Ambiguous Information, Permanent Income, and Consumption Fluctuations

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Abstract

This paper studies asymmetric responses in consumption where the asymmetries are endogenously generated by agents' preferences and incomplete knowledge about information quality. Agents form expectations about the future based on incomplete information which is assumed to be ambiguous and these future expectations, distorted by ambiguity, affect spending asymmetrically. With a noisy signal of uncertain quality, consumption features asymmetric responses: the absolute size of the responses depends on whether the signal delivers good or bad news. I estimate the model on U.S. data by maximum likelihood and the estimates suggest that ambiguity plays a non-negligible role in consumption fluctuations.

Keywords: ambiguity, noisy information, permanent income hypothesis

JEL Classification Codes: D83, E20, E32

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1 Introduction

There is evidence that consumption growth is left-skewed. For example, consumption expenditure growth per capita[1] of the U.S. from 1970 to 2016 exhibits left-skewness with a negative estimate of -0.9015.[2] Van Nieuwerburgh and Veldkamp (2006) suggest that negative skewness of output, investment, employment and consumption represents a gradual boom in economic activities and the sharp and short economic downturn.[3] One potential reason for a left-skewed consumption growth is that positive and negative shocks have asymmetric effects on consumption.[4] It can also be explained by agents reacting symmetrically to asymmetric shocks. While both explanations (or a mixture of them) can easily justify a left-skewed unconditional distribution of consumption growth, it is not so easy to suggest why agents react differently to positive and negative shocks.[5]

Table 1: Left-skewed Consumption Growth

Sample	Country	Skewness
1870-2009	Canada	-0.7829
1820-2009	France	-0.8155
1851-2009	Germany	-7.8171
1861-2009	Italy	-1.3680
1870-2009	Japan	-2.2255
1830-2009	the United Kingdom	-0.7890

Notes: Data is yearly data. Per capital consumption data is from Barro and Ursua (2010) and can be accessed at http://scholar.harvard.edu/barro/publications/barro-ursua-macroeconomic-data

In this paper, by focusing on agents' preferences and information structures I attempt to suggest a possible explanation for the asymmetric effects of (symmetric) exogenous shocks in a simple, forward-looking consumption model where agents' belief formation is the key ingredient to explain consumption dynamics.[6] This follows a view on business cycles em-

[1] Yang (2011) also documents that durable consumption growth in the U.S. is left-skewed.

[2] Table 1 reports skewness of consumption expenditure growth in the rest of the Group of Seven countries. Consumption growth is obtained by taking the first difference of log per-capita consumption.

[3] Van Nieuwerburgh and Veldkamp (2006) explain growth asymmetry in macro aggregates with learning about the aggregate technology level.

[4] The literature suggests that there is ample evidence of the asymmetric effects of shocks on key macro variables. Cover (1992), for example, using the quarterly U.S. data suggests that while positive money supply shocks do not have an effect on output, negative ones do; Kandil (2002), using aggregate data of real output, price, and wage for the United States, provides evidence of the asymmetric effects of aggregate demand shocks; Hussain and Malik (2016) show that the effects of tax increase and decrease are asymmetric.

[5] Imperfect access to credit markets, precautionary saving due to income uncertainty, and loss aversion of households have been suggested in the literature to generate the asymmetry in consumption response.

[6] Cao and Nie (2016) provides an explanation of asymmetric responses of the economy to symmetric exogenous productivity shocks with market incompleteness.

phasizing the role of anticipating the future. Agents form expectations about the future based on incomplete information which is assumed to be ambiguous; such future expectations, distorted by ambiguity, affect spending asymmetrically. An interesting feature of the model is that the possibility of agents responding *symmetrically* is not entirely ruled out such that it is possible to test which hypothesis (asymmetric or symmetric responses to symmetric shocks) fits data better statistically in a simple unified framework and provide a numerical characterisation of the conditional dynamics of consumption.

A common practice of modelling agents' expectations about future outcomes in macroeconomic analysis has been the use of rational expectations, often called model consistent expectations, where it requires, roughly speaking, that agents' beliefs about future variables coincide with expectations predicted by the model. While it has been the main ingredient of most dynamic general equilibrium models currently used, the assumption imposes strong restrictions on agents' behaviours. For example, it is unlikely that consumers are fully aware of the underlying mechanisms governing firms' price-setting practices, technological progress, or other types of uncertainty regarding fundamentals of the economy.[7]

This paper relaxes restrictions imposed on agents' knowledge about the stochastic processes governing the economy: A stochastic signal about a permanent component of productivity is assumed to be not only noisy but also ambiguous in its information quality. As agents' beliefs about the state of the economy critically affect macroeconomic dynamics, how expectations are formed under ambiguity turns out to be very important. In other words, agents face an additional challenge to perceive information of uncertain quality given their preferences. This, in turn, requires to model preferences under ambiguity and I follow the setup axiomatised by Gilboa and Schmeidler (1989) and recently adopted by Epstein and Schneider (2008), Ilut (2012), Ilut, Kehrig, and Schneider (2014), and Baqaee (2016), applying the *maxmin* expected utility decision with multiple priors, where behaviour derived from the decision rule is consistent with experimental evidence such as the Ellsberg Paradox.[8] By assuming that agents exhibit aversion to uncertainty, processing a signal of uncertain quality to update beliefs is equivalent to estimating fundamentals consistent with a worst-case evaluation of ambiguous information, and conditional responses of the agents exhibit asymmetries: The absolute size of the responses depends on whether noisy information delivers good news or bad news.

[7]A number of studies aim to relax such restrictions and to document subsequent macroeconomic outcomes. For instance, Bianchi and Melosi (2016) develop methods to study general equilibrium models where forward looking agents learn about the stochastic properties of realized events following waves of pessimism, optimism, and uncertainty and Adam and Marcet (2011) relax the rationality assumption to capture the notion that agents do not fully understand some underlying statistical properties.

[8]In these models, agents possess multiple priors about the information quality of their signals and act upon their worst case prior to make decisions under ambiguity. In my model, in addition to an ambiguous signal, agents receive an additional signal which is assumed to be unambiguous.

Specifically, the theory is based on a model of business cycles driven by shocks to agents' expectations regarding productivity, where agents form anticipations about the future by observing noisy signals about productivity as in Blanchard, L'Huillier, and Lorenzoni (2013) and Cao and L'Huillier (2015). These signals sometimes turn out to be news and sometimes just noise, and agents need to solve a signal extraction problem to decide their current spending. Later on, if information turns out to be news, agents adjust their expectations upward and the economy gradually adjusts to a new level of activity; if ex-post information turns out to be just noise, the economy returns to its original state of activity. In my version of the model, I modify this information structure such that agents are uncertain about the quality of noisy signals they receive and the uncertainty is captured by *the range of precisions*:

$$1/\sigma_{\nu,t}^2 \in [1/\bar{\sigma}_\nu^2, 1/\underline{\sigma}_\nu^2]$$

where $1/\sigma_{\nu,t}^2$ denotes the *true* signal precision. In such a case, if agents are assumed to exhibit aversion toward ambiguity, they follow the *maxmin* optimisation by which they make decisions that maximize their expected utility under a worst-case belief. The latter depends on the types of signals they receive. For a signal delivering bad news, a worst case is that the signal is very informative. Conversely, for a signal delivering good news, a worst case is that the signal is very noisy.[9] This makes the agents respond more to bad news than to good news such that the size of the response is larger in an absolute value when bad news is delivered. In addition, when information quality becomes more ambiguous and *the range of precision* gets larger, the responses exhibit a larger degree of asymmetries.

Relation to literature:

This paper follows the tradition of a business cycle model where expectations play a significant role; the original thesis laid out in Pigou (1927), which emphasizes that recessions could arise as a result of agents' inability to correctly forecast the economy's need in terms of capital and subsequent investment swings, and a recent work by Beaudry and Portier (2004), where agents receive an imperfect signal about future productivity growth and make decisions about investment based upon these signals, are frequently cited works that started this strand of literature. Distinguishing permanent and transitory productivity shocks is one of the important ingredients in expectation-driven business cycle models and it is closely examined in Boz, Daude, and Durdu (2011), Lorenzoni (2009), Blanchard et al. (2013), and Rousakis (2013). While sharing similar information structures and agent's information processing, I extend the setup to allow for uncertain quality of information and

[9] Differentiating a signal delivering good news from the one delivering bad news is discussed in Section 2.

agents' aversion toward uncertainty.

Many recent papers have adopted ambiguity and study macroeconomic dynamics and asset pricing. Epstein and Schneider (2008) discuss asset markets in which ambiguity averse investors process news of uncertain quality with a worst-case assessment of information quality and the model generates more stronger reaction to bad news than good news which results in asymmetric responses in asset market; Ilut (2012) builds a model of exchange rate determination where an ambiguity averse agent solve a signal extraction problem with uncertain signal precision and take departures from uncovered interest rate parity; in Ilut, Kehrig, and Schneider (2014) firms' hiring decisions is modelled under ambiguous information. Firms receive ambiguous information about productivity of the economy and maximize multiple priors utility, which reflects firms' aversion to ambiguity; Baqaee (2016) attempts to incorporate ambiguous information and the signal extraction problem in order to explain downward wage rigidities where an equilibrium wage is more sensitive to inflation than to disinflation; Ilut and Saijo (2016) build an economy where firms are uncertain about their profitability and the problem is constructed in such a way that the signal precision varies over the cycle. Specifically, the more a firm produces the more precise the signal becomes.[10]

This paper is also related to the random walk behaviour of consumption in which the behaviour of consumption is fully determined by the long-run level of productivity or permanent income.[11] Combined with imperfect information structure, it requires that agents should forecast or estimate the long-run level of productivity to choose consumption such that, since agents' forecasts are distorted by the presence of ambiguity, ambiguity plays an important role in explaining consumption fluctuations.

There are other explanations to generate the asymmetric response of consumption. Carroll (1992) and Carroll (1994) suggest that a household would increase saving and raise consumption spending moderately following a positive income shock in order to build up precautionary savings with income uncertainty. However, with a negative income shock, a household would cut consumption spending considerably. Deaton (1991) instead focus on imperfect access to credit markets such that a household is unable to borrow with a negative income shock, delivering the asymmetric consumption response. Finally, the prospect theory (Kahneman and Tversky 1979) suggests that by weighing the prospect of

[10]Similarly, Ilut and Schneider (2013) build a state-of-the-art ambiguous business cycle model where shocks to confidence, which is modelled as changes in ambiguity, play important role in explaining fluctuations. Whereas Ilut and Schneider (2013) introduce ambiguity in a productivity shock with first-order effects, Masolo and Monti (2015) study the implication of introducing ambiguity in a monetary policy shock.

[11]This random walk behaviour of consumption is discussed in detail in Blanchard et al. (2013) for a baseline New Keynesian model and in Cao and L'Huillier (2015) for a small open economy RBC model.

losses more than that of gains, agents' consumption response would exhibit asymmetry due to the steeper value function for losses than gains.

The rest of the paper is organized as follow. Section 2 illustrates how agents update beliefs under ambiguity, which is the key ingredient to explain the main mechanism of this model. Section 3 presents the model and Section 4 studies quantitative implications of the model. Section 5 concludes.

2 Belief Updating under Ambiguity

One of the key ingredients of this model is to determine how agents update beliefs under ambiguity. In other words, it is related to how agents process ambiguous information and update beliefs about (unobservable) fundamentals. Since consumption is assumed to depend solely on agents' expectation about productivity in the long run, it is essential to study how agents update beliefs about the unobserved state of the economy. With information quality being ambiguous, this task becomes non-trivial.

2.1 The one signal example

Consider the following case in which agents observe a noisy signal s_t about an unobservable fundamental x_t:

$$s_t = x_t + \nu_t$$

where ν_t is an i.i.d. normal shock with variance σ_ν^2. The information quality of the signal is assumed to be ambiguous such that $\sigma_\nu^2 \in [\underline{\sigma}_\nu^2, \bar{\sigma}_\nu^2]$.[12] The fundamental x_t follows a stochastic process:

$$x_t = x_{t-1} + \epsilon_t$$

where ϵ_t is an i.i.d. normal shock with variance σ_ϵ^2. The two innovations are assumed to be independent of each other at all leads and lags. Characterizing agents' belief updating, the one-step ahead prediction of the fundamental at period $t-1$, $x_{t|t-1}$, and its associated error variance $\Sigma_{t|t-1}$ are given by

$$\begin{aligned} x_{t|t-1} &= x_{t-1|t-1} \\ \Sigma_{t|t-1} &= \Sigma_{t-1|t-1} + \sigma_\epsilon^2 \end{aligned}$$

[12] For the rest of this paper, I use both the variance σ_ν^2 and the precision $1/\sigma_\nu^2$ interchangeably to describe information quality. Note that σ_ν^2 is not time-varying, nor are $\underline{\sigma}_\nu^2$ and $\bar{\sigma}_\nu^2$ such that the ambiguity is assumed to be time-invariant.

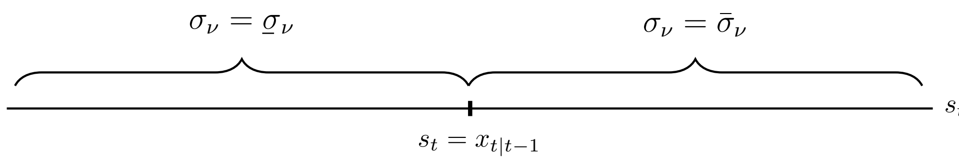

Figure 1: The Cutoff Rule for Belief Updating

Notes: $1/\sigma_\nu^2$ is the precision of information quality which is assumed to be ambiguous such that $1/\sigma_\nu^2 \in [1/\underline{\sigma}_\nu^2, 1/\bar{\sigma}_\nu^2]$.

where $x_{t-1|t-1}$ and $\Sigma_{t-1|t-1}$ are the updated (posterior) belief at period $t-1$ and its associated error variance. Then, by observing the noisy signal at period t, agents update beliefs about the fundamental:

$$\begin{aligned} x_{t|t} &= x_{t|t-1} + Gain_t \left(s_t - x_{t|t-1} \right) \\ \Sigma_{t|t} &= \left(\frac{\sigma_\nu^2}{\Sigma_{t|t-1} + \sigma_\nu^2} \right) \Sigma_{t|t-1} \end{aligned} \tag{1}$$

where $x_{t|t}$ and $\Sigma_{t|t}$ are the updated (posterior) belief at period t and its associated error variance. $Gain_t$ is the Kalman gain defined as

$$Gain_t = \frac{\Sigma_{t|t-1}}{\Sigma_{t|t-1} + \sigma_\nu^2} \tag{2}$$

Assume that, for a given utility function $u(x)$, it is strictly increasing in x and that agents are ambiguity averse in the sense that they maximize expected utility under a worst case belief chosen from the family of priors.[13] This leads to agents updating beliefs according to the cut-off rule shown in Figure 1.

From (1), it is easy to show that whenever $s_t > x_{t|t-1}$, the largest $Gain_t$ minimizes $x_{t|t}$. Similarly, when $s_t < x_{t|t-1}$, minimizing $x_{t|t}$ requires that $Gain_t$ takes the smallest value. As $Gain_t$ is strictly increasing in $1/\sigma_\nu^2$, only the maximum $(1/\underline{\sigma}_\nu^2)$ and the minimum $(1/\bar{\sigma}_\nu^2)$ from the range of precisions become relevant to update beliefs for ambiguity averse agents, which simplifies solving the model.[14] Intuitively, ambiguity averse agents consider a signal very noisy when they receive good signals. On the contrary, they interpret a signal as very informative when receiving bad signals.[15] Obviously, for the limiting case in which a

[13] Here, the family of priors refers to the range of precisions.

[14] As the Kalam gain is given by

$$Gain_t = \frac{\Sigma_{t|t-1}}{\Sigma_{t|t-1} + \sigma_\nu^2}$$

it is straightforward to show that $\partial Gain_t / \partial \sigma_\nu^2 = -\Sigma_{t|t-1} / \left(\Sigma_{t|t-1} + \sigma_\nu^2 \right)^2 < 0$ by holding $\Sigma_{t|t-1}$ constant.

