Bank regulation under fire sale externalities[a]

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Abstract

We examine the optimal design of and interaction between capital and liquidity regulations. Banks, not internalizing fire sale externalities, overinvest in risky assets and underinvest in liquid assets in the competitive equilibrium. Capital requirements can alleviate the inefficiency, but banks respond by decreasing their liquidity ratios. When capital requirements are the only available tool, the regulator tightens them to offset banks' lower liquidity ratios, leading to fewer risky assets and less liquidity compared with the second best. Macroprudential liquidity requirements that complement capital regulations implement the second best, improve financial stability, and allow for more investment in risky assets.

Keywords: Bank capital regulation, liquidity regulation, fire sale externalities, Basel III

JEL Codes: G20, G21, G28.

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1 Introduction

The recent financial crisis led to a redesign of bank regulations. Prior to the crisis, capital adequacy requirements were the dominant tool of bank regulators around the world. Liquidity requirements for internationally active banks were always part of the discussion in the Basel Committee for Banking Supervision, but several factors delayed their introduction until recently. One main factor was the argument that capital and liquidity requirements are substitutes. It was believed that capital requirements would also address liquidity risk by creating incentives for banks to hold assets with lower risk weights, which should have better liquidity characteristics.[1]

The crisis, however, revealed that even well-capitalized banks can experience a deterioration of their capital ratios due in part to illiquid positions (Brunnermeier, 2009). Without the unprecedented liquidity and asset price supports of leading central banks, liquidity problems faced by several financial institutions simultaneously could have resulted in a dramatic collapse of the financial system. The experience brought liquidity into the spotlight and provided the supervisory momentum to introduce harmonized liquidity regulations.[2] As a result, a third generation of bank regulation principles, popularly known as Basel III, strengthens the previous Basel capital adequacy accords by adding macroprudential aspects and liquidity requirements such as the liquidity coverage ratio (LCR) and the net stable funding ratio.

Several countries, including the United States and the countries in the European Union, have already adopted Basel III liquidity requirements together with the enhanced capital requirements. However, guidance from the theoretical literature on the regulation of liquidity and the interaction between liquidity and capital regulations is quite limited, as emphasized by Bouwman (2012) as well. The scarcity of academic guidance is also apparent in a 2011 survey paper on illiquidity by Jean Tirole, in which he succinctly asks, "Can we trust the institutions to properly manage their liquidity, once excessive risk taking has been controlled by the capital requirement?" (Tirole, 2011).

This paper is the first attempt, to the best of our knowledge, to provide an answer to Jean Tirole's question based on microfoundations, and it makes two main contributions. First, we show that banks' choices of capital and liquidity ratios in an unregulated competitive equilibrium are inefficient under fire sale externalities. Both ratios have distinct effects on the extent of fire sale risk that banks take and, hence, on the externalities they impose on each other. Therefore, we argue that implementing the second-best allocations in a decentralized economy requires regulating banks on both channels. Second, the paper contributes to the

[1] See Goodhart (2011) and Bonner and Hilbers (2015) for a review.

[2] See Rochet (2008), Bouwman (2012), Stein (2013), Tarullo (2014), Allen (2014), Bonner and Hilbers (2015) for recent discussions on the regulation of bank liquidity.

literature by analyzing the interaction between capital and liquidity regulations in addressing this inefficiency. In particular, we uncover novel results on the effects of a capital-regulation-only regime on banks' risk-taking and liquidity choices as well as financial stability measures and welfare. We show that banks respond to tightening capital requirements by decreasing their liquidity buffers, a result consistent with the empirical evidence from several developed countries after the introduction of Basel I in 1988 and Basel II in 2004 (Bonner and Hilbers, 2015). Studying the case of capital regulation alone is important because it represents the pre–Basel III era and thus is informative for understanding the development of systemic risk in that period.

We consider a three-period model in which a continuum of banks have access to two types of assets. Banks have to decide at the initial period how many risky and liquid assets to carry in their portfolio. We allow for a flexible balance sheet size, such that banks can increase both their risky and liquid assets at the same time. Banks start with a fixed amount of equity capital and borrow the funds necessary to finance their portfolio from consumers. The risky asset has a constant return but requires, with a known probability, additional investment in the future before collecting returns. This additional investment cost creates a liquidity need, which is proportional to the amount of risky assets on a bank's balance sheet. The liquid asset provides zero net return; however, it can be used to cover the additional investment cost. A combination of limited-commitment and debt-overhang problems prevents banks from raising external finance to cover the additional investment cost. Therefore, if liquidity from the initial period is not enough to offset the shock, banks' only option is to sell some of their risky assets to firms in the traditional sector.[3] This sell-off of risky assets takes the form of fire sales because traditional sector's demand for risky assets is downward-sloping: These firms are less productive in managing the risky asset, and the marginal product of each additional asset is lower under their management. Thus, traditional firms offer a lower price when banks try to sell a higher quantity of risky assets. A lower price, in turn, requires each bank to further increase the quantity of risky assets to be sold, creating an externality that goes through asset prices.

Atomistic banks do not take into account the effect of their initial portfolio choices on the fire sale price. If banks hold more risky assets, the liquidity need in case of an aggregate shock is greater. As a result, there are more fire sales and a lower fire sale price, which in turn requires each bank to sell more risky assets to raise the required liquidity. Similarly, smaller liquidity buffers in the banks' initial portfolios lead to greater fire sales and a lower fire sale price. We compare the unregulated competitive equilibrium in which banks freely

[3]The additional investment cost shock is aggregate in nature; therefore, the liquidity need cannot be satisfied within the banking system, as all the banks are in need of liquidity. This assumption is not crucial for the results. In Section 6.2, we study the case with idiosyncratic shocks.

choose their capital and liquidity ratios to the allocations of a constrained planner. Without internalizing the effect on the fire sale price, banks overinvest in the risky asset (lower capital ratios) and underinvest in the liquid assets in the unregulated competitive equilibrium. The constrained planner, in contrast, is subject to the same contracting constraints as the private agents but internalizes the effect of initial allocations on the fire sale price. We also investigate how the constrained efficient (second-best) allocations can be implemented using quantity-based capital and liquidity regulations, as in the Basel Accords.

Our results indicate that the constrained efficient allocations can be achieved with joint implementation of capital and liquidity regulations (*complete regulation*). The regulation required is macroprudential because it addresses the instability in the banking system by targeting aggregate capital and liquidity ratios. Banks hold liquid assets for precautionary reasons even if there is no regulation on liquidity because they can use these resources to protect against liquidity shocks. Liquidity is advantageous from a macroprudential standpoint as well: Higher liquidity holdings lead to less-severe decreases in asset prices during times of distress. However, banks fail to internalize this macroprudential aspect of liquidity, which results in inefficiently low liquidity ratios when there is no regulation. Similarly, banks neglect the macroprudential effects of capital ratios and end up choosing inefficiently low capital ratios in the competitive equilibrium.

We then use this model to answer Tirole's question, mentioned above, by studying a regulatory framework with capital requirements alone, similar to the pre-Basel III episode, which we call *partial regulation*. In this setup, banks respond to the introduction of capital regulations by decreasing their liquidity ratios further below the already inefficient levels in the competitive equilibrium. If there is no regulation, banks choose a composition of risky and safe assets in their portfolio that reflects their privately optimal level of risk-taking. When the level of risky investment is limited by capital regulations, banks reduce the liquidity of their portfolio in order to get closer to their privately optimal level of fire sale risk. This is, in a sense, an unintended consequence of capital regulation: *Capital regulation improves financial stability by limiting aggregate risky investment, which in turn weakens banks' incentives to hold liquidity because the marginal benefit of liquidity decreases with financial stability.* The regulator tightens capital regulations under a capital ratio regime to offset banks' lower liquidity ratios, reducing socially profitable long-term investments. As a result, bank capital ratios under partial regulation are higher compared to the second-best allocation.

