The Strategy of Conquest

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Abstract

There is a collection of kingdoms. A kingdom shares a common border with other kingdoms, that may in turn share borders with still others. Every kingdom is endowed with resources and is controlled by a ruler. The ruler can choose to fight with neighboring rulers to expand his domain. The winner of a war takes control of the loser's resources and the kingdom. The probability of winning depends on the resources of the combatants and on the technology of fighting. Rulers seek to maximize the size of resources they control. We study the influence of geography, resources, and technology on the dynamics of war and the prospects for peace.

Keywords Hegemony, pre-emption, buffer states, imperial overreach, offensive and defensive realism

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1 Introduction

The study of the causes of wars and their implications dates back to antiquity; today, it is an active subject of research across the social sciences, and also in history and political philosophy. A recurring observation is that conflict between two opponents is shaped by the presence of neighbouring third parties. The existing theoretical work on the dynamics of war has primarily focused on two actor models.[1] The aim of this paper is to develop a framework with interconnected opponents, in order to better understand the motivations for waging war and the prospects for peace.

We consider a collection of kingdoms. A kingdom shares a common border with some kingdoms, who may in turn share a common border with still others, and so forth. Every kingdom is endowed with some resources and is controlled by a ruler. The ruler can choose to fight with neighboring rulers to expand his domain. The winner of a war takes control of the loser's resources and his kingdom; the loser is eliminated. The winner then decides on whether to wage war against other neighbours or to stay peaceful. The neighborhood of kingdoms is reflected in a contiguity network. The probability of winning depends on the resources of the combatants and on the technology of fighting. Rulers seek to maximize the resources they control. We model the interaction between rulers as a dynamic game and study its (Markov Perfect) equilibria.

The technology of fighting is represented by the Tullock contest function: the probability of resource x winning against resource y is given by $p(x, y) = x^\gamma/(x^\gamma + y^\gamma)$, where $\gamma > 0$. Observe that the probability of wining is increasing in own resources and falling in the opponent's resources. Suppose that $x > y$: we observe if $\gamma > 1$, then the expected returns to the rich ruler, $(x + y)p(x, y)$, are larger than his current resources, x; while $(x + y)p(x, y) < x$ if $\gamma < 1$. The converse holds for the poor ruler. So we say that the technology is *rich rewarding* if $\gamma > 1$ and *poor rewarding* if $\gamma < 1$. Classical writers on war emphasized the decisive role of the size of the army in securing victory, Tzu [2008] and Clausewitz [1993]; that would be a setting where γ is large. The possession of nuclear weapons make resource base less important in war, Waltz [1981], Betts [1977]; this is accommodated by setting $\gamma < 1$.[2]

[1]Third parties were important in conflicts in ancient times (in the Pelopennesian War between Athens and Sparta), central to conflicts in medieval times (in the wars between European powers), and remain so today (in the conflict in Syria).
For recent research in this field, see Acemoglu et al. [2012], Caselli et al. [2015] and Novta [2016]. Indeed, Novta [2016] says, "Ideally, the dynamic model would have .. forward looking agents who anticipate the actions of their neighbors.... the model ... becomes prohibitively difficult to solve."

[2]If γ is very large, the richer ruler wins a war with probability close to 1; by contrast, the probability of

Theorem 1 develops two important implications of the technology of war. The first implication pertains to the question of whether to attack a pair of opponents individually now or to wait for them to fight and then to attack the enlarged kingdom. We show that with a rich rewarding technology, *no-waiting* is optimal, and with a poor rewarding technology *waiting* is optimal. The second implication pertains to the question on whom to attack, a rich or a poor kingdom. We show that there is a *monotonicity* in the optimal attack sequence: with a rich rewarding technology, it is optimal to attack opponents in increasing order of resources; the converse holds in case of poor rewarding technology.

Theorem 2 provides a characterization of the dynamics of war when the technology is rich rewarding. In any configuration with three or more kingdoms, all rulers find it optimal to choose a full attacking sequence, i.e., they continue to attack so long as a opponent exists. The no-waiting property leads poor rulers to attack much richer rulers as a preventive measure, due to their fear of the latter becoming even more rich (and less beatable) over time. This is a world of incessant warfare. The violence only stops when all opposition is eliminated: every outcome involves hegemony of a single ruler.

Proposition 1 takes up the case when technology is poor rewarding. It develops conditions for the existence of equilibrium with perpetual war (leading to hegemony) and of equilibrium with peace (the coexistence of multiple kingdoms). Observe that by definition of poor rewarding, the richer ruler loses, on average, by fighting with a poorer opponent; by contrast, the poorer ruler gains from such a war. However, the waiting property (identified in Theorem 1), suggests that the poorer ruler would prefer to wait and allow for opponents to become large before engaging in a fight. This raises the potential for peace. But for peace to be sustained, the incentive for the poor rewarding ruler to wage war must be offset by loses in subsequent wars. Thus peace can be sustained only under the imminent threat of war with other opponents.

We then examine the role of the contiguity network in shaping strategy and the probability of becoming the hegemon. One way to do this is to ask how the differences in resources matter, in a given network. We shall say that a ruler is *strong* if he has a full attacking sequence in which at every point his opponent has less resources; a ruler who does not have such a strategy is said to be *weak*. We show that the strength of a ruler depends both on resources and on the network: a ruler is weak if he is surrounded by richer rulers or lies within a neighborhood that has a boundary of rulers, who each have more resources than the total resources within

winning is relatively insensitive to relative resources, for γ close to 0;

the neighborhood. Figure 4 illustrates this concept of a 'weak' ruler.[3] We show that for sufficiently large γ, the probability of becoming a hegemon is negligible for a weak ruler, and (roughly) proportional to the number of rulers for a strong ruler, Proposition 2.

A second way is to look at the effects of higher resources for a single ruler, keeping resources of all others fixed. A priori, more resources always increase the probability of winning a war; but Theorem 1 suggests that greater resources could lead a neighbour to switch away his attack to a different opponent and this switch is bad news, due to the no-waiting property. Building on this observation, we show that under rich rewarding technology a ruler always gains from more resources if and only if he is the centre of a star network. Similar considerations also arise in the case of poor rewarding technology.

We close this discussion by addressing the question: what types of locations are favorable to a ruler? The answer turns on the nature of the technology. If the technology is rich rewarding then, given any profile of resources, for a ruler the probability of becoming a hegemon is maximized if he is the centre of a star network, i.e, he is connected to everyone (so he can attack in increasing order of resources) and his neighbours are not connected amongst themselves (this obliges them to attack the ruler first). By contrast, if the technology is poor rewarding then peripheral rulers may be at an advantage. Proposition 3 summarizes these observations.

Finally, we examine the circumstances that facilitate peace. In the basic model, there are no costs to fighting: Theorem 2 and Proposition 1 show that peace cannot be sustained if the technology is rich rewarding, but that it is peace is possible, due to strategic considerations alone, if the technology is poor rewarding. Our first remark therefore is that to the extent that the resource base has become less critical for winning wars, the pressures towards war may have declined over time. We then turn to the role of *direct* costs of war – in terms of physical destruction and the loss of human life – as a deterrent to war. Section 4 extends our basic model to accommodate costs: if a ruler with resources x fights a ruler with resources y then the winner only gets to control $(1-\delta)[x+y]$, where $\delta \in [0,1]$ is a measure of the cost of war. We first take up the two ruler problem. In the rich rewarding case, both similar and very dissimilar resources discourage war; by contrast, in the poor rewarding case, similar resources discourage war, but dissimilar resources lead to war. These observations point to a second – and distinct – motivation for peace. We then study a setting with multiple rulers. For small costs of conflict, the results in the basic model carry through, while for large costs

[3] The figure highlights the role of the contiguity network: some 'rich' rulers may be weak, while 'poorer' rulers are strong.

no ruler has an incentive to wage war.[4] In the intermediate cost range, a number of interesting possibilities arise that bring out the role of the contiguity network. We show that a state can sustain peace between two richer rulers *only* if it is *poor* and if it is located between them: this provides an account for *buffer states* (Proposition 4). And we show that fears of *imperial overstretch* can lead to more wars (and a larger empire) or to greater peace (with smaller kingdoms), depending on the circumstances (Examples 1 and 2).

We now place our paper in the context of the literature.

Our results illuminate the basic forces underlying patterns in imperial and military history and provide a theoretical account for key theories in international relations. Theorem 2 describes the expansion of a kingdom through contiguous expansion until it controls the entire geographically feasible area. This emergence of hegemony is consistent with historical experience; Levine and Modica [2013] present a detailed summary of historical experience. We draw on their work in our discussions in section 3. A central tension in the modern literature on international relations concerns the contrasting prescriptions of 'offensive' and 'defensive' realism, see e.g., Betts [2013], Mearsheimer [2001] and Waltz [1979]. Roughly speaking, 'offensive' realism advocates a strategy of persistent combativeness and aggression, while 'defensive' realism favors a strategy of restraint. Our paper reconciles this tension, and locates their justification in observable parameters such as the contiguity network, resources and technology. We relate our work to these theories and a number of related concepts in international relations, in sections 3 and 4.

In bringing together resources and technology and the contiguity network, we build a bridge between the large and sophisticated literature on the economics of conflict and the growing research on networks. An important early contribution on the dynamics of conflict and appropriation is Hirshleifer [1995]; some recent contributions are mentioned in footnote 2 above. For surveys of the literature, see Konrad [2009], and Garfinkel and Skaperdas [2012]. To the best of our knowledge, the present paper is the first attempt at studying the dynamics of interconnected conflict in a multi-actor world. Our model yields two sets of new results. The first concern the relation between technology and strategy of war and the prospects for peace, reflected in Theorems 1, 2 and Proposition 1. The second concerns the different ways in which the contiguity network shapes conflict dynamics, reflected in our results on location advantages, the nature of strong and weak rulers, the role of buffer states, and imperial overreach, Propositions 2-4.

[4] Some technological developments may have large effects on γ *and* on δ: nuclear weapons make the size of the army less relevant and also raise the potential costs of war.

For an overview of the research on conflict and networks, see Dziubiński, Goyal, and Vigier [2016]. Recent work by Franke and Öztürk [2015], Kovenock and Roberson [2011], and König et al. [2014] focuses on one-shot models. Our paper makes an advance on two fronts: we study the dynamics of conflict and we show that that these dynamics are decisively shaped by the the technology parameter, i.e., departures from $\gamma = 1$.

The rest of the paper is organized as follows. Section 2 presents the basic model and Section 3 analyzes it. Section 4 discusses extensions of the basic model and Section 5 concludes. Appendix A presents the proofs of all results in the main text of the paper, while Appendix B contains some additional results and their proofs.

2 The model

We will study a dynamic game in which interconnected rulers decide on whether to wage war or to remain peaceful. We start by describing the three building blocks in our model: one, a collection of interconnected 'kingdoms', two, resource endowment for every kingdom, and three, a technology of war. We will then describe the choices of rulers and the solution concept.

Let $V = \{1, 2, \ldots, n\}$, where $n \geq 2$ be the set of vertices. Every vertex $v \in V$ is endowed with resources, $r_v \in \mathbb{R}_{++}$. The vertices are connected in a network, represented by an undirected graph $G = \langle V, E \rangle$, where $E = \{uv : u, v \in V,\ u \neq v\}$ is the set of edges (or links) in G. Thus a link indicates 'access'.[5]

Every vertex $v \in V$ is owned by one ruler. To begin, there are $N = \{1, 2, \ldots, n\}$ rulers. Let $\mathbb{o} : V \to N$ denote the *ownership function*. The resources of ruler $i \in N$ under ownership function $\mathbb{o}$, is

$$R_i(\mathbb{o}) = \sum_{v \in \mathbb{o}^{-1}(i)} r_v \tag{1}$$

The network together with the ownership function induces a neighbour relation between the rulers: two rulers $i, j \in N$ are *neighbours* in network $G = \langle V, E \rangle$ (under ownership function $\mathbb{o}$) if there exists $u \in V$, owned by i, and $v \in V$, owned by j, such that $uv \in E$. Figure 1 illustrates vertices and resource endowments (within the vertex), the interconnections across vertices; vertices controlled by the same ruler share a common colour. The light line between vertices represents the interconnections, the dotted lines encircling vertices owned by the same

[5] In our context, this access will be determined by the physical layout of kingdoms and by the state of military and transport technology.

ruler indicate the ownership function, and the thick lines between vertices reflect the induced neighbourhood relation between rulers.

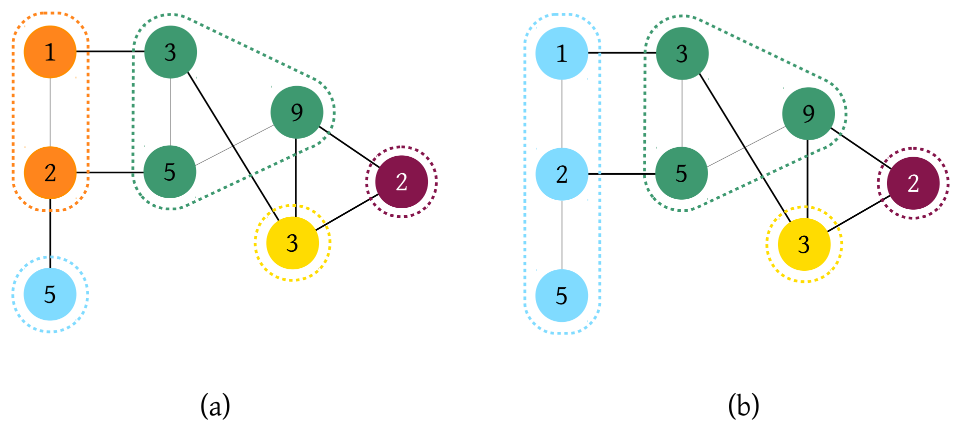

Figure 1: Neighbouring Rulers

If there is a war between rulers i ruler j, then i wins with probability,

$$p(R_i, R_j) = \frac{R_i^\gamma}{R_i^\gamma + R_j^\gamma} \tag{2}$$

where $\gamma > 0$. This is the widely used Tullock Contest Success Function.[6]

We now introduce timing and the choices of rulers. There are rounds, numbered $t = 1, 2, \ldots$.

At the start of a round, one of the rulers is picked at random. The ruler picked (say) i chooses either to be peaceful or to attack one of his neighbours. If he chooses peace, one of the remaining rulers is picked at with equal probability, and asked to choose between war and peace, so forth. If no ruler chooses war, the game ends. We assume that the ruler bases his decision on the ownership function. If the attacker loses, the round ends. Otherwise, the attacker is allowed to attack neighbours until he loses, chooses to stop, or there are no neighbours to attack.[7]

Winning a conflict, the attacker takes over the vertices of the losing ruler (and their

[6]For an axiomatic analysis of contest functions, see Skaperdas [1996].

[7]De Jong, Ghiglino, and Goyal [2014] introduce a model of conflict with resources and a network: the key difference is that conflict is imposed exogenously. Links are picked at random and ruler *must* fight. By contrast, in the present paper, the choice of waging a war or being at peace is the central object of study.

connections), together with the resources owned by them. As he takes over the connections of the newly acquired vertices, the set of neighbours also changes. This is reflected in Figure 1: the Blue kingdom wins the war with the Orange kingdom and expands. This expansion brings it in contact with a new neighbour, the Green Kingdom. The game ends when all rulers choose to be peaceful (the case of a single surviving ruler is a special case, as there is no opponent left to attack).

After every conflict the looser loses all her vertices, so the game ends after at most $n-1$ rounds. It may of course end earlier, if all the rulers choose peace in a round.

In any round, at the point of decision for a ruler, the state is given by the ownership configuration. Given a set of vertices $U \subseteq V$, $G[U] = (U, \{vu \in E : v, u \in U\})$ is the subgraph of G induced by U, i.e. the subgraph of G restricted to vertices in U and links between them. The set of ownerships is denoted by

$$\mathbb{O} = \{\phi \in N^V : \text{ for all } i \in N,\ G[\phi^{-1}(i)] \text{ is connected}\}. \tag{3}$$

Since the graph is fixed, we omit the network as an argument G. So, the set of states is denoted by $\mathbb{O}$.

Given state, $\phi \in \mathbb{O}$, a ruler i chooses a sequence of rulers to attack. A sequence σ is *feasible* at state ϕ if σ is empty, or if $\sigma = j_1, \ldots, j_k$ and for all $1 \leq l < k$, $j_l \notin \{i, j_1, \ldots, j_{l-1}\}$ and j_l is a neighbour of one of the rulers from $\{i, j_1, \ldots, j_{l-1}\}$ under ϕ. Let N^* denote the set of all finite sequences over N (and let us suppose that it includes the empty sequence). Formally, a strategy of ruler i is a function $s_i : \mathbb{O} \to N^*$ such that for each state (ϕ, t), $s_i(\phi, t)$ is feasible.[8]

The probability that ruler 1 with resources x_1 wins a sequence of conflicts with rulers with resources $x_2, \ldots, x_m$, accumulating the resources of the loosing opponents at each step of the sequence is

$$p_{\mathrm{seq}}(x_1, \ldots, x_m) = \prod_{k=2}^{m} p\left(\sum_{j=1}^{k-1} x_j, x_k\right), \tag{4}$$

For convenience, we assume that $p_{\mathrm{seq}}(x) = 1$.

Given a state ϕ, a strategy profile $\boldsymbol{s}$, the probability of ownership ϕ' at the end of game (which may be at round $n-1$ or earlier) is given by $F(\phi' \mid \boldsymbol{s}, \phi)$. And the expected payoff to ruler i from strategy profile $\boldsymbol{s}$ is:

[8] Observe that the only feasible sequence for rulers who do not own any vertices, as well as for the ruler who owns all the vertices, is the empty sequence.

$$\Pi_i(\boldsymbol{s} \mid \mathbb{o}) = \sum_{\mathbb{o}' \in \mathbb{O}} F(\mathbb{o}' \mid \boldsymbol{s}, \mathbb{o}) R_i(\mathbb{o}'). \tag{5}$$

Every ruler seeks to maximize his expected payoff. We study (Markov Perfect) equilibria of the game.

We note that the game is finite. Hence standard results guarantee existence of an equilibrium (possibly in mixed strategies).

Observation 1. *For any graph G, resource vector $\boldsymbol{r} \in \mathbb{R}^V_{++}$, and initial state $(\mathbb{o}, t)$, there exists an equilibrium.*

2.1 Remarks on model

We start with a discussion of the rulers. It is assumed that rulers only care about resources and that the utility is linear. These are simplifying assumptions. The motivations of rulers have been discussed at length in the classical literature, see e.g., Hobbes [1651] and Machiavelli [1992]. Hobbes [1651] discusses three motives for waging a war: greed, glory and fear. Our formulation of ruler's objectives is close to the 'greed' motive, but a larger empire with more resources is also naturally associated with 'glory'.[9] Turning next to the specific functional form: suppose that a ruler has increasing but diminishing returns to resource: utility is given by $u(x)$, with $u(0) = 0$, $u' > 0$ and $u'' < 0$. This means that $u(x+y) < u(x)+u(y)$. Expected payoff to x *vs* y can be written as:

$$p(x, y)u(x + y) = p(x, y)(u(x) + u(y))(1 - d(x, y)) \tag{6}$$

where $d(x, y) = 1 - u(x+y)/(u(x)+u(y))$. So $0 < d(x, y) < 1$: in other words, there is a cost of conflict and the magnitude of this cost depends on x and y. In section 4 below we present a model of costs of conflict: this gives us a first impression of how concave utility will matter. We note that, as things stand, in the basic model, there is no trade-off between consumption and war making. This is again a simplifying assumption. In a richer model, with concerns about consumption and taking the view that resources can be consumed, our main results, will continue to hold, so long as rulers care sufficiently about long run consumption.

