Power Networks – A Network Approach to Voting Theory

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Abstract

In this paper I study elections where the preferences of the voters are derived from their position in the social network. I argue that in some cases studying the ideology of the electorate is not the right way to understand the result of an election. In small electorates or in committees the personal connections can be much more important than political ideology. I develop a voting model where these personal connections add up to a social network and I study what network properties and the social structures lead to a voting equilibrium. I show that single peaked preferences on chain and tree networks are inherent in my model, then I define a set of networks where the equilibrium is robust to changes in the intensity of a connection. In the discussion I relate my approach to the practice of using centrality measures (betweenness or eigenvector centrality) in the analysis and in an illustrative example I calibrate my model to predict the victory of the Medici family in a medieval power struggle.

Key words: political economy, voting, social networks, Condorcet winner

JEL classification: D72, D85

1 Introduction

Voting theory focuses mostly on the relationship between the ideology of the electorate and the creed of the candidates. In this paper I depart from the idea that this approach is successful in large electorates (e.g. national elections) but as the number of voters decreases, the importance of the personal ties increases. In small electorates where every voter knows the candidates personally the personal traits of the candidate are likely to be more important than the political ideology they represent.

The basic idea of my research is that society is not just a collection of individual voters. I believe that if I want to model a social phenomenon, studying the individual characteristics of the people (e.g. general political preferences) gives me only half of the picture, understanding the interpersonal relationships in a social group can be equally important. Members of a group are connected with a great variety of different ties (e.g. family ties, friendship, coinciding interests, regular information exchange) and these relations influence their preferences and (economic/political) decisions just as much as their individual characteristics do. The collection of these ties is the social network.

I want to shed light on the differences between general political preferences and preferences based on personal ties. In large electorates a political outcome, for example the electoral victory of candidate Anne, only affects voter Beth through the distance between the policy ideal for Beth and the political decisions made by Anne. However in small electorates the electoral victory of Anne can have direct effect on the voters regardless of the implemented policy: with a non-infinitesimal probability candidate Anne and voter Beth are friends, relatives or tied with other interpersonal link. Besley and Coate [1997] mention a special example of this direct effect that is related to the candidate and not to her policy: liking a "good-looking" representative. Here I define this direct effect more generally and argue that being *close* to the decision maker either emotionally (e.g. affection, trust) or physically (e.g. to share a hobby or to live next door) has its own value in politics as it can lead to informal influence.

The voter's informal influence is not the only thing that can shape the policy, the elected decision maker's informal influence over some of her peers can be equally important. In constitutional democracies there is a set of decisions that the winner of an election cannot make herself, she can only appoint people to make these decisions autonomously. In these appointment decisions trust, affection and the possibility of an informal influence plays an important part. If Anne and Beth trust each other, Beth can have informal influence over Anne's decisions but in the same time she also can expect to be appointed to an autonomous decision making position as Anne expects to have an informal influence over the decisions of Beth. If I consider that being close to an appointed decision maker has similar perks (on a smaller scale) as being close to the elected one it is easy to see how political power spreads along social ties in the social network.

To study voting decisions and electoral outcomes in small electorates, I develop a model of electoral competition. In my model voting takes place in a society which is characterized by interconnected interests: the (electoral) success of a player yields profit, not only to her, but also to those who are tied to her. The collection of these ties draws a social network and that I use to study the result of a three-stage voting game. In the first stage the players choose whether or not to take the costly effort of running for office. In the second stage the players elect one representative from the self-selected pool of candidates. Finally in the third stage the winner of the election receives a monetary prize (it represents the utility of the political power) that she shares with her neighbors in the network, who then share their profit with their own neighbors and so on as the political power spreads along the social ties.

My paper intends to link the citizen-candidate literature of voting theory e.g. Osborne and Slivinski [1996], Besley and Coate [1997] and Cadigan [2005] with the literature of politics in social networks e.g. Ballester et al. [2006], König et al. [2015], Galeotti and Mattozzi [2011], Elliott et al. [2014], Demange [1982].

There are two empirical papers very close to my work, as both of them uses information on the social network to explain political outcomes. In a case study Padgett and Ansell [1993] describe the political context of the 15th century Florence and explain the way how Cosimo de' Medici took advantage of the position of the Medici family in the marriage network of the Florentine nobility and managed to take over the city-state. Cruz et al. [2015] follow a statistical approach while studying the correlation between the candidate's centrality in the family network and electoral outcome on the 2010 municipal elections in the Philippines. They find that the more central candidates have higher vote share (and probability of winning the election). In this essay I model the behavior of voters to explain the empirical results of Padgett and Ansell [1993] and Cruz et al. [2015].

My first result shows that any social network with a strong Condorcet winner has a pure strategy Nash equilibrium, where she runs uncontested, she gets all the votes and wins. Then I talk about two network properties that can guarantee the existence of equilibria with one or two candidates in some network classes and significantly simplify the problem in others. The first property is single-peakedness of the preferences. The second is a balanced architecture of the network around a core object that can be a link, a *hub player* or a particular group of players. I show that the Condorcet winners exist and that there are voting equilibria with one or two candidates in a wide set of networks. In the chain (from e.g. Besley and Coate [1997]) and the tree (from e.g. Demange [1982]) I show that single peaked preferences are inherent, while in the bridge network (from e.g. Calvó-Armengol and Jackson [2004]) and in the windmill network (first introduced in this paper) I define criteria for balanced architecture. I also show that every network where the equilibrium is robust to the changes in the intensity of the connections falls into one of these network classes.

In the discussion I explore the relationship between centrality measures (betweenness and eigenvector centrality) and the strong Condorcet winner and show that although the strong Condorcet winner tends to have high centrality it is easy to find examples where she is not the player with the *highest* centrality. This result suggests that in some special cases the analysis of the social structure needs to go beyond centrality in order to provide a good prediction of the electoral outcome. Finally I provide an illustration, where I show the predictive power of the model in a real life situation. I use historical data for calibration and I predict the result of a power struggle in the medieval Florence. I find that in the social network of the Florentine nobility the Medici family was in a strong Condorcet winner position, so they had sufficient support to overthrow the ruling clique of the city.

The rest of the paper is organized in the following way: Section 2 introduces the model of the society and the voting game, Section 3 and Section 4 contain the main results of this essay, finally Section 5 concludes. In the Appendix I provide the proofs of the propositions.

2 Model

In this section I present the model of the paper. In the first part I describe voters' preferences in a social network and for this part I use the model by Elliott et al. [2014]. While in the original paper the model was constructed to study default cascades in financial networks I argue that the cross holdings on the financial market and the interconnected interests of a small electorate are qualitatively similar. In the second part I describe the 3-stage citizen-candidate voting game.

2.1 Society

The primitive of the model is a society of n players, where the set of players is denoted by $N = \{1, 2, 3, \ldots n\}$. The society is structured by personal ties between pairs of players. These ties represent informal influences over each other's decisions, they also show how much one player is interested in the success of the other (it is always better to have influence over a successful player). The basic assumption of the model is that the players typically do not enjoy all the perks of success alone – power spreads in the society along the social ties by the nature of politics and not by the decision of any player. In this model success and political power (both formal and informal) are represented by monetary income that enters the society as a prize and spreads through "transfers".

The social network appears as a directed and weighted graph. The arc that points to i from j has a weight of c_{ij} that represents the share j transfers to i of all the income she receives from outside of the society as a prize or from her peers as a transfer – in other words c_{ij} expresses the intensity of the tie between i and j. As c_{ij} is a share its value is always between 0 and 1 where the c_{ij} arc with zero-weight is qualitatively the same as there was no arc between i and j. I assume that the summed weight of the arcs pointing away from a given vertex is always less than 1, so every player j keeps a positive $\widehat{c_j}$ share of her income for herself ($1 - \sum_k c_{kj} = \widehat{c_j}$ for every $j \in N$).

Assumption 1. $\sum_{k \in N} c_{kj} \leq 1$ *for every* $j \in N$.

The $n \times n$ matrix C represents the social graph such that the i, j element of the matrix is c_{ij} ($[C]_{ij} = c_{ij}$) and if there is no arc from one vertex to the other the corresponding matrix element is zero. The diagonal elements of C are also zero as the j to j arcs are not defined in the graph. Matrix C can be interpreted as a weighted adjacency matrix of the social network. By definition every column j of C adds up to $1 - \widehat{c_j}$. The fact that the entries of C are between 0 and 1, and that the columns add up to less than 1 is enough to guarantee that the eigenvalues of C are within the unit circle. Matrix $\widehat{C}$ is a diagonal matrix with all the $\widehat{c_j}$'s in its main diagonal.

Example 1. *Consider the society that consists of three players* ($N = \{1, 2, 3\}$)*, Player* 1 *and* 2 *share 0.2 with each other while Player* 2 *and* 3 *share 0.1 with each other, consequently Player 1 keeps 0.8, Player 2 keeps 0.7, and Player 3 keeps 0.9 share of her income.*

The C matrix (weighted adjacency matrix) and $\widehat{C}$ diagonal matrix that corre-

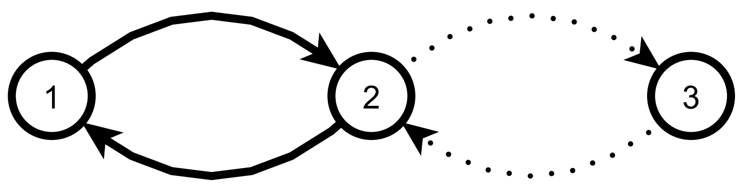

Figure 1: Society of Example 1

spond to Example 1 are:

$$C = \begin{pmatrix} 0 & 0.2 & 0 \\ 0.2 & 0 & 0.1 \\ 0 & 0.1 & 0 \end{pmatrix}; \quad \widehat{C} = \begin{pmatrix} 0.8 & 0 & 0 \\ 0 & 0.7 & 0 \\ 0 & 0 & 0.9 \end{pmatrix}.$$

Example 1 illustrates spill over and feed backs of power in the society. Suppose that Player 1 is the elected representative and she gives up a part of her political power (represented by $c_{12} = 0.2$) appointing Player 2 to a decision making position. In the same time she also wins informal influence over the position she appointed Player 2 (represented by $c_{21} = 0.2$). Similarly Player 2 as appointed decision maker gains informal influence by her appointee (Player 3) over a lower level position which means a second degree informal influence for Player 1.

In the model all these formal and informal influences are represented by a series of money transfers: matrix C describes the transfer rule in each round. The initial income that comes from outside of the society is received by the winner of the election – the identity matrix describes this step for every potential winner. Then the income is distributed according to C (first degree informal influence), for example if Player 1 wins the election and receives a prize of value 1, she transfers 0.2 to Player 2, keeping 0.8 for herself. Then the beneficiaries of the first round share their income in the second round according to matrix C^2 (second degree informal influence), in this example Player 2 transfers $0.2 * 0.2 = 0.04$ back to Player 1 and $0.2 * 0.1 = 0.02$ to Player 3, keeping 0.14 for herself. Matrix C^3 describes the third round (third degree informal influence), and so on. Since the eigenvalues of C are within the unit circle $\sum_{k=0}^{\infty} C^k$ converges to $(I - C)^{-1}$. The final distribution of political power (represented by monetary wealth) can be summarized as:

$$A = \widehat{C}(I - C)^{-1}, \tag{1}$$

where every column represents a different distribution, e.g. column j represents the distribution if j receives the prize of value 1 from outside of the society. This means that in matrix A the element $a_{ij} = [A]_{ij}$ stands for the final income that i receives (after accounting for all the back and forth transfers) if j wins the prize. In the ith row of A player i can directly compare her potential income in case of the victory of each column player and thus she can derive her preferences over the candidates: player i prefers the victory of j to the victory of k if $a_{ij} > a_{ik}$. For further insight about the calculation of matrix A see Elliott et al. [2014].

