Liquidity, Innovation, and Endogenous Growth*

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Abstract

We study optimal liquidity management, innovation, and production decisions for a continuum of firms facing financing frictions and the threat of creative destruction. We show that while financing frictions prompt firms to decrease production, they may also spur investment in innovation. We examine which types of firms substitute production for innovation in response to negative operating shocks that decrease cash availability. We embed our firm dynamics into a model of endogenous growth and show that financing frictions have contrasting effects on economic growth.

Keywords: Innovation; Cash management; Financial constraints; Endogenous growth

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1 Introduction

"Yes, I mean that. Less money raised leads to more success. That is the data I stare at all the time." (*Fred Wilson, co-founder of Union Square Ventures*)[1]

Innovation is pivotal to economic growth. The emergence of new products and improvements in existing goods shape the world in which we live and drive the dynamics of the economy. Nonetheless, investment in innovation (or R&D) is costly and difficult to finance. It requires a (possibly long) gestation period before becoming productive, it is not pledgeable, and its outcome is highly uncertain. To safeguard financial flexibility, innovative firms respond to these financing frictions by hoarding cash (or liquidity).[2] Cash availability then affects firms' production and innovation decisions and factors into economic growth.

The goal of this paper is to build a tractable framework to understand the links among cash accumulation, innovation, and economic growth. To this end, we introduce financing frictions and corporate cash hoarding into a model of endogenous growth. We study an economy in which the innovations of constrained incumbents and entrants are the key drivers of economic growth. Using the model, we first investigate how financing frictions affect the production and innovation mix in relation to various firm characteristics. We then embed the constrained firms' maximization problem into a general equilibrium setting and study the effects on economic growth.

We start by solving the optimization problem of a continuum of incumbent firms facing the threat of creative destruction—that is, of losing their market positions when new firms launch higher quality products. Firms are financially constrained in that they have costly access to external financing. These financial constraints make firms effectively risk-averse and generate an incentive to retain earnings in cash reserves, as highlighted by previous contributions.[3] Additionally, we allow firms to manage the risk-return trade-off

[1] http://avc.com/2013/09/maximizing-runway-can-minimize-success/

[2] Among others, Lyandres and Palazzo (2015), Ma, Mello, and Wu (2013), Falato, Kadyrzhanova, and Sim (2013), Falato and Sim (2014), and Begenau and Palazzo (2015) suggests that investment in R&D is a major determinant of corporate cash reserves.

[3] See, e.g., Décamps, Mariotti, Rochet, and Villeneuve (2011), Bolton, Chen, and Wang (2011, 2013), and Hugonnier, Malamud, and Morellec (2015). In these models, as in ours, the marginal value of cash

by adjusting production and investment in innovation in response to operating shocks that change corporate cash availability.

We show that constrained firms scale down production and increase markups in response to negative operating shocks. These shocks decrease a firm's cash reserves and weaken its financial strength. The firm then seeks to limit cash flow risk and operating costs to avert a costly refinancing. As a result, production is cyclical (and markups countercyclical) to firm-specific shocks.

Despite the tendency to scale down production, we highlight that constrained firms may *increase* their investment in innovation in the face of negative operating shocks. That is, firms may substitute production for innovation when cash reserves decrease. Although investment in innovation is inherently uncertain, it can be highly rewarding as it may lead to technological breakthroughs. When firms attain such breakthroughs, they access monopoly rents from the brand-new technology and can raise fresh funds in light of a success (the achievement of technological improvements) rather than a failure (running out of funds because of bad operating performance).

In our analysis of the relation between innovation and corporate cash, we discriminate between target cash reserves and deviations from this target. Consistent with the available evidence (see footnote 2), we show that innovation prompts firms to hoard cash reserves. Moreover, we predict how firms will adjust their investment in innovation as their cash reserves fall below the target level. We illustrate that firms with low operating margins, volatile profits, or efficient innovation technologies (i.e., more path-breaking or requiring a shorter gestation period) will increase investment in innovation as their cash reserves decrease. In the real world, these characteristics are typical of small and young firms, which substantially contribute to the advancement of the technological frontier despite their financial constraints (see Akcigit and Kerr, 2015, or Acemoglu, Akcigit, Bloom, and Kerr, 2013). As we quote at the beginning, *"Less money [...] leads to more success."* Our results can also rationalize the observation that investment in innovation did not decrease uniformly during the recent financial crisis.[4]

decreases with the level of cash reserves and reaches one at the target level.

[4]For example, Archibugi, Filippetti, and Frenz (2013) report that some firms have actually increased

In our model, production decisions affect cash flow volatility. In turn, cash flow volatility affects the dynamics of stock prices. In contrast to previous dynamic models with financing frictions (see, e.g., Décamps, Mariotti, Rochet, and Villeneuve, 2011), we find that the relation between marginal value of cash and idiosyncratic stock volatility can be positive or negative. While we observe the leverage effect (i.e., a decline in stock price leads to an increase in stock return volatility) for relatively unconstrained firms, the relation between changes in stock prices and volatility of stock returns can become positive when liquidity constraints are tight.

We then embed our firms' dynamics into the general equilibrium setting. We focus on a "balanced growth path" equilibrium in which all aggregate quantities grow at an endogenous, constant rate. In equilibrium, incumbents solve their maximization problem by taking the market interest rate as given. In each industry, incumbent's value and the rate of creative destruction (equivalently, the entry rate) are jointly determined by the free-entry condition. Incumbents' and entrants' innovations aggregate and determine the rate of economic growth. In turn, the rate of economic growth feeds back into firms' decisions through the market interest rate, which is pinned down by the representative household's Euler equation.

We find that financing frictions generate two opposite effects on economic growth. First, financing frictions act as entry barriers and deter innovation by new firms. Second, for this very reason, incumbents increase their investment in innovation in that they expect profits to last longer before losing their market position due to fresh entry.[5] As explained, financing frictions may also urge incumbents to put more emphasis on innovation in response to negative operating performance. Overall, we find that financing frictions decrease the entrants' contribution to economic growth and increase the incumbents' contribution with respect to an identical, but unconstrained, economy. This second (enhancing) effect can dominate in industries featuring large entry costs (for example, those requiring expensive technologies). Interestingly, our analysis reveals that financing frictions may be *not* welfare-decreasing.

their R&D investment despite the toughening constraints.

[5]This effect is similar to Aghion and Howitt (1992).

Related literature Our paper contributes to the literature that studies the determinants of corporate cash reserves, which has been spurred by the increase in cash-to-asset ratios of U.S. firms over the last few decades (see, e.g., Bates, Kahle, and Stulz, 2009). Notable contributions are, e.g., Riddick and Whited (2009), Décamps, Mariotti, Rochet, and Villeneuve (2011), Bolton, Chen, and Wang (2011, 2013), and Hugonnier, Malamud, and Morellec (2015). Within this strand, the papers that analyze the relation between cash reserves and investment do so by considering neoclassical models of investment and capital accumulation, either incremental (as in Bolton, Chen, and Wang 2011, 2013) or lumpy (as in Hugonnier, Malamud and Morellec, 2015). To the best of our knowledge, our paper is the first to study corporate cash management problem in a Schumpeterian framework.[6] This problem is economically important given the documented relations between cash reserves and R&D investment and between R&D and economic growth (see e.g. Caballero and Jaffe, 1993, Akcigit and Kerr, 2015, or Kogan, Papanikolaou, Seru, and Stoffman, 2014). Nonetheless, it is not trivial because it involves solving a problem with an infinite number of (sequentially arriving) stochastic growth options.

The paper also contributes to the literature on the financing of innovation. Lyandres and Palazzo (2015) and Ma, Mello, and Wu (2014) investigate and test the relation among competition, R&D investment, and cash reserves. Falato, Kadyrzhanova, and Sim (2013) study and test the relation between investment in intangible capital and cash holdings in a neoclassical model with no growth. While we abstract from strategic competition within industries, we contribute to this strand by studying the joint dynamics of cash, production, and innovation, as well as their impact on economic growth. As mentioned, our model studies the relation between R&D and target cash reserves *and* between R&D and deviations from this target. While we acknowledge that R&D investment prompts firms to hoard precautionary cash reserves, we characterize how firms will adjust their investment in innovation when operating shortfalls erode cash reserves so that they fall below the target level.

The link between R&D and corporate cash holdings is supported by vast empirical ev-

[6]The key difference in neoclassical capital accumulation models and Schumpeterian innovation models is that, in the former models, investment implies an immediate increase in the output, whereas the payoff from investment in R&D arrives at uncertain (Poisson) times.

idence. Among others, Hall (2005) and Hall and Lerner (2010) document that innovation is best financed through internal funds because it is subject to asymmetric information, it is not pledgeable, and has highly uncertain returns. Brown, Fazzari, and Petersen (2009) report that financing constraints greatly affect innovation at both the firm level and aggregate level, and document that innovation decisions are related to the supply of internal and external equity finance.[7] Consistently, Hall (2005), Hall and Lerner (2010), Rajan (2012), Brown, Martisson, and Petersen (2013), and Acharya and Xu (2015) emphasize the relative importance of equity rather than debt for the financing of innovation, as we do in our theoretical model.

Schumpeter (1942) emphasized the importance of innovation for economic growth by introducing the concept of "creative destruction." The work of Schumpeter has led to the development of a subfield of macroeconomics known as "Schumpeterian models of endogenous growth;" see Aghion, Akcigit, and Howitt (2014) for an overview. Our model belongs to this literature. In particular, we build on the literature on endogenous technological change (see also Romer, 1990, Grossman and Helpman, 1991, or Klette and Kortum, 2004) and assume that both incumbent firms and a continuum of potential entrants pursue investment in innovation, as in the recent contributions of Acemoglu and Cao (2015), Akcigit and Kerr (2015), or Acemoglu, Akcigit, Bloom, and Kerr (2013).

Our paper also relates to the literature studying the effects of financial constraints on economic growth; see Levine (2005) and Beck (2012) for two surveys. Thus far, however, relatively little attention has been paid to the role of corporate liquidity in providing flexibility to constrained firms. Our model seeks to fill this gap.

The paper is organized as follows. Section 2 describes the model. Section 3 solves the model in a benchmark unconstrained economy, which serves as a counterfactual to single out the effects of financing frictions on innovation and growth. Section 4 solves the model in the economy featuring financing frictions. Section 5 provides a quantitative assessment

[7] Brown, Fazzari, and Petersen (2009) report that, in the US, young publicly traded firms in high-tech industries finance R&D investment almost entirely with internal and external equity as debt financing is difficult due to non-pledgeability. Citing from their paper, *"While the large literature on finance and economic growth has good reasons to focus on debt and credit constraints, our results suggest that more attention should be given to equity finance [...] for models that emphasize innovation. [...] External equity is the more relevant substitute for internal cash flow for young high-tech firms."*

of the model's predictions. Section 6 concludes. All proofs are in the Appendix.

2 The Model

Throughout the paper, time is continuous and uncertainty is modeled by a probability space (Ω, F, P), equipped with a filtration $(F_t)_{t\geq 0}$ that represents the information available at time t. We study an economy in which the representative household maximizes constant relative risk aversion (CRRA) preferences:

$$\int_0^{\infty} e^{-\rho t}\, \frac{\mathcal{C}_t^{1-\theta}-1}{1-\theta}\, dt\,, \tag{1}$$

where $\mathcal{C}_t$ is consumption at time t, ρ is the discount rate, and θ is the inverse of elasticity of intertemporal substitution. The household supplies labor inelastically and receives labor wage denoted by W_t. Population is constant at L.

Final good sector. There is one multipurpose final good $\mathcal{Y}_t$ serving as the numeraire of the economy. The final good is produced competitively using labor and a continuum of inputs (or intermediate goods) $j \in [0, 1]$. The production technology of the final good is:

$$\mathcal{Y}_t \;=\; \frac{1}{1-\beta}\int_0^1 L^{\beta}\, \bar{X}_{jt}^{1-\beta}\, q_{jt}^{\beta}\, dj\;, \qquad \beta \in [0,1]. \tag{2}$$

In this equation, $\bar{X}_{jt}$ is the quantity of the input j and q_{jt} is its quality. Initial quality is normalized to one; i.e., $q_{j0} = 1$ for any $j \in [0, 1]$. We assume that only the highest quality version of each input j is used in the production function of the final good. As detailed below, improvements in quality of inputs occur via two channels: innovation by incumbent firms and creative destruction by entrants.

Intermediate goods sector. Each input j is produced by the latest innovator of the industry j. Each innovator enforces a patent on the brand-new version of the input j and becomes the incumbent monopolist of the industry j. The patent is assumed to last forever. Nonetheless, patents do not prevent firms from investing in innovation and improving further the quality of inputs.

Each incumbent monopolist j produces one input j and at the same time invests in innovation to further enhance its quality. We denote the innovation intensity (or innovation rate) of the incumbent j by z_{jt}. Innovation is costly, and its outcome is uncertain. Specifically, we assume that an incumbent firm paying the flow cost

$$\Phi(z_{jt}, q_{jt}) = \zeta \, \frac{z_{jt}^2}{2} \, q_{jt}, \quad \zeta > 0,$$

increases the quality of its input at Poisson rate ϕz_{jt}. These Poisson events represent technological breakthroughs, whose likelihood increases with z_{jt}. When an incumbent j achieves a technological breakthrough, the quality of the input j jumps from q_{jt^-} to

$$q_{jt} \; = \; \lambda q_{jt^-} \, ,$$

where $\lambda > 1$ represents the incremental improvement in quality.[8]

The dynamics of incumbents' cash flows $d\Pi_{jt}$ satisfy

$$d\Pi_{jt} = \left[\bar{X}_{jt}(p_{jt} - 1) - \Phi(z_{jt}, q_{jt}) \right] dt \; + \; \sigma \bar{X}_{jt} dZ_{jt} \, . \tag{3}$$

For any industry j, $\bar{X}_t$ is the production rate, p_t is the selling price, and Z_t is a standard Brownian motion representing random operating shocks. Operating shocks are firm-specific and independent across firms. To ease the notation, the marginal cost of production is normalized to one. The quantities $\bar{X}_{jt}$ and p_{jt} are endogenously determined.

The cash flow process in equation (3) implies that incumbent firms can make operating profits as well as losses. If access to outside financing was frictionless, losses could be covered by raising funds whenever needed. We depart from this assumption and assume

[8] See also Acemoglu and Cao (2015) or Acemoglu et al. (2013). We do not allow incumbents to operate multiple product lines (as instead Klette and Kortum, 2004): each firm innovates only in one product line. As in Aghion, Howitt, and Mayer-Foulkes (2005), Acemoglu, Aghion, and Zilibotti (2006), Acemoglu and Cao (2015), or Lucas and Nicolini (2015), we abstract from skilled labor in the intermediate-good sector.

that firms face financing costs, as Bolton, Chen, and Wang (2011, 2013).[9] Specifically, incumbent firms can cover operating shortfalls incurring the cost $\epsilon > 0$ for any dollar raised (in the following, "routine financing"). Incumbents can also raise funds once they attain technological breakthroughs, by promising a surplus share to the financiers. In particular, financiers can extract a fraction α of the surplus created S.[10] Notably, the way we model financing frictions seeks to capture features that are typical among R&D-intensive firms (see, e.g., Lerner et al., 2012). These firms usually delay refinancing events until they are in the extreme need of funds or they achieve technological breakthroughs. In fact, it is easier to attract financiers when breakthroughs occur in that they act as a positive signal (e.g., the first proof of product feasibility).

To maintain financial flexibility, incumbents can retain earnings in cash reserves. We denote cash reserves at time t by $\bar{C}_{jt}$. Cash reserves earn a rate of interest δ, which is lower than the market interest rate r. Thus, holding cash entails an opportunity cost. The cost $r - \delta$ can be interpreted as an agency cost of free cash flows (as in Décamps, Mariotti, Rochet, and Villeneuve, 2011) or as tax disadvantages (as in Riddick and Whited, 2009). This opportunity cost may also arise when the supply of liquidity in the economy is scarce, see Krishnamurthy and Vissing-Jorgensen (2012) (In Appendix A.5, we extend the model and endogenize δ via this channel, by introducing the government as liquidity provider). Nothwithstanding its microfoundation, the wedge between r and δ leads to a dynamic trade-off between dividend payout and liquidity hoarding. The dynamics of cash reserves are given by

$$d\bar{C}_{jt} = \Big[\delta\bar{C}_{jt} + \bar{X}_{jt}(p_{jt} - 1) - \Phi(z_{jt}, q_{jt})\Big]dt + \sigma\bar{X}_{jt}dZ_{jt} - d\bar{D}_{jt} + d\bar{F}_{jt} + d\bar{F}^{I}_{jt}. \tag{4}$$

The terms in the square brackets represent the return on cash reserves plus *expected cash flows*. $d\bar{D}_{jt} \geq 0$ is the instantaneous flow of payouts, $d\bar{F}_{jt} \geq 0$ is the net inflow from rou-

[9] As Bolton, Chen, and Wang (2011, 2013), financing costs are exogenous in the model. Financing costs could be microfounded through limited enforcement (Albuquerque and Hopenhayn, 2002), asymmetric information (Clementi and Hopenhayn, 2002), or limited pledgeability (Holmstrom and Tirole, 2011).