The aforementioned findings have important policy implications. The lack of complementary liquidity requirements leads to lower levels of bank liquidity and long-term investments, and more severe financial crises compared to the second best, undermining the purpose of capital adequacy requirements. Our results indicate that the pre-Basel III regulatory framework, with its focus on capital requirements, was ineffective in addressing systemic instability caused by fire sales, and that Basel III liquidity regulations are a step in the right direction.

The constrained inefficiency of competitive equilibrium in this paper is due to the existence of pecuniary externalities under incomplete markets. In our framework, this is the only source of inefficiency.[4] The Pareto suboptimality due to pecuniary externalities is well known in the literature. Greenwald and Stiglitz (1986), for instance, show that pecuniary externalities by themselves are not a source of inefficiency but can lead to significant welfare losses when markets are incomplete or when there is imperfect information. More recently, Lorenzoni (2008) shows that the combination of pecuniary externalities in the fire sale market and limited commitment in financial contracts leads to too much investment in risky assets in the competitive equilibrium. If the markets were complete there would not be a reason for fire sales, and the first-best world would be established where there would be no role for regulation.

The paper proceeds as follows. Section 2 contains a brief summary of related literature. Section 3 provides the basics of the model and presents the unregulated competitive equilibrium and the constrained planner's problem. Section 4 compares two alternative regulatory frameworks: complete regulation (both capital and liquidity regulations) and partial regulation (only capital or liquidity regulation). Section 5 considers a few extensions of the model to analyze if the constrained optimal can be implemented using a single linear rule that combines capital and liquidity regulations, the implications of fire sale externalities for shadow banking, and the quantitative implications of capital and liquidity regulations on welfare and financial stability. Section 6 investigates the robustness of the results to some changes in the model environment. Section 7 concludes. The internet appendix contains the proofs and closed-form solutions of the model.

2 Literature review

Even though capital regulations have been studied extensively on their own, we are aware of only a few papers that investigate the jointly optimal determination of capital and liquidity regulations. Kashyap, Tsomocos, and Vardoulakis (2014) consider an extended version of the Diamond and Dybvig (1983) model to investigate the effectiveness of several bank regulations in addressing two common financial system externalities:[5] excessive risk-taking

[4] We do not model agency or information problems that the literature has traditionally used to justify capital or other bank regulations.

[5] The authors consider the following regulations: deposit insurance, loan-to-value limits, dividend taxes, and capital and liquidity ratio requirements.

due to limited liability and bank runs. Their paper does not consider fire sale externalities, and as a result, optimal regulation does not necessarily involve capital or liquidity regulations. Walther (2015) also studies macroprudential regulation in a model characterized by pecuniary externalities due to fire sales. In his setup, the socially optimal outcome is to have no fire sales in equilibrium, whereas in our paper partial fire sales are also optimal. Furthermore, even though banks have two independent choice variables in his model as well, Walther does not study implications of regulating only one channel on banks' investment decisions and financial stability.

A few other studies consider the effectiveness of capital and liquidity requirements separately in addressing a particular market failure, but unlike this paper do not study the jointly optimal design of these regulations or the interaction between the two. Cifuentes, Ferrucci, and Shin (2005) argue that liquidity buffers play a role similar to capital buffers in curbing systemic externalities arising from asset fire sales, and they may even be more effective under severe stress scenarios. Perotti and Suarez (2011) show that banks choose an excessive amount of short term debt in the presence of systemic externalities and analyze the effectiveness of liquidity regulations as in Basel III as opposed to Pigovian taxation in implementing the social optimal level of short term funding. Calomiris, Heider, and Hoerova (2013) argue that the role of liquidity requirements should be conceived not only as an insurance policy that addresses the liquidity risks in distressed times, as proposed by Basel III, but also as a prudential regulatory tool that makes crises less likely.

Repullo (2005) shows, in direct contrast to our result, that a higher capital requirement reduces the attractiveness of risky investment, and hence, causes a bank to increase its investment in safe assets. In his model, the balance sheet size of bank is exogenously fixed, and hence, a decrease in risky investment necessarily implies an increase in safe assets. In contrast, we consider a model with a flexible bank balance sheet in which capital requirement decreases risky investment level, and banks respond by decreasing their liquidity ratios. Farhi, Golosov, and Tsyvinski (2009) consider a Diamond-Dybvig model with unobservable liquidity shocks and unobservable trades. They show that competitive equilibria are inefficient even if the markets for aggregate risk are complete and that optimal allocations can be implemented through a simple liquidity ratio requirement on financial intermediaries. Donaldson, Piacentino, and Thakor (2015) show that under a moral hazard problem a liquidity requirement reduces bank lending below the efficient level while an increase in bank capital requirement might increase bank liquidity creation.

Stein (2012) shows that banks, not internalizing the fire sale externalities, rely too much on short term debt, a cheap form of financing, which in turn supports greater lending. In Stein's setup once the liquidity choice of banks is aligned with the socially optimal level by regulation, the investment decision is also aligned automatically. In our paper, regulating liquidity alone or imposing a tax on it is not sufficient to guarantee the socially optimal level of investment. Both the amount of total liquidity and total investment determine the amount of fire sales, and thus should be regulated.

As in our paper, a few seminal papers have pointed out the inefficiency of liquidity choice of banks in laissez-faire equilibrium under market incompleteness or informational frictions. Bhattacharya and Gale (1987) consider an extended version of Diamond and Dybvig (1983) with several banks and show that when banks face privately observed liquidity shocks, they underinvest in liquid assets and free-ride on the common pool of liquidity in the interbank market. Allen and Gale (2004b) show that when markets for hedging liquidity risk is incomplete, private liquidity hoardings of banks are inefficient. Whether there is too much or too little liquid assets in the laissez-faire equilibrium depends on the coefficient of relative risk aversion: if it is greater than one, the liquidity is inefficiently low.

De Nicoló, Gamba, and Lucchetta (2012) consider a dynamic model of bank regulation where liquidity is only welfare-reducing because unlike our paper, the authors do not consider the role of liquidity in insuring banks against the fire sale risk. Covas and Driscoll (2014) study the introduction of liquidity requirements on top of existing capital requirements in a dynamic stochastic general equilibrium (DSGE) model. They show that imposing a liquidity requirement leads to a decline in both the output and the amount of bank loans in the steady state. Adrian and Boyarchenko (2013), using a DSGE framework as well, find that liquidity requirements are a preferable prudential policy tool relative to capital requirements, as tightening liquidity requirements lowers the likelihood of systemic distress without reducing consumption growth. These studies impose the regulatory constraints and study their implications, whereas in our paper optimal regulatory constraints emerge endogenously to correct for specific market failures.

Even though the literature on the interaction between capital and liquidity requirements is limited, there are studies that examine the interaction between different tools available to regulators. Acharya, Mehran, and Thakor (2016) show that the optimal capital regulation requires a two-tiered capital requirement with some bank capital invested in safe assets. The special capital should be unavailable to creditors upon failure so as to retain market discipline and should be available to shareholders only contingently on good performance in order to contain risk-taking. Arseneau, Rappoport, and Vardoulakis (2015) show that two policy tools (both asset purchases and interest on reserves) are needed to restore the constrained efficiency when agents do not internalize the effects of portfolio allocations in the primary market on the secondary market illiquidity. Nevertheless, they do not study the implications of using only one policy tool or the interaction between the policies. Hellmann, Murdock, and Stiglitz (2000) show that under risk-shifting by banks, Pareto-efficient outcomes can be achieved by adding deposit-rate controls to capital regulations. Such controls restore prudent behavior by increasing franchise values.