Note that matrix C and matrix A are two different tools to describe the same underlying society and there is an unambiguous mapping between them. The columns of matrix A can be interpreted as distributions since A is *column stochastic*, so the a_{ij} entries are between 0 and 1 and every column j add up to 1. Matrix A has an other appealing property that is going to be very useful in the later analysis, it is *diagonally majored*, so the diagonal elements are the maximal elements of each row (see Lemma 1).

Lemma 1. *For every $i, j \in N$: $0 < a_{ij} < 1$, $\sum_i a_{ij} = 1$ and $a_{ii} = \max_{j \in N} a_{ij}$.*

In Example 1, matrix A can be calculated as:

$$A = \widehat{C}(I - C)^{-1} = \begin{pmatrix} 0.834 & 0.168 & 0.017 \\ 0.147 & 0.737 & 0.074 \\ 0.019 & 0.095 & 0.909 \end{pmatrix},$$

which means that if Player 1 wins a prize of value 1 the final wealth (power) distribution is 0.834 to Player 1, 0.147 to Player 2 and 0.019 to Player 3.

By definition matrix A can be decomposed to the diagonal matrix $\widehat{C}$ and the matrix $T = (I - C)^{-1}$. We can be sure that the order of the elements in every row of T is the same as in the corresponding row in A since the left multiplication by the diagonal matrix $\widehat{C}$ multiples every row i with the constant $\widehat{c_i}$. I introduce $t_{ij} = [T]_{ij}$ as the *preference intensity* of i over the victory of j: t_{ij} is proportional to a_{ij} and it shows how much money goes through i if j wins the election – as i keeps a $\widehat{c_i}$ share of all the money that she receives, the factor of proportionality between t_{ij} and a_{ij} is $\widehat{c_i}$. Lemma 2 shows that the preferences of a voter i can be related to the preference intensities of her neighbors in the social network C:

Lemma 2. *The preference intensity of a player i over a candidate j, $t_{ij} = \sum_{k \in N} c_{ik} t_{kj}$.*

Lemma 2 tells us that the preference intensities of a player i can be written as the weighted sum of the preference intensities of the neighbors of i and the weights are equal to the weight of the arcs pointing from every single one of her neighbors towards i. This relationship can be generalized to social groups. Let G be a set of players, then B_G is the bordering set of G if $B_G \subset N$ and $G \cap B_G = \emptyset$ and for every *border player* $b \in B_G$ there is at least one $i \in G$ such that $c_{ib} \neq 0$. The vector of preference intensities of the border players in B_G over a candidate j is t_{Bj}.

Lemma 3. *The preference intensities of any $i \in G$ over a candidate $j \in (N \setminus G)$, is a linear function of the vector t_{Bj}, $t_{ij} = f(t_{Bj})$.*

The most important consequence of Lemma 3 is a special structure of inheritance of preferences in the network. Suppose there is a group G in the society, let B_G be its corresponding bordering set and let j and k be two candidates. If $t_{ij} > t_{ik}$ for every $i \in B_G$ and every border player in B_G prefers j to k then $t_{lj} > t_{lk}$ for every $l \in G$ so all the voters in G will also prefer j to k. This structure of inheritance of preferences is particularly important in networks where it is easy to define G groups whose bordering set is a singleton: if there is no candidate from group G all the voters in G will derive their voting decision from the preferences of a single border player in B_G.

2.2 Voting game

After the careful definition of the social interactions I introduce the voting game to the model. The voting game is a competition for a prize of value 1 where payoff-maximizing and risk neutral actors play strategically and the winner is decided by the votes of all players. The winner of the election receives the prize that she shares with her neighbors in the social network, her neighbors do likewise with the transfers they receive from the winner, and so on, resulting in a final wealth distribution equal to the winner's corresponding column in matrix A. Consequently the voters' preferences over the competing candidates can be derived directly from matrix A.

The model follows the citizen-candidate literature: candidates are a self-selected group of players who decided to pay the cost of participating in the election. This running fee β is positive but substantially less than the prize ($0 < \beta << 1$). The group of candidates is represented by the binary vector $\mathbf{r}$, element $r_i = 1$ if player i decides to run for office and 0 otherwise. The election is decided by simple majority rule, in case of a tie in votes the winner is chosen with equal probability from the tying candidates. The expected outcome of the election is represented by the probability vector $\mathbf{e}$, element $e_i = 1$ if i wins the election alone, it is $e_i = 1/m$ if i is one of the

m tying winners and 0 otherwise.

On the election all players vote sincerely for one of the candidates. Sincere voting means that the voters vote for the candidate that "offers" the most according to the corresponding row of A, and chooses with equal probability in case of a tie, e.g. player i votes for candidate j in an election where j is in the candidate pool R and $a_{ij} = \max_{l \in R} a_{il}$. Since voting is defined as a non-strategic action[1], the only strategic move in the game is the decision on running for office. The social structure, A, and the strategy profile, $\mathbf{r}$, together determine the (expected) political outcome, $\mathbf{e}$. If no player decides to run for office the default outcome is implemented and its corresponding payoff vector is $\boldsymbol{\pi}_\mathbf{0}$ – throughout this paper I assume that the default option is undesired by everyone:

Assumption 2. *The default option $\boldsymbol{\pi}_\mathbf{0} < \boldsymbol{\pi}(\mathbf{r})$ if there is $i \in N$ such that $r_i \neq 0$.*

On the other hand if there is at least one candidate running the payoff vector is given by:

$$\boldsymbol{\pi}(\mathbf{r}) = A\mathbf{e}(\mathbf{r}) - \beta\mathbf{r}. \tag{2}$$

The social structure C is commonly known across the society. The timing of the game is the following: (1) players simultaneously make their running decision; (2) all the players vote for one of the candidates; (3) winner is announced, and the payoff realized. The solution of the model is a correspondence from the social structure represented by A to an equilibrium candidacy vector $\mathbf{r}$. The equilibrium concept in this essay is *pure strategy Nash equilibrium*, where the equilibrium is defined as a situation where no candidate in the candidate pool R could gain by withdrawing from the election and no non-candidate could gain by running on the election, supposing that no other player changes strategy.

3 Results

3.1 Condorcet winners

First I need to introduce the notation of the following *majority relation*: $j \succ_{maj} k$ means that candidate j wins with the majority of the votes against candidate k in a

[1]Sincere voting is not a strategic decision in general, however it is the best response to the opponents voting decisions if noone follows a *weakly dominated strategy* and there are *maximum 2 candidates* on the ballot (see Besley and Coate [1997]).

2-candidate election. Formally $j \succ_{maj} k$ if

$$|\{i : i \in N; a_{ij} > a_{ik}\}| > |\{i : i \in N; a_{ik} > a_{ij}\}|, \tag{3}$$

the number of players preferring j to k is bigger than the number of players preferring k to j. With the help of this relation I can define the *strong* and *weak Condorcet winners*:

Definition 1. *Player i is a* ***strong Condorcet winner*** *in a network C if $i \succ_{maj} k$ for every $k \in N \setminus i$. Player j is a* ***weak Condorcet winner*** *if there is no $k \in N$ such that $k \succ_{maj} j$, but there is at least one player l such that $j \not\succ_{maj} l$.*

Intuitively a player is a strong Condorcet winner if she can *win* and she is a weak Condorcet winner if she can *win or tie* against every other potential candidate in a 2-candidate election. In Example 1 Player 2 is a strong Condorcet winner, if she runs against Player 1 both candidates vote for herself and Player 3 breaks the tie voting for Player 2 ($a_{32} = 0.095$ versus $a_{31} = 0.019$), similarly if she runs against Player 3 Player 1 is the tie breaker who votes for Player 2 ($a_{12} = 0.168$ versus $a_{13} = 0.017$).

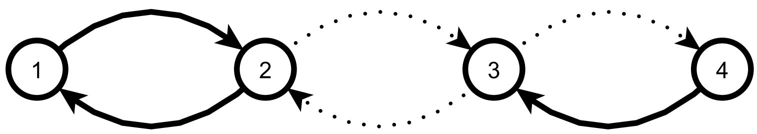

Figure 2: Weak Condorcet winners

Example 2. *Consider the society from Figure 2 with the corresponding C and A matrices:*

$$C = \begin{pmatrix} 0 & 0.2 & 0 & 0 \\ 0.2 & 0 & 0.1 & 0 \\ 0 & 0.1 & 0 & 0.2 \\ 0 & 0 & 0.1 & 0 \end{pmatrix}; \quad A = \begin{pmatrix} 0.834 & 0.168 & 0.017 & 0.003 \\ 0.147 & 0.737 & 0.075 & 0.015 \\ 0.017 & 0.086 & 0.825 & 0.165 \\ 0.002 & 0.009 & 0.083 & 0.817 \end{pmatrix}.$$

In Example 2 Player 2 and 3 are weak Condorcet winners: when they run in a 2-candidate election, Player 1 votes for 2 and Player 4 votes for 3. They are the only weak Condorcet winners: Player 1 would lose against 2 (supported by Player 3 and 4) and Player 4 against 3 (supported by Player 1 and 2) in a 2-candidate election.

Proposition 1. *For sufficiently low values of β, Player i is a **strong Condorcet winner** in a network C if and only if $\mathbf{r} = \{r_l = 1,\ if\ l = i\ and\ r_l = 0\ otherwise\}$ and $\mathbf{e} = \{e_l = 1,\ if\ l = i\ and\ e_l = 0\ otherwise\}$ is a Nash equilibrium.*

Proposition 1 tells us that if there is equilibrium with only one candidate running for office then this candidate is a strong Condorcet winner. Also, if there is a strong Condorcet winner in the society it is always equilibrium that she runs alone and wins the election unopposed (however, not necessarily the only equilibrium). The restrictions on β are only necessary for the "only if" part, the "if" part is also true without the restriction (see proof in Appendix A). Proposition 1 implies that if β is low enough the equilibrium with only one candidate running cannot exist if there is no strong Condorcet winner in the network (e.g. if there is only a weak Condorcet winner there is no 1-candidate equilibrium). Example 1 provides a good illustration of Proposition 1. In that case it is a Nash equilibrium that Player 2 runs alone: Player 1 or 3 has no incentive to destroy the equilibrium by running (since both of them would lose against Player 2), and Player 2 does not have incentive to quit (Assumption 2).

