution stochastic diffusion is strictly positive in the admissible range

2.7 Monopolistic Competition and Adoption Costs in Labor

In most of this paper, we concentrate on a simple formulation with linear profits and adoption costs scaling with M(t) as the economy grows. Richer models of general equilibrium with employment,

[14]In determining whether these α are empirically plausible, consider the crude adjustment to tail indices based on for markups discussed in (59).

labor market clearing, and downward sloping demand are more directly comparable to Luttmer (2007) and provide qualitatively very similar results.[15]

For example, assume a standard setup with a representative consumer of a final good with intertemporal elasticity of substitution (IES) $\varsigma > 0$ and discount rate $\hat{r}$. On a BGP with constant consumption growth, g, the interest rate is $r \equiv \hat{r} + \varsigma g$. The consumer owns a perfectly diversified portfolio of firms, who discount using this rate r. The consumer also provides one unit of labor inelastically.

Assume that intermediate good producing firms are monopolistically competitive with a linear production function in labor. A competitive final good sector combines these intermediate goods using a constant elasticity of substitution aggregator with elasticity $\rho > 1$. Labor can be hired for the production of intermediate goods. Labor can also be hired to be used to adopt a new technology, at a cost of ζ units of labor paid at the market wage.

Proposition 5 (Monopolistic Competition on a BGP)**.** *In the version of the model with monopolistic competition and adoption costs in labor, there exists a continuum of equilibria parameterized by α where,*

$$F(z) = 1 - e^{-\alpha z}. \tag{55}$$

The tail parameter of the underlying productivity distribution is α, while the tail parameter of the profit and firm size distributions, is given by

$$\hat{\alpha} \equiv (\rho - 1)\alpha. \tag{56}$$

With GBM driven innovation, the growth rate, g, solves the following implicit equation

$$0 = \zeta + \frac{\left(\alpha\zeta\left(\alpha\sigma^2 + 2\gamma - 2g\right) + 2\right)\left(\gamma + \sqrt{(g-\gamma)^2 + 2\sigma^2(r-g)} - g + (\rho-1)\sigma^2\right)}{\alpha\left(\alpha\sigma^2 + \gamma + \sqrt{(g-\gamma)^2 + 2\sigma^2(r-g)} - g\right)\left((\rho-1)^2\sigma^2 - 2(-\gamma\rho + \gamma + g(\rho-2) + r)\right)}. \tag{57}$$

In the deterministic case with $\sigma = 0$, the growth rate is independent of ρ,

$$g = \frac{1 - \alpha(\hat{r} - \gamma(1+\alpha))\zeta}{\alpha(\varsigma + \alpha)\zeta}. \tag{58}$$

Proof. See the full general equilibrium derivation in Appendix C. □

Comparing Propositions 4 and 5, two things stand out. First, the functional forms are very similar, suggesting that most of the interesting features of the simpler formulation are preserved in the monopolistic competition framework. This implies that the simpler formulation is sufficient for analyzing many questions. Second, the growth rate in the deterministic case is independent of the elasticity of substitution, ρ, and consequently the markup. One interpretation is that stronger or weaker monopoly protection, (if modeled as increasing the elasticity between products and therefore monopoly rents), has no impact on aggregate growth rates.

From (56), higher markups lead to changes in the tail parameter of the size distribution, $\hat{\alpha}$, compared to the underlying productivity distribution. Therefore, when comparing growth rates to tail parameters of the firm size distribution in the data, it is important to adjust for the elasticity and markup. Define the markup as $\bar{\rho} \equiv (\rho - 1)/\rho > 0$. Given an estimated $\bar{\alpha}$ from the firm size, profits, or revenue empirical distribution, the underlying tail index of the productivity distribution is

$$\alpha = \frac{1 - \bar{\rho}}{\bar{\rho}}\hat{\alpha}. \tag{59}$$

[15]See Perla, Tonetti, and Waugh (2014) for a related derivation of the deterministic equilibrium with richer cost functions and international trade.

This adjustment might explain some of the differences between the calibrated α in our model and those of the firm size distribution in the data. For example, with 33% markups, an empirically estimated $\hat{\alpha} = 1$ in the size or profits distribution corresponds to an underlying $\alpha = 2$ in the productivity distribution. We will use equation (59) to give a rough conversion from the productivity distribution to those of the empirical revenue/size/profit distribution in the rest of the paper.[16]

3 Stochastic Innovation with a (Potentially) Finite Frontier

3.1 Basic Setup

With a stochastic innovation process based on Brownian motion or multiplicative Poisson jumps, the technology frontier becomes infinite immediately. As an alternative, we introduce a discrete two-state Markov process driving the innovation growth rate of the firm. In the high state, the firm is innovating and increasing its productivity. In the low state, its productivity does not grow through innovation. This captures the concept that some times firms have good ideas or projects that generate growth and some times firms are just producing using their existing technology. This innovation status changes according to a continuous time Markov chain.[17]

Stochastic Process for Innovation The two innovative states are $i \in \{\ell, h\}$. The jump intensity from low to high is $\lambda_\ell > 0$ and from high to low is $\lambda_h > 0$.[18] Let $\hat{\lambda} \equiv \frac{\lambda_\ell}{\lambda_h}$, a measure of the relative jump intensity to h vs ℓ, which is part of the determination of the stationary proportion of types in the economy.

The state of a firm is now $\{Z, i\}$. Define the mass of agents at time t with productivity below Z and type i as $\Phi_i(t, Z)$ such that the total mass is still normalized, $\Phi_\ell(t, \infty) + \Phi_h(t, \infty) = 1$. Define the distribution unconditional on type as $\Phi(t, Z) \equiv \Phi_\ell(t, Z) + \Phi_h(t, Z)$.

Since the Markov chain has no absorbing states, and there is a strictly positive flow between the states for all Z, the support of the distribution conditional on ℓ or h is the same (except, perhaps, exactly at an initial condition). Recall that support $\{\Phi(t, \cdot)\} \equiv [M(t), B(t))$. The growth rate of the upper and lower bounds of the support are defined as $g(t) \equiv M'(t)/M(t)$ and $g_B(t) \equiv B'(t)/B(t)$ if $B(t) < \infty$.

The normalized distribution $F_\ell(t, z)$ and $F_h(t, z)$ are defined as in Section 2.3, with $F_\ell(t, 0) = F_h(t, 0) = 0$ and $F_\ell(t, \bar{z}(t)) + F_h(t, \bar{z}(t)) = 1$. Denote the unconditional, normalized distribution with $F(0) = 0$ and $F(\bar{z}) = 1$ as

$$F(z) \equiv F_\ell(z) + F_h(z). \tag{60}$$

Value Functions and the Growth Rate of the Frontier While $i = h$, firms grow at an exogenous innovation rate $\gamma > 0$, and, without loss of generality, do not grow if $i = \ell$. The continuation value functions are now $V_i(t, Z)$ and include the drift in a high state as well as the intensity of

[16] When comparing to Luttmer (2007) and some other papers using monopolistic competition, keep in mind that the stochastic process in those papers was placed on profits or revenue directly rather than the underlying productivity distribution used here. Therefore, the tail parameters from those papers have something like (59) already built in, and require no adjustment.

[17] Luttmer (2011) also emphasizes the need for fast growing firms–driven by differences in the quality of blueprints for size expansion–to account for the size distribution of firms. In his model, firms will stochastically slow down eventually, where here will assume that firms can jump back and forth between the states. In his model, he assumes that a proportion of firms need to start in the high quality blueprint state in order to generate empirically reasonable tails, while in our model we will not require this assumption due to the jumps back and forth.

[18] In Section 4, the growth rate γ will become a control variable for a firm, with the choice subject to a convex cost.

jumps between i. Modifying (2) gives the following Bellman equations in the continuation region,[19]

$$rV_\ell(t,Z) = Z + \lambda_\ell\left(V_h(t,Z) - V_\ell(t,Z)\right) + \boldsymbol{\partial}_t V_\ell(t,Z) \tag{61}$$
$$rV_h(t,Z) = Z + \gamma Z\boldsymbol{\partial}_Z V_h(t,Z) + \lambda_h\left(V_\ell(t,Z) - V_h(t,Z)\right) + \boldsymbol{\partial}_t V_h(t,Z) \tag{62}$$

Using the normalization techniques in Section 2.3 and Appendix B.1, the stationary equations, as shown in Figure 5, are

$$(r-g)v_\ell(z) = e^z - gv_\ell'(z) + \lambda_\ell\left(v_h(z) - v_\ell(z)\right) \tag{63}$$
$$(r-g)v_h(z) = e^z - (g-\gamma)v_h'(z) + \lambda_h\left(v_\ell(z) - v_h(z)\right) \tag{64}$$

Define the net value of adoption as

$$v_s \equiv v_\ell(0) = v_h(0) \tag{65}$$

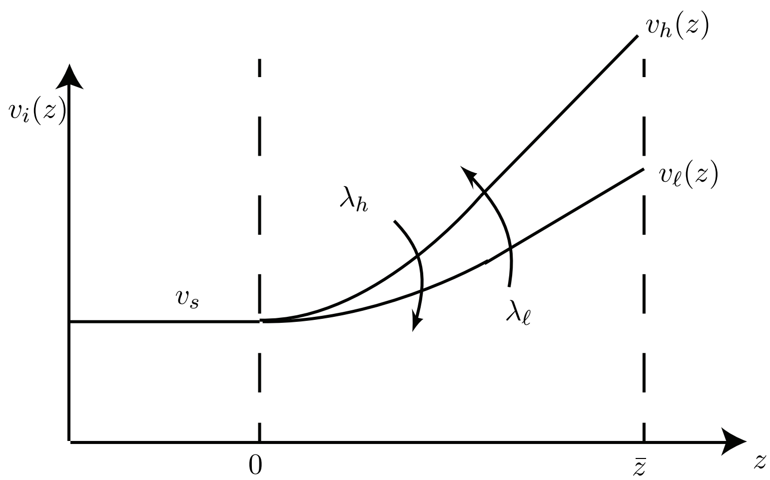

Figure 5: Normalized, Stationary Value Functions

From this process, if $B(0) < \infty$, then $B(t)$ will remain finite for all t, as it evolves from the innovation of firms in the interval infinitesimally close to $B(t)$; that is, $B'(t)/B(t) = \gamma$ if $\Phi_h(t, B(t)) - \Phi_h(t, B(t) - \epsilon) > 0$, for all $\epsilon > 0$. With the continuous measure of firms, there will always be some h firms that have not jumped to the low state for any t, so the growth rate of the frontier is always γ.

In the normalized setup, $\bar{z}(t) \equiv \log(B(t)/M(t))$, and a necessary condition for a stationary equilibrium with $\sup\{\bar{z}(t)|\,\forall t\} < \infty$ is that $g = g_B = \gamma$. This is necessary because if $g < g_B$, then $\bar{z}$ diverges, while if $g > g_B$, the minimum of the support would eventually be strictly greater than the maximum of the support.

[19]Ordering the states as $\{l, h\}$, the infinitesimal generator for this continuous time Markov chain is $\mathbb{Q} = \begin{bmatrix} -\lambda_\ell & \lambda_\ell \\ \lambda_h & -\lambda_h \end{bmatrix}$, with adjoint operator $\mathbb{Q}^*$. The KFE and Bellman equations can be formally derived using these operators and the drift process.

Technology Diffusion Firms upgrading their technology through adoption receive a new i type and a draw of Z from the productivity distribution. The exact specification typically does not affect the qualitative results, so we will write the process fairly generally and then analyze specific cases. Assume that an adopting firm draws a (i, Z) from distributions $\hat{\Phi}_\ell(t, Z)$ and $\hat{\Phi}_h(t, Z)$, both of which will be determined by the equilibrium $\Phi_\ell(t, Z)$ and $\Phi_h(t, Z)$. Assume that this gives a proper cdf, so $\hat{\Phi}_\ell(t, 0) = \hat{\Phi}_h(t, 0) = 0$ and $\hat{\Phi}_\ell(t, B(t)) + \hat{\Phi}_h(t, B(t)) = 1$. We are maintaining a simplification that the gross value of adoption is independent of an agent's current type.

In principle, there may be adopters hitting the adoption threshold with either innovation type. Assume that h and ℓ types have the same adoption threshold $M(t)$, to be proved later. Modifying (9) and recognizing that the λ_i jumps are of measure 0 when calculating the flow across the boundary, this implies

$$S(t) \equiv S_\ell(t) + S_h(t) \tag{66}$$
$$S_\ell(t) = M'(t)\boldsymbol{\partial}_Z \Phi_\ell(t, M(t)) \tag{67}$$
$$S_h(t) \equiv \left(M'(t) - \gamma M(t)\right) \boldsymbol{\partial}_Z \Phi_h(t, M(t)). \tag{68}$$

After normalization, the flow of adopters that are of type h or type l is given by,[20]

$$S_\ell = g F_\ell'(0) \tag{69}$$
$$S_h = (g - \gamma) F_h'(0) \tag{70}$$

Interpreting these equations, S_ℓ is the flow of ℓ agents moving backwards at a relative speed of g across the barrier, while S_h is the flow of h agents moving backwards at the slower relative speed of $g - \gamma$ across the barrier.

Law of Motion Modifying (5) to include the jumps between innovation states gives the following KFEs,

$$\boldsymbol{\partial}_t \Phi_\ell(t, Z) = -\lambda_\ell \Phi_\ell(t, Z) + \lambda_h \Phi_h(t, Z) + (S_\ell(t) + S_h(t))\hat{\Phi}_\ell(t, Z) - S_\ell(t) \tag{71}$$
$$\boldsymbol{\partial}_t \Phi_h(t, Z) = -\gamma Z \boldsymbol{\partial}_Z \Phi_h(t, Z) - \lambda_h \Phi_h(t, Z) + \lambda_\ell \Phi_\ell(t, Z) + (S_\ell(t) + S_h(t))\hat{\Phi}_h(t, Z) - S_h(t). \tag{72}$$

Using $\hat{F}_i(z)$ as the normalized version of $\hat{\Phi}_i(z)$, yields the following equations for the stationary distributions.

$$0 = g F_\ell'(z) - \lambda_\ell F_\ell(z) + \lambda_h F_h(z) + (S_\ell + S_h)\hat{F}_\ell(z) - S_\ell \tag{73}$$
$$0 = (g - \gamma) F_h'(z) - \lambda_h F_h(z) + \lambda_\ell F_\ell(z) + (S_\ell + S_h)\hat{F}_h(z) - S_h \tag{74}$$
$$0 = F_\ell(0) = F_h(0) \tag{75}$$
$$1 = F_\ell(\bar{z}) + F_h(\bar{z}). \tag{76}$$

Adoption Decision Firms choose Z thresholds, below which they adopt a new technology through the diffusion process.[21] Modify (3) and (4) to get the value matching and smooth pasting

[20] As a notation simplification, (68) is assuming that $g \geq \gamma$. Otherwise, an additional constraint or indicator function is necessary to ensure that $S_h(t) \geq 0$.

[21] The choice could also depend on the type i, see Appendix A.3 for a proof that ℓ and h agents choose the same threshold, $M(t)$, if the net value of adoption is independent of the current innovation type.

conditions,

$$V_\ell(t, M(t)) = V_h(t, M(t)) = \int_{M(t)}^{B(t)} V_\ell(t, Z)\mathrm{d}\hat{\Phi}_\ell(t, Z) + \int_{M(t)}^{B(t)} V_h(t, Z)\mathrm{d}\hat{\Phi}_h(t, Z) - \zeta M(t) \quad (77)$$

$$\boldsymbol{\partial}_Z V_\ell(t, M(t)) = 0, \quad \text{if } M'(t) > 0 \quad (78)$$

$$\boldsymbol{\partial}_Z V_h(t, M(t)) = 0, \quad \text{if } M'(t) - \gamma M(t) > 0 \quad (79)$$

The smooth pasting conditions only need to hold if the agents are drifting backwards relative to the adoption threshold. Normalize into a stationary environment,

$$v_\ell(0) = v_h(0) = \int_0^{\bar{z}} v_\ell(z)\mathrm{d}\hat{F}_\ell(z) + \int_0^{\bar{z}} v_h(z)\mathrm{d}\hat{F}_h(z) - \zeta \quad (80)$$

$$v_\ell'(0) = 0, \text{ if } g > 0 \quad (81)$$

$$v_h'(0) = 0, \text{ if } g > \gamma. \quad (82)$$

Terminology for Various Cases of the Normalized Support There are three possibilities for the stationary $\bar{z}$ that we will analyze separately. The first is if $\bar{z} = \infty$, which we will call "infinite support", which happens for any initial condition that starts with $B(0) = \infty$ (i.e., $\sup \text{support}\{F(0, \cdot)\} = \infty$). The second case is when $B(0) < \infty$ (which implies $B(t) < \infty$), but where $\lim_{t\to\infty} \bar{z}(t) = \infty$. We label this case "finite, unbounded support". The final case is when the initial condition has finite support, and $\lim_{t\to\infty} \bar{z}(t) < \infty$, which we will refer to as "finite, bounded support". An important question will be whether the unbounded and infinite support examples have the same stationary equilibrium. It will turn out that this is not the case, suggesting that caution should be used when using an infinite support as an approximation of a finite, but ultimately unbounded, empirical distribution.

An important question is whether a stationary equilibrium with bounded finite support can even exist for a given version of the model. This will be discussed further in Propositions 7 and 8.

3.2 Stationary BGP with Infinite Support

If $\Phi(0, Z)$ has infinite support, $\Phi(t, Z)$ will converge to a stationary distribution as $t \to \infty$. A continuum of stationary distributions, each with its associated aggregate growth rates, are possible from different initial conditions. The intuition for this hysteresis is identical to that discussed in Section 2 and Perla and Tonetti (2014).[22]

This section introduces an important difference from the setup used in Sections 3.3 and 3.4: the adoption technology will instead have firms copying both the type and productivity of the draw, rather than always starting in the ℓ state. While an exactly correlated draw of the type and the productivity is not necessary here, see Appendix D.2 for a proof that independent draws of Z and the innovation type for adopters has only degenerate stationary distributions in equilibrium. The normalized adoption distributions are then, $\hat{F}_\ell(z) \equiv F_\ell(z)$ and $\hat{F}_h(z) \equiv F_h(z)$, which can be verified to yield a proper CDF: $\hat{F}_\ell(\infty) + \hat{F}_h(\infty) = F_\ell(\infty) + F_h(\infty) = 1$.

As $B(t) = \infty$ for all t, unlike in the examples with finite support, there is no requirement that $g = \gamma$ to ensure a stationary non-degenerate productivity distribution. However, $g \geq \gamma$ is neccesary

[22] Uniqueness of related models with Geometric Brownian Motion is discussed in Luttmer (2012).

to ensure that $S_h \geq 0$. Summarizing the equations from Section 3,

$$(r-g)v_\ell(z) = e^z - gv_\ell'(z) + \lambda_\ell\left(v_h(z) - v_\ell(z)\right) \tag{83}$$

$$(r-g)v_h(z) = e^z - (g-\gamma)v_h'(z) + \lambda_h(v_\ell(z) - v_h(z)) \tag{84}$$

$$v_\ell(0) = v_h(0) = \tfrac{1}{r-g} = \underbrace{\int_0^\infty v_\ell(z)\mathrm{d}F_\ell(z) + \int_0^\infty v_h(z)\mathrm{d}F_h(z)}_{\text{Adopt both } i \text{ and } Z \text{ of draw}} - \zeta \tag{85}$$

$$v_\ell'(0) = 0 \tag{86}$$

$$v_h'(0) = 0 \tag{87}$$

$$0 = gF_\ell'(z) - \lambda_\ell F_\ell(z) + \lambda_h F_h(z) + (S_\ell + S_h)F_\ell(z) - S_\ell \tag{88}$$

$$0 = (g-\gamma)F_h'(z) - \lambda_h F_h(z) + \lambda_\ell F_\ell(z) + (S_\ell + S_h)F_h(z) - S_h \tag{89}$$

$$S_\ell = gF_\ell'(0) \tag{90}$$

$$S_h = (g-\gamma)F_h'(0) \tag{91}$$

$$0 = F_\ell(0) = F_h(0) \tag{92}$$

$$1 = F_\ell(\infty) + F_h(\infty) \tag{93}$$

In Propositions 1, 2 and 4 we characterized the stationary distributions in terms of the tail index of the initial distribution of productivities, given in (40) explicitly by $\alpha = \kappa F'(0)$, a scalar.[23] In this section, the stationary distribution is a vector $\vec{F}(z) \equiv \begin{bmatrix} F_\ell(z) \\ F_h(z) \end{bmatrix}$ solving (88) and (89) , a system of linear ODEs. If we define the unconditional distribution $F(z) \equiv F_\ell(z) + F_h(z)$, and if both $F_\ell(z)$ and $F_h(z)$ are power laws, any mixture of these distributions inherits the smallest (i.e.thickest) tail parameter (as discussed in Gabaix (2009)). Since there are now two dimensions of heterogeneity, the tail index, α, is defined as that of the unconditional distribution, $F(z)$. The ODE solution for the vector $\vec{F}(z)$ given in Proposition 6 by (100) will depend on the roots of C (both positive, see Appendix D.1). The smallest root of C, representing the slower rate of decay for both elements of $F(z)$, will now become the tail index α. Hence we can still parameterize the endogenous power law distribution by a single parameter α.

To characterize the continuum of stationary distributions, we define the following expressions that depend on tail index, α, and the growth rate, g,

$$\vec{F}(z) \equiv \begin{bmatrix} F_\ell(z) \\ F_h(z) \end{bmatrix} \qquad v(z) \equiv \begin{bmatrix} v_\ell(z) \\ v_h(z) \end{bmatrix} \tag{94}$$

$$A \equiv \begin{bmatrix} \frac{1}{g} \\ \frac{1}{g-\gamma} \end{bmatrix} \qquad B \equiv \begin{bmatrix} \frac{r+\lambda_\ell - g}{g} & -\frac{\lambda_\ell}{g} \\ -\frac{\lambda_h}{g-\gamma} & \frac{r+\lambda_h-g}{g-\gamma} \end{bmatrix} \tag{95}$$

$$\varphi = \sqrt{(\lambda_h - \alpha\gamma)^2 + 2\lambda_l(\alpha\gamma + \lambda_h) + {\lambda_l}^2} \tag{96}$$

$$C = \begin{bmatrix} \frac{-\alpha\gamma + 2\alpha g + \lambda_h - \lambda_l + \varphi}{2g} & \frac{\lambda_h}{g} \\ \frac{\lambda_l}{g-\gamma} & \frac{-\alpha\gamma + 2\alpha g - \lambda_h + \lambda_l + \varphi}{2(g-\gamma)} \end{bmatrix} \tag{97}$$

$$D = \begin{bmatrix} \frac{\lambda_h(g(\alpha\gamma - \lambda_h + \varphi - \lambda_l) + 2\gamma\lambda_l)}{\gamma g(\alpha\gamma - \lambda_h + \varphi + \lambda_l)} \\ \frac{g(\alpha\gamma - \lambda_h + \varphi - \lambda_l) + 2\gamma\lambda_l}{2\gamma(g-\gamma)} \end{bmatrix} \tag{98}$$

Proposition 6 (Stationary Equilibrium with Infinite Support)**.** *There exists a continuum of equilibria parameterized by the tail index $\alpha > 1$ for $g(\alpha)$ that satisfies*

$$\frac{1}{r-g} + \zeta = \int_0^\infty \left[\left[(I+B)^{-1}\left(e^{Iz} + e^{-Bz}B^{-1}\right)A\right]^T e^{-Cz}D\right]\mathrm{d}z \tag{99}$$

[23] In Propositions 1 and 4 we assumed the adoption draws are from non-distorted distributions so $\kappa = 1$.

and the parameter restrictions given in (D.16) *to* (D.19)*. The stationary distributions and the value functions are given by:*

$$\vec{F}(z) = \left(\mathbf{I} - e^{-Cz}\right) C^{-1} D \tag{100}$$

$$\vec{F}'(z) = e^{-Cz} D \tag{101}$$

$$v(z) = (I + B)^{-1} \left(e^{Iz} + e^{-Bz} B^{-1}\right) A \tag{102}$$

Proof. See Appendix D.1 □

A set of complicated parameter restrictions (D.16) to (D.19) (in Appendix D.1) are necessary to ensure that $r > g$ and that the eigenvalues of B and C are positive. Positive eigenvalues of B ensure that value matching is defined, and the option value of diffusion asymptomatically goes to 0 for large z. Comparing (49) with (100) shows that the positivity of the tail index α is now equivalent to C having positive eigenvalues. For the decomposition of the option value, comparing (50) with (102) shows that positive eigenvalues of B ensure the option values in the vector $v(z)$ converges to 0 as z increases.

Note that in Propositions 1, 2 and 4, the tail index is determined by the single initial condition $F'(0) > 0$, a scalar. In Proposition 6 the initial condition $F'(0)$ is a vector, so in principle this raises the possibility that the continuum of stationary equilibria could be two dimensional, parametrized by $F'_\ell(0) > 0$ and by $F'_h(0) > 0$. However as shown in Appendix D.1 this is not possible since the only initial condition that ensures that $F_\ell(z)$ and $F_h(z)$ remain positive and satisfy (73) and (74) is exactly the eigenvector of C corresponding to its dominant (Frobenius) eigenvalue. Since the eigenvector is determined only up to a multiplicative constant, the continuum of stationary distributions is therefore one dimensional. We use the smallest eigenvalue of C, defined as the tail index α, to solve for $F'_h(0)$, which then determines $F'_\ell(0)$ from the eigenvector restriction. This then allows us to obtain the expressions (97) and (98) in terms of parameters, α and g. Then value matching, (D.9) and (D.27), gives us expression (99) to define g in terms α, so we end up with a continuum of stationary equilibria parametrized by α.

See Figures 6 and 7 for an example of infinite support with $\gamma = .01, r = .05, \lambda_\ell = .0004, \lambda_h = .03$, and $\zeta = 6.0$. In this equilibrium, $g = .029, \alpha = 2.5$ and $F_\ell(\infty) = .988$. The relationship between g and α is shown in Figure 7. The growth rate is decreasing as the tail becomes thinner, and the total number of agents in the h state increases.

Curiously, with other parameters the growth rate can instead be increasing in α, that is as the tail gets thinner, as shown in Figure 8. This change in the monotonicity of $g(\alpha)$ from Figures 7 and 8 is accompanied by a change in the monotonicity of $F_h(\infty)$. Recall that with the draw technology in (85), where both the i and the z are imitated, an increasing proportion of ℓ types means more S_ℓ crossing the adoption boundary that can end up in the growth state h, generating a higher $g(\alpha)$.

3.3 Stationary BGP with Finite Initial Support

In this section we study the stationary distribution, the BGP, when the initial distribution $\phi(0, Z)$ has finite support. Consider for simplicity that the process of adopting new technologies is disruptive to R&D, so the firm starts in the ℓ type regardless of its former type, and the Z is drawn from a distortion of the unconditional distribution i.e., $\hat{\Phi}_h(t, Z) = 0$ and[24]

$$\hat{\Phi}_\ell(t, Z) \equiv (\Phi_\ell(t, Z) + \Phi_h(t, Z))^\kappa \tag{103}$$

[24]Unlike the infinite support case in Section 3.3, the equilibrium are not sensitive to the degree of correlation in the draws, and we have simply chosen the most convenient.