[15] To clarify definition, a signal is said to deliver good news whenever it is greater than agents' ex-ante expectations and vice versa.

signal is related to a single likelihood ($\underline{\sigma}_\nu^2 = \bar{\sigma}_\nu^2 = \sigma_\nu^2$), the gain of observing noisy signals is pinned down by $Gain_t = \Sigma_{t|t-1}/(\Sigma_{t|t-1} + \sigma_\nu^2)$ regardless of whether the signal delivers good ($s_t > x_{t|t-1}$) or bad news ($s_t < x_{t|t-1}$).

Figure 2 plots an asymmetric response to the realisation of the signals. The size of the response is larger in absolute value when a bad signal is delivered. In addition, when information quality becomes more ambiguous, the responses exhibit a larger degree of asymmetries.

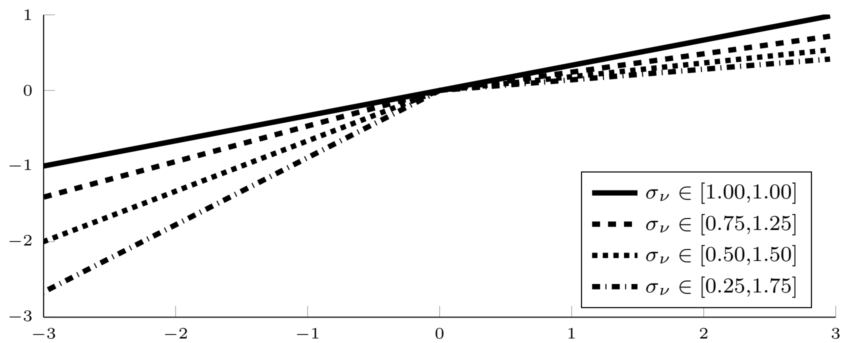

Figure 2: Asymmetric Belief Updating

Notes: The updated belief $x_{t|t}$ is determined by a signal extraction problem where $x_{t|t} = x_{t|t-1} + Gain_t(s_t - s_{t|t-1})$ and $s_t = x_t + \nu_t$. The variance of x is set to 0.5 and the variance of ν varies. The x-axis denotes denotes realized news ($s_t - s_{t|t-1}$) and the y-axis denotes the updated belief ($x_{t|t}$).

2.2 Extension to the multiple signal case

Assume now that agents receive multiple signals about the fundamental where one of the signals is ambiguous. Here, let the number of signals be two.[16] Specifically, in addition to the noisy signal s_t described in the previous section, agents receive an additional signal a_t whose quality is measured by the signal precision $1/\sigma_\eta^2$:

$$a_t = x_t + \eta_t$$

where η_t is an i.i.d. normal shock with variance σ_η^2 and the three innovations are independent of each other at all leads and lags. Agents use the two signals to update beliefs about the fundamental x_t. Let $\mathbb{E}_t[x_t] = \mathbb{E}[x_t|\mathcal{I}_t] = x_{t|t}$ and $\mathbb{E}_{t-1}[x_t] = \mathbb{E}[x_t|\mathcal{I}_{t-1}] = x_{t|t-1}$ respectively represent the estimates of x_t with the current information set ($\mathcal{I}_t$) and the

[16]Extending the discussion to the case where the number of signals is N and $N-1$ of them are unambiguous is trivial.

lagged information set ($\mathcal{I}_{t-1}$):

$$x_{t|t} = x_{t|t-1} + Gain_t(S_t - S_{t|t-1}) \tag{3}$$

where $S_t = (a_t,\ s_t)'$ is a vector of signals and $Gain_t = (G_t,\ H_t)$ is a row vector representing the gains of observing the signals. Specifically, G_t and H_t respectively denote the Kalman gain of observing the ambiguous signal s_t and the unambiguous signal a_t.

From (3) the updated estimate on x_t with two signals can be summarized by a weighted average of the previous period estimate of the fundamental $x_{t|t-1}$ and of revisions based on the surprises associated with the realisation of each shock:

$$x_{t|t} = x_{t|t-1} + G_t(s_t - s_{t|t-1}) + H_t(a_t - a_{t|t-1}) \tag{4}$$

where

$$G_t = \left(\frac{\sigma_\eta^2 \Sigma_{t|t-1}}{\sigma_\nu^2 \sigma_\eta^2 + \sigma_\nu^2 \Sigma_{t|t-1} + \sigma_\eta^2 \Sigma_{t|t-1}} \right)$$

and

$$H_t = \left(\frac{\sigma_\nu^2 \Sigma_{t|t-1}}{\sigma_\nu^2 \sigma_\eta^2 + \sigma_\nu^2 \Sigma_{t|t-1} + \sigma_\eta^2 \Sigma_{t|t-1}} \right)$$

represent the relative importance of the errors (the surprises) with respect to the prior estimate and $\Sigma_{t|t-1}$ denotes the error variance of the one-step ahead prediction of the fundamental. From (4), I can show that ambiguity averse agents update their beliefs according to the following decision criteria:

$$\sigma_\nu^2 = \begin{cases} \underline{\sigma}_\nu^2, & \text{if } a_t > x_{t|t-1} \text{ and } s_t < x_{t|t-1} \\ \bar{\sigma}_\nu^2, & \text{if } a_t < x_{t|t-1} \text{ and } s_t > x_{t|t-1} \end{cases}$$

As G_t (H_t) is increasing (decreasing) in the signal precision ($1/\sigma_\nu^2$), whenever revisions to the previous period estimate of the fundamental following the signals, $(a_t - x_{t|t-1})$ and $(s_t - x_{t|t-1})$, have different signs, it is easy to pin down the signal precision to minimize $x_{t|t}$. For instance, when $a_t > x_{t|t-1}$ and $s_t < x_{t|t-1}$, G_t should take the largest values and H_t smallest value. Thus, $1/\underline{\sigma}_\nu^2$ minimizes the estimate of the fundamental $x_{t|t}$. Similarly, when $a_t < x_{t|t-1}$ and $s_t > x_{t|t-1}$, minimizing $x_{t|t}$ requires G_t to take the smallest value and H_t to take the largest possible value such that $1/\bar{\sigma}_\nu^2$ is chosen to update beliefs. Intuitively, as a given signal becomes less precise, the gain from observing that particular signal is relatively smaller; at the same time, you gain relatively more from observing the other signal. Left panel in Figure 3 depicts this cut-off rule of ambiguity averse agents. When both signals are greater than (the upper-right quadrant) or smaller than (the lower-left quadrant) the previous period estimate of the fundamental, (4) does not seem to produce simple decision criteria.

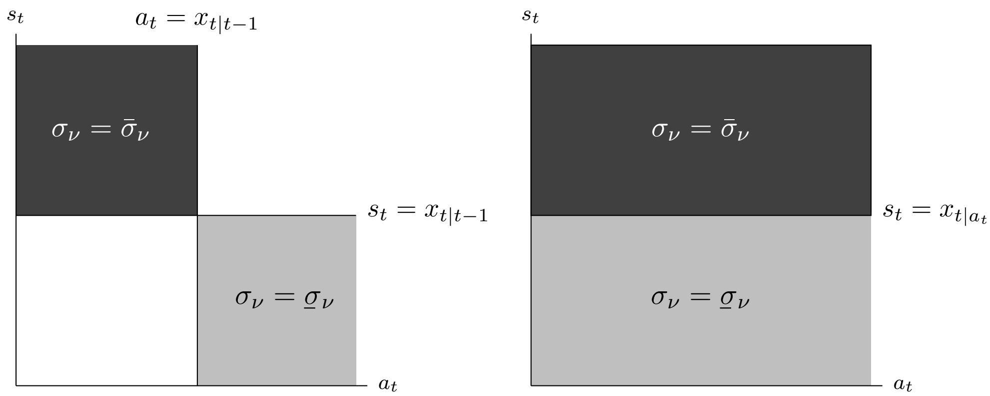

Figure 3: The Cutoff Rules: from (4) (left) and (5) (right)

Notes: Left panel represents the decision rule of simultaneous belief updating, whereas right panel depicts the decision rule of sequential belief updating. In this particular case, it is assumed that $a_t = x_{t|t-1}$.

2.2.1 Sequential belief updating

Now, let the agents update beliefs sequentially such that they first update beliefs with the unambiguous signal a_t and then with the ambiguous signal s_t such that $\mathbb{E}[x_t|\mathcal{I}_{t-1}, a_t] = x_{t|a_t}$ and $\mathbb{E}[x_t|\mathcal{I}_{t-1}, S_t] = x_{t|t}$ respectively represent a belief updated with productivity and a belief updated with both signals:

$$\begin{aligned} x_{t|a_t} &= \left(\frac{\sigma_\eta^2}{\sigma_\eta^2 + \Sigma_{t|t-1}}\right) x_{t|t-1} + \left(\frac{\Sigma_{t|t-1}}{\sigma_\eta^2 + \Sigma_{t|t-1}}\right) a_t \\ x_{t|t} &= x_{t|a_t} + \left(\frac{\Sigma_{t|a_t}}{\Sigma_{t|a_t} + \sigma_\nu^2}\right) \left(s - x_{t|a_t}\right) \end{aligned} \tag{5}$$

where $\Sigma_{t|a_t}$ is the error variance associated with the prediction $x_{t|a_t}$. Then, ambiguity averse agents update beliefs according to the following decision rule:

$$\sigma_\nu^2 = \begin{cases} \bar{\sigma}_\nu^2, & \text{if } s_t > x_{t|a_t} \\ \underline{\sigma}_\nu^2, & \text{if } s_t < x_{t|a_t} \end{cases}$$

From the second term in (5), whenever $s_t > x_{t|a_t}$, $1/\bar{\sigma}_\nu^2$ (low precision) is chosen to update beliefs since the attached weight to the revision based on the surprise associated with the noise shock s_t is decreasing in the signal precision. Similarly, whenever $s_t < x_{t|a_t}$, $1/\underline{\sigma}_\nu$ (high precision) is chosen to update beliefs. Right panel in Figure 3 depicts this cut-off rule for the ambiguity averse agents. This is consistent with the way agents update beliefs such that they would use all information including any unambiguous information contemporaneously available in order to make decisions under ambiguity. In fact, we can easily obtain (5) from (4), and vice versa. Thus, updating beliefs sequentially as described in (5) is just a different way to illustrate simultaneous belief updating.[17]

3 Model

Having illustrated agents' belief updating, which will be a crucial ingredient of the model to be followed, for the rest of the paper, I concentrate of the following simple setup which is analytically convenient and simultaneously provides a good starting point to look at the post-war U.S data.

The model aims to capture the notion that productivity changes follow two type of shocks. The first one, which I call a permanent shock, has a permanent effect on productivity movements and the effects of the second one, a transitory shock, die out gradually. Modelling productivity movements with a permanent and transitory shock is present in Aguiar and Gopinath (2007), Garcia-Cicco, Pancrazi, and Uribe (2010), Boz, Daude, and Durdu (2011), Blanchard et al. (2013), Cao and L'Huillier (2015) among others.

The second ingredient is that consumers' spending decisions are based on their expectations about the future, in particular, about the long-run productivity. I assume that agents observe productivity as a whole but are not able to separately observe two components. Allowing for the idea that agents have more information than merely current and past productivity, agents are assumed to observe an additional signal about the permanent productivity. The novelty of this model is that agents perceive this signal as ambiguous. Given this information structure, agents are to solve the signal extraction problem and, given their expectations, choose consumption spending.

To focus on the informational aspect of the model, the model is deliberately simplified such that consumption is the only endogenous variable to be solved for, and the dynamics of consumption is determined by productivity shocks and a shock to the noise in the signal.

3.1 Information structure

Consider a "news and noise" information structure where productivity (in logs) is composed of two components - a permanent component x_t and a transitory component z_t:

$$a_t = x_t + z_t \tag{6}$$

[17] See Appendix D.1 for discussion.

where agents do not observe the two components separately. Instead, they observe productivity as a whole. The permanent component follows a trend that changes randomly due to permanent productivity shocks and it follows the stochastic process:

$$\Delta x_t = \rho_x \Delta x_{t-1} + \epsilon_t \tag{7}$$

whereas the transitory component follows the stationary stochastic process where it dies out after transitory productivity shocks:

$$z_t = \rho_z z_{t-1} + \eta_t \tag{8}$$

The coefficients ρ_x and ρ_z are assumed to be in $[0,1)$ and $\epsilon_t \sim N(0,\sigma_\epsilon^2)$ and $\eta_t \sim N(0,\sigma_\eta^2)$. Agents are assumed to know the precisions of the productivity shocks and both technologies have an identical persistence such that $\rho_x = \rho_z \equiv \rho$. I assume that the following condition holds:

$$\rho\sigma_\epsilon^2 = (1-\rho)^2\sigma_\eta^2$$

which implies that the univariate process for a_t is a random walk:

$$\mathbb{E}[a_{t+1}|a_t, a_{t-1}, \dots] = a_t$$

In addition to productivity, agents observe a noisy signal concerning the permanent component of productivity:

$$s_t = x_t + \nu_t \tag{9}$$

where ν_t is an i.i.d. normal shock with mean zero and variance σ_ν^2. The processes $\{\epsilon_t\}_{t=0}^\infty$, $\{\eta_t\}_{t=0}^\infty$, and $\{\nu_t\}_{t=0}^\infty$ are assumed to be independent of the process $\{x_t\}_{t=0}^\infty$ and of each other. Following Epstein and Schneider (2008) there is incomplete knowledge about signal quality and the agents treat signals as ambiguous by updating beliefs as if they have multiple likelihoods. Specifically, the noisy signal s_t is related to the process x_t by a family of likelihoods through the signal precision:

$$1/\sigma_{\nu,t}^2 \in [1/\underline{\sigma}_\nu^2, 1/\bar{\sigma}_\nu^2]$$

Therefore, agents depart from Bayesian updating[18] and do not know the exact signal quality. Instead, the quality of information is captured by the range of precisions $[1/\bar{\sigma}_\nu^2, 1/\underline{\sigma}_\nu^2]$. In addition, agents are assumed not to be able to attach subjective probabilities to the priors; if they can, agents would simply form a subjective expectation to update

[18] If $1/\underline{\sigma}_\nu^2 = 1/\bar{\sigma}_\nu^2$, we are back to Bayesian updating.

beliefs.[19]

A signal is said to be more ambiguous if, given the lower $(1/\underline{\sigma}_\nu^2)$ or the upper bound $(1/\bar{\sigma}_\nu^2)$ of the signal precisions, the difference between the two is greater.[20] At the limit $(1/\underline{\sigma}_\nu^2 = 1/\underline{\sigma}_\nu^2 = 1/\sigma_\nu^2)$ agents update beliefs by a Bayesian process in which they use the standard Kalman filter with the signal precision given by $1/\sigma_\nu^2$ to estimate the fundamental. The range of precisions is assumed to remain constant over time and does not depend on other parameters.

3.1.1 Skewed consumption growth

An important implication of this model is that agents' beliefs and consumption responses are negatively skewed. Table 2 documents skewness of consumption expenditure growth per capita of the U.S. from 1970:I to 2016:I conditional on productivity process. It shows that consumption series is skewed more to the left than productivity series is. Dividing the sample into two sub-samples - the one with higher than the average productivity growth and the one with lower than the average productivity growth[21] and computing the skewness of consumption expenditure growth and productivity growth of the two samples, I find that consumption growth is more left-skewed than its productivity growth counterpart in the high productivity sample whereas they are more or less equally skewed in the low productivity sample.[22]

Table 2: Left-skewed Consumption Growth Conditional on (Labor) Productivity

Sample	Consumption skewness	Productivity skewness	Number of observations
Whole sample	-0.9014	-0.1812	185
High (labor) productivity	0.1666	1.7957	93
Low (labor) productivity	-1.2042	-1.2360	92

Notes: The first sample (high productivity) contains observation for those with productivity growth higher than the average of the whole sample where the average productivity growth in the whole sample is 0.0032. Similarly, the second sample (low productivity) includes those observations with productivity growth lower than the average.