Proposition 2. *For sufficiently low values of β if Player i and j are the only 2 **weak Condorcet** winners in a network C and any potential third candidate k (such that $a_{ki} > a_{kj}$) could attract voters only from the electorate of i if entered the competition, then $\mathbf{r} = \{r_l = 1,\ if\ l \in \{i,j\}\ and\ r_l = 0\ otherwise\}$ and $\mathbf{e} = \{e_l = 0.5,\ if\ l = \{i,j\}\ and\ e_l = 0\ otherwise\}$ is a Nash equilibrium.*

According to Proposition 2 if there are two weak Condorcet winners in the society it is equilibrium that the two of them runs against each other and ties in votes. The conditions of Proposition 2 on a third candidate k always hold in the networks where the electorates of i and j are disconnected or only connected by i and j.

Proposition 2 applies to Example 2: it is a Nash equilibrium if Player 2 and 3 run, since (supposing that the running costs are not too high) they are better off tying against each other than quitting and letting the other win, and Player 1 would not run since if she did so she would split the electorate of her favored candidate 2 and would make 3 win (Player 4 will not destroy the equilibrium for the same reason).

In this paper, as in many other papers from the the citizen-candidate literature, the multiplicity of equilibria is an issue to be addressed. In the following sections I will focus on equilibria with Condorcet winners running for office. One reason to believe that these are *focal equilibria* is that they are *socially optimal* in the networks with one or two Condorcet winners. Running fee is a waste from a social point of view, however if nobody runs on the election the resulting situation is even worst (Assumption 2). Consequently the equilibrium with the least possible (but not zero) candidates involved is socially optimal. From Proposition 1 we know that if β is low enough the equilibrium with the strong Condorcet winner running is the only 1-candidate, thus socially optimal, equilibrium. Proposition 1 also tells that there cannot be a 1-candidate equilibrium if there are 2 weak Condorcet winners in the network. As there cannot be a 1-candidate equilibrium, any 2-candidate equilibrium is socially optimal, for example the equilibrium described in Proposition 2.

3.2 Single peaked preferences in chain and tree networks

In this part I study two simple network classes, the chain and the tree networks. I show that in these networks the voters preferences follow a particular pattern: they are single peaked. Single peakedness of the preferences is an important property that guarantees the existence of Condorcet winners in these networks. To follow my agenda first I need to start with some definitions:

Definition 2. *A **path** between the vertices i and j is an ordered set of vertices M (with the order $\psi_M : 1 < 2 < 3 < \ldots < m$ where $i = 1$ and $j = m$) such that $c_{k,k-1} \neq 0$ and $c_{l,l+1} \neq 0$ for every $k \in \{2, 3, \ldots, m\}$ and $l \in \{1, 2, \ldots, m-1\}$.*

*A **walk** from i to j is an ordered set of vertices W (with the order $\psi_W : 1 < 2 < 3 < \ldots < w$ where $i = 1$ and $j = w$) such that $c_{l,l+1} \neq 0$ for every $l \in \{1, 2, \ldots, m-1\}$.*

The *length* of the path (denoted by λ) equals to the number of vertices involved in the path. The vertices i and j are said to be *connected* if there is a walk from i to j and from j to i. A set of vertices is a *connected set* if every pair of vertices in the set is connected. Note that there is a path between i and j if they are connected and the walk from i to j and the one from j to i go through the same set of vertices in reverse order.

Example 3. *Consider the network from Figure 3.*

Figure 3 provides an example of a path between the terminal vertices of 1 and 6. Neither vertex X nor the arcs pointing to and from X are involved in the path. In Example 3 the 1 to 6 path has $\lambda_{16} = 6$ and the network is connected even if there is no path between X and the rest of the vertices.

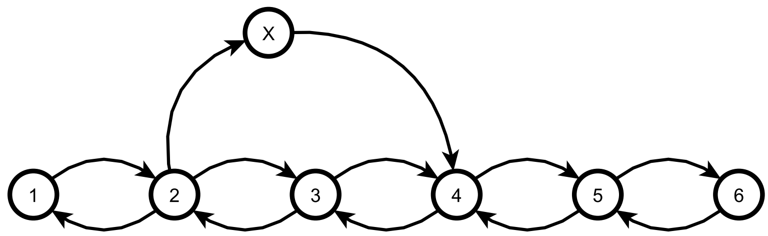

Figure 3: Example of a path

With the help of the path it is very easy to define the chain and the tree networks. The chain networks form one of the simplest network classes.

Definition 3. *A network is a* ***chain*** *if there is an order* $\psi_N : 1 < 2 < 3 < \ldots < n$ *over* N *such that there is a path between Player 1 and Player* n *and all the arcs of the network are involved in it.*

For examples of chain networks see Example 1 and 2.

The tree network is a bit more complex than the chain.

Definition 4. *A network is a* ***tree*** *if between every pair of* $i, j \in N$ *there is exactly one path and the collection of paths contains all the arcs of the network.*

For an illustration of a tree network see Example 4.

Example 4. *Consider the network from Figure 4. The straight links of the graph refer to a pair of arcs pointing in opposite directions and they are uniformly weighted:* $c_{12} = 0.1$, $c_{21} = 0.1$ *and so on.*

After defining the chain and the tree networks I need to introduce the concept of single peaked preferences.

Definition 5. *The preferences of Player* i *are* ***single peaked*** *over a set of potential candidates* R *in a network* C *if there is an order* $\psi_R : 1 < 2 < 3 < \ldots < r$ *over* R*, such that* i *has a most preferred choice* $i^{pref} \in R$ *(*$a_{i,i^{pref}} = \max_{l \in R} a_{il}$*), and* $i^{pref} < j < k$ *or* $k < j < i^{pref}$ *implies that Player* i *prefers* j *to* k *for every* $j, k \in R$*.*

The second condition of the definition says that every i^{pref} splits the order into two halves. The preferences are single peaked if every i prefers the alternative j over k when the two alternatives are on the same half of the order and j is “closer” to i^{pref}. Single peakedness as a network property can be defined in the following

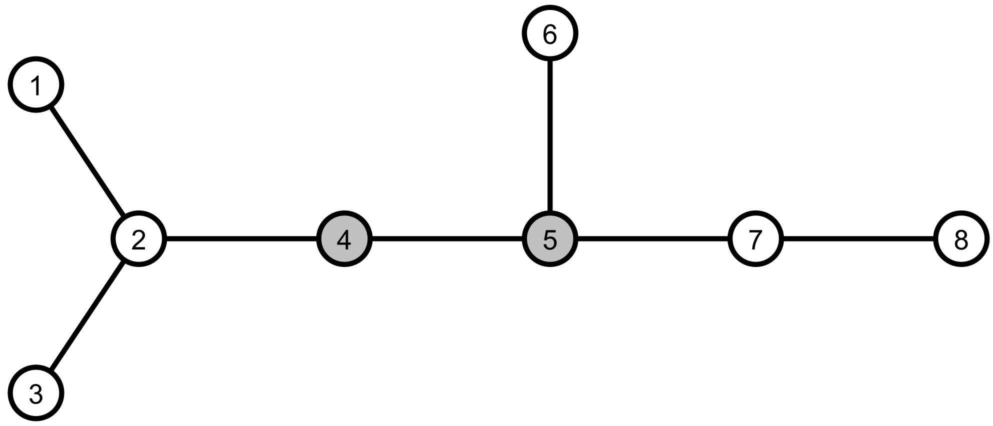

Figure 4: Example of a tree network

way: preferences are single peaked in a chain if every player $i \in N$ has single peaked preferences over N. The preferences are single peaked in a tree if every player $i \in N$ has single peaked preferences over M_{jk} – the set of vertices forming the path between j and k – for every $j, k \in N$.

Proposition 3. *In a chain or a tree network the preferences are single peaked.*

The proof of Proposition 3 is based on the special relationship between the preferences of the neighboring players described in Lemma 3. The consequence of this relationship is that every player prefers a candidate that is "closer" in the order to a candidate that is "further". Note that in this model single peaked preferences emerge naturally – I do not have to impose them.

Thanks to the single peaked preferences I can apply the median voter theorem in a chain network. The theorem says that the player in the median position is a strong Condorcet winner if n is odd and the two neighboring players in the median position are weak Condorcet winners if n is even. The other benefit of having single peaked preferences in the chain is that in this network election is equivalent to the election in the models of the citizen-candidate literature with finite electorate (e.g. Besley and Coate [1997], Cadigan [2005]).

Single peaked preferences are enough to guarantee the existence of Condorcet winners in a tree as well. Demange [1982] in her seminal paper shows that any society can be represented as a tree if the voters have single peaked preferences over all the *available* sets of choices (Theorem 1) and in this case the tree has a *core*, a set of Condorcet winners (Theorem 2). Proposition 4 summarizes the median voter theorem, the existence theorem (Theorem 2) of Demange [1982] and the new result of this paper about the relative position and the number of the weak Condorcet winners in a tree network.

Proposition 4. *In every chain and tree network there is either one strong Condorcet winner or there are two neighboring weak Condorcet winners.*

The proof of the only new result of Proposition 4 (there are maximum two Condorcet winners in a tree and they are neighboring) is based on the idea that if two candidates are not immediate neighbors in the tree then there is a third player on the path between them who can beat at least one of the original candidates in a 2-candidate election. This means that if there are two Condorcet winners they must be neighbors and also that there cannot be more than two Condorcet winners because 3 or more players cannot be immediate neighbors of each other in a tree network.

In Example 4 Player 4 and 5 are the two neighboring weak Condorcet winners. If they run against each other they tie with 4 votes each, while Player 4 can beat players 1 to 3 with the support of 5 to 8, and similarly Player 5 can beat players 6 to 8 with the support of 1 to 4.

At the end of this section I describe the *leaf-to-core navigation* method which is a general algorithm to find the Condorcet winners in a tree network (with $n \geq 3$).

Method 1 (**Leaf-to-core navigation**). *Choose a starting player, she can be any player with only one neighbor. In the first step move to the (only) neighbor of the starting player. By definition the second player separates the tree into several (2 or more) groups of players that are connected only through her. If all these groups are smaller than the rest of the network combined then stop. If there is a giant group that is larger or equal than the rest of the network combined, in the second step move to the neighbor from the giant group. Keep moving by the same rule as long as it is possible without moving back to the previous player.*

Proposition 5. *If $n \geq 3$ the leaf-to-core navigation stops at i because either (1) it separates the network into groups that are smaller than the rest of the network combined if and only if i is the strong Condorcet winner in C; or (2) the next step would be moving to the previous player j if and only if i and j are weak Condorcet winner in C.*

3.3 Bridge and windmill networks

In this part I move on to more complex network structures, but before defining the bridge and the windmill networks I have to introduce the *reach* of a vertex i.

Definition 6. *The set of vertices $\mathcal{R}_i$ is is called the* ***reach*** *of vertex i if there is a walk from i to every $j \in \mathcal{R}_i$.*

Intuitively the reach of a vertex i is the set of all the vertices that can be visited starting from i. The number of vertices in the reach of i is ρ_i. Reach is a more general concept than connectedness, since in this case it is not required to have a way back from the vertices in $\mathcal{R}_i$ to i. The reach of i contains all the vertices that receive money if i wins a prize: $a_{ki} > 0$ for every $k \in \mathcal{R}_i$ and $a_{ki} = 0$ for every $k \notin \mathcal{R}_i$.