[10] Nash bargaining over surplus between the firm and financiers provides a microfoundation for our assumption. Denoting the bargaining power of financiers by $\alpha \in [0, 1]$ and the surplus created by S, the rents extracted by financiers are $\Gamma^* = \arg\max_{\Gamma \geq 0} \Gamma^\alpha \left[S - \Gamma\right]^{1-\alpha} = \alpha S$, whereas the firm retains $(1-\alpha)S$.

tine financing, and $d\bar{F}_t^I \geq 0$ is the net inflow from financing technological breakthroughs.

Each incumbent maximizes its value $V(t, \bar{C}_j, q_j)$ until creative destruction hits. Specifically, it chooses production, innovation, financing, and payout policies to maximize the present value of future dividends subject to the budget constraint (4) and the non-negativity of cash reserves. When a new firm markets a higher quality version of the input j (i.e., when creative destruction hits), the incumbent j loses its market position. As is common in Schumpeterian models, obsolescence drives the liquidation value of the exiting incumbent's productive assets to zero. Thus, shareholders only recover the firm's cash reserves.

Entrants. We assume that there is a mass of entrepreneurs who develop blueprints aimed at improving the quality of inputs. Creative destruction hits an industry j whenever a new firm markets a higher quality version of the input j. Successful blueprints have the potential to improve the quality of inputs by a factor $\Lambda > 1$, i.e. quality may jump from q_{jt^-} to

$$q_{jt} \;=\; \Lambda q_{jt^-}$$

once the blueprint is implemented. We do not impose a priori restrictions on the relation between λ and Λ.[11]

Entrepreneurs do not collaborate with incumbents due to frictions in the sale of ideas (see also Kondo and Papanikolaou, 2013 and Silveira and Wright, 2010). To implement blueprints, entrepreneurs need to set up a firm. Setting up a firm producing the input j at quality level q_{jt} requires a technological cost $\kappa_T q_{jt} L$. To cover this cost, entrepreneurs contact financiers and incur a financing cost $\kappa_I q_{jt} L$. Thus, the total entry cost is

$$K_{jt} = (\kappa_T + \kappa_I) L q_{jt} = \Lambda(\kappa_T + \kappa_I) L q_{jt-}. \tag{5}$$

[11] We allow for different sizes of quality jumps for incumbents and entrants; see Acemoglu and Cao (2015) or Akcigit and Kerr (2015). These contributions assume that $\Lambda > \lambda$ to model the difference between "incremental" innovations by incumbents and "radical" innovations by entrants. The difference between λ and Λ is not essential to our analysis. However, we set $\Lambda > \lambda$ in the numerical analysis.

The ratio $\frac{\kappa_I}{\kappa_T+\kappa_I}$ is thus a measure of financiers' ability to extract rents from entrants.[12]

Balanced growth path. We focus on a balanced growth path equilibrium in which all aggregate quantities grow at constant rate g, which is endogenously determined. For any industry j, the entry rate (equivalently, the rate of creative destruction) is denoted by x_d and determined by the free-entry condition

$$V(t, \bar{C}_j^*, \Lambda q_j) - \bar{C}_{jt}^* = K_{jt}. \tag{6}$$

To preserve tractability, we assume that entrants receive initial financing equal to the entry cost plus cash amounting to their target level of cash reserves.[13] As we show below, this target level always exists and can be characterized explicitly.

To solve the model, we first derive incumbents' optimal policies by taking the market interest rate, growth rate, and entry rate as given. We then aggregate in equilibrium and endogenize these quantities. An equilibrium is an allocation such that: (i) Incumbents set production, innovation, payout, and financing decisions to maximize the expected present value of net dividends; (ii) New firms enter at a rate that makes the free-entry condition binding; (iii) The final good producer maximizes profits; (iv) The representative household maximizes utility from consumption; (v) All markets clear.

3 Benchmark unconstrained economy

We first solve the model in a benchmark economy with no financing frictions ($\epsilon = \alpha = \kappa_I = 0$).[14] In this setting, operating shortfalls can be covered by raising external funds immediately and costlessly. Firms have thus no incentive to keep cash.

[12]Due to the free-entry condition (6), this is the fraction of firm value (net of cash) that financiers extract from entrants. Since information asymmetries reduce entrants' bargaining power relative to that of incumbents, we assume that $\frac{\kappa_I}{\kappa_T+\kappa_I} > \alpha$.

[13]Consistently, Begenau and Palazzo (2015) document that R&D-intensive firms have entered with more and more initial cash reserves over the last decades. One may argue that financiers may be reluctant to provide the target cash amount due to adverse selection problems. We solved a version of the model in which new firms enter with heterogeneous cash endowments. Our main results hold unchanged.

[14]In this unconstrained setting, our model is similar to Acemoglu and Cao (2015).

We start by solving the optimization problem of the final good sector,

$$\max_{\bar{X}_j} \frac{1}{1-\beta} \int_0^1 L^\beta \, \bar{X}_{jt}^{1-\beta} \, q_{jt}^\beta dj \; - \; \int_0^1 p_{jt} \bar{X}_{jt} dj \,, \tag{7}$$

which delivers the demand curve for the highest quality version of each input $j \in [0,1]$:

$$\bar{X}_{jt} = L \left(\frac{q_{jt}^\beta}{p_{jt}} \right)^{\frac{1}{\beta}} . \tag{8}$$

Taking the demand schedule of the final good sector as given, incumbent firms in the intermediate goods sector maximize profits and set the monopoly price[15]

$$p_{jt} = \frac{1}{1-\beta} \equiv p^*. \tag{9}$$

This price implies a constant markup $\frac{\beta}{1-\beta}$ above the marginal cost and is associated with the production rate $\bar{X}_{jt}^* = q_{jt} L X^*$. In this equation,

$$X^* = (1-\beta)^{\frac{1}{\beta}} \tag{10}$$

denotes the time-invariant production rate scaled by quality. In the following, we will often refer to scaled quantities in order to ease the model solution. Substituting (9) and (10) into (2), we obtain the aggregate output

$$\mathcal{Y}_t^* \;=\; L(1-\beta)^{\frac{1}{\beta}-2} \int_0^1 q_{jt} dj \,, \tag{11}$$

and labor wage

$$W_t^* \;=\; L\beta(1-\beta)^{\frac{1}{\beta}-2} \int_0^1 q_{jt} dj \,. \tag{12}$$

[15] As in other contributions, we rule out the case of limit pricing to keep the analysis tractable. For instance, Aghion and Howitt (1992) assume that innovations are always drastic so that a monopolist is unconstrained by potential competition from the previous patents. Akcigit and Kerr (2015) assume that the current incumbent and the former incumbents in a given product line enter a two-stage price-bidding game whereby each firm pays a fee to announce its price. Under this assumption, only the new leader pays the fee and announces its price. We adopt similar assumptions in our setup.

We turn to study incumbents' optimal policies (henceforth, the subscript j will be suppressed when it causes no confusion). We denote incumbents' value in the benchmark economy by $V(t,q)$. Notice that the value function is independent of cash reserves in this unconstrained environment. Following standard arguments, $V(t,q)$ satisfies the Hamilton-Jacobi-Bellman (HJB) equation:

$$r^* V(t,q) - V_t(t,q) = \max_{z^*} \left\{ \beta L q \left(1-\beta\right)^{\frac{1}{\beta}-1} - \frac{(z^*)^2}{2}\zeta q + \phi z^* \left[V(t,\lambda q) - V(t,q) \right] - x_d^* V(t,q) \right\}. \tag{13}$$

In this equation, we denote by r^* and x_d^* the market interest rate and the rate of creative destruction, which are endogenously determined later in the analysis. The terms on the left-hand side represent the return required by investors and the change in firm value as time elapses, respectively. The first two terms on the right-hand side represent operating cash flows net of production and innovation costs. The third term is the probability-weighted change in value when the incumbent markets a technological breakthrough. The fourth term is the value discount due to creative destruction. To solve the maximization problem, we conjecture firm value to be linear in q_t,

$$V(t,q) = V(q_t) = q_t L v^*, \tag{14}$$

for some function $v^* > 0$. In the following, we will refer to v^* as scaled firm value. Substituting (14) into (13), we obtain

$$\max_{z^*} \left\{ \mu^* - \frac{(z^*)^2}{2}\frac{\zeta}{L} + \phi z^* (\lambda v^* - v^*) - x_d v^* \right\} = r v^*, \tag{15}$$

where the auxiliary quantity

$$\mu^* \equiv \beta \left(1-\beta\right)^{\frac{1}{\beta}-1} \tag{16}$$

represents scaled net profits. The maximization of (15) delivers the optimal innovation decision:

$$z^* = \phi \frac{L}{\zeta} (\lambda - 1) v^*. \tag{17}$$

The optimal z^* increases with scaled firm value v^*, with the size of quality improvements

λ, and with the Poisson coefficient ϕ. By substituting (17) into (15), we derive the closed-form expression for v^* reported in Proposition 1. We plug this expression into the free-entry condition,

$$v^*(x_d^*) = \kappa_T\,, \tag{18}$$

which allows us to derive the entry rate x_d^* (or rate of creative destruction).

The expected increase in quality of an input j is given, per time interval dt, by

$$E_{t-}[dq_{jt}] = q_{jt-}\left[(\lambda-1)\phi z^* \;+\; (\Lambda-1)x_d^*\right]dt.$$

The first (respectively, second) term in square brackets is the incumbent's (entrant's) contribution to the quality improvement of the input j. Such improvements occur at independent Poisson times in different industries $j \in [0,1]$. By the law of large numbers, the rate of economic growth satisfies

$$\int_0^1 q_t(j)dj \;=\; e^{g^* t}, \tag{19}$$

where

$$g^* \;=\; (\lambda-1)\phi z^* \;+\; (\Lambda-1)x_d^*\,. \tag{20}$$

Finally, the maximization problem of the representative household delivers the standard Euler equation. Since aggregate output and consumption grow at rate g^*, the equilibrium interest rate satisfies

$$r^* = \rho + \theta g^*. \tag{21}$$

We summarize our findings in Proposition 1.

Proposition 1 *In the unconstrained economy, scaled firm value is given by:*

$$v^* = \frac{2\mu^*}{(x_d^*+r^*) + \sqrt{(x_d^*+r^*)^2 - 2\mu^*\phi^2\frac{L}{\varsigma}(\lambda-1)^2}}\,.$$

The equilibrium rate of creative destruction x_d^*, the growth rate g^*, and the interest rate

r^* *satisfy*

$$
\begin{aligned}
x_d^* &= \frac{1}{1+\theta\ (\Lambda-1)}\left[\frac{\mu^*}{\kappa_T}-\rho+\kappa_T\ \phi^2\,\frac{L}{\zeta}\,(\lambda-1)^2\left(\frac{1}{2}-\theta\right)\right],\\
g^* &= \frac{1}{1+\theta\ (\Lambda-1)}\left([\Lambda-1)\left(\frac{\mu^*}{\kappa_T}-\rho\right)+\kappa_T\ \frac{\Lambda+1}{2}\ \phi^2\,\frac{L}{\zeta}\,(\lambda-1)^2\right],\\
r^* &= \frac{1}{1+\theta\ (\Lambda-1)}\left[\rho+\ \theta\,(\Lambda-1)\,\frac{\mu^*}{\kappa_T}+\kappa_T\theta\frac{\Lambda+1}{2}\phi^2\,\frac{L}{\zeta}\,(\lambda-1)^2\right].
\end{aligned}
$$

Two remarks are worth a mention. First, the growth rate g^* is non-monotonic in the entry cost. On the one hand, a low κ_T allows more entry. On the other hand, greater entry renders incumbents subject to exit due to creative destruction and decreases their innovation rate. Second, it may appear surprising that the innovation rate z^* does not depend on monopoly rents, whose magnitude is determined by the parameter β. The extant literature has placed a strong accent on the link between the incentives to innovate and the monopolistic rents a firm can extract (among others, see the seminal contributions of Arrow, 1962, Romer, 1990, or Aghion and Howitt, 1992). The reason is that this link is offset by the free entry condition: an increase in monopoly rents stimulates entry and increases creative destruction, thereby countervailing the positive effect of β on z^*. We discuss this link further in Section 4.2, i.e. for the constrained economy.

By the household budget constraint, consumption (denoted by $\mathcal{C}_t^*$ in the unconstrained economy) is the sum of labor wage (W_t^*) and dividends (d_t^*) net of financing to the corporate sector (f_t^*):

$$\mathcal{C}_t^*\ =\ W_t^*+d_t^*-f_t^*\,,$$

where

$$d_t^*\ =\ \int_0^1\left(\mu^*Lq_{jt}-\frac{(z^*)^2}{2}\zeta q_{jt}\right)dj\ =\ \left(\mu^*-\frac{(z^*)^2}{2}\ \frac{\zeta}{L}\right)Le^{g^*t}\,, \tag{22}$$

$$f_t^*\ =\ \int_0^1 x_d^*K_{jt}dj\ \ = x_d^*\ \Lambda\kappa_T Le^{g^*t}. \tag{23}$$

Consumption grows at rate g^*; that is $\mathcal{C}_t^* = \mathcal{C}_0^* e^{g^* t}$ where

$$\mathcal{C}_0^* = W_0^* + \mu^* L - \frac{(z^*)^2}{2}\,\zeta - x_d^* \Lambda L \kappa_T\,.$$

Since labor wage also grows at rate g^*, the relation $W_t^* = W_0^* e^{g^* t}$ holds, where

$$W_0^* = L\beta(1-\beta)^{\frac{1}{\beta}-2}$$

by (12). We analyze in detail the properties of these aggregate quantities in Section 4.4 and Section 5.2, where we compare the constrained and unconstrained economies.

4 The constrained economy

We now analyze the constrained economy, in which firms have incentives to retain earnings in precautionary cash reserves. We first study the optimal production, innovation, financing, and dividend decisions of incumbent firms. We then derive the stationary cross-sectional distribution of liquidity in the economy. Using the distribution, we aggregate our firm-level analysis in general equilibrium.

4.1 Deriving the value of incumbent firms

Recall that $V(t,\bar{C},q)$ represents the time-t value of an incumbent firm producing an input at quality level q and with cash reserves $\bar{C}$.[16] The precautionary benefit from an additional dollar saved is decreasing in cash reserves, whereas the marginal cost is constant. We conjecture the existence of a target cash level $\bar{C}^*(q)$ at which the marginal cost and the marginal benefit are equalized. Above this target, it is optimal to pay out excess cash to shareholders.

By assumption, incumbent firms can raise external funds by incurring the proportional cost ϵ. Raising the amount $d\bar{F} > 0$ changes firm value from $V(t,\bar{C},q)$ to $V(t,\bar{C}+d\bar{F},q)-$

[16] As for the unconstrained economy, we drop the index j within our partial equilibrium analysis (when it causes no confusion).

$(1+\epsilon)d\bar{F}$. This financing option is exercised only if its marginal gain $V_c(t, \bar{C}+d\bar{F}, q)$ is larger than the marginal cost $1+\epsilon$. Incumbents find it optimal to delay these refinancing events (and thus incurring the corresponding cost) until the cash buffer is depleted. Marginal gain from external financing equals the marginal cost when the firm is out of cash; that is, the condition $V_c(t,0,q) = 1+\epsilon$ holds.

