Moral Hazard, Informed Trading, and Stock Prices☆

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Abstract

We analyze a dynamic model of informed trading where an activist shareholder accumulates shares in an anonymous market and then expends costly effort to increase the firm value. We find that equilibrium prices are affected by the position accumulated by the activist, because the level of effort undertaken is increasing in the size of his acquired position. In equilibrium, price impact has two components: one due to asymmetric information (as in Kyle (1985) model) and one due to moral hazard (a new source of adverse selection). Price impact is higher when the activist is more productive and when uncertainty about activist's position is large. We thus obtain a trade-off: with more noise trading (less 'price efficiency') the activist can build up a larger stake, which leads to more effort expenditure and higher firm value (more 'economic efficiency'). The model implies that ownership disclosure rules tend to improve market liquidity and economic efficiency and help differentiate productive activists from stock pickers. Shortening the period during which activists can trade, however, hurts economic efficiency.

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1. Introduction

Activist shareholders play an active role in modern corporate governance. Among the most active and visible activists are outside shareholders who identify a firm with the potential for value creation, purchase a significant number of shares in the open market and then publicly announce their intention to influence management.[1] The empirical literature suggests that these activists are often successful in increasing the value of targeted companies (e.g., Brav, Jiang, Partnoy, and Thomas, 2008). The key ingredient to the success of outside activists is their ability to purchase shares in the open market before stock prices reflect their intention to become active, and therefore to increase firm value. The value created presumably depends on the activist's effort expenditure, which in turn depends on the size of the stake that was acquired during the trading period. The larger the stake the higher the incentives for the insider to provide additional effort. Thus, there is a fundamental link between market conditions that influence the block acquisition process, the activist's final position, the activist's effort expenditure, and the firm's value.

In his seminal contribution, Kyle (1985) derives the equilibrium price dynamics in a model where a large trader possesses long-lived private information about the value of a stock that will be revealed at some known date, and optimally trades into the stock to maximize his expected profits. Risk-neutral market makers try to infer from aggregate order flow the information possessed by the insider. Because order flow is also driven by uninformed 'noise traders,' who trade solely for liquidity purposes, prices are not fully revealing. Instead, prices respond linearly to order flow. In Kyle's model the insider has private information about a value that is independent of his actions. In other words, after

[1] Activist hedge funds constitute the core of this group. Carl Icahn, who manages several activist hedge funds, is one prime example of such outside activist shareholder. Activist shareholders also include long-term shareholders who own large stakes of a company, monitor management, and influence management in various ways. Pension funds, mutual funds, and insurance companies are among prominent representatives of this group.

the start of the trading game the insider cannot affect the liquidation value. Whether or not the insider chooses to trade in the stock, this value will be realized.

In this paper we generalize Kyle's model (in the continuous time formulation given by Back, 1992) of informed trading to an activist who can affect the liquidation value of the firm by expending effort at some cost. We solve for the optimal effort level and trading strategy of the activist, as well as for the equilibrium price and corresponding market liquidity. The key feature of the model is that the activist's optimal effort level is increasing in the size of the stake he has accumulated. The endogenous liquidation value of the firm is thus a function of the position accumulated by the insider, which itself depends on the noise trading activity (market liquidity). Unlike in the original Kyle/Back model where the position of the insider is irrelevant for the equilibrium price, in our setting the market maker's estimate of (and uncertainty about) the position of the activist affects the equilibrium price. Therefore, price dynamics are more complex than in the standard Kyle/Back model.

First, in equilibrium, price impact has two components: an asymmetric information component due to the fact that the activist has better information about the exogenous component of the liquidation value (his 'stock picking' ability similar to Kyle's model) and a moral hazard component due to the fact that the position accumulated by the activist will affect the endogenous component of the liquidation value through his effort level (his 'activism' ability). While total price impact (measured by the response of price to order flow) is constant in our model, each component fluctuates over time.

In the earlier stage of the game the dominant part of the price impact is typically due to the exogenous asymmetric information component. In contrast, closer to the end of the trading period the dominant part of the price impact is due to the moral hazard component. The intuition is that the uncertainty about the position of the activist, which governs the moral hazard component of price impact, tends to grow over time in the early part of the trading game, whereas the uncertainty about the exogenous component of the liquidation value always decreases over time.

Second, in equilibrium price impact is higher the more severe the moral hazard problem, which corresponds to cases where the activist is more productive (i.e., can more influence the terminal value), or where the uncertainty about his position is larger. This is a source of price impact that has not, to the best of our knowledge, been studied in previous models.[2] Interestingly, and unlike in the Kyle-Back model, price impact does not go to zero as the prior uncertainty goes to zero. This reflects the fact that even if there is no prior uncertainty about the position of the insider or the fundamental value of the firm, ex-post there will always be uncertainty about the shares accumulated by the insider and therefore some uncertainty about the "endogenously" created value. This implies that price impact cannot go to zero, but instead must reflect the productivity of the insider.[3]

Third, the equilibrium may not fully reveal the components of the liquidation value at maturity. If there is no uncertainty about the exogenous component of firm value and all adverse selection comes from moral hazard, then the market will learn perfectly the value created by the activist. If however, there is uncertainty about the exogenous component of firm value, then there is some 'signal jamming': the market has difficulty separating its estimate of the exogenous and endogenous components of firm value. This seems important given the debate about the social 'usefulness' of activists. Are they simply better stock-pickers or do they really create value for minority shareholders by expending

[2]For example, in Kyle-Back's original model the prior distribution of the initial position of the insider is irrelevant to the equilibrium given a prior for the exogenous component of the liquidation value. In contrast, in our paper information about the activist's position is directly relevant for market liquidity. This also has implications for disclosure regulations which we discuss further below.

[3]This also has interesting consequences for the equilibrium when there are multiple insiders who compete with each other, relative to the standard Kyle-Back setup. This is further investigated in Collin-Dufresne and Fos (2015).

effort? The model suggests that this is very difficult for the market to sort out, if there is uncertainty about the activists holdings.

Fourth, the model shows how ownership disclosure regulation can affect economic efficiency. The model suggests that if we can force the activist to disclose his holdings, i.e., reduce uncertainty about his position, then this makes it easier for the market to sort out his 'stock-picking' from his 'activism' abilities. From that perspective, such ownership disclosure requirements may be useful. Moreover, this also enhances economic efficiency, as it leads them to accumulate more shares and thus expend more effort on average. The model also informs the debate about the optimal duration of the pre-disclosure period (e.g., Bebchuk, Brav, Jackson, and Jiang, 2013). The model shows that shortening the period during which the activist can trade anonymously may have negative consequences for economic efficiency, as it leads the activist to accumulate fewer shares and therefore expend less effort.

Finally, we obtain a trade-off between economic efficiency and price efficiency. With more noise trading (less price efficiency) the activist can build up a larger stake, and then expend more effort to increase firm value, thus leading to higher economic efficiency.

The paper concludes with testing two key predictions of the proposed model. The empirical analysis is based on a novel hand-collected data on trades by Schedule 13D filers (see Collin-Dufresne and Fos, 2014, for detailed description of the dataset) and Schedule 13F filing data on common stock holdings. First, we use data on trading strategies of activist shareholders and show that an activist's trading strategy depends not only on the 'valuation gap', but also on his stake size after controlling for price. This is in contrast to the Kyle (1985) model, in which the trading strategy of the informed trader depends on the 'valuation gap' only. Second, our model predicts a positive relation between shareholders' activism abilities and stock illiquidity. Using data on common stock holdings, we show that stock illiquidity increases when the proportion of shares owned by activist hedge funds increases.

Related Literature

This paper is related to several strands of literature.

First, the paper contributes to the microstructure literature. To the best of our knowledge, we are the first to endogenize the terminal firm value. For example, in Kyle (1985) the terminal firm value is exogenous. Similarly, the literature has maintained the assumption of an exogenous terminal firm value (e.g., Glosten and Milgrom, 1985; Easley and O'Hara, 1987; Back, 1992).

Second, the paper contributes to the corporate governance literature which investigates the role of activist shareholders in monitoring the management.[4] To the best of our knowledge, there is no dynamic model which incorporates anonymous trading by an informed investor who can endogenously change the firm value (i.e., an activist). Shleifer and Vishny (1986) analyze the role of a large minority shareholder in solving the Grossman and Hart (1980a) free-rider problem and show that the large shareholder's return on his own shares suffices to cover his monitoring and takeover costs. In their static model firm value is affected by effort expenditure by the large shareholder, which is increasing in the large shareholder's stake. Admati, Pfleiderer, and Zechner (1994) develop a static model in which a large shareholder has access to a costly monitoring technology affecting securities' expected payoffs. In their model the large shareholder trades off the benefits (more profits from monitoring) and costs (more losses from risk-sharing) of owning a large stake. Similarly to Shleifer and Vishny (1986), monitoring by the large shareholder is costly and is more efficient the higher his stake in the company. DeMarzo and Urosevic (2006) analyze a dynamic market with a blockholder whose actions affect corporate value.

A key distinction between these papers and ours is that the common feature of these

[4]See Edmans (2013) for survey of blockholders and corporate governance literature.

models is that the large shareholder cannot increase his stake by trading anonymously. For example, DeMarzo and Urosevic (2006) assume a full information rational expectations equilibrium. This is the main innovation of our paper, in which the activist can trade in anonymous markets to change his stake size.[5]

We are aware of only one paper that studies a dynamic model with a shareholder who trades in anonymous markets and whose actions affect corporate value. In contemporaneous and independent work Back, Li, and Ljungqvist (2014) also solve a dynamic Kyle model with a shareholder who can expend costly effort to increase firm value. Their version of the Kyle model differs from ours in several respects. First, similarly to Maug (1998), their model assumes that the shareholder can change firm value by an exogenously given amount. In our model, the amount of value creation is endogenously determined by the activist. That is, in their model the activist decides whether to intervene whereas in our model the activist decides how much effort to expend. Second, our model assumes two types of private information: the exogenous component as in Kyle (1985) and the moral hazard component. This allows to distinguish within our model activists who create value from pure stock pickers and to analyze the implications of ownership disclosure rules on market liquidity and economic efficiency. Lastly, Back et al. (2014) follow Stoughton and Zechner (1998) in analyzing IPO mechanisms and argue that the initial inventory of the blockholder should be large. Therefore, this framework seems more suitable to analyze monitoring by long term blockholders (e.g., private equity and pension funds). In contrast, our model also captures activist shareholders who identify

[5]An exception is Noe (2002), who develops a static model in which strategic investors produce private information through their own action. Monitoring induces a fixed private cost to the investor. The microstructure set up of the model is special, as the market maker does not condition quotes on aggregate order flow (see his discussion on page 311). The main reason for making this simplifying assumption is that there is no need to update market maker's prior beliefs, which is non-trivial when the terminal value of stock is endogenous. While in our dynamic model the terminal value is endogenous, we are able to solve the model when the market maker conditions prices on the order flow.

potential targets, accumulate shares in anonymous markets, and then intervene (such as Carl Icahn for example).[6]

Third, the paper contributes to the literature that studies the role of noise trading as a solution to the Grossman and Hart (1980a) free-rider problem. Kyle and Vila (1991) develop a static model in which noise trading provides camouflage that helps an outside shareholder to purchase enough shares at favorable prices so that takeovers become profitable. The main result is that noise trading has a positive impact on profits of large shareholder but has undetermined effects on economic efficiency. Specifically, noise trading encourages fewer (more) takeovers when takeovers would otherwise always (never) occur. However, unlike in our paper, Kyle and Vila (1991) assume that the takeover premium the large shareholder must pay to takeover the firm is exogenous and that the initial stake of the large shareholder is exogenously given, which essentially avoids the moral hazard problem we study. Maug (1998) endogenizes the initial stake of the large shareholder in a static model and shows that market liquidity mitigates the free-rider problem by allowing the informed activist shareholder to purchase shares at a discount from uninformed shareholders. If the activist intervenes, firm value increases by an exogenously given amount. The main innovation of our model is that we study the dynamic relation between noise trading, share accumulation and effort expenditure by the activist.