With the help of the reach I can introduce the class of bridge networks. Bridge networks have been studied both in the literature of sociology and economics (see e.g Calvó-Armengol and Jackson [2004]) as an example of a social structure with two social groups that are minimally connected between each other.

Definition 7. *C is a* ***bridge*** *network if it has a pair of vertices $i, j \in N$ such that $c_{ij} \neq 0$, $c_{ji} \neq 0$, $\mathcal{R}_i = \mathcal{R}_j = N$ and if the i to j and the j to i arcs are removed the resulting network $\widetilde{C}$ ($c_{kl} = \widetilde{c}_{kl}$ for every $\{k, l\} \in N^2 \setminus \{\{i, j\}\{j, i\}\}$ and $\widetilde{c}_{ij} = \widetilde{c}_{ji} = 0$) is such that $\widetilde{\mathcal{R}}_i \cap \widetilde{\mathcal{R}}_j = \emptyset$. In this case Player i and j are called* ***bridge players****.*

A direct consequence of Definition 7 is that for every $k \in N$ such that $i, j \in \mathcal{R}_k$ there is always an $l \in N$ such that every walk from k to l goes through i and j ($i, j \in W_{k \to l}$), and for every $f \in N$ such that $i, j \notin \mathcal{R}_f$ there is always a $h \in N$ ($h \neq i$ and $h \neq j$) such that $h \notin \mathcal{R}_f$. I refer to the sets of players $\widetilde{\mathcal{R}}_i$ and $\widetilde{\mathcal{R}}_j$ as social groups G_1 and G_2, and the number of players involved in the groups is denoted by g_1 and g_2, respectively.

Example 5. *Consider the network from Figure 5.*

Figure 5 contains thick straight links that refer to a pair of arcs pointing in opposite directions e.g. the link between Player 1 and 2 means that $c_{12} \neq 0$ and $c_{21} \neq 0$. It also has arched arrows, that refer to a one-sided relationship, e.g. the arrow between Player 1 and 3 means that $c_{31} \neq 0$ but $c_{13} = 0$. The graph is an example of a bridge network, Player 5 and 6 are the *bridge players*, since they are linked in both ways and they reach the whole network however if the arcs connecting them are removed the network falls into two parts. These two parts are the social

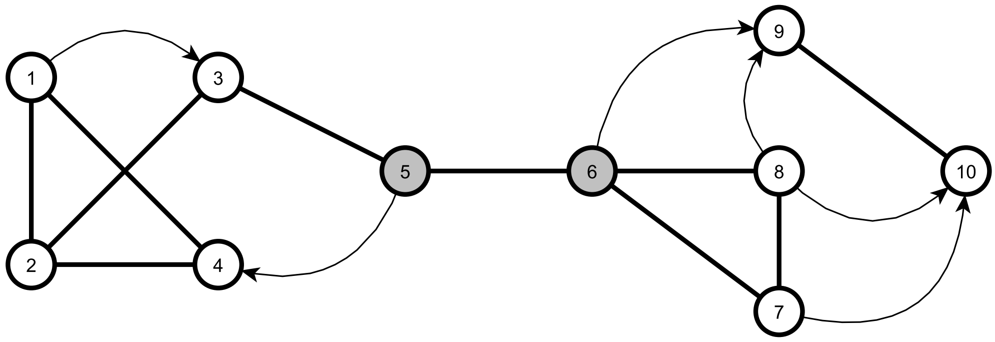

Figure 5: Example of a bridge network

groups in the graph: G_1 consists of Player 1 to 5 and G_2 consists of Player 6 to 10 and each have 5-5 players ($g_1 = g_2 = 5$). Note that players 9 and 10 are not connected to the bridge players, but they are still in their reach.

Proposition 6. *In a bridge network where the two social groups G_1 and G_2 are of equal size ($g_1 = g_2$) the bridge players are the only two weak Condorcet winners.*

Intuitively, both groups delegate its strongest candidate – the candidate that can beat any other player from that group in a 2-candidate election – and if the two groups have the same size the two candidates tie. A bridge network where the two groups have equal size is called *balanced* (e.g. a chain with *even number* of vertices or a tree with two weak Condorcet winners).

In a windmill network the connection among social groups is as limited as in a bridge network.

Definition 8. *C is a **windmill** network if it has a vertex $i \in N$ such that $\mathcal{R}_i = N$ and if i and its incoming and outgoing arcs are removed the resulting network $\widetilde{C}$ ($c_{kl} = \widetilde{c}_{kl}$ and $\widetilde{c}_{ki} = \widetilde{c}_{ik} = 0$ for every $k, l \in N \setminus \{i\} = \widetilde{N}$) is such that $\widetilde{N} \setminus \widetilde{\mathcal{R}}_k \neq \emptyset$ for every $k \in \widetilde{N}$. In this case Player i is called **hub player**.*

A direct consequence of Definition 8 is that for every $k \in N$ such that $i \in \mathcal{R}_k$ there is always an $l \in N$ such that every walk from k to l goes through i ($i \in W_{k \to l}$), and for every $f \in N$ such that $i \notin \mathcal{R}_f$ there is always a $h \in N$ ($h \neq i$) such that $h \notin \mathcal{R}_f$. I refer to the sets of players in the different $\widetilde{\mathcal{R}}_k$'s as social groups $G_1, G_2, \ldots, G_{n-1}$ or the arms of the windmill network. The number of players in group G_k is denoted by g_k.

Example 6. *Consider the network from Figure 6.*

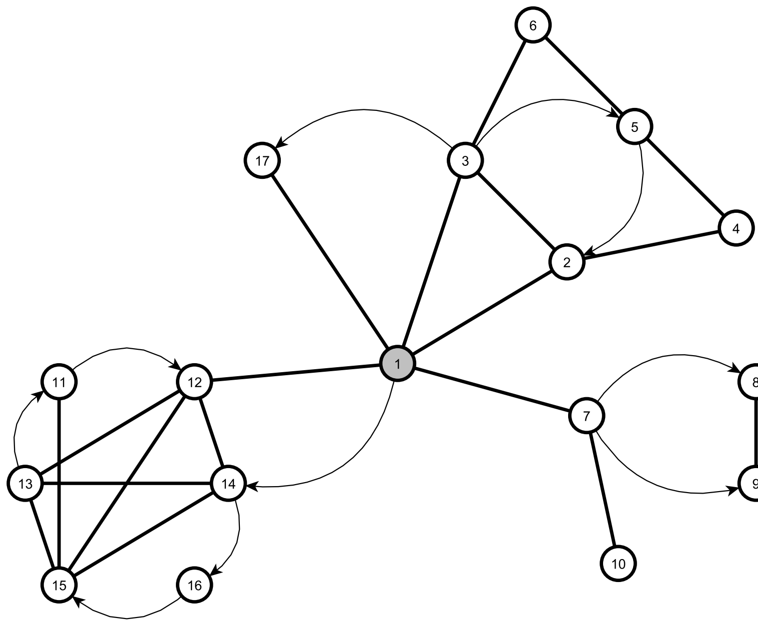

Figure 6: Example of windmill network

In Figure 6 the interpretation of the thick links and the arrows is the same as in Figure 5. The graph is an example of a windmill network, Player 1 is the *hub player* since reaches the whole network and if she is removed none of the players can reach all the remaining players in the resulting network. Note that players 8 and 9 are not connected to the rest of the network but they are still in the reach of Player 1.

Proposition 7. *In a windmill network the hub player is the strong Condorcet winner if* $g_k < n - g_k$ *for every social group* G_k *and player* $k \in \widetilde{N}$.

The hub player is a compromise-candidate in this situation: every group prefers her to any member of the other groups. If none of the groups is large enough to elect one of its members without the support of the other groups ($g_k < n - g_k$ is the sufficient condition), the hub player is a strong Condorcet winner. A windmill network where there is no group bigger than the rest of the network combined is called *balanced* (e.g. a chain with *odd number* of vertices or a tree with one strong Condorcet winner).

3.4 Isostructural networks

In this part I show that the balanced bridge and windmill networks are exceptional as they are the only networks where the equilibrium is robust to changes in the intensity of the connections. I need to introduce first the concept of *structure*.

Definition 9. *The* ***structure*** *of a network C is an $n \times n$ matrix S such that $c_{ij} > 0$ implies that $s_{ij} = 1$, and $c_{ij} = 0$ implies $s_{ij} = 0$ for every $i, j \in N$.*

Matrix S is the unweighted version of C, it is a binary matrix with entry 1 for every arc, independently of its weight.

With the help of structure I can define the *isostructural* networks.

Definition 10. *Networks C' is* ***isostructural*** *to C if they have the same structure S.*

To analyze how much a certain equilibrium depends on the actual weights of the arcs in a network C it is useful to take a look at the collection of all the networks isostructural to C. Proposition 8 and 9 shows that the networks where the equilibrium is robust to the changes in the intensity of the connections can be described by two network classes.

Proposition 8. *Player i is a strong Condorcet winner in C and in every network C' isostructural to C, if and only if $\mathcal{R}_i$ is a balanced windmill where i is the hub player and (1) $\rho_i > \rho_k$ for every $k \in N$ such that $\mathcal{R}_i \cap \mathcal{R}_k = \emptyset$; (2) and $\rho_l > 2\widetilde{\rho}_l$ for every $l \in N$ such that $\mathcal{R}_i \subset \mathcal{R}_l$.*

$\widetilde{\mathcal{R}}_l$ is defined as earlier: it is the reach of l in an alternative network where the hub player i is removed, and $\widetilde{\rho}_l$ is the number of players in $\widetilde{\mathcal{R}}_l$. The second condition of Proposition 8 holds if the reach of l reduces to less than the half if i is removed from the network, or in other words if more than half of $\mathcal{R}_l$ can only be reached from l through i.

Proposition 9. *Player i and j are weak Condorcet winners in C and in every network C' isostructural to C, if and only if $\min(\rho_i, \rho_j) > \rho_k$ for every $k \notin (\mathcal{R}_i \cup \mathcal{R}_i)$ and either (1) $\mathcal{R}_i = \mathcal{R}_j$ is a balanced bridge and i and j are the bridge players; or (2) $\mathcal{R}_i \cap \mathcal{R}_j = \emptyset$ and $\mathcal{R}_i$ and $\mathcal{R}_j$ are balanced windmills, $\rho_i = \rho_j$ and i and j are the hub players; or (3) $\mathcal{R}_i \subset \mathcal{R}_j$, such that $\rho_i = \frac{1}{2}\rho_j$, and it is a balanced bridge-windmill hybrid: from j's point of view it is a balanced bridge as every walk from j to any $l \in \mathcal{R}_i$ goes through the c_{ij} arc, $\mathcal{R}_i$ is a balanced windmill.*

Proposition Proposition 8 and 9 tell us that the only networks where the Condorcet winners are robust to the changes in the intensity of ties are the networks that contain large balanced bridges or balanced windmills.