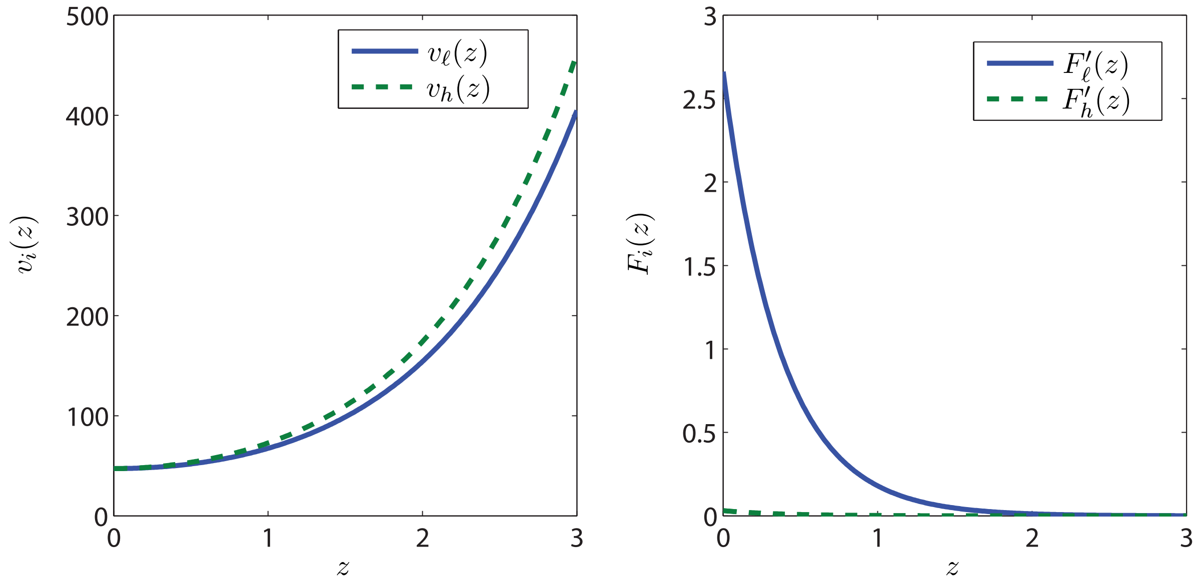

Figure 6: Exogenous $v_i(z)$, and $F'_i(z)$ with a Infinite Frontier

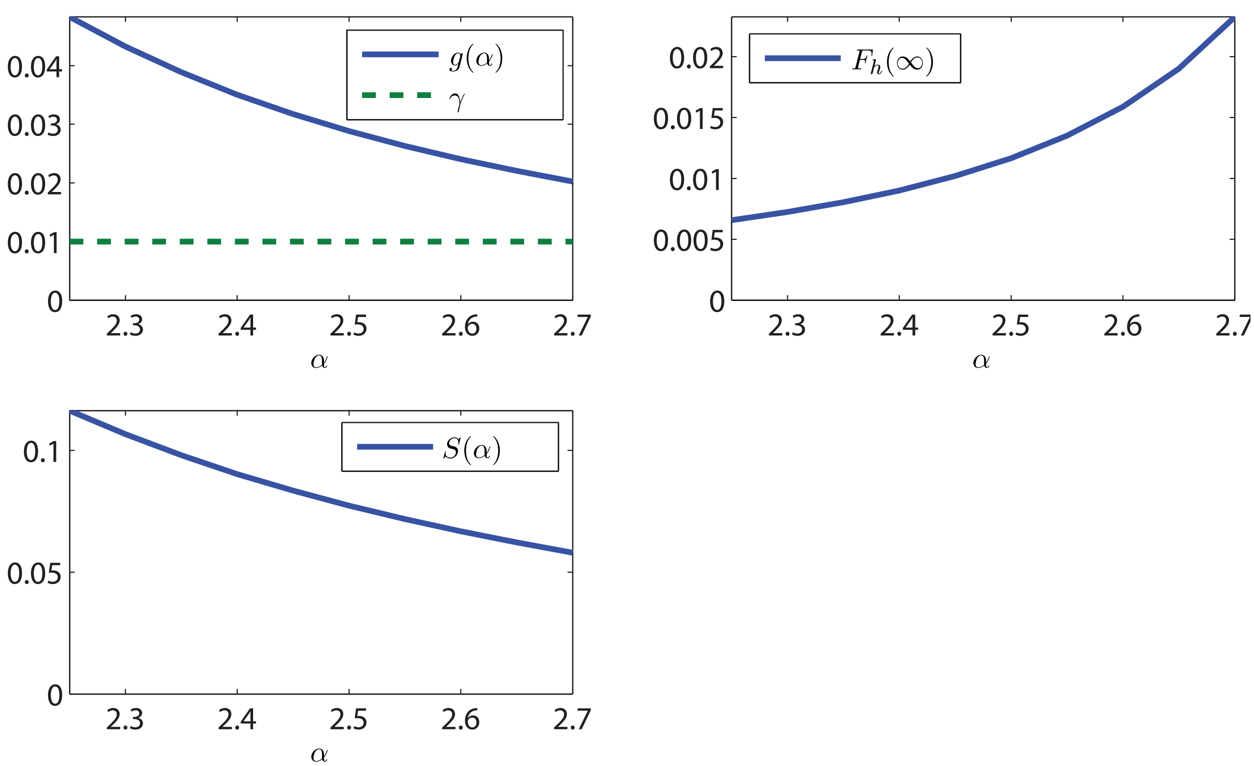

Figure 7: Growth rate as a Function of α with an Infinite Frontier

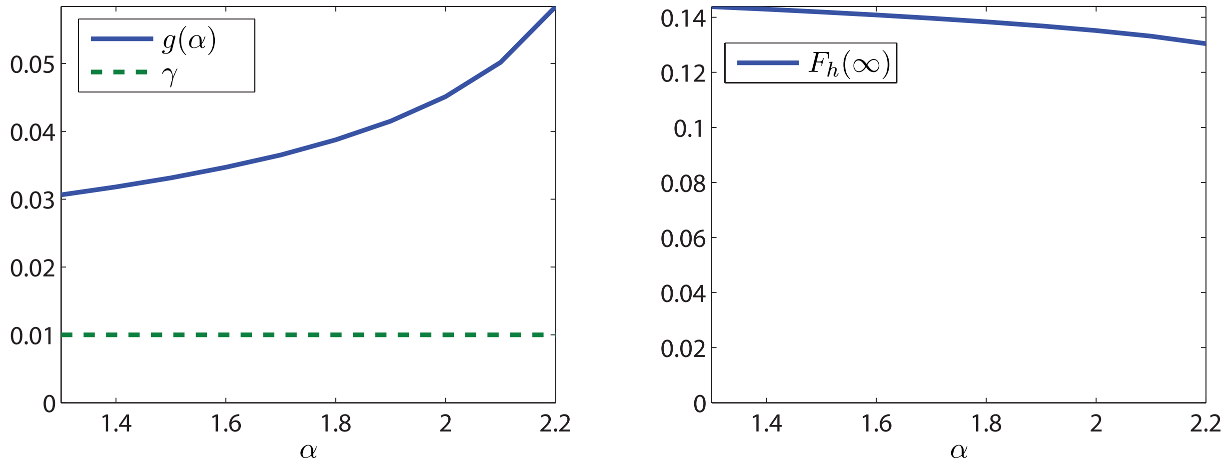

Figure 8: Growth rate as a Function of α with an Infinite Frontier with Alternative Parameters

Normalizing to a stationary draw distribution and then using the definition of the unconditional normalized distribution, $F(z)$, yields adoption distribution

$$\hat{F}_\ell(z) = (F_\ell(z) + F_h(z))^\kappa = F(z)^\kappa \tag{104}$$

We write $F(z)^\kappa$ for the normalized draw process (for which, given the assumptions, all firms end up in the low state).

Due to the bounded growth rates of the Markov process, if the support of $\Phi(0, z)$ is finite, then it remains finite as it converges to a stationary distribution. With an exogenous γ and a finite frontier, a necessary requirement for non-degeneracy of $F_i(z)$ is then $g = \gamma$. Hence, in the stationary equilibrium there are no h type agents hitting the adoption threshold, and the smooth pasting condition for h firms is not a necessary condition.

Necessary conditions for a stationary equilibrium with a finite initial frontier are $v_\ell(z)$, $v_h(z)$, $F_\ell(z)$, $F_h(z)$, S, such that,

$$(r-g)v_\ell(z) = e^z - gv_\ell'(z) + \lambda_\ell(v_h(z) - v_\ell(z)) \tag{105}$$
$$(r-g)v_h(z) = e^z + \lambda_h(v_\ell(z) - v_h(z)) \tag{106}$$
$$v_\ell(0) = \int_0^{\bar{z}} v_\ell(z) \mathrm{d}F(z)^\kappa - \zeta \tag{107}$$
$$v_\ell'(0) = 0 \tag{108}$$
$$0 = gF_\ell'(z) + SF(z)^\kappa + \lambda_h F_h(z) - \lambda_\ell F_\ell(z) - S \tag{109}$$
$$0 = \lambda_\ell F_\ell(z) - \lambda_h F_h(z) \tag{110}$$
$$0 = F_\ell(0) = F_h(0) \tag{111}$$
$$1 = F_h(\bar{z}) + F_\ell(\bar{z}) \tag{112}$$
$$S = gF_\ell'(0) \tag{113}$$

Define the constants, $\hat{\lambda} \equiv \frac{\lambda_\ell}{\lambda_h}, \bar{\lambda} \equiv \frac{\lambda_\ell}{r-\gamma+\lambda_h} + 1$. The following characterizes the equilibrium,

Proposition 7 (Stationary Equilibrium with Continuous Draws and Finite, Unbounded Support)**.** *There does not exist an equilibrium with finite and bounded support (for any $\kappa > 0$). There exists a unique equilibrium, with $g = \gamma$ and $\bar{z} \to \infty$.*

In the case of $\kappa = 1$, the unique stationary distribution is,

$$F_\ell(z) = \tfrac{1}{1+\hat{\lambda}} e^{-\alpha z} \tag{114}$$

$$F_h(z) = \hat{\lambda} F_\ell(z), \tag{115}$$

where α is the tail index of the power law distribution:

$$\alpha \equiv (1 + \hat{\lambda}) F_\ell'(0), \tag{116}$$

and $F_\ell'(0)$ is determined by model parameters:

(117)

$$F_\ell'(0) = \frac{\lambda_h\left(\zeta r(r+\lambda_h+\lambda_\ell)-\sqrt{\zeta((4\gamma+r^2\zeta)(-\gamma+r+\lambda_h)^2+2(-2\gamma+(\gamma-r)r\zeta)(\gamma-r-\lambda_h)\lambda_\ell+(\gamma-r)^2\zeta\lambda_\ell{}^2)}+\zeta\gamma^2 2+\zeta(-\gamma)(3r+2\lambda_h+\lambda_\ell)\right)}{\gamma 2\zeta(\lambda_h+\lambda_\ell)(\gamma-r-\lambda_h)}. \tag{118}$$

The firm value functions are,

$$v_\ell(z) = \frac{\bar{\lambda}}{\gamma + (r-\gamma)\bar{\lambda}} e^z + \frac{1}{(r-\gamma)(\nu+1)} e^{-\nu z} \tag{119}$$

$$v_h(z) = \frac{e^z + \lambda_h v_\ell(z)}{r - \gamma + \lambda_h}, \tag{120}$$

where $\nu > 0$, the rate at which the option value is discounted, is given by

$$\nu \equiv \frac{(r-\gamma)\bar{\lambda}}{\gamma}. \tag{121}$$

Proof. See Appendix D.3. □

So far, while the distributions of productivities z with initial distributions that have finite support also have finite support for $t < \infty$, stationary distributions all have asymptotically infinite relative support, that is $\bar{z} \to \infty$: the ratio of frontier to lowest productivity goes to infinity. In the ℓ state the growth rate is zero and z stays put, but in the h state the growth rate of z is positive. Given the Markov process for ℓ and h, there will be some agents who hit lucky streaks more than others, escape from the pack, and break away. Given a fixed barrier, the logic is similar to the linear or asymptotically linear Kesten processes bounded away from zero with affine terms, for which, under appropriate conditions, the asymptotic tail index can be explicitly computed in terms of the stationary distribution induced by the Markov process. In our model however, the adoption process introduces non-linear jumps that are not multiplicative in productivities, and which do not permit the simple characterization of the tail index. The endogenous absorbing adoption barrier (which acts like a reflecting barrier with stochastic jumps) complicates the analogy since one might think if the frontier is growing rapidly, the endogenous barrier could also move rapidly to keep up with the frontier. However, the incentives for adoption, which drive the speed of the moving barrier, are driven by the mean draw in productivity. Therefore, if the frontier diverges to infinity but the mean doesn't keep growing at the same rate, the frontier technology will diverge. As we will see in Section 3.4, if we strengthen the adoption process by allowing a positive fraction of adopters to leapfrog to the frontier, the multiplicative jump process generating the escape to infinity in relative productivities may in fact be contained.

Note the distinction between Propositions 6 and 7: the stationary equilibrium associated with initially finite vs. initially infinite support are different, even though the initially finite support case of Proposition 7 also has an asymptotically unbounded relative support. This stationary equilibrium is unique and independent of the initial distribution, in contrast to the case of infinite initial support, which featured hysteresis and a continuum of stationary solutions. Figure 9 provides an example where $\gamma = .02, r = .06, \lambda_\ell = .01, \lambda_h = .03$, and $\zeta = 25$.

Comparative statics on the $\bar{z}$ and α are shown in Figure 12 for changes in η, γ, ζ, and λ_h. For example, higher growth rates of innovators leads to a more distant technology frontier, but also to thinner tails. Alternatively, a higher cost of adoption leads to a more distant frontier, and thicker tails.

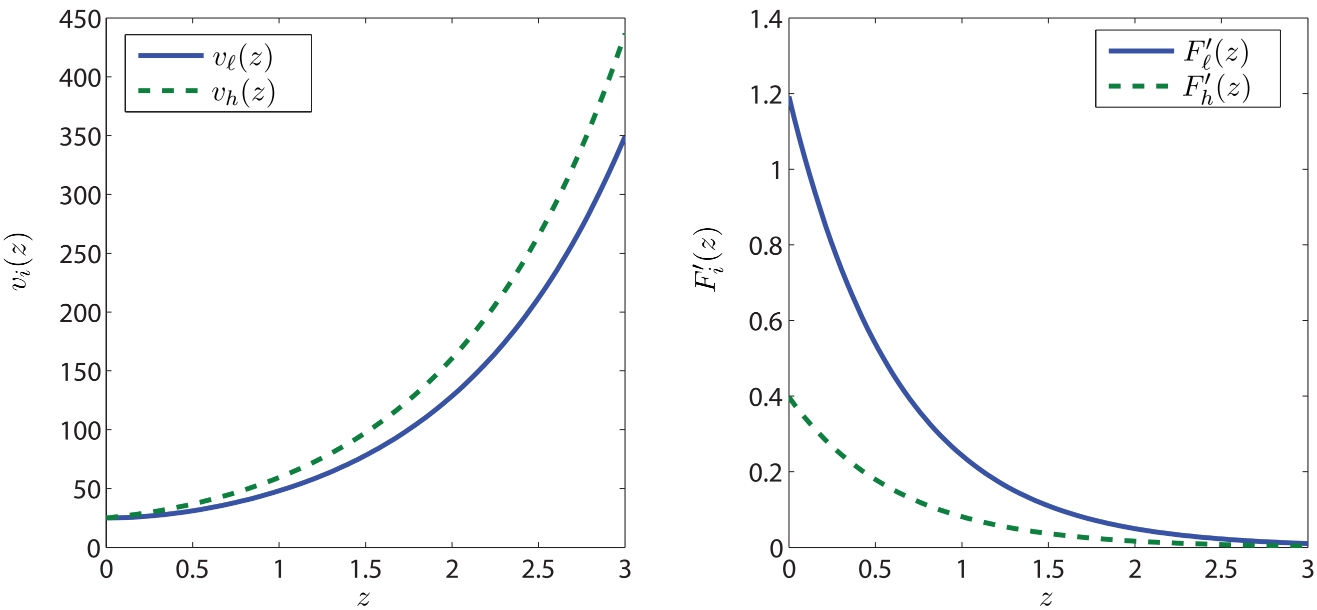

Figure 9: Normalized, Stationary Value Functions and PDFs for the Unbounded Cases

3.4 Stationary BGP with Bounded Relative Support

Proposition 7 shows that while the frontier remains finite for $t < \infty$, the ratio of the frontier productivity to the mean of the distribution is finite, but tends to infinity as $t \to \infty$.[25] This is because a diminishing, yet strictly positive number of firms keep getting lucky and grow at γ forever, but as the mass of agents with extremely high z is thinning out, it doesn't strongly effect the diffusion incentives of those with a low z (i.e., the mean expands slower than the frontier). Since there is no leapfrogging of these perpetually lucky firms, this process will continue forever.

An alternative way to model within-firm productivity change is to assume that firms can leapfrog to the frontier with some probability. This can be interpreted either as an alternative type of innovation or technology adoption process, depending on which firms have access to these jumps. Such leapfrogging is a continuous time version of a quality-ladders model that keeps the frontier bounded.[26] The jumps can occur either for firms adopting through diffusion, or for innovating firms that succesfully leapfrog to the frontier, which can be viewed as positive spillovers from the frontier to innovators.

[25] This result is robust to variations in the diffusion specification including assuming that adopting agents draw from the $F_h(z)$ distribution and start with a h type with $\kappa > 1$, which adds the maximum possible incentives to increase diffusion and compress the distribution.

[26] In a model of leapfrogging arrivals and a multiplicative step above the frontier, in continuous time the frontier would become infinite immediately. Alternatively, it could be recast as a step-by-step innovation model in the spirit of Aghion, Akcigit, and Howitt (2013) with the same qualitative results.

Assume that there is a possibility that during the process of becoming or remaining innovators, the firm invents a significant step using existing methods, and jumps to the frontier.

We model leapfrogging as an innovation that propels firms to the frontier of the productivity distribution.[27] This major innovation can be achieved by all firms operating their existing technology, i.e., both $i = \ell$ and h types. However, since such an innovation is potentially disruptive, those firms that jump to the frontier become $\ell-$types and must wait for the Markov transition to h before they become innovators again.

To accomodate firms jumping to the frontier, we modify the model presented in Proposition 7 by adding an arrival rate for operating firms of jumps to the frontier, $\eta \geq 0$. See Figure 10 for a visualization of the stationary value functions.[28] To consider the case where it is adopters–rather than just innovating firms–who jump to the frontier, see Section 4.2 where an endogenous jump probability is chosen by adopting firms.

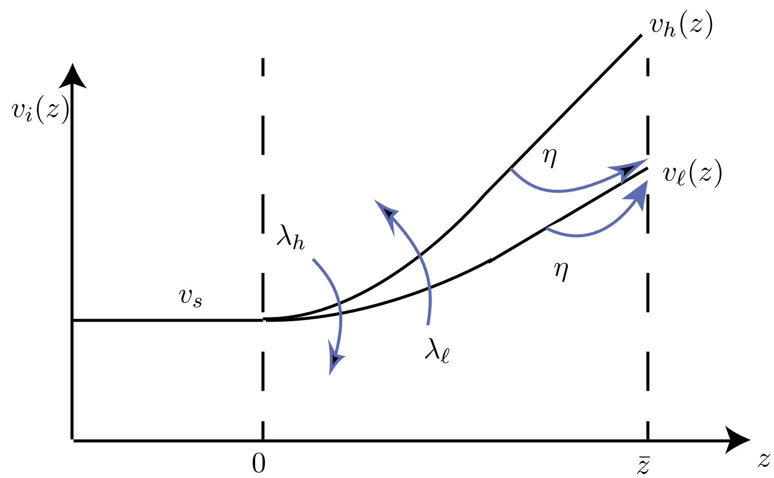

Figure 10: Normalized, Stationary Value Functions with Bounded Support

There now *may* be a jump discontinuity in the right continuous cdf at $\bar{z}$. Due to right continuity of the cdf, the mass at the discontinuity $z = \bar{z}$ is:

$$\Delta_\ell = \lim_{\epsilon \to 0} \left(F_\ell(\bar{z}) - F_\ell(\bar{z} - \epsilon)\right) \tag{122}$$

$$\Delta_h = \lim_{\epsilon \to 0} \left(F_h(\bar{z}) - F_h(\bar{z} - \epsilon)\right) \tag{123}$$

Set $\kappa = 1$ for simplicity, and define $\mathbb{H}(z)$ as the Heaviside operator. The following characterizes

[27]Rather than being the autarkic process improvement of the γ growth, this is leap-frogging and may be viewed as a melding of innovation and diffusion, as the jump is a function of the existing productivity distribution. The intuition here is that while the stochastic, continuous growth of innovators is process improvement, these would be the sorts of innovations that are captured as new patents citing the prior-art.

[28]The assumption of a jump to the ℓ state at the frontier is only for analytical convenience, and this assumption can be changed with no qualitative differences. If some firms jumped to the h state at the frontier instead, then a right discontinuity in $F_h(z)$ would exist, $\Delta_h > 0$, and more care is necessary in solving the KF and integrating the value matching condition. With this specification, a possible downside is that $v_\ell(\bar{z}) < v_h(\bar{z} - \epsilon)$ for some set of small ϵ, and those firm would rather keep the lower z rather than innovate. Intuitively, the idea with this specification is similar to the notion of negative shocks to productivity due to experimentation, and that innovation can be disruptive to a firm. This simplification helps ensure that the values of jumps to the frontier remains identical for both agents, and hence all types have the same adoption threshold as demonstrated in Appendix A.3. If it is adopters, as nested by the endogenous θ probability in Section 4.2, who jump, this doesn't occur.

the necessary conditions for a stationary equilibrium,

$$(r-g)v_\ell(z) = e^z - gv_\ell'(z) + \lambda_\ell(v_h(z) - v_\ell(z))) + \eta(v_\ell(\bar{z}) - v_\ell(z)) \tag{124}$$
$$(r-g)v_h(z) = e^z + \lambda_h(v_\ell(z) - v_h(z)) + \eta(v_\ell(\bar{z}) - v_h(z)) \tag{125}$$
$$v_\ell(0) = \int_0^{\bar{z}} v_\ell(z)(F_\ell'(z) + F_h'(z))\mathrm{d}z - \zeta \tag{126}$$
$$v_\ell'(0) = 0 \tag{127}$$
$$0 = gF_\ell'(z) + \lambda_h F_h(z) - \lambda_\ell F_\ell(z) - \eta F_\ell(z) + \eta\mathbb{H}\left(z - \bar{z}\right) + SF(z) - S \tag{128}$$
$$0 = \lambda_\ell F_\ell(z) - \lambda_h F_h(z) - \eta F_h(z) \tag{129}$$
$$0 = F_\ell(0) = F_h(0) \tag{130}$$
$$1 = F_h(\bar{z}) + F_\ell(\bar{z}) \tag{131}$$
$$S = gF_\ell'(0) \tag{132}$$

Let $r > \gamma$ and define the constants, $\hat{\lambda} \equiv \frac{\lambda_\ell}{\eta+\lambda_h}$, $\bar{\lambda} \equiv \frac{r-\gamma+\lambda_\ell+\lambda_h}{r-\gamma+\lambda_h}$, and $\nu = \frac{r-\gamma+\eta}{\gamma}\bar{\lambda}$. Furthermore, assume that the value of $F_\ell'(0)$ that solves (137) is larger than η/γ.

Proposition 8 (Stationary Equilibrium with a Bounded Frontier)**.** *With the maintained assumptions, a unique equilibrium with $\bar{z} < \infty$ exists with $g = \gamma$ where the stationary distribution is,*

$$F_\ell(z) = \frac{F_\ell'(0)}{(F_\ell'(0) - \eta/\gamma)(1+\hat{\lambda})}(1 - e^{-\alpha z}) \tag{133}$$
$$F_h(z) = \hat{\lambda}F_\ell(z), \tag{134}$$

where

$$\alpha \equiv (1+\hat{\lambda})(F_\ell'(0) - \eta/\gamma) \tag{135}$$
$$\bar{z} = \frac{\log(\gamma F_\ell'(0)/\eta)}{\alpha}. \tag{136}$$

The equilibrium $F_\ell'(0)$ solves the following implicit equation substituting for α and $\bar{z}$,

$$\zeta + \frac{1}{r-\gamma} = \frac{\gamma F_\ell'(0)\alpha\bar{\lambda}\left(-\frac{e^{-\nu\bar{z}}\left(-1+e^{-\alpha\bar{z}}\right)\eta}{(-\gamma+r)\alpha\nu} + \frac{e^{\bar{z}}\eta\left(e^{-\alpha\bar{z}}-1\right)}{\alpha(\gamma-r)} + \frac{-e^{-(\alpha+\nu)\bar{z}}+1}{\nu(\alpha+\nu)} + \frac{-e^{\bar{z}-\alpha\bar{z}}+1}{\alpha-1}\right)}{\gamma(\gamma F_\ell'(0)-\eta)(\nu+1)}. \tag{137}$$

The value functions for the firm are,

$$v_\ell(z) = \frac{\bar{\lambda}}{\gamma(1+\nu)}\left(e^z + \frac{1}{\nu}e^{-\nu z} + \frac{\eta}{r-\gamma}\left(e^{\bar{z}} + \frac{1}{\nu}e^{-\nu\bar{z}}\right)\right) \tag{138}$$
$$v_h(z) = \frac{e^z + (\lambda_h - \eta)v_\ell(z) + \eta v_\ell(\bar{z})}{r-\gamma+\lambda_h}. \tag{139}$$

Proof. See Appendix D.4.

□

In the above, note that $F_i(z)$ are continuous, and $\Delta_\ell = \Delta_h = 0$. This is because leapfrogging firms become type ℓ, at which point they immediately fall back in relative terms. As those firms falling back also jump back and forth to the h type, the $F_\ell(z)$ and $F_h(z)$ distribution smoothly mix, ensuring continuity. If such firms were added as h agents, there would be a jump discontinuity in the cdf exactly at $\bar{z}$.

The above α is an empirical "tail index" that can be estimated from a discrete set of data points. An example for $r = .06, \lambda_\ell = 0.01, \lambda_h = 0.03, \zeta = 25, \eta = 0.001, g = \gamma = .02$ is given in Figure 11. With these parameters, the frontier is $\bar{z} = 2.47$, or converting from logs, the frontier firm is approximately 12 times as efficient as the least productive firm at adoption threshold. The α computed from this example is 1.2, close to the empirical Zipf's law.

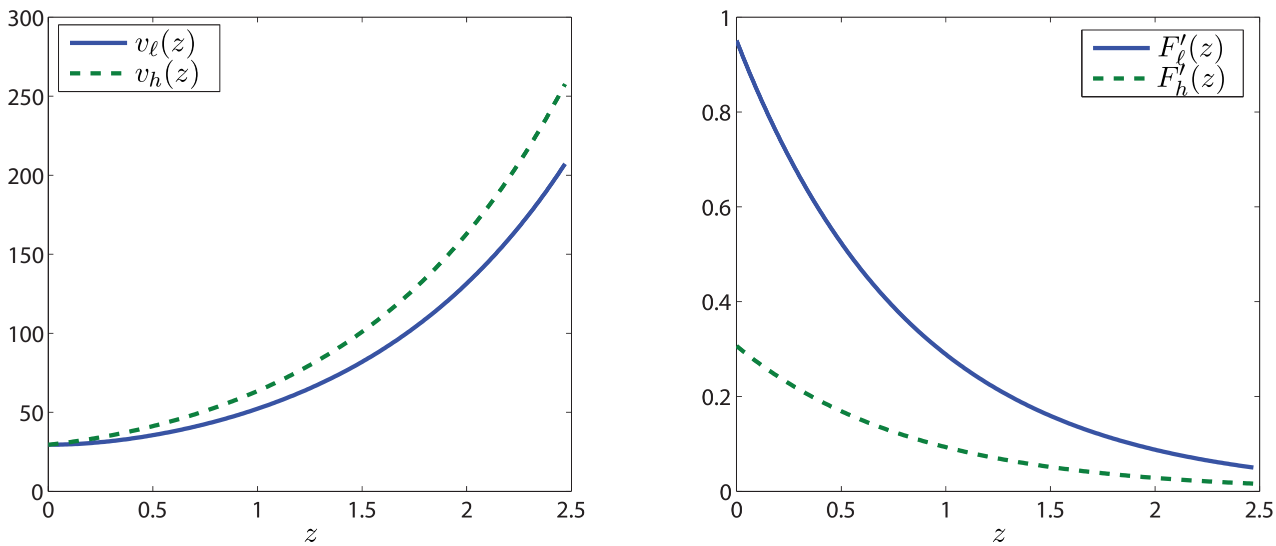

Figure 11: Exogenous $v_i(z)$, and $F'_i(z)$ with a Bounded Frontier

As noted at the end of previous section, leapfrogging to the frontier by a positive mass of agents can contain the escape in relative productivities by lucky firms who get streaks of long sojourns in the high growth group h. Firms which had leapfrogged to the frontier may grow fast until they get a draw that slows them down by putting them back in the ℓ state. They will be overtaken by others who leapfrog to the frontier from anywhere in the productivity distribution and replenish it. This leapfrogging/quality ladder process prevents laggards from remaining as laggards forever. The distribution of relative productivities then remains bounded as the frontier acts as a locomotive in a relay race. Note that this locomotive process is similar to models of technology diffusion where the growth rate of adopters is an increasing function the distance to the frontier, unlike innovators with multiplicative growth in their productivity level (see Benhabib, Perla, and Tonetti (2014)). As adopters fall behind, their growth rate increases to match the growth rate of innovators, so relative productivities remain bounded.

