

# CHOICE AND MATCHING<sup>†</sup>

CHRISTOPHER P. CHAMBERS AND M. BUMIN YENMEZ\*

ABSTRACT. We study *path-independent* choice rules applied to a matching context. We use a classic representation of these choice rules to introduce a powerful technique for matching theory. Using this technique, we provide a deferred acceptance algorithm for many-to-many matching markets with contracts and study its properties. Next, we obtain a compelling comparative static result: If one agent's choice expands, the remaining agents on her side of the market are made worse off, while agents on the other side of the market are made better off. We study the impact of firm mergers using this result.

A choice rule with a capacity that always binds whenever possible is deemed *acceptant*. We provide a constructive proof to show that every path-independent choice rule has an acceptant path-independent expansion with the same maximum cardinality. Finally, we characterize the class of *responsive* choice rules using acceptance.

## 1. Introduction

In this paper, we study *path-independent* choice rules in the context of many-to-many matching markets with contracts (Roth, 1984): There are two sides such as firms and workers. Each firm and worker can be matched with contracts that may include different terms such as wages, working hours, fringe benefits, and so on. This model incorporates labor markets, buyer-seller markets with heterogeneous goods as well as multi-unit assignment problems. There are many specific real-world matching markets that fit in this framework: the staffing problem in which each worker is assigned multiple shifts and each shift is assigned multiple workers; the United Kingdom residency matching market in which each medical student is matched with two residency programs, one medical and one surgical, and each residency program is matched with multiple students (Konishi and Ünver, 2006a); the consulting market in which each firm hires a group of consultants and each consultant can work for multiple firms (Echenique and Oviedo, 2006); couples matching in the National Resident

---

*Date:* July 10, 2013.

<sup>†</sup>We thank Federico Echenique, Tamas Fleiner, Sean Horan, Mark Johnson, Onur Kesten, Michael Ostrovsky, Ilya Segal, Tayfun Sönmez, Utku Ünver and the seminar audiences at the Midwest Economic Theory Conference and the Social Choice and Welfare Conference. Yenmez gratefully acknowledges financial support from National Science Foundation grant SES-1326584.

\*Chambers: Department of Economics, University of California, San Diego, 9500 Gilman Drive, #0508, La Jolla, CA 92093. Yenmez: Carnegie Mellon University, Tepper School of Business, 5000 Forbes Ave, Pittsburgh, PA 15213, USA. Chambers: cpchambers@ucsd.edu. Yenmez: byenmez@andrew.cmu.edu. Keywords: Two-Sided Matching with Contracts, Choice Rule, Acceptance, Path Independence, Stability. JEL: C78, D47, D71, D78.

Matching Program (Hatfield and Kojima, 2010); and course allocation problem in which each student is assigned a set of classes and each class is assigned a set of students (Sönmez and Ünver, 2010).

In our model, each agent has a choice rule. A choice rule selects a subset from a given set of contracts. Choice rules are one of the most basic tools in microeconomics: see for example Moulin (1991), or Chapter 2 in Mas-Colell et al. (1995). In our setting, *path independence* of choice rules guarantees the existence of stable matchings. A choice rule is path independent if the choice over a set of contracts does not change if the set is segmented arbitrarily, the choice applied to each segment and finally the choice applied again to all chosen contracts from the segments (Plott, 1973).

Taking choice rules as primitives of a matching market has many advantages. First and most importantly, in many contexts, it is not natural to assume that a choice rule is rationalizable by a preference. In the environments we have in mind, a single rational decision maker does not determine the choice rule. A firm has many managers, each of whom makes separate hiring decisions. A major theme of our work is that path independence, rather than classical rationality properties based on revealed preference, is more appropriate in such contexts. Even when there is only one agent making decisions, it is not realistic that the choice is rationalizable by an underlying preference. Indeed, cyclical choice is persistently observed in many experiments including very simple settings with binary comparisons over a few alternatives are made.<sup>1</sup>

Further, the concept of stability is inherently a choice-theoretic notion. Even for choice rules that are classically rationalizable, Martínez et al. (2012) show that the set of stable matchings remains the same when an agent's preference relation changes as long as these two preference relations have the same induced choice rule. Moreover, *substitutability*, the primary condition guaranteeing the existence of stable matchings when preferences are primitives, is most naturally described via its choice-theoretic implications. Finally, diversity considerations of schools (Echenique and Yenmez, 2012) or distributional constraints of regions (Kamada and Kojima, 2014) are expressed using choice rules.

Somewhat surprisingly, the theories of path-independent choice rules and stable matchings have developed independently. This is unusual, as the classical representation of path-independent choice rules meshes very well with most matching contexts. We proceed to describe this representation and describe its utility in matching environments.

**1.1. Results.** It is well-known in the choice-theoretic literature, though apparently unrecognized in the matching literature, that path-independent choice rules have a very tractable

---

<sup>1</sup>See, e.g., Tversky (1969); Loomes et al. (1991) for experimental evidence of cyclical choice behavior and Manzini and Mariotti (2007); Masatlioglu et al. (2012); De Clippel and Eliaz (2012); Cherepanov et al. (2013) for theoretical work on non-rationalizable choice.

representation. Indeed, each path-independent choice rule can be associated with a *collection* of preference relations. From any set of contracts, the choice rule chooses the union of best contracts with respect to each preference relation (Aizerman and Malishevski, 1981). Thus, not only are path-independent choice rules mathematically convenient, but also they naturally model a firm with separate positions, each of which has a ranking over contracts, for example.

This representation turns out to be extremely useful for establishing new results; and in fact, provides an intimate connection between the one-to-one matching model and its many-to-many counterpart. To this end, we can view a firm as a collection of *positions*, and a worker as a collection of *personas*, each of whom behave independently of each other. This allows us to treat each agent in the many-to-many matching market as a collection of agents with unit demand acting independently of each other. Such a representation, therefore, allows us to apply standard ideas from the one-to-one case in many-to-many markets.

We illustrate this principle by generalizing some known results and also by establishing new ones. In particular, we obtain two results on stable matchings. First, we show that there is a natural generalization of the deferred acceptance algorithm (DA) (Gale and Shapley, 1962) to many-to-many matching markets with contracts. In one version of the algorithm, firm positions make offers to the workers. In the other version, worker personas make offers to the firms. We show that the firm-proposing DA produces the firm-optimal stable matching when choice rules are path independent (Theorem 1). Second, we establish a useful comparative static result: Expanding the choice rule of an agent makes every other agent on the same side worse off and every agent on the other side better off in a revealed preference sense (Theorem 2). We study the impact of firm mergers on workers and other firms using this result in Section 5. We show that if firms merge and expand, then workers are better off and every remaining firm is worse off. However, if firms merge and consolidate, workers are worse off while other firms are better off.

It should be noted that these results are only a small fraction of what can be established using the representation of path-independent choice rules. Many other related results can be proved; and in most cases, a result in the one-to-one model has a natural counterpart in the many-to-many model via this technique.

Moreover, ideas from matching theory suggest natural research directions for choice theory. We establish several results motivated by matching theory that we believe are of independent interest to choice theorists. We proceed to detail these ideas below.

In a perfect world, firms would admit every worker from any given pool (or schools would admit every student), but they are usually bound by some capacity. Therefore, the capacity acts as a physical constraint on the set of contracts, so that choices must be feasible for that capacity. We say that an agent's choice rule is *q-acceptant* if it always selects  $q$  contracts

when at least  $q$  contracts are available in the pool, and otherwise selects all of them. A choice rule is *acceptant* if it is  $q$ -acceptant for some  $q$ .

Next, we prove several results on path-independent choice rules. We show that every path-independent choice rule for which all choices have cardinality of at most  $q$  can be expanded to a  $q$ -acceptant path-independent choice rule, and we provide an algorithm for doing so (Theorem 4). Thus, acceptance comes essentially for free. This result is important because if the deferred acceptance algorithm is used to assign workers to firms, then all of the workers prefer the outcome with the expanded choices.<sup>2</sup>

Another desirable property of choice rules is *the law of aggregate demand (LAD)*.<sup>3</sup> LAD states that the number of contracts chosen from a set is not smaller than that of any subset. Since LAD is a weaker condition than acceptance, Theorem 4 also implies that every path-independent choice rule can be expanded to satisfy LAD without increasing the maximum cardinality (Corollary 3).

Aside from these, we offer a characterization of a class of prominent path-independent choice rules that are used in practice, such as school choice (Abdulkadiroğlu and Sönmez, 2003) and the National Resident Matching Program (Roth and Peranson, 1999), without directly imposing path independence. The characterization is useful in discussing the normative content of this widely used choice rule. We say that an agent's choice rule is *responsive* if there exist a preference relation over the contracts and a capacity for which the choice rule always selects the highest-ranked available contracts according to the preference relation without going over the capacity (Roth and Sotomayor, 1990). We show that, together with acceptance, it is enough to impose an axiom of Ehlers and Sprumont (2008), the *weakened weak axiom of revealed preference (WWARP)*. This axiom states that for any pair of contracts,  $x, y$ , if  $x$  is chosen when  $y$  is available and  $y$  is not chosen, then it can never be that  $y$  is chosen when  $x$  is available and  $x$  is not chosen. We show that a choice rule is responsive if and only if it is acceptant and satisfies WWARP (Theorem 5). This result is a direct generalization of the classical result that a single-valued choice rule satisfying the *weak axiom of revealed preference* is classically rationalizable. We also establish a characterization of responsive choice rules for capacity  $q$  (Theorem 6).

Finally, we establish the computationally useful fact that every path-independent and  $q$ -acceptant choice rule is uniquely determined by the choices made on sets of cardinality  $q + 1$  (Theorem 7). Therefore, the number of sets whose choices must be reported grows only polynomially in the number of contracts, not exponentially.

---

<sup>2</sup>This follows directly from Theorem 2. DA is used in many such markets; see Roth and Peranson (1999) and Pathak and Sönmez (2013).

<sup>3</sup>LAD has been studied in matching markets with contracts (Fleiner, 2003; Hatfield and Milgrom, 2005).

**1.2. Related Literature.** There is now a substantial literature on many-to-many matching markets with contracts that deals with the existence of stable matchings (Roth, 1984; Fleiner, 2003; Klaus and Walzl, 2009; Hatfield and Kominers, 2012). For example, Roth (1984) introduces a deferred acceptance algorithm when choice rules are rationalizable by preferences. He further assumes that choice rules are *Pareto separable*, a restrictive assumption that we do not need, and he arrives at similar conclusions as in Theorem 1. In contrast, our focus is on the comparative statics of stable matchings and on the theory of path-independent choice rules. In addition, we introduce a new technique based on Aizerman-Malishevski decomposition result to study stable matchings that can be adapted in other studies.

Most of these papers take the preferences of agents as primitives of the model. In this context, Hatfield and Kominers (2012) show that substitutability is not only sufficient but also necessary for the existence of stable matchings in a maximal domain sense. However, when choice rules are primitives of the model, substitutability alone is not enough but additionally imposing *consistency* results in existence (Aygün and Sönmez, 2013). Consequently, in our setting, substitutability and consistency are needed for the existence of stable matchings. The conjunction of substitutability and consistency is equivalent to path independence (Aizerman and Malishevski, 1981).