Fourth, the paper contributes to the literature that studies price informativeness and its effect or real economy. In the seminal paper of Grossman and Stiglitz (1980), market

[6] Our two papers also differ in their empirical analysis. The analysis in our paper is based on activism events that closely correspond to the class of corporate governance mechanisms described. Specifically, we use data on trading strategies of activist investors first introduced by Collin-Dufresne and Fos (2014) and which correspond to initial filers. In addition, we use common stock holdings data from Schedule 13F and show that stock illiquidity is positively associated with shareholders' activism abilities. Back et al. (2014) center their empirical analysis on the role of stock liquidity in predicting governance events which may be executed by long term shareholders who own a significant block at the beginning of the trading period.

participants trade off costs and benefits of becoming informed. In equilibrium, prices depend on the relative weight of informed traders. Firm value, however, is exogenously determined and is not affected by traders' decision to become informed. Our paper suggests a trade off between economic efficiency (i.e., effort expenditure by activist shareholder) and price efficiency. We show that noise traders contribute to economic efficiency by increasing the activist's optimal stake and therefore effort expenditure but have a negative impact on price efficiency by keeping prices from converging to their terminal value.[7]

Finally, the paper is related to the literature that studies optimal disclosure (e.g., Grossman and Hart, 1980b; Fishman and Hagerty, 1992, 1995). Our paper shows that information on activists' positions can be valuable and significantly affect market liquidity, price efficiency and economic efficiency. Indeed, in our model more disclosure about activists positions may improve market liquidity (i.e., reduce price impact) and enhance economic efficiency.

2. Informed Trading with Hidden Action: One Period Model

To highlight the key ingredients of the model, we first solve a simple one-period model and then develop a fully dynamic model of informed trading where an activist shareholder continuously accumulates shares in an anonymous market and then expends costly effort to increase the firm value.

The main new feature of our model relative to Kyle (1985) is that we assume the activist trading in the stock can choose to exert effort w paying a cost $C(w) = \frac{w^2}{2\psi}$ to

[7] This also links our paper to the literature that studies the role of secondary financial markets of the real economy (Bond, Edmans, and Goldstein, 2012). The literature has analyzed the impact of information from secondary market prices on real decisions as well as the impact of the information on decision makers' incentives. For example, Edmans, Goldstein, and Jiang (2012) document a strong effect of market prices on takeover activity.

produce the terminal (liquidation) value of $v+w$ where v is a constant known only to him and w is a choice variable. Timing wise, we assume that the activist chooses his effort level right 'after' the trades are settled. The parameter ψ captures the productivity of the activist. The higher ψ the more productive the activist.[8] We assume that the Market has an initial prior about $v \sim N(V_0, \sigma_v^2 T)$, where $T = 1$ in the one period model. For simplicity of interpretation we focus on the case where $v > V_0$ and the activist typically accumulates a positive number of shares and chooses a positive effort level w. It is natural to think of this activist as a hedge-fund activist for example.

The second important departure from the original Kyle model is that we assume the activist may start with an initial position X_0 which is known only imperfectly by the Market. Indeed we assume the Market has an initial Gaussian prior $X_0 \sim N(Q_0, \sigma_X^2 T)$ which is correlated with its estimate of v and we denote their covariance by $\Sigma_{Xv}(0) = \sigma_{Xv}T$, where $T = 1$ in the one period model.[9] This allows us to interpret the initial date 0 as the date at which the activist becomes informed or as the date at which the market becomes aware of the existence of the activist.[10] We note that the original Kyle model can easily be extended to random initial endowment in stocks by the activist, since the latter plays no role in the equilibrium and indeed, the equilibrium price and market liquidity are unaffected by it. This underlines the difference between both models.

The market maker will set prices so as to break even, i.e., such that his expected profits are zero. The market maker only observes total order flow $Y = \theta + u$, where $u \sim N(0, \sigma_u^2)$ is uninformed noise trading which we assume to be normally distributed. The activist is

[8]Note that if $\psi \to 0$ it becomes optimal to choose $w = 0$ and the problem becomes identical to the original Kyle model.

[9]The only technical requirement we impose is that the prior Covariance matrix be positive definite.

[10]Of course, this is not fully consistent with rational expectation, in that such an interpretation ignores the fact that the fully rational market maker should have been aware prior to date 0 of the possibility of such information becoming available to the activist. Modeling that is possible along the lines of Li (2012), but we leave such an extension for future work.

risk-neutral and maximizes his expected terminal profit:

$$\max_{\theta,w} \mathrm{E}\left[(v + w - P_1)\theta + (w + v - \overline{P_0})X_0 - C(w)|v, \overline{P_0}, X_0\right], \tag{1}$$

where $\overline{P_0}$ denotes the average price the activist paid for his initial stake X_0. We note that unlike in the original Kyle model, it is important to condition on the initial position of the activist shareholder, which plays an important role in our setting. For simplicity, we assume that the discount rate is zero. Because of the assumed timing, the activist's optimal choice of w will maximize:

$$\max_{w} w(\theta + X_0) - C(w), \tag{2}$$

With our choice of cost function this leads to $w^* = \psi(\theta + X_0)$. Plugging back into his objective function we see that the activist is maximizing:

$$(v - \overline{P_0})X_0 + \max_{\theta} \mathrm{E}\left[(v - P_1)\theta + \frac{\psi(\theta + X_0)^2}{2}|v, X_0\right]. \tag{3}$$

We look for a linear equilibrium where price responds linearly to order flow $P_1 = P_0 + \Delta Y$. We obtain (after dropping the constant term and taking expectation):

$$\max_{\theta} v\theta - (P_0 + \Delta\theta)\theta + \frac{\psi(\theta + X_0)^2}{2}. \tag{4}$$

The FOC w.r.t. θ is $v - P_0 - 2\Delta\theta + \psi(\theta + X_0) = 0$, leading to:

$$\theta^* = \frac{v - P_0 + \psi X_0}{2\Delta - \psi}. \tag{5}$$

We see that the optimal trading strategy (assuming a linear price order flow relation) is linear of the form:

$$\theta = \beta\left(v + \psi X_0 - P_0\right) = \alpha + \beta v + \gamma X_0, \tag{6}$$

where $\beta = \frac{1}{2\Delta - \psi}$, $\alpha = -\frac{P_0}{2\Delta - \psi} \equiv -\beta P_0$, and $\gamma = \frac{\psi}{2\Delta - \psi} \equiv \psi\beta$.

Next we show that if the trading strategy is linear of the form (6), then the price order flow relation is indeed linear.

Lemma 1. *If the activist adopts a trading strategy of the form given in (6), then the price order flow relation is linear and of the form $P_1 = P_0 + \Delta Y$ with:*

$$\Delta = \frac{(1+\psi\beta)\beta\sigma_v^2 + \psi(\gamma+1)\gamma\sigma_X^2 + \left(\gamma + \psi\beta(2\gamma+1)\right)\sigma_{Xv}}{\beta^2\sigma_v^2 + \gamma^2\sigma_X^2 + 2\beta\gamma\sigma_{Xv} + \sigma_u^2} \tag{7}$$

and

$$P_0 = \psi\alpha + V_0(1+\psi\beta) + \psi(\gamma+1)Q_0) - \Delta(\alpha + \beta V_0 + \gamma Q_0). \tag{8}$$

Proof 1. *See Appendix.*

It remains to find the fixed point solution for α, β, γ, δ, and P_0, if it exists. Using $\gamma = \psi\beta$ and defining $\omega^2 \equiv \sigma_v^2 + 2\psi\sigma_{Xv} + \psi^2\sigma_X^2$ we obtain $\Delta = \beta\omega^2(1+\psi\beta)/\left(\beta^2\omega^2 + \sigma_u^2\right)$. From $\beta = \frac{1}{2\Delta - \psi}$ we obtain $2\beta\Delta = 1 + \beta\psi$. After substituting it in Δ, we find $\beta = \frac{\sigma_u}{\omega}$. Then we immediately obtain $\alpha = -\beta P_0$, $\gamma = \frac{\psi\sigma_u}{\omega}$, $P_0 = V_0 + \psi Q_0$, and:[11]

$$\Delta = \frac{1}{2}\left(\psi + \frac{\omega}{\sigma_u}\right). \tag{9}$$

The model provides several insights into the relation between shareholder activism and financial markets. First, the total price impact is increasing in the productivity of the activist (ψ). The market maker is concerned not only about the activist having some private information, but also about the activist changing firm value after having aquired

[11] Note that the second order condition in activist's maximization problem is satisfied. The SOC w.r.t. θ is $-2\Delta + \psi = -\frac{\omega}{\sigma_u} < 0$.

shares. Second, price impact is increasing in the signal to noise ratio $\frac{\omega}{\sigma_u}$, which implies that it is increasing in the uncertainty about his position (Σ_X) as well as about the exogenous component of asset value Σ_v. Thus, ownership disclosure in as much as it reduces Σ_X will have implications for market liquidity and value creation. Interestingly, we see that in the limit where prior uncertainty goes to zero, price impact remains positive equal to $\psi > 0$. This reflects the fact that even in the limit where there is no ex-ante prior uncertainty, the possibility of aquiring shares anonymously creates the possibility of ex-post uncertainty and therefore price-impact even in the limit must remain positive. Third, the activist's trading strategy depends not only on the exogenous component of the firm value, but also on his initial stake. This is because the position plays a key role in determining the activist's optimal effort expenditure. Fourth, the price reflects not only the market maker's best estimate of the exogenous component, but also the expected value of the increase in firm value due to effort expenditure by the activist, which depends on his current position.

How does the trading strategy of the activist change over time? What are equilibrium price dynamics? Does illiquidity (price impact) change when insiders can trade more frequently? What is the source of value creation by activists? How is the information about their future actions incorporated into prices? What is the role of ownership disclosure rules for activists and economic efficiency? Unfortunately, a static model is not suitable for addressing these questions. We next develop a dynamic version of the model, where we can address these questions.

3. Informed Trading with Hidden Action: Continuous Time Model

The continuous time version of our model is based on the Back (1992). As before, we assume the activist trading in the stock can choose to exert effort w paying a cost $C(w) = \frac{w^2}{2\psi}$ to produce the terminal (liquidation) value of $v + w$ where v is a constant known only to him and w is a choice variable. The Market has an initial Gaussian prior $X_0 \sim N(Q_0, \sigma_X^2 T)$ which is correlated with its estimate of v and we denote their covariance by $\Sigma_{Xv}(0) = \sigma_{Xv} T$. The activist is risk-neutral and maximizes his expected terminal profit:

$$\max_{\theta_t \in \mathcal{A}, w} \mathrm{E}\left[\int_0^T (v + w - P_t)\theta_t dt + (v + w - \overline{P_0})X_0 - C(w) \,|\mathcal{F}_t^Y, v, X_0, \overline{P_0}\right], \tag{10}$$

where we denote by $\mathcal{F}_t^Y$ the information filtration generated by observing the entire past history of aggregate order flow $\{Y_s\}_{s \le t}$. Timing wise, we assume that the activist chooses his effort level right 'after' the terminal date T, given all past information on prices and trades.[12] In particular, his optimal choice will simply maximize at T:

$$\max_w wX_T - C(w), \tag{11}$$

where $X_T = X_0 + \int_0^T \theta_t dt$ is the activist's initial stake plus the stake accumulated during the trading game by the activist. With our choice of cost function this leads to:

$$w^* = \psi X_T. \tag{12}$$

Plugging back into his objective function we see that the activist is maximizing:

$$(v - \overline{P_0})X_0 + \max_{\theta_t \in \mathcal{A}} \mathrm{E}\left[\int_0^T (v - P_t)\theta_t dt + \frac{\psi X_T^2}{2} \,|\mathcal{F}_t^Y, v, X_0\right]. \tag{13}$$

Following Back (1992) we assume that the activist chooses a trading rule θ in some admissible set $\mathcal{A}$ defined to be the set of absolutely continuous trading strategies which

[12] A richer model might allow the activist to work continuously during the share accumulation phase on the project that eventually will lead to the liquidation value. We leave such a model for future work.

satisfy the technical restriction that $\mathrm{E}[\int_0^T |\theta_s|^2 ds] < \infty$.[13]

The market maker is risk-neutral, but does not observe the terminal value. Instead (given his prior) he only observes the aggregate order flow, which is the sum of informed and uninformed order flow:

$$dY_t = \theta_t dt + \sigma dZ_t, \tag{14}$$

where Z_t is a standard Brownian motion independent of υ. We assume that the uninformed order flow volatility, σ, is constant.[14]

To solve for an equilibrium, we proceed as follows. First, we derive the dynamics of the stock price consistent with the market maker's risk-neutral filtering rule, conditional on a conjectured trading rule for the activist. Then we solve the activist's optimal portfolio choice problem, given the assumed dynamics of the equilibrium price. Finally, we show that the conjectured rule by the market maker is indeed consistent with the activist's optimal choice.