[19]For example, if agents have a subjective belief such that $p(1/\bar{\sigma}_\nu^2) = 1/3$ and $p(1/\underline{\sigma}_\nu^2) = 2/3$, then they can construct a subjective expectation on $1/\sigma_\nu^2$: $\mathbb{E}[1/\sigma_\nu^2] = 1/3\left(1/\bar{\sigma}_\nu^2\right) + 2/3\left(1/\underline{\sigma}_\nu^2\right)$.

[20]Similarly, given $\kappa > 0$ and $\left[-\kappa\left(1/\sigma_\nu^2\right), \kappa\left(1/\sigma_\nu^2\right)\right]$, higher κ corresponds to a signal being more ambiguous.

[21]To define productivity, I simply assume that labour is the only input of the production process:

$$Y_t = A_t N_t$$

and that productivity is defined to be the output divided by labour input and in the first sample, observations are those that the log difference of output, $\log(A_t) - \log(A_{t-1})$, is greater than 0.0032 and for the second sample, I take those such that $\log(A_t) - \log(A_{t-1})$ is less than 0.032.

[22]Conducting the exercise with TFP data (dtfp_util) from Fernald (2014) draws a similar conclusion such that the estimated skewness from the TFP series for the sample period is 0.2967.

Similarly, I regress consumption growth on productivity growth and obtain residuals ϵ_t^{ols} where the residuals can be thought of as the variations in consumption growth not explained by the variations of productivity changes. The skewness of the residuals under different specifications are all estimated to be strictly negative and statistically significant at the 1% level (p-val < 0.01), suggesting that the asymmetries are effective beyond productivity changes.

Table 3: Consumption and Productivity Regressions

Specification	Δa	$\Delta a(-1)$	$\Delta c(-1)$	Skewness of residuals
1	0.4976 (0.0629)			-0.6897
2	0.5037 (0.0618)	0.1936 (0.0621)		-0.6230
3	0.4687 (0.0632)	0.1106 (0.0721)	0.1633 (0.0741)	-0.5139

Notes: Standard errors are in parentheses. A constant term is included in all specifications.

However, it does not provide any concrete evidence on why a noisy signal should be ambiguous instead of productivity observation. Conceptually, a model in which agents are ambiguous about productivity observation can also deliver negatively skewed consumption responses. Although I cannot tell which one of the signals is the underlying source of ambiguity, a sensible first step is to analyse the consequence of agents receiving an ambiguous noise signal about permanent productivity as productivity observation is likely coming from more reliable sources.

3.2 Consumption behaviour

I now go on to describe the behaviour of consumption in the model. Assume that consumption smoothing leads to the consumption Euler equation:

$$c_t = \widehat{\mathbb{E}}\left[c_{t+1}|\mathcal{I}_t\right] = \widehat{\mathbb{E}}_t\left[c_{t+1}\right]$$

where $\widehat{\mathbb{E}}$ to denote the consumers' expectation based on a worst-case belief at period t.[23]

[23]With ambiguous information quality, consumers maximize the multiple prior utility

$$\max_{C_t} \min_{\Omega} \mathbb{E}_t\left[\sum_{t=0}^{\infty} u(C_t)\right]$$

where the prior is on information quality of the signal s_t: $\Omega = \left[1/\bar{\sigma}_\nu^2, 1/\underline{\sigma}_\nu^2\right]$ and expectation is conditional on information available at period t ($\mathcal{I}_t$), subject to a budget constraint. Consumers' information $\mathcal{I}_t$ includes observations up to time t such that $\mathcal{I}_t = \{s_j, a_j\}_{j=0}^t$ and the consumers' utility maximisation adheres to the *maxmin* criterion. More precisely, consumers maximize expected utility under a worst-case evaluation of information quality chosen from Ω. With the *min* operator consumers evaluate different

Thus, consumption at period t is equal to expected consumption under a worst case assessment of information quality at period $t+1$. As $c_t = \widehat{\mathbb{E}}_t\left[c_{t+1}\right], c_{t+1} = \widehat{\mathbb{E}}_{t+1}\left[c_{t+2}\right], \ldots,$ by the law of iterated expectation

$$c_t = \lim_{j\to\infty} \widehat{\mathbb{E}}_t\left[c_{t+j}\right]$$

Having a long-run restriction such that

$$\lim_{j\to\infty} \widehat{\mathbb{E}}_t\left[c_{t+j} - a_{t+j}\right] = 0$$

where the lower case denotes a log transformation of a given variable, I get[24]

$$c_t = \lim_{j\to\infty} \widehat{\mathbb{E}}_t\left[a_{t+j}\right]$$

Thus, consumption only depends on the consumers' expectations of productivity in the long run under a worst case assessment of information quality. Solving the model, consumption becomes a function of consumers' expectations under a worst-case belief:

$$c_t = \frac{1}{1-\rho}\left(\widehat{\mathbb{E}}_t\left[x_t\right] - \rho\widehat{\mathbb{E}}_t\left[x_{t-1}\right]\right) \tag{10}$$

where $\widehat{\mathbb{E}}_t\left[x_t\right]$ and $\widehat{\mathbb{E}}_t\left[x_{t-1}\right]$ represent the consumers' expectations on the current and the lagged permanent components of productivity under a worst-case belief. The essential ingredient in this mechanism is that the long-run productivity estimate is consistent with a worst-case expectation.

This simple permanent income consumption model can be derived from widely used DSGEs under certain limiting conditions. For example, Blanchard et al. (2013) theoretically

scenarios according to their priors and choose the worst case scenario available conditional on their decisions on the choice variables. With the *max* operator consumers maximize the worst case expected utility by choosing over the choice variables. For the rest of the paper, with a slight abuse of notation, I use $\widehat{\mathbb{E}}$ to denote the consumers' expectation based on a worst-case belief and reformulate the consumers' problem as follows:

$$\max_{C_t} \widehat{\mathbb{E}}_t\left[\sum_{t=0}^{\infty} u(C_t)\right]$$

subject to a budget constraint.

[24]The underlying assumption in this stylized model is that the supply side is drastically simplified such that I consider an economy where consumption is the only demand component with no capital and output is perfectly determined by the demand side. This implies that $y_t = c_t$ and to produce output y_t, conditional on the current level of productivity a_t, the labour input adjusts. On the contrary, in an open endowment economy, whenever $y_t \neq c_t$ consumption smoothing households, having access to an internationally traded bond, borrow (lend) against (for) future income.

show (Online Appendix Section 6.4.2) that a baseline New Keynesian (NK) model (without capital and no bells and whistles) converges to a simple permanent income model with a fixed real interest rate in which consumption is equal to the expectations of the long-run level of labour productivity. Allowing for ambiguity on information quality, a baseline NK model converges to a simple permanent income model in which consumption is equal to the expectations of the long-run level of labour productivity where the expectations are conditional on a worst case evaluation of information quality. Similarly, Appendix C shows that the small open economy RBC model converges to a simple permanent income model with a small sensitivity of the country interest-rate premium to the level of external debt.

3.3 Solving the model

As shown in (10), solving the model requires solving for consumption as a function of *beliefs about the long-run productivity* (BLR) under a worst case belief.[25] Consumers derive the expectations on the state vector

$$\mathbf{x}_t = (x_t,\ x_{t-1},\ z_t)'$$

using the Kalman filter. Let $x_{t|t} = \widehat{\mathbb{E}}[x_t|\mathcal{I}_t]$, $x_{t-1|t} = \widehat{\mathbb{E}}[x_{t-1}|\mathcal{I}_t]$ and $z_{t|t} = \widehat{\mathbb{E}}[z_t|\mathcal{I}_t]$ be the worst case current and lagged beliefs on the permanent component of productivity and the worst case current belief on the transitory component of productivity. Given new observations, the previous estimate of the permanent component is updated by applying the Kalman filter:

$$\mathbf{x}_{t|t} = [I - Gain_t \times C]\, A\mathbf{x}_{t-1|t-1} + Gain_t \times S_t$$

where $\mathbf{x}_{t|t} = (x_{t|t},\ x_{t-1|t},\ z_{t|t})'$ and $\mathbf{x}_{t-1|t-1} = (x_{t-1|t-1},\ x_{t-2|t-1},\ z_{t-1|t-1})'$ are the worst case beliefs on $\mathbf{x}_t$ at time t and on $\mathbf{x}_{t-1}$ at time $t-1$ and $S_t = (a_t,\ s_t)'$ is a vector of observables. A and C are functions of underlying parameters of the model, $Gain_t$ is a vector of Kalman gains, and I is the 3×3 identity matrix:

$$A = \begin{bmatrix} 1+\rho & -\rho & 0 \\ 1 & 0 & 0 \\ 0 & 0 & \rho \end{bmatrix},\ C = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$$

[25]Beliefs about the long-run productivity under a worst-case belief refer to

$$\lim_{j\to\infty} \widehat{\mathbb{E}}_t\,[a_{t+j}]$$

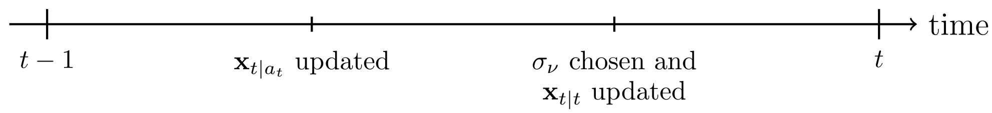

Figure 4: The Timing of Belief Updating

As discussed in Section 2.1, $Gain_t$ depends on the type of news received.

Assumption 1 *Consumers sequentially update beliefs by first updating beliefs with productivity and then with a noisy signal.*

Under Assumption 1, the solution to the model can be tracked down by a simple cut-off rule on the ambiguity parameter σ_ν. As discussed in Appendix D.1, updated beliefs are the same whether consumers update beliefs *sequentially* as in Assumption 1 or consumers update beliefs *simultaneously*. In the simultaneous belief updating, agents would use all available information including current productivity to make decisions under ambiguity, which by definition is exactly the same as updating beliefs sequentially.[26] Figure 4 describes the timing of belief updating.

Proposition 1 (The sequential updating of beliefs) *The sequential updating of consumers' beliefs can be given by*

$$\mathbf{x}_{t|t} = \mathbf{A}_t \mathbf{x}_{t-1|t-1} + \mathbf{B}_t a_t + G_t s_t \tag{11}$$

where $\mathbf{A}_t = [I - G_t C_2] [I - H_t C_1] A$, $\mathbf{B}_t = [I - G_t C_2] H_t$, H_t *is the Kalman gain of observing productivity* a_t, G_t *is the Kalman gain of observing a noisy signal* s_t, *and* A, C_1, *and* C_2 *are the matrices of underlying parameters:*

$$C_1 = \begin{bmatrix} 1 & 0 & 1 \end{bmatrix}, \; C_2 = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}.$$

Proof. See Appendix B.1. ■

The types of news that the noisy signal delivers play a crucial role in updating beliefs in terms of choosing the appropriate signal precision. Formalizing the notion, good and bad news are defined as follows:

Definition 1 (Types of news) *A noisy signal delivers* ***good news*** *when it is greater than the ex-ante belief,* $x_{t|a_t}$, *where the ex-ante belief is an expectation about the permanent component of productivity updated with all "unambiguous information" available contempo-*

[26]This does not imply that there do not exist order effects. See Appendix D.1 for details.

raneously, which is $\widehat{\mathbb{E}}[x_t|a_t, \mathcal{I}_{t-1}]$*. Similarly, it delivers* ***bad news*** *when it is smaller than* $x_{t|a_t}$*.*

A large noisy signal does not necessarily mean that it delivers good news. Instead, the types of news are related to the surprises carried by the signal relative to the ex-ante belief. The three shocks in the model are not identical in terms of the types of news that they deliver. Specifically, while a positive permanent productivity shock and a positive noise shock deliver good news to consumers, a positive transitory shock generates bad news. The intuition for these results is straightforward. A positive permanent shock to productivity increases a noisy signal one-to-one but due to the presence of the transitory component, agents underestimate the productivity increase such that $s_t > x_{t|a_t}$. Consequently, it delivers good news to consumers. Also, as a (positive) noise shock does not affect agents' beliefs, it also delivers good news. Finally, for a positive transitory shock, an increase in η_t positively affects productivity but it has no effect on a noisy signal. As $s_t < x_{t|a_t}$, it delivers bad news to consumers. Table 4 summarizes the relationship between the shocks and the types of news delivered. In addition, as discussed in Section 2, the types of news that signals deliver are directly associated with perceived information quality. For example, signals delivered by permanent productivity shocks and noise shocks are perceived (by the consumers) as low quality whereas transitory shocks to productivity generate signals with high perceived quality.

Table 4: Shocks, News, and Information Quality

Shocks (+)	News type	Information quality	Δ c	Δ a
Permanent tech shock (ϵ)	Good	Low	+	+
Transitory tech shock (η)	Bad	High	+	+
Noise shock (ν)	Good	Low	+	no change

Notes: Assume that the shocks are positive ones. +'s in Δc *and* Δa *refer to the increases in consumption and productivity.*

Proposition 2 (The cut-off rule with good news) *Let* $\mathbf{x}_{t|t}$ *be the beliefs updated with both productivity and a noisy signal and* $\mathbf{x}_{t|a_t}$ *be the beliefs updated with productivity:*

$$\mathbf{x}_{t|t} = \widehat{\mathbb{E}}\left[x_t|\{a_s\}_{s=0}^t, \{s_s\}_{s=0}^t\right]$$

$$\mathbf{x}_{t|a_t} = \widehat{\mathbb{E}}\left[x_t|\{a_s\}_{s=0}^t, \{s_s\}_{s=0}^{t-1}\right]$$

Then, if $s_t > s_{t|a_t}$ *(delivering good news), the following conditions are satisfied:*

(i) $x_{t|t} - x_{t|a_t} > 0$, $x_{t-1|t} - x_{t-1|a_t} > 0$, $z_{t|t} - z_{t|a_t} = 0$

(ii) $c_{t|t} - c_{t|a_t} > 0$

(iii) *for ambiguity averse consumers,* $\sigma_\nu^2 = \bar{\sigma}_\nu^2$

Proof. See Appendix B.2. ■

Proposition 3 (The cut-off rule with bad news) *Similarly, if* $s_t < s_{t|a_t}$ *(delivering bad news), the following conditions are satisfied:*

(i) $x_{t|t} - x_{t|a_t} < 0$, $x_{t-1|t} - x_{t-1|a_t} < 0$, $z_{t|t} - z_{t|a_t} = 0$

(ii) $c_{t|t} - c_{t|a_t} < 0$

(iii) *for ambiguity averse consumers,* $\sigma_\nu^2 = \underline{\sigma}_\nu^2$

Proof. See Appendix B.2. ■

Proposition 2 and 3 suggest that for ambiguity averse agents updating beliefs is consistent with choosing an extremum of the range of precisions. Only the boundaries of the range of precisions ($1/\bar{\sigma}_\nu^2$ and $1/\underline{\sigma}_\nu^2$) need to be evaluated to solve the model and the relevant gains of observing the noisy signal can be either $G_t(1/\underline{\sigma}_\nu^2)$ or $G_t(1/\bar{\sigma}_\nu^2)$, where $G_t(\cdot)$ represents the Kalman gain of observing the noise signal with the given precision of noise at period t. Specifically, noisy signals delivering good news are treated as if they are uninformative and the ones delivering bad news are considered very precise.

Definition 2 (The limit case) *A limit case refers to the specification in which the range of precisions degenerates to* $1/\sigma_\nu^2$.