From (136), a relationship can be found that determines the range of the productivity distribution for any particular $F'_\ell(0)$:

$$\bar{z} = \frac{\eta + \lambda_h}{\eta + \lambda_\ell + \lambda_h} \frac{\log(F'_\ell(0)) - \log(\eta/\gamma)}{F'_\ell(0) - \eta/\gamma}. \tag{140}$$

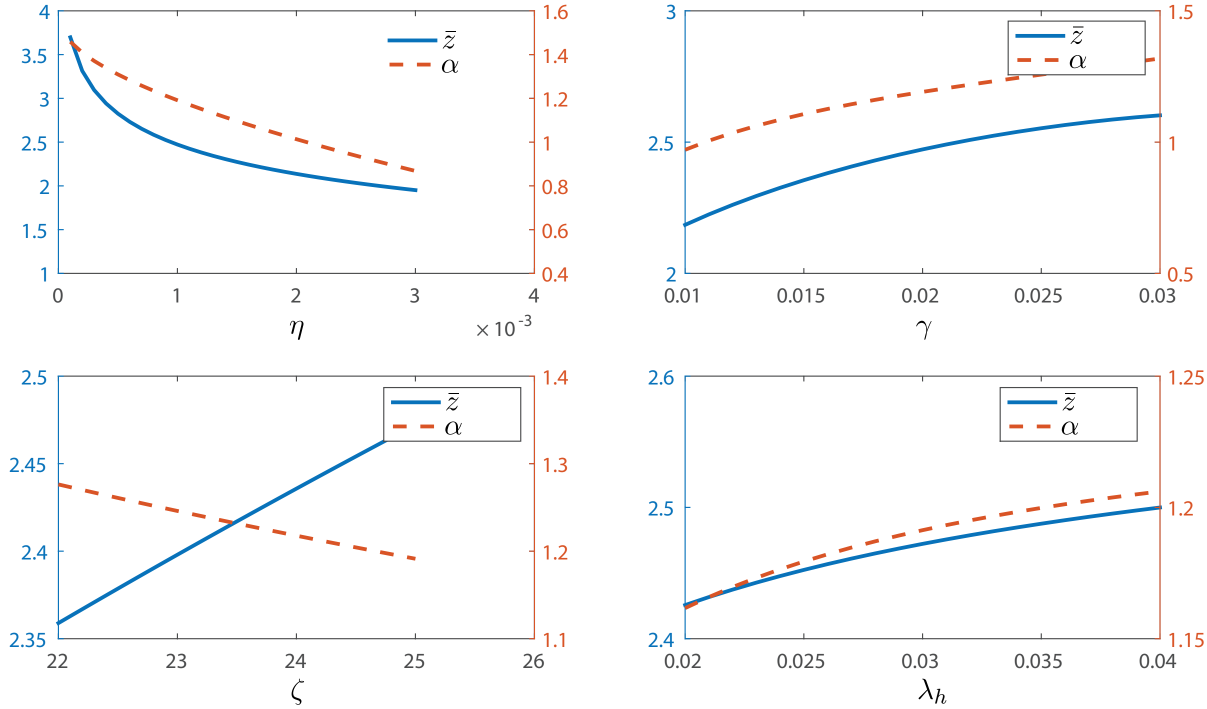

Figure 12: Comparative Statics with a Bounded Frontier

4 Endogenous, Stochastic Innovation

This section introduces endogenous innovation into the stochastic model. We assume that firms can control the drift of their innovation process, as in Atkeson and Burstein (2010) and Stokey (2014). At first we assume that the arrival rate of jumps to the frontier, η, is zero in order to analyze the unbounded case with $\bar{z} \to \infty$. Then in Section 4.2 we move to the model with endogenously chosen innovation rates and jump probabilities and show that the frontier $\bar{z}$ is finite.

We are modeling the innovation choice with no direct spillovers to ensure that it is orthogonal to the technology diffusion process. The interactions are between the tradeoffs in the firm's choices, rather than a coupled innovation and adoption technology.[29]

4.1 Continuous Choice with the Finite, Unbounded, Frontier

Assume that, with a convex cost proportional to its current Z, a firm in the innovative state can choose its own growth rate $\gamma \geq 0$. Let $\chi > 0$ be the productivity of their R&D technology, and the cost quadratic in the growth rate γ. Adapting the equations in Section 3.3 after normalizing the

[29]This is in contrast to approaches such as Chor and Lai (2013), where they are interested in the direct interaction with a dependent innovation process, with aggregate spillovers of knowledge.

innovation cost, and using the result that $\bar{z} \to \infty$ in the absence of jumps to the frontier,

$$(r-g)v_\ell(z) = e^z - gv_\ell'(z) + \lambda_\ell(v_h(z) - v_\ell(z)) \tag{141}$$

$$(r-g)v_h(z) = \max_{\gamma \geq 0}\left\{ e^z - \underbrace{\tfrac{1}{\chi}e^z\gamma^2}_{\text{R\&D cost}} - \underbrace{(g-\gamma)}_{\text{Drift}} v_h'(z) + \lambda_h(v_\ell(z) - v_h(z)) \right\} \tag{142}$$

$$v_s \equiv v_\ell(0) = v_h(0) = \int_0^\infty v_\ell(z) \mathrm{d}F(z)^\kappa - \zeta \tag{143}$$

$$v_\ell'(0) = v_h'(0) = 0 \tag{144}$$

$$0 = gF_\ell'(z) + (S_\ell + S_h)F(z)^\kappa + \lambda_h F_h(z) - \lambda_\ell F_\ell(z) - S_\ell \tag{145}$$

$$0 = \underbrace{(g - \gamma(z))}_{\text{Drift}} F_h'(z) + \lambda_\ell F_\ell(z) - \lambda_h F_h(z) - S_h \tag{146}$$

$$0 = F_\ell(0) = F_h(0) \tag{147}$$

$$1 = F_h(\infty) + F_\ell(\infty) \tag{148}$$

$$S_\ell = gF_\ell'(0) \tag{149}$$

$$S_h = gF_h'(0) \tag{150}$$

Define the constant,

$$\bar{\lambda} \equiv \frac{r + \lambda_\ell + \lambda_h}{r + \lambda_\ell} \tag{151}$$

With this setup, instead of all firms growing at rate γ exogenously, h-type firms are choosing a growth rate γ that is a function of their current productivity level, z. As the choice of γ is increasing in z in equilibrium, agents in the h state will end up crossing the endogenous adoption threshold, as shown in Appendix A.3, and thus the smooth pasting condition for h types is now necessary.

Proposition 9 (Stationary Equilibrium with Continuous Endogenous Innovation and Finite, Unbounded Support)**.** *If $r > \sqrt{\frac{\chi}{\bar{\lambda}}}$, then a unique equilibrium exists with a growth rate of,*

$$g = \bar{\lambda} r \left[1 - \sqrt{1 - \frac{\chi}{\bar{\lambda} r^2}}\right]. \tag{152}$$

The value function of the firm solves the following system of non-linear ODEs,

$$(r-g)v_h(z) = e^z - gv_h'(z) + \tfrac{\chi}{4}e^{-z}v_h'(z)^2 + \lambda_h(v_\ell(z) - v_h(z)) \tag{153}$$

$$(r-g)v_\ell(z) = e^z - gv_\ell'(z) + \lambda_\ell(v_h(z) - v_\ell(z)) \tag{154}$$

$$v_\ell(0) = v_h(0) = \frac{1}{r-g}. \tag{155}$$

Given a solution to this system, the endogenous innovation choice is such that $\gamma(0) = 0$, $\lim_{z\to\infty} \gamma(z) = g$, and

$$\gamma(z) = \tfrac{\chi}{2}e^{-z}v_h'(z). \tag{156}$$

With this $\gamma(z)$, $F_i(z)$ solves the KFEs in (145) to (148).

Proof. See Appendix D.5. A numerical method to compute the equilibrium is described in Appendix E.2

□

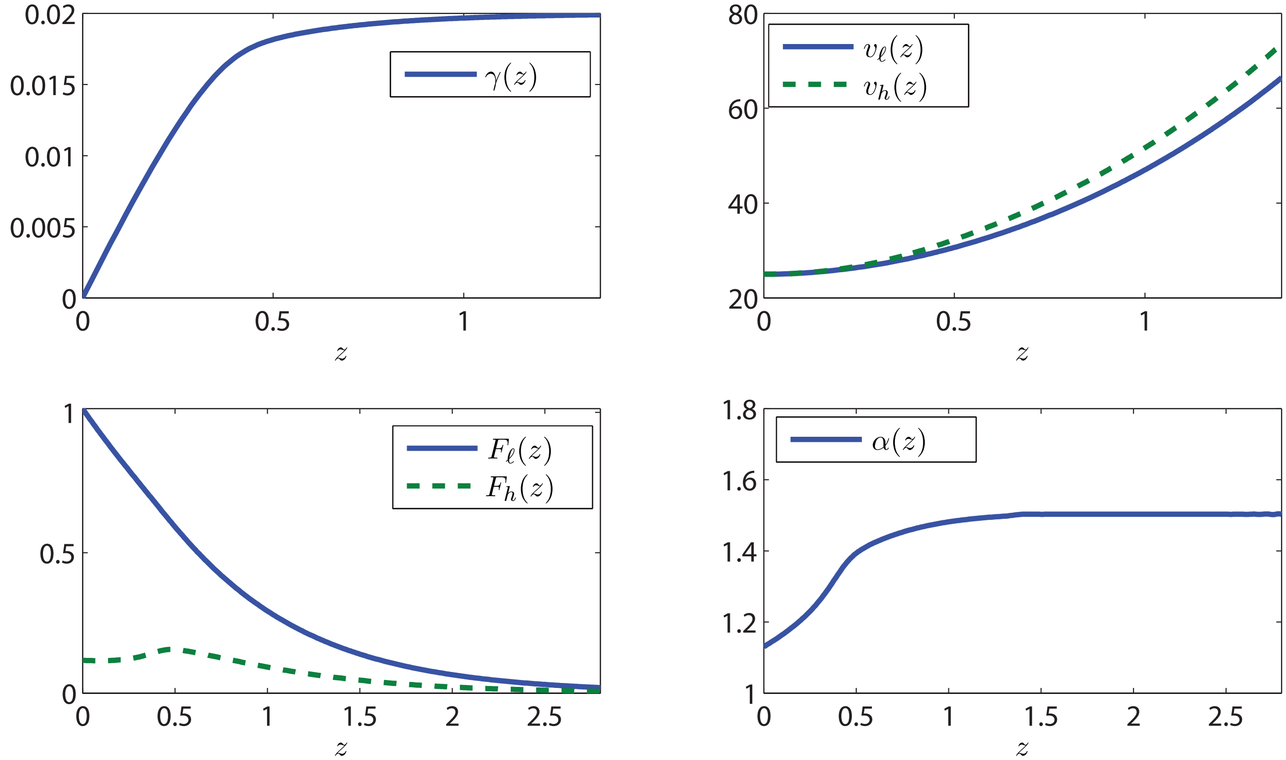

Figure 13: Endogenous $\gamma(z), v_i(z)$, and $F_i(z)$ with an Unbounded Frontier

An example with $r = .06, \lambda_\ell = 0.01, \lambda_h = 0.03, \chi = 0.00212, \kappa = 1, \zeta = 25$ is given in Figure 13. The asymptotic growth rate in this example is calibrated to be 2.0%.

In order to get a sense of the shape of the unconditional distribution, we define the "local" tail index

$$\alpha(z) \equiv \frac{F'(z)}{1 - F(z)} \tag{157}$$

In the standard log-log plot used for estimating power laws, as in Gabaix (2009), this $\alpha(z)$ would be the slope of the non-linear equation at z. Note that with this definition, the "local" tail index of a Pareto distribution is constant and equal to its true tail index. Furthermore, for any distribution with infinite support, the tail index is $\alpha \equiv \lim_{z\to\infty} \alpha(z)$. Figure 13 plots the local tail coefficient, converging to around 1.5. As the tail index is increasing, this shows that there is more productivity variability for firms with lower relative productivity.

Growth Rates Conditional on Size As the intensity of innovation, $\gamma(z)$, is increasing in z, this would suggest that the larger and more productive firms tend to do the most research. While the economics and model are very different, this is related to Acemoglu, Aghion, and Zilibotti (2006) and Benhabib, Perla, and Tonetti (2014), both of which feature weakly increasing innovation in relative productivity. In our model, the intuition comes from an analysis of the option values, where agents closer to the endogenous adoption threshold have less incentive to invest in incremental productivity enhancement and accordingly decrease their endogenous investment in $\gamma(z)$.

Because $\gamma(z)$ is increasing, the growth rate, conditional on being a high type is also increasing in z. While this may appear to contradict Gibrat's law, and some modern evidence on non-Gibrat's growth as surveyed in Sutton (1997) and modeled in Luttmer (2007) and Arkolakis (2011), it is important to remember that (1) technology adoption here is in fact growth of small firms, (2) we have left out the role of selection, which is extremely important for reconciling growth rates of small firms, and (3) the growth process is not an iid random walk, but has auto-correlation due to the Markov chain.

The first consideration is that smaller firms at the adoption threshold are growing rapidly (i.e., in fact, conditional on adoption in continuous time, they are growing at an infinite rate). Therefore, the model does have small firms tending to grow faster than larger firms. Here, we have simplified the model to ensure only a single adoption barrier exists and that firms make immediate productivity jumps. With more frictions and heterogeneity leading to a continuum of adoption barriers, the average growth rates might be more empirically plausible.

Second, many models investigating the empirics of Gibrat's law have emphasized that the higher growth rates for small firms is only conditional on survival. As small firms are more likely to exit, it implies that the average growth rate for smaller firms in the sample is higher. As we have purposely shut off exit in our model, this effect is not present. Davis, Haltiwanger, and Schuh (1996) find that when selection and mean reversion in stochastic processes are taken into account, the inverse relationship can disappear. However, Arkolakis (2011) discusses how the inverse relationship between size and growth tends to still exist even after selection, and describes how the Davis, Haltiwanger, and Schuh (1996) adjustment does not apply to random walks.

Finally, using our Markov chain process for growth, there is auto-correlation of growth rates for firms. This is in contrast to simply being a random walk, and hence the Davis, Haltiwanger, and Schuh (1996) results may still apply (where it doesn't in Arkolakis (2011), as discussed).

4.2 Continuous Innovation Choice with a Bounded Frontier

As discussed in Section 4.1, without leapfrogging $z \to \infty$. To endogenize the choice of leapfrogging, we allow firms that are adopting technology to jump to the frontier with probability $\theta \in [0, 1)$. Furthermore, just as we allow firms to choose their innovation rate γ, subject to a convex cost, we also allow firms to choose the probability of a jump to the frontier with a convex cost. The cost of choosing jump probability θ is $\frac{1}{\varsigma}\theta^2$. When a firm upgrades its technology through adoption, it can discover a state of the art invention and jump to the frontier. We can adapt Section 3.4 to include the same endogenous intensive innovation choice of γ, and a new choice of θ:

$$(r-g)v_\ell(z) = e^z - gv_\ell'(z) + \lambda_\ell(v_h(z) - v_\ell(z)) + \eta(v_\ell(\bar{z}) - v_\ell(z)) \tag{158}$$

$$(r-g)v_h(z) = \max_{\gamma \geq 0}\left\{e^z - \frac{1}{\chi}e^z\gamma^2 - (g-\gamma)v_h'(z) + \lambda_h(v_\ell(z) - v_h(z)) + \eta(v_\ell(\bar{z}) - v_h(z))\right\} \tag{159}$$

$$v_s \equiv v_\ell(0) = v_h(0) = \max_{\theta \geq 0}\left\{(1-\theta)\int_0^{\bar{z}} v_\ell(z)\mathrm{d}F(z)^\kappa + \theta v_\ell(\bar{z}) - \zeta - \frac{1}{\varsigma}\theta^2\right\} \tag{160}$$

$$v_\ell'(0) = v_h'(0) = 0 \tag{161}$$

$$\begin{aligned}0 &= gF_\ell'(z) + \lambda_h F_h(z) - \lambda_\ell F_\ell(z) - \eta F_\ell(z)\\ &\quad + (\eta + \theta(S_\ell + S_h))\mathbb{H}\left(z - \bar{z}\right) + (1-\theta)(S_\ell + S_h)F(z)^\kappa - S_\ell\end{aligned} \tag{162}$$

$$0 = (g - \gamma(z))F_h'(z) + \lambda_\ell F_\ell(z) - \lambda_h F_h(z) - \eta F_h(z) - S_h \tag{163}$$

$$0 = F_\ell(0) = F_h(0) \tag{164}$$

$$1 = F_h(\bar{z}) + F_\ell(\bar{z}) \tag{165}$$

$$S_\ell = gF_\ell'(0) \tag{166}$$

$$S_h = gF_h'(0) \tag{167}$$

Note that when a particular firm is choosing θ, it does not influence the θ chosen by the other agents. Firms will take into account the effects of the aggregate θ choice on $F_i(z)$, and as all adopting firms are a priori identical, each firm will choose the same θ, which will induce a $F_i(z)$

that is consistent with firm beliefs about $F_i(z)$ in equilibrium. Define $\bar{\lambda}$ as

$$\bar{\lambda} \equiv \frac{r + \eta + \lambda_\ell + \lambda_h}{r + \eta + \lambda_\ell}. \tag{168}$$

Proposition 10 (Stationary Equilibrium with Continuous Endogenous Innovation and Bounded Support)**.** *A continuum of equilibria exist, parameterized by a $\bar{z}$. To ensure $g < r$ there is a maximum feasible g—which is decreasing in η—given by*

$$g_{max} = \bar{\lambda}(r + \eta)\left[1 - \sqrt{1 - \frac{\chi}{\bar{\lambda}(r + \eta)^2}}\right]. \tag{169}$$

The accompanying $\gamma(z), v_i(z)$, and $F_i(z)$ solve the system of non-linear ODEs,

$$(r - g)v_\ell(z) = e^z - gv_\ell'(z) + \lambda_\ell(v_h(z) - v_\ell(z)) + \eta(v_\ell(\bar{z}) - v_\ell(z)) \tag{170}$$

$$(r - g)v_h(z) = e^z - gv_h'(z) + \tfrac{\chi}{4}e^{-z}v_h'(z)^2 + \lambda_h(v_\ell(z) - v_h(z)) + \eta(v_\ell(\bar{z}) - v_h(z)) \tag{171}$$

$$v_\ell'(0) = v_h'(0) = 0 \tag{172}$$

$$v_\ell(0) = v_h(0) = \frac{1 + \eta v_\ell(\bar{z})}{r - g + \eta} \tag{173}$$

The endogenous innovation choice is such that $\gamma(0) = 0, \gamma(\bar{z}) \equiv g$ and

$$\gamma(z) = \tfrac{\chi}{2}e^{-z}v_h'(z) \tag{174}$$

Given the innovation intensity, the distribution solves (160)*,* (162) *and* (163) *where the chosen intensity of jumps to the frontier for adopting agents is,*

$$\theta = 1 - \sqrt{1 - \varsigma(v_\ell(\bar{z}) - v_\ell(0) - \zeta)}. \tag{175}$$

Proof. See Appendix D.6. A numerical method to solve for the continuum of equilibria is described in Appendix E.1. □

In the limit, as $\eta \to 0$, the upper bound on g in (169) becomes the limiting case in Proposition 9.

Comparing to Stokey (2014), here the endogenous choice of γ is complicated by the option value. In the case of Proposition 9, the asymptotic γ as $z \to \infty$ becomes unique as the option value disappears, unlike with the bounded $\bar{z}$ in Proposition 10. Hence, a different distribution and $\bar{z}$ induce different growth option values, and give a continuum of self-fulfilling $\gamma(z)$.

Figure 16 plots the maximum growth rate of the set of admissible g as a function of η using (169) and with the same parameters as Figure 13. As $\eta \to 0$, the number of jumps to the frontier approaches 0, and the model in Section 4.2 asymptotically becomes that in Section 4.1. The intuition for a decreasing $\max(g(\eta))$ is that with more jumps to the frontier, the distribution becomes more compressed. As the growth rate of the frontier is determined by the autarkic innovation decision at $\bar{z}$, which takes into account the option value of diffusion, the more compressed the distribution, the lower the innovation rate, as in Figure 15.

4.3 Summary of Hysteresis and Multiplicity

The results of uniqueness of stationary equilibria are summarized in Table 1. Interestingly, hysteresis exists for the two opposites: infinite support with exogenous innovation, and bounded, finite support with endogenous innovation. Figure 16 demonstrates that the unbounded case is the limit of the bounded case (as the maximum at $\eta = 0$ is the g in the unbounded case from (169)), while the differences between Sections 3.2 and 3.3 show that the case of infinite and unbounded support are not identical.

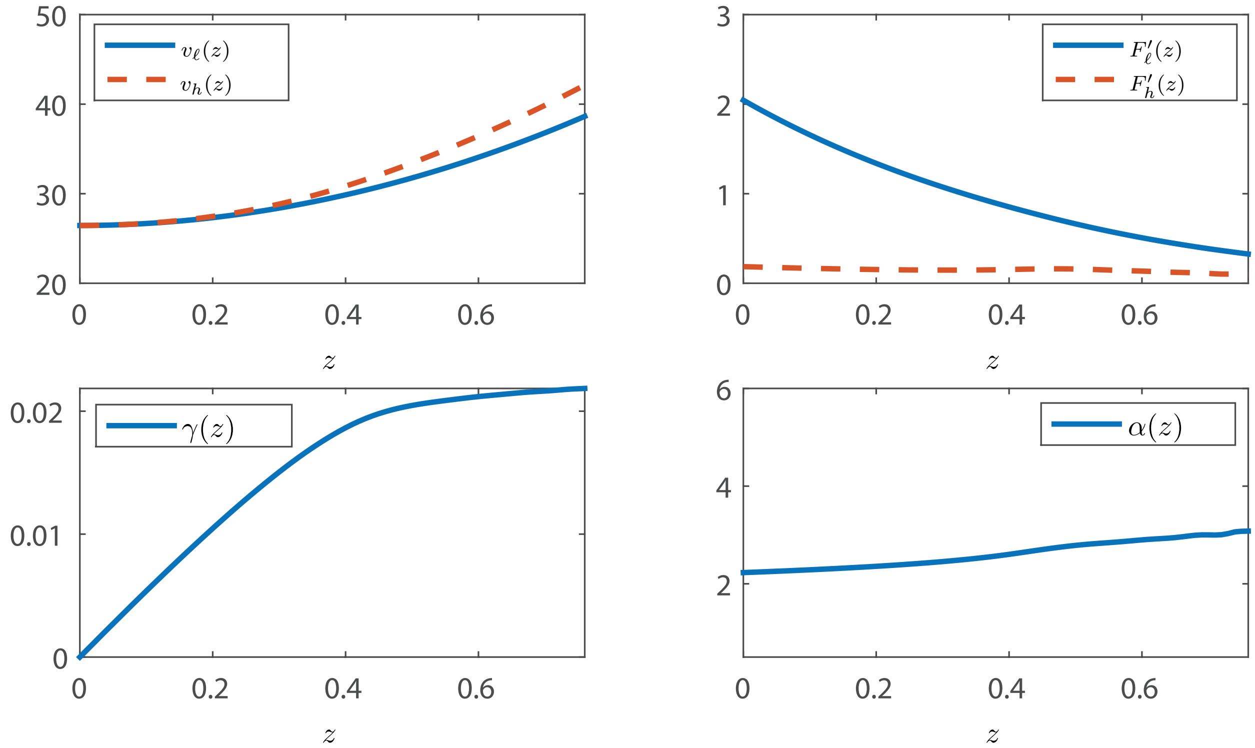

Figure 14: Endogenous $\gamma(z), v_i(z)$, and $F_i(z)$ with an Bounded Frontier

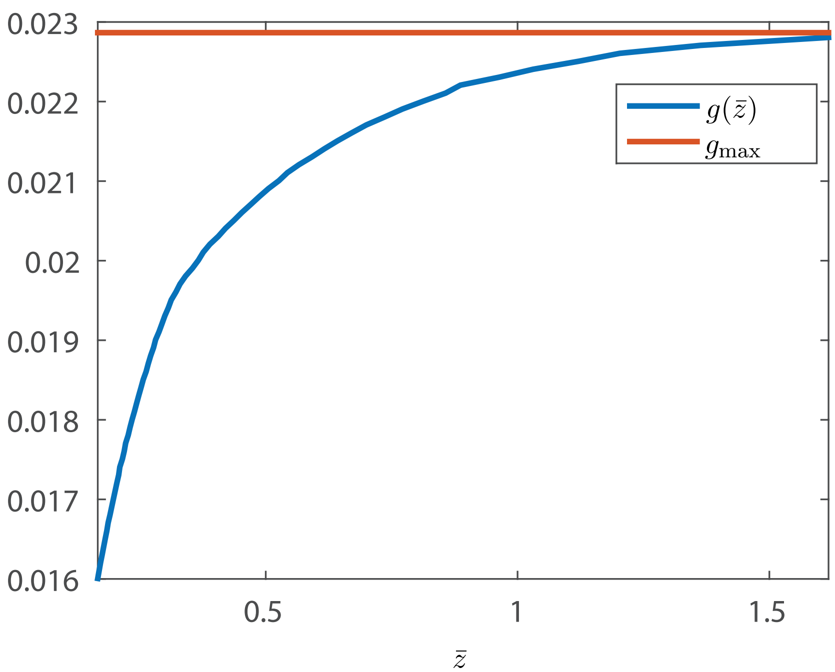

Figure 15: Equilibrium g as a function of $\bar{z}$

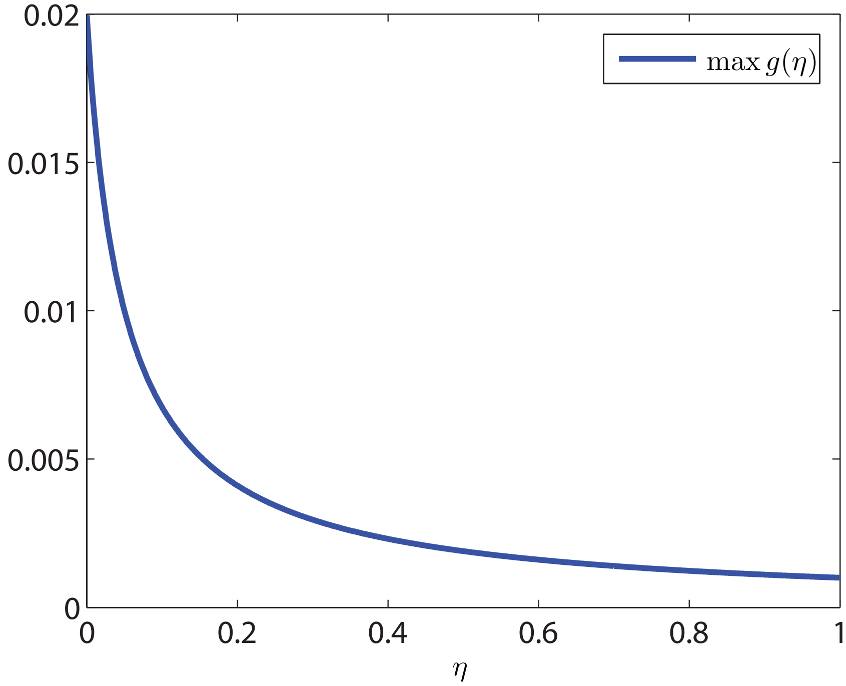

Figure 16: Maximum Equilibrium $g(\eta)$

Support	**Innovation**	
	Exogenous	*Endogenous*
Infinite	Hysteresis	Hysteresis
Unbounded	Unique	Unique
Bounded	Unique	Hysteresis

Table 1: Summary of Hysteresis and Uniqueness

5 Conclusion

Technology adoption, technological innovation and their interaction contribute to economic growth and to the evolution of the productivity distribution. In the various models that we study, growth rates and the distribution of productivities are endogenous, and they depend on the specification of the innovation and adoption processes, as well as the initial distribution of productivities available for adoption. In particular, whether adoption contributes to long-run growth in addition to innovation can depend on the properties (and tail index) of the initial distribution.[30] The specification of the innovation process (as GBM or a Markov process) can determine whether the asymptotic stationary distribution of relative productivities (the ratio of the frontier to the bottom) has finite support or not. We show in Propositions 7 and 8 that quality ladder type innovations driven by discrete Markov processes, under which a positive fraction of innovators leapfrog to the frontier, guarantee a stationary long run distribution with a finite support. We also study the problem of hysteresis: the possible multiplicity of stationary distributions that depend on initial conditions. Multiple stationary distributions occur in cases where (1) the support of the stationary distribution is infinite and adoption contributes to long-run growth (see Propositions 1, 4 and 6) or (2) when the intensity of the innovation is endogenously chosen with a positive probability of leapfrogging to the finite frontier of the stationary distribution (see Proposition 10 and Section 4.3). The various models of innovation and adoption processes that we have studied describe a rich set of long-run productivity distributions and of growth rates that may be useful for empirical work on the evolution of productivities.