Incumbent firms can also raise funds when marketing a higher quality product (i.e., after a technological breakthrough). While this financing channel is not subject to the cost ϵ, financiers extract a fraction α of the surplus created. In this case, it is optimal for the firm to raise an amount that restores the cash reserves at the target level $\bar{C}^*(\lambda q)$. Thus, the total surplus generated by this transaction is $S(t,\bar{C},q) = V(t,\bar{C}^*(\lambda q),\lambda q) - (\bar{C}^*(\lambda q) - \bar{C}) - V(t,\bar{C},q)$ and firm value increases by $(1-\alpha)S(t,\bar{C},q)$.[17]

As in the unconstrained economy, incumbents take the demand schedule of the final good sector as given and choose their value-maximizing production rate. Differently, the optimal production $\bar{X}_t$ and innovation z_t decisions depend on the level of cash reserves. Consider first the region in which it is optimal to retain earnings $c \in [0, C^*]$. Using standard arguments, firm value satisfies the following HJB equation:

$$rV(t,\bar{C},q) - V_t(t,\bar{C},q) = \max_{z,\bar{X}} \Big\{ \frac{\sigma^2}{2}\bar{X}^2 V_{cc}(t,\bar{C},q) + \left[\delta\bar{C} + (p-1)\bar{X} - \frac{z^2}{2}q\zeta\right] V_c(t,\bar{C},q) + \phi z(1-\alpha)\big(V(t,\bar{C}^*(\lambda q),\lambda q) - (\bar{C}^*(\lambda q)-\bar{C}) - V(t,\bar{C},q)\big) + x_d\,(\bar{C} - V(t,\bar{C},q)) \Big\}. \tag{24}$$

The left-hand side admits an interpretation analogous to (13). The right-hand side is the expected change in firm value per time interval. The first and the second terms capture the effect of cash flow volatility and cash accumulation. The third term captures the

[17]If an incumbent firm does not accept the financiers' surplus sharing offer, its value jumps from $V(t,\bar{C},q)$ to $V(t,\bar{C},\lambda q)$ after a breakthrough. Since firm value is concave, marginal value of cash satisfies $V_c(t,\bar{C},\lambda q) < V_c(t,0,\lambda q) = 1+\epsilon$. Thus, raising external funds through the "routine" procedure is never optimal, whereas moving from $V(t,\bar{C},\lambda q)$ to $V(t,\bar{C}^*(\lambda q),\lambda q) - (\bar{C}^*(\lambda q) - \bar{C})$ is optimal, because the marginal value of cash is above one. Hence, it is always optimal for the firm to take the financiers' offer to share the surplus $\widetilde{S}(t,\bar{C},q) = V(t,\bar{C}^*(\lambda q),\lambda q) - (\bar{C}^*(\lambda q) - \bar{C}) - V(t,\bar{C},\lambda q)$, independent of the fraction of the surplus they would like to extract. Note that in our model we assume that financiers are extracting the fraction α of $S = \widetilde{S} + (V(t,\bar{C},\lambda q) - V(t,\bar{C},q))$. While this assumption is made for analytical tractability, one can interpret the term $\alpha(V(t,\bar{C},\lambda q) - V(t,\bar{C},q))$ either as an additional fixed cost of intermediation or as the cost of marketing the technological breakthrough.

change in value upon a technological breakthrough. Finally, the fourth term represents exit due to creative destruction. Exiting shareholders recover the firm's cash reserves.

We conjecture and verify that firm value scales with q_t

$$V(t, \bar{C}, q) \;=\; q_t L\, v\left(\frac{\bar{C}_t}{q_t L}\right) \;\equiv\; Q_t\, v(c)\,, \qquad Q_t = q_t L \tag{25}$$

for some function $v(c)$. The function $v(c)$ represents scaled firm value, whereas c represents scaled cash reserves

$$c \;\equiv\; \bar{C}_t / Q_t.$$

We conjecture that the target cash level satisfies

$$\bar{C}^*(q_t) \;=\; Q_t C^*$$

for some constant $C^* > 0$ (independent of quality). Solving for $v(c)$ rather than for $V(t, \bar{C}, q)$ eases the model solution. In fact, optimal policies are determined by solving the HJB equation for $v(c)$

$$\begin{aligned} &\max_{z,X} \left\{ \frac{\sigma^2}{2} X^2 v''(c) + v'(c) \left[\delta c \;+\; X^{1-\beta} - X - \frac{z^2}{2} \frac{\zeta}{L} \right] \right. \\ &\left. +\;\; (1-\alpha)\phi z \left[\lambda(v(C^*) - C^*) - (v(c) - c) \right] \;+\; x_d(c - v(c)) \;-\; r v(c) \right\} \;=\; 0\,, \end{aligned} \tag{26}$$

where $X(c) \equiv \bar{X}_t / Q_t$ represents scaled production.

We define the curvature (i.e., the effective "risk aversion") of the value function as

$$\gamma(c) = -\sigma^2 \frac{v''(c)}{v'(c)} \tag{27}$$

and we let $F(\gamma)$ be the unique solution to the following auxiliary equation

$$(1-\beta) F^{-\beta} \;=\; 1 + \gamma F \tag{28}$$

(see Appendix A.1 for further details). We conjecture and verify that the marginal value of cash $v'(c)$ is monotone decreasing in c (that is, $v''(c) \leq 0$). Thus, the maximization problem (26) has an interior solution for any c. Specifically, the optimal production $X(c)$ and innovation $z(c)$ decisions respectively satisfy

$$
\begin{aligned}
X(c) &= F(\gamma(c)), && (29)\\
z(c) &= \phi\,(1-\alpha)\,\frac{L}{\zeta}\,\frac{(\lambda w^* - v(c) + c)}{v'(c)}\,, && (30)
\end{aligned}
$$

where

$$
w^* \equiv w^*(C^*) = v(C^*) - C^*
$$

represents the scaled firm value at the target cash level, net of cash reserves. Substituting (29)-(30) into the HJB equation (26), we obtain:

$$
\begin{aligned}
&-\frac{\sigma^2}{2}X(c)^2\,v''(c) + v'(c)\beta X(c)^{1-\beta} + v'(c)c\,\delta \\
&+\ \frac{\varphi^2}{2}\frac{(\lambda w^* - v(c) + c)^2}{v'(c)} + x_d\,(c - v(c))\ -\ rv(c)\ =\ 0\,,
\end{aligned}
\qquad (31)
$$

where the auxiliary quantity

$$
\varphi^2 \ \equiv\ \phi^2\,(1-\alpha)^2\left(\frac{L}{\zeta}\right) \qquad (32)
$$

represents the effective rate at which the incumbent improves its input.

To pin down firm value, we solve (31) subject to the following boundary conditions:

$$
\begin{aligned}
v'(0) &= 1+\epsilon\ , && (33)\\
v'(C^*) &= 1, && (34)\\
v''(C^*) &= 0\,. && (35)
\end{aligned}
$$

Equation (33) means that when cash reserves are depleted, the marginal value of cash equals the marginal cost of raising funds (i.e., $1+\epsilon$). Equation (34) means that the marginal value of one dollar at C^* equals its value if paid out to shareholders. As c approaches the target C^*, the firm's effective risk-aversion γ goes to zero and precautionary concerns are gradually relaxed. Above C^*, excess cash is paid out to shareholders and firm value is linear:

$$v(c) = v(C^*) + c - C^* \quad \text{for} \quad c > C^*.$$

Equation (35) is the super-contact condition at C^*, which guarantees that the dividend threshold C^* maximizes firm value.

By substituting (34) and (35) into (31), we can derive the scaled incumbent value at the target cash level. By straightforward calculations, we obtain

$$w(C^*) \;=\; v(C^*) - C^* = \frac{2\left[\mu^* - (r-\delta)C^*\right]}{x_d + r + \left[(x_d+r)^2 - 2\varphi^2(\lambda-1)^2(\mu^* - (r-\delta)C^*)\right]^{1/2}}, \tag{36}$$

where μ^* is defined in (16). We summarize our findings in the next proposition.[18]

Proposition 2 *Suppose that ϵ is sufficiently small. Then, the value of an incumbent firm is given by (25), where $v(c)$ is the unique concave solution to (31) satisfying the boundary conditions (33)-(35). The value of an incumbent firm holding its target level of cash reserves is given by $w(C^*) + C^*$ as from (36). By the free-entry condition, the following equality holds:*

$$w^*(C^*) = \kappa_T + \kappa_I. \tag{37}$$

Using these results, we better characterize optimal corporate policies in the next section.

4.2 Characterizing corporate policies

Firm value and liquidity hoarding. We start by investigating firm value as a function of cash reserves. The concavity of the value function and the free-entry condition (37) together imply that the inequality $v(c) \leq c + w^*(C^*) = c + \kappa_T + \kappa_I$ holds. To gain a deeper understanding of the impact of liquidity frictions on firm value, we compute a Taylor approximation for $v(c)$ in a neighborhood of C^* in the next Proposition.

[18]While our analytical proof holds for small ϵ, numerical results indicate that firm value exists and is concave for all reasonable values of ϵ.

Proposition 3 *Suppose that ϵ is sufficiently small. For any c in a left neighborhood of C^*, the value of an incumbent firm can be approximated as follows:*

$$v(c) \approx v(C^*) - (C^* - c) - \frac{1}{6}v_3(C^* - c)^3 + \frac{1}{24}v_4(C^*)(c - C^*)^4$$

where

$$\begin{aligned} v_3 &= \frac{2\ (r-\delta)}{\sigma^2 X_0^2}, \\ v_4(C^*) &= -\frac{v_3}{X_0}\left[4\ \frac{r-\delta}{\beta} + \frac{2\ \beta}{\sigma^2(1-\beta)} + \frac{1}{\sigma^2 X_0}\ \left(2\delta C^* - \varphi^2(\lambda-1)^2(\kappa_T + \kappa_I)^2\right)\right]. \end{aligned}$$

In these expressions, $X_0 = X^* = (1-\beta)^{1/\beta}$ represents the firm's production rate at C^*, which equals that of the unconstrained economy (10).

Proposition 3 illustrates that the opportunity cost of cash $r - \delta$ has an unambiguous, negative effect on firm value. Indeed, it increases the cost of the firm's precautionary hedge. Conversely, firm value is non-monotonic in σ conditional on a value of C^*. This effect raises the question of the dependence of C^* on σ. The following proposition addresses this question by investigating the determinants of C^*.

Proposition 4 *Suppose that ϵ is sufficiently small. The target cash level C^* is given by*

$$\begin{aligned} C^* &\approx C_1^* \sqrt{\epsilon} + C_2^* \epsilon \\ C_1^* = & \left(\frac{2}{v_3}\right)^{\frac{1}{2}}, \quad C_2^* = \frac{v_4(0)}{3\ v_3^2} \end{aligned}$$

The first crucial observation is that, to the highest order, the target level C^* is proportional to the square root of the refinancing cost, $\epsilon^{1/2}$. Thus, a small financing friction ϵ can have a large effect on firm value and corporate decisions, as we show in the following. To the highest order, the target level of cash reserves is proportional to the quotient $\frac{\sigma X_0(\beta)}{(r-\delta)^{1/2}}$. Hence, it is largely driven by cash flow volatility and the opportunity cost of cash, in line with previous cash holdings models. The target level also depends on the elasticity β. A larger β means that a firm sets a higher markup and produces less. Proposition 4 then suggests that firms with larger monopolistic rents should hold less cash.

The next Corollary follows by substituting $v_4(0)$ from Proposition 3 into the approximation for C^* in Proposition 4.

Corollary 5 *Suppose that ϵ is sufficiently small. The target cash level C^* is increasing in λ and ϕ and decreasing in ζ.*

To safeguard financial flexibility, firms with more efficient innovation technologies (i.e., technological breakthroughs occur more often or are more path-breaking) set a larger target cash level. This is consistent with the study of Falato, Kadyrzhanova, and Sim (2013) or Falato and Sim (2014), among others. Moreover, the model is able to capture the upward trend in cash reserves driven by innovative firms documented by Begenau and Palazzo (2015). In the model, q_t increases over time and drives the increase of the target (non-scaled) level of cash reserves, $\bar{C}^*(q_t)$.

Production and markups. We next analyze optimal production decisions. Equation (29) highlights that the dynamics of $X(c)$ are fully determined by firm's effective risk aversion $\gamma(c)$ (see equation (27)) and by the function F (which solves equation (28)). Since $F(\gamma)$ is monotone decreasing in γ, so is $X(c)$. The intuition is the following. When liquidity constraints tighten (cash reserves decrease), effective risk aversion increases and the firm becomes reluctant to take on idiosyncratic risk. To limit operating volatility, the firm scales down production. Since effective risk aversion becomes negligible at the target cash level as $v''(C^*) = 0$, production at C^* equals the optimal (constant) production rate (10) of the benchmark economy. As a result, $X(c) < X_0$ for all $c < C^*$.

Recall that each incumbent firm j acts as the monopolist of the industry j. Given the demand schedule of the final good sector, selecting $X(c)$ is equivalent to setting the price

$$p(c) = X(c)^{-\beta} \;\geq X_0^{-\beta} = \frac{1}{1-\beta}.$$

Liquidity frictions then lead incumbents to deviate from the constant price p^* associated with the benchmark economy. In the constrained economy, the incumbent sets a markup equal to $p(c)-1$, which exceeds the markup set by an identical firm operating in the benchmark economy. Following negative shocks (which deplete the cash reserves), the constrained incumbent decreases production and increases markups. Thus, liquidity constraints cause markups to be *countercyclical* to idiosyncratic shocks.

The next proposition better investigates the effects of liquidity frictions on production decisions by computing a Taylor approximation of $X(c)$.

Proposition 6 *Suppose that ϵ is sufficiently small. The optimal production policy can be approximated by*

$$X(c) \approx X_0 + X_1(c - C^*) + \frac{X_2}{2}(c - C^*)^2,$$

where X_0 is defined in Proposition 3, $X_1 = \frac{2(r-\delta)}{\beta}$ and

$$X_2 = \frac{X_1}{X_0}\left[X_1(\beta+5) + \frac{2\beta}{\sigma^2(1-\beta)} + \frac{1}{\sigma^2 X_0}\left(2\delta C^* - \varphi^2(\lambda-1)^2(\kappa_T+\kappa_I)^2\right)\right].$$

This approximation implies that (1) A larger β leads to a lower sensitivity of markups to liquidity shocks; (2) Firms facing a lower rate of creative destruction (smaller x_d) or having more efficient innovation technologies (larger ϕ or larger λ) produce less (lower $X(c)$) and set higher markups.

Proposition 6 suggests that liquidity frictions create a link between production decisions and firm characteristics that is absent in the unconstrained economy. Ceteris paribus, firms with more efficient innovation technologies—e.g. whose technological breakthroughs occur more often (larger ϕ) or are more path-breaking (larger λ)— should invest more in innovation. Thus, such firms deplete cash reserves faster, have a higher effective risk-aversion, and scale down production by a larger rate when constrained.

Extant models of endogenous growth have stressed that monopoly rents generate firms' incentives to invest in innovation.[19] Proposition 6 warns that liquidity constraints

[19] See, e.g., Aghion and Howitt (1992) and Aghion, Akcigit, and Howitt (2014). Similarly, in Romer's (1990) variety model, lower rents for innovators lead to lower R&D incentives. Note, however, that Arrow (1962) argues that the incentive to invest in innovations is lower under monopolistic than under competitive conditions, due to "organizational inertia."

may generate a different, reverse relation. Firms with more efficient innovation technologies are more R&D-intensive, which makes them more financially constrained and leads them to charge higher markups. Notably, the positive effect of innovation on markups is empirically supported by Cassiman and Vanormelingen (2013).

Investment in innovation We now analyze optimal innovation decisions. Proposition 7 follows from (30) and a Taylor expansion of $z(c)$ around C^*.

Proposition 7 *Suppose that ϵ is sufficiently small. The optimal innovation rate can be approximated by*

$$z(c) \;\approx\; \tilde{z}(c) \;\equiv\; z_0 + \frac{1}{2}(C^* - c)^2\, z_2 - \frac{1}{6}(C^* - c)^3 z_3. \tag{38}$$

In this expression, we have defined $z_0 = \phi\,(1-\alpha)\,\frac{L}{\zeta}(\lambda-1)(\kappa_T+\kappa_I)$, $z_2 = -\phi\,(1-\alpha)\,\frac{L}{\zeta}\,v_3\,(\lambda-1)(\kappa_T+\kappa_I)$, and $z_3 = -\phi\,(1-\alpha)\,\frac{L}{\zeta}\,\big(v_3 \;+ v_4(\lambda-1)(\kappa_T+\kappa_I)\big)$. v_3 and v_4 are defined as in Proposition 3.

The next corollary relies on Proposition 7 and on the monotonicity properties of the following auxiliary quantity

$$\mathcal{Z} \equiv \frac{2\,(\lambda-1)(\kappa_T+\kappa_I)}{1 \;-\frac{1}{X_0}\left[4\,\frac{r-\delta}{\beta} + \;\frac{2\,\beta}{\sigma^2(1-\beta)} + \frac{1}{\sigma^2 X_0}\;\left(2\delta C^* - \varphi^2(\lambda-1)^2(\kappa_T+\kappa_I)^2\right)\;\right](\lambda-1)(\kappa_T+\kappa_I)},$$

which we use to characterize the optimal innovation rate $z(c)$.