Our paper is also related to the literature that features financial amplification and asset fire sales, which includes the seminal contributions of Fisher (1933), Bernanke and Gertler (1989), Kiyotaki and Moore (1997), Krishnamurthy (2003, 2010), and Brunnermeier and Pedersen (2009). In our model, fire sales result from the combined effects of asset-specificity and correlated shocks that hit an entire industry or economy. This idea, originating with Williamson (1988) and Shleifer and Vishny (1992), is employed by fire sale models such as Lorenzoni (2008), Davila and Korinek (2017), and Kara (2016). These papers show that under pecuniary externalities arising from asset fire sales, there exists overinvestment in risky assets in a competitive setting compared with the socially optimal solution. Relatedly, in He and Kondor (2016) there is overinvestment in risky assets in boom periods and underinvestment during recessions under pecuniary externalities. However, unlike our paper, none of these papers consider jointly optimal determination of risky investment levels and liquidity.

The constrained inefficiency of competitive markets in this paper is due to the existence of pecuniary externalities under incomplete markets. The Pareto suboptimality of competitive markets when the markets are incomplete goes back at least to the work of Borch (1962). The idea was further developed in the seminal papers of Hart (1975), Stiglitz (1982), and Geanakoplos and Polemarchakis (1986), among others. Greenwald and Stiglitz (1986) extended the analysis by showing that, in general, pecuniary externalities by themselves are not a source of inefficiency but can lead to significant welfare losses when markets are incomplete or there is imperfect information. Pecuniary externalities are categorized into two types by Davila and Korinek (2017): distributive externalities that are due to marginal rates of substitution of different agents not being equalized and collateral externalities that arise from market price affecting the value of collateral. In our case, banks are financially constrained and limited commitment impedes the equalization of the marginal rate of substitutions. The resulting distributive externalities lead to overinvestment in risky assets and underinvestment in liquid assets.

3 Model

The model consists of three periods, $t = 0, 1, 2$; along with a continuum of banks and a continuum of consumers, each with a unit mass. Consumers are risk neutral, and their preferences are represented by the utility function $E[c_0 + c_1 + c_2]$. Bankers are also risk neutral but only consume in period 2.

There are two types of goods in this economy, a consumption good and an investment good (that is, the liquid and illiquid assets). Consumers are endowed with ω units of consumption goods in each period.[6] Banks have two technologies: a storage technology and a technology that converts consumption goods into investment goods one-to-one at $t = 0$. Investment goods that are managed by a bank until the last period will yield $R > 1$ consumption goods per unit. However, investment goods are subject to a restructuring shock at $t = 1$, which we discuss in detail below, and hence we refer to them as the risky assets. Risky assets can be thought as mortgage-backed securities or a portfolio of loans to firms in the corporate sector.[7] Investment goods can never be converted back into the consumption goods, and they fully depreciate after the return is collected at $t = 2$.

Banks choose at $t = 0$ how many risky assets to hold, denoted by n_i, and how many liquid (safe) assets, denoted by b_i, to put aside for each unit of risky assets. The total amount of liquid assets held by each bank is then $n_i b_i$, and b_i can be interpreted as a liquidity ratio. The storage technology allows moving liquid assets from one period to another. Therefore, the total asset size of a bank is $n_i + n_i b_i = (1 + b_i)n_i$. On the liability side, each bank is endowed with e units equity capital at $t = 0$ in terms of consumption goods. The fixed amount of equity capital assumption captures the fact that it is difficult for banks to raise equity in the short-term (see for example, Almazan, 2002; Repullo, 2005; Dell'Ariccia and Marquez, 2006). Hence, each bank raises $l_i = (1 + b_i)n_i - e$ units of consumption goods from consumers at $t = 0$ to finance its portfolio of safe and risky assets. We assume that each bank is a local monopsony in the deposit market so that consumers earn zero net expected interest rate from their lending to the banks. The non-contingent debt is the only allowed contract between banks and consumers at the initial period, and therefore, the asset markets are incomplete.

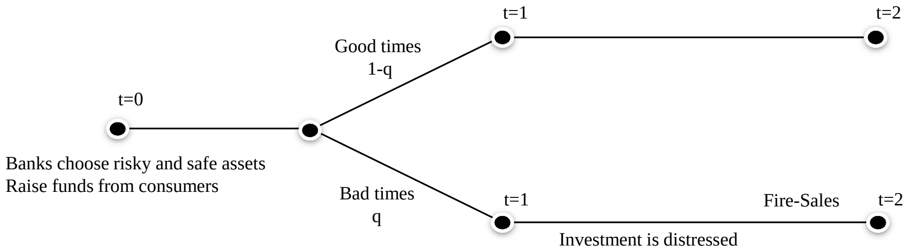
We assume that there is a nonpecuniary cost of operating a bank, captured by $\Phi((1 + b_i)n_i)$. The operational cost is increasing in the size of the balance sheet, $\Phi'(\cdot) > 0$, and it is convex, $\Phi''(\cdot) > 0$. This assumption, similar to the ones imposed by Van den Heuvel (2008) and Acharya (2003, 2009), ensures that the banks' problem is well defined and that there is an interior solution to this problem. It also allows us to have banks with flexible balance sheet size in the model: Banks can increase or decrease the amount of risky and liquid assets simultaneously. Thanks to this flexibility, we can study the interaction between these two independent choices of banks.

Investment and deposit collection decisions are made at time $t = 0$. The only uncertainty

[6] We assume that the initial endowment of consumers is sufficiently large, and it is not a binding constraint in equilibrium.

[7] To simplify the exposition, we abstract from modeling the relationship between banks and firms. Instead, we assume that banks directly invest in physical projects. This assumption is equivalent to assuming that there are no contracting frictions between banks and firms, as more broadly discussed by Stein (2012).

Figure 1: Timing of the model



in the model is about the risky asset and is resolved at the beginning of $t = 1$: The economy lands in good times with probability $1-q$ and in bad times with probability q. In good times, no bank is hit with restructuring shocks, and therefore no further action is taken. Banks keep managing their investment goods and in the final period realize a total return of $Rn_i + n_i b_i$. However, in bad times, the risky assets are distressed and have to be restructured in order to remain productive, as in Holmstrom and Tirole (1998) and Lorenzoni (2008). Restructuring costs are equal to $c \leq 1$ units of consumption goods per unit of the risky asset. If c is not paid, the risky investment is scrapped (that is, it fully depreciates).

A bank can use the liquid assets hoarded from the initial period, $n_i b_i$, to carry out the restructuring of the distressed investment at $t = 1$. If the liquid assets are not sufficient to cover the entire cost of restructuring, the bank needs external finance. However, we assume that because of a combination of debt-overhang and limited-commitment problems, banks cannot borrow the required resources from the household sector.[8] The only way for banks to raise the funds necessary for restructuring is by selling some fraction of the risky asset to firms in the traditional sector, which are owned by consumers.

The asset sales by banks are in the form of fire sales: The risky asset is traded below its fundamental value for banks, and the price decreases as banks try to sell more assets. Banks retain only a fraction, γ, of their risky assets after fire sales, which depends on banks' liquidity shortages as well as on the fire sale price of risky asset. The sequence of events is illustrated in Figure 1.

We first solve the competitive equilibrium of the model when there is no regulation on banks. Then we present the constrained planner's problem and analyze its implementation using quantity-based capital and liquidity requirements as in the Basel Accords.

[8]In section 6.1 we describe the general setting in which banks can pledge only a fraction of their returns in the final period to the lenders. We then derive the parameter region that gives rise to this basic setup in which the pledgeability constraint does not bind in the initial period but it does in the bad state of interim period because of debt overhang. Throughout the paper we focus on that parameter region.