[30] See for example Propositions 1 to 4 and 6 to 8 where he tail index is denoted as α.

Appendix A General Proofs

A.1 Limit of Draw Arrival Process

The following is a rough, heuristic derivation of the law of motion and cost function for searching which yields a conditional draw above the firm's search threshold.

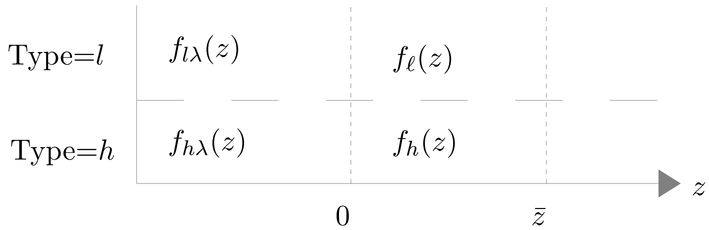

Figure 17: Search Regions PDF Before Taking Limit

Instead of instantaneous draws, assume a firm in this setup choosing to adopt has an arrival rate of $\bar{\lambda} > 0$ of opportunities. While adopting, they pay a normalized flow cost of $\zeta\bar{\lambda}$. Note that we are scaling the flow cost by the arrival rate in order to take the limit and have a finite expected value of search costs.

Furthermore, assume that the firm draws *unconditionally* from the z distribution in the economy (rather than simply those above their current threshold), and starts with a ℓ type. In the stationary equilibrium, as all agents start low, searching firms accept a draw if they get above the normalized cutoff of 0. The proof will construct a limit where agents get a successful draw above 0 in any infinitesimal time period, and hence draw from the conditional distribution of $z \geq 0$.

Define $F_{l\lambda}(z)$ and $F_{h\lambda}(z)$ to be the cdfs of agents in the $z < 0$ region, as in Figure 17, which will be shown to disappear in the limit. See Appendix A.3 for a proof that the thresholds are identical for both types.

As firms in the region $F_{l\lambda}(z)$ are otherwise identical, set the mass of searching agents as $F_{l\lambda}(0)$. Assume that agents have an unconditional draw of all z within the economy, then conditional on a draw, the probability of escaping the $F_{l\lambda}(0)$ mass is $(1 - F_{l\lambda}(0) - F_{h\lambda}(0))$. It is easily shown that the the arrival rate of *successful* draws is then $\bar{\lambda}(1 - F_{l\lambda}(0) - F_{h\lambda}(0))$. The distribution of waiting times until the first success is an exponential distribution with this parameter. The survivor function is therefore: $e^{-\bar{\lambda}(1-F_{l\lambda}(0)-F_{h\lambda}(0))t}$. Due to the total mass one 1 firm, $F_{l\lambda}(0) + F_{h\lambda}(0)) \in (0, 1)$, so the survivor function is decreasing in t. $F_{l\lambda}(0)$ is independent of the $\bar{\lambda}$ arrival rate when taking limits as no agents enter this region from successful searches. Taking the limit for any t, $\lim_{\bar{\lambda}\to\infty} e^{-\bar{\lambda}(1-F_{l\lambda}(0)-F_{h\lambda}(0))t} = 0$. Therefore, in the limit in equilibrium, $F_{l\lambda}(0) = 0$ as measure 1 agents get a successful draw in any strictly positive interval. The same arguments can be used to explain why $F_{h\lambda}(0) = 0$.

To ensure that the expected search costs in this limit are finite, calculate the present discounted value of flow payments until the first success. This is the exponential distribution with parameter $\bar{\lambda}(1 - F_{l\lambda}(0) - F_{h\lambda}(0))$ and flow cost $\zeta\bar{\lambda}$

$$\mathbb{E}\left[\text{search costs}\right] = \int_0^\infty \left(\int_0^t \zeta\bar{\lambda}e^{-rs}\mathrm{d}s\right)\bar{\lambda}(1 - F_{l\lambda}(0) - F_{h\lambda}(0))e^{-\bar{\lambda}(1-F_{l\lambda}(0)-F_{h\lambda}(0))t}\mathrm{d}t \tag{A.1}$$

$$= \frac{\bar{\lambda}\zeta}{r + \bar{\lambda}(1 - F_{l\lambda}(0) - F_{h\lambda}(0))} \tag{A.2}$$

Taking the limit and using the result that $F_{l\lambda}(0)$ and $F_{h\lambda}(0) \to 0$

$$\lim_{\bar{\lambda}\to\infty} \mathbb{E}\left[\text{search costs}\right] = \zeta \tag{A.3}$$

Therefore, in the limit the model can have draws directly from above the current threshold, with measure 0 remaining behind, and a cost for an instantaneous adoption of ζ.

A.2 Recall Doesn't Change the Optimal Stopping Conditions

As in McCall search, the ability to reject a draw and recall the last productivity will have no effect in equilibrium. To see this, assume firms can accept or reject a draw, but still draw from the truncated distribution above $M(t)$ for simplicity (see Appendix A.1 for why this isn't an important assumption). To show that this doesn't change the optimal policy, it is necessary to compare the optimal stopping conditions (i.e. value matching and smooth pasting) to show that they are identical with and without recall.

The value-matching condition is not effected by recall, as conditioning on a successful draw it is above $M(t)$. Hence accept or reject has no effect on the value matching condition for the stopping problem.

The smooth pasting condition is more complicated and needs to be analyzed assuming deviations from the $M(t)$ stopping rule. If an agent draws, then the net value they gain is the probability that they draw above Z and accept, plus the probability they draw below Z and keep Z. Conditional on acceptance, the distribution of $\tilde{Z}$ is the drawing distribution truncated above acceptance, i.e. the current Z.

$$V_s(t,Z) = (1-\Phi(t,Z))\int_Z^\infty V(t,\tilde{Z})\frac{\Phi'(t,\tilde{Z})}{1-\Phi(t,Z)}\mathrm{d}\tilde{Z} + \Phi(t,Z)V(t,Z) \tag{A.4}$$

$$= \int_Z^\infty V(t,\tilde{Z})\Phi'(t,\tilde{Z})\mathrm{d}\tilde{Z} + \Phi(t,Z)V(t,Z) \tag{A.5}$$

Taking the derivative of this with respect to Z and using the fundamental theorem of calculus,

$$\boldsymbol{\partial}_Z V_s(t,Z) = -V(t,Z)\Phi'(t,Z) + \Phi(t,Z)\boldsymbol{\partial}_Z V(t,Z) + \Phi'(t,Z)V(t,Z) \tag{A.6}$$

$$= \Phi(t,Z)\boldsymbol{\partial}_Z V(t,Z) \tag{A.7}$$

The smooth pasting condition is evaluated at $M(t)$, and $\Phi(t,M(t)) = 0$ at points of continuity,

$$\boldsymbol{\partial}_Z V_s(t,M(t)) = 0 \tag{A.8}$$

Therefore, the smooth pasting condition is identical to the original version without recall.

A.3 Common Adoption Threshold for All Idiosyncratic States

Proof. This section proves under what general conditions heterogeneous firms will choose the same adoption threshold.

Allow for some discrete type i, and augment the state of the firm with an additional state γ (which could be a vector or a scalar). Assume that there is some control u which controls the infinitesimal generator $\mathbb{Q}_u$ of the Markov process on type i and potentially γ. Also assume that the agent can control the growth rate $\hat{\gamma}$ at some cost. The feasibility set of the controls is $(u,\hat{\gamma}) \in U(t,i,z,\gamma)$ The cost of the controls for adoption and innovation have several requirements for this general property to hold,

1. The net value of searching, $v_s(t)$ is identical for all types i, productivities z, and additional state γ.

2. The minimum of the cost function is 0 and in the interior of the feasibility set:

$$\min_{(\hat{\gamma},u)\in U(t,i,z,\gamma)} c(t,z,\hat{\gamma},i,\gamma,u) = 0, \text{ for all } t,\gamma,i$$

3. Let $\bar{v}(t)$ be the value of a jump with rate η to the frontier, which are identical for all agent states (e.g. $\bar{v}(t) = v_\ell(t, \bar{z}(t)) = v_h(t, \bar{z}(t))$). Without this requirement, firms may have differing incentives to "wait around" for arrival rates of jumps at the adoption threshold. A slightly weaker requirement is if the arrival rates and value are identical only at the threshold: $\eta(t, 0, \cdot)$ and $\bar{v}(t, 0, \cdot)$ are idiosyncratic states.

Then the normalization of the firm's problem gives the following set of necessary conditions,

$$(r - g(t))v_i(t, z, \gamma) = \max_{(\hat{\gamma}, u) \in U(\cdot)} \Big\{ e^z - c(t, z, \hat{\gamma}, i, \gamma, u) + (\hat{\gamma} - g)\frac{\partial v_i(t, z, \gamma)}{\partial z} + \frac{\partial v_i(t, z, \gamma)}{\partial t}$$
$$+ \mathbf{e_i} \cdot \mathbb{Q}_u \cdot v(t, z, \gamma) + \eta(\bar{v}(t) - v_i(t, z, \gamma)) \} \tag{A.9}$$
$$v_i(t, \underline{z}(t, i, \gamma), \gamma) = v_s(t) \tag{A.10}$$
$$\frac{\partial v_i(t, \underline{z}(t, i, \gamma), \gamma)}{\partial z} = 0 \tag{A.11}$$

Where $\underline{z}(t, i, \gamma)$ is the normalized search threshold for type i and additional state γ. To prove that these must be identical, we will assume that $\underline{z}(t, i, \gamma) = 0$ for all types and additional states, and show that this leads to identical necessary optimal stopping conditions. Evaluating at $z = 0$,

$$v_i(t, 0, \gamma) = v_s(t) \tag{A.12}$$
$$\frac{\partial v_i(t, 0, \gamma)}{\partial z} = 0 \tag{A.13}$$

Differentiate (A.12),

$$\frac{\partial v_i(t, 0, \gamma)}{\partial t} = v_s'(t) \tag{A.14}$$

(A.12) and (A.13) are identical for any i and γ. Substitute (A.12) to (A.14) into (A.9)

$$(r - g(t))v_s(t) = \max_{(\hat{\gamma}, u) \in U(\cdot)} \left\{ 1 - c(t, z, \hat{\gamma}, i, \gamma, u) + \mathbf{e_i} \cdot \mathbb{Q}_u \cdot v_s(t) + \eta(\bar{v}(t) - v_s(t)) + v_s'(t) \right\} \tag{A.15}$$

Since in order to be a valid intensity matrix, all rows in $\mathbb{Q}_u$ add to 0 for any u, the last term is 0 for any i or control u,

$$(r - g(t))v_s(t) = \max_{(\hat{\gamma}, u) \in U(\cdot)} \left\{ 1 - c(t, z, \hat{\gamma}, i, \gamma, u) + v_s'(t) + \eta(\bar{v}(t) - v_s(t)) \right\} \tag{A.16}$$

The optimal choice for any i or γ is to minimize the costs of the $\hat{\gamma}$ and u choices. Given our assumption that the cost at the minimum is 0 and is interior,

$$(r - g(t))v_s(t) = 1 + v_s'(t) + \eta(\bar{v}(t) - v_s(t)) \text{ for all } i \tag{A.17}$$

Therefore, the necessary conditions for optimal stopping are identical for all i, γ, z, confirming our guess. Furthermore, (A.17) provides an ODE for $v_s(t)$ based on aggregate $g(t)$ and $\bar{v}(t)$ changes. Solving this in a stationary environment gives an expression for v_s in terms of equilibrium g and $\bar{v}$,

$$v_s = \frac{1 + \eta\bar{v}}{r - g + \eta} \tag{A.18}$$

□

Appendix B Exogenous Geometric Brownian Innovation

B.1 Normalization and Stationarity

Normalizing the Productivity Distribution Define the normalized distribution of productivity, as the distribution of productivity relative to the endogenous adoption threshold $M(t)$:

$$\Phi(t, Z) \equiv F(t, \log(Z/M(t))) \tag{B.1}$$

Differentiating to obtain the pdf yields

$$\boldsymbol{\partial}_Z \Phi(t, Z) = \frac{1}{Z}\frac{\partial F(t, \log(Z/M(t)))}{\partial z} = \frac{1}{Z}\boldsymbol{\partial}_z F(t, z) \tag{B.2}$$

Differentiating again,

$$\boldsymbol{\partial}_{ZZ} \Phi(t, Z) = \frac{1}{Z^2}\left(\boldsymbol{\partial}_{zz} F(t, z) - \boldsymbol{\partial}_z F(t, z)\right) \tag{B.3}$$

Differentiating (B.1) with respect to t and using the chain rule to obtain the transformation of the time derivative

$$\boldsymbol{\partial}_t \Phi(t, Z) = \frac{\partial F(t, \log(Z/M(t)))}{\partial t} - \frac{M'(t)}{M(t)}\frac{\partial F(t, \log(Z/M(t)))}{\partial z} \tag{B.4}$$

Using the definition $g(t) \equiv M'(t)/M(t)$ and the definition of z,

$$\boldsymbol{\partial}_t \Phi(t, Z) = \boldsymbol{\partial}_t F(t, z) - g(t)\boldsymbol{\partial}_z F(t, z) \tag{B.5}$$

Normalizing the Law of Motion Substitute (B.2), (B.3) and (B.5) into (5),

$$\begin{aligned}\frac{\partial F(t, \log(Z/M(t)))}{\partial t} - g(t)\frac{\partial F(t, \log(Z/M(t)))}{\partial z} &= -(\gamma + \sigma^2/2)\frac{\partial F(t, \log(Z/M(t)))}{\partial z} \\ &+ \frac{\sigma^2}{2}\left(\frac{\partial^2 F(t, \log(Z/M(t)))}{\partial z^2} - \frac{\partial F(t, \log(Z/M(t)))}{\partial z}\right) \\ &+ S(t)F(t, \log(Z/M(t)))^{\kappa} - S(t)\end{aligned} \tag{B.6}$$

Using the definition of z and reorganizing yields the normalized KFE,

$$\boldsymbol{\partial}_t F(t, z) = (g(t) - \gamma)\boldsymbol{\partial}_z F(t, z) + \tfrac{\sigma^2}{2}\boldsymbol{\partial}_{zz} F(t, z) + S(t)F(t, z)^{\kappa} - S(t) \tag{B.7}$$

For $S(t)$, take (9) and substitute from (B.2),

$$S(t) = \left(M'(t) - \gamma M(t)\right)\frac{1}{M(t)}\frac{\partial F(t, \log(M(t)/M(t)))}{\partial z} + \frac{\sigma^2}{2}\frac{M(t)^2}{M(t)^2}\frac{\partial^2 F(t, \log(M(t)/M(t)))}{\partial z^2} \tag{B.8}$$

$$= (g(t) - \gamma)\boldsymbol{\partial}_z F(t, 0) + \tfrac{\sigma^2}{2}\boldsymbol{\partial}_{zz} F(t, 0) \tag{B.9}$$

Normalizing the Value Function Define the normalized value of the firm as,

$$v(t, \log(Z/M(t))) \equiv \frac{V(t,Z)}{M(t)} \tag{B.10}$$

Rearranging,

$$V(t,Z) = M(t)v(t, \log(Z/M(t))) \tag{B.11}$$

Differentiating (B.11) with respect to t

$$\boldsymbol{\partial}_t V(t,Z) = M'(t)v(t, \log(Z/M(t))) - M'(t)\frac{\partial v(t, \log(Z/M(t))}{\partial z} + M(t)\frac{\partial v(t, \log(Z/M(t))}{\partial t} \tag{B.12}$$

Divide by $M(t)$ and use the definition of $g(t)$

$$\frac{1}{M(t)}\boldsymbol{\partial}_t V(t,Z) = g(t)v(t,z) - g(t)\boldsymbol{\partial}_z v(t,z) + \boldsymbol{\partial}_t v(t,z) \tag{B.13}$$

Differentiating (B.11) with respect to Z,

$$\boldsymbol{\partial}_Z V(t,Z) = \frac{M(t)}{Z}\frac{\partial v(t, \log(Z/M(t)))}{\partial z} \tag{B.14}$$

$$\boldsymbol{\partial}_{ZZ} V(t,Z) = \frac{M(t)}{Z^2}\left(-\frac{\partial v(t, \log(Z/M(t)))}{\partial z} + \frac{\partial^2 v(t, \log(Z/M(t)))}{\partial z^2}\right) \tag{B.15}$$

Rearranging,

$$\frac{1}{M(t)}\boldsymbol{\partial}_Z V(t,Z) = \frac{1}{Z}\boldsymbol{\partial}_z v(t,z) \tag{B.16}$$

$$\frac{1}{M(t)}\boldsymbol{\partial}_{ZZ} V(t,Z) = \frac{1}{Z^2}\left(\boldsymbol{\partial}_{zz} v(t,z) - \boldsymbol{\partial}_z v(t,z)\right) \tag{B.17}$$

Divide (2) by $M(t)$ and then substitute from (B.13), (B.16) and (B.17),

$$\begin{aligned} r\frac{1}{M(t)}V(t,Z) = {} & \frac{Z}{M(t)} + (\gamma + \sigma^2/2)\frac{M(t)}{M(t)}\frac{Z}{Z}\boldsymbol{\partial}_z v(t,z) + \frac{\sigma^2}{2}\left(\boldsymbol{\partial}_{zz} v(t,z)) - \boldsymbol{\partial}_z v(t,z)\right) \\ & + g(t)v(t,z) - g(t)\boldsymbol{\partial}_z v(t,z) + \boldsymbol{\partial}_t v(t,z) \end{aligned} \tag{B.18}$$

Use (B.11) and the definition of z and rearrange (B.18),

$$(r - g(t))v(t,z) = e^z + (\gamma - g(t))\boldsymbol{\partial}_z v(t,z) + \tfrac{\sigma^2}{2}\boldsymbol{\partial}_{zz} v(t,z) + \boldsymbol{\partial}_t v(t,z) \tag{B.19}$$

Optimal Stopping Conditions Divide the value matching condition in (3) by $M(t)$,

$$\frac{V(t, M(t))}{M(t)} = \int_{M(t)}^{B(t)} \frac{V(t,Z)}{M(t)}\boldsymbol{\partial}_Z \Phi(t,Z)\Phi(t,Z)^{\kappa-1}\mathrm{d}Z - \frac{M(t)}{M(t)}\zeta \tag{B.20}$$

Use the substitutions in (B.2) and (B.10)

$$v(t,0) = \int_{M(t)}^{B(t)} v(t, \log(Z/M(t)))\frac{1}{Z}\frac{\partial F(t, \log(Z/M(t)))}{\partial z}F(t, \log(Z/M(t)))^{\kappa-1}\mathrm{d}Z - \zeta \tag{B.21}$$

Use the change of variable $z = \log(Z/M(t))$ in the integral, which implies $\mathrm{d}z = \frac{1}{Z}\mathrm{d}Z$. Note that the bounds of integration change to $[\log(M(t)/M(t)), \log(B(t)/M(t))] = [0, \bar{z}(t)]$

$$v(t,0) = \int_0^{\bar{z}(t)} v(t,z)\boldsymbol{\partial}_z F(t,z) F(t,z)^{\kappa-1}\mathrm{d}z - \zeta \tag{B.22}$$

Which can also be written succinctly as,

$$v(t,0) = \int_0^{\bar{z}(t)} v(t,z)\mathrm{d}F(t,z)^{\kappa} - \zeta \tag{B.23}$$

Evaluate (B.16) at $Z = M(t)$ to find that

$$\frac{\partial V(t, M(t))}{\partial Z} = \boldsymbol{\partial}_z v(t,0) \tag{B.24}$$

Substitute this into (4) to give the smooth pasting condition as,

$$\boldsymbol{\partial}_t v(t,0) = 0 \tag{B.25}$$

As a model variation, if the cost is proportional to Z, then the only change to the above conditions is that the smooth pasting condition becomes $\boldsymbol{\partial}_z v(t,0) = -\zeta$. This cost formulation has the potentially unappealing feature that the value is not monotone in Z, as firms close to the adoption threshold would rather have a lower Z to decrease the adoption cost for the same benefit.

B.2 BGP of Model with Deterministic, Exogenous Innovation

Proof of Proposition 1. Solving (26) with $\sigma = 0$ and with the initial condition (27) gives,

$$F(z) = 1 - e^{-\frac{S}{g-\gamma}z} \tag{B.26}$$

From (29), given a $F'(0)$,

$$F(z) = 1 - e^{-F'(0)z} = 1 - e^{-\alpha z} \tag{B.27}$$

From the normalization of the initial condition $\Phi(0, Z)$, it can be shown that it is already in the form of (B.27), and $F'(0) = \alpha$ at time 0, so the stationary distribution remains on a balanced growth path for the $\alpha \equiv F'(0)$ in the initial condition.

To solve (23) and (25), use assume a solution of the form,

$$v(z) = ae^{z} + be^{-\nu z} \tag{B.28}$$

Substitute this guess into (23) and (25) and equate undetermined coefficients to find the following system of equations

$$0 = -a\gamma + ar - 1 \tag{B.29}$$
$$0 = b\gamma\nu - bg\nu - b + br \tag{B.30}$$
$$0 = a - b\nu \tag{B.31}$$

Solving this system, and recognizing that the guess of (B.28) is confirmed and there is no z in the system,

$$\nu = \frac{r-g}{g-\gamma} \tag{B.32}$$

$$a = \frac{1}{r-\gamma} \tag{B.33}$$

$$b = \frac{a}{\nu} \tag{B.34}$$

Substituting these back into the guess gives the value function

$$v(z) = \frac{1}{r-\gamma}e^{z} + \frac{1}{\nu(r-\gamma)}e^{-\nu z} \tag{B.35}$$

Subsituting (B.27) and (B.35) into (24), and integrating and simplifying

$$1 = (\alpha-1)\zeta(-\alpha\gamma + (\alpha-1)g + r) \tag{B.36}$$

Solving for g,

$$g = \frac{1-\zeta(\alpha-1)(r-\alpha\gamma)}{\zeta(\alpha-1)^2} \tag{B.37}$$

To derive the parameter restriction on r, note that $r > g > \gamma$ is necessary for a non-explosive (B.35). Substituting for g and simplifying gives,

$$r > \gamma + \frac{1}{\zeta\alpha(\alpha-1)} \tag{B.38}$$

□

B.3 BGP of Model Without Stochastic Innovation and $\kappa > 0$

Proof of Proposition 2. The solution techniques for $v(z)$ are identical to that in Appendix B.2. However, the KFE is now in general a nonlinear ODE. Rearrange (26),

$$F'(z) = \tfrac{S}{g-\gamma} - \tfrac{S}{g-\gamma}F(z)^{\kappa} \tag{B.39}$$

This non-linear ODE is separable,

$$\mathrm{d}z = \frac{\mathrm{d}F(z)}{\tfrac{S}{g-\gamma} - \tfrac{S}{g-\gamma}F(z)^{\kappa}} \tag{B.40}$$

Integrate,

$$z + C_1 = \int_0^F \frac{1}{\tfrac{S}{g-\gamma} - \tfrac{S}{g-\gamma}q^{\kappa}}\mathrm{d}q \tag{B.41}$$

Define ${}_1\mathbb{F}_2$ as the Hypergeometric function according to (36). Define the following function of $q \in [0,1]$,

$$Q(q) = \frac{g-\gamma}{S} q \; {}_1\mathbb{F}_2\left(1, 1/\kappa, 1+1/\kappa, q^{\kappa}\right) \tag{B.42}$$

Assume that $Q(q)$ has an inverse $Q^{-1}(\cdot)$. Using (36), (B.42) and (D.45),

$$z + C_1 = Q(F(z)) \tag{B.43}$$

$$F(z) = Q^{-1}(z + C_1) \tag{B.44}$$

From (28) and (B.42), $Q^{-1}(C_1) = 0$. From (B.42), $C_1 = Q(0) = 0$. Since $C_1 = 0$ and $F(z) = Q^{-1}(z)$, $Q(q)$ is the quantile function for the random variable z. From (29), we can get the stationary distribution indexed by a $F'(0)$ as in the example with $\kappa = 1$). Writing the stationary quantile function,

$$Q(q) = \frac{q}{F'(0)}\ {}_1\mathbb{F}_2\left(1, 1/\kappa, 1 + 1/\kappa, q^{\kappa}\right) \tag{B.45}$$

From probability theory, using the quantile function of the log of a random variable, the tail index is,[31]

$$\alpha = \lim_{q \to 1} \frac{Q''(q)}{(Q'(q))^2} \tag{B.46}$$

Using results from taking derivatives of special functions described in (38)

$$= \lim_{q \to 1} \kappa F'(0) q^{\kappa - 1}, \tag{B.47}$$

which shows the distortion of the tail parameter by κ relative to the Exponential distribution with $\kappa = 1$

$$\alpha = \kappa F'(0) \tag{B.48}$$

This nests the results of a standard Pareto, $\alpha = F'(0)$.