Since the market maker is risk-neutral, equilibrium imposes that

$$P_t = \mathrm{E}\left[\upsilon + w \,|\, \mathcal{F}_t^Y\right]. \tag{15}$$

Since the market maker is rational he knows the optimal choice of effort by the activist will be a function of his acquired stake, i.e, that $w = \psi X_T$. The market maker will thus

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[13]A shown in Back, it is optimal for the activist to choose an absolutely continuous trading strategy, since, in continuous time, the market maker can immediately infer from the quadratic variation of the order flow the informed component with infinite variation. The square integrability condition is a technical requirement often used in continuous time to rule out specific arbitrage strategies such as 'doubling strategies' (see Harrison and Pliska, 1981; Dybvig and Huang, 1988).

[14]We could easily generalize volatility of uninformed order flow to be a function of posterior co-variance $\sigma(\Sigma_t, t)$ as in Baruch (2002) (though at the expense of closed-form solutions). The extension to arbitrary stochastic processes as in Collin-Dufresne and Fos (2013) is non-trivial and left for future research.

filter the position of the activist. We define his estimate of the activist's position as:

$$Q_t = \mathrm{E}\left[X_t \,|\, \mathcal{F}_t^Y\right] \tag{16}$$

and his estimate of the constant component as:

$$V_t = \mathrm{E}\left[\upsilon \,|\, \mathcal{F}_t^Y\right]. \tag{17}$$

It follows from the definition of conditional expectation that:

$$P_t = \mathrm{E}\left[\upsilon + \psi X_T \,|\, \mathcal{F}_t^Y\right] = V_t + \psi \mathrm{E}\left[Q_T \,|\, \mathcal{F}_t^Y\right]. \tag{18}$$

We conjecture that the trading strategy of the activist will be linear in his 'valuation gap' as well as his position gap:

$$\theta_t = \beta_t(\upsilon - V_t) + \gamma_t(X_t - Q_t), \tag{19}$$

where β_t measures the speed at which the activist decides to close the gap between his assessment of the fundamental value υ (known only to him) and the market maker's estimate V_t and where γ_t is the loading on the activist's position estimation error made by the market maker.

The novel feature relative to most of the literature is that the activist does not only trade because of his valuation gap. Instead, his trading decision is also motivated by the accumulated position. Indeed the price has two components to its value: one component that is independent of the action of the activist, and another that depends on his choice of effort, which itself depends on the position he has accumulated. Thus the price depends in equilibrium on the market's estimate of the activist's position in the stock. His accumulated position has dynamics:

$$dX_t = \theta_t dt. \tag{20}$$

Now, given his conjecture about the trading dynamics of the activist, the market maker's optimal filtered price dynamics follows from standard results in the filtering literature (e.g., Liptser and Shiryaev, 2001, Chapter 12). The novel feature, is that the market maker will estimate both the fundamental value v and the position of the activist X_t from the observed aggregate order flow. And, in equilibrium, the price dynamics will be multi-variate Markov.

Let's denote the posterior covariance matrix of the filtered state $S_t = [v; X_t]'$ by Σ_t ($(2, 2)$ matrix):

$$\Sigma(t) = Var[S_t \,|\, \mathcal{F}_t^Y]. \tag{21}$$

Note that:

$$M_t = \mathrm{E}[S_t \,|\, \mathcal{F}_t^Y] = [V_t; Q_t]$$

and for simplicity we introduce the notation:

$$\Sigma_v(t) = \mathrm{E}\left[(v - V_t)^2 \,|\, \mathcal{F}_t^Y\right] \equiv \Sigma_{11}(t) \tag{22}$$

$$\Sigma_X(t) = \mathrm{E}\left[(X_t - Q_t)^2 \,|\, \mathcal{F}_t^Y\right] \equiv \Sigma_{22}(t) \tag{23}$$

$$\Sigma_{Xv}(t) = \mathrm{E}\left[(v - V_t)X_t \,|\, \mathcal{F}_t^Y\right] \equiv \Sigma_{12}(t), \tag{24}$$

with initial conditions $\Sigma_v(0) = \sigma_v^2 T$, $\Sigma_X(0) = \sigma_x^2 T$, and $\Sigma_{Xv}(0) = \sigma_{Xv} T$. $\Sigma_v(0)$ measures the prior uncertainty about the exogenous component of private information available to the activist, $\Sigma_X(0)$ measures the prior uncertainty about the activist's position, and

$\Sigma_{Xv}(0)$ measures their covariance.

A direct application of known results on conditionally Gaussian filtering gives the following Lemma.

Lemma 2. *If the activist adopts a trading strategy of the form given in (19), then the stock price and the filtered position of the activist given by equations (18) and (16) satisfy:*

$$\begin{aligned} dV_t &= \lambda_t dY_t & (25) \\ dQ_t &= \Lambda_t dY_t, & (26) \end{aligned}$$

where λ_t and Λ_t satisfy:

$$\begin{aligned} \lambda_t &= \frac{\beta_t \Sigma_v(t) + \gamma_t \Sigma_{Xv}(t)}{\sigma^2} & (27) \\ \Lambda_t &= \frac{\beta_t \Sigma_{Xv}(t) + \gamma_t \Sigma_X(t)}{\sigma^2}. & (28) \end{aligned}$$

Further, the dynamics of the posterior covariance matrix is given by:

$$\begin{aligned} d\Sigma_v(t) &= -\lambda_t^2 \sigma^2 dt & (29) \\ d\Sigma_X(t) &= \Lambda_t(2 - \Lambda_t)\sigma^2 dt \\ d\Sigma_{Xv}(t) &= \lambda_t(1 - \Lambda_t)\sigma^2 dt \end{aligned}$$

Proof 2. *This follows directly from an application of theorems 12.6, 12.7 in Liptser and Shiryaev (2001). We provide a simple 'heuristic' motivation of the result using the Gaussian projection theorem in the Appendix.*

We now try to solve the activist's partial equilibrium problem taking as given price dynamics:

$$dV_t = \lambda_t(\theta_t dt + \sigma dZ_t) \qquad (30)$$

$$dQ_t = \Lambda_t(\theta_t dt + \sigma dZ_t) \tag{31}$$
$$dX_t = \theta_t dt. \tag{32}$$

Note that if the activist indeed follows the conjectured trading rule, then both V_t and Q_t are martingales in the market maker's filtration. It follows then from the definition of the equilibrium price in equation (18), that the price can be rewritten simply as:

$$P_t = V_t + \psi Q_t \tag{33}$$

and that its dynamics are:

$$dP_t = (\lambda_t + \psi\Lambda_t)dY_t. \tag{34}$$

Following the intuition from Kyle-Back's original model, we conjecture that in equilibrium the total price impact $\lambda_t + \psi\Lambda_t$ will be constant. Indeed, since the activist is risk-neutral, he would otherwise seek to concentrate all his trading in periods with the lowest total price impact. We shall thus construct an equilibrium with this property. Further, we conjecture that in equilibrium the posterior variance of the price should converge to zero, since otherwise the risk-neutral activist could change his trading strategy to take advantage of positive expected return trades. We first prove that there exists a trading strategy that leads to such an outcome.

Lemma 3. *Suppose that the activist chooses his trading strategy as conjectured in (19), with*

$$\gamma_t = \psi\beta_t \quad \textit{and} \quad \beta_t = \frac{\Delta\sigma^2}{\Omega_t}, \tag{35}$$

where

$$\Omega_t = \Sigma_v(t) + 2\psi\Sigma_{Xv}(t) + \psi^2\Sigma_X(t). \tag{36}$$

Then the total price impact due to Bayesian updating is constant:

$$\lambda_t + \psi \Lambda_t = \Delta \tag{37}$$

and

$$\Omega_t = \Omega_0 + (2\psi - \Delta)\Delta\sigma^2 t. \tag{38}$$

Further, there exists a positive $\hat{\Delta}$ *such that* $\Omega_T = 0$ *given by:*

$$\hat{\Delta} = \psi + \sqrt{\psi^2 + \phi^2} \tag{39}$$

where ϕ *is the 'signal to noise ratio' defined as*

$$\phi = \frac{\omega}{\sigma}, \tag{40}$$

where we define the annualized quantity of initial private information $\omega^2 = \frac{\Omega_0}{T} = \sigma_v^2 + 2\psi\sigma_{Xv} + \psi^2\sigma_X^2$*. For this choice of* $\hat{\Delta}$ *the expression for* Ω_t *simplifies:*

$$\Omega_t = \omega^2 (T - t). \tag{41}$$

Proof 3. *Suppose the activist adopts the conjectured trading strategy. Then using equations (27) and (28) we immediately obtain:*

$$\lambda_t + \psi \Lambda_t = \Delta \ \ \forall t.$$

Further, using the dynamics of the covariance matrix in (29) we find that when the activist follows such a strategy:

$$d\Omega_t = (2\psi - \Delta)\Delta\sigma^2 dt.$$

It follows that $\Omega_t = \Omega_0 + (2\psi - \Delta)\Delta\sigma^2 t$*, and thus the equation* $\Omega_T = 0$ *admits two roots for* Δ *one of which is positive and given by* $\hat{\Delta}$ *in the Lemma.*

By definition $\Omega_t = Var[v + \psi X_t \,|\, \mathcal{F}_t^Y]$. For the conjectured equilibrium Ω_t given in (41) decays linearly, which is reminiscent of the dynamics of the posterior variance of the estimated liquidation value in the original Kyle (1985) model. Note however that the dynamics of Ω_t is affected by the prior uncertainty about the position of the activist, which is irrelevant in the original Kyle model (as can be verified by taking the limit $\psi \to 0$).

Indeed, the total price impact $\hat{\Delta}$ obtained in this model is always greater than that obtained in a model without moral hazard (i.e., when $\psi = 0$) in which case it becomes identical to price impact obtained in the KB model $\hat{\Delta}|_{\psi=0} = \frac{\sigma_v}{\sigma}$. In fact, price-impact in our model depends on only two quantities: the productivity of the activist ψ and the signal to noise ratio $\frac{\omega}{\sigma}$. It is increasing in both. Interestingly, the signal to noise ratio relevant for the moral hazard model depends on the prior uncertainty about the activist's position (σ_X, σ_{Xv}) which is irrelevant in the KB model. Interestingly, we see that, as in the one-period model, price impact remains strictly positive equal to $\psi > 0$ even in the limit where prior uncertainty ω goes to zero. This reflects the fact that even in the limit where there is no ex-ante prior uncertainty, the possibility of acquiring shares anonymously creates the possibility of ex-post uncertainty and therefore price-impact cannot go to zero when prior uncertainty goes to zero. Further, comparing the price impact in the dynamic model we see that it is more than twice the price impact in the one-period model when $\psi > 0$ (note that it is exactly twice as large as in the one period model when $\psi = 0$, i.e., in the Kyle-Back model).

Further, note that Ω_t is not equal to the posterior variance of the liquidation value (the latter is $Var[v + \psi X_T \,|\, \mathcal{F}_t^Y]$). Instead, Ω_t can be interpreted as the variance of the liquidation value of the stock if it were liquidated at the present time t. Indeed, Ω_t converges at maturity to the variance of the terminal stock price liquidation value. In fact, we have that $\Omega_T = \mathrm{E}[(v + \psi X_T - P_T)^2 \,|\, \mathcal{F}_T^Y]$. And since the previous result shows that (for the conjectured equilibrium) $\Omega_T = 0$, this suggests that we should obtain a convergence result for the equilibrium price also in the filtration of the activist. Indeed, we can prove the following.