We next take up the technology of conflict. We assume that there are only two possible

[9] A recent strand of the literature has been concerned with the implications of the difference in objectives between rulers and the ruled, see e.g., Jackson and Morelli [2007].

outcomes, win or lose. Allowing for a draw does not materially alter the key trade-offs but it does complicate the description of the dynamics: should a new ruler be picked to decide on action or should the original active ruler be allowed first choice of move. In the latter case, our current mode of analysis can be extended in a straightforward manner. Turning now to the conflict technology, we assume it is given by the Tullock contest function (this is also known as the ratio model of conflict). An alternative would be the difference formulation, in which the probability of x winning against y is

$$\frac{e^{\gamma x}}{e^{\gamma x} + e^{\gamma y}}$$

Appendix B shows that we can extend the scope of our main result on contest functions, Theorem 1 to cover this alternative formulation. A third feature of the conflict process is that there is no loss of resources during battle. Costs of conflict are often significant and an important factor in sustaining peace; see, for example the early work of Schelling [1960]. We develop an extension of our model in which a fraction $\delta \in [0, 1]$ of resources is lost in a battle. Section 4 explores the implications of the costs of war for the dynamics of conflict.

Finally, we comment on the order of moves. We assume that if an attack is successfully resisted then the round ends. In the next round, a ruler is picked at random from the set of surviving rulers. In principle, the successful defender can choose an attack sequence. This would make the model symmetric between an attacker and a defender. It would also yield a simpler model, as it removes the uncertainty on who will be the next active ruler. Our main results Theorems 1-2 and Propositions 2- 1 continue to hold in this setting. We assume that rulers can gather all their resources and move them into battle. A major concern in military strategy and imperial history has been the costs and the time that it takes to move army and resources from one battle ground or one frontier to another. In section 4 below we extend our model to allow for this friction: this extension sheds light on an important theme in imperial history – the overstretched empire.

3 The Dynamics of Conflict

We begin by developing some implications of the technology for incentives to engage in conflict.

Our first observation is that there exists a close relation between the technology parameter γ and the expected returns to fighting. Suppose $x > y$. Then $(x + y)p(x, y) > x$, if $\gamma > 1$ while $(x + y)p(x, y) < x$ if $\gamma < 1$. When $\gamma > 1$, the expected resources of the richer ruler after the fight are higher than his ex-ante resources; the converse is true for the poorer ruler. By contrast, if $\gamma < 1$ then the richer ruler in a fight has lower expected returns after the fight than his ex-ante resources; the poorer ruler has higher expected returns than his current resources. It is then natural to say that the Tullock contest function is *rich rewarding* for $\gamma > 1$ and that it is *poor rewarding* for $\gamma < 1$.

We now develop two powerful implications of the technology parameter γ, that play a central role in the analysis.

Theorem 1. *For all $x, y, z \in \mathbb{R}_{++}$,*

1. *The timing of attack:*
   - $p(x,y)p(x+y,z) \begin{cases} > p(x,y+z), & \text{if } \gamma > 1, \\ < p(x,y+z), & \text{if } \gamma < 1. \end{cases}$
2. *The order of attack:*
   - *If $y < z$ then* $p(x,y)p(x+y,z) \begin{cases} > p(x,z)p(x+z,y), & \text{if } \gamma > 1, \text{ and} \\ < p(x,z)p(x+z,y), & \text{if } \gamma < 1. \end{cases}$

The first property pertains to the issue of whether it is better to wait or to attack immediately: if $\gamma > 1$ then it is preferable to attack a sequence of two rulers rather than to wait for them to fight and merge and then attack them. The converse is true if $\gamma < 1$. The second property is concerned with the issue of which of two opponents – rich or poor – to attack first. Again the answer turns on the value of γ: it is better to attack the poor followed by the rich opponent in case $\gamma > 1$; the converse holds true in case $\gamma < 1$. This second property can be generalized to say that, for any fixed set of opponents, the optimal attack sequence is monotonically increasing (decreasing) in case $\gamma > 1$ ($\gamma < 1$). We state and prove this result in Appendix B.

We use these properties of technology to study behavior of rulers in the dynamic game. Observe that when the technology of conflict is rich rewarding, the richer ruler (in a pair) has a 'short run' incentive to attack a poorer neighbour. But such a ruler may want to wait and allow the poor neighbor to engage in conflict with other opponents before fighting. Similar trade-offs arise for the poor ruler today: he has no 'short run' incentive to attack a richer opponent, but he may wish to launch a preemptive attack if there is a fear that over time his neighbors become even richer (through conquest).

Given ownership state $\mathbb{o}$, the set of *active* rulers at $\mathbb{o}$ is

$$\mathrm{Act}(\mathbb{o}) = \{i \in N : \varnothing \subsetneq \mathbb{o}^{-1}(i) \subsetneq V\}. \tag{7}$$

A permutation of the elements of the set $\mathrm{Act}(\mathbb{o}) \setminus \{i\}$, σ, such that the sequence σ is feasible for i in G under $\mathbb{o}$ is called a *full attacking sequence* (or f.a.s). Figure 2 illustrates such a sequence.

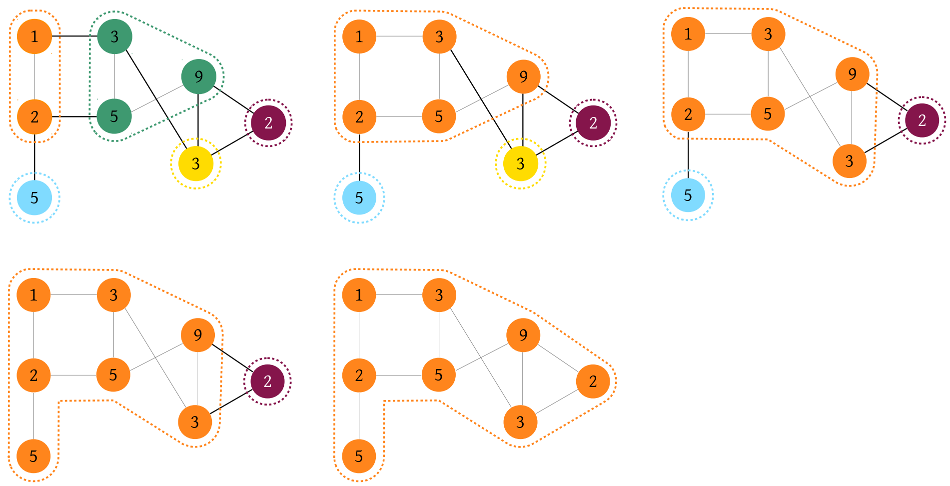

Figure 2: Full Attacking Sequence

We are now ready to state:

Theorem 2. *Suppose that $\gamma > 1$. For any connected network G and for all (generic) resource profiles $\boldsymbol{r} \in \mathbb{R}^V_{++}$, there is hegemony in every equilibrium outcome. In every equilibrium, at any ownership state $\mathbb{o} \in \mathbb{O}$, an active ruler chooses an optimal full attacking sequence (if $|A(\mathbb{o})| \geq 3$), and at least one of the active rulers attacks his opponent (if $|A(\mathbb{o})| = 2$).*

The theorem describes a world characterized by incessant warfare. The violence stops only when all opposition is eliminated. Thus, in equilibrium there is hegemony and the (ex-ante) probability of becoming a hegemon is unique.[10]

[10]We note that the persistent warfare result holds for sub-game perfect equilibria as well. This is because at every state, an optimal full attacking sequence is a dominant strategy for a strong ruler. This is independent of how the state was reached. Given that at least one ruler chooses a full attacking sequence at every state, an optimal full attacking sequence is an optimal strategy for every other ruler at that state, again, independent of how the state was reached.

We now discuss the main steps in the proof. We shall say that aa ruler is 'strong' if he has an attacking sequence $\sigma = i_1, \ldots, i_k$, where for all $l \in \{1, \ldots, k\}$,

$$\sum_{j=0}^{l-1} R_{i_j}(\mathbb{o}) > R_{i_l}(\mathbb{o}) \tag{8}$$

In other words, at every step in the fighting sequence, the ruler has more resources than the next opponent. The set of *strong* rulers at ownership state $\mathbb{o}$ is

$$S(\mathbb{o}) = \{i \in \mathrm{Act}(\mathbb{o}) : i \text{ has a strong f.a.s. } \sigma \text{ at } \mathbb{o}\}. \tag{9}$$

A ruler who is not *strong* is said to be *weak*. Note that in any state, the ruler with the most resources is strong, while the ruler with the least resources is weak. Thus both sets are non-empty in every network and for any (generic) resource profile.

The *first* step shows that, assuming that all other rulers remain passive, it is optimal for a strong ruler to choose a full attacking sequence. This relies on the observation that when the contest success function is rich rewarding, a strong ruler has a full attacking sequence that increases her resources in expectation, at every step, along the sequence.

The *second* step then extends the argument to cover active opponents. This is driven by the no-waiting property developed in Theorem 1. If opponents are active, then it is even more attractive to not give them an opportunity to move, i.e., to employ a full attacking sequence.

The *third* step covers non-strong rulers: in any state with 3 or more active rulers, it is optimal for *every* ruler to choose a full attacking sequence. This step builds on the first two steps: observe that from step two, every ruler knows that he will be facing an attack sooner or later. This means that waiting can only mean that the opposition will become more powerful. The no-waiting property from Theorem 1 then implies that every ruler must attack as soon as possible and choose the full attacking sequence. Finally, if there are only two active rulers, then the richer ruler has a strict incentive to attack the poorer opponent (this follows from the definition of the rich rewarding contest function). Realizing this, the poorer ruler is indifferent between fighting or not fighting.

The Theorem offers a process of expansion of a kingdom through contiguous expansion, until it controls the entire geographically feasible area. By way of illustration, we present in Figure 3 the gradual expansion of the Roman Republic, during the period 500 to 272 BC.

The emergence of hegemony is consistent with broad trends in imperial history. Levine and Modica [2013] present a detailed summary of historical experience and we borrow from their work here. They argue that China is a geographically isolated, being bounded by forests in the South, deserts on the West, wasteland in the North and the Pacific Ocean in the East. So we may take China as a connected network isolated from other parts of the world. During the period starting from 221 BC, the area has been ruled by a hegemonic state roughly three quarters of the time. They also discuss the Roman Empire: Rome gradually expanded and controlled the entire Italian peninsula by 272 BC (as shown in Figure 3), and over time the empire would expand much further afield: Rome would rule over the Mediterranean area as a hegemony for over 400 years from the time of Augustus in 27 BC to the permanent division between Eastern and Western Empires, around 400 AD. In a similar vein, the rapid expansion of the Islamic Caliphate during the period from 624 AD to 733 AD is worth noting; the Caliphate lasted over 500 years until the Mongol invasion in 1258 AD.

In particular, the optimality of attacking strategy for poor rulers highlights the role of preemption and preventive war, and is consistent with arguments in moral philosophy:[11]

> "...a manifest to injure, a degree of active preparation that makes that intent a positive danger, and a general situation in which waiting, or doing anything other than fighting, greatly magnifies the risk." Walzer [1977]

Probably the best known instance of a preventive war is the Pelopennesian War. In his history of the war, Thucydides [1998] argues that Sparta and its allies initiated the war because they feared that any postponement of such an attack would lead to Athens becoming even stronger and more dangerous. A more recent example is the Israeli air strike against Egypt and then Syria, in 1967.

We now turn attention to the poor rewarding contest functions. We begin the analysis with a general observation: (generically) every bilateral conflict is profitable to one of the two sides. So everyone abstaining from fighting can only be sustained if the resulting state, after a war, entails further wars. There is thus a fragility to peace. This is central to the analysis of the dynamics in the setting with a poor rewarding contest function.

Recall that, in the rich rewarding setting, the existence of a strong ruler who gains from each consecutive fight and the no-waiting property were the driving force behind the optimality of a full fighting sequence and this in turn led to hegemony. In the poor rewarding case this

[11]In our model, only one ruler is active at a time. We believe that the pressures towards war and aggression identified in Theorems 1 and 2 are robust and expect that they will get reinforced if multiple rulers can become active at the same time.

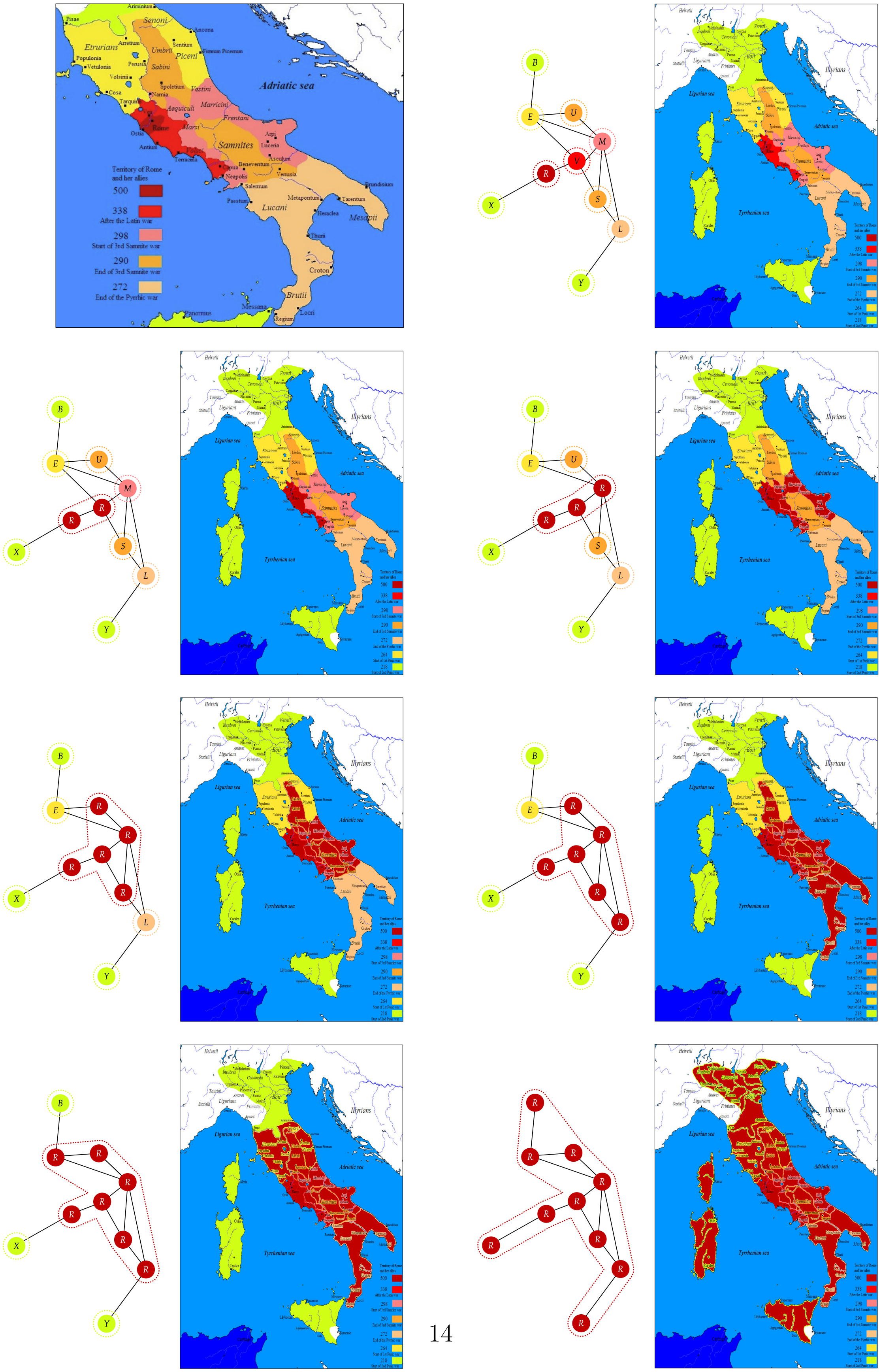

Figure 3: Expansion of the Roman Republic

is not the case, in general, because every ruler prefers to wait for others to move and there may be no ruler who gains from each consecutive fight. But if there is a vertex with resources sufficiently larger than the sum of resources of all the other vertices then, in any state, every ruler, other than the current owner of the rich vertex, prefers an attacking sequence over peace, if all other rulers choose peace. Hence, the outcome must be hegemony.

We then examine the situation where resources are not so unequal. We were unable to carry out an analysis for general networks. So, we focus on a star network. In a star all the fights necessarily involve the central ruler; whoever attacks the centre and wins becomes the central ruler in turn. We develop sufficient conditions on resources for perpetual peace to be the unique equilibrium. The general idea here is as follows. Suppose the centre has x resources and the each of the spokes has y resources. We want x and y to be such that:

1. no ruler wishes to engage in $n-1$ fights in the initial network,
2. for any $i \in 1, ..., n-2$ every 'peripheral' ruler wants to engage in $n-i-1$ fights in the star network arising after i initial fights.

The second condition ensures that if any vertex executes $1 \leq i \leq n-2$ fights, then there will be a fight until hegemony after that, which is not profitable for that vertex. Taken together these two conditions yield a situation in which no one wishes to start a war: upon winning this war, the winner has to be prepared to fight until the finish. But such a fight is not worthwhile given that the resources satisfy the first condition. The key difficulty in making this argument work lies in showing that for *any* $n \geq 3$, and for *any* $\gamma \in (0, 1)$, we can indeed find x and y that permits the construction described above. We note that this example is "tight" in the sense that the resources in the centre are of almost the same order as the high resources leading to fight till hegemony derived in the first part of the analysis.[12]

Next we develop sufficient condition on resources for war followed by peace. Again, we focus on the star network. The construction here builds on the perpetual peace argument above: we propose a sequence of fights that are worthwhile for the rulers involved, and lead to a star where peace is possible. These observations are summarized in the following result.

Proposition 1. *Suppose that G is connected, $\boldsymbol{r} \in R^V_{++}$, and that $\gamma < 1$.*

[12]The argument in the example clearly also works when the resource values are perturbed slightly (in particular, the resources at spokes of the star network do not have to be equal, it is enough that they are sufficiently close).

1. *Fix a resource profile* $\boldsymbol{r}$ *such that for some vertex* $v \in V$*,*

$$r_v \geq 2^{\frac{|V|-1}{1-\gamma}} \sum_{u \in V \setminus \{v\}} r_u. \tag{10}$$

Then every equilibrium outcome is hegemony.