To calculate the value-matching condition, an expectation must be taken over $\hat{F}(z) \equiv F(z)^{\kappa}$. The quantile function solves the equation $F(z) = q$ for z. For the new quantile function solving $\hat{F}(z) = \hat{q}$, use the definition, $\hat{F}(z) = F^{\kappa}(z) = \hat{q}$, and transform to find $F(z) = \hat{q}^{1/\kappa}$. Therefore, the quantile for $\hat{F}$ is $\hat{Q}(q) = Q(q^{1/\kappa})$

Recall that expectations can be written in quantile functions, i.e. given a cdf $F(z)$ and quantile $F^{-1}(p)$, then for $h(\cdot)$,

$$E(h(X)) = \int_{-\infty}^{\infty} h(x) F'(x) dx = \int_0^1 h(F^{-1}(p)) dp \tag{B.49}$$

Using this, integrate over $\hat{F}(z)$ with the quantile $\hat{Q}(q)$ in (24)

$$\frac{1}{r - g} + \zeta = \int_0^1 v(\hat{Q}(q)) \mathrm{d}q \tag{B.50}$$

Substituting for $v(z)$ and $\hat{Q}(\cdot)$ in this function gives an implicit equation in g □

[31] See http://stats.stackexchange.com/questions/104169/limiting-expression-for-power-law-tail-index-from-a-quantile-function for a derivation

B.4 Model with Geometric Brownian Motion

Proof of Proposition 4. Guess that the form of the solution of (26) and (27) is,

$$F(z) = 1 - e^{-\alpha z} \tag{B.51}$$

The boundary value (27) is fulfilled. Substitute this guess into (26) and collect terms to use undetermined coefficients to find

$$0 = S + \alpha(\gamma - g + \alpha\sigma^2/2) \tag{B.52}$$

Solving this equation and choosing the non-explosive root,

$$\alpha = \frac{g-\gamma}{\sigma^2} - \sqrt{\frac{(g-\gamma)^2}{\sigma^4} - \frac{S}{\sigma^2/2}} \tag{B.53}$$

To solve (23) and (25), assume a solution of the form,

$$v(z) = ae^z + \frac{b}{\nu}e^{-\nu z} \tag{B.54}$$

Substitute this guess into (23) and (25) and equate undetermined coefficients to find the following system of equations

$$0 = -a\gamma + ar - \frac{a\sigma^2}{2} - 1 \tag{B.55}$$

$$0 = b\gamma - \frac{bg}{\nu} - bg - \frac{1}{2}b\nu\sigma^2 + \frac{br}{\nu} \tag{B.56}$$

$$0 = a - b \tag{B.57}$$

The absence of z confirms that a solution to this system is a particular solution of the ODE and initial condition. While there could be another exponential term for this 2nd order equation, use a no-bubble condition to eliminate it, and choose the positive ν root in the quadratic. The solution to this system of equations is,

$$a = \frac{1}{r - \gamma - \sigma^2/2} \tag{B.58}$$

$$\nu = -\frac{\gamma - g}{\sigma^2} + \sqrt{\left(\frac{g-\gamma}{\sigma^2}\right)^2 + \frac{r-g}{\sigma^2/2}} \tag{B.59}$$

$$b = a \tag{B.60}$$

And the substituted solution is,

$$v(z) = \frac{1}{r - \gamma - \sigma^2/2}\left(e^z + \frac{1}{\nu}e^{-\nu z}\right) \tag{B.61}$$

Use (B.61) to find,

$$v(0) = \frac{\frac{1}{\nu} + 1}{r - \gamma - \frac{\sigma^2}{2}} \tag{B.62}$$

Substituting (B.51), (B.61) and (B.62) into (24) gives an equation relating α and ν,

$$\frac{\frac{1}{\nu}+1}{r-\gamma-\frac{\sigma^2}{2}} = \int_0^\infty \left(\frac{\alpha}{r-\gamma-\sigma^2/2}e^{-(\alpha-1)z} + \frac{\alpha}{\nu(r-\gamma-\sigma^2/2)}e^{-(\alpha+\nu)z}\right)\mathrm{d}z - \zeta \tag{B.63}$$

$$= \frac{\alpha(\nu+1)(\alpha+\nu-1)}{(\alpha-1)\nu(\alpha+\nu)(r-\gamma-\sigma^2/2)} - \zeta \tag{B.64}$$

We assumed that $\alpha > 1$ and will analyze interior cases where $\nu > 0$ and the integral converges. Simplifying further,

$$0 = (\alpha-1)(\alpha+\nu)\zeta + \frac{\nu+1}{\gamma-r+\sigma^2/2} \tag{B.65}$$

This can be solved further for ν,

$$\nu = \frac{1-(\alpha-1)\alpha\zeta(r-\gamma-\sigma^2/2)}{(\alpha-1)\zeta(r-\gamma-\sigma^2/2)-1} = \frac{\alpha-1}{1-(\alpha-1)\left(r-\gamma-\frac{\sigma^2}{2}\right)\zeta} - \alpha \tag{B.66}$$

Equate (B.59) and (B.66) to find an expression in g and α

$$0 = \gamma - g + \alpha\sigma^2 + \sqrt{(g-\gamma)^2 + 2\sigma^2(r-g)} + \frac{(\alpha-1)\sigma^2}{(\alpha-1)(r-\gamma-\sigma^2/2)\zeta-1} \tag{B.67}$$

Solve for g,

$$g = \frac{-\left((\alpha-1)\alpha\zeta\left(2\gamma+\sigma^2\right)+2\right)\left((\alpha-1)\zeta\left(\alpha\sigma^2+2\gamma\right)+2\right)-4(\alpha-1)^2\zeta^2r^2+2(\alpha-1)\zeta r\left((\alpha-1)\zeta\left(\left(\alpha^2+1\right)\sigma^2+2(\alpha+1)\gamma\right)+4\right)}{2(\alpha-1)^2\zeta\left((\alpha-1)\zeta\left(-2\gamma+2r-\sigma^2\right)-2\right)} \tag{B.68}$$

Decomposing,

$$g = \frac{1-(\alpha-1)\zeta\,(r-\alpha\gamma)}{(\alpha-1)^2\zeta} + \frac{\sigma^2}{2}\frac{\alpha\left(\alpha(\alpha-1)\left(r-\gamma-\frac{\sigma^2}{2}\right)\zeta-2\right)+1}{(\alpha-1)\left((\alpha-1)\left(r-\gamma-\frac{\sigma^2}{2}\right)\zeta-1\right)} \tag{B.69}$$

Define the contibution to growth from catchup diffusion in the decomposition of (B.69) as,

$$g_c = \frac{1-(\alpha-1)\zeta\,(r-\alpha\gamma)}{(\alpha-1)^2\zeta} \tag{B.70}$$

From (B.66), a necessary condition for (B.69) to be the positive solution to (B.67) is,

$$0 < 1-(\alpha-1)\left(r-\gamma-\frac{\sigma^2}{2}\right)\zeta \tag{B.71}$$

Furthermore, assuming (B.71) and manipulating (B.66) gives an upper bound,

$$\frac{\alpha-1}{\alpha} > 1-(\alpha-1)\left(r-\gamma-\frac{\sigma^2}{2}\right)\zeta \tag{B.72}$$

Reorganizing (B.71) and (B.72), solving the quadratic, and choosing the positive root give necessary, but not sufficient, bounds on α,

$$\frac{1}{2}\left(1+\sqrt{\frac{4+\zeta(r-\gamma-\sigma^2/2)}{\zeta(r-\gamma-\sigma^2/2)}}\right) < \alpha < 1 + \frac{1}{\zeta(r-\gamma-\sigma^2/2)} \tag{B.73}$$

Upper Bound on α Take (B.53) and solve for S,

$$S = \alpha\left(g - \gamma - \alpha\frac{\sigma^2}{2}\right) \tag{B.74}$$

The condition for positive S is therefore,

$$g > \gamma + \alpha\sigma^2/2 \tag{B.75}$$

To find a bound, substitute (B.69) into (B.75),

$$\begin{aligned} 0 < &\left((\alpha-1)^2\alpha\zeta^2\sigma^4 - 2(\alpha-1)\zeta\sigma^2\left((\alpha-3)\alpha + \left(\alpha^2-1\right)\zeta(r-\gamma)\right) + 4((\alpha-1)\zeta(r-\gamma)-1)^2\right) \\ &\times\left((\alpha-1)\zeta\left(2\gamma - 2r + \sigma^2\right) + 2\right) \end{aligned} \tag{B.76}$$

Note that from the assumptions in (B.73), the 2nd term is negative, and for negative g from (B.76) it is necessary and sufficient for the first term to be negative.

$$0 > (\alpha-1)^2\alpha\zeta^2\sigma^4 - 2(\alpha-1)\zeta\sigma^2\left((\alpha-3)\alpha + \left(\alpha^2-1\right)\zeta(r-\gamma)\right) + 4((\alpha-1)\zeta(r-\gamma)-1)^2 \tag{B.77}$$

This is a cubic equation, solving for α here gives a complicated expression for a maximum α such that $S > 0$.

□

B.5 Dynamic Solution of the Deterministic Model

De-Trending The γ trend can be removed through a change of variables. Let $\tilde{Z} = e^{\gamma t}Z$, $\tilde{V}(t, e^{-\gamma t}Z) = e^{-\gamma t}V(t,Z)$, and $\tilde{\Phi}(t, e^{-\gamma Z}Z) = \Phi(t,Z)$. Differentiating these, and substituting into the Bellman Equation in (23) gives,

$$\tilde{r}\tilde{V}(t,\tilde{Z}) = \tilde{Z} + \frac{\partial\tilde{V}(t,\tilde{Z})}{\partial t} \tag{B.78}$$

Where $\tilde{r} = r - \gamma$. Substituting into (26) gives,

$$\frac{\partial\tilde{\Phi}(t,\tilde{Z})}{\partial t} = S(t)\tilde{\Phi}(t,\tilde{Z})^{\kappa} - S(t) \tag{B.79}$$

Similarly $\tilde{M}(t) \equiv e^{-\gamma t}M(t)$, $M'(t)/M(t) \equiv \gamma + \tilde{M}'(t)/\tilde{M}(t)$, and $\tilde{B}(t) \equiv e^{-\gamma t}B(t)$ This provides a system of equations in the same form as that without drift. The remainder of this section as well as Appendices B.2, B.5 and B.5.1 solve the model with $\gamma = 0$ drift for notational simplicity to conserve on the tildes, with a transformation done at the end.

General Solution to the Law of Motion in Deterministic Model The general solution will be proved by guess-and-verify

For any initial condition with cdf and pdf $\Phi_0(Z)$, $\Phi_0'(Z)$ guess that the pdf of the solution is

$$\boldsymbol{\partial}_Z\Phi(t,Z) = \frac{\Phi_0'(Z)}{1-\Phi_0(M(t))} \tag{B.80}$$

If we take this equation and plug it into (5), we see that

$$\frac{\Phi_0'(Z)\phi(M(t))M'(t)}{(1-\Phi_0(M(t)))^2} = \frac{\Phi_0'(Z)}{1-\Phi_0(M(t))}\frac{\Phi_0'(M(t))}{1-\Phi_0(M(t))}M'(t) \tag{B.81}$$

Recalling the truncation notation introduced in (42) and (43), this means the general solution is

$$\Phi_{M(t)}'(Z) \equiv \frac{\Phi_0'(Z)}{1-\Phi_0(M(t))} \tag{B.82}$$

which holds for any $M(t)$ and $\Phi_0(Z)$.

Alternative Cost Functions and the Sequential Formulation Given the de-trending above, assume $\gamma = 0$ and drop the tildes.

Given a $V_s(t)$ *gross* value of search at time t, define $T(Z,t)$ as the (relative) time to search. This is related to the optimal adoption threshold through $M(t) \equiv \max\{Z|T(Z,t)=0\}$. In the de-trended setup, a firm operates until it adopts a new technology at calendar time $t+T$.

With a cost function of ζZ, the sequential formulation is

$$V(t,Z) = \max_{T\geq 0}\left\{\int_0^T e^{-r\tau}Z\mathrm{d}\tau + e^{rT}\left[V_s(t+T)-\zeta Z\right]\right\} \tag{B.83}$$

$$= \max_{T\geq 0}\left\{\tfrac{1-e^{-rT}}{r}Z + e^{-rT}\left[V_s(t+T)-\zeta Z\right]\right\} \tag{B.84}$$

Simplification to ODE and Simple Integral Equation For algebraic simplicity, this will solve the version with the cost function ζZ instead of $\zeta M(t)$.

Because the problem is non-stochastic, it is possible to reduce it to a system of ODEs. Taking the first order condition of (B.84) for T.

$$0 = e^{-rT}\left(Z + r\zeta Z - rV_s(t+T) + V_s'(t+T)\right) \tag{B.85}$$

Hence, given $V_s(t)$, the solution to this ODE is a waiting time $T(t,Z)$. Focus on the the indifference point where $T=0$, i.e. $M(t)$, Since the ODE in (B.86) needs to hold for all Z, evaluate it at $M(t)$ which has $T=0$ by definition

$$rV_s(t) = (1+r\zeta)M(t) + V_s'(t) \tag{B.86}$$

To get another equation from (B.84), we will need to find a way to eliminate $V(t,Z)$. Given an Z and the law of motion $M(t)$, we know that the time when $Z = M(t)$ is $M^{-1}(Z)$. By definition, T is the amount of time the agent will wait until they search. But given a solution $M(t)$, this means an agent with $Z > M(t)$ will wait for $T(t,Z) = M^{-1}(Z) - t$ until they search. As this T is the optimal choice, so we can plug it into (B.84) and drop the max

$$V(t,Z) = \tfrac{1-e^{-r(M^{-1}(Z)-t)}}{r}Z + e^{-r(M^{-1}(Z)-t)}\left[V_s(M^{-1}(Z)) - \zeta Z\right] \tag{B.87}$$

Substituting (B.87) into (3), we get

$$V_s(t) = \int_{M(t)}^{B}\left(\tfrac{1}{r}Z - \tfrac{1+\zeta r}{r}e^{-r(M^{-1}(Z)-t)}Z + e^{-r(M^{-1}(Z)-t)}V_s(M^{-1}(Z))\right)\mathrm{d}\Phi_{M(t)}(Z) \tag{B.88}$$

Hence the system has been reduced to two ordinary integro-differential equations, (B.86) and (B.88), in $M(t)$ and $V_s(t)$.

B.5.1 Dynamic Equilibrium with Deterministic Innovation

Proof of Proposition 3. The solution will take an arbitrary $\Phi_0(Z)$. Recall that, in general, $M(0) \neq \inf \operatorname{support}\{\Phi_0\}$.

Transforming the Aggregate State M is a more natural aggregate state that is easier to normalize than t, and is bijective for strictly increasing economies.

Define M as the current threshold from $M(t)$ and assume invertibility due to strictly increasing $M(t)$ (at least until growth stops).

$$t = M^{-1}(M) \equiv q(M) \tag{B.89}$$

Here $q(M)$ is just for notational convenience so we don't need to carry around the inverse. Use the inverse function theorem,

$$M'(t) = \frac{1}{q'(M(t))} = \frac{1}{q'(M)} \tag{B.90}$$

And using the definition of the growth rate,

$$g(t) \equiv M'(t)/M(t) = \frac{1}{q'(M(t))M(t)} \tag{B.91}$$

Finally, defining the post-COV growth rate of the adoption threshold as a function of the current threshold

$$g(q(M)) \equiv \hat{g}(M) = \frac{1}{Mq'(M)} \tag{B.92}$$

Using this q, we can transform the other functions. The derivatives are done using the chain rules.

$$H(M) \equiv V_s(q(M))e^{-rq(M)} \tag{B.93}$$

Differentiating and reorganizing,

$$H'(M) = e^{-rq(M)}q'(M)(V_s'(q(M)) - rV_s(q(M))) \tag{B.94}$$

$$V_s'(q(M)) = \frac{H'(M)e^{rq(M)}}{q'(M)} + rV_s(q(M)) \tag{B.95}$$

Substitute (B.93) and (B.95) into (B.88)

$$rV_s(q(M)) = (1+\zeta r)M + \frac{e^{rq(M)}}{q'(M)}H'(M) + rV_s(q(M)) \tag{B.96}$$

$$H'(M) = -(1+\zeta r)e^{-rq(M)}Mq'(M) \tag{B.97}$$

While the *ODE* is nonlinear, note that the $H(M)$ term has been removed. Take (B.84) and put in the change of variable,

$$V_s(q(M)) = \int_M^B \left(\tfrac{1-e^{-r(q(Z)-q(M))}}{r}Z - \zeta e^{-r(q(Z)-q(M))}Z + e^{-r(q(Z)-q(M))}V_s(q(Z))\right) \mathrm{d}\Phi_M(Z) \tag{B.98}$$

Rearrange (B.98)

$$(1-\Phi_0(M))V_s(q(M))e^{-rq(M)} = \int_M^B \left(\tfrac{1}{r}e^{-rq(M)}Z - \tfrac{(1+\zeta r)}{r}e^{-rq(Z)}Z + e^{-rq(Z)}V_s(q(Z))\right)\mathrm{d}\Phi_0(Z) \tag{B.99}$$

Use the $H(\cdot)$ definition with (B.99)

$$(1-\Phi_0(M))H(M) = \int_M^B \left(\tfrac{1}{r}e^{-rq(M)}Z - \tfrac{(1+\zeta r)}{r}e^{-rq(Z)}Z + H(Z)\right)\mathrm{d}\Phi_0(Z) \tag{B.100}$$

Expanding

$$\begin{aligned}(1-\Phi_0(M))H(M) &= \tfrac{1}{r}e^{-rq(M)}\int_M^\infty Z\mathrm{d}\Phi_0(Z)\\ &\quad - \tfrac{(1+\zeta r)}{r}\int_M^\infty e^{-rq(Z)}Z\mathrm{d}\Phi_0(Z)\\ &\quad + \int_M^B H(Z)\mathrm{d}\Phi_0(Z)\end{aligned} \tag{B.101}$$

Differentiating (B.101) with respect to M and using the rules of differentiation under the integral sign.

$$H'(M) = e^{-rq(M)}\left(\zeta M\Phi'_M(M+) - q'(M)\int_M^\infty Z\Phi'_M(Z)\mathrm{d}Z\right) \tag{B.102}$$

Substitute from (B.97). Note that the $e^{-rq(M)}$ drops out.

$$-(1+\zeta r)Mq'(M) = \zeta M\Phi'_M(M+) - q'(M)\int_M^\infty Z\Phi'_M(Z)\mathrm{d}Z \tag{B.103}$$

Simplifying and solving for $q'(M)$

$$q'(M) = \frac{\zeta M\Phi'_M(M+)}{\int_M^\infty Z\Phi'_M(Z)\mathrm{d}Z - (1+\zeta r)M} \tag{B.104}$$

Or, writing as the growth rate of the threshold

$$\hat{g}(M) = \frac{\int_M^\infty Z\Phi'_M(Z)\mathrm{d}Z - (1+\zeta r)M}{\zeta M^2\Phi'_M(M+)} \tag{B.105}$$

$$= \frac{\frac{1}{M}\int_M^\infty Z\Phi'_M(Z)\mathrm{d}Z - (1+\zeta r)}{\zeta M\Phi'_M(M+)} \tag{B.106}$$

Similar algebra can be done for the $\zeta M(t)$ cost function, but the final ODE in (B.104) ends up complicated and nonlinear.[32]

[32] To sketch the algebra for the alternative cost function, the sequential form is

$$V(t,Z) = \max_{T\geq 0}\left\{\int_0^T e^{-r\tau}Z\mathrm{d}\tau + e^{rT}\left[V_s(t+T) - \zeta Z\right]\right\} \tag{B.107}$$

$$= \max_{T\geq 0}\left\{\tfrac{1-e^{-rT}}{r}Z + e^{-rT}\left[V_s(t+T) - \zeta M(t+T)\right]\right\} \tag{B.108}$$

Thin and Fat Tailed Distributions A simple definition of a power law, or fat tailed, distribution is that in the limit, the pdf or counter-cdf is proportional to a power law for some tail index α. i.e.,

$$\lim_{x\to\infty}(1-F(x)) \propto x^{-\alpha} \tag{B.115}$$

$$\lim_{x\to\infty} F'(x) \propto x^{-1-\alpha} \tag{B.116}$$

This can be made more formal using a definition for regularly varying with index $\alpha > 0$ as

$$\lim_{z\to\infty}\frac{L(tx)}{L(x)} = t^{-a}, \forall t > 0$$

Then, a distribution with a differentiable cdf $F(z)$ and counter-cdf $1-F(z)$ is defined as a power-law with tail index α if $1-F(z)$ is regularly varying with index $\alpha > 0$.

In the case of $\lim_{z\to\infty}\frac{1-F(tx)}{1-F(x)} = 1, \forall t > 0$, this is called slowly-varying. If $\lim_{z\to\infty}\frac{1-F(tx)}{1-F(x)} = \infty, \forall t > 0$, then this is neither a slowly-varying function or a power-law. As an example, the counter-cdf of the Cauchy distribution is slowly varying (i.e., like $\alpha = 0$ above), while for the lognormal, normal, and other distributions the limit is infinite. Intuitively, as α captures the number of moments, then $\alpha = 0$ means the Cauchy has no moments, while $\alpha = \infty$ means the distribution has infinite moments.

If $\zeta M(t)$ is used instead, the equivalent to (B.86) is

$$rV_s(t) = (1-r\zeta)M(t) - \zeta M'(t) + V_s'(t) \tag{B.109}$$

If $\zeta M(t)$ is used instead, the equivalent to (B.88) is

$$V_s(t) = \int_{M(t)}^{B}\left(\tfrac{1}{r}Z - \tfrac{1}{r}e^{-r(M^{-1}(Z)-t)}Z + e^{-r(M^{-1}(Z)-t)}V_s(M^{-1}(Z)) - e^{-r(M^{-1}(Z)-t)}\zeta M(t)\right)\mathrm{d}\Phi_{M(t)}(Z) \tag{B.110}$$

Using (B.90), with the alternative ODE in (B.109), the transformed ODE becomes,

$$H'(M) = e^{-rq(M)}\left(\zeta - (1+\zeta r)Mq'(M)\right) \tag{B.111}$$

Using the alternative cost function,

$$V_s(q(M)) = \int_{M}^{B}\left(\tfrac{1}{r}Z - \tfrac{1}{r}e^{-r(q(Z)-q(M))}Z + e^{-r(q(Z)-q(M))}V_s(q(Z)) - e^{-r(q(Z)-q(M))}\zeta M\right)\mathrm{d}\Phi_M(Z) \tag{B.112}$$

With the alternative specificiation (B.114)

$$\begin{aligned}(1-\Phi_0(M))H(M) = {}& \tfrac{1}{r}e^{-rq(M)}\int_M^\infty Z\mathrm{d}\Phi_0(Z)\\ & - \tfrac{1}{r}\int_M^\infty e^{-rq(Z)}Z\mathrm{d}\Phi_0(Z)\\ & - \zeta M\int_M^\infty e^{-rq(Z)}\mathrm{d}\Phi_0(Z)\\ & + \int_M^B H(Z)\mathrm{d}\Phi_0(Z)\end{aligned} \tag{B.113}$$

$$(1-\Phi_0(M))V_s(q(M))e^{-rq(M)} = \int_M^B\left(\tfrac{1}{r}e^{-rq(M)}Z - \tfrac{1}{r}e^{-rq(Z)}Z + e^{-rq(Z)}V_s(q(Z)) - e^{-rq(Z)}\zeta M\right)\mathrm{d}\Phi_0(Z) \tag{B.114}$$

From Soulier (2009), Corollary 1.9: A continuously differentiable function, $\ell(x)$ is regularly varying at infinity with index $\alpha \in \mathbb{R}$ *if and only if*

$$\lim_{x\to\infty} \frac{x\ell(x)}{\ell'(x)} = \alpha \tag{B.117}$$

Define a thin-tailed distributions as those where this limit diverges and $\alpha = \infty$.

Fat-tailed and Perpetual Growth $\implies$ BGP from Tail Index: First, note that if growth continues forever, then $M \to \infty$, so assume that growth continues forever and the initial distribution is fat-tailed. Use the simple definition, if $\Phi(0, Z)$ is fat tailed, then $\Phi'(0, Z) \propto Z^{-1-\alpha}$ for large Z and some α. Use (43) and fix M,

$$\Phi'_M(Z) \propto \Phi'(0, Z), \quad \text{for } Z > M \tag{B.118}$$

As the fixed M goes to ∞, the domain of Z goes to ∞, and from (B.116),

$$\lim_{M\to\infty} \Phi'_M(Z) \propto Z^{-1-\alpha}, \quad \text{for } Z > M \tag{B.119}$$

Hence, for any fat-tailed initial condition, the asymptotic distribution under perpetual growth is a Pareto, and we can use our solution for a Pareto distribution based on the tail index.

Not Fat-tailed $\implies$ No Growth from Diffusion or Non-Existence: Denote the non-constant part of the denominator of the limit of (44) as $\hat{\alpha}$, and expand using (43),

$$\hat{\alpha} \equiv \lim_{M\to\infty} \frac{M\Phi'(M)}{1 - \Phi(M)} \leq \infty \tag{B.120}$$

Compare (B.117) and (B.120) to see that $\hat{\alpha}$ is finite if and only if the counter-cdf is regularly varying. For regularly varying functions, $\hat{\alpha} = \alpha$, the tail index of the distribution.

For distributions that are rapidly varying (i.e, not regularly or slowly varying), the denominator of (44) diverges to infinity.[33] As the numerator is bounded due to the assumption that expectations exist, this means that the asymptotic growth rate is $\hat{g} = 0$.

Recall that for an increasing, differentiable function $h(z)$ and a differentiable cdf $F(z)$, $z \in [0, \infty)$ (Note: if the lower support of z is $m > 0$, define $F'(z) = 0$ for $0 \leq z < m$):

$$\mathbb{E}\left[h(z)\right] = \int h(z)F'(z)\mathrm{d}z = \int h'(z)(1 - F(z))\mathrm{d}z + h(\min \text{support}\left\{F\right\}) \tag{B.121}$$

To show the numerator is bounded, reorganize (42)

$$1 - \Phi_M(Z) = \frac{1 - \Phi(0, Z)}{1 - \Phi(0, M)} \tag{B.122}$$

Then from (44), (B.121) and (B.122)

$$\frac{1}{M}\mathbb{E}\left[\Phi_M(z)\right] = \frac{1}{M}\int_M^\infty Z\Phi'_M(Z)\mathrm{d}Z \tag{B.123}$$

$$= \frac{1}{M}\frac{\int_M^\infty (1 - \Phi(0, Z))\mathrm{d}z}{1 - \Phi(0, M)} \tag{B.124}$$

[33] See Resnick (2007) page 53 and 67 for a related characterization of rapid variation.

Since $1 - \Phi(0, Z + M)$ is rapidly varying, there exists some M such that it is bounded by a regularly varying function for some $\alpha > 1$, i.e, $1 - \Phi(0, Z + M) < (Z + M)^{-\alpha}$ for all $Z > 0$. As $\frac{1}{M}\int_M^\infty Z\Phi'_M(Z)dZ$ can be calculated for this power-law (with a positive lower support), the numerator is bounded:

$$\frac{1}{M}\frac{\int_0^\infty (1-\Phi(0,Z+M))dZ}{1-\Phi(0,M)} = \frac{1}{M}\frac{\int_M^\infty (1-\Phi(0,Z))dZ}{1-\Phi(0,M)} = \frac{1}{M}\int_M^\infty Z\Phi'_M(Z)dZ < \frac{\alpha}{\alpha-1}$$

Also note the case of the Cauchy where $1 - F(z)$ is slowly varying and $\hat{\alpha} = 0$. In that case, the denominator is 0 and the asymptotic growth rate is diverges and is not-defined. This could also be shown since the expectation of the the numerator of (44) is not defined in the case of the Cauchy.

□

Appendix C Model with Monopolistic Competition and Labor

C.1 Unnormalized Model

The following sets up Geometric Brownian Motion version of the model, with monopolistic competition and technology diffusion costs in labor. The results are qualitatively similar to the simpler specification of the model. For simplicity, this section is written primarily for the balanced growth path. For exposition, drop the t subscript where possible.

Following a closed economy version of Perla, Tonetti, and Waugh (2014), the setup includes the standard setup of representative consumer purchasing a composite good with wages from inelastic labor supply and dividends from corporate profits. The composite good is produced by a competitive sector from a continuum of differentiated intermediates. A monopolistic firm, differentiated by its productivity Z, produces a single intermediate.