Corollary 8 *Suppose that $\mathcal{Z} > 0$. Then,*

- *if $\mathcal{Z} > C^*$, $\tilde{z}(c)$ is monotone increasing;*
- *if $\mathcal{Z} < C^*$, $\tilde{z}(c)$ is decreasing for $c < C^* - \mathcal{Z}$. This pattern is more likely to arise for firms with: (1) larger cash flow volatility σ, (2) more severe financing constraints ϵ, (3) more frequent technological breakthroughs, i.e. larger ϕ, (4) smaller opportunity cost of cash $r-\delta$,*

Corollary 8 illustrates that $z(c)$ might increase when cash reserves decrease. That is, given two firms with the same fundamental characteristics but different cash reserves, the firm with the smaller cash reserves may invest more in innovation. In so doing, this firm seeks to increase the probability of attaining a technological breakthrough. When a breakthrough occurs, the firm earns monopoly rents related to the brand-new technology and can raise outside funds in light of a "success" rather than a "failure" (i.e., running out of funds due to operating losses). This pattern reminds a *"gambling for resurrection"* type of behavior and is largely driven by financial constraints.

The reason why firms may put more emphasis on innovation when cash reserves decrease is as follows. By (30), the optimal innovation policy $z(c)$ is a constant multiple of the quotient $\frac{(\lambda w^* - v(c) + c)}{v'(c)}$. The numerator represents the change in firm value when a technological breakthrough occurs. The denominator is the marginal "cost" of using one dollar to finance innovation. Both the numerator and the denominator are monotone decreasing in c. The numerator is decreasing because $v'(c) \geq 1$. The denominator is decreasing in c because firm value is concave in c. Thus, $z(c)$ is decreasing if and only if the numerator decreases at a higher rate than the denominator. By direct calculation, this is equivalent to

$$\frac{v'(c) - 1}{\lambda w^* - v(c) + c} \;\geq\; \sigma^{-2}\gamma(c).$$

Given the boundary condition (33), $z(c)$ is decreasing for small values of c if and only if $\frac{\epsilon}{\lambda w^* - v(0)} \;\geq\; \sigma^{-2}\gamma(0)$. The effective risk aversion of the firm scales with the coefficient of cash flow volatility ($\gamma \sim \sigma^2$) and hence the right-hand side is not very sensitive to volatility. At the same time, the gain $\lambda w^* - v(0)$ from replenishing cash reserves is modest when volatility is higher, which explains why $z(c)$ tends to be decreasing for firms with volatile cash flows. A similar logic applies to the opportunity cost $r - \delta$: Since v is increasing in δ, the gain $\lambda w^* - v(0) = \lambda(\kappa_T + \kappa_I) - v(0)$ is decreasing in δ. Corollary 8 also predicts that firms with more efficient R&D technologies should substitute production for innovation when constrained, to increase the likelihood of a breakthrough while limiting operating volatility. Furthermore, financing frictions make the optimal innovation rate $z(c)$ dependent on β (recall from (17) that z^* does not depend on this parameter). Specifically, a constrained firm is relatively more likely to substitute production for innovation when β is small. In this case, the markup set by the firm is smaller and, hence, the firm has greater incentives to decrease production and favor innovation. Overall, our analysis suggests that the decreasing pattern of $z(c)$ in c should be particularly relevant for small, constrained firms with low cash flows and high operating risk.

In our model, we study the relation between innovation and *target* cash reserves and between innovation and *deviations* from this target. That is, we distinguish between ex-ante heterogeneity (firms may differ in the parameters ϕ, λ, ζ and thus have different C^*) and ex-post heterogeneity (whereby ex-ante identical firms choosing the same C^* hold different *levels* of cash reserves). Our model predicts that innovation is a prime determinant of corporate cash reserves, as follows from Corollary 5 and the discussion therein. This is consistent with the evidence that R&D-intensive firms tend to old more cash in comparison to "old economy" firms (for example Falato and Sim, 2013, or Begenau and Palazzo, 2015). In addition, we characterize how identical firms should adjust their investment in innovation when operating shortfalls erode the reserves below the target C^*. In particular, we find that small firms might increase their innovation rate in the face of negative operating shocks and decreasing cash reserves.

Idiosyncratic volatility of cash flows and returns. In our model, production decisions affect the volatility of cash flows, which is equal to $\sigma X(c)$. Since production increases with the level of cash reserves, so does cash flow volatility. That is, a firm scales down operating risk when liquidity constraints tighten (i.e., when cash reserves decrease). This result has interesting implications for the relation between cash flow volatility and cash reserves (e.g., Bates, Khale, and Stulz, 2009). As previous contributions do, our model suggests that cash flow volatility affects (namely, increases) the target level of cash reserves. In addition, our model prompts that the level of cash reserves determines cash flow volatility via optimal production decisions. After positive (respectively, negative) operating shocks, cash reserves increase (decrease), the curvature of the value function decreases (increases), and the firm is willing to take on more (less) risk. The optimal production rate rises (decreases) and so does cash flow volatility.

We relate endogenous cash flow volatility to the idiosyncratic volatility of stock returns. By (4), volatility of stock returns $dR(c) = \frac{dv(c)}{v(c)}$ is given by

$$\sigma_R(c) \;=\; \sigma X(c)\, \frac{v'(c)}{v(c)}\,.$$

All cash flow shocks are idiosyncratic in our model, which implies that $\sigma_R(c)$ coincides with the idiosyncratic return volatility. A direct calculation implies that $\sigma_R(c)$ is locally monotone increasing in c if and only if

$$\frac{X'(c)}{X(c)}\frac{v'(c)}{v(c)} \;>\; -\frac{d}{dc}\frac{v'(c)}{v(c)}\,.$$

Since $v(c)$ is increasing and concave in c, the ratio $v'(c)/v(c)$ is decreasing. Thus, if the rate of change in production is higher than the rate of change in firm value, idiosyncratic volatility is locally increasing.

The monotonicity of $\sigma_R(c)$ is closely related to the leverage effect identified by Black (1976), according to which the volatility of stock returns increases after negative shocks to stock prices. In this context, the cash management model of Décamps, Mariotti, Rochet, and Villeneuve (2011) shows that liquidity frictions may generate the leverage effect. In their model, idiosyncratic volatility of returns is monotone decreasing in c, whereas the stock price is increasing. Nonetheless, the evidence on the co-movement between stock returns and idiosyncratic volatility appears ambiguous. Early studies report a positive relation (e.g. Duffee, 1995, or Malkiel and Xu, 2002), whereas more recent works (Ang, Hodrick, Xing, and Zhang, 2006, 2009) report a negative relation. Our model is capable of capturing both the positive and the negative relation. In fact, the instantaneous covariation of returns and idiosyncratic volatility is given by

$$\langle v(c)^{-1}dv(c), d\sigma_R(c)\rangle \;=\; v(c)^{-1}\sigma_C(c)^2 v'(c)\sigma_R'(c)dt\,,$$

so the hump-shaped pattern for $\sigma_R(c)$ leads to a negative (respectively, positive) co-movement for firms with high (low) cash reserves.

4.3 The stationary cross-sectional distribution of liquidity

We next derive the stationary cross-sectional distribution of liquidity in the economy, which we denote by $\eta(c)$. The dynamics of scaled cash reserves of each incumbent satisfy

$$dC_t = \mu(C_t)dt + \sigma(C_t)dZ_t + dF_t + dF_t^I - dD_t + (\lambda C^* - C_t)dN_t. \tag{39}$$

In this equation,

$$\mu(c) = \delta c + X^{1-\beta}(c) - X(c) - \frac{z^2(c)}{2}\frac{\zeta}{L}, \quad \text{and} \quad \sigma(c) = \sigma X(c)\,,$$

follow from Section 4.1, and N_t is a Cox process with stochastic intensity $\phi z(C_t)$ representing quality improvements (technological breakthroughs) in the incumbent's input. Since liquidity shocks are independent and identically distributed (i.i.d.) across firms, the cross-sectional distribution of liquidity satisfies the following Kolmogorov forward equation[20]

$$\frac{1}{2}\left(\sigma^2(c)\eta(c)\right)'' - (\mu(c)\eta(c))' \; - \; x_d\eta(c) \; - \; z(c)\phi\eta(c) \; = \; 0\,.$$

We solve for the distribution by imposing the following boundary conditions:

$$\begin{aligned} 0.5(\sigma^2\eta)'(0) - (\mu(0)\eta(0)) \; &= 0 \\ 0.5(\sigma^2\eta)'(C^*) - (\mu(C^*)\eta(C^*)) \; &= \; \int_0^{C^*} \phi z(c)\eta(c)dc \; + \; x_d. \end{aligned} \tag{40}$$

For any level of cash reserves c, the quantity $0.5(\sigma^2\eta)'(c) - (\mu(c)\eta(c))$ represents the infinitesimal change in the mass of firms due to cash accumulation (as captured by the drift $\mu(c)$) and due to idiosyncratic cash flow shocks (as captured by the volatility $\sigma(c)$). The first equation in (40) is the "mass conservation" condition at zero. It guarantees that there is no loss of mass for firms that run out of liquidity and use external financing to remain solvent. The second equation is the "mass conservation" at C^*. It ensures that loss of mass is offset by the inflow of successful innovators. In the next section, we use

[20] See Appendix A.3 for a formal argument.

this stationary distribution to derive equilibrium aggregate quantities.

4.4 Equilibrium quantities

We embed the preceding analysis into a general equilibrium setting. Along the balanced growth path, aggregate output, consumption, net dividends, and wage grow at the constant, endogenous rate g. As for the unconstrained economy, the law of large numbers implies the following relation:

$$\int q_{jt} dj = e^{gt},$$

where the growth rate g is the sum of two components, as follows:

$$g = (\lambda - 1)\phi \int_0^{C^*} z(c)\eta(c)dc + (\Lambda - 1)x_d \,. \tag{41}$$

Since innovation decisions are *i.i.d.* across firms, the law of large numbers implies that the contribution of incumbents to economic growth is given by the size of their quality improvements multiplied by their average innovation rate (the first term in (41)). Similarly, the contribution of entrants is given by the size of their quality improvements times the entry rate (the second term). In equilibrium, the entry rate (i.e., the rate of creative destruction) is pinned down by the free-entry condition

$$v\left(C^*(x_d); x_d\right) - C^*(x_d) = \kappa_T + \kappa_I.$$

Using (36) and the approximation for C^* in Proposition 4, we solve for x_d.

Proposition 9 *For ϵ sufficiently small, the equilibrium rate of creative destruction satisfies:*

$$x_d \approx \frac{\mu^* - (r-\delta)^{1/2}\sigma X_0 \epsilon^{1/2}}{\kappa_T + \kappa_I} + \varphi^2(\lambda-1)^2 \frac{\kappa_T + \kappa_I}{2} - r.$$

To single out the various effects at play, it is interesting to study the properties of x_d while holding r fixed (we numerically investigate the "full" equilibrium in Section 5.2). This exercise may be relevant within a single industry, whose impact on the interest rate is sufficiently small. Proposition 9 shows that financing frictions faced by incumbents discourage fresh entry, and this effect is stronger when cash flow volatility σ and the opportunity cost of cash $r-\delta$ are larger. Moreover, the entry rate is monotone increasing in the elasticity β since X_0 is monotone decreasing in β.

As for the unconstrained economy, the Euler equation of the representative household pins down the equilibrium interest rate

$$r \;=\; \rho+\theta\left[(\lambda-1)\phi\int_0^{C^*} z(c;r)\eta(c;r)dc+(\Lambda-1)x_d(r)\right]. \tag{42}$$

This equation represents the fixed point equation for $r>\delta$, which we solve numerically.

By the household budget constraint, consumption $\mathcal{C}_t=\mathcal{C}_0e^{gt}$ is the sum of labor income W_t and dividends d_t net of financing f_t. Since these quantities grow at rate g too, the relation $\mathcal{C}_0e^{gt}=(W_0+d_0-f_0)e^{gt}$ holds. In particular, the wage paid by the competitive final good sector is given by

$$W_t=\frac{1}{1-\beta}\int_0^1 L^\beta\,\bar{X}_{jt}^{1-\beta}\,q_{jt}^\beta dj-\int_0^1 p_{jt}\bar{X}_{jt}dj\,.$$

Since $W_t=W_0e^{gt}$, simple calculations imply that W_0 satisfies

$$W_0 \;=\; \frac{\beta L}{1-\beta}\int \eta(c)X(c)^{1-\beta}dc\,.$$

In turn, the derivation of the dividend rate is not a trivial exercise in our continuous time model. Even though the dividend process of every firm is singular,[21] aggregate dividends are smooth and grow at rate g. At every instant, firms with cash reserves close to C^* may move to C^* (following (39)) and eventually pay out dividends. The computation of the dividend rate then requires to keep track of the whole cross-sectional distribution of liquidity in the economy. To address this issue, we proceed as follows.

We define the function $Y(q_{jt},c)$ as the present value of a virtual production unit that we call *"dynasty"*. Namely, a dynasty represents the expected present value of dividends

[21] As in other dynamic cash management models (see, for example, Décamps, Mariotti, Rochet, and Villeneuve, 2011; Bolton, Chen, and Wang, 2011, 2013; and Hugonnier, Malamud, and Morellec, 2015), cash is paid out when hits the threshold C^*.

net of cash injections of all firms that do and will ever operate in a given industry j. As for the other quantities in a balanced growth path equilibrium, $Y(q_{jt}, c)$ is homogeneous in the quality of input j. Hence, the relation

$$Y(q_{jt}, c) \;=\; Lq_{jt}y(c)$$

holds for some function $y(c)$. Specifically, $y(c)$ solves the following equation:

$$\begin{aligned}&\frac{1}{2}\sigma^2(c)y''(c) \;+\; \mu(c)y'(c) \;+\; \phi z(c)\left[\lambda(y(C^*) - C^*) - y(c) + c\right]\\&+ x_d\left[\Lambda y(C^*) - y(c) + c - \Lambda\left(C^* + \kappa_T\right)\right] \;=\; ry(c)\,,\end{aligned}$$

see Appendix A.4 for further details. The first two terms on the left-hand side represent the effect of cash flow volatility and cash savings on the value of the dynasty. The third term represents the probability-weighted change in value due to a quality improvement by the current incumbent. When this happens, net financing to the dynasty amounts to $\lambda C^* - c$. The fourth term represents the probability-weighted change in value due to a quality improvement by a new firm. Upon entry, net financing to the dynasty amounts to $\Lambda\left(C^* + \kappa_T\right) - c$, where the last term represents the cash reserves of the exiting incumbent. Notice that the financing fees ϵ, κ_I, and α are paid in equity shares and not in cash. As a result, the marginal value of cash $y'(c)$ satisfies

$$y'(C^*) = 1 = y'(0)\,.$$

Solving for $y(c)$ allows to determine the net dividend rate. Since net dividends grow at rate g, it follows that

$$d_0 - f_0 \;=\; (r-g)L\int_0^{C^*}\eta(c)y(c)dc\,.$$

The analysis demonstrates that labor wage in the constrained economy is lower than in the unconstrained benchmark. In fact, $W_0 \leq W_0^*$ since $X(c) < X_0$ for any $c < C^*$. Differently, comparing net dividend in the constrained and unconstrained economies is more difficult. On the one hand, the constrained economy features a smaller entry rate (see Proposition 9 and the analysis in Section 5.2), meaning that investors provide financing to new firms less often. On the other hand, the amount provided is larger in the constrained economy. In fact, investors finance not only the entry cost but also provide entrants with precautionary cash reserves. We numerically analyze consumption and welfare in the constrained and unconstrained economies in Section 5.2.

5 Model analysis

In this section, we provide a quantitative assessment of the model implications. Table 1 reports the baseline parametrization. Routine financing entails a cost $\epsilon = 8\%$ for any dollar raised, consistent with the estimation of Hennessy and Whited (2007). Financiers extract a share $\alpha = 6\%$ of surplus when financing an incumbent's breakthrough. The financing component of the entry cost κ_I is 10% of the technological component, which means that financiers extract more rents from entrants than from incumbents (E.g., financing of entrants is more fraught with asymmetric information). By setting $\lambda = 1.04$ and $\Lambda = 1.10$, we capture in a simple fashion that innovations by entrants tend to be path-breaking while innovations by incumbents tend to be incremental; see, e.g., Akcigit and Kerr (2015), and Acemoglu et al. (2013). We set $\sigma = 0.4$, which implies that cash flow volatility $\sigma X(c)$ varies between 9.9% and 12.7%. We set the return on cash δ =4.9%, which implies an opportunity cost $r - \delta$ around 0.5% (We endogenize δ in Appendix A.5).