Solving the model, then, requires consumers to determine *beliefs about the long-run productivity* under a worst case belief, which is the right-hand side of (10), to satisfy consumers' aversion to ambiguity.[27]

3.3.1 The steady-state Kalman gain

In practice of computing the Kalman gain, one often applies the steady state concept in the sense that the economy is assumed to have been in operation long enough that the Kalman gain has converged to its steady state value. When the signal precision is *known*, i.e. $1/\underline{\sigma}_\nu^2 = 1/\bar{\sigma}_\nu^2 = 1/\sigma_\nu^2$, for a reasonable value of signal precision, the convergence of the Kalman gain to its steady state value is achieved relatively quickly. Figure 5 shows that the rate of numerical convergence of the Kalman gain in different signal precisions for the model illustrated in Section 2.1 with *a single likelihood.* It shows that numerical

[27]Specifically, consumers' filtering in (11) is combined with (10) to determine consumption.

convergence of the Kalman gain is achieved relatively quickly. Thus, the assumption of the steady-state Kalman gain is not so restrictive in this case where the variances of the stochastic processes are constant over time.

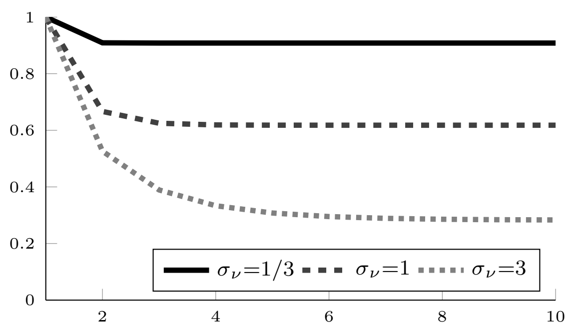

Figure 5: Kalman Filter: Convergence

Notes: The three lines represent the convergence of Kalman gains with the following signal precision ($\sigma_\nu = 1/3$, $\sigma_\nu = 1$, $\sigma_\nu = 3$). $x_{1|0}$ and $\Sigma_{1|0}$ are respectively set to 0 and 10^6.

However, this is not the case when ambiguity is introduced in the model such that the convergence of the Kalman gain is not achieved even in the long-run.[28] Figure 6 (Left Panel) shows the values for the Kalman gain for a 100-period simulation where $\sigma_\epsilon = \sigma_\nu = 1$ and $\bar{\sigma}_\nu = 1.25$ and $\underline{\sigma}_\nu = 0.75$. It shows that the Kalman gain numerically do not converge to the steady state values, the high and low gain respectively for the case of good news ($\underline{\sigma}_\nu$) and bad news ($\bar{\sigma}_\nu$), which are denoted by the dotted lines. This implies that the Kalman gain is not time-invariant even in the long-run as the signal-to-noise ratio does not remain constant and depends on the types of news that agents receive.

The high (low) gain is associated with agents receiving bad (good) news. Moreover, agents tend to overly sensitive to signals they receive: Agents attach more (less) value on the signals they receive in case of bad (good) news than at the steady state. This is due to the fact that the error variance $\Sigma_{t|t-1}$ and the variance of the noise σ_ν^2 move the Kalman gain in (2) in opposite directions.

When a spell of consecutive bad (or good) observations is realized, the Kalman gain indeed converges to the steady state value; however, if agents receive a different type of news, the Kalman gain does not immediately converge to the (other) steady state value. In fact, history dependence is crucially important in this case as past realisations of news affect agents' present belief formation.

The exception is when the variance of the noisy signal is sufficiently high ($\sigma_\nu^2 >> \sigma_\epsilon^2$)

[28]The only exception is when the fundamental x_t is an i.i.d such that $x_t = \epsilon_t$. In that case, $\Sigma_{t|t-1}$ and $\Sigma_{t|t}$ are constants and that $\Sigma_{t|t-1} = \Sigma_{t|t} = \sigma_\epsilon^2$ for all t. Therefore, the Kalman gain takes the value $Gain_t = \underline{Gain} = \frac{\sigma_\epsilon^2}{\sigma_\epsilon^2 + \bar{\sigma}_\nu^2}$ for the good news regime and $Gain_t = \overline{Gain} = \frac{\sigma_\epsilon^2}{\sigma_\epsilon^2 + \underline{\sigma}_\nu^2}$ for the bad news regime.

as shown in Figure 6 (Right Panel) where it shows the values for the Kalman gain for a 100-period simulation where $\sigma_\epsilon = 1$, $\sigma_\nu = 10$, $\bar{\sigma}_\nu = 12.5$, and $\underline{\sigma}_\nu = 7.5$.

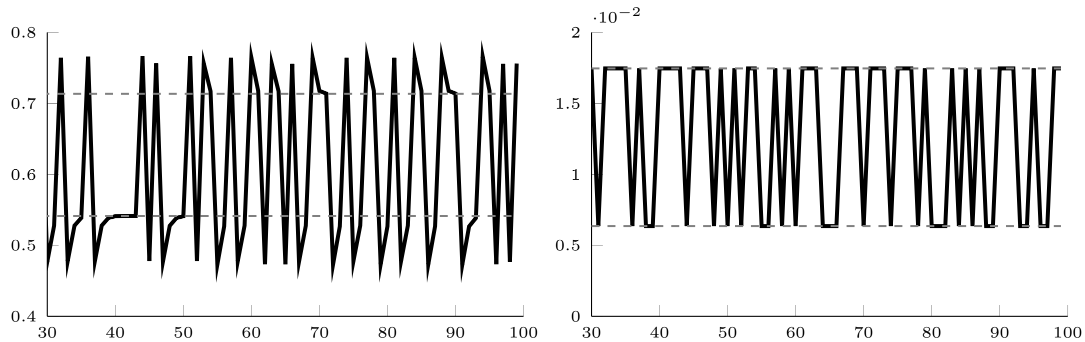

Figure 6: Kalman Filter: Convergence (Ambiguous Signal)

Notes: The black solid line depicts the Kalman gain according to (2) and the gray dotted lines denote the steady state Kalman gains for $\sigma_\nu = \underline{\sigma}_\nu$ (Top) and $\sigma_\nu = \bar{\sigma}_\nu$ (Bottom).

3.3.2 Ambiguity toward the Kalman gain

So far, the assumption has been that agents have beliefs about the information precision and that the information precision is ambiguous:

$$1/\sigma^2_{\nu,t} = [\underline{\sigma}^2_\nu, \bar{\sigma}^2_\nu]$$

While this way of making explicit assumptions about means and variances of stochastic variables and deriving implications for Kalman gains and conditional best forecasts is a standard approach, it is not unreasonable to assume that agents directly form beliefs about how much they learn from one new data point, i.e. beliefs about Kalman gains. Assuming that such beliefs are ambiguous, the uncertainty can be captured by *the range of Kalman gains*:

$$Gain_t = [\underline{Gain}, \overline{Gain}]$$

In other words, agents do not know the Kalman gain due to new information at period t. Furthermore, they do not believe it is constant, nor do they have a subjective probability distribution over it. They are firmly convinced that it falls into the range, $\underline{Gain}$ (low gain) to $\overline{Gain}$ (high gain), and exhibit aversion toward uncertainty. Then, with good news ($s_t > x_{t|t-1}$), the computed Kalman gain takes the smallest value ($Gain_t = \underline{Gain}$) and with bad news ($s_t < x_{t|t-1}$), $Gain_t = \overline{Gain}$.

Assuming that agents are ambiguous about the Kalman gain simplifies the analysis as computing the Kalman gain is no longer history-dependent as illustrated in the previous section. For the rest of the paper, I will stick to the following assumption of the steady state Kalman gains.

Assumption 2 (The steady-state Kalman gains) *The steady-state Kalman gains are achieved with respect to $1/\underline{\sigma}_\nu$ for the case of good news and $1/\bar{\sigma}_\nu$ for the case of bad news.*

4 Quantitative Exercises

In this section I adopt the solution procedure described in Section 3.3 and proceed to analyse asymmetries generated by ambiguous information.

4.1 Asymmetric responses of consumption

Figure 7 reports the responses of consumption to the shocks in two different set-ups: When the noisy signal is *unambiguous* (the limit case) and when it is *ambiguous*. The time unit is one quarter and the impulse responses are one standard deviation positive *and* negative shocks. Responses of the positive shocks are depicted in the first column and those of the negative shocks are depicted in the second column. I use the estimated parameters in Table 6 as parameters. More precisely, the persistence parameter for productivity ρ is set to 0.9754 and σ_u is set to 0.68% which implies that the standard deviations of the technology shocks are given by $\sigma_\epsilon = 0.02\%$ and $\sigma_\eta = 0.67\%$ and that of the noise shock, σ_ν, is set to 3.86% and *the range of precisions* is given by $[4.65\%, 3.51\%]$. Since ambiguity has no effect on the dynamics of productivity, the responses are completely symmetric as depicted in Figure 8. Obviously, as productivity does not depend on the consumers' expectations, a noise shock does not move productivity at all.

In response to a permanent technology shock ϵ_t, consumption increases slowly, which implies that the volatilities of other shocks, which cloud consumers' ability to recognize and adjust consumption, are large. In response to a transitory technology shock η_t, consumption initially increases but then declines. As productivity initially increases and then slowly declines, consumers partly believe that this increase in productivity is due to a permanent increase in productivity. However, consumers do learn over time that the increase in productivity is due to the transitory shock and consumption returns to the original level. For a noise shock ν_t, consumption increases and then returns to normal over time. It is such that the consumption responses are symmetric in the limit case. However, with ambiguity, consumption, in most cases, tends not to move as much as in the limit case.

The effects of ambiguity can be observed from the asymmetric responses of consumption to the signs of the shocks. Under ambiguity, consumers are hesitant to respond to good news

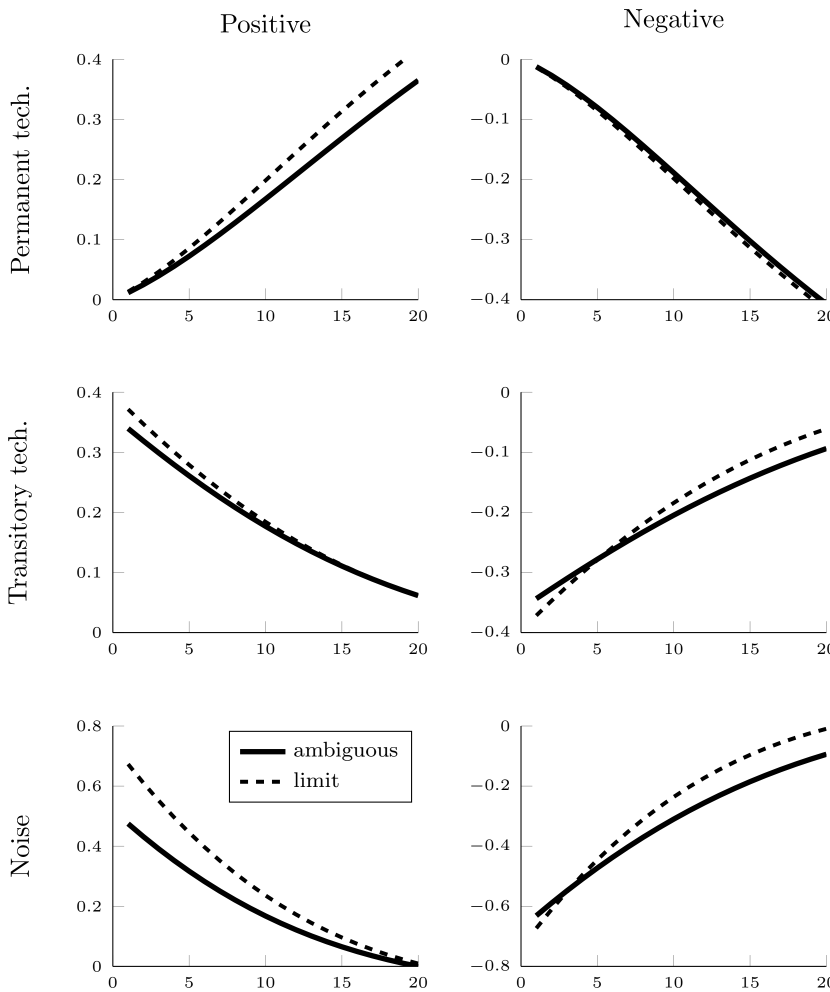

Figure 7: Impulse Responses: Consumption

Notes: Plots in the left column correspond to the IRFs of positive shock of one standard deviation and those in the right column correspond to the IRFs of negative shock of one standard deviation. The solid line corresponds to the case in which the noisy signal is ambiguous whereas the dotted line corresponds to the limit case in which $\underline{\sigma}_\nu^2 = \bar{\sigma}_\nu^2$.

but are more willing to react to bad news. The intuition is that, with ambiguity, consumers become pessimistic about the future (which is not only uncertain but also ambiguous) and that such pessimism directly translates into consumption responses in this setup.

Figure 9 depicts the asymmetric responses of consumption more closely by comparing the size of consumption responses to the shocks with different signs. In the limit case, the magnitudes of the effects are completely symmetric. However, when the noisy signal is ambiguous, the magnitudes of the responses are greater with the negative shocks (solid line) than with the positive shocks (dotted line). To examine how the downward bias of ambiguity averse consumers translate into consumption dynamics, I conduct the following simulation exercise: I keep the underlying (unambiguous) parameters the same as in the

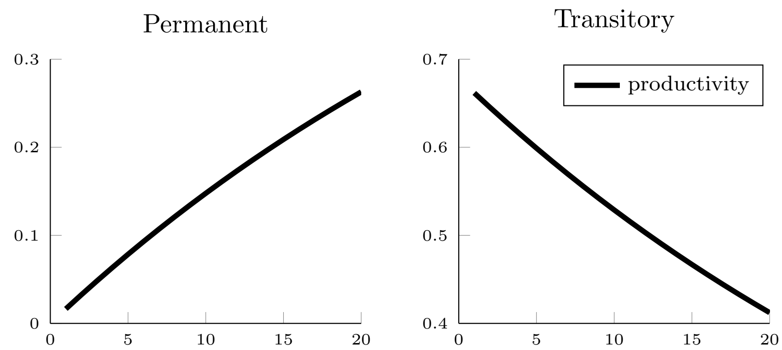

Figure 8: Impulse Responses: Productivity

Notes: The IRF of productivity is identical in both the limit case and ambiguous specification. Productivity does not respond to a noise shock.

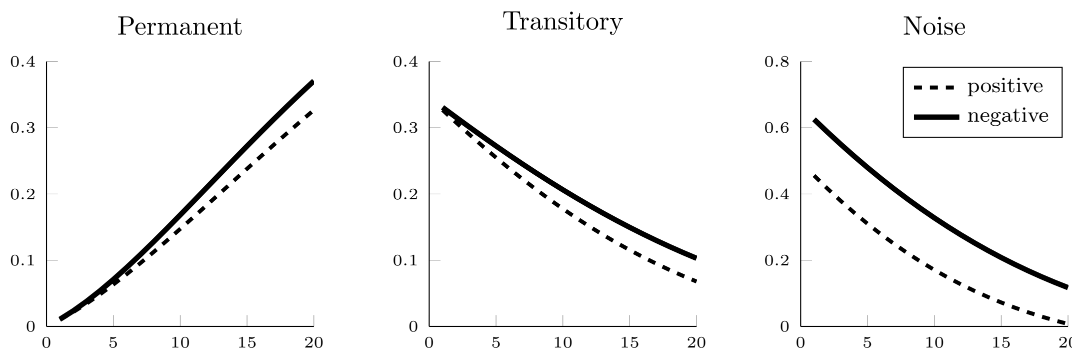

Figure 9: Size of Impulse Responses: Consumption

Notes: In order to compare the magnitude of the responses, I multiply -1 to the responses of consumption with negative shock and plot it with the responses to the positive shock. The solid line corresponds to the case in which the shocks are positive and the dotted line corresponds to consumption responses the following negative shocks.

previous exercise and use different values for the range of precisions of the ambiguity parameter σ_ν to evaluate the effects of ambiguity on consumption dynamics. Specifically, I assume that

$$\sigma_{\nu,t} = [(1-\xi)\sigma_\nu, (1+\xi)\sigma_\nu]$$

and consider the following cases where ξ takes the value 0, 0.1, 0.25, 0.5, 0.75, and 0.9.[29]

Table 5 reports the simulated consumption moments. The first row represents the limit case ($\xi = 0$) and the rest considers the case in which the signal is ambiguous. The degree of ambiguity is in increasing order and the higher degree of ambiguity is associated with more negatively skewed consumption growth and higher volatility.