4 Discussion

4.1 Centrality and Condorcet winners

In this part I want to explain how does the analysis based on centrality measures (as in Cruz et al. [2015] for example) relate to the approach of modeling the voters' behavior. The main advantage of working with centrality measures, like betweenness or eigenvector centrality, is that it is quick and easily to define a measure for all players in any networks, and the measure has predictive power. Cruz et al. [2015] show on Philippine municipal data that villagers with high family network centrality tend to run and win local elections with higher probability. This is a convincing evidence of the *positive correlation* between centrality and electoral victory. However the analysis based on centrality measures does not take into account the candidates' position in the network relative to each other, only the centrality of each candidate separately.

Example 7. *Consider the network Soc_1 from Figure 7.*

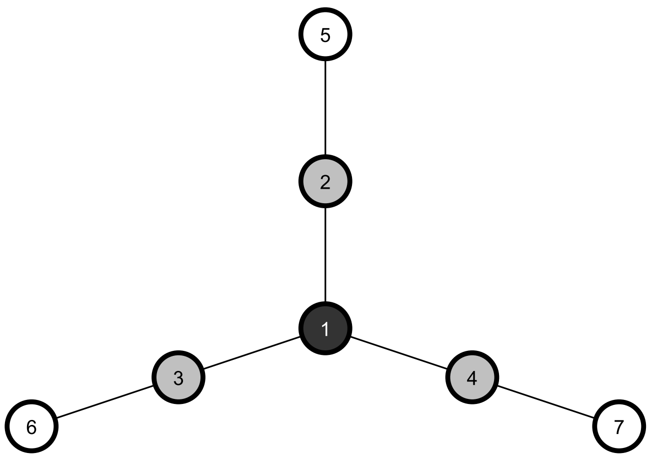

Figure 7: Peculiar ballot in Soc_1

Suppose that in the network Soc_1 Player 1 to 4 runs for office. In this case Player 1 is the most central player (betweenness centrality 12 versus 5; eigenvector centrality 0.25 versus 0.16667) but she is completely cut off from the voters by the other candidates so she will get only 1 vote (her own) and lose while Player 2, 3 and 4 will tie on the first place with 2 votes each.

Although particular sets of candidates can lead to unexpected outcomes, the analysis of centrality is still very efficient in large samples of elections when the influence of these candidate sets is relatively low. For example Cruz et al. [2015] draw their conclusions based on the analysis of electoral information from 15.000 villages and 709 municipalities.

The other concern with the analysis based on betweenness and eigenvector centralities is that there are certain network structures that inflate the centrality of unlikely winners. Neither betweenness nor eigenvector centrality is designed to capture Condorcet winners and so they fail sometimes. In fact the analysis of the electoral situation based on a centrality measure would be a very efficient approach if there was a centrality measure that identified Condorcet winners with certainty. Unfortunately I do not know of any centrality measure that is designed to identify Condorcet winners in a network. To illustrate the problem with betweenness and eigenvector centrality I provide an example of a network where there is a strong Condorcet winner, but she is not the most central player according to either of the centrality measures.

Example 8. *Consider the network Soc_2 from Figure 8.*

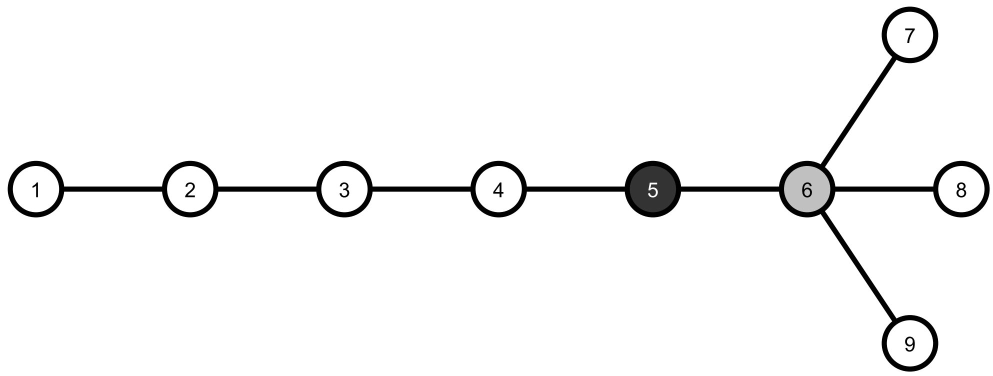

Figure 8: Betweenness centrality in Soc_2

The straight links in Figure 8 refer to a pair of arc pointing to opposite directions and they are uniformly weighted: $c_{12} = 0.1$, $c_{21} = 0.1$ and so on. Thanks to the uniform weights the centrality of the vertices can be calculated by the formula designed for unweighted, undirected networks: *betweenness centrality* of a vertex i is equal to the number of shortest paths from all vertices to all others that go through i. In the society Soc_2 the vector of betweenness centralities is $(0, 7, 12, 15, \mathbf{16}, \mathbf{18}, 0, 0, 0)$, so Player 6 is the most central player according to betweenness centrality. The vector of *eigenvector centralities* in a network is equal to the eigenvector that belongs to the leading eigenvalue of the adjacency matrix. In the network Soc_2 this vector is $(0.02, 0.05, 0.08, 0.11, \mathbf{0.17}, \mathbf{0.24}, 0.11, 0.11, 0.11)$, so Player 6 is the most central according to eigenvector centrality as well. However Player 5 is the strong Condorcet winner in the network, so she can beat Player 6 by 5 votes against 4 in a 2-candidate election.

4.2 Application: Rise of the Medici

In this part I give an example how the model works in a real life situation. I use the story of the Rise of the Medici family to show the predictive power of the approach of modeling the voters' behavior. This case is well documented by Padgett and Ansell [1993], and it has all the key elements of my model. The story takes place is the Florentine city-state in the early 15th century. The electorate is the Florentine nobility that formed a social network: the noble families were linked by marriage, business ventures, friendship, *etc.* The election is the standoff of 1433 when the Florentine nobility voted with their feet about the future of the city-state: the supporters and the opponents of the Medici family gathered with their arms on the Piazza della Signoria to put an end to a long power struggle – the standoff did not escalate to fight since the Medici opponents lost the "vote", they were heavily outnumbered.

To model voters' behavior I need to build and calibrate the social network. The data set I use for the calibration provides information on the marriage and business ties of 16 noble families of Florence who represent the elite of the Florentine nobility in terms of wealth and importance. There is anecdotal evidence by Padgett and Ansell [1993] that the marriage and business ties between families played the role that the model predicts: linked families helped each other and provided informal influence to each other in the same time. For example by coordinated voting Cosimo de' Medici managed to get his trusted friends elected to the key political positions of the city-state, who in turn paved the way for the Medici takeover. However the marital and the business ties had different importance: marriage meant a life-long binding and a symbolical merger of the two families, while business partnership in the 15th century Florence in most of the cases only meant one-time deals and guaranteeing for each other in credit agreements.

One thing that is worth to note about the calibration is that I follow Padgett and Ansell [1993] and take the family as the decision making actor of the society (not the individual). This is quite in line with the reality of the 15th century Florence, where the patriarch of each family had a high influence on the marital, business and political decisions of the household.

In Figure 9 I draw the social network based on the row data: business connections are represented with dotted line, marriage ties with a thin solid line and if there were both business and marriage ties between the families it is represented by a thick solid line. I dropped the Pucci family from the data set as they had neither marital nor business connections to any other family in the set. To build the weighted adjacency matrix C I have to choose the exact weights of each arc. In the election of the weights I considered three things: (1) two arcs connecting the same pair of families in different directions are equally weighted; (2) the sum of the weights pointing away from a family ($\sum_k c_{kj}$) should always be less than 1; and (3) the weight of the marriage tie is higher than the weight of the business tie.

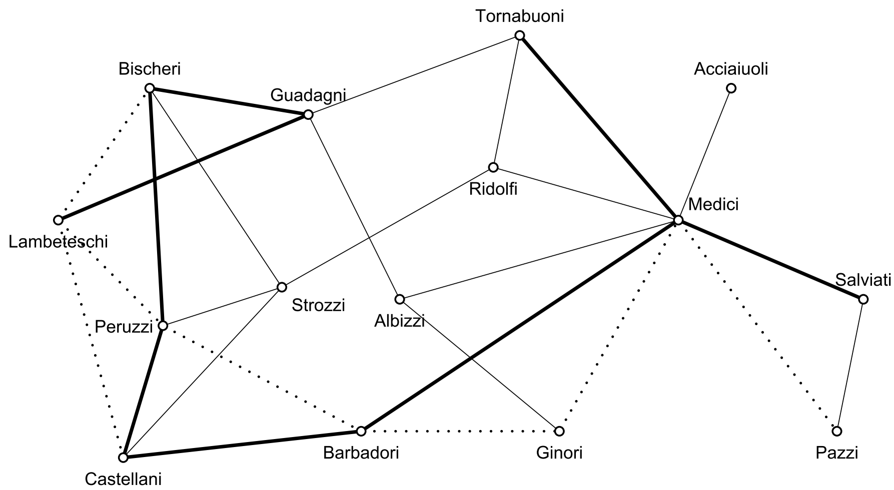

Figure 9: Florentine marriage network and business ties

I calibrated the network several times, trying a large set of different values for the β running fees, and a and b weights for marriage and business respectively (and used $a + b$ weight if the pair of families are connected both by marriage and business). With every different setting I checked all the possible 2-candidate elections and the result was always the same: the Medici family is the strong Condorcet winner in the network.

Many different evidences points to the direction of the Medici success. The Medici family has the highest eigenvector centrality measure in the network: 0.109 versos 0.097, which is the second highest eigenvector centrality of the Peruzzi family. They also have the highest betweenness centrality measure (in an unweighted version of the network): 43.3 versus 16.6, which is the second highest betweenness centrality of the Barbadori family. The high betweenness centrality of the Medici family was also pointed out by Jackson [2008], who uses this story as an illustrative example of the importance of the networks in everyday life.

To see that the Medici family is the strong Condorcet winner in the network I consider the potential challengers one by one. The families Acciaiuoli, Salviati and Pazzi are in the periphery of the network. These families are connected to the rest of the network through the Medici family their preferred candidate from the rest of the network is Cosimo, the head of the Medici family, according to the voting model. The situation is not so clear with the block on the left side of Figure 9 consisted by the Bischeri, Castellani, Lamberteschi, Peruzzi and Strozzi families. In fact these families were the backbone of the oligarchic rule that characterized Florence in the beginning of the 15th century and these were the families that opposed fiercely the rise of the Medici family. However, the oligarchs did not have the numbers to prevent the success of the Cosimo de' Medici. For example, in my model if the parameters are set $a = 0.1$, $b = 0.07$ and $\beta = 0.01$ and Medici runs against a Strozzi candidate the Medici wins the election (10 votes against 5). No closely linked family (from the center of the graph) can successfully challenge the Medici family either. For example the Albizzi family might seem to be in a position to potentially divide and rule between the oligarchs on the left side of Figure 9 and the Medici supporters on the right, but when an Albizzi candidate runs against the Medici candidate he loses election (5 votes against 10) because he cannot secure the full support of the oligarch block on the left of the graph.