Path-independent choice rules are a heavily studied class of choice rules, and much is known about the structure of these objects. In-depth studies of these objects include Plott (1973) and Moulin (1985). These papers have established many important properties of these choice rules some of which we use to prove our results.

**1.3. Organization of the paper.** The rest of the paper is as follows. Section 2 introduces the model and some preliminary results. Section 3 presents the matching results and Section 4 has the results on choice. Section 5 studies the welfare consequences of firm mergers on workers as well as the remaining firms in the market using our framework. Section 6 concludes. Finally, the Appendix has the omitted proofs.

## 2. Model

**2.1. Individual Choice.** Suppose  $\mathcal{A}$  is the set of potential alternatives (or partners, contracts, etc.) and  $\mathcal{P}(\mathcal{A}) = 2^{\mathcal{A}}$  is the powerset of  $\mathcal{A}$ . A *choice rule*  $C : \mathcal{P}(\mathcal{A}) \rightarrow \mathcal{P}(\mathcal{A})$  is such that

- for every  $A \subseteq \mathcal{A}$ ,  $C(A) \subseteq A$  and
- for every non-empty  $A \subseteq \mathcal{A}$ ,  $C(A) \neq \emptyset$ .

The interpretation is if  $A$  is the set of available alternatives, then  $C(A)$  is the set of chosen ones. We require the chosen set to be non-empty if there is at least one alternative available.<sup>4</sup>

---

<sup>4</sup>The assumption that the choice is not empty valued plays a role only in Theorem 5. For the rest of the results, it is not needed.

A **preference relation**  $\succeq$  on  $\mathcal{A}$  is a binary relation on  $\mathcal{A}$  that is complete, transitive, and antisymmetric.<sup>5</sup> Choice rule  $C$  is **responsive** if there exist a preference relation  $\succeq$  on  $\mathcal{A}$  and a positive integer  $q$  such that for any  $A \subseteq \mathcal{A}$

$$C(A) = \bigcup_{i=1}^q \{a_i^*\},$$

where  $a_i^*$  is defined inductively as  $a_1^* = \max_A \succeq$  and, for  $i \geq 2$ ,  $a_i^* = \max_{A \setminus \{a_1^*, \dots, a_{i-1}^*\}} \succeq$  (Roth and Sotomayor, 1990). When  $q$  is fixed exogenously, we call such choice rules **q-responsive**.<sup>6</sup>

Next we consider two properties of a choice rule that guarantee the existence of stable matchings in markets.

**Definition 1.** Choice rule  $C$  is **substitutable** if for every  $a \in A \subseteq B$ ,  $a \in C(B)$  implies  $a \in C(A)$ .

Substitutability requires that if an alternative is chosen from a set, then it must also be chosen from a subset containing it. Kelso and Crawford (1982) were the first to study substitutability in a matching context. However, it was studied in the choice literature earlier and known as Sen's  $\alpha$  or Chernoff's axiom (Moulin, 1985). Hatfield and Kominers (2012) show that when choice rules are constructed from preferences over sets of contracts, the class of choice rules satisfying substitutability forms a maximal domain for which stable matchings exist for many-to-many matching markets with contracts, in the sense that any larger domain possesses a profile with no stable matching.<sup>7</sup>

**Definition 2.** Choice rule  $C$  satisfies **consistency** if  $C(A) \subseteq B \subseteq A$  then  $C(A) = C(B)$ .

Consistency implies that excluding alternatives that are not chosen does not affect the chosen set. Substitutability alone does not guarantee the existence of stable matchings when choice rules are primitives; consistency is also needed in addition to substitutability (Fleiner, 2003; Aygün and Sönmez, 2013).<sup>8,9</sup>

The following axiom has been studied extensively in the choice literature. It was first introduced informally by Arrow (1951), and formally by Plott (1973). Moulin (1985) provides an excellent survey related to the topic. It is important in the study of non-rationalizable choice rules.

<sup>5</sup>Complete: For all  $a, b \in \mathcal{A}$ ,  $a \succeq b$  or  $b \succeq a$ . Antisymmetric: For all  $a, b \in \mathcal{A}$ ,  $a \succeq b$  and  $b \succeq a$  implies  $a = b$ . Transitive: For all  $a, b, c \in \mathcal{A}$ ,  $a \succeq b$  and  $b \succeq c$  implies  $a \succeq c$ .

<sup>6</sup>Here, we assume that  $\succeq$  is over all alternatives excluding an 'outside option.' This plays a role in Theorems 5 and 6.

<sup>7</sup>For many-to-one matching markets with contracts, somewhat weaker substitutability conditions are sufficient for the existence stable matchings (Hatfield and Kojima, 2010).

<sup>8</sup>Aygün and Sönmez (2013) use *irrelevance of rejected contracts* instead of consistency.

<sup>9</sup>Blair (1988) was the first to use consistency in a matching market with wages. See also Alkan and Gale (2003) for the use of consistency in a matching context.

**Definition 3.** *Choice rule  $C$  is **path independent** if for every  $A$  and  $B$ ,  $C(A \cup B) = C(A \cup C(B))$ .*

In words, a choice rule is path independent if the choice over a set of contracts does not change if the set is segmented arbitrarily, the choice applied to each segment and finally the choice applied again to all chosen contracts from the segments.

The two aforementioned conditions required for the existence of stable matchings are equivalent to path independence.<sup>10</sup>

**Lemma 1** (Aizerman and Malishevski (1981)). *Choice rule  $C$  is path independent if and only if it satisfies substitutability and consistency.*

In our analysis below, we only consider choice rules that are path independent.

**2.2. Matching Markets.** There exist finite sets of workers  $\mathcal{W}$ , firms  $\mathcal{F}$ , and contracts  $\mathcal{X}$ . Each contract  $x \in \mathcal{X}$  specifies a relationship between a firm and a worker and it may involve many terms such as wages, working hours, and fringe benefits. For each contract  $x \in \mathcal{X}$ , the firm and worker associated with the contract are denoted by  $f(x)$  and  $w(x)$ , respectively.

For a set of contracts  $X \subseteq \mathcal{X}$ , let  $X_w = \{x \in X : w = w(x)\}$  be the set of contracts associated with worker  $w$  and let  $X_f = \{x \in X : f = f(x)\}$  be the set of contracts associated with firm  $f$ .

Each worker  $w$  has a choice rule  $C_w$  on  $\mathcal{P}(\mathcal{X}_w)$  and each firm  $f$  has a choice rule  $C_f$  on  $\mathcal{P}(\mathcal{X}_f)$ . The domain of each choice rule is extended to all sets of contracts such that for any  $X \subseteq \mathcal{X}$ ,  $C_w(X) = C_w(X_w)$  and  $C_f(X) = C_f(X_f)$ . The profile of workers' choice rules is denoted by  $C_{\mathcal{W}}$  and the profile of firms' choice rules is denoted by  $C_{\mathcal{F}}$ . A **matching market** is a tuple  $\langle \mathcal{W}, \mathcal{F}, \mathcal{X}, C_{\mathcal{W}}, C_{\mathcal{F}} \rangle$ .

For worker  $w$ , a set of contracts  $X \subseteq \mathcal{X}_w$  is **revealed preferred** to a set of contracts  $Y \subseteq \mathcal{X}_w$  if  $C_w(X \cup Y) = X$ . Symmetrically, for firm  $f$ , a set of contracts  $X \subseteq \mathcal{X}_f$  is **revealed preferred** to a set of contracts  $Y \subseteq \mathcal{X}_f$  if  $C_f(X \cup Y) = X$ . This definition reflects the idea that preference in this context is over sets of contracts, rather than individual contracts.

A **matching** is a set of contracts. If  $\mu$  is a matching, the set of contracts assigned to worker  $w$  is  $\mu_w$  and the set of contracts assigned to firm  $f$  is  $\mu_f$ .

For two-sided matching markets, stability has proved to be a useful solution concept (Roth and Sotomayor, 1990). The concept of stability has been extended to matching with contracts setting as well.<sup>11</sup> Matching  $\mu$  is **stable** if

<sup>10</sup>Both Lemma 1 and Lemma 2 appear in Aizerman and Malishevski (1981), which does not include the proofs. However, Moulin (1985) proves both of these results, which are stated as Lemma 6 and Theorem 5 there.

<sup>11</sup>Our definition reduces to the one used in Hatfield and Milgrom (2005) for many-to-one markets and is the same as the one in Hatfield and Kominers (2012). Furthermore, when choice rules are path independent,

- (1) (**Individual rationality**) For every worker  $w$ ,  $C_w(\mu_w) = \mu_w$ , and for every firm  $f$ ,  $C_f(\mu_f) = \mu_f$ .
- (2) (**No blocking**) There does not exist a nonempty blocking set of contracts  $X$  such that  $X \cap \mu = \emptyset$ , for every worker  $w$ ,  $X_w \subseteq C_w(\mu_w \cup X)$ , and for every firm  $f$ ,  $X_f \subseteq C_f(\mu_f \cup X)$ .

In other words, a matching  $\mu$  is individually rational if every agent wants to keep all of her contracts. The second condition rules out the existence of a blocking set  $X$  such that every agent wants to have all of the contracts in  $X$  associated with her. This definition of stability generalizes the corresponding notion of Hatfield and Milgrom (2005) in the context of many-to-one matching markets, see Hatfield and Kominers (2012).

A stable matching  $\mu$  is **worker optimal** if for every stable matching  $\mu'$  and for every worker  $w$ ,  $\mu_w$  is revealed preferred to  $\mu'_w$ . A stable matching  $\mu$  is **worker pessimal** if for every stable matching  $\mu'$  and for every worker  $w$ ,  $\mu'_w$  is revealed preferred to  $\mu_w$ . Firm-optimal and firm-pessimal stable matchings are defined analogously. A priori optimal or pessimal stable matchings need not exist. We establish their existence in the following section assuming that choice rules are path independent.

### 3. Stable Matchings and Comparative Statics

We start with the following decomposition lemma that every path-independent choice rule can be written as the union of choices made by preference relations over individual contracts. Although this result is well known in the choice literature (e.g., Moulin (1985)), it is unrecognized in the matching context. For what follows, it is going to be crucial in establishing our results.

**Lemma 2** (Aizerman and Malishevski (1981)). *Choice rule  $C$  on  $\mathcal{P}(\mathcal{X})$  is path independent if and only if there exists a finite sequence of preference relations on  $\mathcal{X}$ ,  $\{\succeq_i\}_{i \in I}$ , such that for every  $X \subseteq \mathcal{X}$*

$$C(X) = \bigcup_{i \in I} \{x_i^*\},$$

where  $x_i^*$  is defined as  $x_i^* = \max_X \succeq_i$ .

Note that every contract is considered in the maximization of the preference relations rather than the set of remaining contracts. If the choice rule can be empty valued, then this result remains intact when  $\succeq_i$  is over  $\mathcal{X} \cup \{\emptyset\}$  and  $x_i^* = \max_{X \cup \{\emptyset\}} \succeq_i$ .

In the next example, we demonstrate the Aizerman-Malishevski decomposition for a simple choice rule.<sup>12</sup>

which we assume throughout the paper, all of the various stability notions used in the literature are equivalent to each other, see Echenique and Oviedo (2006); Klaus and Walzl (2009).