[29] I fix the length of series to 1000 periods and the number of replication is set to 10000.

Table 5: Consumption Moments Simulation

Specification	ξ	$[\underline{\sigma}, \bar{\sigma}]$	Skewness	Variance (%)	Mean
1.	0	$[3.860, 3.860]$	0.000	0.680	0.000
2.	0.1	$[3.474, 4.246]$	-0.176	0.684	0.000
3.	0.25	$[2.895, 4.825]$	-0.454	0.704	0.000
4.	0.5	$[1.930, 5.790]$	-1.035	0.793	0.000
5.	0.75	$[0.965, 6.755]$	-2.092	1.062	0.000
6.	0.9	$[0.386, 7.334]$	-3.634	1.677	0.000

Notes: The true signal precision is given by $1/(0.0386)^2$. For this exercise I assume that degrees of ambiguity is symmetric such that it is captured by the parameter α where $\sigma_{\nu,t} \in [(1-\xi)\,\sigma, (1+\xi)\,\sigma]$ and $\sigma_\nu = 0.0386$. More precisely, the six cases depicted here correspond to α being 0, 0.1, 0.25, 0.5, 0.75, and 0.9.

4.2 Separation of beliefs from fundamentals

One of the interesting features of this model is when information is assumed to be ambiguous, the relationship between agents' beliefs and fundamentals can potentially become *unrelated* to each other. For example, consider the simple model in Section 2.1. Specifically, take the extreme case where *the range of precisions* is given by

$$1/\sigma_{\nu,t}^2 \in [0, +\infty]$$

which implies that when agents receive good news ($s_t > x_{t|t-1}$), they would consider this information useless, whereas when agents receive bad news ($s_t < x_{t|t-1}$), they would take this information at face value. Since x_t follows a random walk, agents' beliefs can be characterized by

$$x_{t|t} \leq x_{t-j|t-j}, \quad \forall 1 \leq j \leq t \tag{12}$$

implying that agents' beliefs are non-increasing. Given the initial condition such that $x_{0|0} = x_0$ (agents' beliefs are correct to begin with), this amounts to more and more negative beliefs about fundamentals over time. At the same time, the series for agents' beliefs can become flat for extremely long periods as agents' beliefs would become detached from fundamental as in (12). This would lead to the flattened series of agents' beliefs.

Departing from this extreme case example, a presence of ambiguity would still generate separation of beliefs from fundamentals and how severe a detachment is can be attributed to the degree of ambiguity. First, assuming that noisy signals are *unambiguous*, I run the Kalman smoother on U.S. data to extract the sequence of structural shocks and construct productivity series (a_t) and noisy signals (s_t) - the two signals consumers observe. Then, I feed these signals into my benchmark model with ambiguity and reconstruct the series for

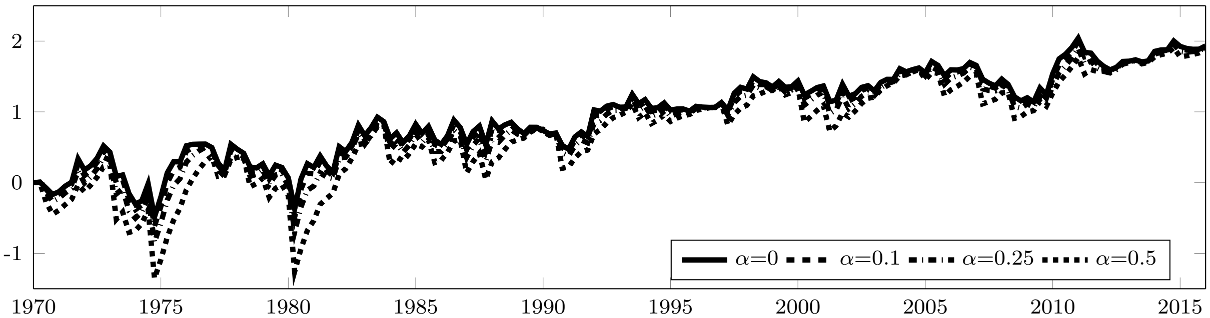

Figure 10: Reconstructed Consumption with Ambiguity

Notes: The figure shows the detrended consumption. The smoothed estimates of productivity and noisy signals are obtained from the U.S data with the limiting assumption ($\underline{\sigma}_\nu = \bar{\sigma}_\nu$). The solid line corresponds to the path of consumption without ambiguity. The dashed lines correspond to the counterfactual sample paths obtained with different degrees of ambiguity. The size of ambiguity is captured by the parameter ξ such that $\sigma_\nu \in [(1-\xi)\,\sigma, (1+\xi)\,\sigma]$ where $\sigma = 4.36\%$.

beliefs about the fundamental and consumption growth. I use four different sets of values on the range of precisions, $\sigma_{\nu,t} \in [(1-\xi)\sigma_\nu, (1+\xi)\sigma_\nu]$ where σ_ν is the estimated standard deviation of the noise shock: (1) $\xi = 0$, (2) $\xi = 0.1$, (3) $\xi = 0.25$, and (4) $\xi = 0.5$.

Figure 10 shows that agents consume less with more ambiguous the signals are. This may provide some interesting welfare consequences. Less (more) willingness to react to good (bad) news translates into underestimating the economy's long run potential, which in turn decreases contemporaneous consumption and is welfare worsening.

4.3 Estimation

4.3.1 Econometrician's filtering

While the econometrician does not observe noisy signals, she observes consumption such that the econometrician's set of observables include productivity and consumption series. The consumers' filtering suggests that only a maximum and minimum of the range of precisions needs to be evaluated for the ambiguity averse consumers' belief updating. The econometrician use this decision rule for consumers' filtering, and the econometrician's filter as well becomes state dependent – the one with *the low precision* $(1/\bar{\sigma}_\nu^2)$ and the one with *the high precision* $(1/\underline{\sigma}_\nu^2)$. The underlying mechanism for constructing the econometrician's filter is based on the fact that even though the econometrician does not observe a noisy signal, she can fully recover the state (or the types of news that consumers received) each period. The econometrician is able to determine whether beliefs have been updated with the high or low precision using information available contemporaneously. In fact, similar to the consumers' cut-off rule to update beliefs, a cut-off rule to determine how consumers have updated beliefs at each period can be applied to the econometrician's filter. Then, a likelihood function can be constructed accordingly and the model is estimated through the maximum likelihood estimation.

Proposition 4 (The econometrician's cutoff rule) *Let* $\mathbf{x}_{t|t}$ *be the beliefs updated with both productivity and a noisy signal and* $\mathbf{x}_{t|a_t}$ *be the beliefs updated with productivity:*

$$\mathbf{x}_{t|t} = \widehat{\mathbb{E}}\left[\mathbf{x}_t | \{a_s\}_{s=0}^{t}, \{s_s\}_{s=0}^{t}\right]$$

$$\mathbf{x}_{t|a_t} = \widehat{\mathbb{E}}\left[\mathbf{x}_t | \{a_s\}_{s=0}^{t}, \{s_s\}_{s=0}^{t-1}\right]$$

where $\mathbf{x}_{t|t} = \left(x_{t|t},\ x_{t-1|t},\ z_{t|t}\right)'$. *Then,*

1. $s_t > x_{t|a_t} \Longleftrightarrow c_t > c_{t|a_t}$

2. $s_t < x_{t|a_t} \Longleftrightarrow c_t > c_{t|a_t}$

where $c_{t|a_t}$ *is the consumption that consumers would have consumed had not observed the noisy signal* s_t*.*

Proof. See Appendix B.3. ■

By observing productivity, the econometrician is able to determine consumption that consumers would have chosen to spend without observing a noisy signal, which is denoted by $c_{t|a_t}$. Comparing this with the observed consumption in the data and applying the cut-off rule in Proposition 4, the econometrician can recover the types of news that consumers received. Essentially, much like the belief updated with productivity $x_{t|a_t}$ is used for the cut-off rule of the consumers, $c_{t|a_t}$ is similarly used for the cut-off rule of the econometrician.

For instance, consider the case in which the observed consumption c_t is greater than $c_{t|a_t}$. From Proposition 4 this implies that consumers have updated beliefs with the signal precision $1/\bar{\sigma}_\nu^2$. Intuitively, as the observed consumption is greater than the consumption consumers would have consumed without having observed a (contemporaneous) noisy signal, consumers must have received good news from the noisy signal and have decided to consume more. At the same time, with good news, consumers must have considered this signal not so informative.

The econometrician's filtering can be obtained with the consumer's filter described in the previous section and the econometrician's cutoff rule in Proposition 4. Let the consumers' belief updating be given by equation (11) and the econometrician's state vector be $\mathbf{x}_t^E = (x_t,\ x_{t-1},\ z_t,\ x_{t|t},\ x_{t-1|t},\ z_{t|t})'$. Furthermore, let $c_{t|a_t}$ be the consumption after observing productivity a_t:

$$c_{t|a_t} = \frac{1}{1-\rho}\left(\widehat{\mathbb{E}}\left[x_t|a_t, \mathcal{I}_{t-1}\right] - \rho\widehat{\mathbb{E}}\left[x_{t-1}|a_t, \mathcal{I}_{t-1}\right]\right)$$

where $\mathcal{I}_{t-1} = \{a_j, s_j\}_{j=0}^{t-1}$. Then, the measurement equation for the econometrician's state vector $\mathbf{x}_t^E$ is

$$\mathbf{x}_t^E = Q\mathbf{x}_{t-1}^E + RV_t^{'} \tag{13}$$

where $V_t^{'} = (\epsilon_t,\ \eta_t,\ \nu_t)$ and R and Q are defined by

$$R = (1-j)R^0 + jR^1$$
$$Q = (1-j)Q^0 + jQ^1$$

The matrices Q^j and R^j depend on the realized news such that $j = 0$ corresponds to the realisation of good news and $j = 1$ indicates the realisation of bad news. According to Proposition 4, $j = 0$ if $c_t > c_{t|a_t}$ and $j = 1$ if $c_t < c_{t|a_t}$. Since the econometrician observes productivity and consumption, the observation equation is

$$(a_t,\ c_t)^{'} = T\mathbf{x}_t^E \tag{14}$$

where

$$T = \begin{bmatrix} 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{1-\rho} & \frac{-\rho}{1-\rho} & 0 \end{bmatrix}$$

The econometricians' filtering problem can, then, be solved with (13), (14), and the cutoff rule in Proposition 4.[30]

[30]For identification, Blanchard et al. (2013) illustrate two special cases, when the signal is perfectly informative or when it is completely uninformative, in which a structural VAR recovers ϵ_t and η_t, and their dynamics effects. In the presence of ambiguity, however, it is not possible, even in these special cases, to simply rely on a structural VAR to recovers shocks in the model. First, consider the case of a fully uninformative signal where $1/\sigma_\nu^2 = 0$ where the range of precisions is given by

$$1/\sigma_\nu^2 = [0, 1/\underline{\sigma}_\nu^2]$$

such that agents would believe either that signal is fully uninformative or that its precision is given by $1/\underline{\sigma}_\nu^2$. In this case, even if the signal is useless, it may affect the agents' belief updating. Similarly, when the signal is fully informative such that $1/\sigma_\nu^2 = \infty$ where the range of precision is defined by

$$1/\sigma_\nu^2 = [1/\bar{\sigma}_\nu^2, \infty],$$

agents might believe that the signal is less than fully informative depending on the types of news they receive. This is an interesting departure from the limit case benchmark. In the limit case, if the signal is fully informative, consumers are able to identify the permanent shock to productivity directly. However, when information quality is given by the range of precisions, even if the signal is fully informative, consumers are not able to recover the permanent shock perfectly.

4.3.2 Structural estimation

The model is estimated through maximum likelihood where a likelihood function depends on the types of news realized. The econometrician can fully recover the regime, i.e., *whether a noisy signal delivers good or bad news*, with the contemporaneously available information as discussed in the previous section. Therefore, the regime can be revealed in each period and the likelihood function can be modified to incorporate that.

Table 6: Parameter Estimates, US 1970:I-2016:I

Parameter	Description	Value	s.e.
ρ	Persistence productivity	0.9754	0.0021
σ_u	Std dev. productivity	0.0068	0.0002
σ_ϵ	Std dev. permanent shock (implied)	0.0002	-
σ_η	Std dev. transitory shock (implied)	0.0067	-
$\underline{\sigma}_\nu$	Std dev. noise shock (lower bound)	0.0351	0.0003
$\bar{\sigma}_\nu$	Std dev. noise shock (upper bound)	0.0465	0.0003
σ_ν	Std dev. noise shock	0.0386	0.0020
	log-likelihood	1336.15	

Notes: σ_ϵ and σ_η are obtained with random walk assumption of (9). Hence, no standard errors are given.

Consumption is constructed by taking the first difference of the logarithm of the ratio of NIPA consumption to population whereas productivity is constructed by taking the first difference of the logarithm of the ratio of GDP to employment. Real personal consumption expenditure (PCECC96), real gross domestic product (GDPC1), population (LNS10000000Q), and employment (LNS12000000Q) series are from 1970:I to 2016:I and are available at the U.S. Bureau of Economic Analysis for the first two series and at the U.S. Bureau of Labor Statistics for the next two series. Following Blanchard et al. (2013), I remove secular drift in the consumption-to-productivity ratio from the consumption series. Table 6 shows the estimation results with US consumption and productivity data.[31]

The qualitative implications of the dynamic effects of each shock are as follows. Permanent shocks on productivity slowly and steadily increase productivity and consumption. Transitory shocks on productivity have slowly decreasing effects on productivity and consumption while noisy shocks generate slowly decreasing effects on consumption only. The main takeaway is that the effects are asymmetric such that as depicted in Figure 9 the absolute size of the responses are larger for negative shocks than for positive ones.

The limit case, as defined in Definition 3, refers to the case in which a noisy signal is assumed to be unambiguous. Table 7 reports the parameters obtained when estimating

[31] For maximum likelihood estimation I initialize the variance covariance matrix of the estimator with a diagonal of 100.

Table 7: Parameter Estimates, US 1970:I-2016:I (the limit case)

Parameter	Description	Value	s.e.
ρ	Persistence productivity	0.9763	0.0044
σ_u	Std dev. productivity	0.0067	0.0002
σ_ϵ	Std dev. permanent shock (implied)	0.0002	-
σ_η	Std dev. transitory shock (implied)	0.0066	-
σ_ν	Std dev. noise shock	0.0436	0.0093
	log-likelihood	1332.68	

Notes: σ_ϵ and σ_η are obtained with random walk assumption. As they are indirectly recovered, no standard errors are given.

the model by assuming that $\underline{\sigma}_\nu = \bar{\sigma}_\nu$. This limit case indicates that consumers are aware of the exact signal precision and the impulse responses are symmetric to the sign of the shocks. The estimated standard deviation of a noise shock, $\hat{\sigma}_\nu$, in the limit case is shown to lie inside the estimated range of precisions in Table 6.

Since this limit model (unambiguous signal) is a special case of the benchmark model (ambiguous signal), a likelihood ratio test can be used to compare the goodness of fit of the two models. Specifically, the "null" model (unambiguous signal) has 3 parameters with a log-likelihood of 1332.68 whereas the "alternative" model (ambiguous signal) has 5 parameters with a log-likelihood of 1336.15 such that the test statistic is $2 \times (1336.15 - 1332.68) = 6.94$ with degrees of freedom equal to 2. Thus, the null model is rejected in favour of the alternative model at a significance level of 0.05.

4.3.3 Recovering states and shocks via the Kalman smoother

A useful exercise is to exploit the fact that the econometrician has access to the whole sample (t=1:T) and she is able to estimate states and shocks using the Kalman Smoother. Having more information available, we are able to have better estimates what the states and shocks were.