5.1 Corporate outcomes

We start our analysis by comparing the optimal corporate policies in the constrained and unconstrained economies. Supporting our analytical results, Figure 1 shows that, in the constrained economy, production is increasing in cash reserves. Moreover, it is lower than in the unconstrained economy for any $c < C^*$. Our analysis also reveals that investment in innovation in the constrained economy may exceed that of the unconstrained economy. Under the baseline parametrization, $z(c)$ is always higher than z^*, being roughly

7.14% higher for c close to (i.e., in a right neighborhood of) zero. The innovation rate is decreasing when cash reserves are small and is almost flat when cash reserves are large.

Figure 2 investigates innovation decisions further. An increase in the refinancing cost ϵ up to 14% makes $z(c)$ steeper around zero. Conversely, an increase in the cost α up to 12% shifts $z(c)$ downwards very sharply. An increase in α erodes the incumbents' surplus from innovation and so the incentives to invest. However, it preserves the decreasing pattern of $z(c)$ in c. An increase in κ_I up to 0.1 (implying a financing component of the entry cost about 14.3% of the technological component) spurs innovation: e.g., $z(0^+)$ is 10.7% larger than z^*. A larger κ_I deters entry and renders incumbents less threatened by creative destruction. This leads to an increase in their innovation rate.

We also explore the impact of cash flow volatility and profitability on $z(c)$. Notably, the parameters σ and β do not affect z^* but do affect $z(c)$. Figure 2 shows that a decrease in β increases the firm's willingness to substitute production for innovation. When precautionary cash is scarce and profitability is low, firms cut on core production and invest more in innovation. Moreover, a decrease in σ leads to a sharp drop in the optimal innovation rate. Notably, $z(c)$ is increasing in c for low-volatility firms, as explained in Section 4.2.

Our analysis reveals that $z(c)$ can be higher or lower than z^*. Moreover, $z(c)$ can be non-monotonic in c. These results may help explain the R&D decisions of small firms. Small firms are more financially constrained (larger ϵ), operate with lower margins (lower β), and their cash flows are volatile (larger σ). Despite the financial constraints, small firms can be very R&D intensive, as discussed in the introduction. Our model can rationalize this pattern.

Finally, Figure 1 shows that incumbent firm value can be larger in the constrained economy. In the unconstrained economy, entry is less costly as it does not involve any financing cost. The entry rate is larger, which implies that incumbent firms are more likely to exit. We investigate the general equilibrium effects of this result in the next section.

5.2 Aggregate quantities

The cross-sectional distribution of liquidity. Figure 3 displays the cross-sectional distribution of liquidity $\eta(c)$ for $c \in [0, C^*]$ under several parametrizations. The distribution is increasing in c as in the model of Bolton, Chen, and Wang (2011). That is, incumbents' cash reserves are relatively large most of the time. Variations in the parameter values affect not only the shape of the distribution but also its interval of definition. For instance, a decrease in the return on cash from 4.9% to 1% or in the coefficient of cash flow volatility from 0.4 to 0.2 both lead to a decrease in the target cash level. The interval of definition becomes smaller, and the distribution gets steeper. Conversely, a decrease in the elasticity β reinforces firms' precautionary policies. The target cash level increases, and the distribution becomes flatter around zero. That is, firms with smaller monopoly rents hold more cash.

The shape of the cross-sectional distribution has important implications for the aggregate effect of firm-specific policies. As an illustration, consider the effect of the decreasing pattern of $z(c)$ on incumbents' innovation $\int \eta(c)z(c)dc$. If the stationary distribution is concentrated close to the target level C^*, the impact of the innovation decisions of liquidity constrained firms (i.e., those with cash reserves close to zero) is small. Even if constrained firms increase their innovation rate when c is close to zero, the effect of their policies may be modest on aggregate. These considerations can help gauge the impact of government interventions subsidizing innovation (for instance, Acemoglu, Akcigit, Bloom, and Kerr, 2013).

Financing frictions, growth, and welfare. We next investigate the effects of financing frictions and corporate cash hoarding on economic growth. Under the baseline parametrization, economic growth is 1.916% (respectively, 2.013%) in the constrained (unconstrained) economy.

Figure 4 displays creative destruction and growth as functions of the technological and financing components of the entry cost, κ_T and κ_I. Creative destruction appears decreasing in κ_T, whereas growth displays a U-shaped pattern.[22] Our model then highlights that an increase in κ_T deters fresh entry but boosts the incumbents' innovation rate. When κ_T is low enough, an increase in κ_T leads to lower growth. In this case, the reduction in entrants' contribution to growth overtakes the increase in the incumbents' one. When κ_T is sufficiently large, conversely, a further increase in κ_T leads to an increase in growth. In this case, the rise in incumbents' contribution to growth more than offsets the decrease in entrants' contribution.

In our model, the entry cost includes a financing component on top of the technological component. By imposing barriers to entrants, financiers slow down creative destruction (i.e., the entry rate is lower). Indeed, Figure 4 shows that x_d decreases in κ_I. If creative destruction hits less often, incumbents expect to enjoy monopoly rents for longer and thus increase their investment in innovation (as illustrated in Section 5.1). Since the growth rate is non-monotonic in the entry cost, g can be larger than g^* *if* κ_T and κ_I are sufficiently large. Thus, our analysis points out that financing frictions may be not detrimental to economic growth.

An interesting question is how financing frictions and corporate cash hoarding affect consumption and welfare. To address this question, we make the following thought experiment. Suppose that the planner intervenes to relax financing frictions. For simplicity, we assume that the planner can remove financing frictions and "switch" the economy from constrained to unconstrained. After this intervention, the equilibrium quantities become those described in Section 3. Importantly, firms stop retaining earnings and pay out their cash reserves amounting to $\tilde{C} \equiv L \int_0^{C^*} c\eta(c)dc$. The resulting intertemporal wealth of the representative household is the sum of consumption and corporate cash reserves. If the planner does not intervene, conversely, cash reserves remain "trapped" in the corporate sector.

Recall that a dynasty represents the expected present value of dividends net of financing in a given industry j. In the unconstrained economy, the scaled value of a dynasty j equals the scaled value of the incumbent j, which is equal to κ_T due to the free-entry

[22]This may not be the case when risk aversion θ is less than (or equal to) 0.5. We do not consider these parametrizations, as they may bring along an interest rate greater than the rate of economic growth.

condition. In the constrained economy, by contrast, the scaled value of the dynasty is larger than the value of the corresponding incumbent; in particular, the following relations $y(C^*) - C^* > v(C^*) - C^* = \kappa_T + \kappa_I \geq \kappa_T = v^* = y^*$ hold (see Section 4.4).

Figure 5 shows consumption and welfare as a function of κ_T and κ_I if the planner does and does not intervene. If the planner does not intervene and the economy remains constrained, the representative household enjoys larger net dividends. If the planner does intervene, the representative household receives a larger income wage as well as corporate cash reserves. The figure shows that welfare is non-monotonic in κ_T and follows a U-shaped pattern as the growth rate. Importantly, the figure shows that financing frictions may *be not* welfare-decreasing. When κ_I is sufficiently large, a firm operating in the constrained economy is more valuable than an identical firm operating in the unconstrained economy. As explained, the former is less threatened by creative destruction. In the constrained economy, aggregate consumption is larger than in the unconstrained economy if the increase in net dividends more than offsets the decrease in labor wage.

6 Concluding remarks

Recent empirical studies document strong relations between corporate cash reserves and innovation, and between innovation and economic growth. In this paper, we develop a tractable framework to study the micro and macro implications of financing frictions and liquidity hoarding on investment in innovation and economic growth. To this end, we embed financing frictions into a Schumpeterian model that features innovations by incumbents and entrants. Our modeling of financial frictions accounts for several features that characterize R&D-intensive industries. In particular, these firms prefer internal funds and delay external (equity) financing until milestones are attained or they are in extreme need of funds. The model generates rich joint dynamics of cash reserves, investment in innovation, and production decisions, which capture documented stylized facts and deliver new testable predictions. One of our main findings is that financing frictions can spur the investment in innovation of small, constrained firms. At the aggregate level, financing frictions and liquidity hoarding have offsetting effects on growth.

We believe that the interactions among liquidity frictions, innovations, and growth are important for the dynamics of the economy and policy-making. Our paper makes only the first step in this direction. First of all, we abstract from aggregate shocks. Nonetheless, it would be interesting to investigate the impact of government liquidity policies on production, innovation, and long-run growth over the business cycle (see also Aghion, Farhi, and Kharroubi, 2012). Second, we take the costs of external financing as exogenous and do not microfound them. Such microfoundations (e.g., based on asymmetric information and limited pledgeability of assets) could allow to investigate fully-fledged dynamic capital structure choices and study their macro (interestingly, welfare) implications. We leave these aspects for future research.

A Appendix

A.1 Deriving incumbents' policies and value

In the presence of liquidity frictions, the incumbents' HJB equation is

$$
\begin{aligned}
0 = \; & V_t + \max_{z,\bar{X}} \left\{ \frac{\sigma^2}{2} \bar{X}^2 V_{cc} + V_c \left(\delta \; \bar{C} + X(p-1) - \frac{z^2}{2} q\zeta \right) \right. \\
& \left. + \; \phi(1-\alpha) z(\lambda(V - \bar{C}^*) - (V - \bar{C})) \; + \; x_d \, (\bar{c} - V) \; - \; rV \right\}.
\end{aligned}
\tag{43}
$$

We conjecture

$$
V(t, \bar{C}, q) \; = \; qL \, v\left(\frac{\bar{C}}{qL} \right) \; \equiv \; Q \, v(c) \quad Q = qL \, .
$$

Having defined $X \; \equiv \; \bar{X}/Q$, $c \; \equiv \; \bar{C}/Q$ so the expression for the price of input becomes

$$
p = \; \left(\bar{X}^{-1} Q \right)^{\beta} = X^{-\beta} \, .
$$

Substituting

$$
V_t \; = 0 \; , \quad V_c \; = \; v'(c) \; , \quad V_{cc} \; = \; \frac{1}{Q} v''(c)
$$

into (43), we get[23]

$$
\begin{aligned}
& \max_{z,\bar{X}} \left\{ \frac{\sigma^2}{2} X^2 v''(c) + v'(c) \left(\delta c \; + \; X^{1-\beta} - X - \frac{z^2}{2} \frac{\zeta}{L} \right) \right. \\
& \left. + \; \phi(1-\alpha) z(\lambda(v(C^*) - C^*) - (v(c) - c)) \; + \; x_d(c - v(c)) \; - \; rv(c) \right\} \; = \; 0 \, .
\end{aligned}
\tag{44}
$$

Let $A(c) \; = \; -\sigma^2 \frac{v''(c)}{v'(c)}$. The first order condition for X takes the form

$$
-A(c)X + (1-\beta)(X)^{-\beta} \; = \; 1. \tag{45}
$$

Define $F(a)$ to be the unique solution to

$$
-aF(a) + (1-\beta)(F(a))^{-\beta} \; = \; 1 \, . \tag{46}
$$

Then, $X(c) \; = \; F(A(c))$. Denote $w^* \; \equiv \; v(C^*) - C^*$. By the first order condition, we have that the optimal innovation rate solves

$$
z \; = \; \phi(1-\alpha) \frac{L}{\zeta} \, \frac{(\lambda w^* - v(c) + c)}{v'(c)}
$$

[23]The true surplus created upon financing a technological breakthrough occurs is given by $\tilde{S}(c) = \lambda\left(v(C^*) - C^*\right) + c - \lambda v(c/\lambda)$. To ease the analysis, we take a slightly modified version of this surplus, i.e. $S(c) = \lambda(v(C^*) - C^*) + c - v(c)$. This is without loss of generality. In fact, we can show that $0 < \tilde{S}(c) < S(c)$. We have $\lambda(v(C^*) - v(c/\lambda)) \; > \; \lambda(C^* - c/\lambda) = \lambda C^* - c$ since $v'(c) \geq 1$. Hence, it is always optimal for the firm to bargain with the financiers.

and the corresponding term in the HJB is

$$\frac{\phi^2(1-\alpha)^2}{2}\frac{L}{\zeta}\frac{(\lambda w^* - v(c) + c)^2}{v'(c)}$$

and we define $\varphi \equiv \phi(1-\alpha)\left(\frac{L}{\zeta}\right)^{1/2}$. Now, by (45), it follows that

$$F^{1-\beta} - F = \beta F^{1-\beta} - \sigma^2 \frac{v''(c)}{v'(c)} F^2$$

and hence we get

$$0.5\sigma^2 F(A(c))^2\, v''(c) \;+\; v'(c)(F^{1-\beta} - F) \;=\; -0.5\sigma^2 F(A(c))^2\, v''(c) \;+\; v'(c)\beta F^{1-\beta}\,.$$

Thus, the HJB equation can be written as follows:

$$\begin{aligned} &-\frac{\sigma^2}{2} F(A(c))^2\, v''(c) + v'(c)\,\beta\, F(A(c))^{1-\beta} + \; v'(c) c\,\delta \\ &+\; \frac{\varphi^2}{2}\frac{(\lambda w^* - v(c) + c)^2}{v'(c)} + x_d\,(c - v(c)) \;-\; rv(c) \;=\; 0\,, \end{aligned} \tag{47}$$

subject to the following set of boundary conditions

$$v'(0) \;= 1 + \epsilon\;, \qquad v'(C^*) \;= 1, \qquad v''(C^*) = 0\,.$$

It follows that $A(C^*) = 0$ and $X(C^*) = X_0$. Then, at C^*, the following equation

$$\mu + \delta C^* + 0.5(\varphi(\lambda-1)w^*)^2 - x_d\, w^* \;-\; r(w^* + C^*) \;=\; 0 \tag{48}$$

holds, where we have defined $\mu = \mu^* \;=\; \beta(X_0)^{1-\beta}$ and $X_0 = \; X(C^*) = \; (1-\beta)^{\frac{1}{\beta}}$. This reveals that the solution to w^* is given by[24]

$$w^* = w(C^*) = \frac{x_d + r - \Big((x_d + r)^2 - 2(\varphi(\lambda-1))^2(\mu - (r-\delta)C^*)\Big)^{1/2}}{\varphi^2(\lambda-1)^2}.$$

[24]The choice of the solution to pick is motivated as follows. Consider the frictionless case $C^* = 0$. In the limit when $\phi \to 0$, $w^* = \frac{\mu}{r + x_d}$, where $\mu \;=\; \beta(X_0)^{1-\beta}$. Our solution can be rewritten as

$$w^* \;=\; \frac{2\mu}{x_d + r \mp \Big((x_d+r)^2 - 2(\varphi(\lambda-1))^2(\mu - (r-\delta)C^*)\Big)^{1/2}}.$$

Obviously, to get the continuous solution in the limit, we need to pick the plus sign, as the other solution blows up. Thus, by continuity, we pick that with the plus sign.