<sup>12</sup>In general, this decomposition is not unique. A method for finding one such decomposition is as follows. Start with any element  $x_1 \in C(\mathcal{X})$ . Then choose an element  $x_2 \in C(\mathcal{X} \setminus \{x_1\})$ . Continue iteratively, choosing



*Example 1.* Let choice rule  $C$  on  $\mathcal{P}(\{1, 2, 3\})$  be as follows:  $C(1, 2, 3) = C(1, 2) = C(1, 3) = \{1\}$ ,  $C(2, 3) = \{2, 3\}$  and  $C(X) = X$  for every  $X \subseteq \{1, 2, 3\}$  with  $|X| = 1$ .<sup>13</sup> Consider the following two preference relations on  $\{1, 2, 3\}$ :

- $\succeq_1$ :  $1 > 2 > 3$ , and
- $\succeq_2$ :  $1 > 3 > 2$ .

We claim that the decomposition for choice rule  $C$  holds with these preference relations. For instance, consider  $\{1, 2\}$ . For this set, we have  $\max_{\{1,2\}} \succeq_1 = 1$ ,  $\max_{\{1,2\}} \succeq_2 = 1$  and  $C(1, 2) = 1$ . Similarly, for  $\{2, 3\}$  we have  $\max_{\{2,3\}} \succeq_1 = 2$ ,  $\max_{\{2,3\}} \succeq_2 = 3$  and  $C(2, 3) = \{2, 3\}$ .

To study stable matchings, we provide a new interpretation of the Aizerman-Malishevski decomposition of path-independent choice rules in the matching context: Each firm can be viewed as a collection of “positions” such that for each position there is a separate preference relation over individual contracts. Therefore, each firm’s choice can be constructed by choosing the best contract for each position. But for each worker, all positions in the same firm are the same. Analogously, each worker can be thought of as the union of some “personas” each of which has its own preference relation over individual contracts. But for each firm all personas of the same worker are the same.

This interpretation allows us to establish some striking new results as well as generalize known results in the literature. For example, we generalize the deferred acceptance algorithm (DA) of Gale and Shapley (1962) to many-to-many matching markets with contracts in which each worker is viewed as a union of personas.<sup>14,15</sup>

### **Worker-Proposing Deferred Acceptance Algorithm (DA)**

**Step 1:** Each persona of every worker considers her most preferred contract and applies to the associated firm with this contract. Each firm  $f$  considers the set of contracts that has been offered to it, say  $X_f^1$ , tentatively accepts  $C_f(X_f^1)$  and rejects the rest. If there are no rejections, stop.

**Step  $k$ :** Each persona whose contract was rejected at Step  $k - 1$  considers her next preferred contract if such a contract exists and applies to the associated firm with this contract. Otherwise, this persona does not apply to any firm. Each firm  $f$  considers all of the new contracts that have been offered to it and the tentatively

---

$x_k \in C(\mathcal{X} \setminus \{x_1, \dots, x_{k-1}\})$ , obtaining a preference  $x_1 > x_2 > \dots$ . By considering all possible sequential choices, we obtain a collection of preferences, which are an Aizerman-Malishevski decomposition.

<sup>13</sup>For ease of notation we denote  $C(\{x, \dots, y\})$  by  $C(x, \dots, y)$ .

<sup>14</sup>Note that this approach is different from decomposing each agent to multiple copies with unit demand with the same preferences, which is done in the case of responsive choice rules (Roth and Sotomayor, 1990). Here, different personas of the same worker are the same for all firms. Echenique (2007) shows that the set of choice rules that are path independent are exponentially more than the set of responsive choice rules.

<sup>15</sup>There are other generalizations of DA based on fixed-point algorithms, see Hatfield and Milgrom (2005); Hatfield and Kojima (2010), for example.

accepted contracts at Step  $k - 1$ , say  $X_f^k$ , accepts  $C_f(X_f^k)$  and rejects the rest. If there are no rejections, stop.

In the worker-proposing DA, each persona of a worker applies to firms with certain contracts. Each firm treats different personas of the same worker the same and makes choices over the offered contracts only. Since there is a finite number of contracts, the algorithm ends in finite time. When choice rules can be empty valued, then DA can be defined analogously, where each persona applies to the next firm if the associated contract is better than being unmatched. Note that we use the Aizerman-Malishevski decomposition to determine the order in which workers propose. The firm-proposing deferred acceptance algorithm can be defined analogously by viewing each firm as the union of some positions with different preferences over contracts.

In the next result, we show that the worker-proposing deferred acceptance algorithm produces the worker-optimal stable matching. This matching is also the firm-pessimal stable matching. The proof, which is provided in the Appendix, illustrates how one can translate ideas from one-to-one matching markets to many-to-many matching markets under the path independence hypothesis. The idea of the proof constitutes a general method: by decomposing individuals into agents with unit demand in their own right (via the Aizerman-Malishevski decomposition), ideas and results from the many-to-many case can be adapted from analogous one-to-one results almost effortlessly.

**Theorem 1.** *Suppose that each agent has a path-independent choice rule. Then the worker-proposing deferred acceptance algorithm produces the worker-optimal stable matching. This matching is also the firm-pessimal stable matching.*

There are a couple of related results in the literature. In particular, Roth (1984) introduces a deferred acceptance algorithm when choice rules are constructed from preferences over sets of contracts. He further assumes that choice rules are *Pareto separable*, a restrictive assumption that we do not need, and he arrives at similar conclusions as in Theorem 1. Another related work is Fleiner (2003) which shows the existence of stable matchings under path-independent choice rules using fixed-point methods. Unlike Fleiner (2003), we do not rely on any such method but provide elementary proofs based on Aizerman-Malishevski decomposition and we further show worker optimality and firm pessimality in addition to existence. Similarly, other papers in the literature provide existence results when choice rules are constructed from preferences (Klaus and Walzl, 2009; Hatfield and Kominers, 2012).

Next we establish a striking result on comparative statics. First, a definition is in order.

**Definition 4.** *Choice rule  $C'$  on  $\mathcal{P}(\mathcal{X})$  is an **expansion** of choice rule  $C$  on  $\mathcal{P}(\mathcal{X})$  if, for every  $X \subseteq \mathcal{X}$ ,  $C'(X) \supseteq C(X)$ .*

Thus, for an agent, if choice rule  $C'$  is an expansion of choice rule  $C$ , then for any set of contracts, every contract chosen by  $C$  is also chosen by  $C'$ .

Choice rules may be expanded in a variety of situations. For example, in the context of residency matching, a residency program's responsive choice rule may be expanded by either increasing the capacity of the program or by having more acceptable doctors (or contracts);<sup>16</sup> in the context of UK residency matching, a doctor's choice rule may be expanded by including more medical or surgical residency programs in the doctor's list; in the context of controlled school choice, a school's choice rule may be expanded by reinterpreting the bounds as soft bounds rather than hard bounds (Hafalir et al., 2013; Ehlers et al., 2014); and in the context of Japanese residency matching, a region's choice rule may be expanded by allowing each hospital to be matched with more than the target capacity of doctors (Kamada and Kojima, 2014). For all of these situations and possibly more, we derive a comparative static result implied by the expansion of an agent's choice rule. In the next section, we study when choice rules can be expanded in such a manner.

**Theorem 2.** *Suppose that each agent has a path-independent choice rule. Fix a firm  $\hat{f}$ . Suppose that  $C'_{\hat{f}}$  is a path-independent expansion of  $C_{\hat{f}}$ . Then the following hold.*

- (1) *For any  $(C_{\mathcal{W}}, C_{\mathcal{F}})$ -stable matching  $\mu$ , there exists a  $(C_{\mathcal{W}}, (C'_{\hat{f}}, C_{-\hat{f}}))$ -stable matching  $\mu'$  such that, every worker revealed prefers  $\mu'$  to  $\mu$  and every firm other than  $\hat{f}$  revealed prefers  $\mu$  to  $\mu'$ .*
- (2) *For any  $(C_{\mathcal{W}}, (C'_{\hat{f}}, C_{-\hat{f}}))$ -stable matching  $\mu'$ , there exists a  $(C_{\mathcal{W}}, C_{\mathcal{F}})$ -stable matching  $\mu$  such that every worker revealed prefers  $\mu'$  to  $\mu$  and every firm other than  $\hat{f}$  revealed prefers  $\mu$  to  $\mu'$ .*

The proof of Theorem 2 is provided in the Appendix and it uses the deferred acceptance algorithm that we define above.

As an immediate corollary, we get the following.

**Corollary 1.** *Suppose that each agent has a path-independent choice rule. Fix a firm  $\hat{f}$ . Suppose that  $C'_{\hat{f}}$  is a path-independent expansion of  $C_{\hat{f}}$ . Then the following hold:*

- (1) *Every worker  $w$  revealed prefers the worker-proposing deferred acceptance algorithm under  $(C_{\mathcal{W}}, (C'_{\hat{f}}, C_{-\hat{f}}))$ , say  $\mu'$ , to the outcome under  $(C_{\mathcal{W}}, C_{\mathcal{F}})$ , say  $\mu$ , and every firm other than  $\hat{f}$  revealed prefers  $\mu$  to  $\mu'$ .*
- (2) *Every worker  $w$  revealed prefers the firm-proposing deferred acceptance algorithm under  $(C_{\mathcal{W}}, (C'_{\hat{f}}, C_{-\hat{f}}))$ , say  $\mu'$ , to the outcome under  $(C_{\mathcal{W}}, C_{\mathcal{F}})$ , say  $\mu$ , and every firm other than  $\hat{f}$  revealed prefers  $\mu$  to  $\mu'$ .*

---

<sup>16</sup>Of course, this expansion can be used for any responsive choice rule not necessarily in the context of residency matching.

Theorem 2 establishes that when an agent's choice rule expands, other agents on the same side are worse off in the revealed preference sense and all agents on the other side are better off in the revealed preference sense. Therefore, when a residency program lists more doctors, all doctors are better off and all other residency programs are worse off. Similarly, residency programs benefit when a doctor in the UK residency matching enlists more programs but the remaining doctors are hurt. In the school choice context, expanding a school's choice rule improves students' matches but worsens schools' matches. Finally, when a region's choice rule expands in the context of Japanese residency matching program, all doctors get better programs and other regions get worse doctor distributions.

This result can be applied to other settings as well. For example, when a firm has a new job opening, then all positions in the other firms are worse off and all workers are better off. Similarly, we can study the welfare consequences of an agent joining the market, because this can be modeled as the agent's choice rule being expanded from the empty choice. Therefore, we get as corollaries the comparative statics results of Kelso and Crawford (1982); Gale and Sotomayor (1985); Crawford (1991); Blum et al. (1997); Konishi and Ünver (2006b).<sup>17</sup> These papers consider the situation in which an agent joins the market or leaves the market, or what happens when a hospital increases its capacity.