Lemma 4. *Suppose the trading strategy followed by the activist is as described in Lemma 3. Then, in the filtration $\mathcal{F}_t$ of the activist, the equilibrium price process is two-factor Markov in state variables P_t and X_t and is given by:*

$$dP_t = \frac{(\hat{\Delta}\sigma)^2}{\omega^2(T-t)}(v+\psi X_t - P_t)dt + \hat{\Delta}\sigma dZ_t \tag{42}$$

$$dX_t = \frac{(\hat{\Delta}\sigma)^2}{\omega^2(T-t)}(v+\psi X_t - P_t)dt. \tag{43}$$

In the filtration of the activist, the price process P_t converges in L^2 to $v+\psi X_T$ at maturity. Note that in the filtration of the market maker, the stock price P_t is a Brownian martingale.

Proof 4. *See Appendix.*

We now turn to the optimal policy of the activist. We want to show that given the conjectured equilibrium price impact process, the strategy conjectured by the market maker is indeed a best response for the activist. Note that the expected profits of the activist given in equation (13) can be rewritten as follows:[15]

$$(v-\overline{P_0})X_0 + \psi\frac{X_0^2}{2} + \max_{\theta_t\in\mathcal{A}} \mathrm{E}\left[\int_0^T (v-P_t+\psi X_t)\theta_t dt \,|\mathcal{F}_t^Y, v, X_0\right]$$

Since the first two terms do not affect his choice,[16] we define the following value function that captures the optimization problem of the insider:

$$J(t) = \max_{\theta_s\in\mathcal{A}} \mathrm{E}\left[\int_t^T (v-P_s+\psi X_s)\theta_s ds \,|\mathcal{F}_t^Y, v, X_t\right]. \tag{44}$$

[15] This follows from the fact that $\frac{X_T^2}{2} = \frac{X_0^2}{2} + \int_0^T X_t dX_t = \frac{X_0^2}{2} + \int_0^T X_t\theta_t dt$.

[16] Note that the sum of these two terms is equal to the expected profit of the insider were he to not accumulate any additional share after time 0 and then provide the optimal effort at maturity (given that $X_T = X_0$).

We show the following.

Proposition 1. *Suppose that prices have dynamics*

$$dP_t = \Delta dY_t \tag{45}$$

for some constant Δ. *Suppose further that there exists an admissible trading strategy* θ^* *such that* $\mathrm{E}[(P_T - v - \psi X_T)^2] = 0$, *then* θ^* *is an optimal trading strategy and the optimal value function is given by:*

$$J(P, X, t) = \frac{(P - v - \psi X_t)^2 + \Delta^2\sigma^2(T - t)}{2(\Delta - \psi)}. \tag{46}$$

Proof 5. *Consider the function*

$$J(P, X, t) = \frac{(P - v - \psi X_t)^2 + \Delta^2\sigma^2(T - t)}{2(\Delta - \psi)}. \tag{47}$$

Applying Itô's lemma we find:

$$\frac{(P_T - v - \psi X_T)^2}{2(\Delta - \psi)} - J(P_0, X_0, 0) + \int_0^T (v + \psi X_t - P_t)\theta_t dt = \int_0^T \frac{(P - v - \psi X)}{(\Delta - \psi)}\Delta\sigma dZ_t.$$

First, note that the right hand side is a square integrable martingale for any admissible trading strategy. Indeed, note that any admissible trading strategy satisfies $\mathrm{E}[\int_0^T X_t^2 dt] < \infty$. *(To see this simply note that by the Cauchy-Schwartz inequality* $X_t^2 = (\int_0^t \theta_s ds)^2 \leq \int_0^t \theta_s^2 ds$.*) Then note that* $P_t = \Delta(X_t - X_0) + \Delta\sigma(Z_t - Z_0)$ *and thus* $E[\int_0^T (P_t - v - \psi X_t)^2 dt] < \infty$ *for any admissible trading strategy.*

Thus, taking expectations it follows that for any admissible trading strategy θ_t *we have:*

$$J(P_0, X_0, 0) = \mathrm{E}\left[\frac{(P_T - v - \psi X_T)^2}{2(\Delta - \psi)} + \int_0^T (v + \psi X_t - P_t)\theta_t dt\right]. \tag{48}$$

Since $\mathrm{E}\left[(P_T - v - \psi X_T)^2\right] \geq 0$*, it follows that for any admissible* θ_t *we have*

$$J(P_0, X_0, 0) \geq \mathrm{E}\left[\int_0^T (v + \psi X_t - P_t)\theta_t dt\right]. \tag{49}$$

And if there exists an admissible trading strategy θ_t^* *such that* $\mathrm{E}\left[(P_T - v - \psi X_T)^2\right] = 0$ *then the (weak) inequality holds with equality. This establishes the optimality of such a trading strategy* θ_t^* *and of the value function.*

Combining the verification theorem with the previous lemmas we have the main result of this paper.

Proposition 2. *There exists an equilibrium characterized by deterministic functions* λ_t *and* Λ_t *such that total price impact is constant:*

$$\lambda_t + \psi\Lambda_t = \hat{\Delta} := \psi + \sqrt{\psi^2 + \frac{\omega^2}{\sigma^2}} \quad \forall t, \tag{50}$$

where $\omega^2 = \sigma_v^2 + 2\psi\sigma_{vX} + \psi^2\sigma_X^2$*. The optimal trading strategy for the activist is:*

$$\theta_t^* = \beta_t(v + \psi X_t - P_t), \tag{51}$$

where $\Omega_t = \omega^2(T - t)$ *and* $\beta_t = \frac{\hat{\Delta}\sigma^2}{\Omega_t}$*. The value function of the activist is:*

$$J(P, X, t) = \frac{(P - v - \psi X_t)^2 + \hat{\Delta}^2\sigma^2(T - t)}{2(\hat{\Delta} - \psi)}. \tag{52}$$

The equilibrium is revealing in that P_t *converges to* $v + \psi X_t$ *at time* T*.*

Proof 6. *Follows immediately from previous results. The admissibility of* θ^* *is easily checked.*

We note that while the dynamics of Ω_t are simple, the separate dynamics of $\Sigma_v(t)$, $\Sigma_X(t)$, and $\Sigma_{Xv}(t)$ are less obvious. Similarly, while the total price impact deriving from both asymmetric information and moral hazard is constant: $\lambda_t + \psi\Lambda_t = \hat{\Delta}$, the individual components λ_t and Λ_t are not. We can however characterize these analytically in closed-form, which provides further insight into the equilibrium.

Lemma 5. *In equilibrium,* $\lambda_t = \lambda_0(\frac{T-t}{T})^{\kappa}$ *with* $\lambda_0 = \frac{\hat{\Delta}}{\omega^2}(\sigma_v^2 + \psi\sigma_{Xv})$ *and* κ *defined in equation (C.6). Thus if* $\lambda_0 = \sigma_v^2 + \psi\sigma_{Xv} > 0$ *(resp.* < 0*) then* λ_t *is positive (resp. negative) and strictly decreasing (resp. increasing) and* $\lim_{t\to T}\lambda_t = 0$.
It follows that if $\lambda_0 > 0$ *(resp.*< 0*) then* $\Lambda_t = \frac{\hat{\Delta}-\lambda_t}{\psi}$ *is strictly increasing (resp. decreasing) and that* $\lim_{t\to T}\Lambda_T = \hat{\Delta}$.
Further, we can solve the system of ODE for all the posterior covariance matrix in closed-form.

$$\Sigma_v(t) = \sigma_v^2 T + \frac{\lambda_0^2\sigma^2\left((T-t)\left(\frac{T-t}{T}\right)^{2\kappa} - T\right)}{2\kappa+1} \tag{53}$$

$$\Sigma_{Xv}(t) = \sigma_{xv}T - \frac{\Sigma_v(t) - \sigma_v^2 T}{\psi} + \frac{\sqrt{\frac{1}{\zeta^2}+1}\lambda_0\sigma^2(T-t)\left(\frac{T-t}{T}\right)^{\kappa}}{\kappa+1} - \frac{\sqrt{\frac{1}{\zeta^2}+1}\lambda_0\sigma^2 T}{\kappa+1} \tag{54}$$

$$\Sigma_X(t) = \left(\Omega(t) - 2\psi\Sigma_{Xv}(t) - \psi^2\Sigma_v(t)\right)/\psi^2 \tag{55}$$

Proof 7. *From its definition* $\lambda_t = \frac{\hat{\Delta}}{\Omega_t}(\Sigma_v(t) + \psi\Sigma_{Xv}(t))$*. Differentiating and using the dynamics of the covariance matrix we obtain* $\dot{\lambda}_t = \frac{\lambda_t\sigma^2}{\Omega_t}(\frac{\omega^2}{\sigma^2} - \hat{\Delta}(\hat{\Delta}-\psi)) = -\frac{\kappa}{\lambda_t}$*, where* κ *has been defined previously. This ODE is easily solved for* λ_t *given its initial condition.*

The results on Λ_t *follow from the fact that* $\Lambda_t = \hat{\Delta} - \psi\lambda_t$*.*

Given the solutions for λ_t, Λ_t *the covariance matrix ODE can be solved explicitly given their initial conditions.*

We can then check the terminal value $\Sigma_v(T)$ *and* $\Sigma_X(T)$ *and observe that* $\Sigma_v(T) = 0 \Leftrightarrow \Sigma_X(T) = 0 \Leftarrow \{\sigma_v = 0 \text{ or } \sigma_X = 0\}$*. This follows immediately from the closed-form solution and using the fact that at any time* t *we have* $\Sigma_{Xv}(t) = 0 \Leftarrow \{\Sigma_v(t) = 0 \text{ or } \Sigma_X(t) = 0\}$*.*

The results show that if either $\sigma_X = 0$ or $\sigma_v = 0$, then $\lim_{t\to T}\Sigma_v(t) = \lim_{t\to T}\Sigma_X(t) =$

0. In other words, if there is prior uncertainty about only one of the two sources of uncertainty (position or exogenous component of asset payoff) then the equilibrium is fully revealing about both the position and the terminal value of the asset (in that both $V_T = v$ and $Q_T = X_T$ at maturity). Instead, if there is prior uncertainty about both sources (both σ_X and σ_v are greater than zero), then the equilibrium reveals the total payoff $v + \psi X_T$ but not the individual components (both $\Sigma_X(T) > 0$ and $\Sigma_v(T) > 0$). In particular, the market cannot infer perfectly the effort expanded by the activist in that case.

We next provide some pictures of the dynamics of these variables for specific parameter choices. We fix $T = \sigma = 1$, and $\psi = 1$ and consider three cases for the initial two sources of adverse selection:

1. Initial position is known: $\sigma_X^2 = 0$, $\sigma_v^2 = 1$.
2. Exogenous component is known: $\sigma_X^2 = 1$, $\sigma_v^2 = 0$.
3. Both are unknown: $\sigma_X^2 = 0.5$, $\sigma_v^2 = 0.5$.

In all cases, we set $\sigma_{Xv} = 0$ so that the total signal to noise ratio $\frac{\omega}{\sigma}$ remains unchanged in all three cases.

Figure 1 shows how the private information available to the activist is revealed through his trading activity over the trading period. $\Sigma_v(t)$ measures the uncertainty about the exogenous component of private information available to the activist. $\Sigma_X(t)$ measures the uncertainty about the activist's position, i.e., the moral hazard component of adverse selection in our model. $\Sigma_{Xv}(t)$ measures their dependence.

[Insert Figure 1 here]

The figure shows that, even though total uncertainty $\Omega(t)$ behaves exactly the same across all three scenarios and whence prices behave similarly, the type of information that gets into prices is very different depending on the initial conditions.