These predictions describe very well the actual sequence of events in the 1430s. First the traditional oligarch rule was challenged when Cosimo de' Medici managed to win the seats of the city council for his supporters (by coordinated voting), then Rinaldo Albizzi, following his own political ambition, led a conspiracy against the Medici family (wanted to purge the city council from the Medici supporters by force) but could not secure the support of Palla Strozzi, the richest man of Florence for his maneuver and eventually had to retreat. Figure 10 shows how Padgett and Ansell [1993] classified the families according to their alliances: the opponents of the Medici are black, the supporters are white and the families with split loyalties (brothers or father-son on different sides) are grey. We can see that the model predicts quite well which family supported whom during the power struggle: the left block is entirely black, they were the main opponents of the Medici rule, the right block is white and the Barbadori and Albizzi families on the frontier of the two competing blocks have split loyalties. The split loyalty of the Salviati family is a bit of a surprise, as it cannot be explained by the ties I have data on. However it can well be the case that this occurs because this network contains only a sample of the families and only two types of ties that connects the families. It is very probable that this calibration misses actors or ties that influenced the political decision of one member of the Salviati family.

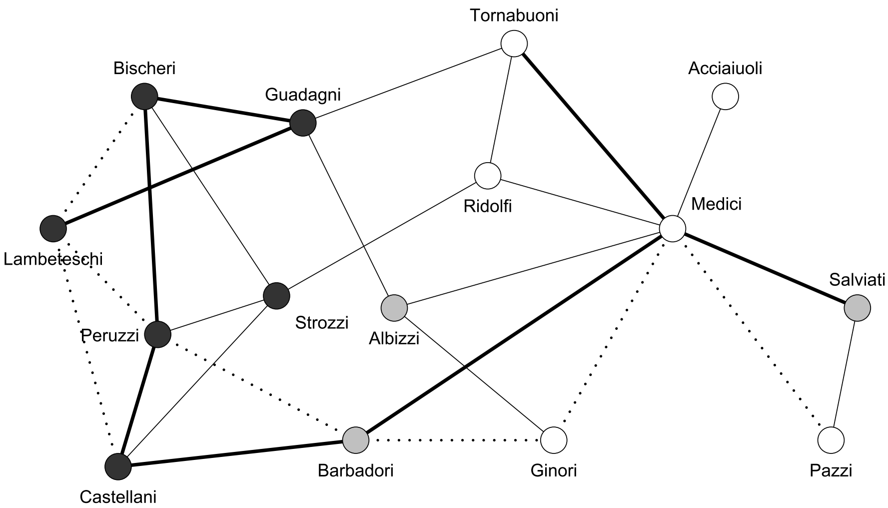

Figure 10: Florentine marriage network, business ties and alliances in the 1430s

5 Conclusion

This paper develops a model of electoral competition where the preferences in the electorate are not based on general political ideology: the voting decision of the players is entirely based on the interpersonal relationships in the society. The collection of these interpersonal relations is represented as a social network. In this framework I study the result of a modified version of the three stage citizen-candidate voting game. I show that the model guarantees single peaked preferences on the chain and the tree network, I identify Condorcet winners in bridge and windmill networks and prove that every network where the equilibrium is robust to the changes in the intensity of the ties should be either a bridge or a windmill network. In the discussion show that the strong Condorcet winner is not always the player with the highest betweenness or eigenvector centrality. Finally I provide a real life example where I use the model to reproduce real political outcome by calibrating the social network with marriage and business ties and finding the strong Condorcet winner of the Florentine elite of the 15th century.

I believe that there are further questions in this topic that can be studied. As political importance seems to be related to the players' position on the social network, it would be natural curiosity to ask how did the social network develop into a given architecture and how did the individuals' social decisions (forming new links or destroying old ones) affect this development. One possible way to answer that question is to endogenize the network formation, introducing a new stage in the voting game, when the self-selected candidates can revise their (endowment of) social connections and offer new links (or destroy existing ones) in order to maximize their probabilities of winning the election.

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A Appendix: Proofs

Proof. Lemma 1.

First part: Column stochastic. Since $\widehat{C}$ is a non-negative matrix and $(I-C)^{-1} = \sum_{i=0}^{\infty} C^i$ where C and all its higher powers are also non-negative matrices and the product of two non-negative matrix is also non-negative, so $a_{ij} \geq 0$ for every $i, j \in N$. Let $\mathbb{1}$ be the row vector of ones, and the size of the vector is determined by the context. The vector of the column sums in A is $\mathbb{1}A$ and $\widehat{C}$ can be written as $\mathbb{1}\widehat{C}$. Consequently:

$$\mathbb{1}A = \mathbb{1}\widehat{C}(I-C)^{-1} = \mathbb{1}(I-C)(I-C)^{-1} = \mathbb{1}. \tag{4}$$

If all the elements in A are non-negative and the columns add up to 1, then $a_{ij} < 1$ for every $i, j \in N$.

Second part: Diagonally majored. Let $\mathfrak{e}_i$ be the ith unit column vector (whose only non-zero element is the ith element which equals to 1) and the size of the vectors is determined by the context. The column sum condition on matrix C can be written as $\mathbb{1}C < \mathbb{1}$ which holds element-wise and the i, j element of matrix C can be written as $c_{ij} = \mathfrak{e}_i^T C \mathfrak{e}_j$ and is $A = \sum_{k=0}^{\infty} C^k$.

For the proof I need to introduce the notation $\tilde{C}$ for the upper left $n-1 \times n-1$ block of C and $\tilde{c}$ for the first $n-1$ elements of the last row of C and $\tilde{\gamma}$ for the first $n-1$ elements of the last column of C:

$$C = \begin{array}{|c|c|} \hline \tilde{C} & \tilde{\gamma} \\ \hline \tilde{c} & c_{nn} \\ \hline \end{array}$$

Furthermore let $\hat{C}$ of size $2n-1 \times 2n-1$ be

$$\hat{C} = \begin{array}{|c||c|}\hline C & \mathfrak{e}_n \tilde{c} \\ \hline 0 & \tilde{C} \\ \hline \end{array} = \begin{array}{|c|c||c|}\hline \tilde{C} & \tilde{\gamma} & 0 \\ \hline \tilde{c} & c_{nn} & \tilde{c} \\ \hline 0 & 0 & \tilde{C} \\ \hline \end{array}.$$

In the following I will refer to the blocks of $\hat{C}$ as $\hat{C}_{11} = \tilde{C}$, $\hat{C}_{21} = \tilde{c}$, and so on, and $[\hat{C}_{ij}]_{kl}$ is the k,l element in the block i,j (e.g. $[\hat{C}_{22}]_{11} = c_{nn}$).

The first step is to prove that

$$\hat{C}^k = \begin{array}{|c|c|}\hline C^k & \sum_{\ell=0}^{k-1} C^\ell \mathfrak{e}_n \tilde{c} \tilde{C}^{k-\ell-1} \\ \hline 0 & \tilde{C}^k \\ \hline \end{array} \tag{5}$$

and

$$\sum_{k=0}^{\infty} \hat{C}^k = \begin{array}{|c|c|}\hline \sum_{k=0}^{\infty} C^k & \sum_{k=0}^{\infty} C^k \mathfrak{e}_n \tilde{c} \sum_{\ell=0}^{\infty} \tilde{C}^\ell \\ \hline 0 & \sum_{k=0}^{\infty} \tilde{C}^k \\ \hline \end{array}. \tag{6}$$

Equation 5 comes from the matrix multiplication, and Equation 6 can be seen by changing the order of summations.

Second step of the proof is to show that

$$[C^k]_{n,i} = [\hat{C}^k]_{n,n+i}$$

for every $i = 1, 2, \ldots, n-1$ and $k \in \mathbb{N}$. From the previous step we know that this property is equivalent to $\hat{C}_{21}^k = \hat{C}_{23}^k$ for every $k \in \mathbb{N}$. Thanks to the property that the block $\hat{C}_{31}^k$ is zero for every k the elements of $\hat{C}_{21}^k$ ($[\hat{C}^k]_{n,i}$ for every $i \in \{1, 2, \ldots, n-1\}$) can be written as the product of the $1 \times n-1$ row vector $\hat{C}_{21}^{k-1}$ and the $n-1 \times 1$ column i of $\hat{C}_{11}^{k-1} = \tilde{C}^{k-1}$. In the same time the elements of $\hat{C}_{23}^k$ can be written as the product of the $1 \times n-1$ row vector $\hat{C}_{21}^{k-1}$ and $n-1 \times 1$ column i of $\hat{C}_{13}^{k-1}$ *plus* the $1 \times n-1$ row vector $\hat{C}_{23}^{k-1}$ and $n-1 \times 1$ column i of $\hat{C}_{33}^{k-1}$. Consequently if $\hat{C}_{21}^{k-1}$ and $\hat{C}_{23}^{k-1}$ are equal it is enough to show that $\hat{C}_{11}^k = \hat{C}_{13}^k + \hat{C}_{33}^k$ for every k.

Using induction first we can say that for $\hat{C}^0 = I_{2n-1}$, where I_h is the identity matrix of dimension $h \times h$, so all the off-diagonal elements of $\hat{C}^0$ are 0 – so $\hat{C}_{21}^0 = \hat{C}_{23}^0$ and $\hat{C}_{11}^0 = \hat{C}_{13}^0 + \hat{C}_{33}^0$. Now assume that for an $\ell \in \mathbb{N}$ the equations $\hat{C}_{21}^\ell = \hat{C}_{23}^\ell$ and

$\hat{C}_{11}^{\ell} = \hat{C}_{13}^{\ell} + \hat{C}_{33}^{\ell}$ hold. The i, j elements of the different blocks of $\hat{C}^{\ell+1}$ are

$$[\hat{C}_{11}^{\ell+1}]_{ij} = \sum_{f=1}^{n} [\hat{C}^{\ell}]_{if} [\hat{C}]_{fj} = [\hat{C}^{\ell}]_{in} [\hat{C}]_{nj} + \sum_{f=1}^{n-1} [\hat{C}^{\ell}]_{if} [\hat{C}]_{fj} \tag{7}$$

$$[\hat{C}_{13}^{\ell+1}]_{ij} = [\hat{C}^{\ell}]_{in} [\hat{C}]_{nj} + \sum_{f=1}^{n-1} [\hat{C}^{\ell}]_{i,f+n} [\hat{C}]_{fj} \tag{8}$$

$$[\hat{C}_{33}^{\ell+1}]_{ij} = \sum_{f=1}^{n-1} [\hat{C}^{\ell}]_{i+n,f+n} [\hat{C}]_{fj}, \tag{9}$$

so $[\hat{C}_{11}^{\ell+1}]_{ij} = [\hat{C}_{13}^{\ell+1}]_{ij} + [\hat{C}_{33}^{\ell+1}]_{ij}$ for every $i, j \in \{1, 2, \ldots, n-1\}$, thus $\hat{C}_{11}^{\ell+1} = \hat{C}_{13}^{\ell+1} + \hat{C}_{33}^{\ell+1}$ and $\hat{C}_{21}^{\ell+1} = \hat{C}_{23}^{\ell+1}$. By induction

$$[C^k]_{n,i} = [\hat{C}^k]_{n,n+i}$$

for every $k \in \mathbb{N}$.