Perhaps the most related result to Theorem 2 is Theorem 6 of Echenique and Yenmez (2012) who also consider expanding choice rules. The result in Echenique and Yenmez (2012) shows only that expanding schools' choice rules makes every student better off in the school choice problem; it does not compare the outcome for schools. In contrast, we expand the choice rule of only one agent and compare the outcomes for all the remaining agents including the ones on the same side of the market using the revealed preference and as a result get a much stronger comparative statics. Second, their model does not have contracts, so their result is silent for some applications like the job matching of Kelso and Crawford (1982) whereas our model incorporates all many-to-many matching markets with contracts that may include wages as well as other contractual terms. Kamada and Kojima (2014) only consider the doctor-proposing deferred acceptance algorithm for a many-to-one matching markets with contracts and show that expanding regions choice rules make doctors better off. As a corollary to our result, we can also say that expanding a region's choice rule makes other regions worse off whereas their result is silent about the welfare implications on regions. In addition, our Corollary 1 also provides comparative statics for the region-proposing deferred acceptance algorithm.

We also establish the following when all firms' choice rules expand.

---

<sup>17</sup>See Alkan and Gale (2003); Ostrovsky (2008) for similar results in different contexts when an additional agent is introduced.

**Corollary 2.** *Suppose that  $\mu$  is a stable matching with respect to  $C = (C_{\mathcal{W}}, C_{\mathcal{F}})$  and  $C'_f$  is an expansion of  $C_f$  for every firm  $f$ . Then there exists a stable matching  $\mu'$  with respect to  $C' = (C_{\mathcal{W}}, C'_{\mathcal{F}})$  such that for every worker  $w$ ,  $C_w(\mu'_w \cup \mu_w) = \mu'_w$ .*

In the school choice context, when each school is matched with a set of students via the student-optimal stable mechanism, this corollary establishes that to maximize students' welfare, schools' choice rules should be expanded as much as possible.

*Remark 1 (Strategyproofness).* Say that a worker has *unit demand* if she does not choose more than one contract from any set of contracts. Hatfield and Milgrom (2005) show that the worker-proposing deferred acceptance algorithm is strategyproof when workers have unit demand and firm choice rules are path independent and satisfy another property called the law of aggregate demand.<sup>18</sup> A natural question to ask is whether the generalized worker-proposing deferred acceptance algorithm that we define is also strategyproof under similar assumptions. Unfortunately, the answer is no. In a more simple setting, Roth and Sotomayor (1990) show that there exists no stable and strategyproof mechanism for workers (Theorem 5.14). They provide a simple example in which firms have unit demand and workers have responsive choice rules and each firm-worker pair defines a unique contract. Consequently, our generalized worker-proposing deferred acceptance algorithm is not strategyproof for workers.

**3.1. When the Modification is not an Expansion.** It is clear from the preceding analysis that expanding the choice rule is a sufficient condition to get the comparative statics. Here, we show that if the choice rule is modified in a different way, then the comparative statics result does not hold anymore.

**Theorem 3.** *Fix a firm  $\hat{f}$ . Suppose that both  $C_f$  and  $C'_f$  are path-independent choice rules. Furthermore,  $C'_f$  is not an expansion of  $C_{\hat{f}}$ . Then there exists path-independent choice rules for workers (and the remaining firms) such that the comparative static result stated in Theorem 2 does not hold.*

Theorems 2 and 3 provide a complete characterization of when we can get the comparative statics result by modifying an agent's choice rule. According to Theorem 2, when a firm's choice rule is expanded then workers are better off and the remaining firms are worse off. On the other hand, according to Theorem 3, when a firm's choice rule is modified in a different way than expansion, then we can construct choice rules for the remaining agents such that the comparative statics result does not hold. This result shows that the expansion property is not only a sufficient condition to get the comparative statics but, in a sense, it is also necessary. The proof of Theorem 3 is provided in the Appendix.

---

<sup>18</sup>We define the law of aggregate demand formally below.

#### 4. Acceptant Path-Independent Choice Rules

We say that a choice rule is acceptant if it admits as many contracts as it can without violating a capacity.

**Definition 5.** *Choice rule  $C$  on  $\mathcal{P}(\mathcal{X})$  is **q-acceptant** if for every  $X \subseteq \mathcal{X}$ ,  $|C(X)| = \min\{q, |X|\}$ . Choice rule  $C$  is **acceptant** if it is q-acceptant for some  $q$ .*

If choice rule  $C$  is  $q$ -acceptant, then it chooses all contracts from a set with cardinality no more than  $q$  and if the set has more than  $q$  contracts, then it chooses exactly  $q$  contracts. Being acceptant can be interpreted as having a capacity constraint but otherwise willing to choose as many contracts as possible.<sup>19</sup>

For example, a school may be rejecting some students because of diversity considerations without filling its seats (Ehlers et al., 2014) and a region may be rejecting some doctors from a hospital because of a target distribution even though the hospital has a higher capacity (Kamada and Kojima, 2014). This leads to the following question. Given a path-independent choice rule, can we always find a path-independent expansion that is also acceptant? We show that this is always possible using the lattice structure of path-independent choice rules below.

**Theorem 4.** *Every path-independent choice rule with maximum cardinality  $q$  has a q-acceptant path-independent expansion.*

Therefore, any path-independent choice rule can be expanded to be acceptant without violating path independence. This means that the school can admit as many students as possible without going over the capacity and the region can admit as many doctors as it can without going over the target capacity of the region. A corollary of this result is that any path-independent choice rule can be expanded to satisfy the *law of aggregate demand* as well without increasing the maximum cardinality.

**Definition 6.** *Choice rule  $C$  on  $\mathcal{P}(\mathcal{X})$  satisfies **the law of aggregate demand** if for every  $X \subseteq Y \subseteq \mathcal{X}$ ,  $|C(X)| \leq |C(Y)|$ .*

The law of aggregate demand is an important property for at least three reasons. First, it guarantees that the set of stable matchings is a lattice (Fleiner, 2003). Second, the worker-proposing deferred acceptance algorithm becomes strategyproof for workers if they have *unit demand* (Hatfield and Milgrom, 2005).<sup>20</sup> Finally, the number of contracts that each agent signs is fixed in any stable matching (Fleiner, 2003).

<sup>19</sup>Acceptance has been used in settings without contracts. For example, Kojima and Manea (2010) use  $q$ -acceptant priorities to characterize DA. In an earlier work, Alkan (2001) calls acceptance *quota filling* and shows the lattice structure of stable matchings under acceptant substitutable choice rules.

<sup>20</sup>A worker has unit demand if she does not choose more than one contract from any set of contracts.

**Corollary 3.** *Every path-independent choice rule with maximum cardinality  $q$  has a path-independent expansion with maximum cardinality  $q$  that satisfies the law of aggregate demand.*

This corollary follows trivially from Theorem 4. To prove Theorem 4, we use the lattice structure of path-independent choice rules. First, some definitions are in order.

A **partial order** is a reflexive, antisymmetric and transitive relation.<sup>21</sup> If  $\succeq$  is a partial order on  $\mathcal{A}$ , we say that the pair  $(\mathcal{A}, \succeq)$  is a **partially ordered set**. A partially ordered set  $(\mathcal{A}, \succeq)$  is a **lattice** if, for every  $a, b \in \mathcal{A}$ , the least upper bound and the greatest lower bound of  $\{a, b\}$  exist in  $\mathcal{A}$  with respect to the partial order  $\succeq$ . We denote the least upper bound of  $\{a, b\}$  by  $a \vee b$ ; and the greatest lower bound of  $\{a, b\}$  by  $a \wedge b$ .

The following lemma is important for establishing a lattice structure for a path-independent choice rule.

**Lemma 3** (Koshevoy (1999)). *Suppose that choice rule  $C$  on  $\mathcal{P}(\mathcal{X})$  is path independent. For any  $X \subseteq \mathcal{X}$ , let  $X_C^\# = \bigcup_{C(Y)=C(X)} Y$ . Then,*

$$\{Y : C(Y) = C(X)\} = \{Y : X_C^\# \supseteq Y \supseteq C(X)\}.$$

Let  $\mathcal{I}(C) = \{C(X) : X \in \mathcal{X}\}$  be the image of a choice rule  $C$  on  $\mathcal{P}(\mathcal{X})$ . Define the following partial order: for every  $X, Y \in \mathcal{I}(C)$ ,  $X \geq Y$  if and only if  $X_C^\# \supseteq Y_C^\#$ .

**Lemma 4** (Johnson and Dean (2001)). *For any path-independent choice rule  $C$  on  $\mathcal{P}(\mathcal{X})$ ,  $(\mathcal{I}(C), \geq)$  is a lattice where for any  $X, Y \in \mathcal{I}(C)$ ,  $X \vee Y = C(X_C^\# \cup Y_C^\#)$  and  $X \wedge Y = C(X_C^\# \cap Y_C^\#)$ .*

The structure of such lattices has been studied at least since Dilworth (1940). These lattices are surveyed by Edelman and Jamison (1985).

Let us demonstrate how to construct this lattice for the choice rule in Example 1.

*Example 2.* Consider the choice rule  $C$  in Example 1. Since  $C(1, 2, 3) = \{1\}$ , then  $\{1\} \in \mathcal{I}(C)$  with  $\{1\}_C^\# = \{1, 2, 3\}$ . This is the greatest element of  $(\mathcal{I}(C), \geq)$ . Next to each element  $X \in \mathcal{I}(C)$  in the lattice, we write  $(X_C^\#)$  if it is different from  $X$ . Note that since  $\{2, 3\}_C^\# \supseteq \{2\}_C^\#$ ,  $\{2, 3\} \geq \{2\}$  in  $\mathcal{I}(C)$ . Moreover,  $\{2\} \vee \{3\} = C(2, 3) = \{2, 3\}$  and  $\{2\} \wedge \{3\} = C(\{2\} \cap \{3\}) = C(\emptyset) = \emptyset$ .

Since  $C$  has such a lattice representation it is path independent.

*Proof of Theorem 4.* The proof is by construction. Consider a path-independent choice rule  $C$ . If  $X = X_C^\#$  holds for every  $X \in \mathcal{I}(C)$  with  $|X| \leq q$  then  $C$  is  $q$ -acceptant and there is nothing to prove because a choice rule is trivially an expansion of itself. Otherwise, consider a minimal such set  $X$  (minimality is with respect to set inclusion). Take any  $x \in X_C^\# \setminus X$ . Then there exists a path-independent expansion  $C'$  of  $C$  where there is an additional element

<sup>21</sup>Reflexive: For all  $a \in \mathcal{A}$ ,  $a \succeq a$ .

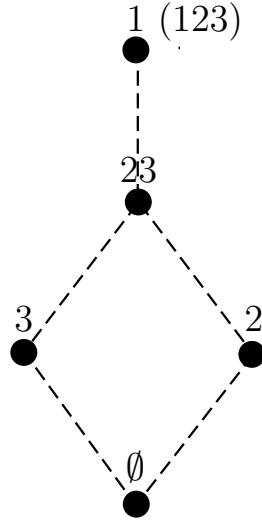


FIGURE 1. Lattice representation of the choice function in Example 1

$X \cup \{x\}$  in  $\mathcal{I}(C')$ , so  $\mathcal{I}(C') = \mathcal{I}(C) \cup \{X \cup \{x\}\}$  (Johnson and Dean, 2001, Theorem 6). By construction  $C'$  is a path-independent expansion of  $C$  where  $(X \cup \{x\})_{C'}^\# = X_C^\#$  and  $X_{C'}^\# = X_C^\# \setminus \{x\}$ . If  $C'$  is  $q$ -acceptant, then we are done. Otherwise, repeat this procedure of expanding the choice rule by adding an element in the lattice. Since  $|\mathcal{I}(C)|$  grows by one at each step, the procedure ends in a finite number of steps because the number of contracts is finite. □

In the proof, we construct an algorithm for finding a  $q$ -acceptant path-independent expansion of any choice rule with maximum cardinality  $q$ . Next we provide the following example to illustrate how this algorithm works.