Figure 11 plots the smoothed estimate of the permanent component of productivity (x_t) and of long-run productivity ($x_{t+\infty}$) for the two models - *the benchmark ambiguous signal model* and *the limit case unambiguous signal model*. The permanent component of productivity (Left panel) is estimated to be higher in the benchmark ambiguous case than in the limit case. Neglecting the presence of ambiguity, thus, implies underestimating the state of the economy (measured by its permanent productivity). Similar results hold for estimating long-run productivity: Misspecification of the model (by not taking into account of the presence of ambiguity) results in underestimating the economy's long-run potential for the sample period.

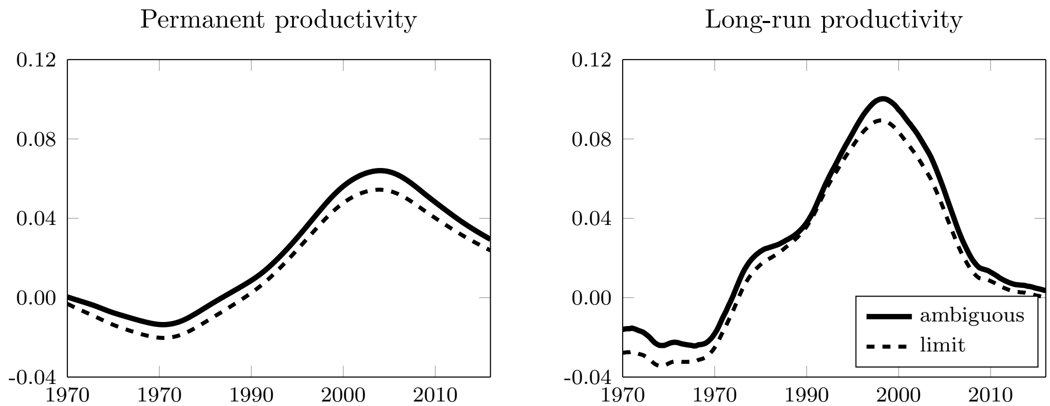

Figure 11: Smoothed Estimates of the Permanent Component of Productivity and of Long-run Productivity for the the Two Specifications

Notes: Left panel depicts the smoothed estimate of x_t and Right panel depicts the smoothed estimate of $x_{t+\infty}$ with ambiguous signals (solid line) and with unambiguous signal (dashed line). I use the parameters estimated in Section 4.3.2, i.e., for the benchmark ambiguous signal model $\sigma_\nu \in [0.0351, 0.0465]$ and for the limit case unambiguous signal model $\sigma_\nu = 0.436$.

5 Concluding Remarks

I have provided a simple theory of asymmetric consumption fluctuations due to ambiguous quality of information and agents' aversion toward ambiguity. Methodologically, I have attempted to solve a simple forward looking model of consumption with ambiguous information quality. It allows one to examine the dynamics of consumption when agents face strong uncertainty. In the stylized permanent income model, the closed form solution of consumption dynamics is obtained where consumption is driven by consumers' beliefs about the long run under a worst case belief. Since the consumers' belief updating is consistent with evaluating the boundaries of the range of information precisions, the econometrician's filtering can be expressed as a regime switching model where the regime is fully retrieved by the econometrician. The structural estimation using U.S. data suggests an asymmetric nature of consumption responses to exogenous shocks.

Throughout the paper, I have made a strict assumption such that there is no learning *about* ambiguity where it remains the same over time and that all agents are alike (no heterogeneity) in terms of their perception toward ambiguity that they face or in the sense that they all receive a common noisy signal. Studies incorporating the interaction between learning and heterogeneity when dealing with strong uncertainty may be the natural extension of this model to explore a future research avenue. For example, a more realistic setup on ambiguity such that agents adhere the α-*maxmin* expected utility with a time varying α may produce some interesting dynamics, e.g. a time-varying skewness of consumption growth.

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A Constructing the Data Series

The following describes the data series used in this paper. The data are from the Federal Reserve Economic Database (FRED) and span the period from the first quarter of 1970 through the first quarter of 2016. All of the series are seasonally adjusted and are quarterly unless otherwise indicated. The two key variables of this model are constructed as follows:

(Labour) Productivity: I measure labour productivity, denoted a_t, as the logarithm of the ratio of GDP to employment such that

$$a_t \equiv \log A_t = \log Y_t - \log N_t$$

where Y_t is *real gross domestic product (in billions of chained 2009 dollars)* and N_t is the *employment level*.

Consumption: I measure consumption, c_t, as the logarithm of the ratio of NIPA consumption to population such that

$$c_t \equiv \log C_t = \log Cons_t - \log Pop_t$$

where $Cons_t$ is *real consumption expenditure (in billions of chained 2009 dollars)* and Pop_t is the population level at period t.

As discussed in Blanchard et al. (2013), there is an issue such that in contrast to any balanced growth model, productivity and consumption have different growth rates over the sample (0.32 percent per quarter for productivity, versus 0.41 percent for consumption), which potentially reflects factors left out of this simple model. To deal with the issue, in the empirical analysis, I allow for a secular drift in the consumption-to-productivity ratio and remove it from the consumption series.

For the TFP variable, I use the utilisation adjusted quarterly TFP series from Fernald (2014). It measures the business section TFP less utilisation of capital and labour.

B Proofs

B.1 Proposition 1

Proof. Conditional on $\mathbf{x}_{t|a_t}$, consumers' filtering is given by

$$\begin{aligned}
\mathbf{x}_{t|t} &= \mathbf{x}_{t|a_t} + G_t(s_t - s_{t|a_t}) && (15)\\
&= \mathbf{x}_{t|a_t} + G_t s_t - G_t C_1 \mathbf{x}_{t|a_t} \\
&= [I - G_t C_1]\mathbf{x}_{t|a_t} + G_t s_t && (16)
\end{aligned}$$

where G_t is the Kalman gain for the following system of equations:

$$\begin{aligned}
\mathbf{x}_t &= A\mathbf{x}_{t-1} + BV_t \\
s_t &= C_1\mathbf{x}_t + D_1 W_t
\end{aligned}$$

and $\mathbf{x}_t = (x_t,\ x_{t-1},\ z_t)'$, $V_t = (\epsilon_t,\ 0,\ \eta_t)'$, $W_t = \nu_t$, $D_1 = 1$,

$$A = \begin{bmatrix} 1+\rho & -\rho & 0 \\ 1 & 0 & 0 \\ 0 & 0 & \rho \end{bmatrix} \quad B = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad C_1 = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}$$

Similarly, conditional on $\mathbf{x}_{t|t-1}$, $\mathbf{x}_{t|a_t}$ is given by

$$\begin{aligned}
\mathbf{x}_{t|a_t} &= \mathbf{x}_{t|t-1} + H(a_t - a_{t|t-1}) \\
&= A\mathbf{x}_{t-1|t-1} + Ha_t - HC_2A\mathbf{x}_{t-1|t-1} \\
&= [I - HC_2]A\mathbf{x}_{t-1|t-1} + Ha_t && (17)
\end{aligned}$$

where H is the Kalman gain for the following system of equations

$$\begin{aligned}
\mathbf{x}_t &= A\mathbf{x}_{t-1} + BV_t \\
a_t &= C_2\mathbf{x}_t + D_2 W_t
\end{aligned}$$

and $\mathbf{x}_t = (x_t,\ x_{t-1},\ z_t)'$, $V_t = (\epsilon_t,\ 0,\ \eta_t)'$, $W_t = \nu_t$, $D_2 = 0$,

$$A = \begin{bmatrix} 1+\rho & -\rho & 0 \\ 1 & 0 & 0 \\ 0 & 0 & \rho \end{bmatrix} \quad B = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad C_2 = \begin{bmatrix} 1 & 0 & 1 \end{bmatrix}$$

Substituting $\mathbf{x}_{t|a,t}$ from (17) into (16)

$$\mathbf{x}_{t|t} = [I - G_t C_1][I - HC_2]A\mathbf{x}_{t-1|t-1} + [I - G_t C_1]Ha_t + G_t s_t$$

■

B.2 Proposition 2 and 3

Proof. From (15), I have

$$\mathbf{x}_{t|t} = \mathbf{x}_{t|a_t} + G_t(s_t - s_{t|a_t})$$

where G_t is a 3×1 column vector such that G_t^i is the Kalman gain associated with the *i-th* component of $\mathbf{x}_{t|t}$. For example, G_t^1 is the gain of observing the noisy signal s_t on $x_{t|t}$. Furthermore, $G = \Sigma_X C_1' [C_1 \Sigma_X C_1' R_t]^{-1}$ where $R_t = Var(W_t) = \sigma_\nu^2(t)$ and $\Sigma_X = Var_{t-1}(\mathbf{x}_t)$:[32]

$$\Sigma_X = \begin{bmatrix} \Sigma_{11} & \Sigma_{12} & \Sigma_{13} \\ \Sigma_{21} & \Sigma_{22} & \Sigma_{23} \\ \Sigma_{31} & \Sigma_{32} & \Sigma_{33} \end{bmatrix}$$

where Σ_{ii} is the $Var_{t-1}(\mathbf{x}_{i,t})$ and Σ_{ij} is the $Cov_{t-1}(\mathbf{x}_{i,t}, \mathbf{x}_{j,t})$. With a little bit of algebra, each component of G can be defined by

$$G_t^1 = \Sigma_{11}(\Sigma_{11} + \sigma_\nu^2(t))^{-1} \tag{18}$$
$$G_t^2 = \Sigma_{21}(\Sigma_{11} + \sigma_\nu^2(t))^{-1} \tag{19}$$
$$G_t^3 = \Sigma_{31}(\Sigma_{11} + \sigma_\nu^2(t))^{-1}$$

Since $\Sigma_{11} > 0$, $0 < \rho < 1$, and $(\Sigma_{11} + \sigma_\nu^2(t))^{-1} > 0$, $G_t^1 > 0$ such that if $s_t > s_{t|a_t}$, $x_{t|t} - x_{t|a_t} > 0$ and that if $s_t < s_{t|a_t}$, $x_{t|t} - x_{t|a_t} < 0$. Similarly, as $\Sigma_{21} > 0$ and $(\Sigma_{11} + \sigma_\nu^2(t))^{-1} > 0$, $G_t^2 > 0$ such that if $s_t > s_{t|a_t}$, $x_{t-1|t} - x_{t-1|a_t} > 0$ and that if $s_t < s_{t|a_t}$, $x_{t-1|t} - x_{t-1|a_t} < 0$. Finally, given that $\Sigma_{31} = 0$, $G_t^3 = 0$ such that $z_{t|t} - z_{t|a_t} = 0$, $\forall$ s_t and $s_{t|a_t}$.

For consumption, c_t and $c_{t|a_t}$ are given by

$$c_t = \frac{1}{1-\rho}\left(x_{t|t} - \rho x_{t-1|t}\right) \tag{20}$$
$$c_{t|a_t} = \frac{1}{1-\rho}\left(x_{t|a_t} - \rho x_{t-1|a_t}\right)$$

[32] Since $\epsilon_t \perp \nu_{t+j}, \eta_t \perp \nu_{t+j}, \forall j$, $\Sigma_{13} = \Sigma_{23} = \Sigma_{31} = \Sigma_{32} = 0$.

Substituting $x_{t|t}$ and $x_{t-1|t}$ from (15) into (20) gives

$$\begin{aligned} c_t &= \frac{1}{1-\rho}\left(x_{t|t} - \rho x_{t-1|t}\right) \\ &= \frac{1}{1-\rho}\left(x_{t|a_t} + G_t^1(s_t - s_{t|a_t}) - \rho x_{t-1|a_t} - \rho G_t^2(s_t - s_{t|a_t})\right) \\ &= c_{t|a_t} + \frac{1}{1-\rho}\left((s_t - s_{t|a_t})(G_t^1 - \rho G_t^2)\right) \end{aligned} \tag{21}$$

From (18) and (19), it gives

$$\begin{aligned} G_t^1 - \rho G_t^2 &= \Sigma_{11}(\Sigma_{11} + \sigma_\nu^2(t))^{-1} - \rho\Sigma_{21}(\Sigma_{11} + \sigma_\nu^2(t))^{-1} \\ &= (\Sigma_{11} - \rho\Sigma_{21})(\Sigma_{11} + \sigma_\nu^2(t))^{-1} \end{aligned}$$

As $\Sigma_{11} > \Sigma_{21}$ and $0 < \rho < 1$, $G_t^1 - \rho G_t^2 > 0$. Therefore, when $s_t > s_{t|a_t}$, the second term on the right-hand side of (21) is positive and when $s_t < s_{t|a_t}$, it is negative. Thus, if $s_t > s_{t|a_t}$, $c_t - c_{t|a_t} > 0$. Similarly, if $s_t < s_{t|a_t}$, $c_t - c_{t|a_t} < 0$.

From (21) when $s_t - s_{t|a_t} > 0$, $G_t^1 - \rho G_t^2$ should be as small as possible to minimize c_t. Since $G_t^1 - \rho G_t^2 = (\Sigma_{11} - \rho\Sigma_{21})(\Sigma_{11} + \sigma_\nu^2(t))^{-1}$, it is such that $\sigma_\nu = \bar{\sigma}_\nu$. Similarly, when $s_t - s_{t|a_t} < 0$, minimizing c_t requires that $G_t^1 - \rho G_t^2$ takes the largest possible values. Thus, when $s_t - s_{t|a_t} < 0$, $\sigma_\nu = \underline{\sigma}_\nu$.

This completes the proof of Proposition 2 and Proposition 3. ■

B.3 Proof to Proposition 4

Proof. Rewrite equation (21):

$$s_t - s_{t|a_t} = \frac{1-\rho}{G_t^1 - \rho G_t^2}\left(c_t - c_{t|a_t}\right)$$

Since $\frac{1-\rho}{G_t^1 - \rho G_t^2} > 0$, $c_t - c_{t|a_t}$ and $s_t - s_{t|a_t}$ should have the same sign: if $c_t > c_{t|a_t}$, $s_t > s_{t|a_t}$ and if $c_t < c_{t|a_t}$, $s_t < s_{t|a_t}$. ■

C SOE-RBC Model

This appendix derives a permanent income consumption model under a worst-case belief from the small open economy *RBC* model as in Cao and L'Huillier (2015).

C.1 Setup

A representative consumer maximizes multiple priors utility:

$$\widehat{\mathbb{E}}_t \left[\sum_{t=0}^{\infty} \beta^t \log C_t \right]$$

where $\widehat{\mathbb{E}}_t$ is the expectation operator under a worst-case belief that is chosen from a set of conditional probabilities on information quality. The maximisation is subject to

$$C_t + B_t = Y_t + Q_t B_{t+1}$$

where B_t is the external debt of the country, Q_t is the price of this debt, and Y_t is the output of the country. Output is produced using only labor

$$Y_t = A_t N$$

where the labour input is assumed constant. The resource constraint is

$$C_t + NX_t = Y_t$$

The price of debt is sensitive to the level of outstanding debt:

$$\frac{1}{Q_t} = R_t = R^* + \psi \left\{ e^{\frac{B_{t+1}}{X_t} - b} - 1 \right\}$$

where b denotes the steady state level of B_{t+1}/X_t.