3.1 Crisis and fire sales

The decision of agents at time t=0 depends on their expectations regarding the events at time $t = 1$. Thus, applying the solution by backwards induction, we first analyze the equilibrium at the interim period in each state of the world for a given set of investment levels. We then study the equilibrium at $t = 0$. Note that if the good state is realized at $t = 1$, banks take no further action and obtain a total return of $\pi_i^{Good} = Rn_i + b_i n_i$ at the final period, $t = 2$. Therefore, for the interim period $t = 1$, studying the equilibrium only for bad times is sufficient. We start with the problem of traditional firms in bad times, then analyze the problem of banks.

3.1.1 Traditional sector

Firms in the traditional sector, owned by consumers, can buy investment goods from banks and manage them. Unlike banks, they have a concave production technology, and they employ y units of investment goods purchased from banks at $t = 1$ to produce $F(y)$ units of consumption goods at $t = 2$. Let P denote the market price of the investment good in bad times at $t = 1$.[9] Each firm in the traditional sector takes the market price as given and chooses the amount of investment goods to buy, y, in order to maximize net returns from investment at $t = 2$:

$$\max_{y \geq 0} F(y) - Py. \tag{1}$$

The first-order condition of this problem, $F'(y) = P$, determines the traditional firms' (inverse) demand function for the investment good. We can define their demand function, $Q^d(P)$, as follows: $Q^d(P) \equiv F'(P)^{-1} = y$.

Assumption 1 (*Efficiency*)**.** *$F'(y) > 0$ and $F''(y) < 0$ for all $y \geq 0$, with $R \geq F'(0) > \nu \equiv qR(1+c)/(R-1+q)$.*

Under the *Efficiency* assumption, firms in the traditional sector have a concave production technology, which yields a downward-sloping demand function for investment goods (see Figure 2). Firms are also less productive than banks at each level of investment goods employed due to $F'(0) \leq R$. As a result, banks have to accept a price lower than the fundamental value, R, to sell any assets to them and accept even lower prices to sell more assets.

The origins of this idea can be found in Williamson (1988) and Shleifer and Vishny (1992), who claim that some assets are industry-specific and, hence, less productive when managed

[9] The price of the investment good at $t = 0$ will be one as long as there is positive investment, and the price at $t = 2$ will be zero because the investment good fully depreciates at this point.

by outsiders.[10] Outsiders do not have the specific expertise to manage these assets well and, thus, they cannot afford to pay the value in best use for the assets of distressed enterprises. For instance, monitoring and collection skills of loan officers greatly affect the value of bank assets, particularly bank loans. The lack of such skills among outsiders creates a deadweight cost when assets are transferred from banks to outsiders via fire sales (Acharya, Shin, and Yorulmazer, 2011).[11] A decreasing returns to scale technology for outsiders, as in the works of Kiyotaki and Moore (1997), Lorenzoni (2008), and Korinek (2011), arises if the industry-specific assets are heterogeneous. Traditional sector would initially purchase assets that are easy to manage, but as they continue to purchase more assets, they would need to buy those that require increasingly sophisticated management and operation skills.

In addition, we assume that $F'(0)$ is not too small—to be exact, $F'(0) > \nu \equiv qR(1+c)/(R-1+q)$. This assumption ensures that a small amount of fire sale does not decrease the price of assets dramatically below the fundamental value, R. Next, we need to impose more structure on the return function of the traditional sector to ensure that the equilibrium of this model exists and is unique.

Assumption 2 (Elasticity)**.**

$$\epsilon^d = \frac{\partial Q^d(P)}{\partial P}\frac{P}{Q^d(P)} = \frac{F'(y)}{yF''(y)} < -1 \quad \textit{for all } y \geq 0$$

The *Elasticity* assumption states that the traditional sector's demand for the investment good is elastic and rules out multiple equilibria in the asset market at $t = 1$. Rewriting the assumption as $F'(y) + yF''(y) > 0$, it implies that banks' proceeds from selling assets to firms in the traditional sector, $Py = F'(y)y$, is strictly increasing in the amount of assets sold, y. Without this assumption, different levels of asset sales would raise the same level of funds, leading to multiple equilibria.[12]

Assumption 3 (*Regularity*)**.** $F'(y)F'''(y) - 2F''(y)^2 \leq 0 \quad \textit{for all } y \geq 0.$

The *Regularity* assumption holds for log-concave functions, yet it is weaker than log-concavity.[13] We use this assumption to guarantee that the objective functions of banks and

[10]Industry-specific assets can be physical or they can be portfolios of financial intermediaries (Gai et al., 2008).

[11]The existence of fire sales for both physical and financial assets is supported by empirical and anecdotal evidence. Pulvino (2002) finds that distressed airlines sell aircraft at a 14 percent discount from the average market price. This discount exists when the airline industry is depressed but not when it is booming. Coval and Stafford (2007) show that fire sales exist in equity markets when mutual funds engage in sales of similar stocks.

[12]This assumption is also imposed by Lorenzoni (2008), Korinek (2011), and Kara (2016) to rule out multiple equilibria under fire sales. Gai et al. (2008) provide an example in which this assumption is not satisfied and thus multiple equilibria exist. The ex-ante beliefs of agents determine the choice of equilibrium, and the authors show that the irrespective of the beliefs, the competitive equilibrium is constrained inefficient and leads to overinvestment.

[13]A function is said to be log-concave if the logarithm of the function is concave. Log-concave demand functions are

the planner are concave and yield interior solutions.[14]

Assumption 4 (*Technology*)**.** $1 + qc < R < 1/(1-q)$.

The first inequality in the *Technology* assumption states that the net expected return on the risky asset is positive. The second inequality, $R < 1/(1-q)$, guarantees that the return in the good state alone is not high enough to make banks' expected profit positive.

3.1.2 Banks' problem in the bad state

Consider the problem of bank i when bad times are realized at $t = 1$. The bank has an investment level, n_i, and liquid assets of $b_i n_i$ chosen at the initial period. If $b_i \geq c$, the bank has enough liquid resources to restructure all of the assets. In this case, the bank obtains a gross return of $Rn_i + (b_i - c)n$ on its portfolio at $t = 2$. However, if $b_i < c$, then the bank does not have enough liquid resources to cover the restructuring cost entirely, and thus, decides what fraction of these assets to sell $(1 - \gamma_i)$. Note that γ_i then represents the fraction of assets that a bank keeps after fire sales.[15] Thus, the bank takes the price of the investment good (P) as given and chooses γ_i to maximize total returns from that point on:

$$\max_{0 \leq \gamma_i \leq 1} R\gamma_i n_i + P(1-\gamma_i)n_i + b_i n_i - cn_i, \tag{2}$$

subject to the budget constraint

$$P(1-\gamma_i)n_i + b_i n_i - cn_i \geq 0. \tag{3}$$

Budget constraint (3) states that the sum of the revenues raised by selling assets and the liquid assets carried from the initial period must at least cover the restructuring cost. By the *Efficiency* assumption, the equilibrium price of assets must satisfy $P \leq F'(0) \leq R$, otherwise the traditional sector would not purchase any assets. In equilibrium, we must also have $P > c$, otherwise in the bad state banks would scrap assets rather than selling them; that is, there would not be any fire sale. However, if there is no supply, then there is an

common in the Cournot games literature and are often used to prove the existence and uniqueness of an equilibrium (Amir, 1996). Let $\phi(y) \equiv F'(y)$ denote the (inverse) demand function of the traditional sector. We can rewrite this assumption as $\phi(y)\phi''(y) - 2\phi'(y)^2 \leq 0$. We can show that the demand function is log-concave if and only if $\phi(y)\phi''(y) - \phi'(y)^2 \leq 0$. Clearly, the *Regularity* assumption holds whenever the demand function is log-concave. However, it is weaker than log-concavity and may also hold if the demand function is log-convex (that is, if $\phi(y)\phi''(y) - \phi'(y)^2 \geq 0$).