*Example 3.* Consider choice rule  $C$  in Example 1. The lattice representation of  $C$  is shown on Figure 2 (the leftmost lattice). The maximum cardinality of  $C$  is 2 but  $C$  is not 2-acceptant because  $|C(123)| = 1$ . According to the algorithm, we pick a minimal set  $X$  (with respect to set inclusion) such that  $X_C^\# \neq X$  and  $|X| \leq q$ . The unique such set for  $C$  is  $\{1\}$  with  $\{1\}_C^\# = \{1, 2, 3\}$ . We pick any contract  $x$  in  $\{1, 2, 3\} \setminus \{1\} = \{2, 3\}$ . Let  $x = 2$ . In the lattice representation, we add the node  $X \cup \{x\} = \{1, 2\}$  such that  $\{1, 2\}^\# = \{1, 2, 3\}$ . In other words, we modify the choice rule so that  $C(1, 2, 3) = \{1, 2\}$ . In the new lattice representation,  $\{1\}^\# = X_C^\# \setminus \{x\} = \{1, 3\}$  (the middle lattice on Figure 2). This new path-independent choice rule, say  $C'$ , is still not 2-acceptant because  $|C'(13)| = 1$ . Thus, we find a minimal set  $Y$  such that  $Y_{C'}^\# \neq Y$  and  $|Y| \leq q$ . The unique such set is  $\{1\}$  with  $\{1\}^\# = \{1, 3\}$ . We pick  $y \in \{1, 3\} \setminus \{1\} = \{3\}$  and create an additional node in the lattice



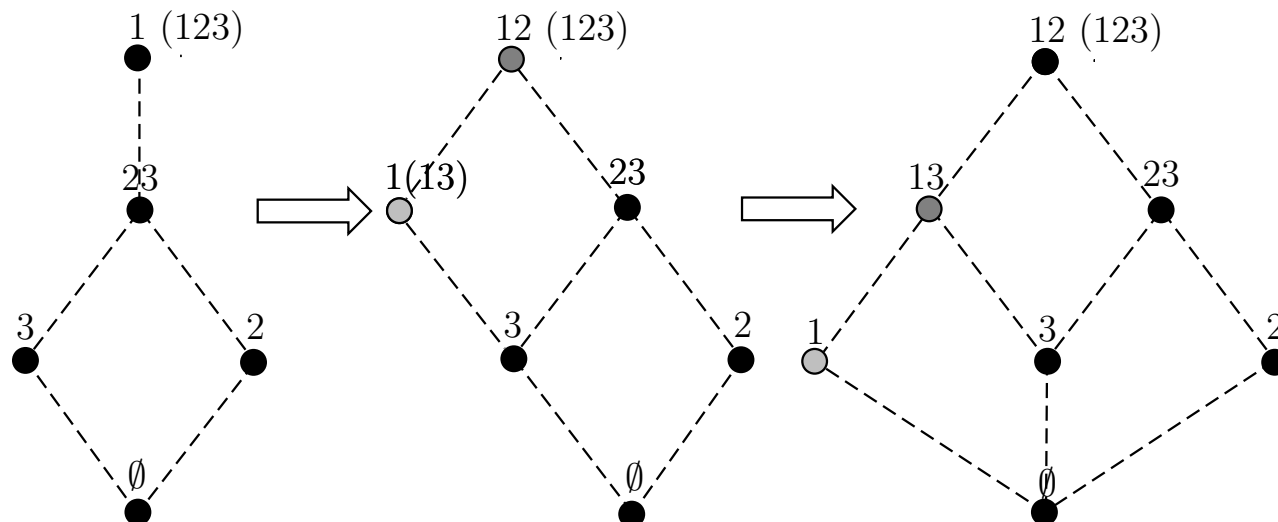


FIGURE 2. Expansion of the choice function in Example 1

for  $\{1, 3\}$ . The new choice rule is a 2-acceptant path-independent expansion of  $C$  and  $C'$  (the rightmost lattice on Figure 2).

One particular class of path-independent choice rules used in the literature is the class of responsive choice rules. They have been used extensively in matching theory including the seminal work of Gale and Shapley (1962) and real matching markets such as the National Resident Matching Program (Roth and Peranson, 1999) and the school choice problem (Abdulkadiroğlu and Sönmez, 2003). Here, we provide an axiomatic characterization of these rules. The characterization is useful because it allows us to discuss the normative content of this widely used choice rule.

Responsive choice rules satisfy the following rationality axiom.

**Definition 7.** *Choice rule  $C$  on  $\mathcal{P}(\mathcal{X})$  satisfies the **weakened weak axiom of revealed preference (WWARP)** if, for every  $x, y \in \mathcal{X}$ ,  $X \subseteq \mathcal{X}$ , and  $Y \subseteq \mathcal{X}$  such that  $x, y \in X \cap Y$ ,*

$$x \in C(X) \text{ and } y \in C(Y) \setminus C(X) \text{ imply } x \in C(Y).$$

WWARP rules out cycles of length two if we use the revealed preference ordering over individual contracts. To be more precise, say that  $x$  is **chosen over**  $y$  if there exists a set containing both such that  $x$  is chosen from the set and  $y$  is not. Then WWARP states that whenever  $x$  is chosen over  $y$ ,  $y$  is not chosen over  $x$ . WWARP alone does not give us a preference relationship over individual contracts because we need to rule out cycles of any length not only two. However, WWARP together with acceptance allow us to rule out all cycles (see Theorem 5).

WWARP was first introduced by Ehlers and Sprumont (2008) in the context of nonrationalizable choice, though it was implicitly discussed in Wilson (1970), who analyzed the class of choice rules satisfying it (calling them “Q cuts”). We show that a choice rule is responsive if and only if it satisfies acceptance and WWARP. Since responsive choice rules are path independent, this result also implies that path independence follows from acceptance and WWARP.

**Theorem 5.** *Choice rule  $C$  is responsive if and only if it satisfies acceptance and WWARP.*

*Proof.* It is obvious that any responsive choice rule satisfies acceptance and WWARP. Conversely, suppose that  $C$  satisfies acceptance and WWARP. Define  $\succ^*$  by  $x \succ^* y$  if there exists  $X$  for which  $\{x, y\} \subseteq X$ ,  $x \in C(X)$  and  $y \notin C(X)$ . WWARP is equivalent to asymmetry of the relation  $\succ^*$ . We claim that  $\succ^*$  is transitive.

Suppose that  $x \succ^* y \succ^* z$ . Associated with acceptance is  $k \in \mathbb{N}_+$  such that  $|C(X)| = \min\{k, |X|\}$ . Since  $x \succ^* y$ , there exists  $X$  for which  $\{x, y\} \subseteq X$ ,  $x \in C(X)$  and  $y \notin C(X)$ . Hence,  $|C(X)| = k$ . Consequently, there exists  $\{a_1, \dots, a_{k-1}\} \subseteq X$  for which  $C(X) = \{x, a_1, \dots, a_{k-1}\}$ . This implies  $a_i \succ^* y$  for all  $i$ . Obviously,  $a_i \neq y$  for all  $i$ . Now, consider  $Y = \{x, y, z, a_1, \dots, a_{k-1}\}$ . Suppose, by means of contradiction, that  $z \in C(Y)$ . Then  $y \in C(Y)$ ; otherwise, we have  $y \succ^* z$  and  $z \succ^* y$  contradicting asymmetry. Now, since  $y \in C(Y)$ , we have  $x \in C(Y)$  and  $a_i \in C(Y)$  for all  $i$ ; otherwise, we would have either  $x \succ^* y$  and  $y \succ^* x$  or  $a_i \succ^* y$  and  $y \succ^* a_i$  for some  $i$ . But  $|Y| \geq k + 1$ , so that  $|C(Y)| \geq k + 1$ , contradicting the assumption that  $C$  is acceptant. Similarly we show that  $y \notin C(Y)$ . Since  $C$  is  $q$ -acceptant, this implies that  $C(Y) = \{x, a_1, \dots, a_{k-1}\}$ . So  $x \succ^* z$ .

The rest is now standard; by the Szpilrajn Theorem (see for example Duggan (1999)), there is a preference relation  $\succeq$  for which  $x \succ^* y$  implies  $x \succ y$ . Clearly, if  $x \in C(X)$  and  $y \succeq x$  and  $y \in X$ , we have  $y \in C(X)$ . Otherwise, we would have  $x \succ^* y$ , and  $y \succ x$ , a contradiction. By definition,  $C$  is responsive with respect to  $\succeq$ . This kind of construction was first introduced by Aleskerov et al. (2007) and Tyson (2008).  $\square$

Responsivity in the case of  $|C(X)| = 1$  is equivalent to the standard notion of rationalizability by a preference relation. And moreover, when  $|C(X)| = 1$  for all  $X$ , WWARP is equivalent to the classical weak axiom of revealed preference (see, e.g., Uzawa (1956) or Arrow (1959)).<sup>22</sup> Thus, Theorem 5 is a generalization of the well-known result that the weak axiom of revealed preference characterizes rationalizability when choice is single valued and all sets of contracts are available.

Theorem 5 provides a characterization of responsive choice rules for any  $q$ . Now, we analyze  $q$ -responsive choice rules for a given  $q$ . For this characterization, a weaker version of WWARP is useful.

<sup>22</sup> The *weak axiom of revealed preference* states that if there exists  $X \subseteq \mathcal{X}$  such that  $x \in C(X)$  and  $y \in X \setminus C(X)$  then there does not exist any  $Y \subseteq \mathcal{X}$  such that  $y \in C(Y)$  and  $x \in Y$ .

**Definition 8.** *Choice rule  $C$  satisfies the  $q$ -weakened weak axiom of revealed preference ( $q$ -WWARP) if, for every  $x, y \in \mathcal{X}$ ,  $X \subseteq \mathcal{X}$ , and  $Y \subseteq \mathcal{X}$  such that  $x, y \in X \cap Y$  and  $|X| = |Y| = q$ ,*

$$x \in C(X) \text{ and } y \in C(Y) \setminus C(X) \text{ imply } x \in C(Y).$$

This axiom is weaker than WWARP and it is derived by restricting the sets for which WWARP applies.

Finally, we provide a characterization of  $q$ -responsiveness.

**Theorem 6.** *Choice rule  $C$  is  $q$ -responsive if and only if it satisfies substitutability,  $q + 1$ -WWARP, and  $q$ -acceptance.*

The proof of Theorem 6 is provided in the Appendix.