A.2 Approximations

In this section, we derive $F^i(c)$, $A^i(c)$, $v_i = \ v^{(i)}$, where F, A, v are the functions derived in the previous section. We start with the function $A(c)$. By simple calculations, it follows that

$$A'(c) \ = \ -\sigma^2\Big(\frac{v'''(c)}{v'(c)} - \frac{(v''(c))^2}{(v'(c))^2}\Big), \quad \Rightarrow \ A'(C^*) = \ -\sigma^2 v'''(C^*),$$

and

$$A''(c) \ = \ -\sigma^2\Big(\frac{v''''(c)}{v'(c)} - 3\frac{v''(c)v'''(c)}{(v'(c))^2} + 2\frac{(v''(c))^3}{(v'(c))^3}\Big), \quad \Rightarrow \ A''(C^*) = \ -\sigma^2 v''''(C^*)\,.$$

We turn to F. By differentiating (46), we have

$$-F - aF' - \beta(1-\beta)F^{-\beta-1}F' = \ 0$$

so

$$F'(A(c)) \ = \ -\frac{F(A(c))}{A(c) + \beta(1-\beta)F(A(c))^{-\beta-1}} \quad \Rightarrow \ F'(A(C^*)) = \ -\frac{X_0^{\beta+2}}{\beta(1-\beta)}. \tag{49}$$

To arrive at F'', we differentiate again the equation above

$$\begin{aligned} & -2F' - aF'' - \beta(-\beta-1)(1-\beta)F^{-\beta-2}(F')^2 - \beta(1-\beta)F^{-\beta-1}F'' = \ 0 \qquad (50)\\ & \Rightarrow \ F''(A(C^*)) = \ \frac{X_0^{2\beta+3}}{\beta^2(1-\beta)^2}(\beta+3) \end{aligned}$$

Summarizing, the following relations hold at C^*

$$\begin{aligned} A'(C^*) \ &= -\ \sigma^2 \ v_3 \\ A''(C^*) \ &= -\ \sigma^2 \ v_4 \end{aligned}$$

and

$$\begin{aligned} F'(A(C^*)) = \ -\ \frac{X_0^{\beta+2}}{\beta(1-\beta)} \ &= \ -\Big(\ (1-\beta)^{1/\beta}\ \Big)^{\beta/2} \\ F''(A(C^*)) \ &= (\beta+3) \ \frac{X_0^{2\beta+3}}{\beta^2(1-\beta)^2}. \end{aligned}$$

By simple calculations, we also obtain

$$\begin{aligned} X(C^*) = \ X_0 &= F(A(C^*)) = \ (1-\beta)^{\frac{1}{\beta}}, \\ X'(C^*) = \ X_1 &= F'(A(C^*))A'(C^*) = \ \frac{2(r-\delta)X_0^{\beta}}{\beta(1-\beta)} = \ \frac{2(r-\delta)}{\beta}, \\ X''(C^*) = \ X_2 &= F''(A(C^*))(A'(C^*))^2 + F'(A(C^*))A''(C^*) \\ &= \frac{X_0^{2\beta-1}(2(r-\delta))^2}{\beta^2(1-\beta)^2}(\beta+3) - \frac{X_0^{\beta+2}}{\beta(1-\beta)}(\sigma^2 v_4), \end{aligned}$$

where we omit the arguments of the functions to ease the notation. We will provide an expression for the term $v_4 = v^{(4)}(c)$ in the following.

We now differentiate equation (47), obtaining

$$
\begin{aligned}
&-\frac{\sigma^2}{2}v'''(c)F(A(c))^2 - \sigma^2 v''(c)F(A(c))F'(A(c))A'(c) + v''(c)\beta(F(A(c)))^{1-\beta} \\
&+ v'(c)\beta(1-\beta)F^{-\beta}(A(c))F'(A(c))A'(c) + v''(c)c\delta + \ v'(c)\delta - \frac{\varphi^2}{2}\frac{(\lambda w^* - v(c) + c)^2 v''(c)}{(v'(c))^2} \\
&+ \varphi^2(\lambda w^* - v(c) + c)\left(\frac{1}{v'(c)} - 1\right) + x_d\left(1 - v'(c)\right) \ - \ rv'(c) \ = \ 0.
\end{aligned}
$$

Using the boundary conditions, we obtain an expression for the third derivative of v at the target cash level

$$
v'''(C^*) \equiv \ v_3 = \ \frac{2\ (r-\delta)}{\sigma^2}X_0^{-2}.
$$

Differentiating again the ODE, we obtain

$$
\begin{aligned}
&-\frac{\sigma^2}{2}v''''(c)F^2 - 2\sigma^2 v'''(c)FF'A' - \sigma^2 v''(c)(F')^2(A')^2 - \sigma^2 v''(c)FF''(A')^2 - \sigma^2 v''(c)FF'A'' \\
&+ v'''(c)\beta F^{1-\beta} + 2v''(c)\beta(1-\beta)F^{-\beta}F'A' - v'(c)\beta^2(1-\beta)F^{-\beta-1}(F')^2(A')^2 \\
&+ v'(c)\beta(1-\beta)F^{-\beta}F''(A')^2 + v'(c)\beta(1-\beta)F^{-\beta}F'A'' + v'''(c)c\delta \ + 2v''(c)\delta \\
&+ \varphi^2\frac{(-v'(c)+1)^2}{v'(c)} \ - \ \varphi^2\frac{(\lambda w^* - v(c) + c)v''(c)}{(v'(c))^2}(2 - v'(c)) \\
&+ \varphi^2\ \frac{(\lambda w^* - v(c) + c)^2(v''(c))^2}{(v'(c))^3} - \frac{\varphi^2}{2}\ \frac{(\lambda w^* - v(c) + c)^2 v'''(c)}{(v'(c))^2} - \ (r + x_d)v''(c) \ = \ 0.
\end{aligned}
$$

At C^*, the above equation becomes:

$$
\begin{aligned}
&-\frac{\sigma^2}{2}v''''(C^*)X_0^2 \ - \ 2\sigma^2 v'''(C^*)XF'A' + \ v'''(C^*)\beta X^{1-\beta} \\
&- \beta^2(1-\beta)X^{-\beta-1}(F')^2(A')^2 \ + \ \beta(1-\beta)X^{-\beta}F''(A')^2 \ + \ \beta(1-\beta)X^{-\beta}F'A'' \\
&+ v'''(C^*)C^*\delta - \frac{\varphi^2}{2}\ (\lambda w^* - v(C^*) + C^*)^2 v'''(C^*) = \ 0.
\end{aligned}
$$

Therefore, we also obtain an expression for the fourth derivative of v at C^*,

$$
v^{(4)}(C^*) \equiv \ v_4 \ = \ \frac{2v_3}{\sigma^2 X_0^2}\left(\frac{\varphi^2}{2}\ (\lambda-1)^2(w^*)^2 - \beta X_0^{1-\beta} - \delta C^* - v_3(\sigma^2)^2\ \frac{X_0^{\beta+3}}{\beta(1-\beta)}\right).
$$

Suppose that ϵ is sufficiently small. For c close to C^*, firm value can be approximated by the following expression

$$
\begin{aligned}
v(c) \ &\approx \ v(C^*) + v'(C^*)(c - C^*) + \frac{v''(C^*)}{2}(c - C^*)^2 + \frac{v'''(C^*)}{6}(c - C^*)^3 \\
&+ \frac{v''''(C^*)}{24}(c - C^*)^4 + \ O((c - C^*)^5) \approx \\
&w^* \ + \ c \ + \ \frac{v_3}{6}(c - C^*)^3 \ + \ \frac{v_4}{24}\,(c - C^*)^4 \ + \ O((c - C^*)^5) \qquad (51)
\end{aligned}
$$

while the first and the second derivative of v satisfy

$$
\begin{aligned}
v'(c) \ &\approx \ 1 + v''(C^*)(c - C^*) + \frac{v'''(C^*)}{2}(c - C^*)^2 + \frac{v''''(C^*)}{6}(c - C^*)^3 = \\
&1 \ + \ \frac{v_3}{2}(c - C^*)^2 \ + \frac{v_4}{6}(c - C^*)^3 \qquad (52) \\
v''(c) \ &\approx \ v''(C^*) + v'''(C^*)(c - C^*) + \frac{v''''(C^*)}{2}(c - C^*)^2 = \\
&v_3(c - C^*) \ + \frac{v_4}{2}(c - C^*)^2 \ . \qquad (53)
\end{aligned}
$$

Using these approximations, it follows that

$$
\begin{aligned}
v'(0) \ &= 1 \ + \ \frac{v_3}{2}(C^*)^2 \ - \frac{v_4}{6}(C^*)^3, \\
v''(0) \ &= \ - \ v_3 \ C^* + \ \frac{v_4}{2}(C^*)^2. \qquad (54)
\end{aligned}
$$

The threshold C^* is then obtained by exploiting the boundary condition at zero, i.e.

$$
C^* : \ \ 1 \ + \ \frac{v_3}{2}(C^*)^2 \ - \frac{v_4}{6}(C^*)^3 \ = \ 1 + \ \epsilon.
$$

We use

$$
C^* \approx \left(\frac{2}{v_3}\right)^{\frac{1}{2}} \ \epsilon^{\frac{1}{2}} \ + \ a\epsilon
$$

where a can be found by solving the following equation

$$
\frac{v_3}{2}\epsilon \left(\left(\frac{2}{v_3}\right)^{\frac{1}{2}} + a\epsilon^{\frac{1}{2}} \right)^2 \ - \frac{v_4}{6}\epsilon^{\frac{3}{2}} \left(\left(\frac{2}{v_3}\right)^{\frac{1}{2}} + a\epsilon^{\frac{1}{2}} \right)^3 \ = \ \epsilon.
$$

Ignoring the terms of order higher than $\frac{1}{2}$, we obtain

$$
a \ = \ \frac{v_4}{3 \ v_3^2}.
$$

Then, the approximation for the target level of cash holdings is given by

$$\begin{aligned} C^* &\approx C_1^* \sqrt{\epsilon} + C_2^* \epsilon, \\ C_1^* &= \left(\frac{2}{v_3}\right)^{\frac{1}{2}}, \\ C_2^* &= \frac{v_4}{3\ v_3^2}. \end{aligned}$$

We also calculate the approximation for $z(c)$, as follows:

$$z(c) \approx\ z(C^*)\ +\ z'(C^*)(c - C^*) + \ \frac{z''(C^*)}{2}(c - C^*)^2\ + \frac{z'''(C^*)}{6}(c - C^*)^3.$$

To ease the notation, we define the auxiliary quantity $\phi^* \equiv \phi\,(1 - \alpha)\,\frac{L}{\zeta}$. The derivatives of $z(c)$ at C^* are given by

$$\begin{aligned} z'(c) &= \phi^* \left(-1 + \frac{1}{v'(c)}\right) - \ \frac{\phi^*\ v''(c)\ (\lambda w^* - v(c) + c)}{(v'(c))^2}, \\ z''(c) &= \frac{\phi^* v''(c)}{v'(c)}\ - \frac{\phi^*\,(2v''(c) + v'''(c)\,(\lambda w^* - v(c) + c))}{(v'(c))^2}\ +\ \frac{2\phi^*\ (v''(c))^2\ (\lambda w^* - v(c) + c)}{(v'(c))^3}, \\ z'''(c) &= \frac{2\phi^* v'''(c)}{v'(c)}\ -\ \frac{\phi^*}{(v'(c))^2}\Big(3v''(c)^2 + 3v'''(c) + v^{(4)}(c)(\lambda w^* - v(c) + c)\Big)\ + \\ &\quad +\ \frac{6\phi^*}{(v'(c))^3}\big(v''(c)^2 + v''(c)v'''(c)(\lambda w^* - v(c) + c)\big)\ -\ \frac{6\phi^*}{(v'(c))^4}v''(c)^3(\lambda w^* - v(c) + c). \end{aligned}$$

So,

$$\begin{aligned} &z(C^*) = \ z_0 = \ \phi^*(\lambda - 1)w^*, && z'(C^*) = z_1 = \ 0, \\ &z''(C^*) = \ z_2\ =\ -\phi^*\ v_3\ (\lambda - 1)w^*, && z'''(C^*) = \ z_3\ =\ -\phi^*\ \big(v_3\ + v_4(\lambda - 1)w^*\big), \end{aligned}$$

and the resulting approximation for $z(c)$ is therefore

$$z(c) \approx z_0 + \ \frac{z_2}{2}(c - C^*)^2\ + \frac{z_3}{6}(c - C^*)^3. \tag{55}$$

Finally, the approximation for $X(c)$ is

$$X(c) \approx\ X(C^*) +\ X'(C^*)(c - C^*)\ + \frac{X''(C^*)}{2}(c - C^*)^2,$$

where $X^{(i)}(C^*)$ are calculated as above. Then, we have

$$X(c) \approx\ X_0 +\ X_1(c - C^*)\ + \frac{X_2}{2}(c - C^*)^2.$$

A.3 The cross-sectional distribution of liquidity

For any incumbent firm, the dynamics of scaled cash holdings satisfy

$$dC_t \;=\; \mu(C_t)dt \;+\; \sigma(C_t)dZ_t \;+\; dF_t \;+\; dF_t^I - \;dD_t \;+\; (\lambda C^* - c)dN_t$$

where N_t is a Cox process with stochastic intensity $\phi\, z(C_t)$. Since cash is i.i.d. across firms, the cross-sectional distribution of firms will satisfy the Kolmogorov Forward equation

$$\frac{1}{2}\left(\sigma^2(c)\eta(c)\right)'' - (\mu(c)\eta(c))' \;- x_d\eta(c) \;-\; z(c)\phi\eta(c) \;=\; 0\,.$$

By calculations, it follows:

$$\begin{aligned}(\mu(c)\eta(c))' &= \mu'(c)\eta(c) + \;\mu(c)\eta'(c)\\ \left(\sigma^2(c)\eta(c)\right)' &= 2\sigma(c)\sigma'(c)\eta(c) + \sigma^2(c)\eta'(c)\\ \left(\sigma^2(c)\eta(c)\right)'' &= 2(\sigma'(c))^2\eta(c) + 2\sigma(c)\sigma''(c)\eta(c) + 4\sigma(c)\sigma'(c)\eta'(c) + \sigma^2(c)\eta''(c).\end{aligned}$$

Now, we need to determine the boundary conditions. By assumption, firms never vanish and are only replaced by new entrants. So the mass always stays constant, equal to 1. At zero, the reflection boundary condition implies that the equality

$$0.5(\sigma^2(c)\eta(c))'(0) - (\mu(0)\eta(0)) \;=\; 0$$

holds. With reflection at C^*, and with innovating firms jumping to C^*, we will also have an additional term

$$\frac{1}{2}\left(\sigma^2(c)\eta(c)\right)'' - (\mu(c)\eta(c))' \;- x_d\eta(c) \;-\; z(c)\phi\eta(c) \;=\; 0\,.$$

By integrating, it follows that

$$\begin{aligned}0 \;&=\; \int_0^{C^*}\left(\frac{1}{2}\left(\sigma^2(c)\eta(c)\right)'' - (\mu(c)\eta(c))' \;-\; x_d\eta(c)\right)dc \;-\; \int_0^{C^*}\phi z(c)\eta(c)dc\\ &=\; \left(\frac{1}{2}(\sigma^2\eta)'(C^*) - (\mu(C^*)\eta(C^*))\right) - \left(\frac{1}{2}(\sigma^2\eta)'(0) - (\mu(0)\eta(0))\right)\\ &\quad - x_d\int_0^{C^*}\eta(c)dc \;-\; \int_0^{C^*}\phi z(c)\eta(c)dc\,.\end{aligned} \tag{56}$$

Thus, mass conservation

$$\int_0^{C^*}\eta(c)dc \;=\; 1$$

is equivalent to

$$\frac{1}{2}(\sigma^2\eta)'(C^*) - (\mu(C^*)\eta(C^*)) \;-\; \int_0^{C^*}\phi z(c)\eta(c)dc \;-\; x_d \;=\; 0\,,$$

or equivalently,

$$\Big(\sigma'(C^*)\sigma(C^*) - \mu(C^*)\Big)\eta(C^*) + 0.5\sigma^2(C^*)\eta'(C^*) \;=\; \Psi(\epsilon). \tag{57}$$

In this equation, we have defined

$$\Psi(\epsilon) \;\equiv\; \int_0^{C^*} \phi z(c;\epsilon)\eta(c;\epsilon)dc \;+\; x_d$$

Having derived the cross-sectional distribution of liquidity, which we solve numerically, we turn to analyze the general equilibrium properties of the model.

A.4 General equilibrium analysis

Using the notation introduced in Section 4.4, we derive the scaled value of the dynasty $y(c)$, the equilibrium net dividends, labor wage, and aggregate consumption. After a technological breakthrough at time t, outside investors inject the amount $q_{jt-}(\lambda C^* - c)$ of liquidity into the firm. As compensation, they receive a fraction of the surplus, $\alpha q_{jt-}(\lambda v(C^*) - v(c))$ in shares of the firm. Note that compensation cannot be in cash because this would mean they had to inject more than C^* and would then receive their own cash back immediately. Thus, after this transaction, the cash buffer rises to $q_{jt-}\lambda C^*$ whereas the value of the firm changes from $q_{jt-}v(c)$ to $q_{jt-}(v(c)+(1-\alpha)(\lambda v(C^*)-v(c)))$. However, the effect of bargaining is a pure share dilution, and has no effect on the outside liquidity in the economy.