2. *For any number of vertices,* $|V| \geq 3$*, there exist resource profile* $\boldsymbol{r}$ *such that perpetual peace is an equilibrium outcome in the star network.*

3. *For any number of vertices,* $|V| \geq 4$*, there exist resource profile* $\boldsymbol{r}$ *such that an initial phase of warfare followed by peace is an equilibrium outcome, in the star network.*

A comparison of Theorem 2 with Proposition 1 reveals contrasting optimal strategies (full attacking sequence versus no war) and outcomes (hegemony versus multiple kingdoms). It highlights the role of technology of war and offers one possible resolution to a key tension in the modern literature on international relations: whether nations should be offensive or defensive? The optimality of full attacking sequence – both for rich and poor rulers – and the search for hegemony echoes the central arguments for 'offensive' realism:

> "Given the difficulty of determining how much power is enough for today and tomorrow, great powers recognize that the best way to ensure their security is to achieve hegemony now, thus eliminating any possibility of a challenge by another great power. Only a misguided state would pass up an opportunity to be the hegemon in the system because it thought it already had sufficient power to survive." Mearsheimer [2001]

By contrast, our analysis of the dynamics of conflict under a poor rewarding technology offers a potential foundation for the thesis of 'defensive realism'. This thesis argues that:

> "... the first concern of states is not to maximize power but to maintain their position in the system." Waltz [1979]

The key reason for peace therefore is fear of conflict escalation: any attack by a ruler leads to a state with war, that is not profitable. This is consistent with the basis for peace identified in Proposition 1.

We now examine the role of the contiguity network more closely. We do this in three different ways: First, we study how the prospects of rich and poor rulers are affected by the contiguity network. Second, we examine the effects of greater resources for a single ruler on his long run probability of becoming the hegemon. Thirdly, we study the question: what types of networks and which location within a network is advantageous for a ruler?

From Theorem 2 we know that every strong ruler has an optimal full attacking sequence and at every stage of the sequence the next conflict increases his resources in expectation. Observe that for sufficiently high γ, it is never optimal to attack a richer ruler if other options are available. Thus, the optimal strategy for a strong ruler involves a strong sequence of attacks. This is clearly not an option for a weak ruler: so the probability of a weak ruler becoming a hegemon falls to zero as we raise γ. Whether a ruler is strong or weak depends on his own resources but also on the distribution of resources and on the position of the ruler in the contiguity network. The boundary of a set of vertices $U \subseteq V$ in G is

$$B_G(U) = \{v \in V \setminus U : \text{ there exists } u \in U \text{ s.t. } uv \in E\} \tag{11}$$

Given a graph, $G = \langle V, E \rangle$ and resource endowment $\mathbf{r}$, a set of vertices, U, is *weak* if $G[U]$ is connected, $B_G(U) \neq \varnothing$, and for all $v \in B_G(U)$, $r_v > \sum_{u \in U} r_u$. A weak set of rulers is surrounded by a boundary, constituted of rulers, each of whom is endowed with more resources than the sum of resources of vertices within the set. Weak sets are illustrated in Figure 4.

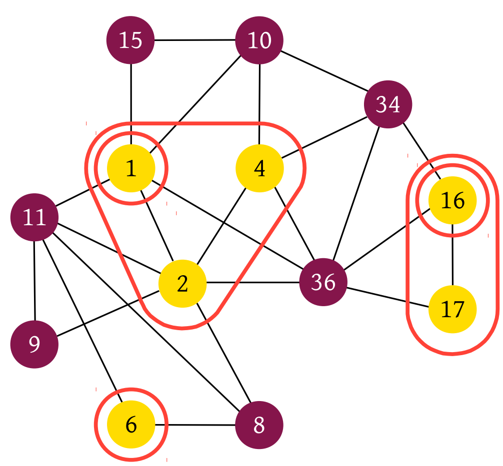

Figure 4: Weak rulers (surrounded by thick lines) and strong rulers

Given state ϖ, technology γ, and resources $\boldsymbol{r}$, let $\text{Prob}_i(\boldsymbol{r}, \gamma | \varpi)$ be the probability of ruler i becoming the hegemon. We can now state:

Proposition 2. *Fix a connected network G and suppose that $\gamma > 1$.*

- *The probability of becoming a hegemon:*

$$\lim_{\gamma \to +\infty} \text{Prob}_i(\boldsymbol{r}, \gamma | \mathbb{O}) \begin{cases} \geq \frac{1}{|\text{Act}(\mathbb{O})|}, & \text{if } i \in S(\mathbb{O}) \\ = 0, & \text{otherwise.} \end{cases} \tag{12}$$

- *For any initial state* $\mathbb{O}$ *a ruler is weak if his vertex belongs to a weak set; otherwise, the ruler is strong.*

Links indicate access: developments in transport (and in technology more generally) will improve access. In our model, this can be studied in terms of additional links in the contiguity network. We note that adding a link between two vertices within the same weak set or between two vertices outside the weak sets has no effect on the partition between weak and strong rulers. Adding a link between a vertex in a weak set to a vertex outside the weak set (be it in another weak set or outside the weak sets) weakly increases the number of strong rulers. Thus, for any resource endowment, the number of strong rulers is maximized in the complete network, and it is minimized when the strongest ruler is at the centre of a star network.

Next consider the effects of increasing the resources of one ruler, keeping everything else fixed. A first intuition would be that greater resources for a ruler must be an advantage as they improve the prospects of winning a battle. However, we know from Theorem 1 that greater resources also make a ruler a less attractive target for attack. This may mean that when a ruler has larger resources, this diverts attack to a neighbor. This *switch* in order of attack means that when the ruler finally engages in a conflict, he may face a larger opponent (that arises from a merger of two kingdoms). We know from part 1 of Theorem 1 that the ruler prefers to face a sequence of rulers rather than the combined forces of the sequence. Thus, the effects of an increase in resources will depend on the possibility of 'switching', which in turn depends on the architecture of the network. Our analysis reveals that more resources for a ruler always raise the probability of becoming a hegemon if and only if there is no switching: in other words if the ruler is a hub in the star network. The basic arguments underlying this proposition can be developed in an example with $n = 3$ vertices (as in Figure 5).

Consider rulers a, b and c, and suppose that $R_b > R_c$. Let G be the clique in Figure 4(i). When $R_a \in (0, R_c)$, a is the poorest ruler, and so rulers b and c choose to attack a first. Increasing the resources of a within the interval $(0, R_c)$ does not change the equilibrium; thus it raises a's probability of becoming a hegemon. However, when R_a crosses R_c, it is now optimal for b to attack c first. This switch in optimal attack lowers the probability of a

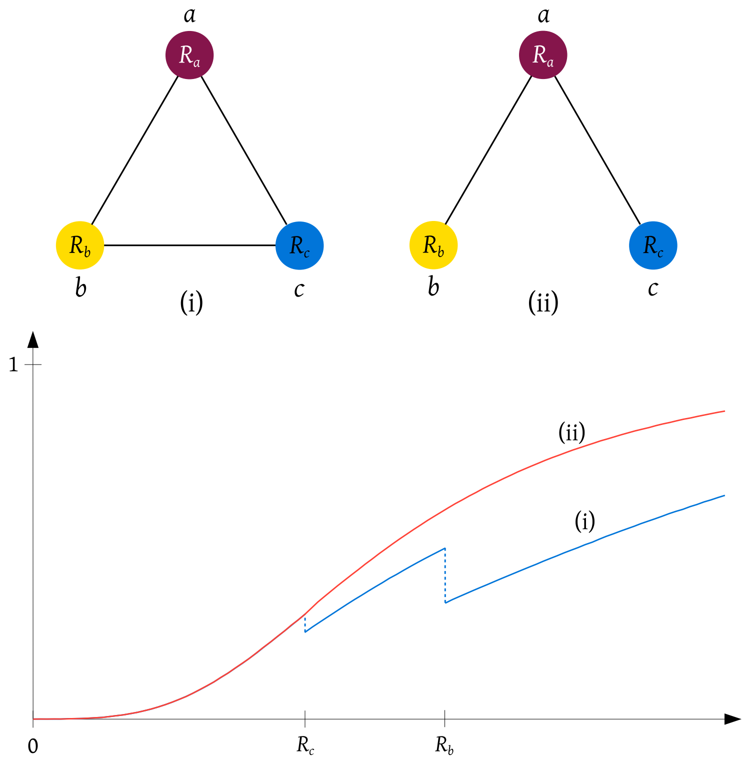

Figure 5: Resources, Networks, and Hegemony

becoming the hegemon (in an interval after R_c). A similar argument comes into play as R_a increases further and goes beyond R_b. There is a local fall in probability of a becoming a hegemon again. It is worth noting that the network structure is important in this argument: if a controls the center of a star network then opponents cannot switch order of attack and, as a consequence, the probability of becoming the hegemon is always increasing in resources. This is illustrated in Figure 5. The general point to note is that so long as the equilibrium strategy remains unchanged, an increase in resources will always enhance the probability of becoming a hegemon. So restrictions imposed by networks matter because they limit the strategic options.

From the above argument, it is clear that being central in a star is *sufficient*: opponents have no possibility of switching and so there is no potential downside to having more resources. To see that being central in a star is *necessary* observe that in any other location or any other network, resource endowments exist for which the construction in the above example can be replicated: a switch in attack can arise, that lowers the probability of becoming the hegemon. Similar arguments can be developed, for the poor rewarding technology, to show that switching generates a non-monotonicity in the probability of becoming a hegemon; again switching of attack sequence is central to the construction. This establishes necessity of being central in a star network for monotonicity in resources. However, as we do not have a complete characterization of equilibrium in the poor rewarding case, we are unable to show that being central in a star is sufficient.

Finally, we take up the issue of location advantage. Generally speaking, adding a link between neighbours of ruler a gives them the possibility of attacking each other, prior to attacking ruler a. With a rich rewarding technology, this is potentially bad news for ruler a, due to the no-waiting property developed in Theorem 1. Consider next a new link between Ruler a and Ruler b. This link offers Ruler a the opportunity of attacking b earlier and this is potentially good news as it may enable Ruler a to attack in preferred order. Moreover, it brings b closer to a and the no waiting property suggests that a would prefer to be attacked earlier rather than later. To develop the implication of these pressures in the simplest manner, we consider a slight variation of our model in which we allow the winning ruler to keep his resources mobile and be allowed to move immediately – *independently of whether he is the attacker or the defender*. Theorem 2 holds in this model. In addition, we establish that for any resource profile $\boldsymbol{r} \in \mathbb{R}^V_{++}$, the probability of ruler i becoming a hegemon is maximized if he is located at the centre of a star network.[13] We next take up the poor rewarding case. Theorem 1 tells us that rulers prefer to wait and let the opponents grow before fighting. Building on this preference for waiting it is possible to show that peripheral rulers may be better off than centrally located rulers. Thus the issue of what is an advantageous location turns on the question of whether the technology is rich or poor rewarding.

We summarize the discussion in the next result.

Proposition 3.

- *Consider an increase in resources of ruler a:*
  - *Rich rewarding: raises his probability of becoming the hegemon for all initial resource profiles* $\boldsymbol{r} \in \mathbb{R}^V_{++}$ *if and only if the ruler is the centre of a star network.*
  - *Poor Rewarding: raises his probability of becoming the hegemon for all initial resource profiles* $\boldsymbol{r} \in \mathbb{R}^V_{++}$ *only if the ruler is the centre of a star network.*

[13] Appendix B presents a discussion of some complications that arise in the basic model – in which the victorious aggressor can continue attacking neighbors, while a victorious defender is obliged to wait his turn in a subsequent round.

- *Location advantage: Suppose that winning ruler retains right of attack.*
  - *Rich rewarding: for every resource profile* $\boldsymbol{r} \in \mathbb{R}^V_{++}$ *the probability of a ruler becoming the hegemon is maximized if and only if he is the centre of a star network.*
  - *Poor rewarding: peripheral location in a network may maximize the probability of becoming the hegemon.*

4 Extension: Costs of Conflict

Wars entail destruction of materials and infrastructure and loss of lives; this section extends the basic model to take these costs into account. We start with a consideration of the two ruler case. In the rich rewarding case, both similar and very dissimilar resources discourage war. In the poor rewarding case, similar resources discourage war, but dissimilar resources lead to war. We then locate these considerations within a contiguity network to show that if the costs of conflict are small, then our main results carry over while if the costs of conflict are large then no ruler wishes to wage war, and perpetual peace is the outcome. The attention then turns to intermediate costs of conflict and casts light on important concepts in international relations and military history. Our analysis shows that a small state can sustain peace between two richer rulers *only* if it is located between them: this provides an account for *buffer states*. And we show that fears of *imperial overstretch* can lead to more wars or greater peace, depending on the architecture of the network.

Suppose that a conflict between two rulers entails a loss of a fixed fraction $\delta \in (0, 1)$ of resources. The expected payoff to a ruler with x resources from a conflict with a ruler with y resources is given by

$$\Pi(x, y) = p(x, y)(x + y)(1 - \delta). \tag{13}$$

When $\delta = 0$, there is a zero cost to war: this is the benchmark model. At the other extreme, when $\delta = 1$, a war leads to complete destruction of resources of both the rulers involved.

As in the benchmark model, we start with a consideration of the two ruler situation. Suppose that there are two rulers, with resources x and y, respectively; let $x > y$. First consider $\gamma < 1$. As the contest success function is poor rewarding, it is never profitable for the richer ruler to attack. Attack is profitable for the poorer ruler if and only if $p(y, x)(x + y)(1 - \delta) > y$. Define $\rho = y/x$, and we can rewrite this inequality as;

$$\delta < \frac{1 - \rho^{\gamma+1}}{1 + \rho}. \tag{14}$$

We plot the value of δ as a function of ρ in Figure 6. Given a δ, there is a threshold resource $\hat{y} \in [0, x]$: waging war is attractive for the poorer ruler if he is poor relative to the other ruler, i.e., if $y \in (0, \hat{y})$, but not otherwise.

Next consider the case where $\gamma > 1$: it is never profitable for a poorer ruler to attack. For the richer ruler it is profitable to attack if and only if $p(x, y)(x + y)(1 - \delta) > x$, i.e.,

$$\delta < \frac{\rho - \rho^{\gamma}}{1 + \rho}. \tag{15}$$

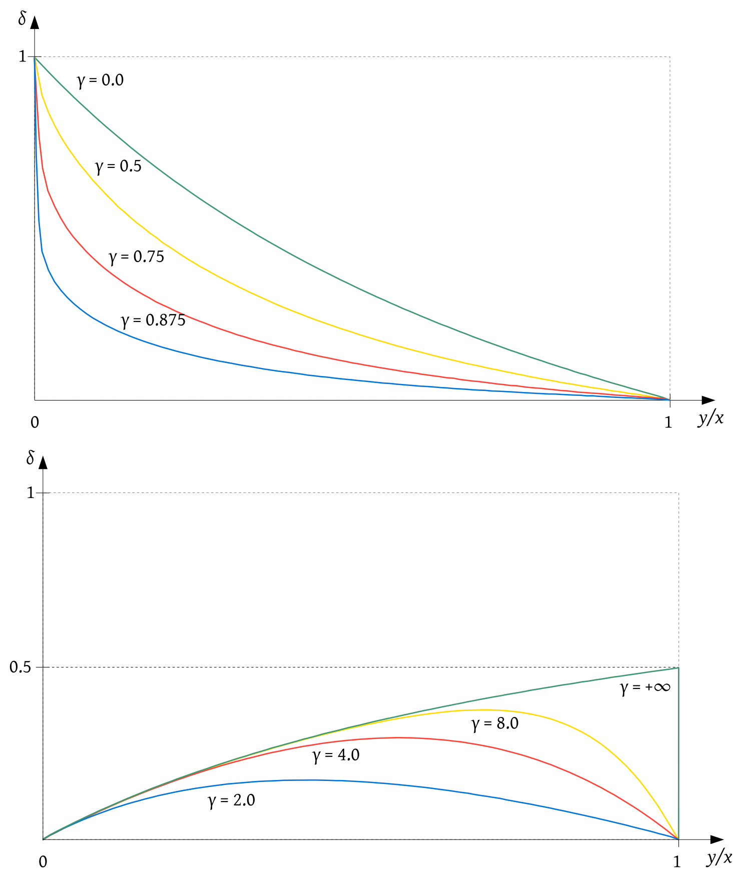

Figure 6: Incentives to Attack. Top: $\gamma < 1$ Bottom: $\gamma > 1$

We plot the threshold value of δ as a function of ρ in Figure 6. For appropriately low values of δ, there are two thresholds: y_{L} and y_{H}. If $y \in (0, y_{\mathrm{L}})$, then attack is not profitable, because the loss in own resources is large relative to the new resources potentially acquired. For large resources, $y \in (y_{\mathrm{H}}, x)$, there is an even chance of winning, that is close to 1/2, and the positive costs of conflict render an attack unattractive. This leaves the intermediate range, $(y_{\mathrm{L}}, y_{\mathrm{H}})$: here the expected gains are sufficiently high and may exceed the cost of conflict, $\delta(x + y)$. Finally, observe that since $(\rho - \rho^{\gamma})/(1 + \rho) < \rho/(1 + \rho) < 1/2$, an attack is never profitable if $\delta > 1/2$. To summarize: if $\delta > 1/2$, then there is no war, irrespective of the resources; if $\delta < 1/2$ then there can be war if the resources are not too similar or too dissimilar.

We now locate these incentives for waging war within a network. Observe that the expected payoff to a ruler with x_0 resources from a sequence of conflicts with m rulers with resources $x_1, \ldots, x_m$ is given by

$$\begin{aligned}
&\Pi_{\mathrm{seq}}(x_0, x_1, \ldots, x_m) \\
&\quad =(1-\delta)^m \left(x_0 + \sum_{j=1}^{m} \frac{x_j}{(1-\delta)^{j-1}} \right) \prod_{i=1}^{m} p \left(x_0 + \sum_{j=1}^{i-1} \frac{x_j}{(1-\delta)^{j-1}}, \frac{x_i}{(1-\delta)^{i-1}} \right) \\
&\quad =(1-\delta)^m \left(x_0 + \sum_{j=1}^{m} \frac{x_j}{(1-\delta)^{j}} \right) p_{\mathrm{seq}} \left(x_0, \frac{x_1}{(1-\delta)^0}, \ldots, \frac{x_m}{(1-\delta)^{m-1}} \right).
\end{aligned} \tag{16}$$

So, the (expected) payoff to the ruler with resources x_0 fighting a sequence of rulers with resources $x_1, \ldots, x_m$ is the same as fighting a sequence of rulers with resources $\frac{x_1}{(1-\delta)^0}, \ldots, \frac{x_m}{(1-\delta)^{m-1}}$. As the payoff varies continuously with δ, it is immediate that, other parameters being fixed, our earlier results reflected in Theorem 2 and Propositions 1 and 2, will continue to hold if δ is small. On the other hand, there will be no attacks at all and perpetual peace is the outcome, in case δ is large (as the costs of conflict are then prohibitive).

We now consider intermediate costs of conflict. We will show that a combination of networks and resources – reflected in a *buffer state*, can help sustain peace. Consider a line with three vertices (and rulers), a, b and c (in that order). Suppose that $\gamma = 16$ and the cost of conflict is $\delta = 0.2$. Let us look at incentives to wage war as we vary resources of ruler b. If $R_b \in (0, 1.79)$, then b serves as a buffer state: ruler a does not find it profitable to attack b and then c, nor attack b only. Similarly, c does not find attacking b or attacking b and then a profitable. Therefore peace in the initial state is the equilibrium outcome. We now turn to the setting when $R_b \in (1.79, 10)$: one of the rulers always has an incentive to attack, and the outcome is either two rulers or a single hegemon. The details of the computations are presented in the Appendix A. It is interesting to note that if there was a link between a and c, then ruler a would definitely attack c. So the small kingdom must be located *between* the opposing powers.