[14] Many regular return functions satisfy conditions given by the *Efficiency*, *Elasticity* and *Regularity* assumptions. Two examples that satisfy all three of these assumptions are $F(y) = R\ln(1+y)$ and $F(y) = \sqrt{y + (1/2R)^2}$.

[15] Following Lorenzoni (2008) and Gai et al. (2008), we assume that banks have to restructure an asset before selling it. Basically, this means that banks receive the asset price P from the traditional sector, use a part, c, to restructure the asset, and then deliver the restructured assets to the firms. Therefore, banks sell assets only if P is greater than the restructuring cost, c. We could assume, without changing our results, that it is the responsibility of the traditional sector to restructure the assets that they purchase.

incentive for each bank to deviate and to sell some assets to outsiders. The deviating bank would receive a price close to $F'(0)$, which is greater than the cost of restructuring, c by the *Efficiency* and *Technology* assumptions.[16] Having $P > c$ together with the *Technology* assumption implies that investment goods are never scrapped in equilibrium.

Banks want to choose the highest possible γ_i because they receive R by keeping assets on the balance sheet, whereas by selling them they get $P \leq R$. Therefore, banks sell just enough assets to cover their liquidity shortage, $cn_i - b_i n_i$. This means that the budget constraint binds, from which we can obtain $\gamma_i = 1 - (c - b_i)/P$. As a result, the fraction of investment goods sold by each bank is

$$1 - \gamma_i = \frac{c - b_i}{P} \in (0, 1), \tag{4}$$

The fraction of assets sold, $1 - \gamma_i$, is decreasing in the price of the investment good, P, and in liquidity ratio, b_i, and increasing in the cost of restructuring, c. The supply of investment goods by each bank, i, is then equal to

$$Q_i^s(P, n_i, b_i) = (1 - \gamma_i)n_i = \frac{c - b_i}{P} n_i \tag{5}$$

for $c \leq P \leq R$. This supply curve is downward-sloping and convex, which is standard in the fire sales literature (see Figure 2, left panel). A negative slope implies that if there is a decrease in the price of assets, banks have to sell more assets in order to generate the resources needed for restructuring.

We can substitute the optimal value of γ_i using (4) into (2) and write the maximized total returns of banks in the bad state at $t = 1$ as $\pi_i^{Bad} = R\gamma_i n_i = R(1 - \frac{c - b_i}{P})n_i$ for a given n_i and b_i. Note that the sum of the last three terms in (2) is zero at the optimal choice of γ_i because of the binding budget constraint.

3.1.3 Asset market equilibrium at date 1

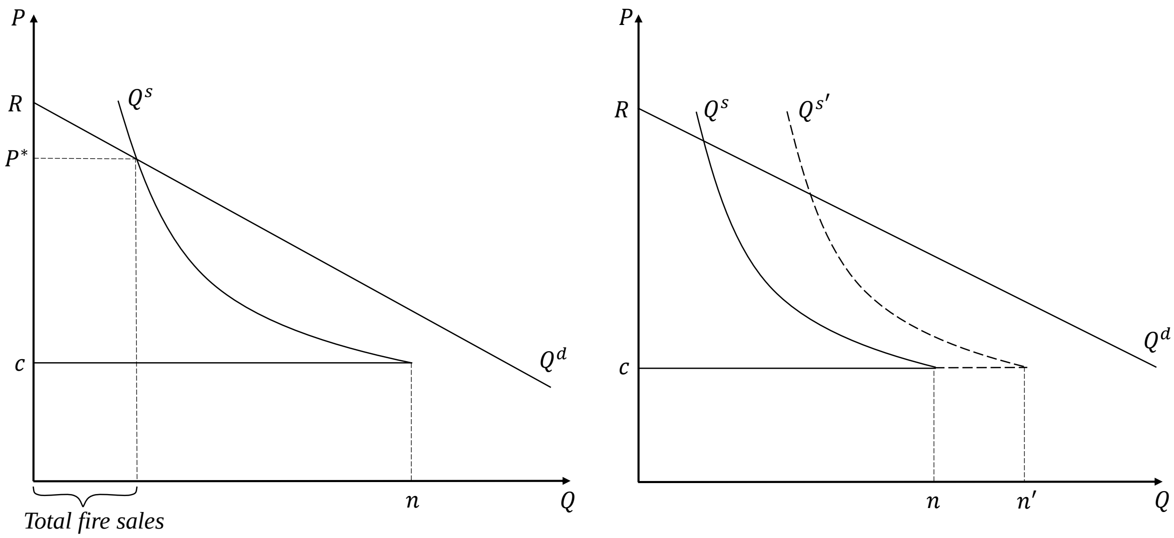
We consider a symmetric equilibrium where $n_i = n$ and $b_i = b$ for all banks. Therefore, the aggregate risky investment level is given by n and the liquidity ratio is given by b as there is a continuum of banks with a unit mass. The equilibrium price of investment goods in the bad state, P, is determined by the market clearing condition

$$Q^d(P) - Q^s(P; n, b) = 0. \tag{6}$$

This equilibrium is illustrated in the left panel of Figure 2. Note that the equilibrium price of the risky asset and the amount of fire sales at $t = 1$ are functions of the initial total

[16] Note that $F'(0) > \nu \equiv qR(1 + c)/(R - 1 + q)$ together with $R < 1/(1 - q)$ implies that $F'(0) > c$.

Figure 2: Equilibrium in the investment goods market and comparative statics



investment in the risky asset and the aggregate liquidity ratio. Therefore, we denote the fire sale price in terms of state variables as $P(n, b)$.

Lemma 1. *The fire sale price of risky asset, $P(n, b)$, is decreasing in n and increasing in b. The fraction of risky assets sold, $1 - \gamma(n, b)$, is increasing in n and decreasing in b.*

Lemma 1 states that higher investment in the risky asset or a lower liquidity ratio increases the severity of the financial crisis by lowering the asset prices. This effect is illustrated in the right panel of Figure 2. Suppose that the banks enter the interim period with larger holdings of risky assets. In this case, banks have to sell more assets at each price, as shown by the supply function given by (5), because the aggregate liquidity shortage, $(c-b)n$, is increasing in the amount of initial risky assets, n. Graphically, the aggregate supply curve shifts to the right, as shown by the dotted-line supply curve in the right panel of Figure 2, which causes a decrease in the equilibrium price of investment goods. A lower initial liquidity ratio has a similar effect by increasing the liquidity shortage in the bad state, $(c - b)n$, and hence causing a larger supply of risky assets to the market. Lower asset prices, by contrast, induce more fire sales by banks because of the downward-sloping supply curve, and hence, making financial crises more costly.

3.2 Competitive equilibrium

As a benchmark, we first study the competitive equilibrium. In the bad state, if banks face a liquidity shortage, their only option is to sell some assets to the traditional sector. At the initial period, each bank, i, chooses the amount of investment in the risky asset, n_i, and the liquidity ratio, b_i, to maximize its expected profits:

$$\Pi_i(n_i, b_i) = (R + b_i - qc)n_i - D(n_i(1 + b_i)) - I(b_i < c)q(R - P)Q_i^s(P, n_i, b_i), \qquad (7)$$

subject to its budget constraint, $e + l_{i0} \geq n_i + b_i n_i$, at $t = 0$. Let $\Gamma(n_i, b_i) \equiv (R + b_i - qc)n_i - D(n_i(1 + b_i))$ represent the basic profits that would be obtained if there were no fire sales. $D(n_i(1 + b_i)) = n_i(1 + b_i) + \Phi(n_i(1 + b_i))$ is the sum of the initial cost of funds and the operational costs of a bank. Note that because consumers earn zero net expected return on their lending to banks, the cost of funds to a bank is $e + l_{i0} = n_i(1 + b_i)$.[17] The last term is the expected cost of fire sales: If liquidity hoarded at $t = 0$ is not sufficient to cover the shock in the bad state at $t = 1$, that is $b_i < c$ as shown by the indicator function, $I(\cdot)$, then the bank sells $Q_i^s(P, n_i, b_i)$ units of assets and loses $R - P \geq 0$ on each unit sold. The amount of assets sold, $Q_i^s(P, n_i, b_i)$, is a function of the initial portfolio allocations and the price of assets, as shown by (5).

