Will yield factors tell more? a generalized affine HJM model with unspanned stochastic volatility

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Motivation and Features

This paper investigates the relationship between yields and volatility factors. It is motivated by the literature that incorporate features of curve-fitting models, like Diebold and Li (2006) and that study the unspanned stochastic volatility, like Collin-Dufresne and Goldstein (2002). Stochastic volatility factors in this paper indirectly affect yields through yield factors.

The model has several distinctive features.

- The yield factors exhibit level, slope, and curvature shapes.
- The volatility factors are partially spanned by yields.
- The volatility factors are the kernels of the shadow means of yield factors.

The Model

The model follows Heath et al. (1992) forward rate[a] framework by specifying a general form of forward rate volatility.

The paper develops a three factor model ($n = 3$), for $i = 1, 2, 3$, $x = (T - t)$, $dW_i(t)$ and $dB_i(t)$ are independent Brownian motions under Q-measure.

Model Specification

The forward rate volatility is defined as

$$\sigma_{0,i}(t,x) = [\sigma_{i,1} + (\sigma_{i,2} + \sigma_{i,3}x)e^{-a_i x}]\sqrt{v_i(t)}, \tag{1}$$

and the market price of risk as a simplified extended affine form,

$$dW_i(t) = dW_i^P(t) + \frac{\lambda_{W_i,0} + \lambda_{W_i,z} z_{i,1}(t) + \lambda_{W_i,v} v_i(t)}{\sqrt{v_i(t)}}dt, \tag{2}$$

[a] Given the identities among forward rates, bond yields, and bond prices, they are used interchangeably

Forward Rate Process

Based on the consistent condition in Bjork and Christensen (1999), the affine forward-rate process[a] can be derived as

$$f(t,x) = \sum_{i=1}^{n} \begin{pmatrix} z_{i,1}[\sigma_{i,1} + (\sigma_{i,2} + \sigma_{i,3}x)e^{-a_i x}] \\ +z_{i,2} + z_{i,3}x + z_{i,4}e^{-a_{i,1}x} + z_{i,5}e^{-2a_{i,1}x} + z_{i,6}e^{-a_{i,1}x} \\ +z_{i,7}xe^{-a_{i,1}x} + z_{i,8}x^2e^{-a_{i,1}x} + z_{i,9}x^2e^{-2a_{i,1}x} \end{pmatrix}. \tag{3}$$

[a] $z_i(t)$ and $v_i(t)$ are denoted as $z_{i,t}$ and $v_{i,t}$ for convenience.

Factor Dynamics

Under physical measure ($P-$measure), the yield factors are derived as

$$dz_{i,1}(t) = (-\kappa^P_{z_{i,1}})\left(\frac{\theta^P_{z_{i,1}} + \kappa^P_{z_{i,1},v}v_i(t)}{-\kappa^P_{z_{i,1}}} - z_{i,1}(t)\right)dt + \sqrt{v_i(t)}dW_i^P(t), \tag{4}$$

volatility factors as

$$dv_i(t) = \kappa^P_{vi}(\theta^P_{vi} - v_i(t))dt + \sigma_{v_i}\sqrt{v_i(t)}\left(\rho_i dW_i^P(t) + \sqrt{1-\rho_i^2}dB_i^P(t)\right). \tag{5}$$

Unspanned Stochastic Volatility

The correlation coefficient, ρ_i, captures how much of the volatility $v_i(t)$ can be spanned with yield factor $z_{i,1}(t)$ and thus with yields. Therefore, $(1 - \rho_i)$ reflects unspannedness. A coefficient $0 < |\rho_i| < 1$ indicates that the volatility can only be partially spanned by yields.

Shadow Mean of Yield Factor

In equation (4), the shadow mean of yield factor $z_{i,1}(t)$ is defined as:

$$\frac{\theta^P_{z_{i,1}} + \kappa^P_{z_{i,1},v}v_i(t)}{-\kappa^P_{Z_{i,1}}}. \tag{6}$$