If there is no uncertainty about the exogenous component of firm value ($\Sigma_v(t) = 0 \ \forall t$) and all adverse selection comes from moral hazard, then Σ_X decays over time and the market knows why the firm value increased (upper panel). That is, the market knows that the activist expends effort to increase firm value. If however, there is uncertainty about the exogenous component of firm value (that not affected by the activist), then the moral hazard component of adverse selection tends to first grow over time as the activist is expected to accumulate more shares and there is some 'signal jamming': the market has difficulty separating its estimate of exogenous and endogenous component of firm value. In the end when maturity approaches the market will learn why firm value increased only if there is no initial uncertainty about his position (i.e., if $\Sigma_X(0) = 0$ as in the middle panel). If the market is at the outset uncertain about both position and exogenous firm value (i.e., if $\Sigma_X(0)\Sigma_v(0) > 0$ as in the lower panel), then the market will never be able to separate the endogenous from the exogenous firm value. In other words, the market will not be able to differentiate between an activist who expends effort to increase firm value and an activist who trades based on exogenous information. For the calibration presented the results are pretty dramatic. Starting from $\Sigma_X(0) = \Sigma_v(0) = 0.5$ we end up at $\Sigma_X(T) = \Sigma_v(T) \approx 0.3$, so there is considerable uncertainty about both components remaining.

This seems important given the debate about the social 'usefulness' of activists. Are these better stock-pickers or do they really create value for minority shareholders by expending effort? The model presented suggests that this is very difficult for the market to sort out, if at the outset there is some uncertainty about the activists holdings. Given our interpretation of date 0 in the model as the first time the market becomes aware of the existence of a potential activist present in the market place, this scenario seems the more likely. Instead, the model suggests that if we can force the activist to disclose information on his holdings, i.e., reduce $\Sigma_X(0)$, then this makes it easier for the market to sort out stock pickers from value creators. From that perspective, such disclosure requirements may be useful. As we discuss further below, the disclosure may also enhance economic efficiency, as it leads the activist to accumulate more shares and thus expend more effort.

Figure 2 presents the equilibrium behavior of both components of price impact Λ_t and λ_t (see Lemma 5). Recall that total price impact $\lambda_t + \psi\Lambda_t = \hat{\Delta}$ (see equation (50)) is constant in this model. However, the figure shows that the 'position impact' Λ_t increases monotonically closer to maturity when $\Sigma_v(0) > 0$. This is consistent with information about the activist's position (and therefore effort expenditure) becomes more important as maturity approaches. Instead, λ_t decreases monotonically and converges to zero at maturity.

[Insert Figure 2 here]

Next, we present the optimal trading strategy of the activist. Equation (51) presents the optimal trading strategy of the activist as a function of his valuation gap. $\beta(t)$ in Figure 3 plots the speed at which the activist reacts to the gap between his assessment of the fundamental value and the market maker's estimate of the terminal firm value. Note that as in the KB model the activist becomes more 'aggressive' as maturity approaches so as to not leave any money on the table. Since he is risk-neutral, any difference between price and expected payoff must lead to aggressive trading on the part of the activist. In equilibrium, however, his actual trading is the product of this rate and of the valuation gap, which disappears as maturity approaches.

[Insert Figure 3 here]

Interestingly, when we focus on the actual trading rate θ_t, its behavior is quite different than in the traditional Kyle model. Indeed, we can compute explicitly the unconditional expected trading rate of the activist in his filtration $\mathrm{E}[\theta_t|\mathcal{F}_0, v, X_0]$. We find:

$$\mathrm{E}[\theta_t|\mathcal{F}_0, v, X_0] = (v + \psi X_0 - P_0)(\frac{T-t}{T})^{\kappa}\hat{\Delta}/(T\phi^2) \tag{56}$$

Clearly the unconditional trading rate is expected to decrease over time. Instead, when there is no moral hazard ($\psi = 0$) the model reverts to the traditional Kyle model and the unconditional expected trading rate is constant (equal to the initial gap normalized by price impact and time to maturity: $\mathrm{E}[\theta_t|\mathcal{F}_0, v, X_0]|_{\psi=0} = \frac{v-P_0}{\frac{\sigma_v}{\sigma}T}$). So the presence of moral hazard changes the trading strategy of the activist unconditionally leading him to be more aggressive early on, even though his total price impact ($\hat{\Delta}$) is constant over time. Figure 4 below plots both expected trading rates.

[Insert Figure 4 here]

Moral hazard and expected profits of the activist

Turning to the profits of the activist investor, we plot his value function on Figure 5 as a function of his valuation gap for different levels of his productivity parameter ψ and on Figure 6 as a function of the productivity parameter for different levels of the valuation gap. Recall that:

$$J(P, X, 0) = \frac{(P - v - \psi X_0)^2 + \hat{\Delta}^2\sigma^2 T}{2(\hat{\Delta} - \psi)}.$$

We see that for small deviations of the price from its 'fundamental' value, the expected profits of the activist are increasing in his ability to create more value for shareholders. However, when deviations are very large, then it may become decreasing in his ability. A possible intuition for this surprising result is that increasing his ability ψ has two effects. On the one hand, for a given stake at maturity it raises the payoff of the activist. On the other hand, it increases price-impact, which reduces the insider's profits. For a very large initial valuation gap, the second effect dominates.

[Insert Figure 5 here]

[Insert Figure 6 here]

We can also calculate the unconditional expected profits of the activist by integrating his value function.

Lemma 6. *The unconditional expected profits of an activist is*

$$U(\omega, \psi, \sigma, T) = \omega\sigma T\left(\zeta + \sqrt{1+\zeta^2}\right)$$

with $\zeta = \frac{\psi\sigma}{\omega}$ *and* $\omega^2 = \sigma_v^2 + 2\psi\sigma_{Xv} + \psi^2\sigma_X^2$ *as before.*

Proof 8. *Taking unconditional expectation of the value function of the activist we see that its unconditional profits are given by:*

$$\mathrm{E}[J] = \frac{\omega^2 T + \hat{\Delta}^2\sigma^2 T}{2(\hat{\Delta} - \psi)}.$$

Now recall that $\omega^2 + (2\psi - \hat{\Delta})\hat{\Delta}\sigma^2 = 0$. *Substituting and rearranging we get the expression in the lemma.*

We see that total unconditional profits of the activist converge to $U(\omega, \psi = 0, \sigma, T) = \sigma\sigma_v T$ identical to KB in the absence of moral hazard. Further, unconditional profits $U(\omega, \psi, \sigma, T)$ of an activist are increasing in (a) activism ability (ψ), (b) (annualized) noise trading volatility (σ), (c) initial level of (annualized) private information (ω), (d) and length of the trading period (T).

Now suppose that an investor needs to pay a fixed cost to establish the potential under or over-valuation of a given firm ($v + \psi X_0 - P_0$), based on which he would decide to accumulate more shares and eventually become an activist shareholder. These costs would be related to his research time, legal costs, etc. Say these costs amount to I. Then it is clear that an investor would from an ex-ante perspective only spend these initial costs if his unconditional expected profits exceed these costs, i.e., if $U(\omega, \psi, \sigma, T) > I$. This suggests that all else equal we would expect shareholder activism to target firms with greater (a) uncertainty about its fundamental value, (b) greater potential for activism impact, (c) higher stock price liquidity, and (d) longer trading period.

4. Discussion

The model suggests an interesting relation between economic efficiency and market (or price) efficiency.

Economic Efficiency

As in many models with noise-traders, since their utility is not defined, the notion of Pareto efficiency is not well-defined. Our notion of economic efficiency refers instead to the value added by the activist through his effort expenditure. Because effort expenditure is determined by the activist's stake size, we use the number of shares accumulated by the activist, X_T, as a measure of economic efficiency.

Lemma 7. *In equilibrium the shares accumulated by the activist X_T are given by*

$$X_T = X_0 + \frac{v + \psi X_0 - P_0}{\sqrt{\psi^2 + \phi^2}} - \sigma(1 + \frac{1}{\sqrt{1 + \frac{1}{\zeta^2}}})Z_T.$$

Thus X_T is normally distributed with mean $\mathrm{E}[X_T \mid \mathcal{F}_0, v, X_0] = X_0 + \frac{v+\psi X_0 - P_0}{\sqrt{\psi^2+\phi^2}}$ and variance $V[X_T] = \sigma^2 T(1 + \frac{1}{\sqrt{1+\frac{1}{\zeta^2}}})^2$, where $\zeta = \frac{\sigma\psi}{\omega}$.

Proof 9. *This follows from $dP_t = \hat{\Delta} dY_t = \hat{\Delta}(dX_t + \sigma dZ_t)$. Integrating and using the fact that $P_T = v + \psi X_T$ at maturity we find:*

$$v + \psi X_T - P_0 = \hat{\Delta}(X_T - X_0) + \hat{\Delta}\sigma(Z_T - Z_0).$$

Solving for X_T and using (50) gives the result.

Lemma 7 has several interesting implications. First, it suggests that when the firm appears under-valued initially to the activist and he would like to accumulate more shares, then the expected number of shares he accumulates is actually decreasing in the signal to noise ratio, $\phi = \omega/\sigma$. This of course reflects the increased price impact. So typically the more noise trading (as measured by σ), the larger the expected number of shares the activist will accumulate (on a positive NPV venture). As a result, the activist will expend more effort and will create more value for shareholders. It implies a positive impact of noise trading on Economic efficiency.

This result also implies that the expected number of shares the activist accumulates is decreasing in the uncertainty about the activist's initial position, $\Sigma_X(0)$. It happens because higher $\Sigma_X(0)$ has a positive impact on the signal to noise ratio, $\phi = \omega/\sigma$. As we discussed earlier, reducing $\Sigma_X(0)$ also makes it easier for the market to sort out stock pickers from value creators. Ownership disclosure could be one mechanism to achieve this goal. Lemma 7 highlights an additional benefit of ownership disclosure: it leads the activist to accumulate more shares (due to lower price impact) and thus expend more effort on average.

Third, when the firm appears under-valued initially to the activist and he would like to accumulate more shares, then the expected number of shares he accumulates can be either decreasing or increasing in his marginal productivity, ψ. If the activist's initial position is positive ($X_0 > 0$) but small (large), then the expected number of shares he accumulates is decreasing (increasing) in his marginal productivity. The intuition that if his initial position is small (e.g., zero) then increasing his marginal productivity increases price impact and thus reduces the expected number of shares the insider expects to accumulate. However, if his initial position is large, then an increase in his productivity also increases his expected profit on any additional share accumulated therefore leading him to trade more aggressively and accumulate more shares (despite the increased price impact).

Fourth, the ex post number of shares purchased by the activist depends on the realized noise trading activity, Z_T. The higher the order flow from noise traders, the fewer shares the activist purchases. That is, when noise traders end up being net buyers (sellers), the activist ends up with fewer (more) shares. Thus, the model highlights the positive impact of selling by noise traders on shareholder activism and economic efficiency.[17]

Lastly, note that the expected number of shares purchased is independent of maturity T. This is true for a given *annualized* level of private information ω. If instead, we hold fixed the total initial amount of private information (i.e., $\Omega(0)$ is constant), then increasing the trading period T will decrease the signal to noise ratio ϕ (since $\omega^2 := \Omega(0)/T$) leading to an increase in the expected number of shares accumulated by the activist. Therefore, for a given amount of initial private information, the longer the period when the activist can accumulate shares anonymously, the larger the expected number of shares the activist will accumulate (on a positive NPV venture). As a result, the activist will expend more effort and will create more value for shareholders. It implies a positive impact of a longer pre-disclosure period on economic efficiency. This result has implications for the current regulatory debate on the allowable discretionary trading period for Schedule 13-D filers prior to their filing with the SEC (e.g., Bebchuk et al., 2013).[18] Specifically, the result implies that shortening the discretionary trading would hurt economic efficiency.

Price efficiency

[17] This prediction finds support in the empirical literature. For example, Gantchev and Jotikasthira (2013) show that activist shareholders accumulate shares when other institutions heavily sell the targets.

[18] See also a discussion of the beneficial ownership reporting rules under Section 13(d) of the Securities Exchange Act of 1934: http://blogs.law.harvard.edu/corpgov/2014/03/31/activist-abuses-require-sec-action-on-section-13d-reporting/.

In our model prices exhibit the semi-strong form of price efficiency in sense that they always reveal all the public information (available to the market maker). Therefore, we are interested in the strong form of price efficiency to indicate the degree to which prices reveal the private information available to the activist.