Whether or not fire sales take place in the competitive equilibrium depends on the initial liquidity ratios of banks . If banks fully insure themselves against the fire sale risk—that is, if they choose $b_i \geq c$ for all $i \in [0, 1]$ at $t = 0$—then fire sales in the bad state are avoided. However, if banks purchase less than full insurance—that is, if $b_i < c$—then fire sales exist. The following lemma shows that in the competitive equilibrium, banks optimally choose less than full insurance and, hence, fire sales take place.

Lemma 2. *Under the Efficiency and Technology assumptions, banks always take fire sale risk in equilibrium; that is, $b_i < c$ for all banks.*

Even though both the amount (c) and frequency (q) of the aggregate liquidity shock are exogenous in the model, whether and to what extent a fire sale takes place are endogenously determined. In Lemma 2 we show that perfect insurance is never optimal and that banks take some amount of fire sale risk; that is, they choose $b_i < c$. The intuition of the proof is as follows: The expected marginal return on liquid assets exceeds unity as long as there are fire sales, and it decreases with the amount of liquidity. Perfect insurance guarantees that no fire sale takes place and, as a result, the expected marginal return on liquid assets is equal to one, which is dominated by the expected marginal return on risky assets. In other words, there is no need to hoard any liquidity when there is no fire sale risk. Therefore, there is an optimal interior level of liquidity ratio for which the private marginal return and cost of liquidity are equalized.

Although banks take some fire sale risk, the main issue is whether banks take the socially

[17] In Section 6.1 we explicitly consider consumers' participation constraint.

optimal amount of fire sale risk. Lemma 2 allows us to focus on the imperfect insurance case; that is, $b_i < c$. We can write banks' profit function under this result as:

$$\Pi_i(n_i, b_i) \quad = \quad \Gamma(n_i, b_i) - q(R - P)Q_i^s(P, n_i, b_i). \tag{8}$$

The unique symmetric equilibrium in which $n_i = n$ and $b_i = b$ for all banks $i \in [0, 1]$ is determined by the first-order conditions of banks' and traditional firms' problems and market clearing:

$$\begin{aligned} \frac{\partial \Gamma}{\partial x_i} - q(R - P)\frac{\partial Q_i^s}{\partial x_i} &= 0, \quad \forall x_i \in \{n_i, b_i\} && (9) \\ F'(y) &= P, && (10) \\ y &= Q^s(P, n, b), && (11) \end{aligned}$$

where $Q_i^s(P, n_i, b_i) = \frac{c-b_i}{P} n_i$. We first show that the closed-form solution for the competitive equilibrium price, P, is independent of the functional form of the traditional sector's demand and the operational cost of banks.

Proposition 1. *Under the Efficiency, Elasticity, Regularity, and Technology assumptions, the competitive equilibrium price of assets is given by*

$$P = \frac{qR(1 + c)}{R - 1 + q}. \tag{12}$$

The equilibrium price, P, is increasing in the probability of the liquidity shock, q, and the size of the shock, c, but decreasing in the return on the risky assets, R.

Proposition 1 shows that the price of assets in the bad state increases in the expected size of the liquidity shock, qc. When banks expect to incur a larger additional cost for the investment, or when they face this cost with a higher probability, they reduce risky investment levels and increase liquidity buffers, as we show in the next proposition. As a result, there are fewer fire sales and a higher price for risky assets in the competitive equilibrium.

3.2.1 A closed-form solution for the competitive equilibrium

In order to obtain closed-form solutions for the equilibrium values of n and b, we need to make functional-form assumptions for the traditional sector's production technology, F, and the operational cost of banks, Φ. Suppose that the operational cost of a bank is given by $\Phi(x) = \phi x^2$, and hence $\Phi^{'}(\cdot)$ is increasing; that is, $\Phi^{'}(x) = 2\phi x$. Note that marginal cost of funds is increasing in parameter ϕ. On the demand side, suppose that the traditional sector's return function is given by $F(y) = R\ln(1+y)$. It is easy to verify that this function satisfies the *Efficiency*, *Elasticity*, and *Regularity* assumptions. We will refer to these two functional form assumptions as the "log-quadratic assumptions" in the remainder of the text and clarify whenever a result is obtained under these assumptions. In Section B.1 in the internet appendix, we provide the closed-form solutions for the competitive equilibrium investment level and liquidity ratios. Proposition 2 presents the comparative statics for the competitive equilibrium.

Proposition 2. *Under the log-quadratic functional form assumptions, the comparative statics for the competitive equilibrium risky investment level, n, and liquidity ratio, b, are as follows:*

1. *The risky investment level (n) is increasing in the return on the risky asset (R) and decreasing in the size of the liquidity shock (c), probability of the bad state (q), and the marginal cost parameter (ϕ).*

2. *The liquidity ratio (b) is increasing in the return on the risky asset (R), size of the liquidity shock (c), and the probability of the bad state (q), and decreasing in the marginal cost parameter (ϕ).*

Proposition 2 shows that b and n increase (decrease) *simultaneously* as a response to a change in R (in ϕ), which is thanks to the flexible bank balance sheet size. Proposition 2 implies that banks act less prudently (by increasing risky investment and reducing liquidity) if they expect financial shocks to be less frequent (a lower q) or less severe (a lower c), which in turn leads to more severe disruption to financial markets (through lower asset prices and more fire sales) if shocks do materialize, as shown by Proposition 1. Stein (2012) obtains a similar result as well.[18]

3.3 Constrained planner's problem

A constrained planner is subject to the same market constraints as the private agents. In particular, the planner takes the limited commitment in financial contracts between banks and consumers as given. However, unlike banks, the constrained planner takes into account the effect of initial portfolio allocations on the price of assets in the bad state. The constrained planner maximizes banks' expected profits subject to a constraint that, after the

[18] This result is reminiscent of the financial instability hypothesis of Minsky (1992), who suggests that "over periods of prolonged prosperity, the economy transits from financial relations that make for a stable system to financial relations that make for an unstable system."

transfers, consumers are at least as well off as they are in the decentralized equilibrium:

$$\max_{n,b,y} \Gamma(n,b) - I(b<c)q(R-P)Q^s(P,n,b) - (1-q)T_2, \tag{13}$$

$$\text{subject to } y = Q^s(P,n,b),$$

$$F'(y) = P,$$

$$(1-q)T_2 + 3\omega + q[F(y) - Py] \geq U_i^{CE}. \tag{14}$$

where $I(\cdot)$ is the indicator function and $Q^s(P,n,b) = \frac{c-b}{P}n$ is the amount of assets sold by banks in the bad state at $t=1$. The last constraint states that consumers' utility must be at least as much as U_i^{CE}, their expected utility in the competitive equilibrium. The term, $q[F(y) - Py]$, gives consumers' expected profits from fire sales through the firms in the traditional sector. Consumers earn net zero expected return from their lending to banks in the initial period. Therefore, this lending only alters the timing of their consumption across periods, but not their utility.

The planner also makes compensatory transfers between banks and consumers to ensure that reallocation of resources leads to a Pareto improvement. We assume that transfers happen only in good times and in the final period—that is, when the pledgeability constraint of banks does not bind. In other words, the planner cannot use transfers to circumvent the financial constraints of bankers. After the planner has determined allocations and transfers in period 0, private agents follow their optimal strategies in the following periods. The first question is whether the constrained planner would avoid fire sales completely by setting $b \geq c$. The next lemma addresses this question.