Auxiliary State Variables

The auxiliary state variables maintain no-arbitrage and have no contemporaneous randomness. Therefore, they do not bear risk premium. They can be derived as, with $z_i^a(t) = (z_{i,1}(t), \ldots, z_{i,9}(t))'$,

$$dz_i^a(t) = \begin{pmatrix} a_i\sigma_{i,1}z_{i,1} + z_{i,3} \\ \sigma^2_{i,1}v_i(t) \\ -\frac{\sigma_{i,1}(a_i\sigma_{i,2}+\sigma_{i,3})}{a_i^2}v_i(t) + \sigma_{i,3}z_{i,1} - a_i z_{i,4} + z_{i,6} \\ -\frac{\sigma_{i,2}(a_i\sigma_{i,2}+\sigma_{i,3})}{a_i^2}v_i(t) - 2a_i z_{i,5} + z_{i,7} \\ \frac{\sigma_{i,1}(a_i\sigma_{i,2}-\sigma_{i,3})}{a_i}v_i(t) - a_i z_{i,6} + 2z_{i,8} \\ -\frac{\sigma_{i,3}(2a_i\sigma_{i,2}+\sigma_{i,3})}{a_i^2}v_i(t) - 2a_i z_{i,7} + 2z_{i,9} \\ \sigma_{i,1}\sigma_{i,3}v_i(t) - a_i z_{i8} \\ -\frac{\sigma^2_{i,3}}{a_i}h_i(t) - 2a_i z_{i,9} \end{pmatrix} dt, \tag{7}$$

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Empirical Analysis

The model is tested with weekly data of USD LIBOR/Swap rates from Jan. 2002 to Nov. 2011. Extended Kalman filter is implemented in the quasi-maximum likelihood estimation.

The fitted and one-period ahead forecast errors are highly comparable to other studies. Yield and volatility factors are extracted with the extended Kalman filter.

Yield Factor and Shadow Mean

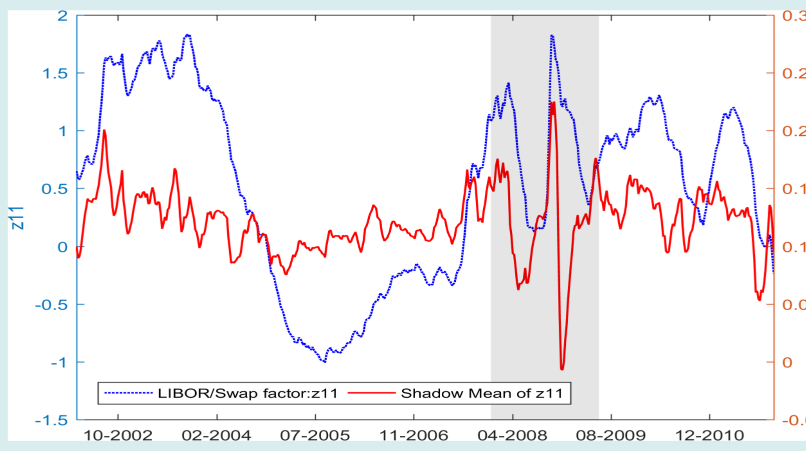

Figure 1: LIBOR/Swap factor $z_{1,1}$ and its shadow mean in equation (6)

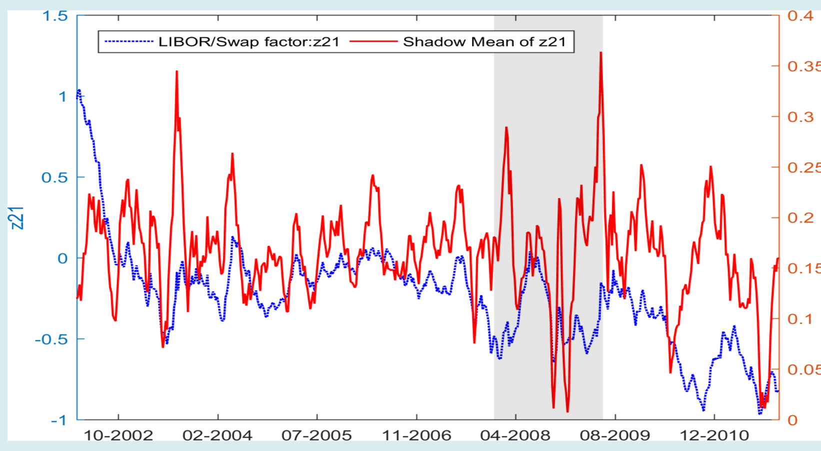

Figure 2: LIBOR/Swap factor $z_{2,1}$ and its shadow mean in equation (6)

The yield and volatility factors are extracted with extended Kalman filter. The comovement patterns of the yield factors and their shadow means (equation (6)) changed during and after the Great Recession in 2007-09 (Figure 1, 3).

Yield factors $z_{1,1}$ and $z_{3,1}$ widely swung around shadow means before the recession. But during the recession, they both followed their shadow means more closely.