Note that in the Kyle model, $\psi = 0$ and markets are most efficient in the absence of noise traders, i.e., when $\sigma = 0$. In that case, the profits of the activist are driven to zero, since even an infinitesimal trade will reveal all of his information. So given that the informed is indifferent, prices might as well jump to v immediately.[19] The larger the amount of noise trading volatility the larger the activist's profits and the less prices reveal his information. When there is moral hazard and the activist can affect prices, the activist will accumulate fewer shares in expectation when there is less noise trading. Thus, he will create less wealth (since his effort is proportional to the size of his acquired stake). Thus somewhat paradoxically, more 'price efficiency' (in the sense of less noise trading activity and thus more informative prices) will lead to less economic efficiency in the sense that the activist will create less shareholder value. This is a dynamic version of the Grossman and Hart (1980a) free-rider problem when there are noise traders, who offer a partial solution to that problem, as also emphasized in a one period model by Kyle and Vila (1991) and Maug (1998).

5. Empirical Regularities

We developed a model of informed trading where an activist shareholder accumulates shares in an anonymous market and then expends costly effort to increase the firm value. The model implies two key testable hypotheses. First, the model predicts that an activist's trading strategy should depend not only on the stock price, but also on

[19]Of course, with an infinitesimal cost, he might not want to trade at all, leading to a Grossman-Stiglitz style paradox.

his stock ownership. Specifically, Proposition 2 (equation (51)) shows that an activist's trading rate is increasing in the size of his acquired position even after controlling for the price level. The intuition is that a larger acquired position leads to a larger effort expenditure and therefore to a larger increase in firm value. Thus, a larger position enhances an activist's incentives to purchase additional shares. We summarize it in the following hypothesis.

Hypothesis 1. *An activist's trading strategy depends positively on the activist's stake size after controlling for the price level.*

Another key predictions of the model is that price impact should increase in shareholders' activism abilities. This result is summarized in Proposition 2 (see equation (50)). The intuition is that the market maker is concerned about selling shares to a more productive shareholder (higher ψ) because these shares will encourage the shareholder to expend effort and increase firm value later on. This result is summarized in the next hypothesis.

Hypothesis 2. *There is a positive relation between the marginal productivity of the activist and stock illiquidity.*

In the next section we present some empirical evidence that seems to support these two hypothesis.

5.1. Sample Description

As we discussed in the introduction, it is natural to think of the insider in our model as a hedge-fund activist. An activist hedge fund typically accumulates shares of a public company in the open market and then expends effort to improve the company value.

Our first data source is the novel data-set built by Collin-Dufresne and Fos (2014) on individual trades by activist shareholders (most of time hedge funds) as identified from

Schedule 13D filings. These authors exploit a disclosure requirement to identify trades that rely on valuable private information. Schedule 13D filings reveal the date and price at which all trades by the Schedule 13D filer were executed during the 60 days that precede the filing date.[20] For each event, the authors extract the following information from the Schedule 13D filings: CUSIP of the underlying security, date of every transaction, transaction type (purchase or sell), transaction size, transaction price, filing date, and the beneficial ownership of the Schedule 13D filer at the filing date. The final sample consists of 3,126 Schedule 13D filings from 1994 to 2010.

Our second data source is Thomson-Reuters Institutional Holdings Database, where we obtain Schedule 13F filing data on common stock holdings. For each firm-quarter, we use common stock ownership information to calculate the percentage of shares outstanding owned by activist hedge funds.[21]

These two databases are then merged with stock-level and firm-level data. Stock returns, volume, and prices come from the Center for Research in Security Prices (CRSP). Firm-level accounting information comes from Compustat.

5.2. Trading Strategies of Activist Shareholders

The dataset on individual trades by activist shareholders reveals that activist shareholders purchase a significant number of shares in targeted companies. For instance, the average (median) stock ownership of a Schedule 13D filer on the filing date is 7.51% (6.11%). The average (median) filer purchases 3.8% (2.8%) of outstanding shares during the sixty-day period prior to the filing date. It corresponds to an average (median)

[20]Rule 13d-1(a) of the 1934 Securities Exchange Act requires investors to file with the SEC within 10 days of acquiring more than 5% of any class of securities of a publicly traded company if they have an interest in influencing the management of the company. In particular, Item 5(c) of Schedule 13D requires the filer to "... describe any transactions in the class of securities reported on that were effected during the past sixty days or since the most recent filing of Schedule 13D, whichever is less."

[21]We thank Wei Jiang for providing the list of activist hedge funds.

purchase of 899,692 (298,807) shares at an average (median) cost of \$16.4 (\$2.5) million.

We begin from testing Hypothesis 1, which suggests that an activist's trading strategy depends on the activist's stake size (equation (51)). This is in contrast to Kyle (1985) and Back (1992), where an activist's trading strategy depends on the 'valuation gap' (i.e., the difference between exogenously given terminal value of the stock and the current stock price) and not on the activist's stake size. To test whether this prediction is supported by data, we estimate the following regression for every event:

$$\theta_{it} = a_{0i} + a_{1i} X_{it-1} + a_{2i} P_{it} + \varepsilon_{it}, \tag{57}$$

where θ_{it} is the number shares purchased by a Schedule 13D filer in company i on date t, X_{it-1} is the number shares owned by the Schedule 13D filer on date $t-1$, and P_{it} is the closing price of the stock i on date t. The analysis is based on daily observations from 60 days before the filing date to the filing date.

Consistently with Hypothesis 1, we find that an activist's trading strategy is positively associated with the stake size in 97% of events and is negatively associated with the stock price in 91% of events. The cross-sectional mean of a_1 is 4.5 with t-stat of 41.05. The cross-sectional mean of a_2 is -18.4 with t-stat of -14.66. Thus, an activist's stock ownership seems to have a significant effect on the activist's trading strategy.

5.3. Activist Stock Ownership and Stock Liquidity

We next turn to Hypothesis 2 and analyze the relation between shareholders' activism abilities and stock illiquidity. Our model predicts the relation to be positive: equation (50) shows that the price impact is increasing in shareholders' activism abilities. To test this prediction, we use the percentage of shares owned by activist hedge funds as a proxy for shareholders' activism abilities. That is, we assume that activist hedge funds' activism abilities are higher than activism abilities of the average investor.

To test this hypothesis, we estimate the following regression:

$$illiq_{it} = \alpha + \beta AHF_{it} + X_{it}\gamma + \eta_i + \eta_t + \epsilon_{it}, \tag{58}$$

where $illiq$ is the average level of Amihud (2002) illiquidity measure, AHF_{it} is the percentage of shares outstanding owned by activist hedge funds, X_{it} is a vector of control variables, η_i are event fixed effects, and η_t are year-quarter fixed effects. X_{it} includes (lagged) illiquidity and (lagged) natural logarithm of average trading volume. The quarterly averages of stock illiquidity and trading volume are calculated using monthly data. The analysis is based on firm-quarter observations during 1990-2010. For each firm-quarter, we use beneficial ownership information from 13F filings to calculate the percentage of shares outstanding owned by activist hedge funds. When we merge data on illiquidity and other controls, we make sure that the information on beneficial ownership is publicly available when the illiquidity is measured. Because the specification includes both firm and time fixed effects, the evidence is based on the within firm variation in illiquidity and activist ownership and therefore is not driven by changes in either firm-invariant or aggregate variables.

Table 1 reports the results. The evidence reveals positive and significant relation between activist ownership and illiquidity. The results are robust for controlling for lagged illiquidity and lagged trading volume. Overall, the evidence is consistent with Hypothesis 2 and suggests that illiquidity increase when shareholders' activism abilities increase.

[Insert Table 1 here]

6. Conclusion

We have proposed a model of activist share-holder that extends Kyle (1985) and Back (1992) to allow for an endogenous liquidation value that is determined by the effort level chosen by the informed activist. In equilibrium, price impact reflects two sources of information asymmetry: one related to the insider's pure informational advantage (his stock-picking ability) as in the original Kyle model, and one that is related to moral hazard (his shareholder activism).

We find that while, in equilibrium, prices eventually reveal the total value of the firm, in many cases, the market cannot identify the actual source of value 'creation.' Forcing activists to disclose information on their position helps in separating stock-pickers from activists. Moreover, more ownership disclosure also leads to more share-accumulation by the activist who then exerts more effort and creates more value. Indeed, in general the model shows that less price efficiency (i.e., higher noise-trading volatility) allows the insider to accumulate more shares and thus to exert more effort to generate more shareholder value.

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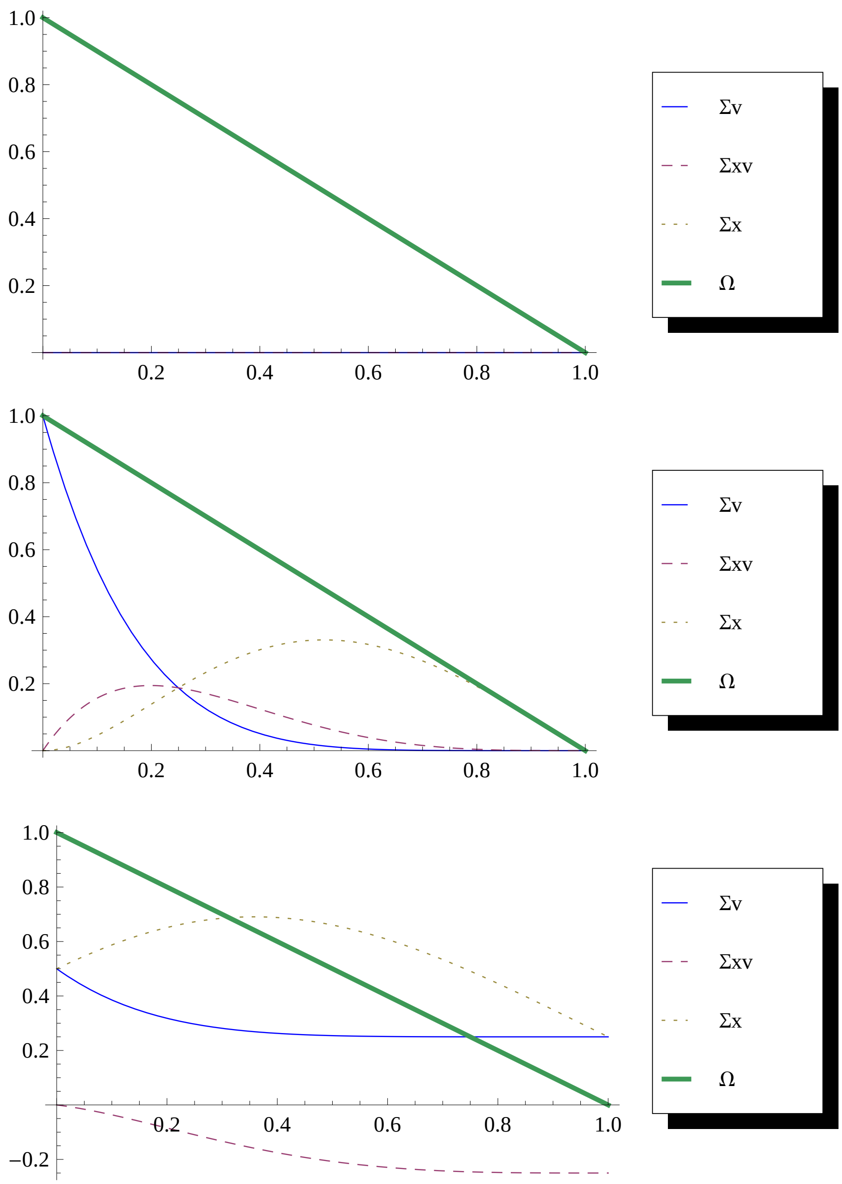

Figure 1: **Flow of private information into Prices.** This figure presents the flow of private information into prices. $\Sigma_v(t)$ summarizes the residual uncertainty about the exogenous terminal value. $\Sigma_X(t)$ the residual uncertainty about the activist's position. $\Sigma_{Xv}(t)$ is the covariance between both. The upper panel corresponds to the case with $\Sigma_v(0) = 0$, the middle panel to $\Sigma_X(0) = 0$, and the lower panel to $\Sigma_X(0) = \Sigma_v(0) = 0.5$. In all cases we set $\Sigma_{Xv}(0) = 0$.