Consumer The consumer gains flow utility from consumption of final goods, $\frac{1}{1-\varsigma}C^{1-\varsigma}$, for $\varsigma \geq 0$. Future utility is discounted at a rate $\hat{r} > 0$. The consumer purchases the final consumption goods purchased by supplying 1 unit of labor inelastically at real wage $W(t)$ and gaining profits from a perfectively diversified portfolio $\bar{\Pi}(t)$ in real profits. The consumer's welfare at time $\tilde{t}$ is then,

$$\begin{aligned} U(\tilde{t}) &= \int_{\tilde{t}}^\infty \tfrac{C(t)^{1-\varsigma}}{1-\varsigma} e^{-\hat{r}(t-\tilde{t})}\mathrm{d}t \\ &\text{s.t. } P(t)C(t) = P(t)W(t) + P(t)\bar{\Pi}(t) \end{aligned} \tag{C.1}$$

From standard asset pricing, the elasticity of inter-temporal substitution implies the interest rate used by the firm to discount future earnings,

$$r(t) \equiv \hat{r} + \varsigma\tfrac{C'(t)}{C(t)} \tag{C.2}$$

On a balanced growth path, this gives the familiar interest rate with CRRA preferences,

$$r \equiv \hat{r} + \varsigma g \tag{C.3}$$

Final Goods Following the standard results from monopolistic competition, a competitive final goods sector produces a good with elasticity of substitution $\rho > 1$ between all available products, given prices and final goods revenue denoted by PC. The solution follows from maximizing the following final goods production function,

$$\max_{Q}\left[\int_{M}^{\infty} Q(Z)^{(\rho-1)/\rho}\mathrm{d}\Phi(Z)\right]^{\rho/(\rho-1)} \tag{C.4}$$

$$\text{s.t.} \int_{M}^{\infty} p(Z)Q(Z)\mathrm{d}\Phi(Z) = PC \tag{C.5}$$

Where $Q(Z)$ is the demand for a product with productivity Z. Defining a price index P, standard CES algebra gives the optimal intermediate good demand, price index, and final goods production as,

$$Q(Z) = \left(\tfrac{p(Z)}{P}\right)^{-\rho} C \tag{C.6}$$

$$P = \left[\int_{M}^{\infty} p(Z)^{1-\rho}\mathrm{d}\Phi(Z)\right]^{\frac{1}{1-\rho}} \tag{C.7}$$

$$C = \left[\int_{M}^{\infty} Q(Z)^{(\rho-1)/\rho}\mathrm{d}\Phi(Z)\right]^{\rho/(\rho-1)} \tag{C.8}$$

Static Variable Profits A monopolist, subject to the demand function in (C.6), maximizes real profits $\Pi(Z)$ by choosing the price $p(Z)$ and labor demand $\ell(Z)$

$$P\Pi(Z) \equiv \max_{p,\ell}\{\ (pZ\ell - PW\ell)\} \text{ s.t. (C.6)} \tag{C.9}$$

Define the markup $\bar{\rho} \equiv \rho/(\rho-1)$. The optimal solution to (C.9) gives $p(Z), \ell(Z)$, and $\Pi(Z)$,

$$\tfrac{p(Z)}{P} = \bar{\rho}\tfrac{W}{Z} \tag{C.10}$$

$$\ell(Z) = \tfrac{Q(Z)}{Z} \tag{C.11}$$

Take (C.9) and divide by P to get $\Pi(Z) = \frac{p(Z)}{P}Q(Z) - \frac{Q(Z)}{Z}W$. Substitute from (C.10) as $W = \frac{Zp(Z)}{\bar{\rho}P}$ to get $\Pi(Z) = \frac{1}{\rho}\frac{p(Z)}{P}Q(Z)$. Finally, use $Q(Z)$ from (C.6) to get real profits

$$\Pi(Z) = \tfrac{1}{\rho}\left(\tfrac{p(Z)}{P}\right)^{1-\rho} C \tag{C.12}$$

Diffusion Costs, Market Clearing, and Aggregate Profits In order to upgrade its technology, a firm must hire ζ units of labor at real wage rate $W(t)$. If in equilibrium a flow of S agents are choosing to upgrade, then they will require ζS units of total labor. Given inelastic supply of 1 unit of labor,

$$1 = \underbrace{\int_{M}^{\infty} \ell(Z)\mathrm{d}\Phi(Z)}_{\text{Variable Production}} + \underbrace{\zeta S}_{\text{Upgrades}} \tag{C.13}$$

As all costs are paid in labor and all final goods are all used for consumption, the consumer simply eats final goods output $C(t)$.

The aggregate real profits for the portfolio of all firm include variable profits and subtracts for real wages from the equity investment in upgrades,

$$\bar{\Pi} = \int_{M}^{\infty} \Pi(Z)\mathrm{d}\Phi(Z) - \zeta SW \tag{C.14}$$

Firm's Dynamic Problem The firm maximizes the present discounted value of profits, using the discount rate of the consumer, $r(t)$. Then,

$$r(t)V(t,Z) = \Pi(t,Z) + (\gamma+\sigma^2)Z\,\boldsymbol{\partial}_Z V(t,Z) + \frac{\sigma^2}{2}Z^2\,\boldsymbol{\partial}_{ZZ}V(t,Z) + \boldsymbol{\partial}_t V(t,Z) \tag{C.15}$$

$$V(t,M(t)) = \int_{M(t)}^{\infty} V(t,\tilde{Z})\mathrm{d}\Phi(t,\tilde{Z}) - \zeta W(t) \tag{C.16}$$

$$\boldsymbol{\partial}_Z V(t,M(t)) = 0 \tag{C.17}$$

C.2 Static Normalization

Follow a similar normalized approach to Appendix B.1 for $z \equiv \log(Z/M)$, and $F(t,z)$ using (B.1). Normalize the real wage and aggregate consumption as,

$$w(t) \equiv \frac{W}{M} \tag{C.18}$$

$$c(t) \equiv \frac{C}{M} \tag{C.19}$$

For the value and real profits, further normalize to make them realtive to the real normalized wage

$$\pi(t,z) \equiv \frac{\Pi(t,Z)}{w(t)M} \tag{C.20}$$

$$v(t,z) \equiv \frac{V(t,Z)}{w(t)M} \tag{C.21}$$

Using (C.10) and (C.18),

$$\tfrac{p(Z)}{P} = \bar{\rho}\tfrac{w}{Z/M} = \bar{\rho}we^{-z} \tag{C.22}$$

Substitute into (C.6) and (C.11)

$$q(Z) = \bar{\rho}^{-\rho}w^{-\rho}c\left(\tfrac{Z}{M}\right)^{\rho} = \bar{\rho}^{-\rho}w^{-\rho}ce^{\rho z} \tag{C.23}$$

Writing $(Z/M)^{\rho} = e^{\rho z}$ directly for simplicity, combine (C.6), (C.11) and (C.23),

$$q(Z) = \bar{\rho}^{-\rho}w^{-\rho}ce^{\rho z} \tag{C.24}$$

$$\ell(Z) = \bar{\rho}^{-\rho}w^{-\rho}ce^{(\rho-1)z} \tag{C.25}$$

Divide (C.7) by $P^{1-\rho}$ and then substitute from (C.22) for $p(Z)/P$ to obtain

$$1 = \bar{\rho}^{1-\rho}w^{1-\rho}\int_{M}^{\infty}\left(\tfrac{Z}{M}\right)^{\rho-1}\mathrm{d}\Phi(Z) \tag{C.26}$$

Simplify (C.26) by defining $\bar{Z}$, a measure of effective aggregate productivity. Then use an integral change of variables from Z to z to give normalized real wages in terms of parameters, $\bar{Z}$, and the productivity distribution

$$\bar{Z} \equiv \left[\int_0^{\infty} e^{(\rho-1)z}\mathrm{d}F(z)\right]^{\frac{1}{\rho-1}} = \mathbb{E}\left[e^{(\rho-1)z}\right]^{\frac{1}{\rho-1}} \tag{C.27}$$

$$w = \bar{Z}/\bar{\rho} \tag{C.28}$$

Intuitively, this says that normalized real wages are proportional to normalized aggregate productivity, where the monopolistic friction lowers the wage share of aggregate output

Divide (C.12) and by Mw and substitute with (C.28) to obtain normalized profits,

$$\pi(Z) = \tfrac{1}{\rho}\left(\tfrac{p(Z)}{P}\right)^{1-\rho}\tfrac{c}{w} = \tfrac{1}{\rho}\tfrac{c}{w}\bar{Z}^{1-\rho}e^{(\rho-1)z} \tag{C.29}$$

Divide (C.14) Mw and use (C.27) and (C.29) to find aggregate profits,

$$\bar{\pi} = \tfrac{1}{\rho}\tfrac{c}{w}\bar{Z}^{1-\rho}\mathbb{E}\left[z^{\rho-1}\right] - \zeta S \tag{C.30}$$

$$= \tfrac{1}{\rho}\tfrac{c}{w} - \zeta S \tag{C.31}$$

Combine (C.13), (C.25) and (C.27) to obtain normalized aggregate labor demand

$$1 = \bar{\rho}^{-\rho}w^{-\rho}c\bar{Z}^{\rho-1} + \zeta S \tag{C.32}$$

Multiply and divide the 2nd term of (C.32) by $w\bar{\rho}$, then use (C.28),

$$1 = \frac{1}{\bar{\rho}}\frac{c}{w}\bar{\rho}^{1-\rho}\bar{Z}^{\rho-1}w^{1-\rho} + \zeta S \tag{C.33}$$

$$\frac{c}{w} = \bar{\rho}(1-\zeta S) \tag{C.34}$$

Substitute (C.34) into (C.29), define a constant, and write the firm's profits (relative to wages)in terms of z parameters and the aggregates $\bar{Z}$ and S,

$$\pi(z) = \frac{1}{\rho-1}(1-\zeta S)\bar{Z}^{1-\rho}e^{(\rho-1)z} \tag{C.35}$$

C.3 Dynamic Normalization

Given the equilibrium distribution and flow of adopters $S(t)$, define the following to decompose into the aggregate and idiosyncratic components of profits,

$$\tilde{\pi}(t) \equiv \frac{1-\zeta S(t)}{(\rho-1)\mathbb{E}_t\left[e^{(\rho-1)z}\right]} \tag{C.36}$$

$$\pi(t,z) = \tilde{\pi}(t)e^{(\rho-1)z} \tag{C.37}$$

All of the general equilibrium conditions in the model have now been reduced to (C.28), (C.36) and (C.37). Normalize the value function by the normalized real wage and the scale,

$$v(t,z) \equiv \frac{V(t,Z)}{M(t)w(t)} \tag{C.38}$$

Or,

$$V(t,Z) = w(t)M(t)v(t,Z/M(t)) \tag{C.39}$$

The derivatives of (C.39) with respect to z are identical to those in Appendix B.1, but with the new multiplicative term. Differentiate the continuation value $V(t,Z)$ with respect to t in (C.39) and divide by $w(t)M(t)$, using the chain and product rule,

$$\frac{1}{w(t)M(t)}\boldsymbol{\partial}_t V(t,Z) = \frac{M'(t)}{M(t)}v(z,t) - \frac{M'(t)}{M(t)}\frac{Z}{M(t)}\boldsymbol{\partial}_z v(t,z) + \frac{M(t)}{M(t)}\boldsymbol{\partial}_t v(t,z) + \frac{w'(t)}{w(t)}v(z,t) \tag{C.40}$$

The growth rate is $g(t) \equiv M'(t)/M(t)$. Further define the growth rate of the relative wage, $g_w(t) \equiv w'(t)/w(t)$, which may be non-zero off of a BGP. Substitute these into (C.40), cancel out $M(t)$, and group $z = Z/M(t)$ to give

$$= (g(t) + g_w(t))v(z,t) - g(t)z\partial_z v(t,z) + \partial_t v(t,z) \tag{C.41}$$

Divide (C.15) by $M(t)w(t)$, use (B.16), (B.17), (C.35) to (C.37) and (C.40), and simplify,

$$(r(t) - g(t) - g_w(t))v(t,z) = \tilde{\pi}(t)e^{(\rho-1)z} + (\gamma - g(t))\partial_z v(t,z) + \frac{\sigma^2}{2}\partial_{zz}v(t,z) \tag{C.42}$$

Further divide, (C.16) and (C.17) by $W(t)$,

$$v(t,0) = \int_0^\infty v(t,z)\mathrm{d}F(t,z) - \zeta \tag{C.43}$$

$$\partial_z v(t,0) = 0 \tag{C.44}$$

The normalization of the KFE and distribution is identical to that in Appendix B.1 and (26) to (28) and (B.74).

C.4 Stationary Equilibrium Equations

In a stationary solution, $S(t)$ and $w(t)$ will be constant, so $g_w(t) = 0$. Furthermore, $\tilde{\pi}$ is constructed to be stationary. The complete set of equations for the stationary equilibrium is,

$$(r - g)v(z) = \tilde{\pi}e^{(\rho-1)z} + (\gamma - g)v'(z) + \tfrac{\sigma^2}{2}v''(z) \tag{C.45}$$

$$v(0) = \int_0^\infty v(z)\mathrm{d}F(z) - \zeta \tag{C.46}$$

$$v'(0) = 0 \tag{C.47}$$

$$0 = (g - \gamma)F'(z) + \tfrac{\sigma^2}{2}F''(z) + SF(z) - S \tag{C.48}$$

$$F(0) = 0 \tag{C.49}$$

$$F(\infty) = 1 \tag{C.50}$$

$$\tilde{\pi} \equiv \frac{1 - \zeta S}{(\rho - 1)\mathbb{E}\left[e^{(\rho-1)z}\right]} \tag{C.51}$$

If the equilibrium distribution is a power law with tail parameter α and $1 + \alpha > \rho$, then,

$$\tilde{\pi} \equiv \frac{1 + \alpha - \rho}{\alpha(\rho - 1)}(1 - \zeta S) \tag{C.52}$$

$$S = \alpha\left(g - \gamma - \alpha\tfrac{\sigma^2}{2}\right) \tag{C.53}$$

C.5 Stationary Equilibrium

Follow the techniques of Appendix B.4, and note that the KFE is identical. Hence,

$$F(z) = 1 - e^{-\alpha z} \tag{C.54}$$

Where α is related to S and g through (B.53). To solve (C.45) and (C.47), assume a solution of the form,

$$v(z) = ae^{(\rho-1)z} + \frac{(\rho - 1)b}{\nu}e^{-\nu z} \tag{C.55}$$

Substitute this guess into (C.45) and (C.47) and equate undetermined coefficients to find that $a = b$, ν is still given by (B.59), and

$$a = \frac{\tilde{\pi}}{r - g - (\rho - 1)(\gamma - g + (\rho - 1)\sigma^2/2)} \tag{C.56}$$

Use (C.55) and (C.56) to find,

$$v(0) = \frac{\tilde{\pi}(\nu + \rho - 1)}{\nu(-\gamma\rho + \gamma + g(\rho - 2) + r) - \nu(\rho - 1)^2\sigma^2/2} \tag{C.57}$$

Substituting (B.51), (C.55) and (C.57) into (C.46) gives an equation relating α and ν,

$$0 = \frac{2\tilde{\pi}(-\alpha(\rho - 2) + \nu + \rho - 1)}{(\rho - 1)^2\sigma^2 - 2(-\gamma\rho + \gamma + g(\rho - 2) + r)} + (\alpha - 1)\zeta(\alpha + \nu) \tag{C.58}$$

This can be solved further for ν

$$\nu = \frac{4\tilde{\pi}\alpha - 2(\alpha - 1)\rho\left(\tilde{\pi} + \alpha\zeta(g - \gamma)\right) - 2\tilde{\pi} + (\alpha - 1)\alpha\zeta(\rho - 1)^2\sigma^2 + 2(\alpha - 1)\alpha\zeta(-\gamma + 2g - r)}{(\alpha - 1)\zeta\left(2(-\gamma\rho + \gamma + g(\rho - 2) + r) - (\rho - 1)^2\sigma^2\right) - 2\tilde{\pi}} \tag{C.59}$$

Equate (B.59) and (C.59)to find an expression in g and α

$$0 = -g + \frac{2\tilde{\pi}(\alpha - 1)(\rho - 1)\sigma^2}{(\alpha - 1)\zeta\left(2(-\gamma\rho + \gamma + g(\rho - 2) + r) - (\rho - 1)^2\sigma^2\right) - 2\tilde{\pi}} + \alpha\sigma^2 + \gamma + \sqrt{(g - \gamma)^2 + 2\sigma^2(r - g)} \tag{C.60}$$

Substitute with (C.52), (C.53) and (C.60) to find an implicit equation in g given the α. The set of $\{\alpha, g\}$ fulfilling this equation is the set of admissible stationary solutions.

$$0 = \zeta + \frac{\left(\alpha\zeta\left(\alpha\sigma^2 + 2\gamma - 2g\right) + 2\right)\left(\gamma + \sqrt{(g - \gamma)^2 + 2\sigma^2(r - g)} - g + (\rho - 1)\sigma^2\right)}{\alpha\left(\alpha\sigma^2 + \gamma + \sqrt{(g - \gamma)^2 + 2\sigma^2(r - g)} - g\right)\left((\rho - 1)^2\sigma^2 - 2(-\gamma\rho + \gamma + g(\rho - 2) + r)\right)} \tag{C.61}$$

In some examples this can be simplified in closed form. For example, with $\gamma = 0$ and $\rho = 2$, define a constant

$$a \equiv \sqrt{\alpha\zeta\left(\alpha((\alpha - 6)\alpha + 1)\zeta\sigma^4 + 4\sigma^2(\alpha + (\alpha + 1)\alpha\zeta r - 1) + 4r(\alpha\zeta r - 2)\right) + 4} \tag{C.62}$$

$$g = \frac{\alpha\left((\alpha(\alpha + 4) - 1)\zeta\sigma^2 - a - 2(\alpha + 1)\zeta r + 2\right) + a + 2}{4\alpha^2\zeta} \tag{C.63}$$

In case of $\sigma = 0$ with an arbitrary γ and ρ, the solution is,

$$g = \frac{1 - \alpha(\hat{r} - \gamma(1 + \alpha))\zeta}{\alpha(\varsigma + \alpha)\zeta} \tag{C.64}$$

And in the baseline example with $\sigma = 0$ and no risk aversion or drift, $\varsigma = \gamma = 0$,

$$g = \frac{1 - \alpha r\zeta}{\alpha^2\zeta} \tag{C.65}$$

Note the independence of ρ in the growth rates of (C.64) and (C.65). However, the stationary distribution of profits, $\pi(z)$, still depends on the ρ elasticity. Doing a change of variables: if the z distribution has tail parameter α, then the distribution of profits has tail parameter $\alpha/(\rho - 1)$ through (C.35)

Appendix D Innovation with a Markov Chain

D.1 Stationary Stochastic Innovation Equilibrium with Infinite Support

Proof of Proposition 6. Define $\mathbf{0}, \mathbf{1}, \mathbf{I}$ as a vector of 0, 1, and the identity matrix and the following:

$$A \equiv \begin{bmatrix} \frac{1}{g} \\ \frac{1}{g-\gamma} \end{bmatrix} \qquad B \equiv \begin{bmatrix} \frac{r+\lambda_\ell-g}{g} & -\frac{\lambda_\ell}{g} \\ -\frac{\lambda_h}{g-\gamma} & \frac{r+\lambda_h-g}{g-\gamma} \end{bmatrix} \tag{D.1}$$

$$C \equiv \begin{bmatrix} \frac{gF'_\ell(0)+(g-\gamma)F'_h(0)-\lambda_\ell}{g} & \frac{\lambda_h}{g} \\ \frac{\lambda_\ell}{g-\gamma} & \frac{gF'_\ell(0)+(g-\gamma)F'_h(0)-\lambda_h}{g-\gamma} \end{bmatrix} \qquad D \equiv \begin{bmatrix} F'_\ell(0) \\ F'_h(0) \end{bmatrix} \tag{D.2}$$

$$\vec{F}(z) \equiv \begin{bmatrix} F_\ell(z) \\ F_h(z) \end{bmatrix} \qquad v(z) \equiv \begin{bmatrix} v_\ell(z) \\ v_h(z) \end{bmatrix} \tag{D.3}$$

Then the equilibrium conditions can be written as a linear set of ODEs:

$$v'(z) = Ae^z - Bv(z) \tag{D.4}$$

$$v'(0) = \mathbf{0} \tag{D.5}$$

$$\vec{F}'(z) = -C\vec{F}(z) + D \tag{D.6}$$

$$\vec{F}(0) = \mathbf{0} \tag{D.7}$$

$$\vec{F}(\infty) \cdot \mathbf{1} = 1 \tag{D.8}$$

$$v_\ell(0) = v_h(0) = \int_0^\infty \left(v(z)^T \cdot \vec{F}'(z) \right) \mathrm{d}z - \zeta \tag{D.9}$$

Solve these as a set of matrix ODEs, where e^{Az} is a matrix exponential. Start with (D.4) and (D.5) to get,[34]

$$v(z) = (I+B)^{-1} \left(e^{Iz} + e^{-Bz}B^{-1} \right) A \tag{D.12}$$

Evaluate at $z = 0$,

$$v(0) = B^{-1}A = \begin{bmatrix} 1/(r-g) & 1/(r-g) \end{bmatrix} \tag{D.13}$$

Then (D.6) and (D.7) gives

$$\vec{F}(z) = \left(\mathbf{I} - e^{-Cz} \right) C^{-1} D \tag{D.14}$$

Take the derivative,

$$\vec{F}'(z) = e^{-Cz} D \tag{D.15}$$

[34] The equation $\vec{F}'(z) = AF[z] + b$ subject to $\vec{F}(0) = \mathbf{0}$ has the solution,

$$\vec{F}(z) = \left(e^{Az} - \mathbf{I} \right) A^{-1} b \tag{D.10}$$

The derivation of these results uses that $\int_0^T e^{tA} \mathrm{d}t = A^{-1} \left(e^{TA} - \mathbf{I} \right)$. With appropriate conditions on eigenvalues, this implies that $\int_0^\infty e^{tA} \mathrm{d}t = -A^{-1}$
Equations of the form,$v'(z) = Ae^z - B \cdot v(z)$ with the initial condition $v'(0) = \mathbf{0}$ have the solution,

$$v(z) = (I+B)^{-1} \left(e^{Iz} + e^{-Bz}B^{-1} \right) A \tag{D.11}$$

This derivation exploits commutativity, as both e^{Bz} and $(I+B)^{-1}$ can be expanded as power series of B.

For (D.12) and (D.14) to be well defined as $z \to \infty$, we have to impose parameter restrictions that constrain the growth rate g so that the eigenvalues of B and C are positive or have positive real parts. S_l and S_h are defined in equations (90) and (91) in terms of $F_l'(0)$ and $F_h'(0)$, C and B will have roots with positive real parts iff their determinant and their trace are strictly positive. For C it is straightforward to compute that the conditions for a positive trace and determinant are

$$S_l + S_h > \frac{(g-\gamma)\lambda_l + g\lambda_h}{(g-\gamma)+g} \tag{D.16}$$

$$S_h + S_l > \lambda_h + \lambda_l \tag{D.17}$$

and for B the corresponding conditions are

$$r > g > \gamma \tag{D.18}$$

$$r - g + \lambda_h + \lambda_l > 0 \tag{D.19}$$

With these conditions imposed, we can proceed to characterize the solutions to the value functions and the stationary distribution.

Evaluate (D.15) at $z = 0$,

$$\vec{F}'(0) = D \tag{D.20}$$

Take the limit of (D.14)

$$\vec{F}(\infty) = C^{-1}D \tag{D.21}$$

And (D.8) becomes

$$1 = C^{-1}D \cdot \mathbf{1} \tag{D.22}$$

We can check that, by construction, with the C and D defined by (D.2), (D.22) is fulfilled for any $F_\ell'(0), F_h'(0), \lambda_\ell$, and λ_h.

For $\vec{F}'(z)$ to define a valid pdf it is necessary for $\vec{F}'(z) > 0$ for all z. It can be shown, that for $C > 0$, the only $D > 0$ fulfilling this requirement is one proportional to the eigenvector associated with the dominant eigenvalue of C.[35] The unique constant of proportionality is determined by (D.22). The two eigenvectors of C fulfilling this proportionality are,

$$\nu_i \equiv \begin{bmatrix} -\frac{F_h'(0)\left(\gamma(g-\gamma)F_h'(0) \pm \sqrt{2(g-\gamma)\lambda_l\left(\gamma\left(g\left(F_h'(0)+F_l'(0)\right)-\gamma F_h'(0)\right)+g\lambda_h\right)+\left(\gamma\left(g\left(F_h'(0)+F_l'(0)\right)-\gamma F_h'(0)\right)-g\lambda_h\right)^2+(g-\gamma)^2{\lambda_l}^2}+\gamma g F_l'(0)-g\lambda_h+g\lambda_l-\gamma\lambda_l\right)}{2g\lambda_l} \\ F_h'(0) \end{bmatrix} \tag{D.23}$$

Denote ν as the eigenvector with both positive elements–which is associated with the dominant eigenvalue–then as discussed above, $D \propto \nu$. Using (D.2) and (D.23), and noting the eigenvector has already been normalized to match the second parameter.

$$D = \nu \tag{D.24}$$

[35]Since $C > 0$ and irreducible (in this case off diagonals not zero), then by Perron-Frobenius it has a simple dominant real root α and an associated eigenvector $\nu > 0$. Hence, as $\vec{F}(0) = 0, F_\ell(\infty) + F_h(\infty) = 1$, and $\vec{F}'(z) > 0$, we have a valid pdf. This uniqueness of the ν solution only holds if the other eigenvector of C has a positive and negative coordinate, which always holds in our model.

The 2nd coordinate already holds with equality by construction, for the first coordinate equating (D.2) and (D.23). Equating the first parameter and choosing the positive eigenvector,

$$F_\ell'(0) = -\frac{F_h'(0)\left(\gamma(g-\gamma)F_h'(0)-\sqrt{2(g-\gamma)\lambda_l\left(\gamma\left(g\left(F_h'(0)+F_l'(0)\right)-\gamma F_h'(0)\right)+g\lambda_h\right)+\left(\gamma\left(g\left(F_h'(0)+F_l'(0)\right)-\gamma F_h'(0)\right)-g\lambda_h\right)^2+(g-\gamma)^2\lambda_l{}^2}+\gamma g F_l'(0)-g\lambda_h+g\lambda_l-\gamma\lambda_l\right)}{2g\lambda_l} \tag{D.25}$$

Solving this equation for $F_\ell'(0)$ and choosing the positive root

$$F_\ell'(0) = \frac{F_h'(0)\lambda_h}{\gamma F_h'(0) + \lambda_\ell} \tag{D.26}$$

We can check that with the C and D defined by (D.2), (D.22) is fulfilled by construction. The value matching in (D.9) becomes,

$$\frac{1}{r-g} + \zeta = \int_0^\infty \left[\left[(I+B)^{-1}\left(e^{Iz} + e^{-Bz}B^{-1}\right)A\right]^T e^{-Cz}D\right] \mathrm{d}z \tag{D.27}$$

Note that if B has positive eigenvalues, then $\lim_{z\to\infty} v(z) = (1+B)^{-1}(e^z)A$. Therefore, as long as C has a minimal eigenvalue (defined here as α), strictly greater than one, the integral is defined.