There is a growing literature on buffer states. A detailed discussion including a list of examples of such states is provided by Chay and Ross [1986] who attribute three characteristics to them,

> '*..they are small countries, in both area and population; they are adjacent to two larger rival powers; and they are geographically located between these opposing powers*'. Chay and Ross [1986]

Belgium was regarded as a buffer state between France and Germany in the period until the first World War; and indeed, Belgian resistance to the German army played a role in slowing down their attack on France.

We summarize the discussion in the following result.

Proposition 4. *Fix a resource endowment $\boldsymbol{r} \in \mathbb{R}^V_{++}$ and $\gamma > 1$. Then there exist threshold values of the cost of conflict $0 < \delta_1 < \delta_2$ such that*

1. *If $\delta < \delta_1$ then Theorem 2 and Propositions 1 and 2 hold.*

2. *If $\delta > \delta_2$ then the equilibrium outcome is perpetual peace.*

3. *If $\delta \in (\delta_1, \delta_2)$, then 'buffer states' can help sustain peace.*

The first part of the proposition follows from the continuity and monotonicity of the contest success function. We would like to emphasize that the threshold value for cost of conflict that ensures peace is not large: in particular, $\delta_2 \geq 1/2$ is sufficient. The second part of Proposition 4 says that sufficiently high cost of conflict leads to peace. We conclude by noting that a rising cost of conflict below that threshold may actually lead to greater conflict. This is because one reason for peace is the fear of conflict escalation. A higher cost of conflict may prevent conflict escalation, and this may encourage rulers to attack their neighbours! We develop this argument in Appendix B.

4.1 Time to Mobilization and Targeted Attacks

In the basic model, both the attacker and the defending ruler can mobilize all their resources in a war. However, the movement of military equipment and troops takes time and effort. In this section we extend the model to study these considerations. We suppose that defending rulers pre-allocate their resources across their kingdom, war is localized (on a vertex), and the target of attack must defend itself with the resources that are available at that vertex only.

Consider a variant of our model with the following order of moves. At the start of a round, all rulers decide on the distribution of resources across their vertices. Then a ruler is picked and he can decide to remain peaceful or to start a sequence of attacks. If the ruler chooses to attack then he can mobilize all his resources. Every attack in the sequence specifies not only the ruler to be attacked but also a vertex, bordering the current territory of the attacker, that the attack is launched at. At every attack in the sequence, the attacked ruler must defend using the resources allocated to the attacked vertex only. The winner of the battle takes over all the resources and the entire territory of the loser. If it is the attacker who wins, then he may proceed and attack the next ruler (on selected vertex) in the sequence. Otherwise, the round ends and a new round begins.

To bring out the effects of change of model clearly, we focus on the situation where $\gamma > 1$. The first observation is that Theorem 2 continues to hold in this setting: this is because every ruler now has a (weakly) higher probability of winning as he only has to contend with the resources allocated to the vertex he attacks. To study the possibility of peace, we now introduce the cost of conflict parameter δ. A general analysis of this model is outside the scope of the present paper; but we show how this model can be used to study the pressures generated by an *overstretched empire*. The traditional view has been that an awareness of 'overstretch' would restrain rulers and would limit the expansion of empires. The idea of an 'overstretch' goes back to ancient antiquity; the withdrawal of Roman armies from Mesopotamia by the Emperor Hadrian in 117 AD is often cited as an instance of imperial containment. Kennedy [1987] presents a wide ranging exploration of the idea in the period from 1500 AD. In recent years, this idea of overreach has been prominent in discussions on the foreign commitments of the United States, see. e.g., Mearsheimer and Walt [2016].

The next example shows that the traditional argument is valid in some circumstances: the fear of overreach leads to less wars and smaller empires. It also shows how, in other contexts, far sighted rulers may actually seek to address the dangers of 'overreach' through the elimination of *additional* potential opponents.

Example 1. Imperial Overreach: fewer wars and smaller empire

Consider the network in Figure 7, a line of four vertices controlled by rulers a, b, c, and d, respectively. Let the resources be 15, 10, 5.5, and 9, respectively. The technology of conflict parameter $\gamma = 2.0$ and the cost of conflict parameter $\delta = 0.15$.

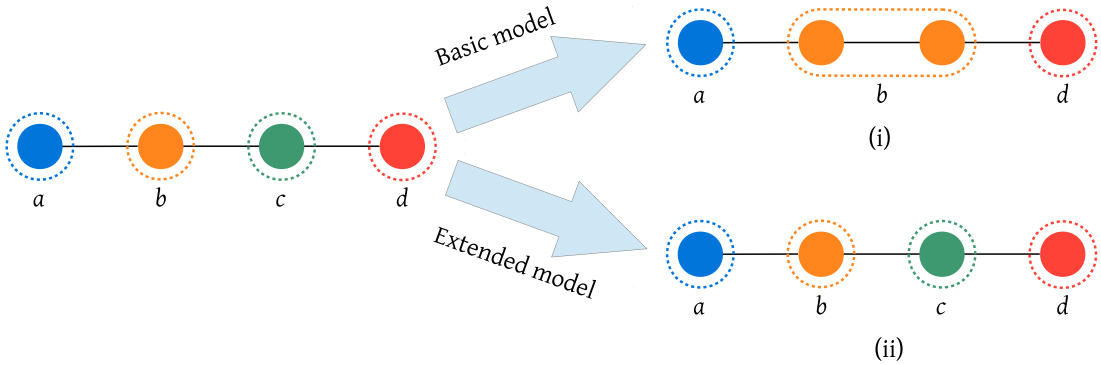

Figure 7: Fewer Wars and Smaller Empire

To bring out the role of order of moves clearly, we first discuss the equilibrium of the basic model (the defender can use all his resources). It may be verified that there is an equilibrium in which ruler a, c, and d choose peace, while ruler b chooses to attack c. In the resulting state, no ruler can benefit from war. So this is the unique equilibrium outcome.

Now consider the model in which rulers first allocates resources and the target kingdom can only use resources at a vertex to defend itself. As before rulers a, c and d choose peace in the initial state. The interesting change is that ruler b also prefers peace to any attacking sequence. This is because the only attacking sequence yielding payoff above the initial resource holding, 10, is attacking c only. However, after this attack succeeds, b is obliged to allocate his resource across the two vertices b and c. The total resources b owns after the conflict are $0.85 \cdot (10 + 5.5) = 13.175$: so at least one of the two vertices will now be less protected than in the initial state. But ruler a finds it profitable to attack b and in the event of a victory goes on to attack d, if the vertex b holds 8.5 resources or less. Similarly, Ruler d finds it profitable to attack c, and in the event of a victory attack a, if the vertex c holds 4.675 resources or less. Neither of these scenarios is profitable for ruler b and, anticipating them, he chooses to stay peaceful in the initial state. Thus the fear of being overstretched discourages a ruler from attacks, and sustains a more peaceful world. □

The desire to avoid being overstretched can however also create greater pressure to eliminate potential opponents and this can in turn lead to greater conflict. To see this, consider the following variation of the above example.

Example 2. Imperial overreach: more wars and larger empire

Suppose the cost of conflict is slightly lower, $\delta = 0.14$. It can be verified that in the basic model, the original equilibrium is sustainable: ruler b attacks c and there is peace after that.

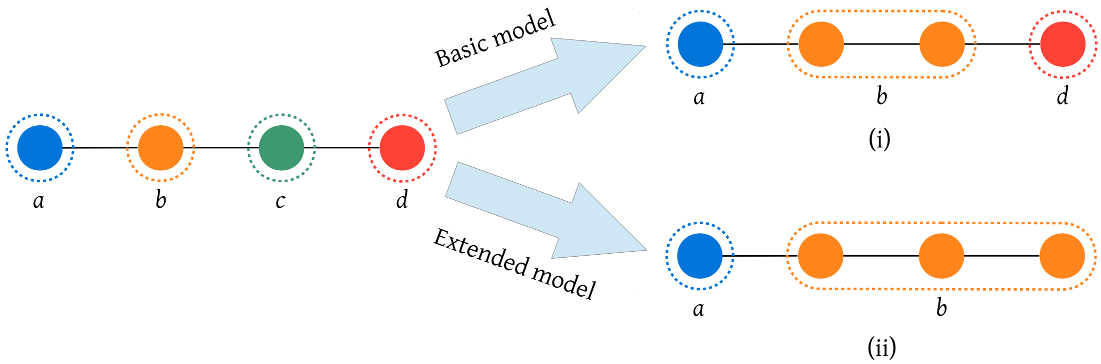

Figure 8: More Wars and Larger Empire

Now let us examine the dynamics under the new order of moves. For Ruler b it is optimal to attack c and then d, in a sequence. In case of b winning, this leads to a state where his empire spans b, c and d. The ruler then moves all his resources to the vertex bordering the territory of a. This outcome is represented in in Figure 8.[14] Thus Ruler b chooses to attack d to reduce the length of his border; in other words, the pressure to avoid overstretching leads to greater warfare and a larger empire.

□

5 Concluding Remarks

This paper develops a framework for the study of the incentives to wage war with a view to conquer territory and resources. The theoretical innovation is a model of interconnected conflict. This offers us a unified framework within which we can reconcile competing theories in international relations (offensive versus defensive realism), and provide a theoretical account for key concepts such as hegemony, preemptive war, buffer state, and the overstretched empire.

[14]The cost $\delta = 0.14$ is chosen so that it is indeed the best choice for b to attack c and then d, in a sequence (i.e. the sequence c, d yields higher payoff than the sequence c). Then after b successfully beats c and d, at the beginning of every subsequent round, he allocates all his resources at vertex b, which makes the attack by a unprofitable (in fact, to make the attack by a unprofitable it is enough that b moves ≈ 14.7 resources to vertex b). The attack by b on a is unprofitable as well: observe that after the two fights his total resources are 19.2 and the expected payoff from conflict with a is ≈ 18.8). So both rulers choose peace.

In our model, we assumed that links were undirected: in practice, kingdoms may well have an asymmetric relationship with one being more or less vulnerable due to the physical layout. This asymmetry can be accommodated by considering directed links and by enriching the technology to be asymmetric between defence and attack. Our analysis highlights the role of technology in shaping conflict: but we have assumed that all rulers have access to the same technology. In future work, it would be interesting to study the incentives to improve their technology. We model kingdoms as unitary actors and find that hegemony arises in many settings. History presents us examples of hegemony that is destroyed from within; it would be important to understand when a kingdom's focus should shift from the threat posed by other nations to the threat from within.[15]

[15] See Hoffman [2015] for a recent study of the key role of technology in European imperial history; for an early discussion on the nature of dual threats to rulers, see Machiavelli [1992].

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Appendix A: Proofs

Proof of Theorem 1. For point 1, first consider the case where $\gamma > 1$: in this case the function $h(x) = x^\gamma$ is strictly convex and $h(0) = 0$. By strict convexity of h, for any $y \in \mathbb{R}_{++}$, $h(y) - h(0) < h(x+y) - h(x)$ and, since $h(0) = 0$, so $h(x+y) > h(x) + h(y)$. Thus for any $y, z \in \mathbb{R}_{++}$,

$$(y+z)^\gamma > y^\gamma + z^\gamma. \tag{17}$$

Take any $x, y, z \in \mathbb{R}_{++}$. Multiplying both sides of (17) by $(x+y)^\gamma$ we get

$$(x+y)^\gamma (y+z)^\gamma > (x+y)^\gamma (y^\gamma + z^\gamma). \tag{18}$$

Since, by (17), $(x+y)^\gamma > x^\gamma + y^\gamma$ so

$$(x+y)^\gamma (y+z)^\gamma > (x+y)^\gamma y^\gamma + (x^\gamma + y^\gamma) z^\gamma. \tag{19}$$

Adding $x^\gamma (x+y)^\gamma$ to both sides we get

$$(x+y)^\gamma \left(x^\gamma + (y+z)^\gamma\right) > \left(x^\gamma + y^\gamma\right)\left((x+y)^\gamma + z^\gamma\right) \tag{20}$$

which can be rewritten as

$$\left(\frac{1}{x^\gamma + y^\gamma}\right)\left(\frac{(x+y)^\gamma}{(x+y)^\gamma + z^\gamma}\right) > \frac{1}{x^\gamma + (y+z)^\gamma}. \tag{21}$$

Multiplying both sides by x^γ we get

$$\left(\frac{x^\gamma}{x^\gamma + y^\gamma}\right)\left(\frac{(x+y)^\gamma}{(x+y)^\gamma + z^\gamma}\right) > \frac{x^\gamma}{x^\gamma + (y+z)^\gamma}. \tag{22}$$

This completes the proof for $\gamma > 1$.

Next consider $\gamma < 1$: so the function $h(x) = x^\gamma$ is strictly concave and $h(0) = 0$. By strict concavity of h, for any $y \in \mathbb{R}_{++}$, $h(y) - h(0) > h(x+y) - h(x)$ and, since $h(0) = 0$, so $h(x+y) < h(x) + h(y)$. Thus for any $y, z \in \mathbb{R}_{++}$,

$$(y+z)^\gamma < y^\gamma + z^\gamma. \tag{23}$$

The remaining part of the proof is analogous the case for $\gamma > 1$, and omitted.

For point 2, take any $x, y, z \in \mathbb{R}_{++}$ such that $y < z$. Consider the case of $\gamma > 1$ first. After some rearrangement, the inequality to prove for this case can be rewritten as

$$z^\gamma (x+z)^\gamma \left((x+y)^\gamma - (x^\gamma + y^\gamma)\right)) > y^\gamma (x+y)^\gamma \left((x+z)^\gamma - (x^\gamma + z^\gamma)\right). \tag{24}$$

Dividing both sides by $z^\gamma (x+z)^\gamma y^\gamma (x+y)^\gamma$ we get

$$\frac{(x+y)^\gamma - (x^\gamma + y^\gamma)}{y^\gamma (x+y)^\gamma} > \frac{(x+z)^\gamma - (x^\gamma + z^\gamma)}{z^\gamma (x+z)^\gamma}. \tag{25}$$

Define

$$\varphi(x,y) = \frac{(x+y)^\gamma - (x^\gamma + y^\gamma)}{y^\gamma (x+y)^\gamma}. \tag{26}$$

We will show that for any $x, y > 0$, $\varphi(x,y)$ is decreasing in y. Taking derivative wrt y, we get

$$\begin{aligned} \frac{\partial \varphi(x,y)}{\partial y} &= \frac{\gamma x^\gamma (x+2y)}{(y^{\gamma+1}(x+y)^{\gamma+1}} + \frac{\gamma}{(x+y)^{\gamma+1}} - \frac{\gamma}{y^{\gamma+1}} \\ &= \frac{2x^\gamma y - \left((x+y)^{\gamma+1} - (x^{\gamma+1} + y^{\gamma+1})\right)}{y^{\gamma+1}(x+y)^{\gamma+1}}. \end{aligned} \tag{27}$$

The numerator of (27) can be rewritten as

$$2xy\left(x^{\gamma-1} - (x+y)^{\gamma-1}\right) + x^2\left(x^{\gamma-1} - (x+y)^{\gamma-1}\right) + y^2\left(y^{\gamma-1} - (x+y)^{\gamma-1}\right), \tag{28}$$

which is negative when $\gamma > 1$. Hence, for $x, y \in \mathbb{R}_{++}$,

$$\frac{\partial \varphi(x,y)}{\partial y} < 0 \tag{29}$$

It follows that for $0 < y < z$,

$$\varphi(x,y) > \varphi(x,z) = \frac{(x+z)^\gamma - (x^\gamma + z^\gamma)}{z^\gamma (x+z)^\gamma}. \tag{30}$$

This shows that $\gamma > 1$ implies the result.

Suppose now that $\gamma < 1$. Analogously to the case of $\gamma > 1$, we reduce the problem to showing that

$$\frac{\partial \varphi(x,y)}{\partial y} > 0, \tag{31}$$

for $x, y \in \mathbb{R}_{++}$. This follows from the fact that (28) is positive in the case of $\gamma < 1$. $\square$

Part 1 of Theorem 1 implies that in the case of $\gamma > 1$ attacking opponents in a sequence gives higher chance of winning than waiting for them to merge and attack them afterwards. This is stated in the corollary below.

Corollary 1. *Let $m \geq 3$, $x_1, \ldots, x_m \in \mathbb{R}_{++}$, and $1 \leq i < j \leq m$ such that $i \neq 1$ or $j \neq m$. Then*

$$p_{\text{seq}}(x_1, \ldots, x_{i-1}, x_i, \ldots, x_j, x_{j+1}, \ldots, x_m) > p_{\text{seq}}\left(x_1, \ldots, x_{i-1}, \sum_{l=i}^{j} x_l, x_{j+1}, \ldots, x_m\right) \tag{32}$$

Proof. We first show that, for any $m \geq 3$ and $x_1, \ldots, x_m \in \mathbb{R}_{++}$,

$$p_{\text{seq}}(x_1, x_2, \ldots, x_m) > p_{\text{seq}}\left(x_1, \sum_{l=2}^{m} x_l\right). \tag{33}$$

The proof is by induction on m. The induction basis, $m = 3$, holds by point 1 of Theorem 1. Now suppose that $m > 3$ and that the claim holds for all $3 \leq m' < m$. By the induction hypothesis and point 1 of Theorem 1,

$$\begin{aligned} p_{\text{seq}}(x_1, x_2, \ldots, x_m) &= p(x_1, x_2) p_{\text{seq}}(x_1 + x_2, x_3, \ldots, x_m) > p(x_1, x_2) p_{\text{seq}}\left(x_1 + x_2, \sum_{l=3}^{m} x_l\right) \\ &= p_{\text{seq}}\left(x_1, x_2, \sum_{l=3}^{m} x_l\right) > p_{\text{seq}}\left(x_1, \sum_{l=2}^{m} x_l\right). \end{aligned} \tag{34}$$

Here the first inequality follows from the induction step and the second inequality follows from Theorem 1. We now proceed with the main inequality in the claim:

$$\begin{aligned} & p_{\text{seq}}(x_1, \ldots, x_{i-1}, x_i, \ldots, x_j, x_{j+1}, \ldots, x_m) \\ & = p_{\text{seq}}(x_1, \ldots, x_{i-1}) p_{\text{seq}}\left(\sum_{l=1}^{i-1} x_l, x_i, \ldots, x_j\right) p_{\text{seq}}\left(\sum_{l=1}^{j} x_l, x_{j+1}, \ldots, x_m\right) \\ & > p_{\text{seq}}(x_1, \ldots, x_{i-1}) p_{\text{seq}}\left(\sum_{l=1}^{i-1} x_l, \sum_{l=i}^{j} x_l\right) p_{\text{seq}}\left(\sum_{l=1}^{j} x_l, x_{j+1}, \ldots, x_m\right) \\ & = p_{\text{seq}}(x_1, \ldots, x_{i-1}, x_i + \ldots + x_j, x_{j+1}, \ldots, x_m). \end{aligned} \tag{35}$$

The inequality follows from the intermediate argument above. $\square$

Proof of Theorem 2. The proof proceeds in three steps.