C.2 Optimality conditions

The first order condition from the optimisation problem delivers

$$\frac{1}{C_t} = \beta R_t \widehat{\mathbb{E}} \left[\frac{1}{C_{t+1}} \right]$$

For log-linearisation, we define endogenous variables c_t, y_t, q_t, b_{t+1}, nx_t as

$$\widehat{c}_t = \log(C_t/X_{t-1}) - \log(\bar{C}/\bar{X})$$

$$y_t = \log(Y_t/X_{t-1}) - \log(\bar{Y}/\bar{X})$$

$$q_t = \log Q_t - \log \bar{Q}$$

$$b_{t+1} = \frac{B_{t+1}}{X_t} - \frac{\bar{B}}{\bar{X}}$$

$$nx_t = \frac{NX_t}{Y_t} - \frac{\bar{NX}}{\bar{Y}}$$

For notational convenience, I also define:

$$C \equiv \frac{\bar{C}}{\bar{X}}, \qquad Y \equiv \frac{\bar{Y}}{\bar{X}}, \qquad B \equiv \frac{\bar{B}}{\bar{X}}, \qquad Q \equiv \bar{Q}, \qquad NX \equiv \frac{\bar{NX}}{\bar{Y}}$$

C.3 Log-linearisation

Start from

$$NX_t = B_t - Q_t B_{t+1}$$

Diving both sides by Y_t:

$$NX_t/Y_t = \frac{B_t}{X_{t-1}}\frac{X_{t-1}}{Y_t} - Q_t \frac{B_{t+1}}{X_t}\frac{X_{t-1}}{Y_t}\frac{X_t}{X_{t-1}}$$

leads to

$$nx_t = \frac{1}{Y}b_t - \frac{GQ}{Y}b_{t+1} - \frac{B}{Y}GQ(q_t - y_t + \Delta x_t) - \frac{B}{Y}y_t \tag{22}$$

Dividing both sides of the resource constraints by Y_t:

$$\frac{C_t}{X_{t-1}}\frac{X_{t-1}}{Y_t} + \frac{NX_t}{Y_t} = 1$$

leads to

$$\frac{C}{Y}(\widehat{c}_t - y_t) + nx_t = 0 \tag{23}$$

Substituting nx_t from (22) into (23), I get

$$\frac{C}{Y}(\widehat{c}_t - y_t) + \frac{1}{Y}b_t - \frac{GQ}{Y}b_{t+1} - \frac{B}{Y}GQ(q_t - y_t + \Delta x_t) - \frac{B}{Y}y_t = 0 \tag{24}$$

Dividing the both sides of the production function by X_{t-1}:

$$\frac{Y_t}{X_{t-1}} = \frac{X_t}{X_{t-1}}Z_t N$$

leads to

$$y_t = z_t + \Delta x_t \tag{25}$$

Multiplying both sides of the first order condition by X_{t-1}:

$$\frac{X_{t-1}}{C_t} = \beta R_t \widehat{\mathbb{E}} \left[\frac{X_t}{C_{t+1}} \frac{X_{t-1}}{X_t} \right]$$

leads to

$$\widehat{c}_t = q_t + \widehat{c}_{t+1} + \Delta x_t \tag{26}$$

where $q_t = -r_t$.

Rest of the model is specified similar to Cao and L'Huillier (2015):

$$q_t = -\psi Q b_{t+1} \tag{27}$$

$$c_t = \widehat{c}_t + x_{t-1} \tag{28}$$

Thus, (24) to (28) along with technology processes comprise the log-linearised version of the model.

C.4 Steady states

The following steady state relations hold:

$$Q = \beta$$

$$(1 - \beta) \frac{B}{Y} = 1 - C/Y$$

C.5 Closed-form solution and limit result for consumption

Define a new variable $\hat{b}_t$:

$$\hat{b}_t = b_t + B x_{t-1}$$

Using the definition of the log-deviation of consumption:

$$c_t = \widehat{c}_t + x_{t-1}$$

I make a following conjecture:

$$
\begin{aligned}
c_t &= D_b \hat{b}_t + D_x \mathbf{x}_t \\
&= D_b \hat{b}_t + D_{x,1} x_t + D_{x,2} x_{t-1} + D_{x,3} z_t
\end{aligned} \tag{29}
$$

where $\mathbf{x}_t = [x_t,\ x_{t-1},\ z_t]'$. I claim that as $\beta \to 1$ and $\frac{\psi}{(1-\beta)} \to 0$, consumption c_t is only a function of belief about the long-run (BLR) under a worst-case belief:

$$
c_t = \frac{1}{1-\rho} \left(\widehat{\mathbb{E}}_t [x_t] - \rho \widehat{\mathbb{E}}_t [x_{t-1}] \right)
$$

expressed in the following proposition.

Proposition 5 (Limit consumption) *As $\beta \to 1$ and $\frac{\psi}{(1-\beta)} \to 0$,*

$$
\begin{aligned}
\lim_{\beta \to 1} \lim_{\psi \to 0} D_b &= 0 \\
\lim_{\beta \to 1} \lim_{\psi \to 0} D_{x,1} &= \frac{1}{C/Y} \frac{1}{1-\rho} \\
\lim_{\beta \to 1} \lim_{\psi \to 0} D_{x,2} &= \frac{1}{C/Y} \frac{-\rho}{1-\rho} \\
\lim_{\beta \to 1} \lim_{\psi \to 0} D_{x,3} &= 0
\end{aligned}
$$

Proof. From (24) and (25), I have

$$
0 = Y(z_t + \Delta x_t) + \beta B(\Delta x_t - \psi \beta b_{t+1}) + \beta b_{t+1} - b_t - C\widehat{c}_t
$$

and with the definition of c_t and $\widehat{b}_t$, I get

$$
\widehat{b}_{t+1} = \frac{1}{(1-\psi\beta B)\beta} \left[\widehat{b}_t + C c_t - Y z_t - (Y + \beta B \psi \beta B) x_t \right] \tag{30}
$$

The Euler equation (26) and the debt equation (27) imply that

$$
\widehat{c}_{t+1} - \widehat{c}_t + \Delta x_t - \psi Q b_{t+1} = 0
$$

which by using the definition of c_t and $\widehat{b}_t$ becomes

$$
c_{t+1} - c_t - \psi \beta \widehat{b}_{t+1} + \psi \beta B x_t = 0 \tag{31}
$$

Using the conjecture (29), (31) becomes

$$(D_b - \psi\beta)\widehat{b}_{t+1} + D_x A\mathbf{x}_t + \psi\beta B x_t - c_t = 0$$

and combined with (30)

$$\left[1 - \frac{(D_b - \psi\beta)C}{(1-\psi\beta B)\beta}\right] c_t = \frac{(D_b - \psi\beta)}{(1-\psi\beta B)\beta}\left(\widehat{b}_t - Y z_t\right) + D_x \mathbf{A}\mathbf{x}_t + K x_t \tag{32}$$

where

$$K = -\left[\frac{(D_b - \psi\beta)}{(1-\psi\beta B)\beta}\left(Y + \beta B \psi\beta B\right)\right] + \psi\beta B$$

and

$$\mathbf{A} = \begin{pmatrix} 1+\rho & -\rho & 0 \\ 1 & 0 & 0 \\ 0 & 0 & \rho \end{pmatrix}$$

Rearranging (32) leads to

$$(1-\bar{x})\, c_t = \frac{\bar{x}}{C}\widehat{b}_t - \frac{\bar{x}}{C} Y z_t + D_x \mathbf{A}\mathbf{x}_t - \left[\frac{\bar{x}}{C}\left(Y + \beta B\psi\beta B\right)\right] x_t + \psi\beta B x_t \tag{33}$$

where

$$\bar{x} = \frac{(D_b - \psi Q)\, C}{(1-\psi Q B) G Q}$$

Finding D_b in the limit leads to $D_b = 0$.[33]

[33] I get the following quadratic equation in D_b:

$$D_b^2 + \left[\frac{1}{C} - (1-\psi\beta B)\frac{\beta}{C} - \psi\beta\right] D_b - \frac{\psi\beta}{C} = 0$$

where I pick the negative root to ensure the stability of the dynamic system and in the limit $D_b \to 0$:

$$D_b = \frac{-\left(\frac{1}{C} - (1-\psi\beta B)\frac{\beta}{C} - \psi\beta\right) - \sqrt{\left(\frac{1}{C} - (1-\psi\beta B)\frac{\beta}{C} - \psi\beta\right)^2 + \frac{4\psi Q}{C}}}{2}$$

For $C = 1$ in the limit:

$$D_b = \frac{-(1-1-0) - \sqrt{(1-1)^2 + 0}}{2} = 0$$

Similarly, for $C \neq 1$ in the limit:

$$D_b = \frac{-\left(\frac{1}{C} - \frac{\beta}{C}\right) - \sqrt{\left(\frac{1}{C} - \frac{\beta}{C}\right)^2 + 0}}{2} = 0$$

From (33), collecting the terms for x_t:

$$(1-\bar{x})\, D_{x,1} = -\frac{\bar{x}}{C}\,(Y+\beta B\psi\beta B) + \psi\beta B + (1+\rho)D_{x,1} + D_{x,2} \tag{34}$$

Similarly, collecting the terms for x_{t-1}:

$$(1-\bar{x})\, D_{x,2} = -\rho D_{x,1} \tag{35}$$

Finally, for z_t:

$$(1-\bar{x})\, D_{x,3} = \rho D_{x,3}$$

Thus, $D_{x,3} = 0$ and from (35)

$$D_{x,2} = \frac{-\rho}{1-\bar{x}} D_{x,1}$$

Substituting $D_{x,2}$ into (34),

$$(1-\bar{x})D_{x,1} = -\frac{\bar{x}}{C}\,(Y+\beta B\psi\beta B) + \psi\beta B + (1+\rho)D_{x,1} - \frac{\rho}{1-\bar{x}}D_{x,1}$$

Then, I can solve for $D_{x,1}$:

$$D_{x,1} = \left(\frac{1-\bar{x}}{1-\rho-\bar{x}}\right)\left(\frac{1}{\bar{x}}\right)\left[\frac{\bar{x}}{C}\,(Y+\beta B\psi\beta B) - \psi\beta B\right]$$

With the limit conditions,

$$\begin{aligned}\lim_{\beta\to 1}\lim_{\psi\to 0} D_{x,1} &= \left(\frac{1-\bar{x}}{1-\rho-\bar{x}}\right)\left(\frac{1}{C}\right)(Y+\beta B\psi\beta B) - \left(\frac{1-\bar{x}}{1-\rho-\bar{x}}\right)\left(\frac{1}{\bar{x}}\right)\psi\beta B \\ &= \frac{1}{C/Y}\left(\frac{1}{1-\rho}\right) \end{aligned}\tag{36}$$

as $\left(\frac{1-\bar{x}}{1-\rho-\bar{x}}\right)\left(\frac{1}{\bar{x}}\right)\psi\beta B$ goes to zero in the limit.[34] Given $D_{x,1}$ in (36), with the limit condi-

[34] In the limit,

$$\frac{1-\bar{x}}{1-\rho-\bar{x}} = \frac{1}{1-\rho}.$$

tions, I find $D_{x,2}$:

$$\lim_{\beta\to 1}\lim_{\psi\to 0} D_{x,2} = \lim_{\beta\to 1}\lim_{\psi\to 0} \frac{-\rho}{(1-\bar{x})} D_{x,1} = \frac{1}{C/Y}\left(\frac{-\rho}{1-\rho}\right)$$

Thus, it shows that Proposition 5 holds and when $C/Y = 1$:

$$c_t = \frac{1}{1-\rho}\left(\widehat{\mathbb{E}}_t\left[x_t\right] - \rho\widehat{\mathbb{E}}_t\left[x_{t-1}\right]\right)$$

■

Thus, only need to show that $\frac{1}{\bar{x}}\psi\beta B \to 0$ in the limit. Using the definition of $\bar{x}$, I have

$$\frac{1}{\bar{x}}\psi\beta B = \frac{(1-\psi\beta B)\,\beta}{(D_b-\psi\beta)\,C}\left(\frac{1-C}{1-\beta}\right)\psi\beta$$

As $D_b = 0$ in the limit, I have

$$\frac{\psi\beta\,(1-C)\,(1-\psi\beta B)\,\beta}{-\psi\beta C(1-\beta)} = -\frac{1-C}{C}\left(\frac{(1-\psi\beta B)\,\beta\psi\beta}{1-\beta}\right) = 0$$

as long as $C \neq 0$. Since $\frac{(1-\psi\beta B)\beta\psi\beta}{1-\beta} \to 0$ and $\frac{1-C}{C} \leq \infty$.

D Online Appendix (not for publication)

D.1 Sequential Belief Updating under Ambiguity

Updating beliefs sequentially or simultaneously makes no difference on ex-post revised beliefs when the quality of information is certain. However, under ambiguity, such claim may not necessarily be materialized.

The aim of this section is two-fold. First, I show that the simultaneous and sequential belief updating do not necessarily generate identical revised beliefs under ambiguity. Second, I show that the simultaneous belief updating under ambiguity can be described as updating beliefs sequentially when the agents first update beliefs with an unambiguous signal. The discussion is based on the case in which there are two signals and one of which is ambiguous. But it is easy to generalize the discussion with N signals where $N > 2$ and there are $N-1$ unambiguous signals.

Let the process $x \sim N(\theta, \sigma_x^2)$ and agents receive two signals about x:

$$a = x + \eta$$
$$s = x + \nu,$$

where η and ν are i.i.d. Gaussian shocks such that $\eta \sim N(0, \sigma_\eta^2)$, $\nu \sim N(0, \sigma_\nu^2)$. Agents update beliefs about x with the two signals. The key assumption here is that the signal s is ambiguous such that $\sigma_\nu^2 \in [\underline{\sigma}_\nu^2, \bar{\sigma}_\nu^2]$.

Consider the three alternative belief updating schemes. First, agents update beliefs first with the unambiguous signal and then with the ambiguous signal. Second, agents update beliefs simultaneously. Finally, agents first update beliefs with the ambiguous signal and then with the unambiguous signal.

D.1.1 Sequential updating [seq-1]:

Assume that agents update beliefs sequentially - first with the unambiguous signal and then with the ambiguous signal. Conditional on an ex-ante expectation on x, initial step is to update belief with the observed signal a. For simplicity, assume that the agents' utility is strictly increasing in x.[35] and agents maximize the multiple priors utility:

$$\max_{x \in X} \min_{\omega \in \Omega} \mathbb{E}\left[u(x; \omega)\right]$$

[35] The results may not hold if U is a non-monotonic function of x.

where the set Ω, the priors, is on the range of precisions such that $\Omega = [1/\bar{\sigma}_\nu^2, 1/\underline{\sigma}_\nu^2]$ and $\partial u/\partial x > 0$. While the agents' utility is strictly increasing in x, it also depends on the ambiguity parameter ω as a belief on x is a function of a signal precision. Then, the *maxmin* operation suggests that ω which minimizes the expected utility is chosen to satisfy agent' aversion toward ambiguity.

The procedure to update beliefs is summarized as follows.

updating with a: the updating beliefs with a is given by

$$x|a = \frac{\sigma_\eta^2}{\sigma_x^2 + \sigma_\eta^2}\theta + \frac{\sigma_x^2}{\sigma_x^2 + \sigma_\eta^2}a \tag{37}$$
$$Var(x|a) = \sigma_{x\eta}^2 = \frac{\sigma_x^2 \sigma_\eta^2}{\sigma_x^2 + \sigma_\eta^2}$$

In other words, the revised belief with the first signal is just a weighted average of an ex-ante expectation (the unconditional expectations of x) and the observed signal where the weights depends on the precision of fundamental and the noise.

updating with s: conditional on $x|a$, the updating beliefs with the ambiguous signal is given by

$$x|a, s = x|a + \left(\frac{\sigma_{x\eta}^2}{\sigma_{x\eta}^2 + \sigma_\nu^2}\right)(s - x|a) \tag{38}$$

Since $x|a$ and $\sigma_{x\eta}^2$ do not depend on σ_ν^2, the following simple cut-off rule can be applied to update beliefs:

Proposition 6 *With sequential belief updating, when $s > x|a$, agents update beliefs with $\sigma_\nu^2 = \bar{\sigma}_\nu^2$. When $s < x|a$, agents update beliefs with $\sigma_\nu^2 = \underline{\sigma}_\nu^2$.*

Proposition 6 can be proved just by checking the second term in the right-hand side of (38). When the observed signal s is greater than $x|a$, the weight attached to the signal should be as small as possible since the weight is inversely related to σ_ν^2 such that $\sigma_\nu^2 = \bar{\sigma}_\nu^2$. Same logic applies when $s < x|a$.