Lemma 3. *Under the risk neutrality, Efficiency, and Technology assumptions, it is optimal for the constrained planner to take fire sale risk; that is, the constrained optimal liquidity ratio satisfies $b < c$.*

The lemma states that it is optimal for the constrained planner to expose the banking sector to some amount of fire sale risk. In other words, full insurance is not constrained optimal. Lemma 3 allows us to focus on the $b < c$ case when analyzing the constrained planner's problem. As we derive in Appendix B.2, we can simplify the optimality conditions for planner's problem to:

$$\frac{\partial \Gamma}{\partial x} - q(R-P)\frac{\partial Q^s}{\partial x} - q(R-P)\frac{\partial Q^s}{\partial P}\frac{\partial P}{\partial x} = 0, \quad \forall x \in \{n, b\}. \tag{15}$$

We denote the constrained efficient allocations by n^{**}, b^{**}, and the associated price of assets in the bad state by P^{**}. Section B.2 in the appendix presents the closed-form solutions for n^{**}, b^{**} and P^{**}.

These first-order conditions are similar to the first-order conditions of the banks' problem in Section 3.2, shown in (9), except that each condition contains an additional term: $-q(R-P)\frac{\partial Q^s}{\partial P}\frac{\partial P}{\partial x}$ for $x \in \{n, b\}$. The difference arises because unlike the individual banks, the constrained planner takes into account how changing the initial risky investment level and liquidity ratio affects the price of assets, P, and hence, the amount of assets sold to the traditional sector, Q^s. In other words, the constrained social planner internalizes the fire sale externalities, that is, the planner internalizes the fact that larger risky investments or lower liquidity ratios lead to a lower asset price and more fire sales in the bad state. We can show that the competitive equilibrium is constrained inefficient under some general conditions and compare the competitive equilibrium level of risky assets and liquidity ratios with the constrained efficient allocations. To perform the comparison, we use the closed-form solutions of equilibrium outcomes presented in the appendix B.2.

Proposition 3. *Under the risk neutrality, Efficiency, Elasticity, and Technology assumptions, the competitive equilibrium is constrained inefficient. Furthermore, under the log-quadratic functional form assumptions, competitive equilibrium allocations compare to the constrained efficient allocations as follows:*

1. *Risky investment levels:* $n > n^{**}$
2. *Liquidity ratios:* $b < b^{**}$

The inefficiency of the competitive equilibrium allocations is because of a combination of market incompleteness and banks' failure to internalize the fire sale externalities. Second part of Proposition 3 shows that in the competitive equilibrium, unregulated banks overinvest in the risky asset, $n > n^{**}$, and inefficiently insure against liquidity shocks by holding low liquidity ratios, $b < b^{**}$. The first result of the proposition is reminiscent of Lorenzoni (2008) and Korinek (2011), who show that there is excessive risky investment under fire sale externalities. The latter, meanwhile, is reminiscent of Bhattacharya and Gale (1987) and Allen and Gale (2004b), who show that private holdings of liquid assets are inefficient under incomplete markets. Allowing banks to invest in both the risky illiquid asset and liquid asset, we show that the pecuniary externality manifests itself in both choices of banks and distorts both margins. Together with the flexible balance sheet size, this setup allows us to study the interaction between the two as well.

3.4 Implementing the constrained efficient allocations: complete regulation

The constrained efficient allocations (n^{**}, b^{**}) can be implemented by applying simple quantity regulations to banks—in particular, by imposing a minimum liquidity ratio as a fraction of risky assets ($b_i \geq b^{**}$) and a maximum level of risky investment ($n_i \leq n^{**}$). The latter corresponds to a minimum risk-weighted capital ratio; that is, $e/n_i \geq e/n^{**}$, because the inside equity of banks, e, is fixed in our model. For analytical convenience, we use the upper bound on risky investment formulation for capital regulation in the rest of the paper.

The quantity-based rules can be mapped to the capital and liquidity regulations in the Basel III accord. First, the risk-weighted capital ratio, e/n_i, corresponds to the Basel definition, as it gives liquid assets, $n_i b_i$, a zero risk weight while giving risky assets, n_i, a weight of one in the denominator. In reality, banks carry several risky assets on their balance sheet for which Basel Accords require different risk weights. However, introducing assets with different risk profiles to our setup would complicate the analysis without adding further insight.

Second, our liquidity regulation mimics the liquidity coverage ratio (LCR) requirement proposed in Basel III. The LCR requires banks to hold high-quality liquid assets against the outflows expected in the next 30 days under a stress scenario. In our setup, the expected cash outflow in the bad state is the liquidity need, c, per each risky asset. Therefore, the liquidity requirement in our setup can be equivalently written as $b_i n_i / c n_i \geq b^{**} n^{**} / c n^{**}$. It is true that the LCR focuses on liquidity shocks on the liability side whereas here we consider liquidity shocks on the asset side. However, this modeling choice is not essential to our result; all we need is a liquidity requirement in some states of the world that cannot be fully met with raising external finance. If we instead model liquidity shock as a proportion of deposits, we would then need capital regulation to limit the size of deposits and liquidity requirement to increase the high quality liquid assets (cash).

4 Partial regulation: regulating only capital ratios

The liquidity requirement was missing in the pre-Basel III era. In order to understand whether Basel III regulations are a step in the right direction, one needs to compare them to the pre-Basel III era. For this purpose, in this section we consider an economy in which the capital ratios of banks are regulated but there is no requirement on their liquidity ratios. Hence, we consider banks that are free to choose their liquidity ratios for a given capital requirement. This setup also allows us to study the interaction of banks' capital and liquidity ratios and to provide an answer to Tirole's question quoted in the introduction: What happens to banks' liquidity when their capital ratios are regulated? Do banks manage their liquidity in an efficient way, or does capital regulation distort their choice of liquidity?

We consider the problem of a planner who is endowed with only one tool. In particular, the planner chooses the level of risky investment, n, at $t = 0$ in a Pareto efficient way but allows banks to freely choose their liquidity ratio, b_i. As in the previous section, the planner is subject to the same contracting constraints as the private agents but takes into account the effect of the initial risky investment level on the price of assets in the bad state. Because banks choose inefficiently high risky investment levels in the competitive equilibrium, the planner's preferred risky investment level has to be lower than the competitive equilibrium level.[19] Therefore, the optimal level can be implemented in a decentralized economy because when it is introduced as a regulatory constraint, banks will set their risky investment at this optimal level. We call this case a "partially regulated economy" and compare it to the competitive equilibrium and second-best allocations studied in the previous section. We start by studying the banks' problem in this case. For a given regulatory upper bound on investment level, n, banks set $n_i = n$ and choose the liquidity ratio, b_i, to maximize their expected profits:

$$\max_{b_i} \Pi_i(b_i; n) = \max_{b_i} \, (R + b_i - qc)n - D(n_i(1+b)) - q(R-P)Q_i^s(P, n, b_i). \tag{16}$$

From the first-order condition of the banks' problem (16) with respect to b_i, we can obtain banks' (implicit) reaction function to the regulatory investment level—that is, the liquidity ratio, b_i, that banks choose for each given risky investment level, n—as follows:

$$b_i(n) = \frac{D'^{-1}(1 - q + q\frac{R}{P})}{n} - 1. \tag{17}$$

The planner takes this reaction function into account while choosing the risky investment level to maximize the expected bank profits subject to the constraint that consumers' utility after transfers is at least as high as in the competitive equilibrium:

$$\begin{aligned}
\max_{n,y} \, \Gamma(n, b(n)) - q\{(R-P)Q^s(P, n, b(n)) - (1-q)T_2\}&, \\
\text{subject to } y = Q^s(P, n, b(n))&, \\
F'(y) = P&, \\
\frac{d\Pi_i(b_i; n)}{db_i} = 0&, \\
(1-q)T_2 + 3\omega + q[F(y) - Py] \geq U_i^{CE}&.
\end{aligned}$$

As we derive in Appendix B.3, we can simplify the condition for planner's choice of n to:

$$\frac{\partial \Gamma}{\partial n} + \frac{\partial \Gamma}{\partial b} b'(n) - q(R-P)\left(\frac{\partial Q^s}{\partial n} + \frac{\partial Q^s}{\partial b} b'(n)\right) - q(R-P)\frac{\partial Q^s}{\partial P}\frac{dP}{dn} \;=\; 0. \tag{18}$$

We denote the optimal risky investment level that solves the first-order condition (18) by

[19] We prove this claim formally in the next section.

n^*, the associated optimal liquidity choice of banks under partial regulation by b^*, and the price of assets in the bad state by P^*. Section B.3 in the appendix presents the closed-form solutions for n^*, b^*, and P^*.