The tail index of the unconditional distribution, $F(z) \equiv F_\ell(z) + F_h(z)$ can be calculated from the C matrix in (D.15). As sums of power law variables inherit the smallest tail index, the endogenous power law tail is minimum eigenvalue of C. After the substitution for $F_\ell'(0)$ from above, the smallest eigenvalue of C is

$$\alpha \equiv \frac{((g-\gamma)F_h'(0) - \lambda_l)\left(\gamma(g-\gamma)F_h'(0) + g(\lambda_h + \lambda_l) - \gamma\lambda_l\right)}{g(g-\gamma)\left(\gamma F_h'(0) + \lambda_l\right)} \tag{D.28}$$

Solving (D.28) for $F_h'(0)$ as a function of α,

$$F_h'(0) = \frac{g\left(\alpha\gamma - \lambda_h + \sqrt{(\lambda_h - \alpha\gamma)^2 + 2\lambda_l(\alpha\gamma + \lambda_h) + \lambda_l{}^2} - \lambda_l\right) + 2\gamma\lambda_l}{2\gamma(g-\gamma)} \tag{D.29}$$

Substituting for $F_i'(0)$ into C and D gives a function in terms of g and α,

$$C = \begin{bmatrix} \frac{-\alpha\gamma+2\alpha g+\lambda_h+\sqrt{(\lambda_h-\alpha\gamma)^2+2\lambda_l(\alpha\gamma+\lambda_h)+\lambda_l{}^2}-\lambda_l}{2g} & \frac{\lambda_h}{g} \\ \frac{\lambda_l}{g-\gamma} & \frac{-\alpha\gamma+2\alpha g-\lambda_h+\sqrt{(\lambda_h-\alpha\gamma)^2+2\lambda_l(\alpha\gamma+\lambda_h)+\lambda_l{}^2}+\lambda_l}{2(g-\gamma)} \end{bmatrix} \tag{D.30}$$

$$D = \begin{bmatrix} \frac{\lambda_h\left(g\left(\alpha\gamma-\lambda_h+\sqrt{(\lambda_h-\alpha\gamma)^2+2\lambda_l(\alpha\gamma+\lambda_h)+\lambda_l{}^2}-\lambda_l\right)+2\gamma\lambda_l\right)}{\gamma g\left(\alpha\gamma-\lambda_h+\sqrt{(\lambda_h-\alpha\gamma)^2+2\lambda_l(\alpha\gamma+\lambda_h)+\lambda_l{}^2}+\lambda_l\right)} \\ \frac{g\left(\alpha\gamma-\lambda_h+\sqrt{(\lambda_h-\alpha\gamma)^2+2\lambda_l(\alpha\gamma+\lambda_h)+\lambda_l{}^2}-\lambda_l\right)+2\gamma\lambda_l}{2\gamma(g-\gamma)} \end{bmatrix} \tag{D.31}$$

As in the example with Geometric Brownian Motion, there are multiple stationary equilibria. While both $F_i'(0)$ could conceivably parameterize a set of solutions for each g, they are constrained by the eigenvector proportionality condition, which ensures that the manifold of solutions is 1 dimensional. □

D.2 Independent Draw of Z and Type with Infinite Support

Write the adoption process more generally to allow some correlation between the draw of the type and productivity, for some constants a_ℓ, b_ℓ, a_h, b_h (constrained with some equilibrium conditions on the masses of $F_\ell(\infty)$ and $F_h(\infty)$,

$$0 = gF_\ell'(z) - \lambda_\ell F_\ell(z) + \lambda_h F_h(z) + a_\ell F_\ell(z) + b_\ell F_h(z) - S_\ell \tag{D.32}$$

$$0 = (g-\gamma)F_h'(z) - \lambda_h F_h(z) + \lambda_\ell F_\ell(z) + a_h F_\ell(z) + b_h F_h(z) - S_h \tag{D.33}$$

With this, the C matrix for the KFE in (D.6) is,

$$C \equiv \begin{bmatrix} \frac{\lambda_\ell}{g_\ell} - \frac{a_\ell}{g_\ell} & -\frac{b_\ell}{g_\ell} - \frac{\lambda_h}{g_\ell} \\ -\frac{a_h}{g_h} - \frac{\lambda_\ell}{g_h} & \frac{\lambda_h}{g_h} - \frac{b_h}{g_h} \end{bmatrix} \tag{D.34}$$

The determinant of C is,

$$\det(C) = \frac{a_\ell(b_h - \lambda_h) - a_h(b_\ell + \lambda_h) - \lambda_\ell(b_h + b_\ell)}{g_h g_\ell} \tag{D.35}$$

A necessary condition for the convergence of the KFE is that the eigenvalues of C are both negative. As the determinant is the product of the eigenvalues, the determinant must be positive. For this determinant, a necessary condition is therefore

$$a_\ell b_h > a_h(b_\ell + \lambda_h) + a_\ell \lambda_h + \lambda_\ell(b_h + b_\ell) \tag{D.36}$$

For independent draws of the type and productivity, for some θ, a, b, the parameters must be related through $a_\ell = \theta a, b_\ell = \theta b, a_h = (1-\theta)a, b_h = (1-\theta)b$. It can be shown that (D.36), is false for any a, b, θ. Therefore, the adoption technology must have correlation between the draw of i and the draw of Z.

D.3 Stationary BGP with a Finite, Unbounded Technology Frontier

Proof of Proposition 7. Define the following to simplify notation,

$$\alpha \equiv (1+\hat{\lambda})\frac{S}{g} \tag{D.37}$$

$$\hat{\lambda} \equiv \frac{\lambda_\ell}{\lambda_h} \tag{D.38}$$

$$\bar{\lambda} \equiv \frac{\lambda_\ell}{r - g + \lambda_h} + 1 \tag{D.39}$$

$$\nu = \frac{(r-g)\bar{\lambda}}{g} \tag{D.40}$$

No Bounded Equilibrium for any κ First we demonstrate that no bounded support equilibrium exists for any κ. From (110),

$$F_h(z) = \hat{\lambda} F_\ell(z) \tag{D.41}$$

Substitute into (109) to form a first-order non-linear ODE in $F_\ell(z)$

$$0 = S(1+\hat{\lambda})^\kappa F_\ell(z)^\kappa + gF_\ell'(z) - S \tag{D.42}$$

Rearrange,

$$F_\ell'(z) = \tfrac{S}{g} - \tfrac{S}{g}(1+\hat{\lambda})^\kappa F_\ell(z)^\kappa \tag{D.43}$$

This non-linear ODE is separable,

$$\mathrm{d}z = \frac{\mathrm{d}F_\ell(z)}{\frac{S}{g} - \frac{S}{g}(1+\hat{\lambda})^\kappa F_\ell(z)^\kappa} \tag{D.44}$$

Integrate,

$$z + C_1 = \int_0^{F_\ell} \frac{1}{\frac{S}{g} - \frac{S}{g}(1+\hat{\lambda})^\kappa q^\kappa}\mathrm{d}q \tag{D.45}$$

Define the following function of $q \in [0,1]$,

$$Q_\ell(q) = \frac{g}{S} q \; {}_1\mathbb{F}_2\left(1, 1/\kappa, 1+1/\kappa, (1+\hat{\lambda})^\kappa q^\kappa\right) \tag{D.46}$$

Assume that $Q_\ell(q)$ has an inverse $Q_\ell^{-1}(\cdot)$. Use (36), (D.45) and (D.46),

$$z + C_1 = Q_\ell(F_\ell(z)) \tag{D.47}$$
$$F_\ell(z) = Q_\ell^{-1}(z + C_1) \tag{D.48}$$

From (111) and (D.46), $Q^{-1}(C_1) = 0$. From (D.46), $C_1 = Q_\ell(0) = 0$. Since $C_1 = 0$ and $F_\ell(z) = Q_\ell^{-1}(z)$, $Q_\ell(q)$ is the quantile function for the random variable z.

To show that there doesn't exist a bounded support solution, assume a $\bar{z} < \infty$. Use (112) and (D.41) to get:

$$F_\ell(\bar{z}) = \tfrac{1}{1+\hat{\lambda}} \tag{D.49}$$

Using the definition of Q_ℓ

$$\bar{z} = Q_\ell^{-1}(\tfrac{1}{1+\hat{\lambda}}) \tag{D.50}$$

From (D.46)

$$\bar{z} = \tfrac{g}{(1+\hat{\lambda})S} \, {}_1\mathbb{F}_2\left(1, 1/\kappa, 1+1/\kappa, 1\right) \tag{D.51}$$

But, for any parameters, the Hypergeometric function is only defined on $(-1,1)$ and goes to infinity on the boundaries. Hence, $\bar{z} \to \infty$ and there can be no stationary equilibrium with bounded support.

Equilibrium for $\kappa = 1$ Take (110) and solve for $F_h(z)$

$$F_h(z) = \hat{\lambda} F_\ell(z) \tag{D.52}$$

Substitute into (109)

$$S = gF_\ell'(z) + \left(\hat{\lambda} + 1\right) SF_\ell(z) \tag{D.53}$$

Solve this as an ODE in $F_\ell(z)$, subject to the $F_\ell(0) = 0$ boundary condition in (111)

$$F_\ell(z) = \tfrac{1}{1+\hat{\lambda}} e^{-\alpha z} \tag{D.54}$$

We can check that if $\alpha > 0$ the right boundary conditions hold

$$\lim_{z\to\infty} \left(F_\ell(z) + F_h(z)\right) = 1 \tag{D.55}$$

Differentiating (D.54),

$$F'_\ell(z) = \tfrac{\alpha}{1+\hat{\lambda}} e^{-\alpha z} \tag{D.56}$$

With (D.52), the pdf for the unconditional distribution, $F(z)$,

$$F'(z) = \alpha e^{-\alpha z} \tag{D.57}$$

Solve (106) for $v_h(z)$

$$v_h(z) = \frac{e^z + \lambda_h v_\ell(z)}{r - g + \lambda_h} \tag{D.58}$$

Substituting into (105) gives the following ODE in $v_\ell(z)$

$$(r-g)v_\ell(z) = e^z + \lambda_h \hat{\lambda} \left(-v_\ell(z) + \frac{e^z + \lambda_h v_\ell(z)}{r - g + \lambda_h}\right) - g v'_\ell(z) \tag{D.59}$$

Using the constant definitions and simplifying

$$(r-g)v_\ell(z) = e^z - \frac{g v'_\ell(z)}{\bar{\lambda}} \tag{D.60}$$

Solve this ODE subject to the smooth pasting condition in (108) and simplify,

$$v_\ell(z) = \frac{\bar{\lambda}}{g + (r-g)\bar{\lambda}} e^z + \frac{1}{(r-g)(\nu+1)} e^{-z\nu} \tag{D.61}$$

Using the definitions of the constants and (D.61)

$$v_\ell(0) = \tfrac{1}{r-g} \tag{D.62}$$

Substitute (D.57), (D.61) and (D.62) into the value matching condition in (107) and simplify

$$\frac{1}{r-g} = \int_0^\infty \left[\frac{e^{z\left(\bar{\lambda} - \alpha - \frac{r\bar{\lambda}}{g}\right)} \alpha g}{(g-r)\left(-r\bar{\lambda} + g\left(\bar{\lambda} - 1\right)\right)} + \frac{e^{z - z\alpha} \alpha \bar{\lambda}}{g + r\bar{\lambda} - g\bar{\lambda}}\right] \mathrm{d}z - \zeta \tag{D.63}$$

Evaluating the integral,

$$\zeta = \frac{\alpha\left(-r\bar{\lambda} + g\left(\bar{\lambda} - \alpha + 1\right)\right)}{(g-r)\left(r\bar{\lambda} + g\left(\alpha - \bar{\lambda}\right)\right)(\alpha - 1)} - \frac{1}{(r-g)(\nu+1)} - \frac{\bar{\lambda}}{g + r\bar{\lambda} - g\bar{\lambda}} \tag{D.64}$$

Substitute for α gives an implicit equation in S

$$0 = \zeta + \frac{g\left(\frac{1}{r-g} + \frac{\bar{\lambda}}{S-g+S\hat{\lambda}} - \frac{\bar{\lambda}}{S-g\bar{\lambda}+r\bar{\lambda}+S\hat{\lambda}}\right)}{-r\bar{\lambda} + g\left(\bar{\lambda}-1\right)} + \frac{1}{(r-g)(\nu+1)} \tag{D.65}$$

As $g = \gamma$ in equilibrium, only S is unknown. This equation is a quadratic in S, and can be analytically in terms of model parameters as,

$$S = \frac{\lambda_h\left(\zeta r(r+\lambda_h+\lambda_\ell) - \sqrt{\zeta((4g+r^2\zeta)(-g+r+\lambda_h)^2+2(-2g+(g-r)r\zeta)(g-r-\lambda_h)\lambda_\ell+(g-r)^2\zeta\lambda_\ell{}^2)}+\zeta g^2 2+\zeta(-g)(3r+2\lambda_h+\lambda_\ell)\right)}{2\zeta(\lambda_h+\lambda_\ell)(g-r-\lambda_h)} \tag{D.66}$$

From this S, α can be calculated through (D.37) and the the rest of the equilibrium follows. $\square$

D.4 Bounded Support

Proof of Proposition 8. Define the following to simplify notation,

$$\alpha \equiv (1+\hat{\lambda})\frac{S-\eta}{g} \tag{D.67}$$

$$\hat{\lambda} \equiv \frac{\lambda_\ell}{\eta+\lambda_h} \tag{D.68}$$

$$\bar{\lambda} \equiv \frac{r-g+\lambda_\ell+\lambda_h}{r-g+\lambda_h} \tag{D.69}$$

$$\nu = \frac{r-g+\eta}{g}\bar{\lambda} \tag{D.70}$$

Solve for $F_h(z)$ in (129),

$$F_h(z) = \hat{\lambda} F_\ell(z) \tag{D.71}$$

Substitute this back into (128) to get an ODE in F_ℓ

$$0 = gF_\ell'(z) + (S-\eta)(1+\hat{\lambda})F_\ell(z) + \eta H\left(z-\bar{z}\right) - S \tag{D.72}$$

Solve this ODE with the boundary condition $F_\ell(0) = 0$

$$F_\ell(z) = \begin{cases} \frac{S}{(S-\eta)(1+\hat{\lambda})}(1-e^{-\alpha z}) & 0 \le z < \bar{z} \\ \frac{S}{(S-\eta)(1+\hat{\lambda})}(1-e^{-\alpha \bar{z}}) & z = \bar{z} \end{cases} \tag{D.73}$$

This function is continuous at $z = \bar{z}$, and therefore so is $F_h(z)$. The unconditional distribution is,

$$F(z) = (1+\hat{\lambda})F_\ell(\bar{z}) \tag{D.74}$$

$$= \frac{S}{S-\eta}(1-e^{-\alpha z}) \tag{D.75}$$

Using the boundary condition that $F(\bar{z}) = 1$, and solving for $\bar{z}$ with the assumption that $S > \eta$,

$$\bar{z} = \frac{\log(S/\eta)}{\alpha} \tag{D.76}$$

The pdf of the unconditional distribution is,

$$F'(z) = \frac{\alpha S}{S - \eta} e^{-\alpha z} \tag{D.77}$$

$$\tag{D.78}$$

To solve for the value solve (125) for $v_h(z)$,

$$v_h(z) = \frac{e^z + (\lambda_h - \eta) v_\ell(z) + \eta v_\ell(\bar{z})}{r - g + \lambda_h} \tag{D.79}$$

Substitute into (124) and simplify

$$(r - g + \eta) v_\ell(z) = e^z + \eta v_\ell(\bar{z}) - \frac{g}{\bar{\lambda}} v'_\ell(z) \tag{D.80}$$

Solving (127) and (D.80) and simplifying,

$$v_\ell(z) = \frac{\bar{\lambda}}{g + (r + \eta - g)\bar{\lambda}} e^z + \frac{\eta}{r - g + \eta} v_\ell(\bar{z}) + \frac{1}{(r + \eta - g)(\nu + 1)} e^{-z\nu} \tag{D.81}$$

Evaluate (D.81) at $\bar{z}$ and solve for $v_\ell(\bar{z})$,

$$v_\ell(\bar{z}) = \left(-\frac{\eta}{g - r} + 1\right) \left(\frac{e^{\bar{z}} \bar{\lambda}}{g + (\eta + r - g)\bar{\lambda}} + \frac{e^{-\nu \bar{z}}}{(\eta + r - g)(\nu + 1)}\right) \tag{D.82}$$

Subtitute (D.82) into (D.81) to find an expression for $v_\ell(z)$

$$v_\ell(z) = \frac{\bar{\lambda}}{g(1 + \nu)} \left(e^z + \frac{1}{\nu} e^{-\nu z} + \frac{\eta}{r - g} \left(e^{\bar{z}} + \frac{1}{\nu} e^{-\nu \bar{z}}\right)\right) \tag{D.83}$$

Substitute (D.77) and (D.83) into the value matching condition in (126) and evaluate the integral,

$$\zeta + \frac{1}{r - g} = \frac{S\alpha\bar{\lambda}\left(-\frac{e^{-\nu\bar{z}}\left(-1+e^{-\alpha\bar{z}}\right)\eta}{(-g+r)\alpha\nu} + \frac{e^{\bar{z}}\eta\left(e^{-\alpha\bar{z}}-1\right)}{\alpha(g-r)} + \frac{-e^{-(\alpha+\nu)\bar{z}}+1}{\nu(\alpha+\nu)} + \frac{-e^{\bar{z}-\alpha\bar{z}}+1}{\alpha-1}\right)}{g(S - \eta)(\nu + 1)} \tag{D.84}$$

To find an implicit equation for the equilibrium S, take (D.84) and substitute for α and $\bar{z}$ from (D.67) and (D.76)

$$\zeta + \frac{1}{r-g} = \frac{S\bar{\lambda}\left(\hat{\lambda}+1\right)\left(\frac{-\left(\frac{S}{\eta}\right)^{-1-\frac{g\nu}{(S-\eta)\left(1+\hat{\lambda}\right)}}+1}{\nu\left(\nu+\frac{(S-\eta)\left(\hat{\lambda}+1\right)}{g}\right)} + g\left(\frac{1}{-g+(S-\eta)\left(\hat{\lambda}+1\right)} + \frac{\eta\left(-\frac{\left(\frac{S}{\eta}\right)^{-\frac{g\nu}{(S-\eta)\left(1+\hat{\lambda}\right)}}}{\nu} + \frac{\left(\frac{S}{\eta}\right)^{\frac{g}{(S-\eta)\left(\hat{\lambda}+1\right)}}\left(\eta+r-S+\hat{\lambda}(\eta+r-g-S)\right)}{-g+(S-\eta)\left(\hat{\lambda}+1\right)}\right)}{S(g-r)\left(\hat{\lambda}+1\right)}\right)\right)}{g^2(\nu+1)} \tag{D.85}$$

□

D.5 Stationary BGP with a Endogenous, Continuous Choice, and a Finite, Unbounded Technology Frontier

Proof of Proposition 9. Assuming an interior solution, taking the first order necessary condition of the Hamilton-Jacobi-Bellman equation in (142), and reorganizing

$$\gamma(z) = \tfrac{\chi}{2} e^{-z} v_h'(z) \tag{D.86}$$

Substituting this back into (142) gives a non-linear ODE,

$$(r-g)v_h(z) = e^z - g v_h'(z) + \tfrac{\chi}{4} e^{-z} v_h'(z)^2 + \lambda_h (v_\ell(z) - v_h(z)) \tag{D.87}$$

As the equilibrium has finite support, the growth rate of the economy will be the growth rate of the frontier,

$$g \equiv \lim_{z\to\infty} \gamma(z) \tag{D.88}$$

For large z, the option value of diffusion approaches 0, and the choice of γ can be done assuming only a transversality condition, rather than the initial value problem with optimal stopping. Guess asymptotic solutions of the following form,[36]

$$v_\ell^\infty(z) = c_\ell e^z \tag{D.89}$$

$$v_h^\infty(z) = c_h e^z \tag{D.90}$$

Substituting this guess into (141) and (D.87) and simplifying leads to the following system of quadratic equations after using undetermined coefficients,

$$(r+\lambda_\ell)c_\ell = 1 + \lambda_\ell c_h \tag{D.91}$$

$$(r+\lambda_h)c_h = 1 + \lambda_h c_\ell + \tfrac{\chi}{4} c_h^2 \tag{D.92}$$

Solving this system for c_ℓ and c_h gives a pair of roots,

$$c_\ell = \frac{\lambda_\ell \left(\frac{\chi}{2} \pm \sqrt{(\lambda_h+\lambda_\ell+r)\left(r^2(\lambda_h+\lambda_\ell+r) - \chi(\lambda_\ell+r)\right)}\right) + r\left(\lambda_\ell(\lambda_h+\lambda_\ell) + \frac{\chi}{2}\right) + r^2\lambda_\ell}{\frac{\chi}{2}(\lambda_\ell+r)^2} \tag{D.93}$$

$$c_h = \frac{r(\lambda_h+\lambda_\ell+r) \pm \sqrt{(\lambda_h+\lambda_\ell+r)\left(r^2(\lambda_h+\lambda_\ell+r) - \chi(\lambda_\ell+r)\right)}}{\frac{\chi}{2}(\lambda_\ell+r)} \tag{D.94}$$

The two roots to the system have matching plus (or minus) signs for these constants. The verification step shows that this is a particular solution of this system. Use of the transversality conditions would eliminate terms for the general solution. Using the solution to this system of equations and (D.88) and (D.90),

$$g = \tfrac{\chi}{2} e^{-z}(c_h e^z) \tag{D.95}$$

Substituting from (D.94) and simplifying

$$g = \frac{r(\lambda_h+\lambda_\ell+r) \pm \sqrt{(\lambda_h+\lambda_\ell+r)\left(r^2(\lambda_h+\lambda_\ell+r) - \chi(\lambda_\ell+r)\right)}}{\lambda_\ell+r} \tag{D.96}$$

[36] More rigorously, do a change of variables $w_i(z) \equiv v_i'(z)e^{-z}$. This is a stationary ODE subject to the new initial condition $w_i(z) = 0$, where the solutions finds the stationary value is $w_i(\infty) = c_i$ and $g \equiv \gamma(\infty) = \frac{\chi}{2} c_h$. To convert back to $v_i(z)$, use (143).

Define

$$\bar{\lambda} \equiv \frac{r + \lambda_\ell + \lambda_h}{r + \lambda_\ell} \tag{D.97}$$

Then, choose the negative root (to ensure $g < r$) and simplify,

$$g = \bar{\lambda} r \left[1 - \sqrt{1 - \frac{\chi}{\bar{\lambda} r^2}}\right] \tag{D.98}$$

As $\gamma(0) = 0$, Evaluating (145) and (146) at $z = 0$ gives,

$$S_\ell = g F_\ell'(0) \tag{D.99}$$

$$S_h = g F_h'(0) \tag{D.100}$$

For the initial values, note that if $v_\ell'(0) = v_h'(0) = 0$, then to fulfill the ODE, the initial value must be,

$$v_\ell(0) = v_h(0) = \tfrac{1}{r-g} \tag{D.101}$$

Parameter Requirements From (D.98), $g > 0$ always holds, and a necessary condition for $r > g$ is,

$$r > \sqrt{\frac{\chi}{\bar{\lambda}}} \tag{D.102}$$

□

D.6 Stationary BGP with a Endogenous, Continuous Choice, and a Finite, Bounded Technology Frontier

Proof of Proposition 10. Note that Section 4.2 nests Section 4.1 when $\eta = 0$

Nested Derivation of Stationary HBJE Assuming an interior solution, take the first order necessary condition of the Hamilton-Jacobi-Bellman equation in (159), and reorganize

$$\gamma(z) = \tfrac{\chi}{2} e^{-z} v_h'(z) \tag{D.103}$$

Substitute this back into (159) to get a non-linear ODE,

$$(r-g)v_h(z) = e^z - g v_h'(z) + \tfrac{\chi}{4} e^{-z} v_h'(z)^2 + \lambda_h(v_\ell(z) - v_h(z)) + \eta\left(v_\ell(\bar{z}) - v_h(z)\right) \tag{D.104}$$

To create a stationary solution for the value function define a change of variables,

$$w_i(z) \equiv e^{-z} v_i'(z) \tag{D.105}$$

From (161),

$$w_\ell(0) = w_h(0) = 0 \tag{D.106}$$

Differentiate and reorganize (D.105),

$$e^{-z} v_i''(z) = w_i'(z) + w_i(z) \tag{D.107}$$

Differentiate (158),

$$(r-g)v_\ell'(z) = e^z - g v_\ell''(z) + \lambda_\ell(v_h'(z) - v_\ell'(z)) - \eta v_\ell'(z) \tag{D.108}$$

Multiply by e^{-z} and use (D.105) and (D.107)

$$(r+\lambda_\ell+\eta)w_\ell(z) = 1 - g w_\ell'(z) + \lambda_\ell w_h(z) \tag{D.109}$$

Note that,

$$e^{-z}\boldsymbol{\partial}_z\left(e^{-z}v_h'(z)^2\right) = 2e^{-z}v_h''(z)e^{-z}v_h'(z) - \left(e^{-z}v_h'(z)\right)^2 \tag{D.110}$$

$$= 2w_h(z)w_h'(z) + w_h(z)^2 \tag{D.111}$$

Differentiate (D.104), multiply by e^{-z}, and use (D.105), (D.107) and (D.111)

$$(r+\lambda_h+\eta)w_h(z) = 1 - \left(g - \tfrac{\chi}{2}w_h(z)\right)w_h'(z) + \lambda_h w_\ell(z) + \tfrac{\chi}{4}w_h(z)^2 \tag{D.112}$$

From (D.103),

$$\gamma(z) = \tfrac{\chi}{2}w_h(z) \tag{D.113}$$

$$g \equiv \tfrac{\chi}{2}w_h(\bar{z}) \tag{D.114}$$

Integrate (D.105) with the initial value from (160) to get,

$$v_i(z) = v_s + \int_0^z e^{\tilde{z}}w_i(\tilde{z})\mathrm{d}\tilde{z} \tag{D.115}$$

Substitute (D.115) into (A.18) and rearrange to get an expression for v_s in terms of w_ℓ and intrinsics,

$$v_s = \frac{1+\eta v_\ell(\bar{z})}{r-g+\eta} = \frac{1+\eta\int_0^{\bar{z}} e^{\tilde{z}}w_\ell(\tilde{z})\mathrm{d}z}{r-g} \tag{D.116}$$

Endogenous choice of θ: Take the value matching condition for the choice of the idiosyncratic $\hat{\theta}$ given equilibrium θ choices of the other firms.

$$v_s \equiv v_\ell(0) = v_h(0) = \max_{\hat{\theta}\geq 0}\left\{(1-\hat{\theta})\int_0^{\bar{z}} v_\ell(z;\theta)\mathrm{d}F(z;\theta)^\kappa + \hat{\theta}v_\ell(\bar{z};\theta) - \zeta - \tfrac{1}{\varsigma}\hat{\theta}^2\right\} \tag{D.117}$$

Crucially, if the firm chooses a $\hat{\theta}\neq\theta$, they are infinitesimal and have no influence on the value or equilibrium distributions. Taking the first order condition and then letting $\hat{\theta}=\theta$ in equilibrium gives,

$$\theta = \varsigma\left(v_\ell(\bar{z};\theta) - \int_0^{\bar{z}} v_\ell(z;\theta)\mathrm{d}F(z;\theta)^\kappa\right) \tag{D.118}$$

Reorganize,

$$\int_0^{\bar{z}} v_\ell(z)\mathrm{d}F(z)^\kappa = v_\ell(\bar{z}) - \frac{\theta}{\varsigma} \tag{D.119}$$

Substitute into (D.117) at the optimal θ

$$v_\ell(0) = (1-\theta)\left(v_\ell(\bar{z}) - \frac{\theta}{\varsigma}\right) + \theta v_\ell(\bar{z}) - \zeta - \tfrac{1}{\varsigma}\theta^2 \tag{D.120}$$

Solve the quadratic for θ and choose the interior (i.e., negative) root,

$$\theta = 1 - \sqrt{1 - \varsigma(v_\ell(\bar{z}) - v_\ell(0) - \zeta)} \tag{D.121}$$

Substitute from (D.115),

$$\theta = 1 - \sqrt{1 - \varsigma\left(\int_0^z e^{\tilde{z}} w_\ell(\tilde{z}) \mathrm{d}\tilde{z} - \zeta\right)} \tag{D.122}$$

This equation provides an equilibrium expression for the optimal choice of θ.