After the recession, yield factor $z_{1,1}$ and $z_{2,1}$ displayed larger deviation from their shadow means (Figure 1, 2).

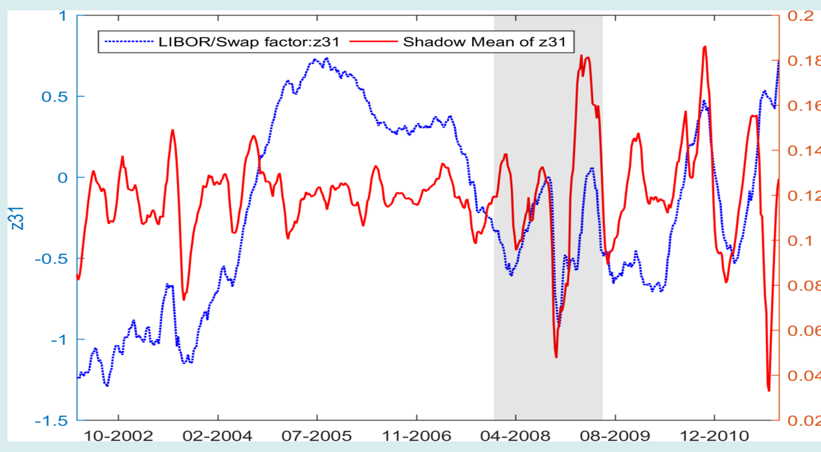

Figure 3: LIBOR/Swap factor $z_{i,1}$ and its shadow mean in equation (6)

Unspannedness

On average, two-thirds of the stochastic volatility can not be spanned with yields as indicated by ρ_i in Table (1). There are about 35% of the first and second volatility factors, $v_1(t)$ and $v_2(t)$, spanned by yields. The Third volatility factor, $v_3(t)$, can only be spanned by less than 30%.

Correlation Coefficients	
ρ_1 $(z_{1,1}, v_1)$	0.344
ρ_2 $(z_{2,1}, v_2)$	0.355
ρ_3 $(z_{3,1}, v_3)$	0.288

Table 1: Unspannedness of Stochastic Volatility

Information Utilization

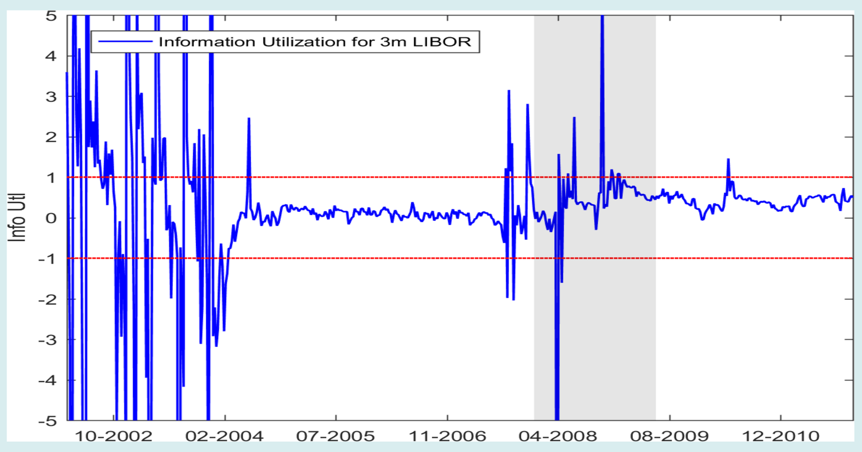

Figure 4: Information Utilization for 3-month LIBOR rate (6)

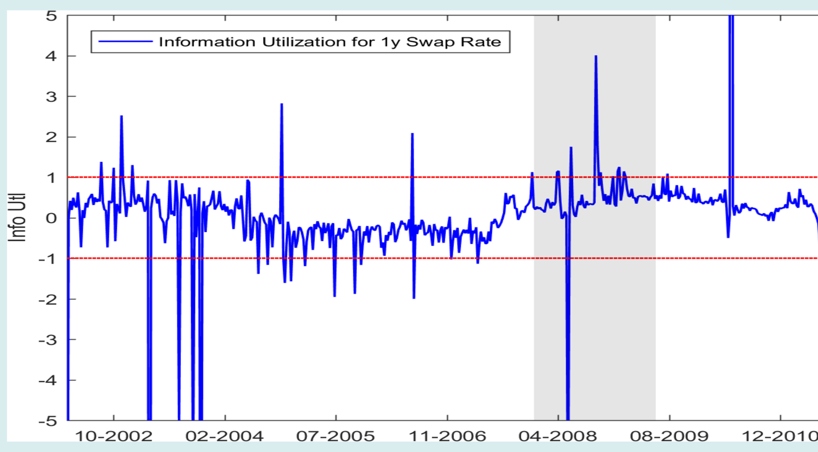

Figure 6: Information Utilization for 10-year swap rate (6)