The scaled value of the dynasty is given by the difference between the expected present value of the dividends paid (denoted by the function $D(c)$) and the expected present value of the financing received by all of the firms that will ever operate in the dynasty j. That is, $y(c) = D(c) - F(c)$. In particular, the function $D(c)$ satisfies

$$\begin{aligned}&\frac{1}{2}\sigma^2(c)D''(c) \;+\; \mu(c)D'(c) \;+\; \phi z(c)\Big(\lambda D(C^*) - D(c)\Big)+ \\ &+ x_d\Bigg(c + \Lambda D(C^*) - D(c)\Bigg) \;=\; rD(c)\;,\;\; D'(0) = 0,\;\; D'(C^*) = 1\,,\end{aligned}$$

whereas the function $F(c)$ satisfies

$$\begin{aligned}&\frac{1}{2}\sigma^2(c)F''(c) \;+\; \mu(c)F'(c) \;+\; \phi z(c)\Big(\lambda F(C^*) - F(c) + (\lambda C^* - c)\Big)+ \\ &+ x_d\Bigg(\Lambda F(C^*) - F(c) + \Lambda\left(C^* + \kappa_T\right)\Bigg) \;=\; rF(c)\;,\;\; F'(0) = -1,\;\; F'(C^*) = 0\,.\end{aligned}$$

Taking the difference, it follows that $y(c)$ satisfies[25]

$$\frac{1}{2}\sigma^2(c)y''(c) \;+\; \mu(c)y'(c) \;+\; \phi z(c)\Big(\lambda(y(C^*)-C^*)-y(c)+c\Big)+$$
$$+\,x_d\left(\Lambda y(C^*)-y(c)+c-\Lambda\left(C^*+\kappa_T\right)\right) \;=\; ry(c)$$

with

$$y'(C^*)=1=y'(0)\,.$$

The present value of dividends net of financing is then given by

$$\int_0^\infty e^{-rt}(d_t-f_t)dt \;=\; \int_0^\infty e^{-rt}(d_0-f_0)e^{gt}dt$$

Since $q_{j0}=1$, it follows:

$$\frac{d_0-f_0}{r-g} \;=\; \int_0^\infty e^{-rt}(d_t-f_t)dt \;=\; \int Y(q_{j0},c_j)dj \;=\; L\int \eta(c)y(c)dc\,.$$

Turning to the wage, straightforward calculations deliver

$$W_t \;=\; \frac{1}{1-\beta}\int_0^1 L^\beta\,\bar{X}_{jt}^{1-\beta}\,q_{jt}^\beta dj \;-\; \int_0^1 p_{jt}\bar{X}_{jt}dj\,.$$

By the results in Section 4, the term $\bar{X}_{jt}^{1-\beta}$ in the above equation is given by

$$\bar{X}_{jt}^{1-\beta} \;=\; q_{jt}\left(p(c_j)^{-1/\beta}L\right)^{1-\beta}\;,\;\bar{X}_{jt} \;=\; (p(c_j))^{-1/\beta}\,q_{jt}\,L.$$

Moreover, as discussed in the main text, the following relation

$$W_t \;=\; W_0\int_0^1 q_{jt}dj$$

holds. In this equation, the expression for W_0 is given by

$$W_0 \;=\; \frac{L}{1-\beta}\int \eta(c)p(c)^{-(1-\beta)/\beta}dc-L\int \eta(c)p(c)^{1-1/\beta}dc\,,$$

[25] For completeness, we also define the dynasty in the frictionless benchmark, denoted by y^* and satisfying the following relation

$$\mu-\frac{(z^*)^2}{2}\frac{\zeta}{L}-x_d^*\kappa_T\Lambda=\Big(r^*-x_d^*(\Lambda-1)-\phi z^*\left(\lambda-1\right)\Big)y^*$$

Therefore, it follows that

$$d_0^*-f_0^*=L\left(\mu-\frac{(z^*)^2}{2}\frac{\zeta}{L}-x_d^*\kappa_T\Lambda\right)=L(r^*-g^*)y^*.$$

that is equivalent to

$$W_0 \;=\; \frac{\beta L}{1-\beta}\int \eta(c)X(c)^{1-\beta}dc\,.$$

Furthermore,

$$\int_0^1 q_{jt}dj \;=\; e^{gt}.$$

Then, $\mathcal{C}_t \;=\; \mathcal{C}_0 e^{gt}$, with

$$\mathcal{C}_0 \;=\; W_0 + (d_0 - f_0)\;.$$

Finally, total welfare is given by the utility of the representative consumer, i.e.

$$\int_0^\infty e^{-\rho t}\,\frac{\mathcal{C}_t^{1-\theta}-1}{1-\theta}\,dt = \frac{1}{\theta-1}\left(\frac{1}{\rho}-\frac{\mathcal{C}_0^{1-\theta}}{\rho+g(\theta-1)}\right).$$

A.5 Government debt and the liquidity premium

The wedge between the market interest rate and the return on cash (i.e., the opportunity cost $r-\delta$) plays a major role in determining the impact of financing frictions on corporate decisions. It affects the target level of cash reserves and, thus, the firms' financial resilience. In the extant cash literature, this wedge is typically attributed to agency frictions (the free cash flow problem) or to tax disadvantages.[26] In this section, we propose a general equilibrium channel to endogenize the wedge $r-\delta$, which is based on the interplay between the demand (from the corporate sector) and the supply (from the government) of liquidity. We follow Woodford (1990) and assume that liquidity can only be stored by holding government bonds.[27] When the supply by the government is scarce, bonds trade at a premium that results in a positive wedge $r-\delta$.

As in previous contributions, we assume that the government faces an exogenous expenditure stream. We denote this expenditure stream by $\mathcal{G}_t$ and assume that it constitutes a fixed fraction $\mathcal{G}\in[0,1]$ of the final good output, i.e. $\mathcal{G}_t=\mathcal{G}\mathcal{Y}_t$. To finance these expenditures, the government levies income taxes and issues public debt. We denote the amount of taxes collected at time t by T_t. We assume that taxes represent a fraction τ of wages, i.e. $T_t=\tau W_t L$. Moreover, we denote the public debt outstanding at time t by B_t. We assume that public debt promises a return of δ, which is endogenously determined in equilibrium. Under these assumptions, the budget constraint of the government is given by

$$\mathcal{G}_t + \delta B_{t-1} = \tau W_t + B_t \tag{58}$$

at any time t. The left-hand side of this equation denotes the government outflow, i.e., the expenditure stream plus the repayment of the maturing debt. The right-hand side denotes the government inflow, i.e., the tax inflow plus the newly issued debt.

We assume that public debt grows at the equilibrium rate g, so the debt-to-output ratio is

[26] See also Zucchi (2015), who shows how this wedge may arise endogenously due to stock illiquidity.

[27] See also Holmstrom and Tirole (1998), Bansal, Coleman, and Lundblad (2011), Krishnamurthy and Vissing-Jorgensen (2012, 2013), Greenwood, Hanson, and Stein (2015), and Nagel (2014).

constant. Under this assumption, if the budget constraint of the government holds at time zero,

$$\mathcal{G}\mathcal{Y}_0 + \delta B_0 = \tau W_0 + g B_0,$$

it does hold at any t. In this equation, $\mathcal{Y}_0$ denotes the final good output at time zero, which is given by

$$\mathcal{Y}_0 = \frac{L}{1-\beta}\int_0^{C^*} X(c)^{1-\beta}\eta(c)dc\,.$$

The equilibrium return δ^* makes the government debt market clear, i.e.,

$$L\int_0^{C^*(\delta)} c\,\eta(\delta, c)\,dc = B_0\,.$$

That is, the demand for liquidity in the economy equals the supply of bonds issued by the government.[28] Fixing the supply B_0 amounts to determining the equilibrium tax rate $\tau(B_0)$ from the government budget constraint, which is then given by

$$\tau(B_0) \;=\; \frac{\mathcal{G}\mathcal{Y}_0(\delta^*) - B_0(g(\delta^*) - \delta^*)}{W_0(\delta^*)}\,.$$

In the absence of liquidity frictions, government bonds trade at the market rate r and have no impact on the dynamics of the economy. By contrast, the supply of government bonds has real effects in the presence of liquidity frictions. By affecting the liquidity premium, debt supply influences corporate policies. In particular, as B_0 increases, liquidity frictions vanish, and the economy converges with the frictionless one. We numerically analyze the interplay between liquidity demand and supply in the next section.

We provide a quantitative assessment of our implications. We use again the benchmark parametrization and additionally set $\mathcal{G} = 10\%$. The equilibrium tax rate τ is uniquely pinned down by the government's budget constraint. Consistent with Bansal, Coleman, and Lundblad (2011) and Krishnamurthy and Vissing-Jorgensen (2012), Figure 6 shows that an increase in B_0 leads to a surge in δ and a decrease in the liquidity premium $r - \delta$. At the same time, a larger δ makes debt more costly for the government, which results in a higher tax rate τ. As a result, higher levels of government debt imply higher taxes, as in Woodford (1990).

Figure 6 also shows that a larger B_0 is associated with a bigger growth rate (and to a larger rate of creative destruction, not displayed in the figure). That is, when firms are constrained, policies that increase liquidity in the economy might effectively enhance growth by stimulating the incumbents' and entrants' innovation rate. The mechanism is the following. An increase in B_0 decreases the cost of holding cash and thus leads to an increase in the target level of cash C^* (Figure 7, right panel). As cash is the resource used to finance R&D, the incumbents' innovation rate also increases in B_0 (see Figure 7, left panel). The value of incumbent firms then rises, which makes it more attractive for new firms to enter the industry. As a result, the rate of creative destruction increases. Although larger liquidity supply is associated with a larger tax rate, we find (in unreported results) that an increase in B_0 can enlarge aggregate consumption and welfare via the positive effect on firms' innovation and growth.

[28]Note that the representative agent does not hold government bonds because $\delta < r$.

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Table 1: Baseline parametrization.

Symbol	Description	Value
ρ	Discount rate	1.5%
θ	Risk aversion	2.00
σ	Cash-flow volatility coefficient	0.40
ϕ	Innovation rate coefficient	3.00
ζ	Innovation cost coefficient	1.00
β	Elasticity of substitution	0.25
δ	Return on cash	4.9%
ϵ	Proportional refinancing cost	0.08
α	Surplus haircut	0.06
λ	Quality improvement (incumbents)	1.04
Λ	Quality improvement (entrants)	1.10
κ_T	Entry cost (technological component)	0.70
κ_I	Entry cost (financing component)	0.07
L	Labor supply	1.00

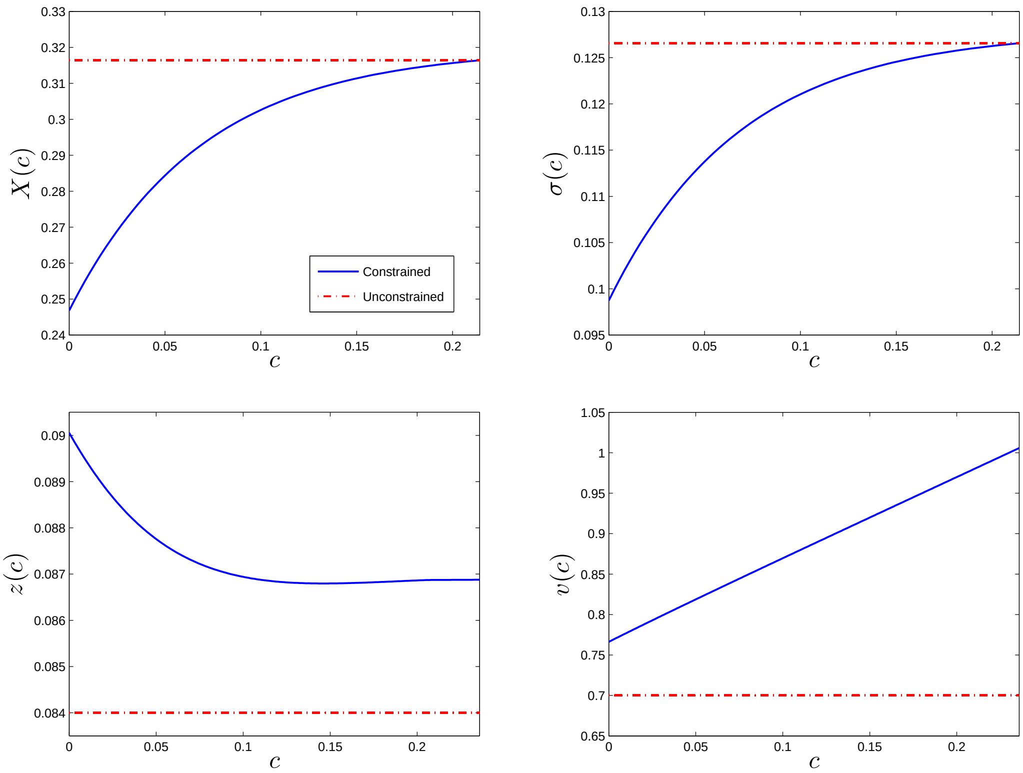

Figure 1: CORPORATE POLICIES.

The figure illustrates the production decision $X(c)$, cash flow volatility $\sigma(c)$, the innovation rate $z(c)$, and the scaled value of an incumbent firm $v(c)$ as functions of cash reserves $c \in [0, C^*]$, under the baseline parametrization. The red line represents the benchmark unconstrained economy, whereas the blue line represents the constrained economy.

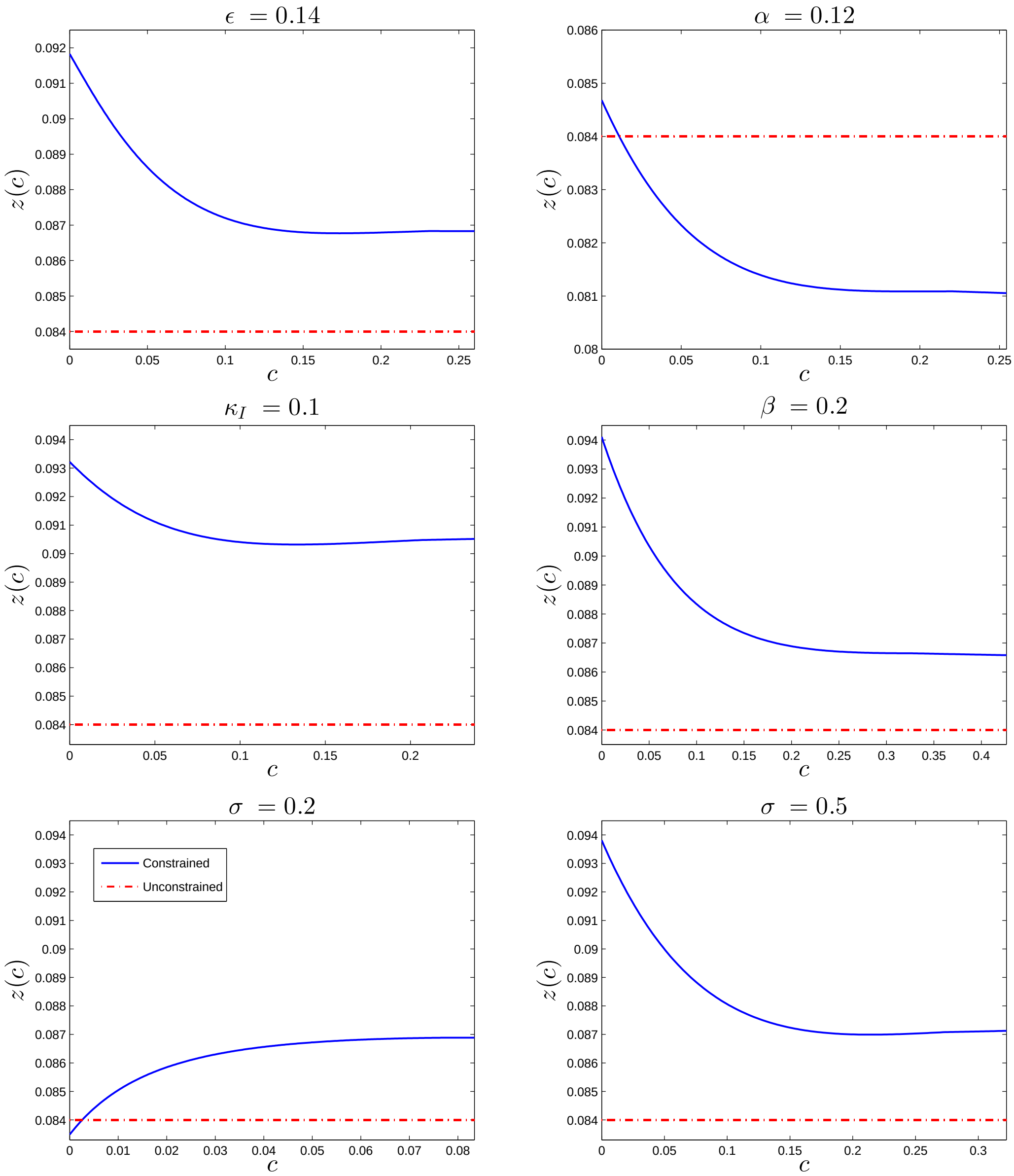

Figure 2: INNOVATION, FINANCING FRICTIONS, AND FIRM CHARACTERISTICS.