KFE and Value Matching From (162), for $z < \bar{z}$ the KFE is,

$$0 = gF'_\ell(z) + \lambda_h F_h(z) - \lambda_\ell F_\ell(z) - \eta F_\ell(z) + (1-\theta)(S_\ell + S_h)F(z)^\kappa - S_\ell,\ z < \bar{z} \tag{D.123}$$

Note that since $\gamma(0) = 0$, $S_\ell = gF'_\ell(0)$ and $S_h = gF'_h(0)$. For the value matching, first use the generic result from (B.121) that,

$$\mathbb{E}\left[v_\ell(z)\right] = \int_0^{\bar{z}} v_\ell(z) \mathrm{d}F(z) \mathrm{d}z = \int_0^{\bar{z}} v'_\ell(z)(1 - F(z)^\kappa)\mathrm{d}z + v_\ell(0) \tag{D.124}$$

Substitute (D.105) and (D.124) into (160) at the optimal θ

$$v_s = (1-\theta)\left(\int_0^{\bar{z}} e^z w_\ell(z)(1-F(z)^\kappa)\mathrm{d}z + v_s\right) + \theta\left(v_s + \int_0^{\bar{z}} e^z w_\ell(z)\mathrm{d}z\right) - \zeta - \tfrac{1}{\varsigma}\theta^2 \tag{D.125}$$

Simplifying gives an expression in terms of $F(z)$ and $w(z)$,

$$\zeta = \begin{cases} \int_0^{\bar{z}} e^z w_\ell(z)(1-F(z)^\kappa)\mathrm{d}z & \text{if } \theta = 0 \\ \int_0^{\bar{z}} e^z w_\ell(z)\mathrm{d}z - (1-\theta)\int_0^{\bar{z}} e^z w_\ell(z) F(z)^\kappa \mathrm{d}z - \frac{1}{\varsigma}\theta^2 & \text{if } \theta > 0 \end{cases} \tag{D.126}$$

For the unbounded case where $\eta = \theta = 0$, and $\bar{z} \to \infty$, we can check the asymptotic value comes from (D.90), (D.93), (D.94) and (D.105)

$$\lim_{z\to\infty} w_i(z) = c_i \tag{D.127}$$

Matrix Formulation of KFE In the sample case where $\kappa = 1$, write the KFE as a matrix ODE, similar to Appendix D.1,

$$C(z) \equiv \begin{bmatrix} (1-\theta)(F'_\ell(0) + F'_h(0)) - (\lambda_\ell + \eta)/g & (1-\theta)(F'_\ell(0) + F'_h(0)) + \lambda_h/g \\ \lambda_\ell/(g - \gamma(z)) & -(\lambda_h + \eta)/(g - \gamma(z)) \end{bmatrix} \tag{D.128}$$

$$D(z) \equiv \begin{bmatrix} F'_\ell(0) \\ \frac{g}{g-\gamma(z)} F'_h(0) \end{bmatrix} \tag{D.129}$$

$$\vec{F}(z) \equiv \begin{bmatrix} F_\ell(z) \\ F_h(z) \end{bmatrix} \tag{D.130}$$

Then, from (163), (166), (167) and (D.123) writing the KFE as a matrix ODE,

$$\vec{F}'(z) = -C(z) \cdot \vec{F}(z) + D(z) \tag{D.131}$$
$$\vec{F}(0) = \mathbf{0} \tag{D.132}$$

Checking the initial condition, since $\gamma(0) = 0$,

$$\vec{F}'(0) = D(0) = \begin{bmatrix} F_\ell'(0) \\ F_h'(0) \end{bmatrix} \tag{D.133}$$

Solving the KFE, define,

$$q(z) \equiv \int_0^z \frac{1}{g - \gamma(\hat{z})} \mathrm{d}\hat{z} \tag{D.134}$$
$$\hat{C}(z) \equiv \begin{bmatrix} [(1-\theta)(F_\ell'(0) + F_h'(0)) - (\lambda_\ell + \eta)/g]\, z & [(1-\theta)(F_\ell'(0) + F_h'(0)) + \lambda_h/g]\, z \\ \lambda_\ell q(z) & -(\lambda_h + \eta) q(z) \end{bmatrix} \tag{D.135}$$

Then, the solution to the KFE in (D.131) and (D.132) is[37]

$$\vec{F}(z) = e^{-\hat{C}(z)} \int_0^z e^{\hat{C}(\hat{z})} D(\hat{z}) \mathrm{d}\hat{z} \tag{D.136}$$

The $A(z)$ matrix will be invertible for $0 < z < \bar{z}$, but as $\gamma(\bar{z}) = g$, it will not be invertible exactly at $\bar{z}$. But in the limit as $z \to \bar{z}$, we know both $\gamma(z)$ and $F(z)$ are continuous. Assume that $\lim_{z\to\bar{z}} F_h'(z) < \infty$ and use $g - \gamma(\bar{z}) = 0$ in (163) and (165). This gives the system of equations,

$$0 = (\lambda_h + \eta) F_h(\bar{z}) - \lambda_\ell F_\ell(\bar{z}) + g F_h'(0) \tag{D.140}$$
$$1 = F_\ell(\bar{z}) + F_h(\bar{z}) \tag{D.141}$$

Solving gives a boundary condition for $F(\bar{z})$ for a given $F_h'(0)$

$$\vec{F}(\bar{z}) = \frac{1}{\lambda_\ell + \lambda_h + \eta} \begin{bmatrix} g F_h'(0) + \eta + \lambda_h \\ -g F_h'(0) + \lambda_\ell \end{bmatrix} \tag{D.142}$$

The boundary condition from (D.142) ensures continuity of $\vec{F}(z)$ discussed in the exogenous bounded example of Appendix D.4 and embeds the integrability constraint as in the proof of Proposition 6 (see Appendix D.1 and equation (D.24)). Hence, given a fixed $F_h'(0)$, there will be a $F_\ell'(0)$ consistent with this boundary value.

[37] This solution comes from the "method of variation of constants", or the Lagrange method. for some $X(t)$ defined on $t > a$, the equation

$$X'(t) = A(t)X(t) + f(t) \tag{D.137}$$

has the general solution,

$$X(t) = \Phi(t) \int_a^t \Phi^{-1}(\tau) f(\tau) \mathrm{d}\tau \tag{D.138}$$
$$\Phi(t) \equiv \exp\left[\int_a^t A(\tau) \mathrm{d}\tau\right] \tag{D.139}$$

Also note that the inverse of a matrix exponential fulfills, $\left(e^A\right)^{-1} = e^{-A}$

Upper bound on g: To find an upper bound on g, note that as $w_i(z)$ is increasing, the maximimum growth rate is as $\bar{z} \to \infty$. In the limit, $\lim_{z\to\infty} w_i'(z) = 0$ as $w_i(z)$ have been constructed to be stationary. Therefore, looking at the asymptotic limit of (D.109) and (D.112),

$$(r + \lambda_\ell + \eta)w_\ell(\infty) = 1 + \lambda_\ell w_h(\infty) \tag{D.143}$$
$$(r + \lambda_h + \eta)w_h(\infty) = 1 + \lambda_h w_\ell(\infty) + \tfrac{\chi}{4} w_h(\infty)^2 \tag{D.144}$$

Comparing this system of 2 equations in $w_\ell(\infty)$ and $w_h(\infty)$ to (D.91) and (D.92) shows the equations to be of the identical form, except r has been augmented by η. Using the solution procedure in that section, define an equivalent to (D.97),

$$\bar{\lambda} \equiv \frac{r + \eta + \lambda_\ell + \lambda_h}{r + \eta + \lambda_\ell} \tag{D.145}$$

Where an equivalent to (D.98) gives an upper bound on the growth rate

$$g < \bar{\lambda}(r + \eta)\left[1 - \sqrt{1 - \frac{\chi}{\bar{\lambda}(r + \eta)^2}}\right] \tag{D.146}$$

□

D.7 Nested Summary of Equations for Endogenous Innovation

Summarizing the full set of equations to solve for $F_i(z)$ and $w_i(z)$ from (163) to (165), (D.99), (D.100), (D.106), (D.109), (D.112) to (D.116), (D.122), (D.123), (D.126), (D.142) and (D.145). First, choose a g where

$$\bar{\lambda} \equiv \frac{r + \eta + \lambda_\ell + \lambda_h}{r + \eta + \lambda_\ell} \tag{D.147}$$
$$g < \bar{\lambda}(r + \eta)\left[1 - \sqrt{1 - \frac{\chi}{\bar{\lambda}(r + \eta)^2}}\right] \tag{D.148}$$

Then solve the HBJE as an initial value problem using the chosen g

$$(r + \lambda_\ell + \eta)w_\ell(z) = 1 - g w_\ell'(z) + \lambda_\ell w_h(z) \tag{D.149}$$
$$(r + \lambda_h + \eta)w_h(z) = 1 - \left(g - \tfrac{\chi}{2} w_h(z)\right) w_h'(z) + \lambda_h w_\ell(z) + \tfrac{\chi}{4} w_h(z)^2 \tag{D.150}$$
$$w_i(0) = 0 \tag{D.151}$$

Find $\bar{z}$ solving the following equation given the g

$$g = \tfrac{\chi}{2} w_h(\bar{z}) \tag{D.152}$$

From the solution, calculate,

$$\gamma(z) = \tfrac{\chi}{2} w_h(z) \tag{D.153}$$
$$\hat{w}_i(z) \equiv \int_0^z e^{\tilde{z}} w_i(\tilde{z}) \mathrm{d}z \tag{D.154}$$
$$\theta = 1 - \sqrt{1 - \varsigma\left(\hat{w}_\ell(\bar{z}) - \zeta\right)} \tag{D.155}$$

Then, the KFE can be solved as a boundary value problem subject to value matching,

$$S_\ell \equiv gF_\ell'(0) \tag{D.156}$$
$$S_h \equiv gF_h'(0) \tag{D.157}$$
$$0 = F_\ell(0) = F_h(0) \tag{D.158}$$
$$\vec{F}(\bar{z}) = \frac{1}{\lambda_\ell + \lambda_h + \eta}\begin{bmatrix} S_h + \eta + \lambda_h \\ -S_h + \lambda_\ell \end{bmatrix} \tag{D.159}$$
$$0 = gF_\ell'(z) + \lambda_h F_h(z) - (\lambda_\ell + \eta)F_\ell(z) + (1-\theta)(S_\ell + S_h)\left(F_\ell(z) + F_h(z)\right)^\kappa - S_\ell \tag{D.160}$$
$$0 = (g - \gamma(z))F_h'(z) + \lambda_\ell F_\ell(z) - (\lambda_h + \eta)F_h(z) - S_h \tag{D.161}$$
$$\zeta = \begin{cases} \int_0^\infty e^z w_\ell(z)(1 - (F_\ell(z) + F_h(z))^\kappa)\mathrm{d}z & \text{if } \theta = \eta = 0 \\ \hat{w}_\ell(\bar{z}) - (1-\theta)\int_0^{\bar{z}} e^z w_\ell(z)(F_\ell(z) + F_h(z))^\kappa \mathrm{d}z - \frac{1}{\varsigma}\theta^2 & \text{if } \theta \text{ or } \eta > 0 \end{cases} \tag{D.162}$$

The non-stationary value functions can be calculated as

$$v_s \equiv \frac{1 + \eta\hat{w}_\ell(\bar{z})}{r - g} \tag{D.163}$$
$$v_i(z) = v_s + \hat{w}_i(z) \tag{D.164}$$

In the case of $\kappa = 1$, given $\gamma(z)$ and $\hat{w}_\ell(z)$, parameterize the solution to the KFE with $\vec{F}'(0) = \{F_\ell'(0), F_h'(0)\}$, and use (D.129) and (D.134) to (D.136)

$$q(z) \equiv \int_0^z \frac{1}{g - \gamma(\hat{z})}\mathrm{d}\hat{z} \tag{D.165}$$
$$\hat{C}(z; \vec{F}'(0)) \equiv \begin{bmatrix} [(1-\theta)(F_\ell'(0) + F_h'(0)) - (\lambda_\ell + \eta)/g]\, z & [(1-\theta)(F_\ell'(0) + F_h'(0)) + \lambda_h/g]\, z \\ \lambda_\ell q(z) & -(\lambda_h + \eta)q(z) \end{bmatrix} \tag{D.166}$$
$$D(z; \vec{F}'(0)) \equiv \begin{bmatrix} F_\ell'(0) \\ \frac{g}{g - \gamma(z)}F_h'(0) \end{bmatrix} \tag{D.167}$$
$$F(z; \vec{F}'(0)) = e^{-\hat{C}(z;\vec{F}'(0))}\int_0^z e^{\hat{C}(\hat{z};\vec{F}'(0))}D(\hat{z}; \vec{F}'(0))\mathrm{d}\hat{z} \tag{D.168}$$

Given this, the equilibrium $\{F_\ell'(0), F_h'(0)\}$ must fulfill the following set of equations (where the two equations from $\vec{F}(\bar{z})$ are collinear from (D.126) and (D.142)

$$\vec{F}(\bar{z}; \vec{F}'(0)) = \frac{1}{\lambda_\ell + \lambda_h + \eta}\begin{bmatrix} gF_h'(0) + \eta + \lambda_h \\ -gF_h'(0) + \lambda_\ell \end{bmatrix} \tag{D.169}$$
$$\zeta = \hat{w}_\ell(\bar{z}) - (1-\theta)\int_0^{\bar{z}} e^z w_\ell(z)\left[F(z; \vec{F}'(0)) \cdot \mathbf{1}\right]\mathrm{d}z - \frac{1}{\varsigma}\theta^2 \tag{D.170}$$

Appendix E Numerical Methods

E.1 Endogenous, Continuous Choice, and a Finite, Bounded Technology Frontier

Recall that there is multiplicity in $\bar{z}$ and g, given properties of the initial distribution. In order to find the set of admissible growth rates, the numerical method here fixes a g, solves the Hamilton-Jacobi-Bellman Equation for the associated $\bar{z}$ and $w_i(z)$, and then finds the $F_i(z)$ which rationalizes the g and $\bar{z}$ subject to the endogenous growth rates from the HJBE and the value matching condition.

E.1.1 HJBE and Pre-Calculations

Fix a g within the bounds of (D.148).[38] The ODE for $w_i(z)$ can be solved, and the $\bar{z}$ determined which fulfills (D.152). To solve the ODE and the $\bar{z}$,

1. Setup an operator and initial condition using (D.149) to (D.151)
2. Solve this initial value problem using finite differences, but add in an event to stop calculations before it becomes singular
   - The key is that the ODE should stop at the $\bar{z}$ such that $g - \frac{\chi}{2}w_h(\bar{z}) \approx 0$ from (D.152). For stability choose a small threshold value, but ensure that at the chosen $\bar{z}$, $\frac{\chi}{2}w_h(\bar{z}) < g$, which will ensure that singularities in the next step don't occur.
   - In matlab, on can add to the ODE settings of 'Events'. See `http://www.mathworks.com/help/matlab/ref/odeset.html#f92-1017470`
3. For the given $g, \bar{z}, w_i(z)$, calculate
   - $\gamma(z)$ from (D.153)
   - $w_i^e(z) \equiv e^z w_i(z)$ from multiplying the results.
   - $\hat{w}_i(z)$ by doing a cumulative integral of $e^z w_i(z) = w_i^e(z)$, defined on $z \in [0, \bar{z}]$. Matlab can use `cumtrapz`, which will be accurate enough for large numbers of grid points in the ODE.
   - Use some simple interpolation of the results for $\gamma(z)$, $\hat{w}_i(z)$ and $w_i^e(z)$. With a large number of grid points, linear interpolation is likely sufficient and is consistent with the cumulative trapezoidal integration above.
   - If θ is intended to be endogenous, then calculate θ from $\hat{w}_\ell(\bar{z})$ using (D.155), define it as the fixed value.

E.1.2 Combined Quadrature and Projection Method Setup

Given a $\bar{z}$, define the approximations with $N-1$ order Chebyshev polynomials, $T_k(z)$ adapted to the support $[0, \bar{z}]$. Then, assume that the cdfs can be approximated by,

$$F_i(z) \approx \sum_{k=0}^{N-1} d_{ik} T_k(z) \tag{E.1}$$

Denote the vectors of coefficients as d_i. Define the Chebyshev polynomial nodes as $z_1, \ldots z_{N-1}$, and the complete set of nodes as

$$\vec{z} \equiv \{0, z_1, \ldots z_{N-1}, \bar{z}\} \in \mathbb{R}^{N+1} \tag{E.2}$$

The set of interior nodes to be used by quadrature as

$$\vec{z}_q \equiv \{z_1, \ldots z_{N-1}\} \in \mathbb{R}^{N-1} \tag{E.3}$$

The set of interior nodes for the KFE ODE to hold is

$$\vec{z}_I \equiv \{0, z_1, \ldots z_{N-1}\} \in \mathbb{R}^{N} \tag{E.4}$$

[38] In principle, one could choose the $\bar{z}$ and then solve the BGP for the g that rationalizes it. However, this seems to be more sensitive to numerical errors.

Define the basis matrices for these nodes, B, by stacking stacking the evaluated polynomials at each node, Similarly, define B, B', B_q, B'_q, for the subsets and the derivatives [39]

$$B \equiv \begin{bmatrix} T(z_0) \\ \dots \\ T(z_{N-1}) \end{bmatrix} \in \mathbb{R}^{N \times N} \tag{E.5}$$

$$B' \equiv \begin{bmatrix} T'(z_0) \\ \dots \\ T'(z_{N-1}) \end{bmatrix} \in \mathbb{R}^{N \times N} \tag{E.6}$$

$$\vec{F}'_i \equiv \left\{F'_i(z_j)\right\}_{j=0}^{N-1} = B' \cdot d_i \in \mathbb{R}^N \tag{E.7}$$

Define the slice of the basis to interior nodes as B_I, then

$$B_I \equiv B_{1:N-2} = \{B_{jk}\}_{j=1}^{N-2} \tag{E.8}$$

$$B_q \equiv B_{1:N-1} = \{B_{jk}\}_{j=1}^{N-1} \tag{E.9}$$

$$B'_I \equiv B'_{1:N-2} \tag{E.10}$$

$$B'_q \equiv B'_{1:N-1} \tag{E.11}$$

Given Chebyshev quadrature weights ω on $\vec{z}_q$, and the quadrature rule,

$$\int_0^{\bar{z}} h(z)\mathrm{d}z \approx \omega \cdot \vec{h} \tag{E.12}$$

Build the completed weights for quadrature, using the ω from (E.12) and combine all static elements of the value-matching integral as,

$$\Omega \equiv \omega \times w^e_\ell(\vec{z}_q) \tag{E.13}$$

Collecting all of the constant costs in the VM condition,

$$C_{vm} \equiv -\zeta + \hat{w}_\ell(\bar{z}) - \tfrac{1}{\varsigma}\theta^2 \tag{E.14}$$

E.1.3 Stationary KFE

Define $x \equiv \{d_\ell, d_h\} \in \mathbb{R}^{2N}$ and an approximate operator, $\mathcal{L} : R^{2N} \to R^{N_d}$ for some $N_d \geq 2N$. The solution method is to setup a non-linear system of equations to solve for the vector x which minimizes the residual of the operator. The operator contains the ODEs, value matching conditions, and boundary values for the points in $\vec{z}$. As we may consider cases where $N_d > 2N$, this will be done as a non-linear least squares projection.

First setting up some calculations of the values given the x_d and then implementing the equations,

[39] With `compecon`, the basis can be calculated for nodes `all_nodes` and function space `fspace` as `B = funbas(fspace, all_cheb_nodes, 0)`.

The basis of the derivative is simply `B_p = funbas(fspace, all_cheb_nodes, 1)`. With compecon, to calculate the quadrature weights for Chebyshev quadrature use `qnwcheb(N - 1,0,z_bar)`

Setup: From x_d, find,

$$F_\ell \equiv B \cdot d_\ell \tag{E.15}$$
$$F_h \equiv B \cdot d_h \tag{E.16}$$
$$F'_\ell \equiv B' \cdot d_\ell \tag{E.17}$$
$$F'_h \equiv B' \cdot d_h \tag{E.18}$$
$$\hat{F} \equiv (F_\ell + F_h)^\kappa \tag{E.19}$$
$$S_\ell \equiv g F'_{\ell 0} \tag{E.20}$$
$$S_h \equiv g F'_{h0} \tag{E.21}$$

Equation 1 **to** 2**:** From (D.158), the initial values are

$$0 = F_{\ell 0} \tag{E.22}$$
$$0 = F_{h0} \tag{E.23}$$

Equation 3 **and** 4**:** From (D.159) boundary value,

$$0 = -F_{\ell N+1} + \frac{S_h + \eta + \lambda_h}{\lambda_\ell + \lambda_h + \eta} \tag{E.24}$$
$$0 = -F_{hN+1} + \frac{-S_h + \lambda_\ell}{\lambda_\ell + \lambda_h + \eta} \tag{E.25}$$

Equation 5**:** From (D.162) and (E.12) to (E.14), the value matching becomes

$$0 = C_{vm} - (1-\theta)\,\Omega \cdot \hat{F}_q \tag{E.26}$$

Equation 6 **to** $N+5$**:** From (D.160) evaluating F_ℓ evaluated at the z_I nodes,[40]

$$0 = gF'_{\ell I} + \lambda_h F_{hI} - (\lambda_\ell + \eta)F_{\ell I} + (1-\theta)(S_\ell + S_h)\hat{F}_I - S_\ell \tag{E.27}$$

Equation $N+6$ **to** $2N+5$**:** From (D.161),evaluated at the z_I nodes

$$0 = (g - \gamma_I) \times F'_{hI} + \lambda_\ell F_{\ell I} - (\lambda_h + \eta)F_{hI} - S_h \tag{E.28}$$

Equation $2N+6$ While (D.159) should theoretically be equivalent to $F_\ell(\bar{z}) + F_h(\bar{z}) = 1$ when equilibrium exist, it is often useful to add in this constraint directly for convergence, and to look for cases where an equilibrium cannot exist

$$1 = F_{\ell N+1} + F_{hN+1} \tag{E.29}$$

Solve for the x root of this approximate operator.[41]

[40] Note that F_ℓ is undefined exactly at $\bar{z}$ due to the Heaviside operator. Since $\bar{z}$ is the maximum of support and cdfs are right continuous, the active Heaviside part of (162) does not need to be evaluated in the system of equations. This result requires the continuity of F_ℓ and F_h at $\bar{z}$, and that the mass is held constant through (165). See Appendix D.4 for a proof of continuity.

[41] When $N_d = 2N$, this is collocation. If $2N$ is less than the number of equations, the approximate x can be solved using a non-linear least squares projection. For example, in Matlab, an overdetermined system in `fsolve` uses Levenberg-Marquardt algorithm (see `http://www.mathworks.com/help/optim/ug/nonlinear-least-squares-with-full-jacobian.html`). When $\kappa = 1$, this system appears to be quadratic in x. Hence, solvers with quadratic approximations should converge quickly.

E.2 Endogenous, Continuous Choice, and a Finite, Unbounded Technology Frontier

Following a similar approach to Appendix E.1, first solve the HJBE and then the KFE. The main difference is that a unique solution exists rather than a continuum of solutions.

E.2.1 HJBE and Pre-Calculations

Set $\eta = 0$ and $\theta = 0$ within the referenced nested equations.

1. Use (D.148) to find the g analytically.
2. Solve this initial value problem using standard finite differences, but add in an event to stop calculations
   - The key is that the ODE should stop at the $z_{\max}$ when $\gamma(z_{\max}) \approx g$. Checking that this halts while $\frac{\chi}{2} w_h(z_{\max}) < g$, will ensure that singularities in the next step don't occur.
   - In matlab, on can add to the ODE settings of 'Events'. See `http://www.mathworks.com/help/matlab/ref/odeset.html#f92-1017470`
3. For the given g and $w_i(z)$, calculate
   - $\gamma(z)$ from (D.153)
   - $w_i^e(z) \equiv e^z w_i(z)$ from multiplying the results.
   - $\hat{w}_i(z)$ by doing a cumulative integral of $e^z w_i(z) = w_i^e(z)$, defined on $z \in [0, z_{\max}]$. Matlab can use `cumtrapz`, which will be accurate enough for large numbers of grid points in the ODE.
   - Use some simple interpolation of the results for $\gamma(z)$, $\hat{w}_i(z)$ and $w_i^e(z)$. With a large number of grid points, linear interpolation is likely sufficient and is consistent with the cumulative trapezoidal integration above.

E.2.2 Combined Quadrature and Projection Method Setup

As the support is infinite, and the integral in the value-matching condition is taken over $[0, \infty)$, the chosen polynomial basis could be Laguerre polynomials with Gauss-Laguerre quadrature.[42]

Define the approximations with $N-1$ order polynomials, $T_k(z)$ adapted to the support $[0, z_{\max})$, where $z_{\max} = \infty$ if Laguerre polynomials are used. Then, assume that the cdfs can be approximated by,

$$F_i(z) \approx \sum_{k=0}^{N-1} d_{ik} T_k(z) \tag{E.30}$$

Denote the vectors of coefficients as d_i. Define the polynomial nodes as $z_1, \ldots z_{N-1}$, and the complete set of nodes as

$$\vec{z} \equiv \{0, z_1, \ldots z_{N-1}, N\} \in \mathbb{R}^{N+1} \tag{E.31}$$

[42] Another approach is to choose a large $z_{\max}$ to form the support as $[0, z_{\max}]$ and use Chebyshev polynomials and Gauss-Chebyshev quadrature. This is a reasonable first approximation, but a finite bounded integration of the value matching condition over a power law can lead to significant bias. Furthermore, Lagurerre polynomials have clustering of nodes closer to the minimum of support, which has the highest curvature in this model.

The set of interior nodes to be used by quadrature is

$$\vec{z}_q \equiv \{z_1, \dots z_N\} \in \mathbb{R}^N \tag{E.32}$$

The set of interior nodes for the KFE ODE to hold is

$$\vec{z}_I \equiv \{0, z_1, \dots z_{N-1}\} \in \mathbb{R}^N \tag{E.33}$$

Define the basis matrices for these nodes, B, by stacking stacking the evaluated polynomials at each node, Similarly, define B, B', B_q, B'_q, for the subsets and the derivatives [43]

$$B \equiv \begin{bmatrix} T(z_0) \\ \dots \\ T(z_{N-1}) \end{bmatrix} \in \mathbb{R}^{N\times N} \tag{E.34}$$

$$B' \equiv \begin{bmatrix} T'(z_0) \\ \dots \\ T'(z_{N-1}) \end{bmatrix} \in \mathbb{R}^{N\times N} \tag{E.35}$$

$$\vec{F}'_i \equiv \left\{F'_i(z_j)\right\}_{j=0}^{N-1} = B' \cdot d_i \in \mathbb{R}^N \tag{E.36}$$

Define the slice of the basis to interior nodes as B_I, then

$$B_I \equiv B_{1:N-2} = \{B_{jk}\}_{j=1}^{N-2} \tag{E.37}$$

$$B_q \equiv B_{1:N-1} = \{B_{jk}\}_{j=1}^{N} \tag{E.38}$$

$$B'_I \equiv B'_{1:N-2} \tag{E.39}$$

$$B'_q \equiv B'_{1:N-1} \tag{E.40}$$

Given the Chebyshev or Laguerre quadrature weights ω on $\vec{z}_q$, and the quadrature rule,

$$\int_0^{z_{\max}} h(z)\mathrm{d}z \approx \omega \cdot \vec{h} \tag{E.41}$$

Build the completed weights for quadrature, using the ω from (E.41) and combine all static elements of the value-matching integral as,

$$\Omega \equiv \omega \times w_\ell^e(\vec{z}_q) \tag{E.42}$$

E.2.3 Stationary KFE

Define $x \equiv \{d_\ell, d_h\} \in \mathbb{R}^{2N}$ and an approximate operator, $\mathcal{L} : R^{2N} \to R^{N_d}$ for some $N_d \geq 2N$.

First setting up some calculations of the values given the x_d and then implementing the equations,

[43] With `compecon`, the basis can be calculated for nodes `all_nodes` and function space `fspace` as `B = funbas(fspace, all_cheb_nodes, 0)`.

The basis of the derivative is simply `B_p = funbas(fspace, all_cheb_nodes, 1)`. With compecon, to calculate the quadrature weights for Chebyshev quadrature use `qnwcheb(N - 1,0,z_bar)`

Setup: From x_d, find,

$$F_\ell \equiv B \cdot d_\ell \tag{E.43}$$
$$F_h \equiv B \cdot d_h \tag{E.44}$$
$$F'_\ell \equiv B' \cdot d_\ell \tag{E.45}$$
$$F'_h \equiv B' \cdot d_h \tag{E.46}$$
$$\hat{F} \equiv (F_\ell + F_h)^\kappa \tag{E.47}$$
$$S_\ell \equiv gF'_{\ell 0} \tag{E.48}$$
$$S_h \equiv gF'_{h0} \tag{E.49}$$

Equation 1 **to** 2**:** From (D.158), the initial values are

$$0 = F_{\ell 0} \tag{E.50}$$
$$0 = F_{h0} \tag{E.51}$$

Equation 3**:** From (D.141) boundary value,

$$0 = F_{\ell N+1} + F_{hN+1} - 1 \tag{E.52}$$

Equation 4**:** From (D.162), (E.12) and (E.13), the value matching becomes

$$0 = \Omega \cdot (\mathbf{1} - \hat{F}_q) - \zeta \tag{E.53}$$

Equation 5 **to** $N+5$**:** From (D.160) evaluating F_ℓ evaluated at the z_I nodes,

$$0 = gF'_{\ell I} + \lambda_h F_{hI} - \lambda_\ell F_{\ell I} + (S_\ell + S_h)\hat{F}_I - S_\ell \tag{E.54}$$

Equation $N+6$ **to** $2N+5$**:** From (D.161), evaluated at the z_I nodes

$$0 = (g - \gamma_I) \times F'_{hI} + \lambda_\ell F_{\ell I} - \lambda_h F_{hI} - S_h \tag{E.55}$$

Solve for the x root of this approximate operator.

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