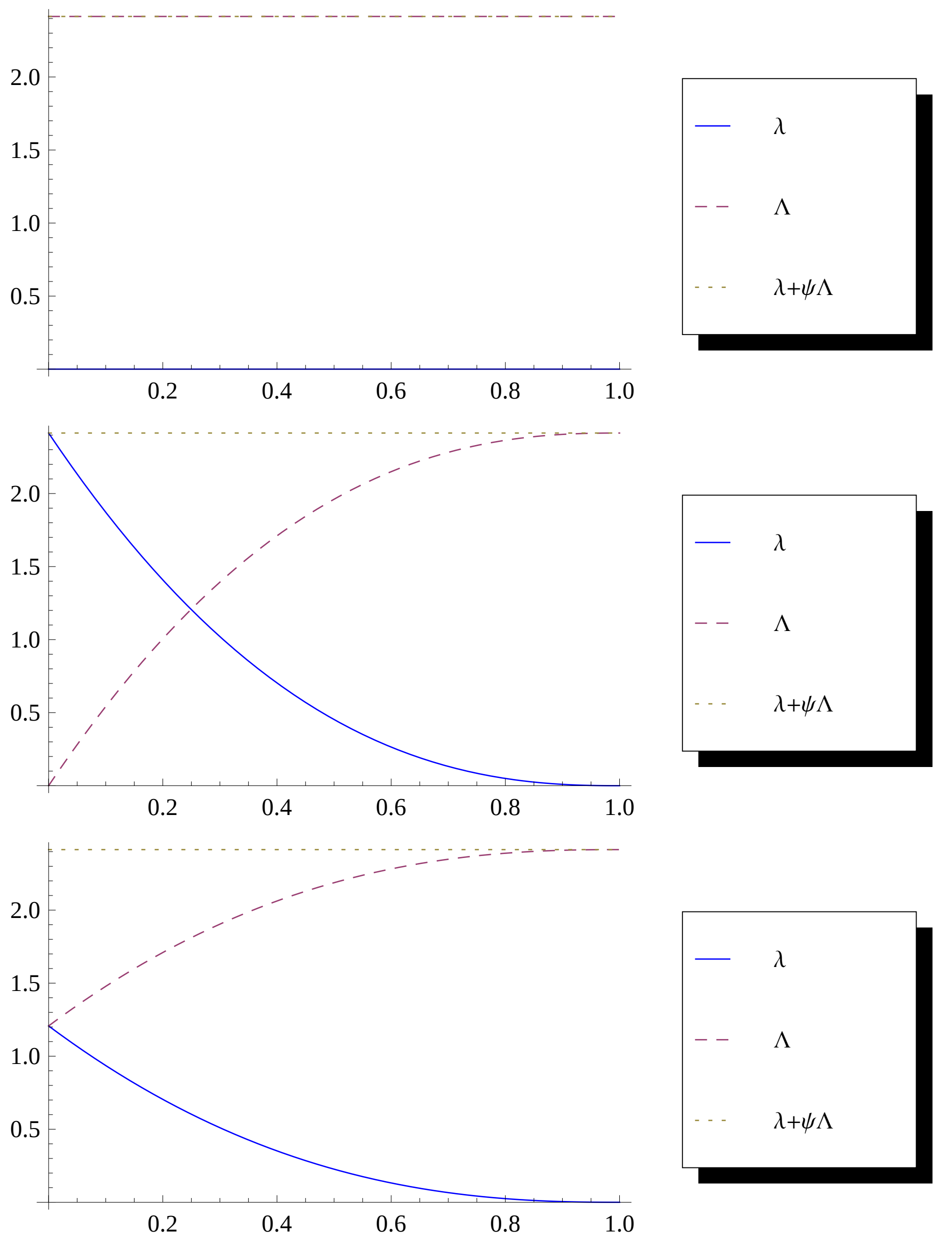

Figure 2: **Components of price impact.** This figure plots the equilibrium pathes of the position-impact Λ_t and price-impact λ_t factors. The upper panel corresponds to the case with $\Sigma_v(0) = 0$, the middle panel to $\Sigma_X(0) = 0$, and the lower panel to $\Sigma_X(0) = \Sigma_v(0) = 0.5$. In all cases we set $\Sigma_{Xv}(0) = 0$.

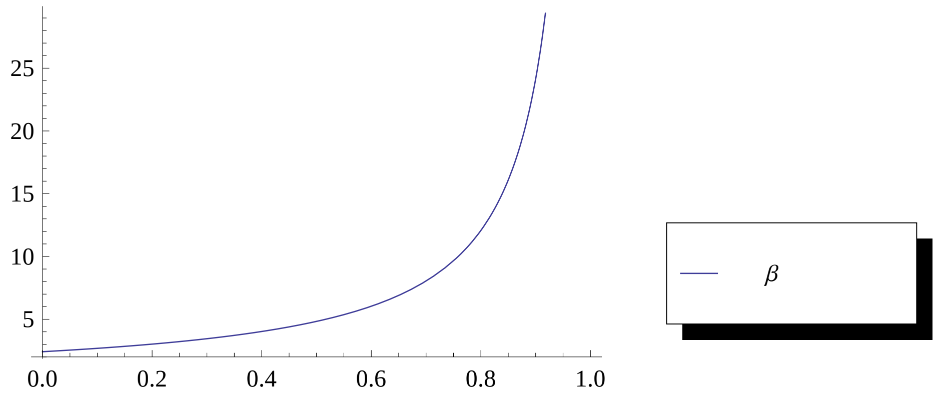

Figure 3: **Optimal trading strategy.** This figure plots the optimal trading strategy of the activist (see equation (19)). $\beta(t)$ plots the speed at which the activist decides to close the gap between his assessment of the fundamental value and the market maker's estimate of the terminal firm value. The figure is drawn for $\Omega(0) = 1$.

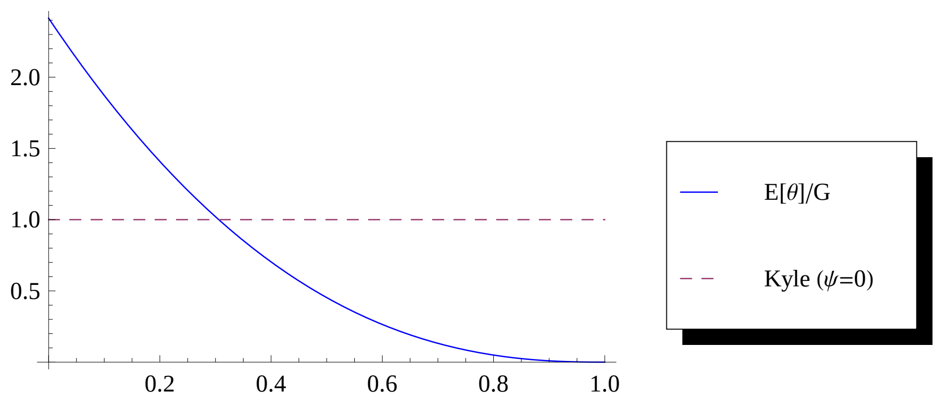

Figure 4: **Optimal trading strategy.** This figure shows the unconditional expected trading rate of the activist shareholder normalized by the initial valuation gap $\mathrm{E}[\theta_t|\mathcal{F}_0, v, X_0]/G_0$ with $G_0 = (v + \psi X_0 - P_0)$ as a function of time and compares that to the expected trading rate in the absence of moral hazard, i.e., when $\psi = 0$.

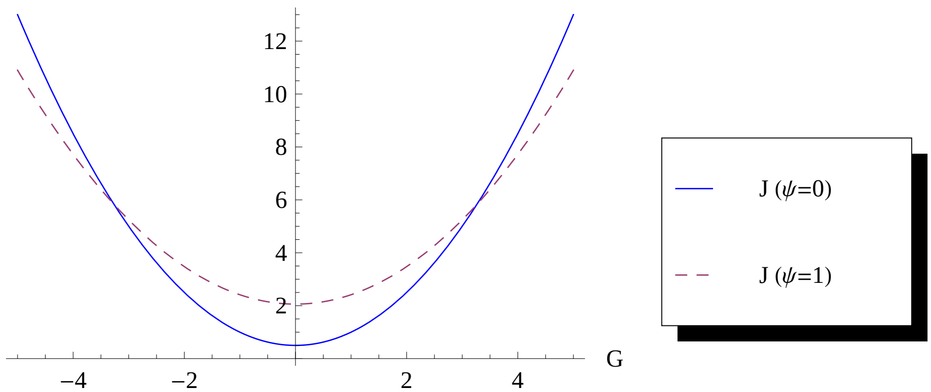

Figure 5: **Value function as function of initial valuation gap.** This figure plots the optimal value function as a function of the initial valuation gap $G = v + \psi X_0 - P_0$ for different values of the productivity level of the activist: $\psi = 0$ which corresponds to the Kyle-Back model with no moral hazard, and $\psi = 1$.

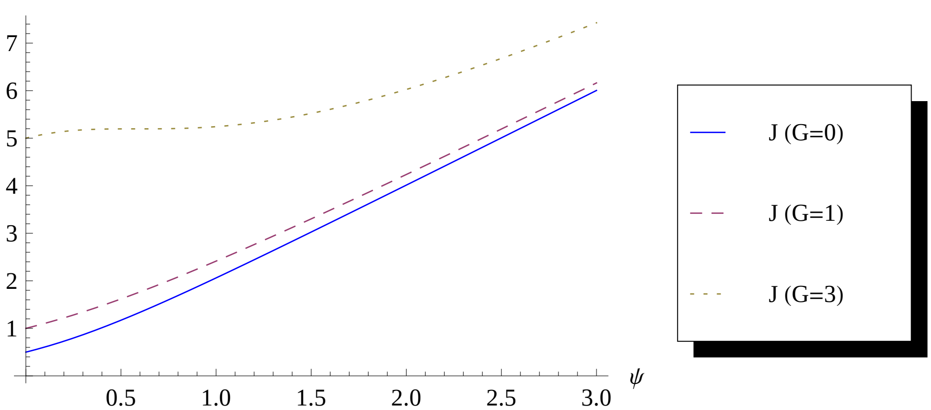

Figure 6: **Value function as a function of productivity.** This figure plots the optimal value function as a function of the productivity parameter ψ for different levels of the initial valuation gap $G = v + \psi X_0 - P_0$: $G = 0$, $G = 1$, $G = 3$.

Table 1: **Activist Ownership and Stock Illiquidity.** This table presents the relation between stock liquidity and stock ownership by activist shareholders. We estimate the following specification: $illiq_{it} = \alpha + \beta AHF_{it} + X_{it}\gamma + \eta_i + \eta_t + \epsilon_{it}$, where $illiq$ is the average level of Amihud (2002) illiquidity measure, AHF_{it} is the percentage of shares outstanding owned by activist hedge funds, X_{it} is a vector of control variables, η_i are firm fixed effects, and η_t are year-quarter fixed effects. X_{it} includes (lagged) illiquidity and (lagged) natural logarithm of average trading volume. The quarterly averages of stock illiquidity and trading volume are calculated using monthly data. The analysis is based on firm-quarter observations during 1990-2010. In each column, we report estimated coefficients and their t-statistics, calculated using heteroscedasticity robust standard errors clustered by firm. *, **, and *** indicate statistical significance at the 10%, 5%, and 1% levels.