An earlier result, due to Eliaz et al. (2011), provides another characterization of  $q$ -responsive rules. While obviously the two characterizations are mathematically equivalent, there are a few differences. First, our characterization is based on the classical revealed preference relation, while the result of Eliaz et al. (2011) does not make direct use of the concept. Our result can also be extended to a case in which the choice function is not defined on all of  $\mathcal{P}(\mathcal{X})$ , but rather on a subset. It would be enough to postulate that the revealed preference relation is acyclic in this case, as will be evident from the proof.

Now, we ask whether a path-independent and acceptant choice rule is uniquely determined by its behavior on some subclass of sets. Indeed, it turns out that a  $q$ -acceptant path-independent choice rule is characterized completely by its behavior on sets of cardinality  $q + 1$ . While this result is not difficult to prove, it can be very useful computationally. Reporting the value of a choice rule on every admissible set requires reporting the value of the choice rule on  $2^{|\mathcal{X}|} - 1$  sets, a quantity that is exponential in the number of contracts. On the other hand, reporting the value of the choice rule on sets of cardinality  $q + 1$  requires a report only for  $\binom{|\mathcal{X}|}{q+1}$  sets; this is polynomial of degree  $q + 1$ . In fact, one only needs to report the rejected contract from sets of cardinality of  $q + 1$ .

**Theorem 7.** *Suppose  $C$  and  $C'$  are two  $q$ -acceptant and path-independent choice rules on  $\mathcal{P}(\mathcal{X})$  such that for all  $X \subseteq \mathcal{X}$  with  $|X| = q + 1$ , we have  $C(X) = C'(X)$ . Then  $C = C'$ .*

*Proof.* For any  $X$  for which  $|X| \leq q$ , we know that  $C(X) = C'(X)$  since both of  $C$  and  $C'$  are  $q$ -acceptant. Let  $l > q + 1$  and suppose we have shown that for all  $X$  such that  $|X| < l$ ,  $C(X) = C'(X)$ . Let  $X$  be a set for which  $|X| = l$ . Let  $x \in X$ . By path independence,  $C(X) = C(C(X \setminus \{x\}) \cup \{x\})$ . Now, we claim that  $|C(X \setminus \{x\}) \cup \{x\}| < l$ . Otherwise, we must have  $C(X \setminus \{x\}) = X \setminus \{x\}$ . However, this contradicts the assumption that  $C$  is  $q$ -acceptant, as  $|X \setminus \{x\}| = l - 1 > q$ . Similarly,  $C'(X) = C'(C'(X \setminus \{x\}) \cup \{x\})$  and  $|C'(X \setminus \{x\}) \cup \{x\}| < l$ .

Since  $|X \setminus \{x\}| < l$ , by the induction hypothesis,  $C(X \setminus \{x\}) \cup \{x\} = C'(X \setminus \{x\}) \cup \{x\}$ , which therefore implies that  $C(X) = C'(X)$ .  $\square$

For a more general result, including a characterization of the implications of path independence on the sets of cardinality  $q + 1$ , see Dietrich (1987).

## 5. Firm Mergers

In this section, we provide a new model of firm mergers for two-sided markets using our framework and we establish welfare consequences of firm mergers for both workers and the remaining firms in the market. Our results have important implications beyond economics such as antitrust, merger litigation, and law.

To study firm mergers, we introduce a new agent to the market who can be thought of as a ‘representative’ for multiple firms. For simplicity, we consider the merger of two firms, say  $f$  and  $f'$ . Suppose that firms  $f$  and  $f'$  have a phantom representative denoted by  $r$ . We consider a modified matching model where the set of firms is given by

$$\hat{F} \equiv (F \setminus \{f, f'\}) \cup \{r\}.$$

In  $\hat{F}$ , we replace firms  $f$  and  $f'$  by representative  $r$ . However, we do not modify the set of workers or the set of contracts. When these two firms are separate, the representative is just an intermediary handling the offers made or received by these two firms. As a result, the choice rule of the representative  $r$  is given by

$$C_r(X) \equiv C_f(X) \cup C_{f'}(X).$$

When the firms merge then firm  $r$  may potentially have a different choice rule. We look at two possible cases in which the unified firm can expand or consolidate.

**Merge and Expand:** After the two firms merge, they can decide to expand. This can be done by hiring new employees or by making the unified firm more efficient. In terms of the Aizerman-Malishovski decomposition, the merged firm has all the positions of firms  $f$  and  $f'$  and possibly more. This means that the choice rule of the representative expands. That is the new choice rule, say  $\hat{C}_r$ , is an expansion of the old choice rule  $C_r$ :

$$\text{for any set of contracts } X, \hat{C}_r(X) \supseteq C_r(X).$$

Now we can study the welfare implications of this merger for workers as well as firms using Corollary 1.

**Theorem 8.** *Suppose that all choice rules are path independent and that either version of the deferred acceptance algorithm is used. Then when firms merge and expand, every worker is better off and every other firm in the market is worse off.*

Therefore, when firms merge and keep all of their contracts, then all workers benefit from the merger and all the remaining firms are worse off.

**Merge and Consolidate:** After the two firms merge, they can decide to consolidate. This can be achieved by laying off workers. Or, in terms of the Aizerman-Malishevski decomposition, the merged firm can remove some positions of firms  $f$  and  $f'$ . This means that the choice rule of the representative shrinks. Therefore, the old choice rule, say  $C_r$ , is an expansion of the new choice rule  $\hat{C}_r$ :

$$\text{for any set of contracts } X, \hat{C}_r(X) \subseteq C_r(X).$$

Now we can study the welfare implications of this merger for workers as well as firms using Corollary 1.

**Theorem 9.** *Suppose that all choice rules are path independent and that either version of the deferred acceptance algorithm is used. Then when firms merge and consolidate, every worker is worse off and every remaining firm in the market is better off.*

When firms merge and only keep a subset of their contracts, then workers are hurt by the merger and other firms benefit.

## 6. Conclusion

We have studied path-independence of choice rules that guarantees the existence of stable matchings in the context of many-to-many matching markets with contracts. Unlike most of the earlier matching literature that constructs choice rules from preferences over sets of contracts, we have taken choice rules as primitives of our model. This new approach has enabled us to make the connection between path-independent choice rules and stable matchings.

We have established several results on the theory of path-independent choice rules and stable matchings. First, we have shown how classical characterizations of path independence allow a new, simplified proof of a classical existence result for stable matching and a new result on comparative statics of stable matchings. We have applied these results to a model of firm mergers. Further, we have developed characterizations of responsive choice rules, as well as shown how to construct acceptant and path-independent choice rules from ones that are merely path independent without increasing the maximum cardinality. This implies that any path-independent choice rule can be expanded to satisfy the law of aggregate demand without increasing the maximum cardinality of the choice rule. Finally, we have investigated some of the implications of our results on practical matching markets such as residency matching and school choice.

We have also brought forth some results from the choice literature to prove our results that can be useful in matching. In particular, we have used a decomposition result of Aizerman and Malishevski (1981). This decomposition result can further be used in the matching context and establishes a natural link between many-to-many and one-to-one matching markets.

### Appendix: Omitted Proofs

In this Appendix, we include the omitted proofs.

*Proof of Theorem 1.* Let  $\mu^W$  be the matching produced by the worker-proposing deferred acceptance algorithm. First, we show that  $\mu^W$  is individually rational for all agents. By construction of the algorithm, each firm  $f$  chooses  $\mu_f^W$  at the last step of the algorithm. By path independence, we get  $C_f(\mu_f^W) = \mu_f^W$ , so  $\mu$  is individually rational for firms. For each persona of worker  $w$ , the contract held at the end of the algorithm is the best among the set of contracts who have not been rejected by firms yet, so this contract is also the best among  $\mu_w^W$  according to this persona's preference. Therefore,  $C_w(\mu_w^W) = \mu_w^W$ , so  $\mu$  is also individually rational for workers.

Second, we show that there does not exist a nonempty blocking set  $X$ . Suppose, for contradiction, that such a set exists. Since choice rules are path independent, for any  $x \in X$ ,  $\{x\}$  is also a blocking set. Fix  $x \in X$ . Let  $w = w(x)$  and  $f = f(x)$ . Since  $x \notin \mu^W$  and at least one persona of worker  $w$  prefers  $x$  over the contract that she holds at  $\mu^W$ ,  $x$  must have been rejected by firm  $f$  at some step of the algorithm. At that step, each position of firm  $f$  gets a better contract. Since each position gets a weakly better contract at each step of the algorithm, each position of firm  $f$  must have a better contract at  $\mu^W$  than  $x$ . A contradiction to the assumption that  $\{x\}$  is a blocking set because this implies  $x \in C_f(\mu_f^W \cup \{x\})$ , i.e., at least one position of firm  $f$  prefers  $x$  over the contract that it is holding at  $\mu^W$ .

Therefore,  $\mu^W$  is a stable matching. Next, we show that  $\mu^W$  is the worker-optimal stable matching. Consider worker  $w$ . Suppose that  $\{\succeq_i\}_{i \in I}$  is the Aizerman-Malishevski decomposition of  $C_w$ . Let  $w^i$  be the persona associated with the preference relation  $\succeq_i$ . Say that contract  $x$  is **achievable** for worker persona  $w^i$  if there exists a stable matching  $\mu$  such that  $x = \max_{y \in \mu_w} \succeq_i$ , i.e., if worker persona  $w^i$  holds contract  $x$  at some stable matching. We show that no achievable contract is rejected in the worker-proposing deferred acceptance algorithm.

**Lemma 5.** *No achievable contract is rejected in the worker-proposing deferred acceptance algorithm.*

*Proof.* Suppose that the claim holds for steps 1 through  $k$ . We show that it also holds for step  $k+1$ . Assume, for contradiction, that achievable contract  $x$  is rejected by firm  $f \equiv f(x)$  at step  $k+1$ . This means that all positions of firm  $f$  have better contracts than  $x$ . Let  $X$  denote the set of contracts that firm  $f$  is holding at the end of step  $k+1$ .

Now consider the stable matching, say  $\mu$ , in which worker persona  $w^i$  holds contract  $x$ . Let  $f^j$  be the firm position which holds this contract. Let  $y$  be the contract that firm position  $f^j$  is holding at  $X$ . Let  $w^k$  be the worker persona holding contract  $y$  at the end of step  $k+1$ . Contract  $y$  is preferred over contract  $x$  by firm position  $f^j$ . Since  $\mu$  is a stable matching, it

must be that worker persona  $w^k$  is holding a contract  $z$  that it prefers over contract  $y$  at stable matching  $\mu$ . But this is a contradiction because achievable contract  $z$  must have been rejected before step  $k + 1$ .  $\square$

Lemma 5 implies that each worker position is holding the best achievable contract at the end of the algorithm. As a result,  $\mu^W$  is the worker-optimal stable matching. Finally, we show that  $\mu^W$  is the firm-pessimal stable matching. Consider  $x \in \mu^W$ . Let  $w \equiv w(x)$  and  $w^i$  be a worker persona associated with this contract. Let  $y$  be any other stable matching contract that  $w^i$  gets. Worker persona  $w^i$  prefers contract  $x$  over contract  $y$  since  $\mu^W$  is the worker-optimal stable matching. As a result, any stable matching outcome is better for persona of  $f(x)$  associated with  $x$  in  $\mu^W$ . That means each firm position gets the worst achievable outcome in the worker-proposing deferred acceptance algorithm. Consequently,  $\mu^W$  is the firm-pessimal stable matching.  $\square$

*Proof of Theorem 2.* Since we have two profiles of choice rules, we prefix the choice rule profiles to stability, individual rationality, and no blocking to avoid confusion. Let  $C = (C_{\mathcal{W}}, C_{\mathcal{F}})$  and  $C' = (C_{\mathcal{W}}, (C'_{\hat{f}}, C_{-\hat{f}}))$ .