Using the results of the previous two steps I have that

$$a_{ni} = \mathfrak{e}_n^T \sum_{k=0}^{\infty} C^k \mathfrak{e}_i = \mathfrak{e}_n^T \sum_{k=0}^{\infty} \hat{C}^k \mathfrak{e}_{n+i} = \underbrace{\mathfrak{e}_n^T \sum_{k=0}^{\infty} C^k \mathfrak{e}_n}_{a_{nn}} \tilde{c} \underbrace{\sum_{\ell=0}^{\infty} \tilde{C}^{\ell}}_{(I-\tilde{C})^{-1}} \mathfrak{e}_i \tag{10}$$

$$= a_{nn} \tilde{c} (I - \tilde{C})^{-1} \mathfrak{e}_i. \tag{11}$$

From $\mathbb{1}C < \mathbb{1}$ we have

$$\mathbb{1}\tilde{C} + \tilde{c} < \mathbb{1}$$

and consequently

$$\tilde{c} < \mathbb{1} - \mathbb{1}\tilde{C} = \mathbb{1}(I - \tilde{C})$$

Substituting this relation we further have

$$a_{ni} = a_{nn} \tilde{c} (I - \tilde{C})^{-1} \mathfrak{e}_i < a_{nn} \mathbb{1} (I - \tilde{C})(I - \tilde{C})^{-1} \mathfrak{e}_i = a_{nn} \mathbb{1} \mathfrak{e}_i = a_{nn} \tag{12}$$

for every $i = 1, 2, \ldots, n-1$. $\square$

Proof. Lemma 2. We can multiply the equation $T = (I - C)^{-1}$ by $(I - C)$ from the left, and the rearrange the equation to get

$$T = I + CT. \tag{13}$$

From Equation 13 I can write i's preference intensity over j as

$$t_{ij} = 0 + \sum_{k=1}^{n} c_{ik} t_{kj}. \tag{14}$$

Since c_{ik} is different from zero only if k is a neighbor of i, t_{ij} is equal to the weighted sum of preference intensities of the neighbors of i over j and the linear weights are the c_{ik} weights of the arcs. $\square$

Proof. Lemma 3. I introduce the scalar g as the number of players in G, and the $1 \times g$ vector t_{Gj} as the vector of the preference intensities of the players in G over j. If C and T are defined in a way that the first g rows and columns are dedicated to the players in G and the following rows and columns to the players in B_G then from Equation 13 we write

$$t_{Gj} = C_{GG} t_{Gj} + C_{GB} t_{Bj}, \tag{15}$$

where C_{GG} is the $g \times g$ upper left corner of matrix C that describes the connection structure of the social group G, C_{GB} is the $g \times b$ part of C that describes the set of arcs pointing towards the social group G from B_G and t_{Bj} stands for the vector of preference intensities of people in B_G. Equation 15 can be solved for t_{Gj} and the result is

$$t_{Gj} = (I - C_{GG})^{-1} C_{GB} t_{Bj}. \tag{16}$$

This expression tells us that the preference intensities of players in G only depend on the linear combination of the preference intensities of players in B_G. $\square$

Proof. Proposition 1.

Part 1: If Player i is a strong Condorcet winner, then her running alone is equilibrium. The only candidate i has no incentive to quit the race (Assumption 2). No other player j has incentives to run and destroy the equilibrium: Player i is a strong Condorcet winner so she would win independently of the candidacy of j. In this situation j would only waste the running fee but would not achieve any change in the outcome if she decided to run. If Player i does not want to quit and no other player wants to enter, the situation is equilibrium.

Part 2: If Player i running alone is equilibrium, then she is a strong Condorcet winner. For the second part of the proof I have to impose a restriction on the running fee:

$$\beta < \frac{a_{jj} - \max_{k \neq j} a_{jk}}{2} \tag{17}$$

for every $j \in N$. The intuition behind this restriction is that the running fee is low enough so every player j would rather pay it if she could tie on the election with any other candidate k (even with the one that offers her the most) than to abstain and to let k to win (this is $\frac{a_{jj}+a_{jk}}{2} - \beta$ compared to a_{jk}). As the running fee is sufficiently low any candidate that could beat or tie against Player i would have an incentive to destroy the equilibrium and run on the election. As there is no such player we can conclude that Player i would beat any other potential opponent in a 2-candidate election. This means that she is the only strong Condorcet winner in the network. $\square$

Proof. Proposition 2. If $\beta < \frac{a_{hh}-a_{hf}}{2}$ for $h = i$ and $f = j$, and for $h = j$ and $f = i$ then the running fee is low enough so none of the weak Condorcet winners wants to quit the competition to break the tie and let the other player win. Further more there is no third candidate k that wants to enter the competition as she could only gain voters from the electorate of her favored candidate i. However Player k cannot win all the votes of the electorate of i and tie against j in a 3-candidate election, at least i would vote for herself. In this case the only thing k can achieve by running is splitting the electorate of her favored candidate i and letting j to win the election alone, which is clearly worse for her than not running. As none of the weak Condorcet winners wants to quit nor a third candidate wants to enter the situation is equilibrium. $\square$

Proof. Proposition 3. In case of the chain $R = N$ and the order ψ is the order of vertices in the chain from one end to the other. Every i has a preferred choice, herself ($i = i^{pref}$), as $a_{ii} > a_{ij}$ for every $j \neq i$ and this preferred choice splits the chain into two halves. Let j and k be two choices on the same half of the chain with respect to i^{pref} and let be $k < j < i^{pref}$. Note that j also splits the chain into two halves (k-half is the one that contains k and i-half is the one that contains i^{pref}). I can define a set G as the set of players on i-half of the order and its bordering set is a singleton $B_G = \{j\}$, and I can apply Lemma 3 that tells us that i would adopt j's preference ordering over j and k. As j prefers herself to k so does i, and the preferences are single peaked.

In case of the tree R is the set of vertices along a path between f and h (for every $f, h \in N$) and the order ψ_{fh} if the order of vertices along the path. Every player i has a most preferred candidate in R: if i is on the f to h path, the most preferred candidate is herself ($i = i^{pref}$), if i is not on the path, then there is a vertex l, such that $\lambda_{il} = \min_{r \in R} \lambda_{ir}$. Since l is the bordering set of a connected group G such that $i \in G$ and $R \cap G = \emptyset$ so we can apply Lemma 3, and we get that i adopts the preferences of l over the potential candidates in R, so $i^{pref} = l$. The rest of the proof is the same as in case of the chain: l splits the path into two halves and if there are two alternatives j and k on the same half but j is "closer" to l, she prefers j and so does i. $\square$

Proof. Proposition 4. In case of a chain the proof come directly from the median voter theorem. It says that the players in the median position of the chain are Condorcet winners – if there is one median player she is a strong Condorcet winner, if there are two neighboring median players they are both weak Condorcet winners.

In case of a tree Theorem 2 of Demange [1982] shows that the set of Condorcet winner players is non-empty. I prove here by contradiction that if there are two players in the set, they are neighbors; and that there are maximum two players in the set.

Suppose that i and j are weak Condorcet winners in a network C but not immediate neighbors. In this case there is at least one player k on the i to j path such that $a_{ki} > a_{kj}$. Note that k is the bordering set for a group of players G that contains all the players who originally voted for j: $\{k\} = B_G$ and $l \in G$ if $a_{lk} > a_{li}$. If k runs against i in a 2-candidate election, all the original electorate of j would support k and she wins the election against i (at least $\frac{n}{2} + 1$ votes against at most $\frac{n}{2} - 1$) so i is not a weak Condorcet winner. *Contradiction.*

Suppose that the set $S = \{i_1, i_2, i_3, \ldots, i_m\}$ contains all the weak Condorcet winners of network C, such that $m \geq 3$. As C is a tree there is a pair $i_x, i_y \in S$ such that i_x and i_y are not immediate neighbors, so there is at least one player k on the i_x to i_y path. Using the analogy with the previous case if k originally preferred i_x to i_y, she can beat i_x in a 2-candidate election with the support of the electorate of i_y. Thus i_x is not a weak Condorcet winner. *Contradiction.* $\square$

Proof. Proposition 5.

Part 1 Let G be a group that is connected to the rest of the network by i, player $k \in G$ and $\overline{G} = N \setminus (G \cup \{i\})$. The bordering set $B_{\overline{G}} = \{i\}$ so by Lemma 3 if i and k runs in a 2-candidate election i get the support of all $\overline{G}$. Supposing that every G is smaller than $\overline{G}$, i could beat any $k \in N$. So i is the strong Condorcet winner. On the other hand if i is the strong Condorcet winner in the tree she always belongs to the groups that is larger than the rest of the network combined, and she separates the network into groups where none is larger than the rest of the network combined: if there was a group larger than the rest of the network combined by Lemma 3 the neighbor of i who belongs to that group could beat i in a 2-candidate election.

Part 2 If the next move would be moving back to the previous player is because the i-half (G_i) and the j-half (G_j) of the network are equal in size. As the bordering set of G_j is $B_j = \{i\}$ i can beat any player from the $k \in G_i$ in a 2-candidate election and since the bordering set of $G_i \setminus \{i\}$ is $B'_i = \{i\}$ she can at least tie with any candidate $l \in G_j$, so she is a weak Condorcet winner. On the other hand if i and j are weak Condorcet winners in the tree the group they both belong is always larger than the rest of the network and from i the next move is j is vice versa: otherwise one of them could beat the other in a 2-candidate election and she was not weak Condorcet winner. □

Proof. Proposition 6. Let b_1 and b_2 be the bridge players in C and their corresponding social groups are G_1 and G_2, respectively. For G_2 the bordering set is $B_{G2} = \{b_1\}$. We can apply Lemma 3 and get that all the players in G_2 prefers b_1 to any $j \in G'_1 = G_1 \setminus \{b_1\}$, so b_1 would have at least $g_2 + 1$ votes against every candidate j. On the other hand b_1 is also the bordering set for the group G'_1 so following the same logic we can see that b_1 would have at least the votes of the players in G'_1, and her own – g_1 votes – against any candidate $k \in G_2$. As $g_1 = g_2$ candidate b_1 can win against any candidate $j \in G'_1$ and win or tie against any candidate $k \in G_2$, so she is a weak Condorcet winner. Changing the roles of G_1 and G_2 we can see that b_2 is also a weak Condorcet winner. As b_1 beats every $j \in G'_1$ and b_2 does so with every $k \in G'_2$ we can be sure that there are no more weak Condorcet winners in the network. □

Proof. Proposition 7. Let h^* be the hub player in C. For every player k such that $h^* \notin \mathcal{R}_k$, in this case $\mathcal{R}_k = \widetilde{\mathcal{R}}_k$. Since for every $i \in N \setminus \mathcal{R}_k$ $a_{ik} = 0$ while $a_{ih^*>0}$ in a 2-candidate election h^* would have the votes of all such players against k, and since $g_k < n - g_k$ this support would be enough to beat k. On the other hand for every social group G_l and player l such that $h^* \in \mathcal{R}_l$ the bordering set of $\overline{G_l} = N \setminus (G_l \cup \{h^*\})$ is $B_{\overline{Gl}} = \{h^*\}$. In this case we can apply Lemma 3 and get that against candidate l the hub player h^* would have the votes of all the players $i \in N \setminus G_l$. Since $g_l < n - g_l$ candidate l would lose in a 2-candidate election against h^* even if she could secure all the votes of her own group. As there is no candidate that could win or tie against h^* she is a strong Condorcet winner. □

Lemma 4. *Given a network C and the vertices $i, j, k \in N$ such that there is a walk from j to k and i is not on the walk ($i \notin W_{j \to k}$) then there is $\epsilon > 0$ and a network C' isostructural to C such that $c'_{hi} \leq \epsilon$ for every $h \in N$ and $a'_{ki} < a'_{kj}$.*

Proof. Lemma 4. On one hand

$$a_{xy} \leq 1 - \widehat{c}_y = \sum_h c_{hy} \tag{18}$$

for every $x, y \in N$ ($x \neq y$), since after receiving the prize Player y keeps $\widehat{c}_y$ that never circulates in the network. On the other hand if there is a walk $W_{y \to x}$ and it is ordered set of vertices with the order $\psi_W : 1 < 2 < 3 < \ldots < w$ where $y = 1$ and $x = w$ then

$$a_{xy} \geq \widehat{c}_x \prod_{l=1}^{w-1} c_{l,l+1} = (1 - \sum_h c_{hx}) \prod_{l=1}^{w-1} c_{l,l+1} \tag{19}$$

for every $x, y \in N$, since the walk $W_{y \to x}$ is only one of the many direct and indirect ways to reach x from y and only along this walk arrives at least $\prod_{l=1}^{w-1} c_{l,l+1}$ of which x keeps $\widehat{c}_x$ for herself.