Step 1: Fix some state ϕ with $|\mathrm{Act}(\phi)| \geq 2$. For a strong ruler i, the optimal full attacking sequence maximizes his payoffs across all attack sequences. Moreover, in generic case, it is a unique maximizer.

Let ϕ be a state with $|\mathrm{Act}(\phi)| = m \geq 2$. Take an active ruler $j_0 \in \mathrm{Act}(\phi)$ with maximal amount of resources $R_{j_0}(\phi)$. For generic resource values, such a ruler is unique. Pick a full attacking sequence $j_1, \ldots, j_{m-1}$ consisting of rulers in $\mathrm{Act}(\phi) \setminus \{j_0\}$ that is feasible for j_0 in G under ϕ (clearly such a sequence exists because G is connected). Since j_0 has maximal amount of resources so, for all $1 \leq k \leq m-1$, we have

$$\sum_{l=0}^{k-1} R_{j_l}(\phi) \geq R_{j_k}(\phi). \tag{36}$$

The expected payoff to ruler j_0 from the attacking sequence is

$$\begin{aligned}\pi_{j_0}(\phi \mid j_1, \ldots, j_{m-1}) &= \left(\sum_{l=0}^{m-1} R_{j_l}(\phi)\right) \prod_{k=1}^{m-1} p\left(\sum_{l=0}^{k-1} R_{j_l}(\phi), R_{j_k}(\phi)\right) \\ &= R_{j_0}(\phi) \prod_{k=1}^{m-1} p\left(\sum_{l=0}^{k-1} R_{j_l}(\phi), R_{j_k}(\phi)\right)\left(\frac{\sum_{l=0}^{k} R_{j_l}(\phi)}{\sum_{l=0}^{k-1} R_{j_l}(\phi)}\right).\end{aligned} \tag{37}$$

Since p is rich rewarding, so

$$p\left(\sum_{l=0}^{k-1} R_{j_l}(\phi), R_{j_k}(\phi)\right)\left(\frac{\sum_{l=0}^{k} R_{j_l}(\phi)}{\sum_{l=0}^{k-1} R_{j_l}(\phi)}\right) \geq 1, \tag{38}$$

with equality only if $k = 1$ and $R_{j_0}(\phi) = R_{j_1}(\phi)$.

At every step in the sequence, the expected resources are growing. So, for generic resource values, there is a full attacking sequence that dominates any partial attacking sequence. By definition, the optimal full attacking sequence maximizes payoffs across all attack sequences.

The first step has a powerful implication: in any state with 2 or more active rulers there is at least one ruler who has a strict incentive to attack, given that other rulers do not attack. Hence, in equilibrium, there must exist a hegemon.

In the dynamic game, in principle, a strong ruler may prefer to wait and allow others to move and then attack later. The next step shows that an optimal full attacking sequence dominates all such waiting strategies.

Step 2: Fix some state $\mathbb{O}$ with $|\mathrm{Act}(\mathbb{O})| \geq 2$. For any strong ruler at $\mathbb{O}$, an optimal full attacking sequence is a dominant choice. Moreover, the choice is strictly dominant if $|\mathrm{Act}(\mathbb{O})| \geq 3$.

Fix some state $\mathbb{O}$. Let $\sigma_i(\mathbb{O})$ be the optimal myopic choice of ruler i in state $\mathbb{O}$. Let $\bar{\pi}_i(\mathbb{O}) = \pi_i(\mathbb{O} \mid \sigma_i(\mathbb{O}))$ denote the optimal myopic payoff ruler i can attain at $\mathbb{O}$.

Claim. *The optimal myopic payoff is the highest that ruler i can hope to attain, i.e., $\bar{\pi}_i(\mathbb{O}) \geq \Pi_i(\boldsymbol{s} \mid \mathbb{O})$ for any feasible strategy profile $\boldsymbol{s}$, starting at state $\mathbb{O}$. Moreover, if i is strong and there are at least three active rulers, then the inequality is strict.*

The proof is by induction on the number of active rulers. For the induction basis, we show that the claim holds for 2 active rulers. If i is the richer ruler then, from the rich rewarding property, his myopic optimal strategy is to attack. It is also clear that attacking yields strictly higher payoffs if other ruler does not attack, and weakly higher payoffs if the other ruler does attack. If i is the poorer ruler, then not attacking is the optimal myopic strategy. In case the richer ruler attacks, the expected payoff to i is less due to the rich rewarding property. That completes the argument for 2 active rulers.

For the induction step, suppose that the claim holds for all $y \leq X$, where $X \geq 2$, active rulers: we will show that it also holds for $X+1$ active rulers. Fix some state $\mathbb{O}$ with $X+1$ active rulers. Take an active ruler i and any strategy profile $\boldsymbol{s}$. If there are no attacking rulers under $\boldsymbol{s}$ then the claim follows, because $\sigma_i(\mathbb{O})$ is at least as good as the empty sequence, ε,[16] at $\mathbb{O}$:

$$\bar{\pi}_i(\mathbb{O}) \geq \pi_i\left(\mathbb{O} \mid \sigma_i(\mathbb{O})\right) \geq \pi_i\left(\mathbb{O} \mid \varepsilon\right) = \Pi_i(\boldsymbol{s} \mid \mathbb{O}). \tag{39}$$

Moreover, by Step 1, the inequality is strict if i is strong.

For the remaining part of the argument assume that there is at least one attacking ruler under $\boldsymbol{s}$. We will establish that $\bar{\pi}_i(\mathbb{O}) \geq \Pi_i(\boldsymbol{s} \mid \mathbb{O})$. Given an attacking ruler j_0, let $\Pi_i(\boldsymbol{s} \mid \mathbb{O}, j_0)$ denote the expected payoff to ruler i from strategy profile $\boldsymbol{s}$ conditional on ruler j_0 being selected to move. Then

$$\Pi_i(\boldsymbol{s} \mid \mathbb{O}) = \left(\frac{1}{|\mathrm{Atck}(\boldsymbol{s}, \mathbb{O})|}\right) \sum_{j_0 \in \mathrm{Atck}(\boldsymbol{s}, \mathbb{O})} \Pi_i\left(\boldsymbol{s} \mid \mathbb{O}, j_0\right). \tag{40}$$

[16] Throughout the proofs we use the standard notation, ε, to denote empty sequences.

Thus to show the claim, it is enough to show that

$$\bar{\pi}_i(\mathbb{o}) \geq \Pi_i\left(\boldsymbol{s} \mid \mathbb{o}, j_0\right), \tag{41}$$

for each attacking ruler $j_0 \in \text{Atck}(\boldsymbol{s}, \mathbb{o})$, with strict inequality for at least one $j_0 \in \text{Atck}(\boldsymbol{s}, \mathbb{o})$ in the case of i being strong.

So take any ruler j_0 attacking at $\mathbb{o}$ under $\boldsymbol{s}$. Three cases are possible:

(i). $j_0 \neq i$ and i is not in the attacking sequence $s_{j_0}(\mathbb{o})$ of j_0,

(ii). $j_0 \neq i$ and i is in the attacking sequence $s_{j_0}(\mathbb{o})$ of j_0,

(iii). $j_0 = i$.

Case (i). Ruler j_0 is different to i and does not have i in his attacking sequence $s_{j_0}(\mathbb{o})$. Let $F(\mathbb{o}' \mid \boldsymbol{s}, \mathbb{o}, j_0)$ be the probability of reaching ownership state $\mathbb{o}'$ from state $(\mathbb{o}, t)$ under strategy profile $\boldsymbol{s}$ after j_0 is selected to move and chooses attacking sequence $s_i(\mathbb{o})$. Then

$$\Pi_i\left(\boldsymbol{s} \mid (\mathbb{o}, t), j_0\right) = \sum_{\mathbb{o}' \in \mathbb{O}} F\left(\mathbb{o}' \mid \boldsymbol{s}, (\mathbb{o}, t), j_0\right) \Pi_i(\boldsymbol{s} \mid \mathbb{o}'). \tag{42}$$

To show (41) it is enough to show that

$$\bar{\pi}_i(\mathbb{o}) \geq \Pi_i(\boldsymbol{s} \mid \mathbb{o}') \tag{43}$$

for each state $\mathbb{o}'$ that can be reached with positive probability from $\mathbb{o}$ when j_0 plays the attacking sequence $s_{j_0}(\mathbb{o})$. We will show that the inequality is strict when i is strong.

Ownership state $\mathbb{o}'$ is reached after at least one fight and so has at most X active rulers. Hence, by the induction hypothesis, $\bar{\pi}_i(\mathbb{o}') \geq \Pi_i(\boldsymbol{s} \mid \mathbb{o}')$, and so to show (43) it is enough to show that

$$\bar{\pi}_i(\mathbb{o}) \geq \bar{\pi}_i(\mathbb{o}'). \tag{44}$$

Take an optimal myopic sequence, $\sigma_i(\mathbb{o}')$, of i at $\mathbb{o}'$. There are two sub-cases to be considered.

(a) Sequence $\sigma_i(\mathbb{o}')$ does not contain the rulers in the sequence of fights that leads to $\mathbb{o}'$. This means, in particular, that $\sigma_i(\mathbb{o}')$ is not a full attacking sequence. Hence, by Step 1, i is not strong.

Since $\sigma_i(\mathbb{o}')$ does not contain the rulers in the sequence of fights that leads to $\mathbb{o}'$, it can be executed at state $(\mathbb{o})$. By optimality of $\sigma_i(\mathbb{o})$ at $(\mathbb{o})$

$$\bar{\pi}_i(\mathbb{o}) = \pi_i\left((\mathbb{o} \mid \sigma_i(\mathbb{o})\right) \geq \pi_i\left(\mathbb{o} \mid \sigma_i(\mathbb{o}')\right) = \pi_i\left(\mathbb{o}' \mid \sigma_i(\mathbb{o}')\right) = \bar{\pi}_i(\mathbb{o}'). \tag{45}$$

(b) Sequence $\sigma_i(\mathbb{o}')$ contains at least one ruler in the sequence of fights that leads to $\mathbb{o}'$. This is true, in particular, when i is strong because, by Step 1, $\sigma_i(\mathbb{o}')$ must be a full attacking sequence then.

Since $\sigma_i(\mathbb{o}')$ contains at least one ruler in the sequence of fights that leads to $\mathbb{o}'$, so $\sigma_i(\mathbb{o}') = \sigma_i^1(\mathbb{o}'), k, \sigma_i^2(\mathbb{o}')$, where k is the ruler who won the sequence of fights leading to $\mathbb{o}'$. We can construct a sequence $\sigma' = \sigma_i^1 \tau \sigma_i^2$ that is feasible for i at $\mathbb{o}$, with τ being a sequence of rulers involved in the sequence of fights leading to $\mathbb{o}'$. By point 1 of Theorem 1 and Corollary 1, σ' yields a strictly higher payoff than $\sigma_i(\mathbb{o}')$. By construction, $\sigma_i(\mathbb{o})$ is an optimal myopic strategy for i at $\mathbb{o}$ and so payoff dominates σ' at $\mathbb{o}$. Hence

$$\bar{\pi}_i(\mathbb{o}) = \pi_i\left(\mathbb{o} \mid \sigma_i(\mathbb{o})\right) \geq \pi_i(\mathbb{o} \mid \sigma') > \pi_i\left(\mathbb{o}' \mid \sigma_i(\mathbb{o}')\right) = \bar{\pi}_i(\mathbb{o}'). \tag{46}$$

Hence (44) and, consequently, (43) hold with strict inequality.

Case (ii). Ruler j_0 is different to i and has i in his attacking sequence $s_{j_0}(\mathbb{o})$. Let $s_{j_0}(\mathbb{o}) = j_1, \ldots, j_m$ be the sequence selected by j_0 at $\mathbb{o}$ under strategy s_{j_0}. Then $i = j_k$ for some $1 \leq k \leq m$. Given $l \in \{1, \ldots, m\}$, let $\mathbb{o}^l$ be the state reached after j_0 looses the l'th fight in the sequence. The expected payoff to i from $\boldsymbol{s}$ at $\mathbb{o}$ given that j_0 is selected to move is equal to

$$\begin{aligned}\Pi_i\left(\boldsymbol{s} \mid \mathbb{o}, j_0\right) = &\sum_{l=1}^{k-1} F\left(\mathbb{o}^l \mid \boldsymbol{s}, \mathbb{o}, j_0\right) \Pi_i\left(\boldsymbol{s} \mid \mathbb{o}^l\right) + \\ &\left(1 - \sum_{l=1}^{k-1} F\left(\mathbb{o}^l \mid \boldsymbol{s}, \mathbb{o}, j_0\right)\right) p\left(r_i(\mathbb{o}), \sum_{l=0}^{k-1} r_{j_l}(\mathbb{o})\right) \Pi_i\left(\boldsymbol{s} \mid \mathbb{o}^k\right),\end{aligned} \tag{47}$$

where $j_1, \ldots, j_{k-1}$ are the rulers attacked by j_0 prior to attacking i.

Hence to show (41) it is enough to show that (43) holds for all $\mathbb{o}' = \mathbb{o}^l$, $l \in \{1, \ldots, k-1\}$, reachable after a sequence of fights of j_0 in which j_0 looses before facing i, and that

$$\bar{\pi}_i(\mathbb{o}) \geq p\left(r_i(\mathbb{o}), \sum_{l=0}^{k-1} r_{j_l}(\mathbb{o})\right) \Pi_i\left(\boldsymbol{s} \mid \mathbb{o}^k\right). \tag{48}$$

holds for ϕ^k, reachable by a sequence of fights of j_0 in which i is attacked by j_0 and wins. (43) is shown by the same arguments as in point (ii) above. In particular, the inequality in (43) is strict when i is strong. For (48), let τ be a sequence of rulers $\{j_0, \ldots, j_{k-1}\}$ feasible to i at ϕ (clearly such a sequence exists). Then sequence $\sigma' = \tau\sigma_i(\phi^k)$, consisting of τ and an optimal myopic sequence of i at ϕ^k, is feasible for i at ϕ. By point 1 of Theorem 1 and Corollary 1, τ yields at least the same payoff to i as the sequence of fights that leads to ϕ' (the inequality is strict, unless $k = 1$). Combining this with the induction hypothesis we get

$$\begin{aligned}\bar{\pi}_i(\phi) &\geq \pi_i(\phi \mid \tau\sigma_i(\phi^k)) \geq p\left(R_i(\phi), \sum_{l=0}^{k-1} R_{j_l}(\phi)\right) \pi_i\left(\phi^k \mid \sigma_i\left(\phi^k\right)\right) \\ &\geq p\left(R_i(\phi), \sum_{l=0}^{k-1} R_{j_l}(\phi)\right) \Pi_i\left(\boldsymbol{s} \mid \phi^k\right), \end{aligned} \tag{49}$$

with strict inequality, unless $k = 1$.

Case (iii). Ruler i is picked to move at ϕ. The strategy chosen by i under strategy profile $\boldsymbol{s}$ at state ϕ is $s_i(\phi)$. Let ϕ' be the state that is reached if i wins all the attacks in sequence $s_i(\phi)$. Then sequence $\sigma' = s_i(\phi)\sigma_i(\phi')$, consisting of the $s_i(\phi)$ and an optimal myopic sequence of i at ϕ', is feasible for i at ϕ. State ϕ' is reached after at least one fight and has at most X active rulers. Hence, by the induction hypothesis, $\bar{\pi}_i(\phi') \geq \Pi_i(\boldsymbol{s} \mid \phi')$ and it follows that

$$\bar{\pi}_i(\phi) \geq \pi_i\left(\phi \mid s_i(\phi)\sigma_i(\phi')\right) \geq \pi_i\left(\phi' \mid \sigma_i(\phi')\right) = \bar{\pi}_i(\phi') \geq \Pi_i(\boldsymbol{s} \mid \phi'). \tag{50}$$

The inequality is strict unless the sequence $s_i(\phi)\sigma_i(\phi')$ is the same as the optimal myopic sequence of i at ϕ.

To complete the proof of the claim, we argue that $\bar{\pi}_i(\phi) > \Pi_i(\boldsymbol{s} \mid \phi)$ if i is strong and there are at least 3 active rulers at ϕ. As we established above, if i is strong then (41) holds with equality in two cases only: $j_0 = i$ and $s_i(\phi)$ is the optimal myopic sequence of i at ϕ, or $j_0 = j \neq i$, j_0 attacks i first under $s_{j_0}(\phi)$ and j_0 is the first ruler to be attacked by i under his optimal myopic sequence of attacks. Generically the second case is possible for at most one ruler other then i. Hence with at least three active rulers there is at least one for which the inequality in (41) is strict. This completes the proof of the claim.

From Step 1, we know that in any state ϕ, there exists a strong ruler for whom the full attacking sequence is the optimal stand alone strategy. It now follows from the claim above that for this strong ruler the optimal full attacking sequence dominates all other strategies, and the domination is strict if there are at least three active rulers at $\mathbb{o}$. The final step in the proof takes up non-strong rulers. We show that faced with rulers such that at every state at least one of them attacks, every ruler will find it profitable to choose an optimal full attacking sequence.

Step 3: *For every ruler i and every strategy profile of other rulers, $\boldsymbol{s}_{-i}$, such that at every state with at least two active rulers there exists an attacking ruler in $N \setminus \{i\}$, choosing an optimal full attacking sequence at every state where i is active maximises payoffs. Moreover, the maximisation is strict if there are at least three active rulers at the state.*

Let $\boldsymbol{s}_{-i}$ be a strategy profile such that at every state with at least two active rulers there exists an attacking ruler in $N \setminus \{i\}$. Let s_i be a strategy such that at every state $\mathbb{o}$ where ruler i is active, $s_i(\mathbb{o})$ is an optimal full attacking sequence for i. We show that for any other strategy, s_i', of ruler i and every state $\mathbb{o} \in \mathbb{O}$,

$$\Pi_i\left((s_i, \boldsymbol{s}_{-i}) \mid \mathbb{o}\right) \geq \Pi_i((s_i', \boldsymbol{s}_{-i}) \mid \mathbb{o})), \tag{51}$$

with strict inequality if there are at least three active rulers at $\mathbb{o}$.

The argument is by induction on the number of active rulers, and proceeds along the similar lines as Step 2. With 2 active rulers, if i is active and the ruler other than i chooses attack, then i is indifferent between attacking or restraining himself. So the claim holds for $X = 2$ active rulers.

Suppose next that the claim holds for all $y \leq X$, for some $X \geq 2$. By assumption, there exists an attacking ruler j_0 other than i and sequence of attacks, $s_{j_0}(\mathbb{o})$, is chosen by j_0 at $\mathbb{o}$. By Equation (40), to show the claim, it is enough to show that

$$\Pi_i\left((s_i, \boldsymbol{s}_{-i}) \mid \mathbb{o}\right) \geq \Pi_i\left((s_i', \boldsymbol{s}) \mid \mathbb{o}, j_0\right), \tag{52}$$

with strict inequality for at least one $j_0 \in \operatorname{Act}(\mathbb{o})$. Analogously to Step 2, two cases are considered separately: (i) $s_{j_0}(\mathbb{o})$ contains i, and (ii) $s_{j_0}(\mathbb{o})$ does not contain i.