**Proof of Part 1:** Since  $\mu$  is  $C$ -stable, we have  $C_i(\mu_i) = \mu_i$  for every  $i \in \mathcal{W} \cup \mathcal{F}$ . In particular,  $C_{\hat{f}}(\mu_{\hat{f}}) = \mu_{\hat{f}}$ . Since  $C'_{\hat{f}}$  is an expansion of  $C_{\hat{f}}$ , we have  $C'_{\hat{f}}(\mu_{\hat{f}}) \supseteq C_{\hat{f}}(\mu_{\hat{f}}) = \mu_{\hat{f}}$ . This implies  $C'_{\hat{f}}(\mu_{\hat{f}}) = \mu_{\hat{f}}$  because  $C'(X) \subseteq X$  for every set  $X$ . Therefore,  $\mu$  is  $C'$ -individually rational.

If there are no  $C'$ -blocking sets for  $\mu$ , then  $\mu$  is  $C'$ -stable and we can take  $\mu' = \mu$  to get the result. Suppose, otherwise, that there are  $C'$ -blocking sets. For each contract in the blocking set, the associated firm with every contract must be  $\hat{f}$  because  $C_i = C'_i$  for every  $i \neq \hat{f}$  and  $\mu$  is  $C$ -stable.

Consider  $X = \{x | f(x) = \hat{f}, x \in C_{w(x)}(\mu_{w(x)} \cup \{x\})\}$ , the set of contracts associated with firm  $\hat{f}$  that workers would like to add. Consequently, firm  $\hat{f}$  can choose from  $X$ . Note that  $X \supseteq \mu_{\hat{f}}$  because  $\mu$  is  $C$ -individually rational for workers. Let  $\hat{X} = C'_{\hat{f}}(X)$ . Because  $C'_{\hat{f}}$  is an expansion of  $C_{\hat{f}}$ , we have  $\hat{X} \supseteq C_{\hat{f}}(X)$ . Since  $\mu$  is a  $C$ -stable set,  $C_{\hat{f}}(X) = \mu_{\hat{f}}$ . As a result, we get  $\hat{X} \supseteq \mu_{\hat{f}}$ . The assumption that there exists a  $C'$ -blocking pair implies  $\hat{X} \not\supseteq \mu_{\hat{f}}$ .

Consider the following matching:

$$\mu_0 = \hat{X} \cup_{f \neq \hat{f}} \mu_f.$$

In  $\mu_0$ , firm  $\hat{f}$  gets  $\hat{X}$  while the rest of the firms get their assigned contracts in  $\mu$ . By construction,  $\mu_0$  is individually rational for firms. We show that the firm-proposing DA starting at  $\mu_0$  produces a  $C'$ -stable matching.

**Step 1:** Each position in firm  $f$  considers its most preferred contract in  $\mu_0$  and applies to the associated worker with this contract. Each worker  $w$  considers the set of

contracts that has been offered to it, say  $X_w^1$ , tentatively accepts  $C_w(X_w^1)$  and rejects the rest. If there are no rejections then stop.

**Step  $k$ :** Each position whose contract was rejected at Step  $k - 1$  considers its next preferred contract if such a contract exists and applies to the associated worker with this contract. Otherwise, this position does not apply to any worker. Each worker  $w$  considers all of the new contracts that have been offered to it and the tentatively accepted contracts at Step  $k - 1$ , say  $X_w^k$ , accepts  $C_w(X_w^k)$  and rejects the rest. If there are no rejections, stop.

The algorithm ends in finite time since there is a finite number of contracts and at least one rejection at every step of the algorithm except the last one. Note that at the end of Step 1, each worker persona is weakly better off compared to  $\mu$  since  $\mu_0 \supseteq \mu$ .

Let  $\mu'$  be the outcome of this algorithm. Suppose, for contradiction, that  $\mu'$  is not  $C'$ -stable. By construction,  $\mu'$  is  $C'$ -individually rational, so there must exist a nonempty blocking set. Choose any contract  $x$  in this blocking set. Since choice rules are path independent,  $\{x\}$  is also a blocking set. Let  $f = f(x)$  and  $w = w(x)$ .

We consider two cases depending on whether  $f = \hat{f}$  or not.

**Case 1 ( $f = \hat{f}$ ):** Since  $\{x\}$  is a blocking set,  $x \in C_w(\mu'_w \cup \{x\})$ . Because each worker persona gets a better firm at each step of DA, we must have  $x \in C_w(\mu_w \cup \{x\})$  and as a result  $x \in X$ . Note that worker  $w$  must have never received  $x$  as an offer in DA. If  $x \in C'_{\hat{f}}(X) = \hat{X}$ , then at least one position of firm  $\hat{f}$  offers  $x$  to worker  $w$  at Step 1, a contradiction. If  $x \notin \hat{X}$ , then at least one position of firm  $f$  must have offered  $x$  to worker  $w$  later in the algorithm since  $x \in C'_f(\mu'_f \cup \{x\})$ , a contradiction.

**Case 2 ( $f \neq \hat{f}$ ):** Since  $\{x\}$  is a blocking set for  $\mu'$ ,  $x \in C_w(\mu'_w \cup \{x\})$ . Moreover, each worker persona gets a weakly better contract at each step of DA. Therefore,  $x \in C_w(\mu_w \cup \{x\})$ . Since  $\mu$  is a  $C$ -stable matching, we must have  $x \notin C_f(\mu_f \cup \{x\})$ . That means, each position of firm  $f$  has a better contract at  $\mu$ . But since  $\{x\}$  is a blocking set for  $\mu'$ ,  $x \in C_f(\mu'_f \cup \{x\})$ . In other words, there exists a position of firm  $f$  who has a worse contract at  $\mu'$  than  $x$ . This implies that this position must have been rejected by worker  $w$  for contract  $x$  at some step of the algorithm, which is a contradiction since there exists a worker persona which has a worse contract at  $\mu'$ .

Therefore,  $\mu'$  is a  $C'$ -stable matching. In addition, each persona of every worker is getting a weakly better firm at every step of the algorithm, so  $C_w(\mu'_w \cup \mu_w) = \mu'_w$ . Similarly, each position of every firm  $f \neq \hat{f}$  is getting a weakly worse worker at every step of the algorithm, so  $C_f(\mu'_f \cup \mu_f) = \mu'_f$ .

**Proof of Part 2:** The proof is analogous to the proof of the first part and it works as follows. Start with  $C'$ -stable matching  $\mu'$ . Let  $\mu_0 \equiv C_{\hat{f}}(\mu'_{\hat{f}}) \cup_{f \neq \hat{f}} \mu'_f$ . Note that  $\mu_0$  is individually rational for workers since each worker gets a subset of contracts that she gets



in  $\mu'$ . Therefore, we can start the worker-proposing deferred acceptance algorithm at  $\mu^0$  for  $C$ . Let  $\mu$  be the outcome. We first show that matching  $\mu$  is  $C$ -stable. By construction, it is individually rational for all agents. Suppose, for contradiction, that there exists a blocking set. Choose any contract  $x$  in this blocking set. Since choice rules are path independent,  $\{x\}$  is also a blocking set. Let  $f = f(x)$  and  $w = w(x)$ . We consider two cases depending on whether  $f = \hat{f}$  or not.

**Case 1 ( $\mathbf{f} = \hat{\mathbf{f}}$ ):** Since each firm position gets a weakly better contract at every step of the algorithm, contract  $x$  could not have been offered to firm  $\hat{f}$  in the algorithm. Therefore,  $x \in C_w(\mu_w \cup \{x\})$  implies  $x \in C_w(\mu'_w \cup \{x\})$ . Since  $\mu'$  is a  $C' \equiv (C_{\mathcal{W}}, (C'_{\hat{f}}, C_{-\hat{f}}))$  stable set, we get  $x \notin C'_{\hat{f}}(\mu'_{\hat{f}} \cup \{x\})$ . Since  $C'_{\hat{f}}$  is an expansion of  $C_{\hat{f}}$ ,  $x \notin C_{\hat{f}}(\mu'_{\hat{f}} \cup \{x\})$ . Furthermore, each firm position gets weakly better in the algorithm, so  $x \notin C_{\hat{f}}(\mu'_{\hat{f}} \cup \{x\})$  implies  $x \notin C_{\hat{f}}(\mu_{\hat{f}} \cup \{x\})$ , a contradiction.

**Case 2 ( $\mathbf{f} \neq \hat{\mathbf{f}}$ ):** Like the previous case,  $x \in C_w(\mu_w \cup \{x\})$  implies  $x \in C_w(\mu'_w \cup \{x\})$ . In addition, each firm position gets a weakly better contract, so  $x \in C_f(\mu_f \cup \{x\})$  implies  $x \in C_f(\mu'_f \cup \{x\})$ . Therefore,  $\{x\}$  is a blocking set for  $mu'$  when choice rule profile is  $C'$ , which is a contradiction.

Thus,  $\mu$  is a stable matching. Consequently, each firm other than  $\hat{f}$  revealed prefers  $\mu$  to  $\mu'$  and every worker revealed prefers  $\mu'$  over  $\mu$ .  $\square$

*Proof of Theorem 3.* Since  $C'_{\hat{f}}$  is not an expansion of  $C_{\hat{f}}$ , there exists a set of contracts  $X$  such that  $C'_{\hat{f}}(X) \not\supseteq C_{\hat{f}}(X)$ .

Say that a choice rule is responsive if there exist a preference relation over the contracts and a capacity for which the choice rule always selects the highest-ranked available contracts according to the preference relation without going over the capacity. A contract  $x$  is acceptable if it is ranked above the empty contract  $\emptyset$ . Construct workers' choice rules as follows.

If there does not exist  $x \in X$  such that  $w = w(x)$ , then worker  $w$  has the empty choice rule in which no contract is ever chosen. However, if there exists  $x \in X$  such that  $w = w(x)$ , then worker  $w$  has a responsive choice rule in which the capacity is  $|\{x \in X : w = w(x)\}|$  and the preference is any ranking of  $|\{x \in X : w = w(x)\}|$  where other contracts are not acceptable.

Regardless of choice rules for the remaining firms there exists a unique stable matching. Suppose that the choice rule profile is  $(C_{\mathcal{W}}, C_{\mathcal{F}})$ . Then  $\mu \equiv C_{\hat{f}}(X)$  is the unique stable matching. The reason is as follows. Any individually rational matching is a subset of  $\mu$ . Furthermore, for any strict subset of  $\mu$ , the set of contracts  $x \in \mu$  not included is a blocking set. Likewise, there exists a unique stable matching for the choice rule profile  $(C_{\mathcal{W}}, (C'_{\hat{f}}, C_{-\hat{f}}))$ , which is  $\mu \equiv C_{\hat{f}}(X)$ .