Applying the inequalities to the situation given in the lemma we have that $a_{ki} \leq \sum_h c_{hi}$ and $a_{kj} \geq (1 - \sum_h c_{hx}) \prod_{l=1}^{w-1} c_{l,l+1}$. Since there is a j to k walk that does not go through i Equation 19 is independent from all the c_{hi} arc weights pointing away from i. This means that in an isostructural network C' by decreasing the value of the c'_{hi}'s the upper bound for a'_{ki} decreases but the lower bound for a'_{kj} is not affected. As the lower bound for a'_{kj} is positive in case $W_{j \to k}$ exists and the upper bound for a'_{ki} can be arbitrarily low there is an ϵ such that$\epsilon \geq c'_{hi} \geq 0$ for every $h \in N$ and the upper bound for a'_{ki} is lower than the lower bound for a'_{kj} and thus $a'_{ki} < a'_{kj}$. $\square$

Proof. Proposition 8.

Part 1: i is a strong Condorcet winner. For every $k \in N$ such that $\mathcal{R}_i \cap \mathcal{R}_k = \emptyset$ $a_{xk} = 0$ if $x \notin \mathcal{R}_k$ and $a_{yi} = 0$ if $y \notin \mathcal{R}_i$ so if i runs against k they both get the support of their corresponding reaches and since $\rho_i > \rho_k$ i would beat k in a 2-candidate election. For every $l \in N$ such that $\mathcal{R}_i \subset \mathcal{R}_l$ the bordering set of $\overline{G}_l = \mathcal{R}_l \setminus (\widetilde{\mathcal{R}}_l \cup \{i\})$ is $B_{\overline{G}l} = \{i\}$ so we can apply Lemma 3 and we get that every player in $\overline{G}_l$ prefers i to l and since $\rho_l > 2\widetilde{\rho}_l$ i would beat l in a 2-candidate election. Consequently there is no player outside of the reach of i that could beat her in a 2-candidate election. From Proposition 7 we know that there is no player in the reach of i either that could beat her in a 2-candidate election if $\mathcal{R}_i$ is a balanced windmill and i is the hub player. Consequently i is a strong Condorcet winner in C.

Part 2: $\mathcal{R}_i$ is a balanced windmill. I prove this part by contradiction. Suppose that $\mathcal{R}_i$ is not a windmill, so $\mathcal{R}_i \setminus (\widetilde{\mathcal{R}}_k \cup \{i\}) = \emptyset$ for some $k \in \mathcal{R}_i$ ($k \neq i$) – in other words there is a player $k \in \mathcal{R}_i$ such that there is a walk from k to every $x \in (\mathcal{R}_i \setminus \{i\})$

that does not go though i. From Lemma 4 we know that there is an $\epsilon > 0$ and a network C' isostructural to C such that $c'_{hi} \leq \epsilon$ for every $h \in N$ and $a'_{xi} < a'_{xk}$ for every $x \in (\mathcal{R}_i \setminus \{i\})$. In this network C' if k runs against i in a 2-candidate election, i loses (getting only 1 vote against $\rho_i - 1$ (while the disconnected players $(n - \rho_i)$ are indifferent), so i is not a strong Condorcet winner in all the C' networks. *Contradiction.* $\mathcal{R}_i$ must be a windmill.

Suppose that $\mathcal{R}_i$ is a windmill but it is not balanced: in this case there is a player $k \in \mathcal{R}_i$ such that $\widetilde{\rho}_k > \frac{1}{2}\rho_i$, so one of the social groups ($\widetilde{\mathcal{R}}_k$ or G_k) is larger than the rest of the reach of i. As every player $j \in \widetilde{\mathcal{R}}_k$ can be reached from k without going through i we can apply Lemma 4 and we know that there is an $\epsilon > 0$ and a network C' isostructural to C such that $c'_{hi} \leq \epsilon$ for every $h \in N$ and $a'_{ji} < a'_{jk}$ for every $j \in \widetilde{\mathcal{R}}_k$. As this set is larger than the half of the reach of i, so if k runs against i in a 2-candidate election i loses. This means that i is not a strong Condorcet winner in all the C' networks. *Contradiction.* $\mathcal{R}_i$ must be a balanced windmill. $\square$

Proof. Proposition 9.

Part 1: i and j are weak Condorcet winners. For every $k \in N$ and such that $\mathcal{R}_i \cap \mathcal{R}_k = \emptyset$ and $\mathcal{R}_j \cap \mathcal{R}_k = \emptyset$ $a_{xk} = 0$ for every $x \notin \mathcal{R}_k$ and $a_{yi} = 0$ for every $y \notin \mathcal{R}_i$ so if i runs against k in a 2-candidate election they both get the support of their corresponding reach and since $\min(\rho_i; \rho_j) > \rho_k$ i would beat k. Since for every $k \in N$ $\rho_i \geq \rho_k$ there is no player k such that $\mathcal{R}_i \subset \mathcal{R}_k$. Consequently there is no player outside of the reach of i that could beat her in a 2-candidate election. From Proposition 6 we know that there is no player in the reach of i either that could beat her in a 2-candidate election if $\mathcal{R}_i$ is a balanced bridge and i is one of the bridge players or it is a balanced windmill i is the hub player. The same argument holds for j. As no player in C can beat i and j in a 2-candidate election they cannot beat each other either, consequently they are weak Condorcet winners.

Part 2: $\mathcal{R}_i$ and $\mathcal{R}_j$ form a balanced bridge, 2 balanced windmills or a hybrid. There are three possible relation between $\mathcal{R}_i$ and $\mathcal{R}_j$: (1) $\mathcal{R}_i = \mathcal{R}_j$, (2) $\mathcal{R}_i \cap \mathcal{R}_j = \emptyset$, or (3) $\mathcal{R}_i \subset \mathcal{R}_j$. I prove this part by contradiction.

First suppose that $\mathcal{R}_i = \mathcal{R}_j$, but $\mathcal{R}_i$ is not a balanced bridge: in this case there is a player $k \in \mathcal{R}_i$ ($k \neq i, k \neq j$) such that there is a walk from k to every $x \in (\mathcal{R}_i \setminus \{i, j\})$ that does not go though i (or j or neither). From Lemma 4 we know that there is an $\epsilon > 0$ and a network C' isostructural to C such that $c'_{hi} \leq \epsilon$ for every $h \in N$ and $a'_{xi} < a'_{xk}$ for every $x \in (\mathcal{R}_i \setminus \{i\})$. In this network C' if k runs against i in a 2-candidate election, i loses (getting only 1 vote against $\rho_i - 1$

(while the disconnected players $(n - \rho_i)$ are indifferent), so i is not a weak Condorcet winner in all the C' networks. *Contradiction.* $\mathcal{R}_i$ must be a bridge if $\mathcal{R}_i = \mathcal{R}_j$. Now suppose that $\mathcal{R}_i = \mathcal{R}_j$, and $\mathcal{R}_i$ is a bridge but it is not balanced. Since the bordering set of $\widetilde{\mathcal{R}}_i \setminus \{i\}$ is i and the bordering set of $\widetilde{\mathcal{R}}_j \setminus \{j\}$ is j if $\widetilde{\mathcal{R}}_i$ is smaller than $\widetilde{\mathcal{R}}_j$ j would beat i in a 2-candidate election. *Contradiction.* $\mathcal{R}_i$ must be a balanced bridge if $\mathcal{R}_i = \mathcal{R}_j$.

Then suppose that $\mathcal{R}_i \cap \mathcal{R}_j = \emptyset$ but at least $\mathcal{R}_i$ is not a balanced windmill with hub i. From the Part 2 for the proof of Proposition 8 we know that in this case there is a player $k \in \mathcal{R}_i$ that can beat i in some C' isostructural to C. *Contradiction.* If $\mathcal{R}_i \cap \mathcal{R}_j = \emptyset$ both $\mathcal{R}_i$ and $\mathcal{R}_j$ must be windmills

Finally consider the case where $\mathcal{R}_i \subset \mathcal{R}_j$ and $\widetilde{\mathcal{R}}_j = \mathcal{R}_j \setminus \mathcal{R}_i$. From the previous case we know that $\mathcal{R}_i$ is a balanced windmill and i is the hub, otherwise player i it could be beaten in one of the C' networks by a candidate $k \in \mathcal{R}_i$ in a 2-candidate election. Now suppose that there is an arc from $k \in \widetilde{\mathcal{R}}_j$ to $l \in \mathcal{R}_i$ ($l \neq i$, but k might be equal to j). This means that there is a walk from j to l that does not go through i and from Lemma 4 we know that there is an $\epsilon > 0$ and a network C' isostructural to C such that $c'_{hi} \leq \epsilon$ for every $h \in N$ and $a'_{li} < a'_{lj}$, and in this case if i and j runs in a 2-candidate election j beats i. *Contradiction.* Every walk from j to $l \in \mathcal{R}_i$ must go through i. Now suppose that there is an arc from $k \in \widetilde{\mathcal{R}}_j$ ($k \neq j$) to i. This means that there is a walk from k to every $l \in \mathcal{R}_i$ that does not go through j and from Lemma 4 we know that there is an $\epsilon > 0$ and a network C' isostructural to C such that $c'_{hj} \leq \epsilon$ for every $h \in N$ and $a'_{lj} < a'_{lk}$ for every $l \in \mathcal{R}_i$, and in this case if k and j runs in a 2-candidate election k beats j. *Contradiction.* Every walk from $k \in \widetilde{\mathcal{R}}_j$ to any $l \in \mathcal{R}_i$ must go through j. Summing up the last case $\mathcal{R}_i$ is a balanced windmill and the walk from every $k \in \widetilde{\mathcal{R}}_j$ to any $l \in \mathcal{R}_i$ (if exists) should go through the c_{ij} arc. $\square$