The figure illustrates the innovation rate $z(c)$ as a function of cash reserves $c \in [0, C^*]$ when varying the financing costs ϵ, α, and κ_I, the elasticity β, and the coefficient of cash flow volatility σ. The red line represents the benchmark unconstrained economy, whereas the blue line represents the constrained economy.

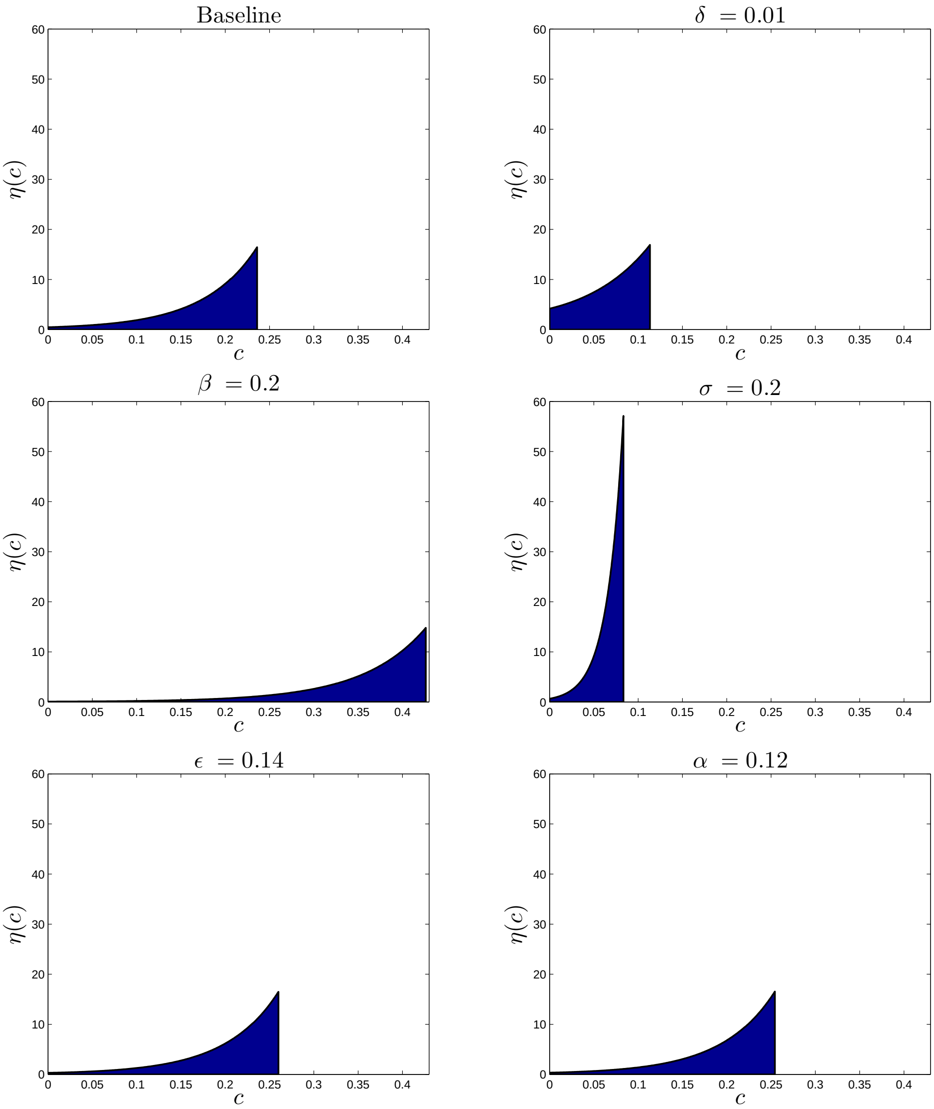

Figure 3: STATIONARY DISTRIBUTION OF LIQUIDITY.

The figure illustrates the stationary cross-sectional distribution of liquidity $\eta(c)$ as a function of cash reserves $c \in [0, C^*]$ in the baseline parametrization and when varying the return on cash δ, the elasticity β, the coefficient of cash flow volatility σ, and the financing costs ϵ and α.

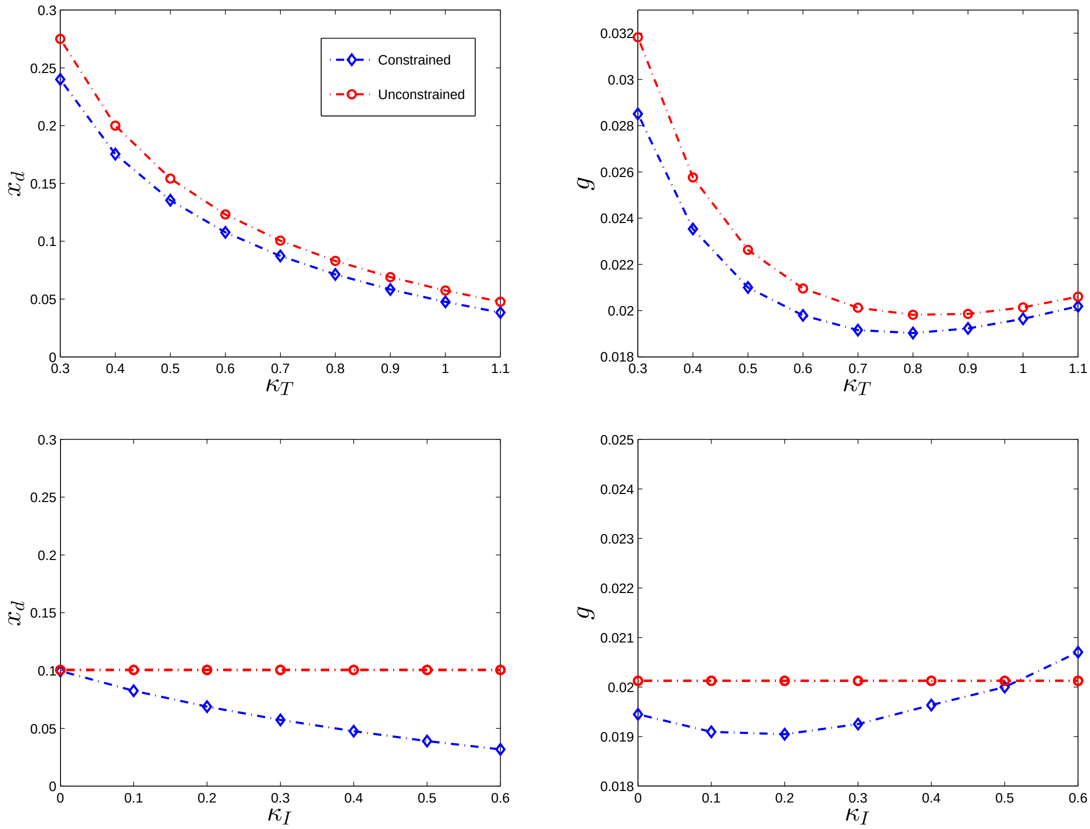

Figure 4: Creative destruction and Growth.

The figure shows the equilibrium rate of creative destruction x_d and the equilibrium growth rate g as functions of the technological κ_T (top panel) and financing κ_I (bottom panel) components of the entry cost. In the top panels, we vary κ_T while setting κ_I to be 10% of κ_T. In the bottom panels, we vary κ_I while keeping κ_T as in the baseline. The red line represents the benchmark unconstrained economy, while the blue line represents the constrained economy.

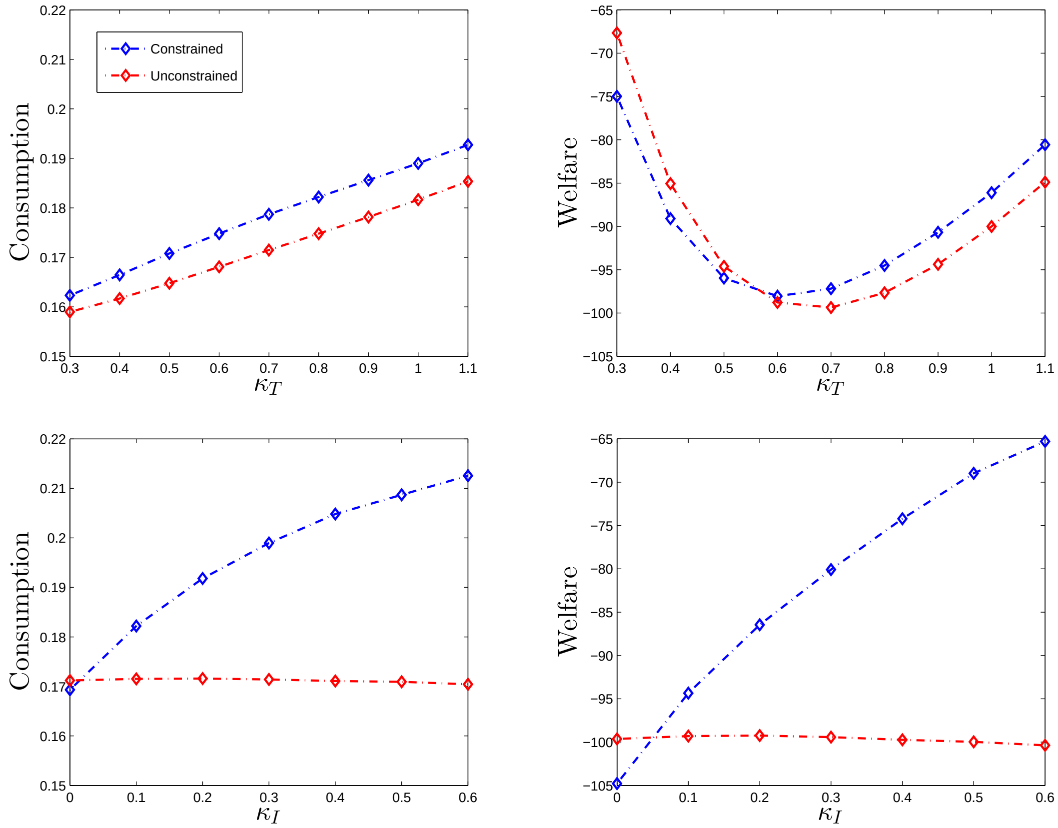

Figure 5: Aggregate consumption and welfare.

The figure shows aggregate consumption and welfare as functions of the technological κ_T (top panel) and financing κ_I (bottom panel) components of the entry cost. In the top panels, we vary κ_T while setting κ_I to be 10% of κ_T. In the bottom panels, we vary κ_I while keeping κ_T as in the baseline. The red line represents the benchmark unconstrained economy, while the blue line represents the constrained economy.

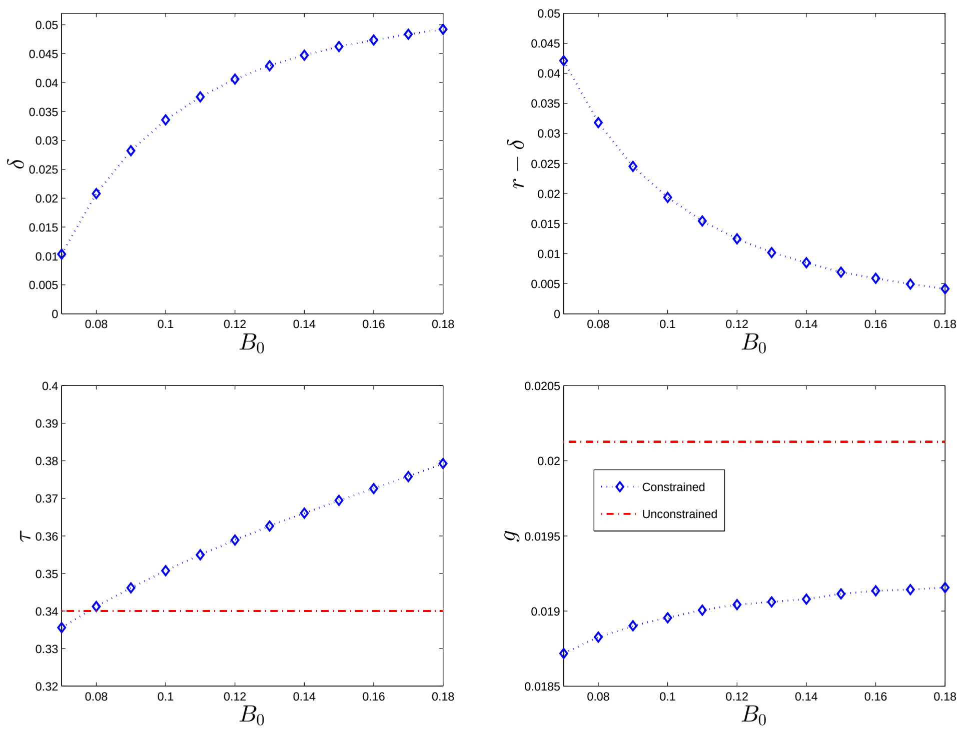

Figure 6: Liquidity supply and equilibrium quantities.

The figure shows the equilibrium return on government debt δ, the liquidity premium $r - \delta$, the tax rate τ, and the economic growth rate g as functions of liquidity supply B_0. The red line represents the benchmark unconstrained economy, while the blue line represents the constrained economy.

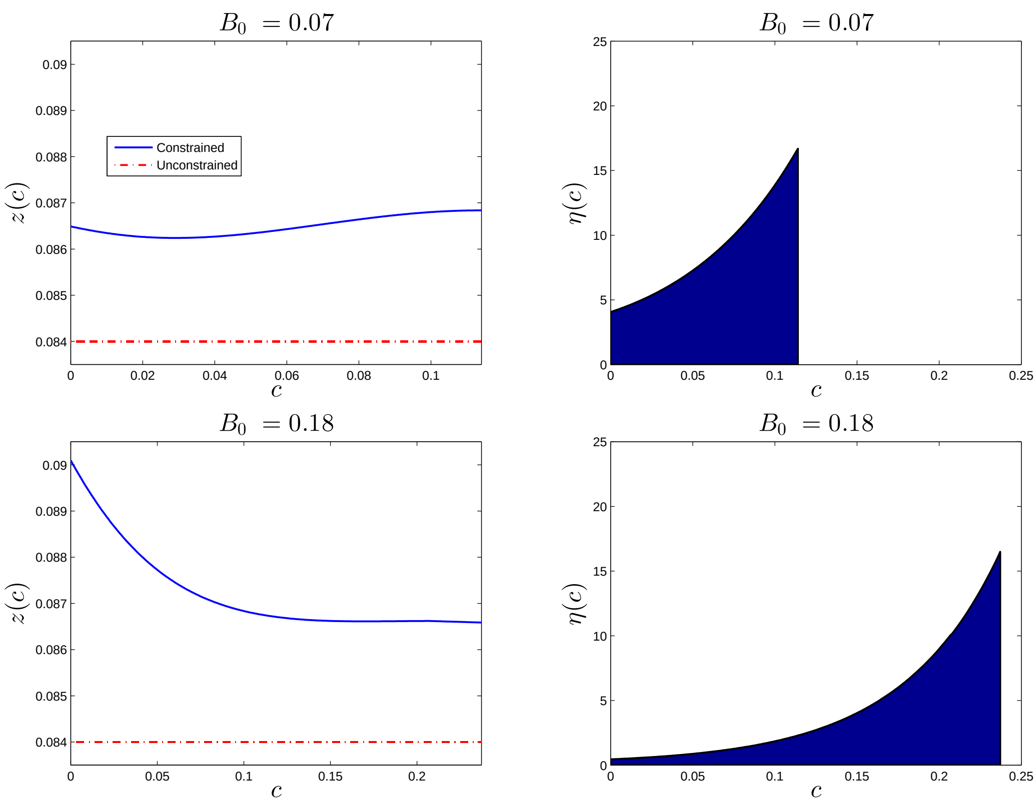

Figure 7: Liquidity supply and innovation.

The figure illustrates the innovation rate $z(c)$ (left panel) and the stationary cross-sectional distribution of liquidity $\eta(c)$ (right panel) as functions of cash reserves $c \in [0, C^*]$ for low supply of liquidity ($B_0 = 0.07$, top panel) and high supply of liquidity ($B_0 = 0.18$, bottom panel). The red line represents the benchmark unconstrained economy, whereas the blue line represents the constrained economy.