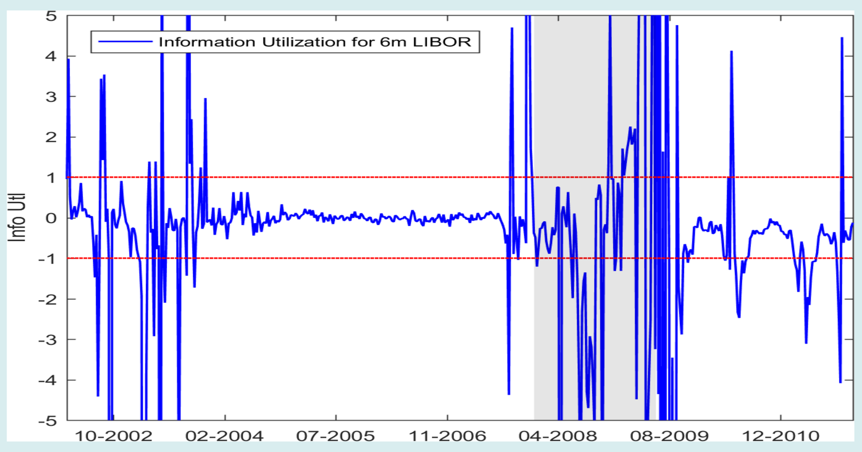

Figure 5: Information Utilization for 6-month LIBOR rate (6)

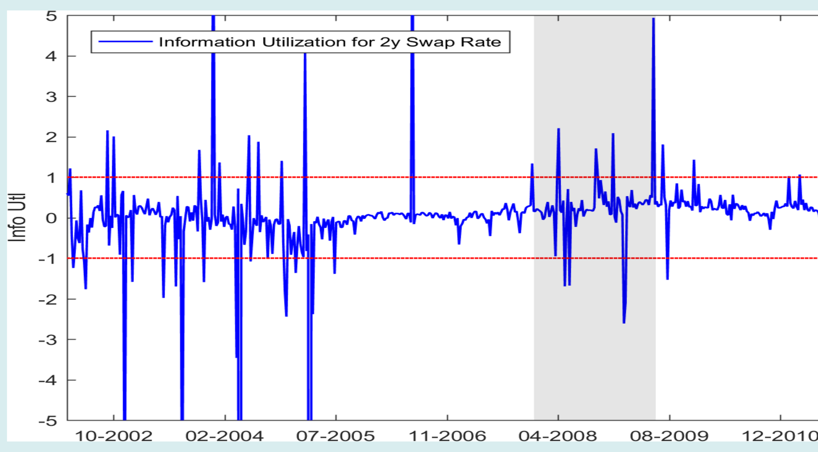

Figure 7: Information Utilization for 15-year swap rate (6)

Information Utilization

Information utilization (*info_utl*) is defined in equation (8) to examine market responses to information flows.

$$info\_utl = \frac{y_t(\tau) - \hat{y}^{fit}_t(\tau)}{y_t(\tau) - \hat{y}^{forcast}_{t|t-1}(\tau)} \tag{8}$$

where, $y_t(\tau)$, $\hat{y}^{fit}_t(\tau)$, and $\hat{y}^{forcast}_{t|t-1}(\tau)$ are the time-t τ-yield, fitted τ-yield, and one-period ahead forecast τ-yield, respectively.

Market Responses

When the market responds normally, the fitted error should be smaller than the forcast error, or $|info\_utl| < 1$. Otherwise, market overreacts to new information when $|info\_utl| > 1$.

The market experienced long quiet normal reactions before the Greet Recession in 2007-09. Immediately before the recession did both 3mth and 6mth rates show large overreactions. After the recession, they went back to normal again.

Conclusions

The model provides a convenient way to diagnose the dynamics of yield and volatility factors. It shows that two-thirds of stochastic volatility can not be spanned with yields. Yield factors and their shadow means changed comovement patterns during and after the Great Recession. It also reveals market overreactions during the recession time.

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