In case (i), by Equation (42), to show (51) it is enough to show that

$$\Pi_i((s_i, \boldsymbol{s}_{-i}) \mid \mathbb{o}) > \Pi_i((s_i', \boldsymbol{s}_{-i}) \mid \mathbb{o}') \tag{53}$$

for each state $\mathbb{o}'$ that can be reached with positive probability from $\mathbb{o}$ when j_0 plays the attacking sequence $s_{j_0}(\mathbb{o})$. State $\mathbb{o}'$ is reached after at least one fight and so has at most X active rulers. Hence, by the induction hypothesis, $\Pi_i((s_i, \boldsymbol{s}) \mid \mathbb{o}') \geq \Pi_i((s_i', \boldsymbol{s}) \mid \mathbb{o}')$, and so to show (53) it is enough to show that

$$\Pi_i((s_i, \boldsymbol{s}) \mid \mathbb{o}) > \Pi_i((s_i, \boldsymbol{s}) \mid \mathbb{o}'). \tag{54}$$

This is shown by constructions analogous to those used in case (i) of Step 2 (in particular, the argument uses the fact that $s_i(\mathbb{o})$ is a full attacking sequence and an argument analogous to case (b) applies).

In case (ii), by Equation (47), to show (52), it is enough to show that (53) holds for all states $\mathbb{o}'$ reachable after a sequence of fights of j_0 in which j_0 looses before facing i, and that

$$\Pi_i((s_i, \boldsymbol{s}) \mid \mathbb{o}) \geq p\left(R_i(\mathbb{o}), \sum_{l=0}^{k-1} R_{j_l}(\mathbb{o})\right) \Pi_i\left(\boldsymbol{s} \mid \mathbb{o}'\right). \tag{55}$$

holds for $\mathbb{o}'$, reachable by a sequence of fights $j_1, \ldots, j_k$ of j_0 in which $i = j_k$ is attacked by j_0 and wins. This is shown by constructions analogous to those used in case (ii) of Step 2. In particular, if there are at least three active rulers, then the inequality is strict for at least one ruler other than i. This completes the proof of Step 3.

To show that a strategy profile where every active ruler chooses an optimal full attacking sequence at every state is an equilibrium, take such a strategy profile $\boldsymbol{s}$. By Step 2, choosing an optimal full attacking sequence at every state is a dominant strategy for strong rulers, and it is strictly dominant for all states with at least three active rulers. By Step 1, a strong ruler exists at every state and so there is an attacking ruler at every state. Consequently, by Step 3, it is a best reply for all other rulers to choose an optimal full attacking sequence at every state. The reply is strictly best if there are at least three active rulers. Notice that an equilibrium is uniquely determined for all states with at least three active rulers. In the states with two rulers, in every equilibrium there is a conflict between these rulers leading two an equilibrium. Therefore for any equilibrium and any ruler i in the initial state, the probability of i becoming a hegemon is the same. $\square$

Proof of Proposition 1. **Part 1:** Let V be a set of vertices. Take any resource vector $\boldsymbol{r}$ such that for some vertex $v \in V$,

$$r_v \geq 2^{\frac{|V|-1}{1-\gamma}} \sum_{u \in V \setminus \{v\}} r_u. \tag{56}$$

Take any connected network $G = (V, E)$ and any ownership state $\mathbb{o} \in \mathbb{O}$. If there is a ruler who owns all the vertices under $\mathbb{o}$ then we are done. Assume otherwise. There are at least two active rulers under ϕ, $|\mathrm{Act}(\phi)| \geq 2$. Let i be the ruler owning vertex v, $\phi(v) = i$, and let $j \in \mathrm{Act}(\phi)$ be any active neighbour of i under ϕ. Let σ be a permutation of $\mathrm{Act}(\phi)\setminus\{j\}$ starting with i. Sequence σ is a full attacking sequence of j at ϕ. We will show that $\Pi(j, \phi; \sigma) > R_j(\phi)$. Let $x = R_j(\phi)$ be the resources of j at ϕ and $y = R_i(\phi)$ be the resources of i at ϕ. By our assumptions,

$$y \geq 2^{\frac{|V|-1}{1-\gamma}} x. \tag{57}$$

After winning a conflict with i, in every subsequent conflict in the sequence j has higher resources than her opponent. Hence the probability of winning each these conflicts is more than $1/2$. In the event of winning all the conflicts in the sequence, ruler j owns at least $x + y$ resources. By these observations

$$\Pi(j, \phi; \sigma) \geq \left(\frac{1}{2^{m-1}}\right)\left(\frac{x^\gamma}{x^\gamma + y^\gamma}\right)(x + y), \tag{58}$$

where $m = |\mathrm{Act}(\phi)|$ is the number of active rulers at ϕ. Let $q = (|V| - 1)/(1 - \gamma)$. Since $|V| \geq m$ so $q \geq (m - 1)/(1 - \gamma)$ and reorganizing we get

$$q + 2 - m \geq \gamma q + 1. \tag{59}$$

Hence

$$2^{q+2-m} \geq 2^{\gamma q+1} > 2^{\gamma q} + 1 - \frac{1}{2^{m-2}}. \tag{60}$$

This can be rewritten as

$$\frac{1 + 2^q}{1 + 2^{\gamma q}} \geq 2^{m-2}. \tag{61}$$

Since $h(z) = (1 + z)/(1 + z^\gamma)$ is increasing on $\mathbb{R}_{++}$ for $\gamma \in [0, 1)$ so, by (57), the inequality above implies

$$\frac{1 + \frac{y}{x}}{1 + \left(\frac{y}{x}\right)^\gamma} \geq 2^{m-2} \tag{62}$$

which can be rewritten as

$$\left(\frac{1}{2^{m-1}}\right)\left(\frac{x^\gamma}{x^\gamma + y^\gamma}\right)(x + y) > x. \tag{63}$$

Thus

$$\Pi(j, \phi; \sigma) > x \tag{64}$$

and so choosing σ ruler j strictly increases her expected payoff.

Since at every ownership state $\obar$ with active rulers there exists a ruler who can increase his expected resources by choosing attack, so every equilibrium outcome is hegemony.

Proof of Parts 2 and 3: Let $v \in V$ be a vertex and let G be a star network with centre v. Take any $y > 0$. Let the resource vector $\boldsymbol{r}$ be such that $r_u = y$, for each spoke $u \in V \setminus \{v\}$, and $r_v = x$, for the centre. We will show that there exists (a range of values of) x such that there is an equilibrium where each ruler chooses peace in the initial ownership state, and and that there exists (a range of values of) x where each ruler at a spoke chooses a sequence of fights that leads to a ownership state with peace.

Let

$$\begin{aligned}\varphi(x, y, m) &= (x + my)p(y, x)\prod_{i=1}^{m-1} p(x + iy, y)\\ &= (x + my)\left(\frac{y^\gamma}{x^\gamma + y^\gamma}\right)\prod_{i=1}^{m-1}\left(\frac{(x + iy)^\gamma}{(x + iy)^\gamma + y^\gamma}\right) \qquad (65)\end{aligned}$$

be the expected payoff from a full attacking sequence of m fights to a ruler owning a spoke in a star over at least $m + 1$ vertices, when each spoke is endowed with y resources and the centre is endowed with x resources. As we show in the Appendix, for all $m \geq 2$, $\gamma \in [0, 1)$, and $y > 0$, there exists a unique $x_m^* = x_m^*(y, \gamma) > y$, such that

$$\varphi(x, y, m)\begin{cases} < y & \text{if } x \in (y, x_m^*),\\ = y & \text{if } x = x_m^*,\\ > y & \text{if } x > x_m^*.\end{cases} \qquad (66)$$

Moreover, $x_{m+1}^*(y, \gamma) > x_m^*(y, \gamma)$.

Taking any $x \in (0, y)$, in the case of $n = 3$, and any $x \in (\max(y, x_{n-2}^* - y), x_{n-1}^*)$, in the case of $n \geq 4$, guarantees that no ruler has incentives to engage in a full attacking sequence (and the interval for the case of $n \geq 4$ is non-empty, as $x_{n-2}^* > y$). Moreover, after at least one fight, every ruler at a spoke has incentives to fight if no other ruler fights, as a full attacking sequence yields him expected payoff higher than y. Thus any ruler deviating from peaceful strategy profile leads to fight till hegemony, which is not profitable for the deviating ruler. Therefore there is an equilibrium where all rulers choose peace in the initial ownership state.

Taking any $x \in (\max(0, x_{n-3}^* - 2y), x_{n-2}^* - y)$, in the case of $n \geq 4$, guarantees that after one fight by a spoke a ownership state with resources at the centre as described above will be reached. Moreover, no ruler has incentives to engage in a full attacking sequence. Thus there is an equilibrium where (1) in the initial state each ruler owning a spoke chooses to attack the centre and the ruler owning the centre chooses peace, (2) in the state with $n-1$ vertices every vertex chooses peace, and (3) in any state with at most $n-2$ at least one vertex chooses attack. In this equilibrium there is one conflict followed by peace.

Notice that the two constructions used to show point 2 and 3 are generic. Analogous argument could be conducted if spokes were endowed with resource sufficiently close to each other and the centre was endowed with resources within a range close to the range given in the construction above.

Fix any $\gamma \in [0, 1)$. We show first that, for all $x, y \in \mathbb{R}_{++}$ and $m \geq 3$,

$$\varphi(x, y, m) < \varphi(x, y, m-1). \tag{67}$$

First, notice that,

$$y^{1-\gamma} \leq (x + (m-1)y)^{1-\gamma}. \tag{68}$$

Multiplying both sides by $y^{\gamma}(x + (m-1)y)^{\gamma}$ we get

$$y(x + (m-1)y)^{\gamma} < y^{\gamma}(x + (m-1)y). \tag{69}$$

Reorganizing, we obtain

$$(x + my)(x + (m-1)y)^{\gamma} < ((x + (m-1)y)^{\gamma} + y^{\gamma})(x + (m-1)y). \tag{70}$$

Dividing both sides by the RHS we get

$$\left(\frac{x + my}{x + (m-1)y}\right)\left(\frac{(x + (m-1)y)^{\gamma}}{(x + (m-1)y)^{\gamma} + y^{\gamma}}\right) < 1. \tag{71}$$

This, together with the fact that

$$\varphi(x, y, m) = \varphi(x, y, m-1)\left(\frac{(x + (m-1)y)^{\gamma}}{(x + (m-1)y)^{\gamma} + y^{\gamma}}\right)\left(\frac{(x + my)}{(x + (m-1)y)}\right) \tag{72}$$

yields (67).

Second, we show that φ is strictly increasing in x for $x > y$. First derivative of φ with respect to x is

$$\frac{\partial\varphi}{\partial x} = \left(\frac{\gamma y^{\gamma}}{x^{\gamma}+y^{\gamma}}\right)(x+my)\prod_{i=1}^{m-1}\left(\frac{(x+iy)^{\gamma}}{(x+iy)^{\gamma}+y^{\gamma}}\right)$$
$$\left(\left(\frac{1}{\gamma(x+my)}\right)-\left(\frac{x^{\gamma-1}}{x^{\gamma}+y^{\gamma}}\right)+\sum_{j=1}^{m-1}\frac{y^{\gamma}}{(x+jy)((x+jy)^{\gamma}+y^{\gamma})}\right). \tag{73}$$

Since $\gamma \in [0,1)$ so

$$(1-\gamma)(x+y) > 0. \tag{74}$$

Reorganizing we get

$$x+y+(m-1)\gamma y > \gamma(x+my). \tag{75}$$

Dividing both sides by $\gamma(x+y)(x+my)$ we get

$$\frac{1}{\gamma(x+my)}+\frac{(m-1)y}{(x+y)(x+my)} > \frac{1}{x+y} \tag{76}$$

Since $x > y$ and $\gamma \in [0,1)$ so $(x/y)^{1-\gamma} > 1$ and so

$$\frac{1}{x+y} > \frac{1}{x+y\left(\frac{x}{y}\right)^{1-\gamma}} = \frac{x^{\gamma-1}}{x^{\gamma}+y^{\gamma}}. \tag{77}$$

Hence

$$\frac{1}{\gamma(x+my)}+\frac{(m-1)y}{(x+y)(x+my)} > \frac{x^{\gamma-1}}{x^{\gamma}+y^{\gamma}}. \tag{78}$$

Notice that

$$\begin{aligned}\frac{(m-1)y}{(x+y)(x+my)} &= \left(\frac{1}{x+y}\right)-\left(\frac{1}{x+my}\right)\\ &= \sum_{i=1}^{m-1}\left(\frac{1}{x+iy}\right)-\sum_{i=2}^{m}\left(\frac{1}{x+iy}\right)\\ &= \sum_{i=1}^{m-1}\left(\left(\frac{1}{x+iy}\right)-\left(\frac{1}{x+(i+1)y}\right)\right)\\ &= \sum_{i=1}^{m-1}\left(\frac{y}{(x+iy)((x+iy)+y)}\right)\end{aligned} \tag{79}$$

Moreover, for $\gamma \in [0,1)$, $x > y$, and $i \geq 1$,

$$\begin{aligned}\frac{y}{(x+iy)((x+iy)+y)} &= \frac{1}{(x+iy)\left(\left(\frac{x}{y}+i\right)+1\right)} \\ &< \frac{1}{(x+iy)\left(\left(\frac{x}{y}+i\right)^{\gamma}+1\right)} \\ &= \frac{y^{\gamma}}{(x+iy)((x+iy)^{\gamma}+y^{\gamma})} \end{aligned} \tag{80}$$

Thus

$$\frac{(m-1)y}{(x+y)(x+my)} < \sum_{i=1}^{m-1}\left(\frac{y^{\gamma}}{(x+iy)((x+iy)^{\gamma}+y^{\gamma})}\right) \tag{81}$$

which, together with (78), implies

$$\frac{1}{\gamma(x+my)} + \sum_{i=1}^{m-1}\left(\frac{y^{\gamma}}{(x+iy)((x+iy)^{\gamma}+y^{\gamma})}\right) > \frac{x^{\gamma-1}}{x^{\gamma}+y^{\gamma}}. \tag{82}$$

Therefore, by that and (73), $\partial\varphi/\partial x > 0$ for all $x > y$ and so φ is increasing in x on $(y, +\infty)$.

Third, we show that for all y and m, satisfying our assumptions, $\lim_{x\to+\infty}\varphi(x,y,m) = +\infty$. To see that notice that

$$\lim_{x\to+\infty}\prod_{i=1}^{m-1} p(x+iy, y) = 1 \tag{83}$$

and

$$\lim_{x\to+\infty} p(y,x)(x+my) = \left(\frac{y^{\gamma}}{1+\left(\frac{y}{x}\right)^{\gamma}}\right)\left(x^{1-\gamma}+m\left(\frac{y}{x^{\gamma}}\right)\right) = +\infty, \tag{84}$$

and so the result follows.

Fourth, we show that

$$\varphi(y,y,m) < y. \tag{85}$$

To see that we start with

$$\varphi(y,y,m) = \left(\frac{1}{2}\right)(m+1)y\prod_{i=1}^{m-1}\left(\frac{(i+1)^{\gamma}}{(i+1)^{\gamma}+1}\right) = \left(\frac{1}{2}\right)(m+1)y\prod_{i=2}^{m}\left(\frac{i^{\gamma}}{i^{\gamma}+1}\right). \tag{86}$$

Since

$$\frac{i^{\gamma}}{i^{\gamma}+1} = 1 - \left(\frac{1}{i^{\gamma}+1}\right), \tag{87}$$

$\gamma \in [0, 1)$, $i \geq 1$, so $i^\gamma/(i^\gamma + 1)$ is increasing in γ. Hence

$$\varphi(y, y, n) < \left(\frac{1}{2}\right)(m+1)y\prod_{i=2}^{m}\left(\frac{i}{i+1}\right) = \left(\frac{1}{2}\right)y\left(\frac{n!}{n!}\right)2 = y. \tag{88}$$

By the four facts established above, for all $m \geq 2$, $\gamma \in [0, 1)$, and $y > 0$, there exists a unique $x_m^* = x_m^*(y, \gamma) > y$, such that

$$\varphi(x, y, m)\begin{cases} < 0 & \text{if } x \in (y, x_m^*), \\ = 0 & \text{if } x = x_m^*,, \\ > 0 & \text{if } x > x_m^*. \end{cases} \tag{89}$$

Moreover, by (67) and by the fact that φ is increasing in x for $x > y$, $x_{m+1}^*(y, \gamma) > x_m^*(y, \gamma)$. $\square$

Proof of Proposition 2. **Part 1:** A sequence $\sigma \in \mathbb{R}^*$ is *strong* if either $\sigma = \varepsilon$ or $\sigma = x_0, \ldots, x_m$ and for all $k \in \{1, \ldots, m\}$, $\sum_{j=0}^{k-1} x_j > x_k$. A sequence $\sigma \in \mathbb{R}^*$ is *weak* if it is not strong.

Let $p(x, y \mid \gamma) = \frac{x^\gamma}{x^\gamma + y^\gamma}$. Since

$$\frac{\partial p}{\partial \gamma} = \left(\frac{x^\gamma y^\gamma}{x^\gamma + y^\gamma}\right)(\ln(x) - \ln(y)) \tag{90}$$

and

$$\lim_{\gamma \to +\infty} \frac{x^\gamma}{x^\gamma + y^\gamma} = \lim_{\gamma \to +\infty} \frac{1}{1 + \left(\frac{y}{x}\right)^\gamma} = \begin{cases} 1, & \text{if } x > y \\ 0, & \text{if } x < y. \end{cases} \tag{91}$$

so for $x > y$, $p(x, y \mid \gamma)$ is increasing and converges to 1 when $\gamma \to +\infty$, and for $x < y$, $p(x, y \mid \gamma)$ is decreasing and converges to 0 when $\gamma \to +\infty$. Consequently, for any non-empty sequence $\sigma = x_0, \ldots, x_m$,

$$\lim_{\gamma \to +\infty} p_{\text{seq}}(\sigma \mid \gamma) = \begin{cases} 1, & \text{if } \sigma \text{ is strong} \\ 0, & \text{if } \sigma \text{ is weak.} \end{cases} \tag{92}$$

In addition, for any strong sequence σ, $p_{\text{seq}}(\sigma; \gamma)$ is increasing when γ is increasing. This is because for all $k \in \{1, \ldots, m\}$, $\sum_{j=0}^{k-1} x_j > x_k$, and so $\lim_{\gamma \to +\infty} \prod_{k=1}^{m} p\left(\sum_{j=0}^{k-1} x_j, x_k \mid \gamma\right) = 1$ and $\prod_{k=1}^{m} p\left(\sum_{j=0}^{k-1} x_j, x_k \mid \gamma\right)$ is increasing when γ is increasing. On the other hand, for any weak $\sigma = x_0, \ldots, x_m$, $\lim_{\gamma \to +\infty} p_{\text{seq}}(\sigma \mid \gamma) = 0$. This is because there exists $k \in \{1, \ldots, m\}$ such that $\sum_{j=0}^{k-1} x_j < x_k$ and for any such k, $\lim_{\gamma \to +\infty} p\left(\sum_{j=0}^{k-1} x_j, x_k \mid \gamma\right) = 0$. Since for all other $k \in \{1, \ldots, m\}$, $p\left(\sum_{j=0}^{k-1} x_j, x_k \mid \gamma\right) \leq 1$ so $\lim_{\gamma \to +\infty} \prod_{k=1}^{m} p\left(\sum_{j=0}^{k-1} x_j, x_k \mid \gamma\right) = 0$.