Dependent variable: stock illiquidity	(1)	(2)	(3)	(4)
Activist Ownership	0.7951***	0.3288***	0.4588***	0.2372***
	[4.76]	[3.87]	[2.94]	[2.80]
lag Illiquidity		0.5445***		0.5026***
		[96.05]		[83.84]
lag Volume (log)			-0.6741***	-0.2505***
			[-60.66]	[-40.29]
Constant	2.7267***	0.6860***	7.8498***	2.8104***
	[47.21]	[12.81]	[72.20]	[35.67]
Year-Month FE	Yes	Yes	Yes	Yes
Firm FE	Yes	Yes	Yes	Yes
Observations	455,315	435,281	435,281	435,281

Supplemental Internal Materials for the paper

"Shareholder Activism, Economic Efficiency, and Stock Prices"

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Appendix A. Proof of Lemma 1

Recall that the market maker is risk-neutral and sets prices such that:

$$
\begin{aligned}
P_1 &= \mathrm{E}[v + w \,|\, Y] = \mathrm{E}[v + \psi(\theta + X_0) \,|\, Y] \\
&= \mathrm{E}[\psi\alpha + v(1+\psi\beta) + \psi(\gamma+1)X_0 \,|\, Y].
\end{aligned}
\tag{A.1}
$$

Using normality of the random variables we obtain:

$$
P_1 = \mathrm{E}[\psi\alpha + v(1+\psi\beta) + \psi(\gamma+1)X_0] + \frac{\mathrm{Cov}(v(1+\psi\beta) + \psi(\gamma+1)X_0, Y)}{\mathrm{Var}(Y)}\left(Y - \mathrm{E}[Y]\right).
$$

Now, note that

$$
\begin{aligned}
\mathrm{E}[\psi\alpha + v(1+\psi\beta) + \psi(\gamma+1)X_0] &= \psi\alpha + V_0(1+\psi\beta) + \psi(\gamma+1)Q_0, \\
\mathrm{Cov}(v(1+\psi\beta) + \psi(\gamma+1)X_0, Y) &= (1+\psi\beta)\beta\sigma_v^2 + \psi(\gamma+1)\gamma\sigma_X^2 + \left(\gamma + \psi\beta(2\gamma+1)\right)\sigma_{Xv}, \\
\mathrm{Var}(Y) &= \beta^2\sigma_v^2 + \gamma^2\sigma_X^2 + 2\beta\gamma\sigma_{Xv} + \sigma_u^2.
\end{aligned}
$$

Thus we have shown that if the optimal strategy θ is linear, then the price-order flow is indeed linear and of the form $P_1 = P_0 + \Delta Y$ with:

$$
\Delta = \frac{(1+\psi\beta)\beta\sigma_v^2 + \psi(\gamma+1)\gamma\sigma_X^2 + \left(\gamma + \psi\beta(2\gamma+1)\right)\sigma_{Xv}}{\beta^2\sigma_v^2 + \gamma^2\sigma_X^2 + 2\beta\gamma\sigma_{Xv} + \sigma_u^2}
\tag{A.2}
$$

and

$$
P_0 = \psi\alpha + V_0(1+\psi\beta) + \psi(\gamma+1)Q_0) - \Delta(\alpha + \beta V_0 + \gamma Q_0).
\tag{A.3}
$$

Appendix B. Proof of Lemma 2

Here we give a heuristic derivation of the filtering equations given in lemma 1 based on the Gaussian projection theorem, a discrete time approximation of the continuous time model and taking the limit as the time step dt goes to zero.

$$\begin{aligned}
P_{t+dt} &= \mathrm{E}\left[v \,|\, Y^t, Y_{t+dt}\right] \\
&= \mathrm{E}\left[v \,|\, Y^t\right] + \frac{Cov(v, Y_{t+dt} - Y_t \,|\, Y^t)}{V(Y_{t+dt} - Y_t \,|\, Y^t)}(Y_{t+dt} - Y_t - \mathrm{E}[Y_{t+dt} - Y_t \,|\, Y^t]) \\
&= V_t + \frac{\beta\Sigma_v dt + \gamma\Sigma_{Xv} dt}{\beta^2\Sigma_v dt^2 + \gamma^2\Sigma_X dt^2 + 2\beta\gamma\Sigma_{Xv} dt^2 + \sigma^2 dt}(Y_{t+dt} - Y_t) \\
&\approx V_t + \frac{\beta\Sigma_v + \gamma\Sigma_{Xv}}{\sigma^2}(dY_t).
\end{aligned}$$

The third line uses the fact that the expected change in order flow is $\mathrm{E}[Y_{t+dt} - Y_t \,|\, Y^t] = 0$ for the conjectured policy. The last line follows from going to the continuous time limit (with $dt^2 \approx 0$).

Thus we also find:

$$\lambda = \frac{\beta\Sigma_v + \gamma\Sigma_{Xv}}{\sigma^2}$$

Similarly, by the projection theorem, we have:

$$Var\left[v \,|\, Y^t, Y_{t+dt}\right] = Var\left[v \,|\, Y^t\right] - (\lambda_t)^2 Var\left[Y_{t+dt} - Y_t \,|\, Y^t\right], \tag{B.1}$$

which gives (when keeping only order dt terms or lower):

$$\Sigma_{t+dt} \approx \Sigma_t - \lambda_t^2\sigma^2 dt. \tag{B.2}$$

A similar 'proof' applies for the optimal filter for X_t.

$$
\begin{aligned}
Q_{t+dt} &= \mathrm{E}\left[X_{t+dt}\,|\,Y^t, Y_{t+dt}\right] \\
&= \mathrm{E}\left[X_{t+dt}\,|\,Y^t\right] + \frac{Cov(X_{t+dt}, Y_{t+dt} - Y_t\,|\,Y^t)}{V(Y_{t+dt} - Y_t\,|\,Y^t)}(dY_t) \\
&= Q_t + \frac{\beta\Sigma_{Xv}dt + \gamma\Sigma_X dt}{\beta^2\Sigma_v dt^2 + \gamma^2\Sigma_X dt^2 + 2\beta\gamma\Sigma_{Xv}dt^2 + \sigma^2 dt} dY_t \\
&\approx Q_t + \frac{\beta\Sigma_{Xv} + \gamma\Sigma_X}{\sigma^2} dY_t.
\end{aligned}
$$

since

$$
\begin{aligned}
\mathrm{E}\left[X_{t+dt}\,|\,Y^t\right] &= \mathrm{E}\left[X_t + (\beta(v - V_t) + \gamma_t(X_t - Q_t))\,dt\,|\,Y^t\right] \\
&= Q_t
\end{aligned}
$$

and

$$
\begin{aligned}
Cov(X_{t+dt}, Y_{t+dt} - Y_t\,|\,Y^t) &= Cov(X_t + (\beta(v - V_t) + \gamma_t(X_t - Q_t))\,dt, (\beta(v - V_t) + \gamma_t(X_t - Q_t))\,dt\,|\,Y^t) \\
&= \beta_t\Sigma_{Xv}dt + \gamma_t\Sigma_X dt + [...]dt^2 \\
&\approx (\beta_t\Sigma_{Xv} + \gamma_t\Sigma_X)\,dt.
\end{aligned}
$$

We thus obtain:

$$
\Lambda = \frac{\beta\Sigma_{Xv} + \gamma\Sigma_X}{\sigma^2}.
$$

Similarly, we have:

$$
Var\left[X_{t+dt}\,|\,Y^t, Y_{t+dt}\right] = Var\left[X_t + \beta_t(v - V_t)dt + \gamma_t X_t dt\,|\,Y^t\right] - \Lambda_t^2 Var\left[Y_{t+dt} - Y_t\,|\,Y^t\right], \tag{B.3}
$$

which gives:

$$
\Sigma_X(t + dt) \approx \Sigma_X(t) + 2\beta_t\Sigma_{Xv}dt + 2\gamma_t\Sigma_X dt - \Lambda_t^2\sigma^2 dt \tag{B.4}
$$

$$= \quad \Sigma_X(t) + \sigma^2 \Lambda(2-\Lambda)dt. \tag{B.5}$$

Lastly, we can compute the covariance between the two filters from:

$$Cov\left[X_{t+dt}, v \,|\, Y^t, Y_{t+dt}\right] = Cov\left[X_{t+dt}, v \,|\, Y^t\right] - \lambda_t \Lambda_t Var\left[dY \,|\, Y^t\right] \tag{B.6}$$

$$\approx \Sigma_{Xv} + \sigma^2 \lambda(1-\Lambda)dt. \tag{B.7}$$

Since

$$Cov\left[X_{t+dt}, v \,|\, Y^t\right] = Cov\left[X_t + \beta_t(v - V_t)dt + \gamma_t(X_t - Q_t)dt, v \,|\, Y^t\right] \tag{B.8}$$

$$\approx \Sigma_{Xv} + \beta \Sigma_v dt + \gamma \Sigma_{Xv} dt \tag{B.9}$$

$$= \Sigma_{Xv} + \sigma^2 \lambda dt \tag{B.10}$$

Appendix C. Proof of Lemma 4

Proof 10. *By definition $P_t = V_t + \psi Q_t$. Thus:*

$$dP_t = \hat{\Delta} dY_t \tag{C.1}$$

$$= \hat{\Delta}\beta_t((v - V_t) + \psi(X_t - Q_t))dt + \hat{\Delta}\sigma dZ_t \tag{C.2}$$

$$= \frac{(\hat{\Delta}\sigma)^2}{\omega^2(T-t)}(v + \psi X_t - P_t)dt + \hat{\Delta}\sigma dZ_t. \tag{C.3}$$

Of course, in the filtration of the market maker price is a martingale, since $\mathrm{E}[dY_t \,|\, \mathcal{F}_t^Y] = 0$, *and* $d[Y,Y]_t = \sigma^2 dt$ *and* $dP_t = \hat{\Delta} dY_t$.

Now, define $h_t = P_t - v - \psi X_t$. We want to show that h_t converges to zero. Its dynamics are:

$$dh_t = dP_t - \psi dX_t \tag{C.4}$$

$$= -\frac{1+\kappa}{T-t} h_t dt + \hat{\Delta}\sigma dZ_t, \quad \text{(C.5)}$$

where:

$$\kappa = \zeta^2 + \zeta\sqrt{1+\zeta^2} > 0 \quad \text{(C.6)}$$

$$\zeta = \frac{\psi\sigma}{\omega}. \quad \text{(C.7)}$$

We can thus compute

$$h_t = h_0 e^{-A_t} + e^{-A_t} M_t, \quad \text{(C.8)}$$

where $A_t = \int_0^t \frac{(1+\kappa)}{T-u} du = \log(\frac{T}{T-t})^{1+\kappa}$ *and* $M_t = \int_0^t e^{-A_s}\hat{\Delta}\sigma dZ_s$ *is an* $\mathcal{F}_t$*-adapted Brownian martingale. We can calculate the quadratic variation of* M_t *from*

$$< M >_t = \int_0^t e^{-2A_s}(\hat{\Delta}\sigma)^2 ds \quad \text{(C.9)}$$

$$= e^{2A_t}\Omega_t - \Omega_0 \quad \text{(C.10)}$$

Thus for any $t < T$ *we see that* M_t *is a square integrable martingale and that* $\lim_{t\to T} < M >_t = \infty$*. If follows that* $\mathrm{E}[h_t] = h_0 e^{-A_t}$ $\forall t < T$ *and that since* $\kappa > 0$ *we obtain* $\lim_{t\to T}\mathrm{E}[h_t] = 0$*. Further, for any* $t < T$ *we have* $\mathrm{E}[h_t^2] = h_0^2 e^{-2A_t} + e^{-2A_t} < M >_t = (h_0^2 - \Omega_0)e^{-2A_t} + \Omega_t$*. Since* $\Omega_T = 0$ *and* $\kappa > 0$ *we have* $\lim_{t\to T}\mathrm{E}[h_t^2] = 0$*. This establishes* L^2 *convergence of* h_t *to* 0 *at* T*.*

Remark 1. *The previous lemma implies that* $(T-t)^\kappa(P_t - v - \psi X_t)$ *converges almost surely to zero at* T *with* κ *defined in equation (C.6). Indeed, note that there exists an* $\mathcal{F}_t$ *adapted Brownian motion* B_t *such that* $M_t = B_{<M>_t}$*. Further, by the Strong law of large numbers for Brownian motions (Karatzas and Shreve, 1988, p. 104) we have* $\lim_{u\to\infty}\frac{B_u}{u} = 0$ *a.s.. Combining we see that* $\lim_{t\to T} e^{-2A_t}M_t = \lim_{t\to T}\frac{B_{<M>_t}}{<M>_t}e^{-2A_t} < M >_t = 0$ *a.s.. It follows that* $(\frac{T-t}{T})^{1+\kappa}h_t = h_0 e^{-2A_t} + e^{-2A_t}M_t$ *converges almost surely to* 0 *at time* T*. We note that if* $\psi = 0$ *then* $\kappa = 0$*. So this proves almost sure convergence of the price to the* *terminal value v if there is no Moral Hazard. We conjecture (but have not yet been able to prove) that $P_t - v - \psi X_t$ converges a.s. to zero also when $\psi > 0$.*