Intuitively, in terms of comparison between the signal observed (s) and the agents ex-ante expectations ($x|a$), when good news arrives, agents are hesitant to believe that the signal is precise. On the contrary, when bad news are delivered, agents would believe that the signal is very informative.

D.1.2 Simultaneous updating [sim]:

Agents update beliefs with the signals $S = (a,\ s)'$ where

$$S = Ax + \epsilon$$

with $A = [1,\ 1]'$, $\epsilon = [\eta,\ \nu]'$. $V = Var(\epsilon)$ is a diagonal matrix with σ_η^2 and σ_ν^2 being the diagonal components. Then, the updating of beliefs is given by

$$x|S = \frac{\sigma_\nu^2\sigma_\eta^2}{\sigma_x^2\sigma_\nu^2 + \sigma_x^2\sigma_\eta^2 + \sigma_\nu^2\sigma_\eta^2}\theta + \frac{\sigma_x^2\sigma_\eta^2}{\sigma_x^2\sigma_\nu^2 + \sigma_x^2\sigma_\eta^2 + \sigma_\nu^2\sigma_\eta^2}s + \frac{\sigma_x^2\sigma_\nu^2}{\sigma_x^2\sigma_\nu^2 + \sigma_x^2\sigma_\eta^2 + \sigma_\nu^2\sigma_\eta^2}a \tag{39}$$

where the multiplicative terms for s and a are relative gains of observing the signal s and a, respectively, and σ_ν^2 is chosen to minimize $x|S$. Let $\tilde{s} = s - \theta$ and $\tilde{a} = a - \theta$, then (39) becomes

$$x|S = \theta + \frac{\sigma_x^2\sigma_\eta^2}{\sigma_x^2\sigma_\nu^2 + \sigma_x^2\sigma_\eta^2 + \sigma_\nu^2\sigma_\eta^2}\tilde{s} + \frac{\sigma_x^2\sigma_\nu^2}{\sigma_x^2\sigma_\nu^2 + \sigma_x^2\sigma_\eta^2 + \sigma_\nu^2\sigma_\eta^2}\tilde{a}$$

Lemma 1 *$x|S$ is a monotonic function of σ_ν^2*

Proof. To prove Lemma 1, it is sufficient to show that $\frac{\partial x|S}{\partial \sigma_\nu}$ does not change the sign for $\forall \sigma_\nu^2 \in [\underline{\sigma}_\nu^2, \bar{\sigma}_\nu^2]$ given $\Psi = \{\sigma_x^2, \sigma_\eta^2, \theta, s, a\}$. Taking the derivative of $x|S$ with respect to σ_ν^2,

$$\begin{aligned}
\frac{\partial x|S}{\partial \sigma_\nu^2} &= \left(\sigma_x^2\sigma_\nu^2\right)\tilde{s}\left(\sigma_x^2\sigma_\nu^2 + \sigma_x^2\sigma_\eta^2 + \sigma_\eta^2\sigma_\nu^2\right)^{-2}(-1)\left(\sigma_x^2 + \sigma_\eta^2\right) + \\
&+ \left(\sigma_x^2\sigma_\nu^2\right)\tilde{a}\left(\sigma_x^2\sigma_\nu^2 + \sigma_x^2\sigma_\eta^2 + \sigma_\eta^2\sigma_\nu^2\right)^{-2}(-1)\left(\sigma_x^2 + \sigma_\eta^2\right) + \\
&+ \sigma_x^2\tilde{a}\left(\sigma_x^2\sigma_\nu^2 + \sigma_x^2\sigma_\eta^2 + \sigma_\eta^2\sigma_\nu^2\right)^{-2}\left(\sigma_x^2\sigma_\nu^2 + \sigma_x^2\sigma_\eta^2 + \sigma_\eta^2\sigma_\nu^2\right) \\
&= \frac{\sigma_x^4\sigma_\eta^2\left(\tilde{a} - \tilde{s}\right) - \sigma_x^2\sigma_\eta^4\tilde{s}}{\left(\sigma_x^2\sigma_\nu^2 + \sigma_x^2\sigma_\eta^2 + \sigma_\eta^2\sigma_\nu^2\right)^2}
\end{aligned} \tag{40}$$

From (40), it is easy to see that the denominator is always positive while the numerator can either be positive or negative depending on the parameters. As σ_ν^2 does not enter into numerator, the sign of $\partial x|S/\partial\sigma_\nu^2$ does not depend on σ_ν^2. $x|S$, therefore, is a monotonic function of σ_ν^2. ■

Proposition 7 *With the simultaneous belief updating, agents update beliefs as if they do it sequentially described in Proposition 6. Specifically, when $s > x|a$, agents update beliefs with $\sigma_\nu^2 = \bar{\sigma}_\nu^2$. When $s < x|a$, the agents update beliefs with $\sigma_\nu^2 = \underline{\sigma}_\nu^2$.*

Proof. From (37), $x|a$ can be written as $x|a = \theta + \left(\frac{\sigma_x^2}{\sigma_x^2+\sigma_\eta^2}\right)\tilde{a}$, where, as before, $\tilde{a} = a - \theta$. It is sufficient to show that when $s > x|a$, $\frac{\partial x|S}{\partial \sigma_\nu^2} < 0$, whereas when $s < x|a$, $\frac{\partial x|S}{\partial \sigma_\nu^2} > 0$. Since the denominator of (40) is always positive, I only need to consider the numerator of (40), which is

$$\sigma_x^4 \sigma_\eta^2 (\tilde{a} - \tilde{s}) - \sigma_x^2 \sigma_\eta^4 \tilde{s} \tag{41}$$

Divide (41) by $\sigma_x^2 \sigma_\eta^2$ gives

$$\sigma_x^2 \tilde{a} - \left(\sigma_x^2 + \sigma_\eta^2\right) \tilde{s} \tag{42}$$

Dividing (42) by $(\sigma_x^2 + \sigma_\eta^2)$ gives

$$\frac{\sigma_x^2}{\left(\sigma_x^2 + \sigma_\eta^2\right)} \tilde{a} - \tilde{s}$$

Therefore, the sign of $\frac{\partial x|S}{\partial \sigma_\nu^2}$ depends on whether s is greater or less than $\left[\sigma_x^2 / \left(\sigma_x^2 + \sigma_\eta^2\right)\right] \tilde{a}$. Since the denominator of (40) is always positive when $\frac{\sigma_x^2}{\left(\sigma_x^2+\sigma_\eta^2\right)} \tilde{a} > \tilde{s}$,

$$\frac{\partial x|S}{\partial \sigma_\nu^2} > 0$$

On the contrary, when $\frac{\sigma_x^2}{\left(\sigma_x^2+\sigma_\eta^2\right)} \tilde{a} < \tilde{s}$,

$$\frac{\partial x|S}{\partial \sigma_\nu^2} < 0$$

The first case coincides with $s < x|a$ and the second case with $s > x|a$. Since $x|S$ is a monotonic function and $\frac{\partial x|S}{\partial \sigma_\nu^2} < 0$, when $s > x|a$, agents update beliefs with $\sigma_\nu^2 = \bar{\sigma}_\nu^2$. Similarly, when $s < x|a$, agents update beliefs with $\sigma_\nu^2 = \underline{\sigma}_\nu^2$.

Thus, simultaneously updating beliefs can be represented as sequentially updating beliefs [seq-1] described in the previous section. ■

D.1.3 Sequential updating [seq-2]:

Consider the case in which agents update beliefs first with the ambiguous signal and then with the unambiguous signal.

<u>updating with s</u>: the updating of beliefs with s can be given by

$$x|s = \theta + \frac{\sigma_x^2}{\sigma_x^2 + \sigma_\nu^2}(s - \theta)$$
$$Var(x|s) = \hat{\sigma}_{xs}^2 = \frac{\sigma_x^2 \sigma_\nu^2}{\sigma_x^2 + \sigma_{nu}^2}$$

Therefore, the following cutoff criteria would apply to update beliefs:

$$s > \theta \Rightarrow \sigma_\nu^2 = \bar{\sigma}_\nu^2 \tag{43}$$

$$s < \theta \Rightarrow \sigma_\nu^2 = \underline{\sigma}_\nu^2 \tag{44}$$

<u>updating with a</u>: conditional on $x|s$, updating beliefs with the unambiguous signal a can be given by

$$x|s, a = \frac{\sigma_\eta^2}{\hat{\sigma}_{xs}^2 + \sigma_\eta^2} x|s + \frac{\hat{\sigma}_{xs}^2}{\hat{\sigma}_{xs}^2 + \sigma_\eta^2} a$$

where $\hat{\sigma}_{xs}^2$ and $x|s$ are chosen with cutoff rule in (43) and (44).

Proposition 8 *Updating beliefs sequentially defined as above (*seq-2*) does not necessarily produce the identical updated beliefs as in the other schemes.*

Belief updating under ambiguity crucially depends on how agents apply cut-off rules. In the previous two cases (*sim* and *seq-1*), the cutoff rules apply with the reference level $s = x|a$. However, in *seq-2*, the reference level to apply cut-off rule is $s = \theta$. Unless $x|a = \theta$, therefore, the updated beliefs (in *seq-2*) are not the same as the ones obtained from the other cases.

D.2 Consumption

Productivity is assumed to have two components, a permanent component x_t and a transitory component z_z:

$$a_t = x_t + z_t$$

and since consumption depends on *agents' beliefs about the long-run* under a worst-case belief, it can be solved by

$$c_t = \lim_{j \to \infty} \widehat{\mathbb{E}}_t \left[a_{t+j} \right] = \lim_{j \to \infty} \widehat{\mathbb{E}}_t \left[x_{t+j} + z_{t+j} \right]$$

such that

$$
\begin{aligned}
c_t &= \lim_{j\to\infty} \widehat{\mathbb{E}}_t \left[\Delta x_{t+j} + \Delta x_{t+j-1} + \cdots + \Delta x_{t+1} + x_t + z_{t+j}\right] \\
&= \lim_{j\to\infty} \widehat{\mathbb{E}}_t \left[\rho^j \Delta x_{t+1} + \rho^j \Delta x_t + \cdots + \Delta x_{t+1}\right] + \lim_{j\to\infty} \widehat{\mathbb{E}}_t \left[x_t\right] + + \lim_{j\to\infty} \widehat{\mathbb{E}}_t \left[\rho^{j+1} z_t\right] \\
&= x_{t|t} + \rho \lim_{j\to\infty} \widehat{\mathbb{E}}_t \left[\left(1+\rho+\cdots+\rho^j\right)\Delta x_t\right] \\
&= x_{t|t} + \frac{\rho}{1-\rho}\widehat{\mathbb{E}}_t \left[\Delta x_t\right] \\
&= x_{t|t} + \frac{\rho}{1-\rho}\left(x_{t|t} - x_{t-1|t}\right) = \frac{1}{1-\rho}\left(x_{t|t} - \rho x_{t-1|t}\right)
\end{aligned}
$$

where $x_{t|t} = \widehat{\mathbb{E}}_t\left[x_t\right]$ and $x_{t-1|t} = \widehat{\mathbb{E}}_t\left[x_{t-1}\right]$ are the worst case beliefs on current and lagged permanent productivity.

D.3 Econometrician's filtering

Let the state vector $\mathbf{x}_{t|a_t}$ be given by

$$
\mathbf{x}_{t|a_t} = \left(x_{t|a_t},\ x_{t-1|a_t},\ z_{t|a_t}\right)'
$$

and the dynamics of consumers' beliefs on $\mathbf{x}_{t|a_t}$ be summarized by

$$
\begin{bmatrix} x_{t|a_t} \\ x_{t-1|a_t} \\ z_{t|a_t} \end{bmatrix} = \left[I - HC_2\right] A \begin{bmatrix} x_{t-1|t-1} \\ x_{t-2|t-1} \\ z_{t-1|t-1} \end{bmatrix} + H \begin{bmatrix} 1+\rho & -\rho & -\rho \end{bmatrix} \begin{bmatrix} x_{t-1} \\ x_{t-2} \\ z_{t-1} \end{bmatrix} + H\epsilon_t + H\eta_t
$$

where A and C_2 are given in Proposition 1 and H represents the gains of observing productivity. Similarly, conditional on expectations $\mathbf{x}_{t|a_t}$, the econometrician's state vector $\mathbf{x}_{t|t}$ becomes

$$
\begin{bmatrix} x_{t|t} \\ x_{t-1|t} \\ z_{t|t} \end{bmatrix} = \left[I - G_t C_1\right] \begin{bmatrix} x_{t|a_t} \\ x_{t-1|a_t} \\ z_{t|a_t} \end{bmatrix} + G_t \begin{bmatrix} 1+\rho & -\rho & 0 \end{bmatrix} \begin{bmatrix} x_{t-1} \\ x_{t-2} \\ z_{t-1} \end{bmatrix} + G_t\epsilon_t + G_t\eta_t + G_t\nu_t
$$

where C_1 is given in Proposition 1 and G_t represents the gains of observing a noisy signal at period t. Substituting $\mathbf{x}_{t|a_t}$ into $\mathbf{x}_{t|t}$, it gives

$$
\begin{bmatrix} x_{t|t} \\ x_{t-1|t} \\ z_{t|t} \end{bmatrix} = \mathbf{A} \begin{bmatrix} x_{t-1|t-1} \\ x_{t-2|t-1} \\ z_{t-1|t-1} \end{bmatrix} + \mathbf{B} \begin{bmatrix} x_{t-1} \\ x_{t-2} \\ z_{t-1} \end{bmatrix} + \left[[I - G_t C_1]H + G_t\right]\epsilon_t + \left[[I - G_t C_1]H + G_t\right]\eta_t + G_t\nu_t
$$

where

$$\mathbf{A} = [I - G_t C_1][I - HC_2] A$$

and

$$\mathbf{B} = \left(H \begin{bmatrix} 1+\rho & -\rho & -\rho \end{bmatrix} + G_t \begin{bmatrix} 1+\rho & -\rho & 0 \end{bmatrix} \right)$$

Denoting the econometrician's state vector as $\mathbf{x}_t^E$ such that

$$\mathbf{x}_t^E = \left(x_t,\ x_{t-1},\ z_t,\ x_{t|t},\ x_{t-1|t},\ z_{t|t} \right)$$

the transition equation can be summarized by

$$\mathbf{x}_t^E = Q\mathbf{x}_{t-1}^E + R(\epsilon_t,\ \eta_t,\ \nu_t) \tag{45}$$

where Q and R are given respectively by

$$Q = \begin{bmatrix} \begin{matrix} 1+\rho & -\rho & 0 \\ 1 & 0 & 0 \\ 0 & 0 & \rho \end{matrix} & \mathbf{0} \\ & \\ \bar{\mathbf{Q}} & \mathbf{A} \\ & \end{bmatrix}$$

$$R = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \\ & & \\ & \bar{\mathbf{R}} & \\ & & \end{bmatrix}$$

with

$$\bar{\mathbf{Q}} = \mathbf{B} \begin{bmatrix} 1+\rho & -\rho & \rho \\ 1+\rho & -\rho & 0 \end{bmatrix}$$

and

$$\bar{\mathbf{R}} = \mathbf{B} \begin{bmatrix} 1+\rho & 0 & 0 \\ 1+\rho & 0 & 0 \end{bmatrix} + \mathbf{B} \begin{bmatrix} 1+\rho & 0 & 0 \\ 1+\rho & 0 & 0 \end{bmatrix} + \mathbf{B} \begin{bmatrix} 1+\rho & 0 & 0 \\ 1+\rho & 0 & 0 \end{bmatrix}$$

As the econometrician observes productivity a_t and consumption c_t, the observation equation is

$$(a_t,\ c_t) = T\mathbf{x}_t^E \tag{46}$$

where

$$T = \begin{bmatrix} 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1/(1-\rho) & \rho/(1-\rho) & 0 \end{bmatrix}$$

Thus, the econoemtrician's filtering problem can be solved by (45) and (46) and the decision rule stated in Proposition 4.