Fix a graph, G, and an resource endowment $\boldsymbol{r}$. Take any ownership state, $\mathbb{o}$ and any ruler $i \in \mathrm{Act}(\mathbb{o})$. Suppose that $i \notin \mathrm{Str}(G, \mathbb{o})$. Since for any weak sequence $\sigma \in \mathbb{R}^*$, $\lim_{\gamma \to \infty} p_{\mathrm{seq}}(\sigma) = 0$, every full fighting sequence of i is not strong, and there are finitely many such sequences, so there exists $\gamma^*_{i,\mathbb{o}}$ such that for all $\gamma > \gamma^*_{i,\mathbb{o}}$ and each full fighting sequence σ of i at $\mathbb{o}$, $p_{\mathrm{seq}}(\sigma) R < R_i(\mathbb{o})$. Thus $i \notin \mathrm{Adv}(G, \mathbb{o} \mid \gamma^*_{i,\mathbb{o}})$. On the other hand, as we observed earlier, $\mathrm{Str}(G, \mathbb{o}) \subseteq \mathrm{Adv}(G, \mathbb{o} \mid \gamma)$ for all $\gamma > 1$. Since there is a finite number of states and a finite number of rulers, so there exists $\gamma^* = \max_{i \in N, \mathbb{o} \in \mathbb{O}} \gamma^*_{i,\mathbb{o}}$ such that for all $\mathbb{o} \in \mathbb{O}$ and $i \in \mathrm{Act}(\mathbb{o})$, $\mathrm{Adv}(G, \mathbb{o}' \mid \gamma^*) = \mathrm{Str}(G, \mathbb{o})$. The claim on probability of hegemony for strong rulers now follows.

Part 2: Fix a network G, resource endowment $\boldsymbol{r}$, and an initial ownership state $\mathbb{o}$. At an initial ownership state every ruler owns exactly one vertex so, given a ruler $k \in N$, we will use v_k to denote the vertex own by k.

Suppose that $i \in N$ is a ruler who owns a vertex v_i in a weak set $U \subseteq V$. Assume, to the contrary, that i is strong. Then i has a strong full attacking sequence. Pick any such sequence, $j_2, \ldots, j_n$, and let j_m be the first ruler on that sequence owning a vertex in the neighbourhood of U, $v_{j_m} \in N_U(G)$. Let $U' = \{v_i, v_{j_2}, \ldots, v_{j_{m-1}}\}$ be the set of vertices owned by the rulers i and $j_2, \ldots, j_{m-1}$. Then $U' \subseteq U$, and, since the sequence is strong, $r_{v_{j_m}} < \sum_{u \in U'} r_u \leq \sum_{u \in U} r_u$, which contradicts the assumption that U is weak. Thus i must be weak.

Suppose now that $i \in N$ is a ruler who owns a vertex v_i that does not belong to any weak set. Assume to the contrary that i is weak. Let $j_2, \ldots, j_m$ be the longest strong attacking sequence of i. Let $U = \{v_i, v_{j_2}, \ldots, v_{j_m}\}$ be the set of vertices owned by the rulers i and $j_2, \ldots, j_m$. Clearly $U \subsetneq V$, as otherwise the sequence would be full, which would contradict the assumption that i is weak. Moreover, $G[U]$ is connected. In addition, for any $v \in N_U(G)$, it must be that $r_v > \sum_{u \in U} r_u$, as otherwise the sequence could be extended to be a longer attacking sequence. Thus U is a weak set, which contradicts the assumption that v_i does not belong to a weak set. Hence i must be strong. $\square$

Proof of Proposition 3. **Part 1:** The proof for rich rewarding technology follows from the example with three vertices worked out in the main text. We present an example for the poor rewarding technology: this example proves the 'necessity' of being central in a star network.

Consider a network with $n = 3$ nodes (c.f. Figure 5). Fix some $\gamma \in (0, 1)$ and suppose that $R_b > R_c$. Following part 1 of Proposition 1 we know that if R_b is sufficiently large (given γ and R_c) then ruler c prefers his optimal full attacking sequence to peace, regardless of the amount of resources of ruler a.[17] Thus, for any positive value of R_a and any connected network over three nodes, any ex post equilibrium outcome is hegemony. Moreover, by Theorem 1, every ruler prefers to fight the ruler with higher resources first (if allowed by the network). This, together with the fact that the equilibrium outcome is hegemony implies that in every equilibrium, on any network with three nodes, there is a unique sequence of fights: in the first round the there is a war between the strongest ruler and his strongest neighbour while the third player waits, and in the second round the waiting player fights with the winner.

This observation has two consequences. Firstly, the ex-ante equilibrium outcome is unique, and we can conduct meaningful comparative statics analysis of the effects of ruler a's resources. Secondly, as with strong rewarding technology, there are non-monotonic effects to increasing the resources of ruler a, that result from switching in order of attacks. These effects depend on the topology of the network, in particular, on the links of the strongest ruler. We now present the computations.

Take ruler a and suppose that $R_a \in (0, R_c)$. Increasing the resources of a within the interval $(0, R_c)$ has no impact on equilibrium strategies and therefore increases a's probability of becoming a hegemon. When R_a crosses R_c, there is a switch in strategies in the clique and in the star with b in the centre: now b fights a at the initial state. This leads to a fall in the probability of a becoming a hegemon. There is no switch possible in case a is centre of the star network. Similar considerations arise as R_a grows further. Ruler a's probability of becoming a hegemon is summarized in Figure 9.

Part 2: Let V be a set of vertices, $\boldsymbol{r} \in \mathbb{R}^V_{++}$ a resource endowment, and $i \in V$ a ruler. Let S be a star over with centre i. Take a connected network G which is not a star. We will show that the expected payoff to i at the initial ownership state is not higher in G than in S. To this end, it is enough to show, for every $j \in V$, that conditional on j being picked to move at the initial ownership state, expected payoff to i in G is at most as high as payoff to i in S. Take any $j \in V$. If $j = i$ that the statement holds, because i can attack the remaining rulers in increasing order with respect to their resources. By Proposition 5, this is optimal across all possible sequences of fights that i may start.

Suppose that $j \neq i$. Since p is rich rewarding and has a Non-Waiting property, in every equilibrium, every ruler chooses an optimal full attacking sequence whenever he has a chance to move (c.f. the arguments in proof of Theorem 2). Thus either j executes his optimal full

[17] For example, if $\gamma = 1/3$ and $R_c = 2$, then $R_b \geq 5$ is sufficient to make c prefer attack to peace at the initial state.

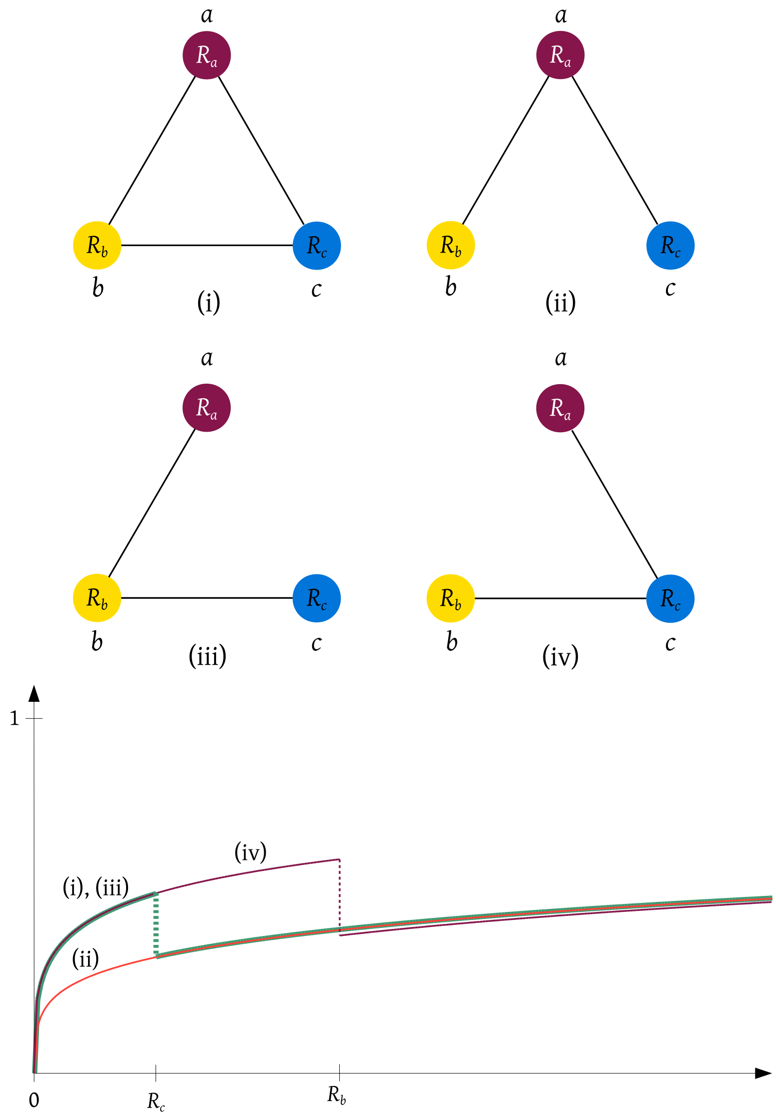

Figure 9: Resources, Geography and Probability of Winning

fighting sequence and reaches i or it looses a fight before reaching i, in which case the winner takes on the fight to either reach i or loosing to some other ruler, who takes on the fight, etc. Thus conditioned on j being picked to move, the probability of i getting attacked and winning is a convex combination of probabilities of the form $p(r_i, \sum_{l \in X} r_l)$ where X is a set of subsequent rulers in the sequence of conflicts that lead to an attack on i. In particular, $j \in X$. Therefore, conditioned on j being picked to move, the probability of i becoming a hegemon is a convex combination of probabilities of the form $p(r_i, \sum_{l \in X} r_l) p_{\text{seq}}(r_i + \sum_{l \in X} r_l, r_{k_1}, \dots, r_{k_q})$, where X is as described above and $r_{k_1}, \ldots, r_{k_q}$ is an optimal full attacking sequence of i in the state reached by winning an attack initiated by j and consisting of the rulers in $V \setminus (\{i\} \cup X)$. By points 1 and 2 of Theorem 1,

$$p\left(r_i, \sum_{l \in X} r_l\right) p_{\mathrm{seq}}\left(r_i + \sum_{l \in X} r_l, r_{k_1}, \ldots, r_{k_q}\right) \leq p_{\mathrm{seq}}(r_i, r_j, r_{j_1}, \ldots, r_{j_m}), \tag{93}$$

where $j_1, \ldots, j_m$ is a sequence of rulers in $V \setminus \{i, j\}$ ordered in an increasing order with respect to their resources. Since the RHS is the probability of i becoming a hegemon in S conditioned on j being picked to move and attacking i, so the probability of i becoming a hegemon in an equilibrium, conditioned on j being picked to move, is at least as high in S and in G, for any $j \in V \setminus \{i\}$.

We now take up the case of poor rewarding technology. Figure 9 shows that the probability of ruler a becoming a hegemon is strictly larger when he is peripheral in a star (for the resource range $R_a \in (R_b, R_c)$). The computations presented there can be easily extended to the setting where the winning ruler retains the initiative of attack (irrespective of whether he is the aggressor or the defender). □

Proof of Proposition 4. Given set of vertices $U \subseteq V$ and resource endowment $\boldsymbol{r}$ let $r_U = \sum_{v \in U} r_v$. Also, given a resource endowment $\boldsymbol{r}$ over a set of vertices V let $Z(\boldsymbol{r}) = \{(r_U, r_{U'}) : U, U' \in 2^V \setminus \{\varnothing\}, U \cap U' = \varnothing\}$.

Part 1: Fix the set of vertices V and resource endowment $\boldsymbol{r}$. Assume first that p is rich rewarding. Take any $x, y \in \mathbb{R}_{++}$. Since $p(x,y)(x+y) > x$ so $\delta^1_{x,y} = 1 - x/((x+y)p(x,y)) > 0$ and for any $\delta \in (0, \delta^1_{x,y})$, $p(x,y)(1-\delta)(x+y) > x$. Next, assume that p has No-Waiting property. Take any $x, y, z \in \mathbb{R}_{++}$. By monotonicity and continuity of p, $p(x,y)p((1-\delta)(x+y), z)((1-\delta)(x+y)+z)$ is continuous and decreasing δ. Moreover, by point 1 of Theorem 1, $p(x,y)p(x+y,z)(x+y+z) > p(x,y+z)(x+y+z)$. Thus $\delta^2_{x,y} = \sup\{0 < \delta \leq 1 : p(x,y)p((1-\delta)(x+y), z)((1-\delta)(x+y)+z) > p(x,y+z)(x+y+z)\}$ is well defined and for all $\delta \in (0, \delta^2_{x,y})$, $p(x,y)p((1-\delta)(x+y), z)(1-\delta)((1-\delta)(x+y)+z) > p(x,y+z)(1-\delta)(x+y+z)$. Let $\delta^1(\boldsymbol{r}) = \min_{x,y \in Z(\boldsymbol{r})} \delta^1_{x,y}$, $\delta^2(\boldsymbol{r}) = \min_{x,y \in Z(\boldsymbol{r})} \delta^2_{x,y}$, and $\delta(\boldsymbol{r}) = \min(\delta^1(\boldsymbol{r}), \delta^2(\boldsymbol{r}))$. Sine $Z(\boldsymbol{r})$ is finite and non-empty so $\delta(\boldsymbol{r})$ is well defined and positive. For any $\delta \in (0, \delta(\boldsymbol{r})$ and amounts of resources from $Z(\boldsymbol{r})$, expected payoff from attacking a poorer side is higher then current resource holding and expected payoff from attacking two opponents in a sequence is higher than payoff from letting them fight and attacking them afterwards. Hence the argument in proof of Theorem 2 works and part 1 holds.

Part 2: Take any connected network G over a set of vertices V ($|V| \geq 2$) and a resource endowment $\boldsymbol{r}$. First we show, for all $x, y \in \mathbb{R}_{++}$ and $\delta > 1/2$ (in the case of rich rewarding p), $\delta > 1 - \min(x, y)/\max(x, y)$ (in the case of poor rewarding p), that

$$(x + y)(1 - \delta)p(x, y) < x. \tag{94}$$

Suppose that p is rich rewarding. Let $x \geq y$. With $\delta > 1/2$, $(x + y)(1 - \delta)p(x, y) < (x+y)/2p(x, y) < x$. On the other hand, let $x \leq y$. Then $(x+y)p(x, y) = (x+y)(1-p(y, x)) \leq y + x - y = x$. Hence $(x + y)p(x, y)(1 - \delta) < x$. Suppose that p is poor rewarding. Let $x \geq y$. Since $(x + y)p(x, y) = (x + y)(1 - p(y, x)) \leq x + y - y = x$ so, with $\delta < 1$, $(x + y)p(x, y)(1 - \delta) < x$. On the other hand, let $x \leq y$. Then $(x + y)p(x, y) \leq (x + y)/2 \leq y$. Hence $(x + y)(x/y)p(x, y) \leq x$ and so with $\delta > 1 - x/y$, $(x + y)(1 - \delta)p(x, y) < x$.

Second, we show, for any resource endowment $\boldsymbol{r}$ over a set of vertices V, $|V| \geq 2$, that if $\delta > \delta_2$, then (94) holds for any $x, y \in Z(\boldsymbol{r})$. The case of rich rewarding p is immediate. In the case of poor rewarding p, take any $x, y \in Z(\boldsymbol{r})$. If $x \geq y$, then (94) holds for any $\delta > 0$ and we are done. Suppose that $x < y$. Notice that $x \geq \min_{v\in V} r_v$ and $y \leq \sum_{v\in V} r_v - \min_{v\in V} r_v$ so $1 - x/y \leq \delta_2$. Thus if $\delta > \delta_2$ then $\delta > 1 - x/y$ and (94) holds.

Third, we show that if $\delta > \delta_2$ then for any $x_0, x_1, \ldots, x_k \in Z(\boldsymbol{r})$ such that, for all $l \in \{1, \ldots, k\}$ $\sum_{j=0}^{l} x_j \in Z(\boldsymbol{r})$, we have

$$\Pi_{\mathrm{seq}}(x_0, x_1, \ldots, x_k) < x_0. \tag{95}$$

Notice that the expected payoffs in the costs of conflict model can be rewritten as

$$\Pi_{\mathrm{seq}}(x_0, x_1, \ldots, x_m) =$$
$$x_0 \prod_{i=1}^{m} p\left((1 - \delta)^{i-1}x_0 + \sum_{j=1}^{i-1}(1 - \delta)^{i-j}x_j, x_i\right)(1 - \delta)\left(\frac{(1 - \delta)^{i-1}x_0 + \sum_{j=1}^{i}(1 - \delta)^{i-j}x_j}{(1 - \delta)^{i-1}x_0 + \sum_{j=1}^{i-1}(1 - \delta)^{i-j}x_j}\right). \tag{96}$$

Thus if δ is sufficiently high so that each bilateral conflict in the sequence is not profitable, i.e for all $i \in \{1, \dots, m\}$,

$$p\left((1-\delta)^{i-1}x_0 + \sum_{j=1}^{i-1}(1-\delta)^{i-j}x_j, x_i\right)(1-\delta)\left((1-\delta)^{i-1}x_0 + \sum_{j=1}^{i}(1-\delta)^{i-j}x_j\right) < (1-\delta)^{i-1}x_0 + \sum_{j=1}^{i-1}(1-\delta)^{i-j}x_j, \quad (97)$$

then the whole sequence is not profitable for the ruler with x_0 resources. Resources in each bilateral conflict in the sequence are elements of $Z(\boldsymbol{r})$. As we have shown above, any such bilateral conflict is unprofitable for either side if $\delta > \delta_2$. Hence (95) follows.

Lastly, we show that with cost of conflict $\delta > \delta_2$ a strategy profile $\boldsymbol{s}$ such that at every ownership state $\mathbb{o} \in \mathbb{O} = \mathbb{O}(G)$ every active ruler chooses peace is an equilibrium. The proof is by induction on the number of active rulers at an ownership state.

For the induction basis, take a ownership state $\mathbb{o} \in \mathbb{O}$ with $|\mathrm{Act}(\mathbb{o})| = 2$ active rulers. Let x and y be the resources of the two active rulers. Since $x, y \in Z(\boldsymbol{r})$ and $\delta > \delta_2$ so none of the two rulers finds it profitable to fight when the other one is peaceful. Thus claim holds.

For the induction step, take any ownership state $\mathbb{o} \in \mathbb{O}$ with $|\mathrm{Act}(\mathbb{o})| = m \geq 3$ and suppose that the hypothesis holds for any ownership state with $2 \leq m' \leq m-1$ active rulers. Assume, to the contrary, that there exists an active ruler $j_0 \in Act(\omega)$ who prefers to choose an attacking sequence $\sigma = j_1, \dots, j_k$ when all other rulers choose peace. Let $\mathbb{o}'$ be the ownership state reached by j_0 winning the sequence. By the induction hypothesis, choosing peace is an equilibrium at the ownership state $s(\mathbb{o}')$. Moreover, R_{j_l}, for all $l \in \{0, \dots, k\}$, and $\sum_{i=0}^{l} R_i$, for all $l \in \{1, \dots, k\}$, are elements of $Z(\boldsymbol{r})$. Hence, by (95), deviating to σ is not profitable for i. This completes the proof of part 2.