Take any  $x \in \mu \setminus \mu'$ . Let  $w \equiv w(x)$ . Then worker  $w$  does not revealed prefer  $\mu'$  to  $\mu$  because worker  $w$  does not fill her capacity in  $\mu'$  and there exists at least one contract in  $\mu \setminus \mu'$  associated with her, which implies  $C_w(\mu' \cup \mu) \neq \mu'(w)$ .  $\square$

*Proof of Theorem 6.* First we prove the following lemma.

**Lemma 6.** *If choice rule  $C$  satisfies substitutability,  $q + 1$ -WWARP, and  $q$ -acceptance, then it satisfies WWARP.*

*Proof.* Suppose for contradiction that  $C$  satisfies substitutability,  $q + 1$ -WWARP,  $q$ -acceptance but not WWARP. Therefore, there are  $X, Y, x$ , and  $y$  for which  $x, y \in X \cap Y$ ,  $x \in C(X)$ ,  $y \notin C(X)$ ,  $y \in C(Y)$ ,  $x \notin C(Y)$ . By  $q$ -acceptance, it follows that  $|X| \geq q + 1$  and  $|Y| \geq q + 1$ , and by  $q + 1$ -WWARP, at least one of these inequalities is strict.

Now,  $C(C(X) \cup \{y\}) = C(X)$  by substitutability and  $q$ -acceptance; likewise,  $C(C(Y) \cup \{x\}) = C(Y)$ . But note that  $|C(X) \cup \{y\}| = |C(Y) \cup \{x\}| = q + 1$ , and that  $x \in C(C(X) \cup \{y\})$ ,  $y \notin C(C(X) \cup \{y\})$ ,  $y \in C(C(Y) \cup \{x\})$ , and  $x \notin C(C(Y) \cup \{x\})$ , a contradiction to  $q + 1$ -WWARP.  $\square$

The proof follows immediately from this lemma and Theorem 5.  $\square$

## References

- Abdulkadiroğlu, Atila and Tayfun Sönmez**, “School choice: A mechanism design approach,” *American Economic Review*, June 2003, *93* (3), 729–747.
- Aizerman, Mark A. and Andrew V. Malishevski**, “General theory of best variants choice: Some aspects,” *Automatic Control, IEEE Transactions on*, 1981, *26* (5), 1030–1040.
- Aleskerov, Fuad, Denis Bouyssou, and Bernard Monjardet**, *Utility maximization, choice and preference*, Springer Verlag, 2007.
- Alkan, Ahmet**, “On preferences over subsets and the lattice structure of stable matchings,” *Review of Economic Design*, March 2001, *6* (1), 99–111.
- and **David Gale**, “Stable schedule matching under revealed preference,” *Journal of Economic Theory*, 2003, *112* (2), 289 – 306.
- Arrow, Kenneth J.**, *Social choice and individual values*, Yale Univ Pr, 1951.
- , “Rational choice functions and orderings,” *Economica*, 1959, *26* (102), 121–127.
- Aygün, Orhan and Tayfun Sönmez**, “Matching with Contracts: Comment,” *American Economic Review*, 2013, *103* (5), 2050–2051.
- Blair, Charles**, “The Lattice Structure of the Set of Stable Matchings with Multiple Partners,” *Mathematics of Operations Research*, November 1988, *13* (4), 619–628.

- Blum, Yosef, Alvin E. Roth, and Uriel G. Rothblum**, “Vacancy Chains and Equilibrium in Senior-Level Labor Markets,” *Journal of Economic Theory*, October 1997, 76 (2), 362–411.
- Cherepanov, Vadim, Timothy Feddersen, and Alvaro Sandroni**, “Rationalization,” *Theoretical Economics*, 2013, 8 (3), 775–800.
- Clippel, Geoffroy De and Kfir Eliaz**, “Reason-based choice: A bargaining rationale for the attraction and compromise effects,” *Theoretical Economics*, 2012, 7 (1), 125–162.
- Crawford, Vincent P.**, “Comparative statics in matching markets,” *Journal of Economic Theory*, 1991, 54 (2), 389 – 400.
- Dietrich, Brenda L.**, “A circuit set characterization of antimatroids,” *Journal of Combinatorial Theory, Series B*, 1987, 43 (3), 314–321.
- Dilworth, Robert P.**, “Lattices with unique irreducible decompositions,” *The Annals of Mathematics*, 1940, 41 (4), 771–777.
- Duggan, John**, “A general extension theorem for binary relations,” *Journal of Economic Theory*, 1999, 86 (1), 1–16.
- Echenique, Federico**, “Counting combinatorial choice rules,” *Games and Economic Behavior*, 2007, 58 (2), 231 – 245.
- and **Jorge Oviedo**, “A Theory of Stability in Many-to-Many Matching Markets,” *Theoretical Economics*, June 2006, 1 (2), 233–273.
- and **M. Bumin Yenmez**, “How to Control Controlled School Choice,” 2012. Caltech SS Working Paper 1366.
- Edelman, Paul H. and Robert E. Jamison**, “The theory of convex geometries,” *Geometriae dedicata*, 1985, 19 (3), 247–270.
- Ehlers, Lars and Yves Sprumont**, “Weakened WARP and top-cycle choice rules,” *Journal of Mathematical Economics*, 2008, 44 (1), 87–94.
- , **Isa E. Hafalir, M. Bumin Yenmez, and Muhammed A. Yildirim**, “School choice with controlled choice constraints: Hard bounds versus soft bounds,” *Journal of Economic Theory*, 2014, 153, 648 – 683.
- Eliaz, Kfir, Michael Richter, and Ariel Rubinstein**, “Choosing the two finalists,” *Economic Theory*, 2011, 46 (2), 211–219.
- Fleiner, Tamás**, “A Fixed-Point Approach to Stable Matchings and Some Applications,” *Mathematics of Operations Research*, February 2003, 28 (1), 103–126.
- Gale, David and Lloyd S. Shapley**, “College Admissions and the Stability of Marriage,” *The American Mathematical Monthly*, jan 1962, 69 (1), 9–15.
- and **Marilda Sotomayor**, “Ms. Machiavelli and the Stable Matching Problem,” *The American Mathematical Monthly*, April 1985, 92 (4), 261–268.

- Hafalir, Isa E., M. Bumin Yenmez, and Muhammed A. Yildirim**, “Effective affirmative action in school choice,” *Theoretical Economics*, May 2013, 8 (2), 325–363.
- Hatfield, John William and Fuhito Kojima**, “Substitutes and stability for matching with contracts,” *Journal of Economic Theory*, 2010, 145 (5), 1704 – 1723.
- \_\_\_\_\_ and **Paul R. Milgrom**, “Matching with Contracts,” *American Economic Review*, September 2005, 95 (4), 913–935.
- \_\_\_\_\_ and **Scott Duke Kominers**, “Contract Design and Stability in Many-to-Many Matching,” 2012. Working Paper.
- Johnson, Mark R. and Richard A. Dean**, “Locally complete path independent choice functions and their lattices,” *Mathematical Social Sciences*, 2001, 42 (1), 53 – 87.
- Kamada, Yuichiro and Fuhito Kojima**, “General Theory of Matching Under Distributional Constraints,” 2014. Working paper.
- Kelso, Alexander S. and Vincent P. Crawford**, “Job Matching, Coalition Formation, and Gross Substitutes,” *Econometrica*, 1982, 50, 1483–1504.
- Klaus, Bettina and Markus Walzl**, “Stable many-to-many matchings with contracts,” *Journal of Mathematical Economics*, 2009, 45 (78), 422 – 434.
- Kojima, Fuhito and Mihai Manea**, “Axioms for Deferred Acceptance,” *Econometrica*, 2010, 78 (2), 633–653.
- Konishi, Hideo and M. Utku Ünver**, “Credible group stability in many-to-many matching problems,” *Journal of Economic Theory*, 2006, 129 (1), 57 – 80.
- \_\_\_\_\_ and \_\_\_\_\_, “Games of Capacity Manipulation in Hospital-intern Markets,” *Social Choice and Welfare*, 2006, 27 (1), 3–24.
- Koshevoy, Gleb A.**, “Choice functions and abstract convex geometries,” *Mathematical Social Sciences*, 1999, 38 (1), 35 – 44.
- Loomes, Graham, Chris Starmer, and Robert Sugden**, “Observing violations of transitivity by experimental methods,” *Econometrica*, 1991, 59 (2), 425–439.
- Manzini, P. and M. Mariotti**, “Sequentially rationalizable choice,” *American Economic Review*, 2007, 97 (5), 1824–1839.
- Martínez, Ruth, Jordi Massó, Alejandro Neme, and Jorge Oviedo**, “On the invariance of the set of Core matchings with respect to preference profiles,” *Games and Economic Behavior*, March 2012, 74 (2), 588–600.
- Mas-Colell, A., M. D. Whinston, and J. R. Green**, *Microeconomic Theory*, Oxford University Press, New York, 1995.
- Masatlioglu, Yusufcan, Daisuke Nakajima, and Erkut Y. Ozbay**, “Revealed Attention,” *American Economic Review*, 2012, 102 (5), 2183–2205.
- Moulin, Herve**, “Choice functions over a finite set: A summary,” *Social Choice and Welfare*, 1985, 2 (2), 147–160.

- , *Axioms of cooperative decision making*, Vol. 15, Cambridge University Press, 1991.
- Ostrovsky, Michael**, “Stability in Supply Chain Networks,” *American Economic Review*, June 2008, *98* (3), 897–923.
- Pathak, Parag A. and Tayfun Sönmez**, “School Admissions Reform in Chicago and England: Comparing Mechanisms by Their Vulnerability to Manipulation,” *American Economic Review*, September 2013, *103* (1), 80–106.
- Plott, Charles R.**, “Path independence, rationality, and social choice,” *Econometrica*, 1973, *41* (6), 1075–1091.
- Roth, Alvin E.**, “Stability and Polarization of Interests in Job Matching,” *Econometrica*, January 1984, *52* (1), 47–57.
- and **Elliott Peranson**, “The Redesign of the Matching Market for American Physicians: Some Engineering Aspects of Economic Design,” *American Economic Review*, sept 1999, *89* (4), 748–780.
- and **Marilda Sotomayor**, *Two-sided Matching: A Study in Game-Theoretic Modelling and Analysis*, Vol. 18 of *Econometric Society Monographs*, Cambridge University Press, Cambridge England, 1990.
- Sönmez, Tayfun and Utku Ünver**, “Course Bidding at Business Schools,” *International Economic Review*, 2010, *51* (1), 99–123.
- Tversky, Amos**, “Intransitivity of preferences,” *Psychological Review*, January 1969, *76* (1), 31–48.
- Tyson, Christopher J.**, “Cognitive constraints, contraction consistency, and the satisficing criterion,” *Journal of Economic Theory*, 2008, *138* (1), 51–70.
- Uzawa, Hirofumi**, “Note on preference and axioms of choice,” *Annals of the Institute of Statistical Mathematics*, 1956, *8* (1), 35–40.
- Wilson, Robert B.**, “The finer structure of revealed preference,” *Journal of Economic Theory*, 1970, *2* (4), 348–353.