Proof of Part 3 Fix a resource endowment $\boldsymbol{r}$ over a set of vertices V. Let $\delta^* = 1 - \min_{v \in V} r_v / \max_{v \in V} r_v$. Notice that for any $u, v \in V$ and $x = r_u$, $y = r_v$, Equation (94) holds if $\delta > 1 - \min(x,y)/\max(x,y)$. Since for any such x and y, $\min(x,y) \geq \min_{v \in V} r_v$ and $\max(x,y) \leq \max_{v \in V} r_v$, so $\delta \leq \delta^*$. Thus if $\delta > \delta^*$ then, for any $u, v \in V$, $\Pi(r_u, r_v) < r_u$. Consequently, by analogous arguments to those used in proof of Proposition 4 to show (95), for any sequence $v_0, v_1, \dots, v_k$ of different elements in V,

$$\Pi_{\mathrm{seq}}(r_{v_0}, r_{v_1}, \dots, r_{v_k}) < v_0. \quad (98)$$

We will show that if $\delta > \delta^*$ then a strategy profile where all rulers choose peace in the first round is an equilibrium on a star network G over V. Let $\boldsymbol{s}$ be a strategy profile such that all rulers choose peace in the first round. Let $\oplus$ be the initial ownership state and take any ruler j_0 and any attacking sequence σ of j_0 different to staying peaceful. Since G is a star, deviating to σ results in j_0 having a sequence of fights with rulers owning single vertices only. By (98, any such sequence yields expected payoff strictly below r_{j_0} and so is unprofitable to j_0. Thus any strategy profile $\boldsymbol{s}$ such that each ruler chooses peace at the initial ownership state $\oplus$ and at any other ownership state $\oplus'$, $\boldsymbol{s}$ is an equilibrium, is an equilibrium (by Observation 1 we know that such a strategy profile exists). $\square$

Computations on buffer state: If $R_b \in (1.79, 3.07)$, then the outcome is hegemony. For ruler a, attacking b and then c is the most preferred sequence. Attacking the winner of conflict between b and c is also profitable for a. Depending on the value of R_b, b and c react differently. They either stay peaceful (for lower values of R_b) or choose to attack each other in the first round. In either case, hegemony is the equilibrium outcome. If $R_b \in (3.07, 3.94)$ then attacking b and then c is the most preferred sequence for a. Attacking the winner of conflict between b and c is not profitable for a. Anticipating this, b and c choose to attack each other in the first round. The winner of this fight prefers staying peaceful to attacking a. Thus the outcome is hegemony of a or two surviving kingdoms. Finally, if $R_b \in (3.943, 10)$ then every player chooses a full attacking sequence.

Appendix B: Additional Results

The Difference Contest Function

Assume the difference form of CSF:

$$p(x, y) = \frac{\exp(\gamma x)}{\exp(\gamma x) + \exp(\gamma y)}, \tag{99}$$

where $\gamma > 0$.

Although there is no $\gamma > 0$ such that p has the strong rewarding property and the No-Waiting property for all $x, y, z \in \mathbb{R}_{++}$, it has them conditionally, i.e. for all $x, y, z > \frac{1}{\gamma}$. From that it follows that Theorem 1 holds for any resource endowment $\boldsymbol{r}$ (and any connected network), if $\gamma \geq \frac{1}{\min_{i \in V} r_i}$. The analysis below shows that these properties are indeed conditionally satisfied.

Strong rewarding. Given $x, y \in \mathbb{R}_{++}$, $x > y$,

$$\left(\frac{\exp(\gamma x)}{\exp(\gamma x) + \exp(\gamma y)}\right)(x + y) > x, \tag{100}$$

if $\gamma \geq \frac{1}{y}$. To see that notice that:

$$\gamma(x - y) > \ln(x) - \ln(y) \tag{101}$$

if $\gamma \geq \ln'(y) = \frac{1}{y}$. From that we get:

$$\frac{y}{x} > \exp(\gamma(y - x)) \tag{102}$$

and further

$$\frac{1}{1 + \exp(\gamma(y - x))} > \frac{1}{1 + \frac{y}{x}}. \tag{103}$$

This is equivalent to

$$\left(\frac{\exp(\gamma x)}{\exp(\gamma x) + \exp(\gamma y)}\right)(x + y) > x. \tag{104}$$

No-Waiting. Given $x, y, z \in \mathbb{R}_{++}$,

$$p(x, y)p(x + y, z) > p(x, y + z) \tag{105}$$

if $\gamma \geq \frac{1}{\min(x,y,z)}$. To see that, notice that

$$\exp(-1) + 2\exp(-2) > \exp(-\gamma 2y) + \exp(-\gamma z) + \exp(-\gamma(x + y)), \tag{106}$$

if $\gamma \geq \frac{1}{\min(x,y,z)}$. Since

$$1 > \exp(-1) + 2\exp(-2) \tag{107}$$

so

$$1 > \exp(-\gamma 2y) + \exp(-\gamma z) + \exp(-\gamma(x + y)) \tag{108}$$

and, multiplying both sides by $\exp(\gamma(x + y))$,

$$\exp(\gamma(x + y)) > \exp(\gamma(x - y)) + \exp(\gamma(x + y - z)) + 1. \tag{109}$$

Further, multiplying both sides by $\exp(\gamma(y+z))$, we get

$$\exp(\gamma(x+2y+z)) > \exp(\gamma(x+z)) + \exp(\gamma(x+2y)) + \exp(\gamma(y+z)). \tag{110}$$

Adding $\exp(\gamma(2x+y))$ to both sides and reorganizing we get

$$\exp(\gamma(2x+y))+\exp(\gamma(x+2y+z)) > \exp(\gamma(2x+y))+\exp(\gamma(x+z))+\exp(\gamma(x+2y))+\exp(\gamma(y+z)), \tag{111}$$

which is equivalent to

$$\exp(\gamma(x+y))(\exp(\gamma x)+\exp(y+z)) > (\exp(\gamma x)+\exp(\gamma y))(\exp(\gamma(x+y))+\exp(\gamma z)). \tag{112}$$

Multiplying both sides by

$$\frac{\exp(\gamma x)}{(\exp(\gamma x)+\exp(y+z))(\exp(\gamma x)+\exp(\gamma y))(\exp(\gamma(x+y))+\exp(\gamma z))}$$

we get

$$\left(\frac{\exp(\gamma x)}{\exp(\gamma x)+\exp(\gamma y)}\right)\left(\frac{\exp(\gamma(x+y))}{\exp(\gamma(x+y))+\exp(\gamma z)}\right) > \frac{\exp(\gamma x)}{\exp(\gamma x)+\exp(\gamma(y+z))} \tag{113}$$

that is

$$p(x,y)p(x+y,z) > p(x,y+z) \tag{114}$$

Monotonicity in Optimal attack sequence

From Property 2 in Theorem 1 we know that a ruler must attack the poorer opponent first and then the richer opponent if the technology is rich rewarding; the converse is true in case the technology is poor rewarding. We build on this property and establish a simple but powerful property of optimal attacks: they order opponents in increasing order of resources, in the case of $\gamma > 1$ and in decreasing order in the case of $\gamma < 1$.

Proposition 5. *Let $m \geq 3$ and $x_0, x_1, \ldots, x_m \in \mathbb{R}_{++}$, be such that $x_1 < \ldots < x_m$. Then, for any permutation $\pi : \{1,\ldots,m\} \to \{1,\ldots,m\}$,*

$$p_{\text{seq}}\left(x_0, x_{\pi(1)}, \ldots, x_{\pi(m)}\right) \leq \begin{cases} p_{\text{seq}}(x_0, x_1, \ldots, x_m), & \textit{if } \gamma > 1, \\ p_{\text{seq}}(x_0, x_m, \ldots, x_1), & \textit{if } \gamma < 1, \end{cases} \tag{115}$$

with equality only if the permutations on both sides are the same.

Proof of Proposition 5. Assume $\gamma > 1$. Let $\pi : \{1, \ldots, m\} \to \{1, \ldots, m\}$ be a permutation of $\{1, \ldots, m\}$. A pair of indices $(i, j) \in \{1, \ldots, m\}$ such that $i < j$ and $\pi(i) > \pi(j)$ is called an *inverse* of π. We will show that for any permutation π of $\{1, \ldots, m\}$ with at least one inverse there exists a permutation π' of $\{1, \ldots, m\}$ with less inverses that yields higher p_{seq}:

$$p_{\mathrm{seq}}(x_0, x_{\pi(1)}, \ldots, x_{\pi(m)}) < p_{\mathrm{seq}}(x_0, x_{\pi'(1)}, \ldots, x_{\pi'(m)}). \tag{116}$$

Since the identity is the unique permutation of $\{1, \ldots, m\}$ with no inverses, this implies the proposition.

So take any permutation π on $\{1, \ldots, m\}$ with at least one inverse, (i, j). Then there exists $i \leq k < j$ such that $(k, k+1)$ is also an inverse of π. Let π' be a permutation of $\{1, \ldots, m\}$ obtained from π by exchanging $\pi(k)$ and $\pi(k+1)$, i.e. $\pi'(k) = \pi(k+1)$, $\pi'(k+1) = \pi(k)$, and $\pi'(l) = \pi(l)$ for $l \in \{1, \ldots, m\} \setminus \{k, k+1\}$. There is at least one inverse less in π' than in π. Moreover,

$$\begin{aligned}
&p_{\mathrm{seq}}\left(x_0, x_{\pi'(1)}, \ldots, x_{\pi'(m)}\right) \\
= \quad & p_{\mathrm{seq}}\left(x_0, x_{\pi'(1)}, \ldots, x_{\pi'(k-1)}\right) \cdot p\left(\sum_{l=1}^{k-1} x_{\pi'(l)}, x_{\pi'(k)}\right) \cdot p\left(\sum_{l=1}^{k} x_{\pi'(l)}, x_{\pi'(k+1)}\right) \cdot \\
& p_{\mathrm{seq}}\left(\sum_{l=1}^{k+1} x_{\pi'(l)}, x_{\pi'(k+2)}, \ldots, x_{\pi'(m)}\right) \\
= \quad & p_{\mathrm{seq}}\left(x_0, x_{\pi(1)}, \ldots, x_{\pi(k-1)}\right) \cdot p\left(\sum_{l=1}^{k-1} x_{\pi'(l)}, x_{\pi'(k)}\right) \cdot p\left(\sum_{l=1}^{k} x_{\pi'(l)}, x_{\pi'(k+1)}\right) \cdot \\
& p_{\mathrm{seq}}\left(\sum_{l=1}^{k+1} x_{\pi(l)}, x_{\pi(k+2)}, \ldots, x_{\pi(m)}\right)
\end{aligned} \tag{117}$$

By point 2 of Theorem 1,

$$\begin{aligned}
& p_{\text{seq}}\left(x_0, x_{\pi(1)}, \ldots, x_{\pi(k-1)}\right) \cdot p\left(\sum_{l=1}^{k-1} x_{\pi'(l)}, x_{\pi'(k)}\right) \cdot p\left(\sum_{l=1}^{k} x_{\pi'(l)}, x_{\pi'(k+1)}\right) \cdot \\
& p_{\text{seq}}\left(\sum_{l=1}^{k+1} x_{\pi(l)}, x_{\pi(k+2)}, \ldots, x_{\pi(m)}\right) \\
> \quad & p_{\text{seq}}\left(x_0, x_{\pi(1)}, \ldots, x_{\pi(k-1)}\right) \cdot p\left(\sum_{l=1}^{k-1} x_{\pi(l)}, x_{\pi(k)}\right) \cdot p\left(\sum_{l=1}^{k} x_{\pi(l)}, x_{\pi(k+1)}\right) \cdot \\
& p_{\text{seq}}\left(\sum_{l=1}^{k+1} x_{\pi(l)}, x_{\pi(k+2)}, \ldots, x_{\pi(m)}\right) = p_{\text{seq}}\left(x_0, x_{\pi(1)}, \ldots, x_{\pi(m)}\right) \qquad (118)
\end{aligned}$$

This completes the proof of the case $\gamma > 1$. The case of $\gamma \in [0, 1)$ follows by analogous arguments, using point 2 of Theorem 1 and duality of $<$ and $>$. $\square$

Alternative formulation on continuation of game, after a loss

In the basic model studied in the paper, a ruler who is picked to move has his resources mobile and can use them to execute a sequence of attacks. When such a ruler looses a conflict with another ruler, the winner needs to wait for being picked to be able to move and attack other rulers. In this setting adding links between neighbours can sometimes be beneficial.

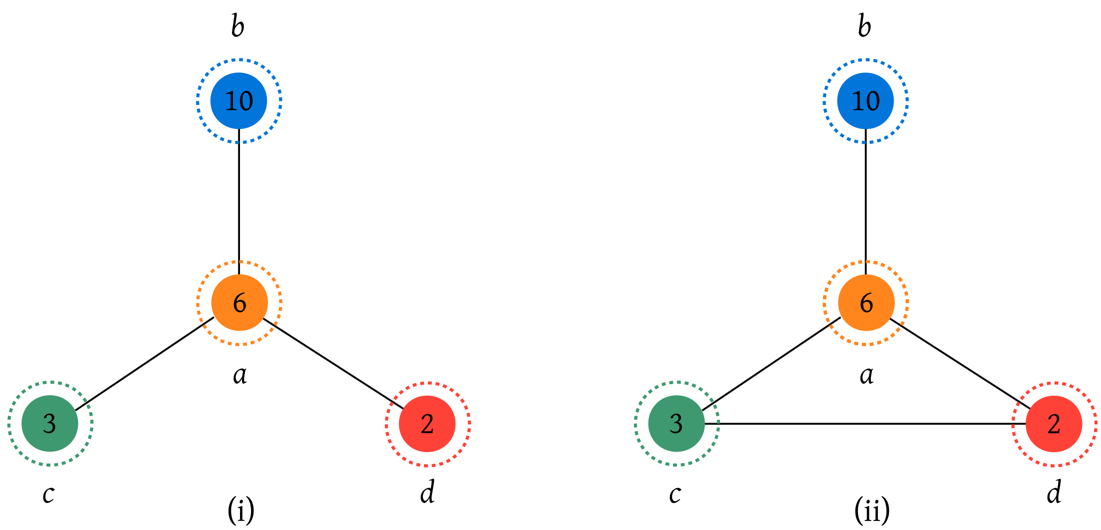

Figure 10: Networks illustrating that link between neighbour can be profitable.

Example 3 (Link between neighbors can be profitable)**.** Consider two networks over 4 vertices, as presented in Figure 10 and assume the main model of the paper, where at the beginning of a round all active rulers have to wait for being picked to move. It can be verified that with $\gamma \in (1, 15.287)$, the equilibrium expected payoff to ruler a is higher in the star network, (i), while with $\gamma > 15.288$, it is higher in the star network with an additional link between the vertices owned by c and d, (ii). To see why (ii) can be better for a than (i), suppose that γ is very high, so that the ruler with more resources wins with probability close to 1. In this case it is crucial for ruler a to face both the vertices c and d before facing b. Moreover, in equilibrium on network (ii), c attacks d first and then a, and d attacks c first and then a. On network (ii) every vertex must attack the centre first. ruler a wins on network (i) in equilibrium if one of the following events occurs: a is picked in the first round, or b or c is picked in the first round and then a or the remaining of b and c is picked in the second round. This event has probability $1/4 + 1/2 \cdot 2/3 = 7/12$. On the other hand, ruler a wins on network (ii) if one of the following events occurs: a or c is picked in the first round, or d is picked in the first round and then a or c is picked in the second round. This event has probability $1/2 + 1/4 \cdot 2/3 = 8/12$. This example illustrates the tension between being able to attack the opponents before they fight with each other and being able to attack the opponents in the right order. With sufficiently large γ the latter becomes more significant and an additional link that helps facing opponents in the right order becomes profitable. Careful calculation shows that γ close to 15.288 is sufficiently high for this to be observed in the example above. ■

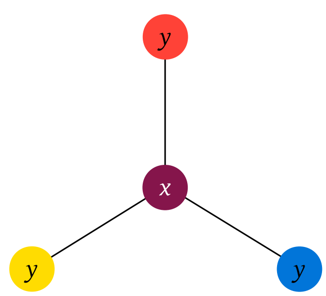

Figure 11: Network where increasing cost of conflict leads to war.

Example 4 (Higher cost of conflict leads to war)**.** Consider a star network over 4 vertices, as presented in Figure 11. Assume Tullock CSF with $\gamma = 0.5$. Every spoke is endowed with y resources and the centre is endowed with x resources. Let $y = 1.0$ and $x \in (2.1, 2.9)$.

Suppose that cost of conflict $\delta = 0.0$. The expected payoff to a spoke ruler with 1.0

resources from executing an attacking sequence of length $m \leq 2$ when the centre ruler has $z \in (3.1, 3.9)$ resources is $\varphi(z, 1.0, m) \geq \varphi(z, 1.0, 2) \in (1.23, 1.37)$ (recall function φ as defined in (65)). Hence after any spoke ruler attacks the centre at the initial ownership state, there will be fight till hegemony in any equilibrium. Payoff to the spoke ruler from executing an attacking sequence of length 3 at the initial state is $\varphi(x, 1.0, 3) \in (0.889, 0.999)$. Thus it is not profitable for a spoke ruler to attack the centre at the initial state. Since $\gamma < 1$ and $x < y$ so it is not profitable for the centre ruler to attack as well. Hence there is an equilibrium with peace at the initial state.

Suppose now that cost of conflict $\delta = 0.2$. The expected payoff to a spoke ruler with y resources from executing an attacking sequence of length m when the centre ruler has x resources is

$$\psi(x, y, m \mid \delta) = \\ \left(x(1-\delta)^m + \sum_{j=1}^{m}(1-\delta)^j y\right) p(y, x) \prod_{i=1}^{m-1} p\left((1-\delta)^i x + \sum_{j=1}^{i}(1-\delta)^j y), y\right). \quad (119)$$

Consider the ownership state resulting from two attacks by spoke on a centre: there are two active rulers, one with 1.0 resources and another one with $z = 0.8(0.8(1.0 + x) + 1.0) \in (2.784, 3.296)$ resources. Expected payoff to the poorer vertex from attacking the richer vertex is $\psi(z, 1.0, 1 \mid 2.0) \in (1.13, 1.23)$. Hence the poorer vertex finds it profitable for the poorer vertex to attack the richer one. Consider now the ownership state resulting from one attack by a spoke ruler on the centre. There are two spokes, each endowed with 1.0 resources and the centre endowed with $z = 0.8(1.0 + x) \in (2.48, 3.12)$ resources. Any attacker anticipates two fights after an attack. Expected payoff to a spoke from two fights is $\psi(z, 1.0, 2 \mid 2.0) \in (0.73, 0.81)$. Thus a spoke ruler does not want to attack and (with $\gamma < 1$ and $z > y$) the centre ruler does not want to attack as well. Lastly, consider the initial ownership state. Payoff to a spoke from attacking the centre is $\psi(x, 1.0, 1 \mid 2.0) \in (1.01, 1.16)$. This leads to a ownership state with peace. Hence every spoke finds it profitable to attack the centre and so there is no peace at the initial ownership state in any equilibrium outcome. ■