Changing n has an indirect effect on the asset price in the bad state in addition to its direct effect because banks change their liquidity ratios in response to changes in n, which then changes the price of assets in the equilibrium. The main question in this case is how banks respond to a tightening of capital regulations.

Proposition 4. *Let the operational cost of a bank be given by $\Phi(x) = \phi x^2$. Then, banks decrease their liquidity ratio as the regulator tightens capital requirements; that is, $b^{'}(n) \geq 0$ for any concave technology function for the traditional sector, $F(\cdot)$, that satisfies the Elasticity and Regularity assumptions along with either:*
(i) $F'(0) = R$, or (ii) $F'(0) \leq R$ and $R < \frac{F'(F'+yF'')}{F'+2yF''}$ for all $y \geq 0$.

Proposition 4 shows that banks reduce their liquidity ratios as the regulator tightens the risky investment level. The regulator attempts to correct banks' excessive risk-taking by requiring a higher risk-weighted capital ratio. However, because this regulation prevents banks from reaching their privately optimal level of risk, they react by reducing their liquidity ratios. In other words, banks undermine the purpose of capital regulations by carrying less-liquid portfolios. It would not be surprising to observe banks holding fewer liquid assets after they have been asked by the regulator to decrease their risky asset holdings. However, what is stated in Proposition 4 goes beyond that: Banks also decrease their liquidity ratios—that is, banks hoard less liquidity per unit of risky asset.

The proof of the proposition provides sufficient conditions by showing that there is strategic complementarity between the regulatory risky investment level, n, and the liquidity ratio, b_i, for each bank. The intuition of the proof is as follows: The marginal return to the liquidity ratio, b_i, is $(1-q)n_i + q\frac{R}{P}n_i$, and it is decreasing in the fire sale price, P. Capital regulation lowers the amount of risky investment, and hence increases the fire sale price as we show in Lemma 1. As a result, liquidity hoarding becomes less attractive for banks, and they decrease their liquidity ratio, b_i. Banks' ability to decrease their liquid assets while capital regulation is limiting their risky assets is possible due to the flexible balance sheet size of banks in our model.

We can also use an analogy from automobile safety regulations to explain Proposition 4. Peltzman (1975) and Crandall and Graham (1984) show that whether regulations such as safety belts and airbags reduce the fatality rate depends upon the response of drivers to the increased protection. They provide empirical evidence that drivers do indeed increase their driving intensity as a response to safety regulations, resulting in a less than expected reduction in fatality rates. Similarly, in our setup, capital regulations intend to make the financial system safer, but individual banks respond by taking more risk on the liquidity channel. As a result, there is a less than expected increase in financial stability and welfare from regulation as we show in the next section. In a sense, Proposition 4 reveals an unintended consequence of capital regulation when it is applied in isolation.

4.1 Complete versus partial regulation: do we need liquidity requirements?

In this section, we investigate whether capital regulation alone can restore the second-best allocations. For this reason, we compare the equilibrium outcomes (level of risky assets, liquidity ratios, asset prices, and the amount of fire sales) in three different settings: a decentralized equilibrium without any regulation, a partially regulated economy in which there is only capital regulation, and a complete regulation (second-best) case that has both capital and liquidity regulations. To perform the comparison, we use the closed-form solutions presented in the appendix B. Proposition 5 summarizes the results.

Proposition 5. *Under the log-quadratic functional form assumptions, risky investment levels, liquidity ratios, and financial stability measures under competitive equilibrium* (n, b, P, $1-\gamma$, $(1-\gamma)n$)*, partial regulation equilibrium* (n^*, b^*, P^*, $1-\gamma^*$, $(1-\gamma^*)n^*$)*, and complete regulation equilibrium* (n^{**}, b^{**}, P^{**}, $1-\gamma^{**}$, $(1-\gamma^{**})n^{**}$) *compare as follows:*

1. *Risky investment levels:* $n > n^{**} > n^*$
2. *Liquidity ratios:* $b^{**} > b > b^*$
3. *Financial stability measures*
   - (a) *Price of assets in the bad state:* $P^{**} > P^* > P$
   - (b) *Fraction of assets sold:* $1-\gamma > 1-\gamma^* > 1-\gamma^{**}$
   - (c) *Total fire sales:* $(1-\gamma)n > (1-\gamma^*)n^* > (1-\gamma^{**})n^{**}$

In a partially regulated financial system, unlike the competitive economy, the overinvestment problem does not arise. On the contrary, in Proposition 5 we show that the investment in risky assets under partial regulation is lower compared to the constrained optimum: $n^* < n^{**}$. The underinvestment is related to the liquidity choice of banks: The problem of unregulated banks having low liquidity ratios is exacerbated with the introduction of capital regulation in isolation. In Proposition 5 we show that banks are less liquid under partial regulation than they were in the competitive equilibrium, that is, $b^* < b$. As suggested in the discussion of Proposition 4, when capital regulation limits the risky investment, banks optimally choose less-liquid portfolios, which partially offsets the positive impact of the reduction in risky investment on financial stability and welfare. Lower liquidity ratios expose the financial system to excessive fire sales and asset price decreases. The precautionary behavior of the regulator is then to implement the capital regulation in a more restrictive way, which increases the fire sale price but leads to a lower level of investment in the profitable risky asset.

Higher liquidity ratios are why more investment in long-term assets is allowed under complete regulation. Proposition 5 shows that the constrained optimal level of liquidity under complete regulation, b^{**}, is higher than the liquidity chosen by banks under the partial regulation, b^{*}. Higher liquidity ratios allow banks to hold more risky assets without increasing the fire sale risk.

In order to see the interaction between the capital and liquidity requirements, consider the following scenario: A country transitions from partial regulation to complete regulation by imposing new liquidity rules in addition to existing capital rules. To be specific, this transition can be compared to moving from the Basel I/II regulatory approach to the Basel III regulatory approach. *Assuming that capital regulation had been set optimally during the pre-Basel III period*, capital requirements can be relaxed after the introduction of liquidity requirements. Therefore, our results would predict that more long-term profitable risky investment can be financed via the banking system after the implementation of liquidity requirements.

How effective is capital regulation in addressing financial instability caused by fire sales when it is not accompanied by liquidity requirements? To answer this question, we can compare the measures of financial instability across the two regulatory regimes. More fire sales and a lower price of the risky asset in the bad state are associated with greater financial instability, and they imply that the externalities have a stronger presence in the economy. Proposition 5 shows that the introduction of capital regulation in isolation increases the fire sale price compared to the competitive equilibrium price. However, the price is still below the constrained optimal price level, which can be achieved with the addition of liquidity requirements. The message is the same when we compare both the fraction and the total amount of risky assets that must be sold to withstand the liquidity shock under the two regulatory regimes, as shown in items 3-a and 3-b in Proposition 5. In general, minimum capital requirements may actually serve several purposes, such as countering moral hazard problems generated by the existence of limited liability and deposit insurance, that we do not analyze in this paper. However, what we show here is that, under fire sale externalities, capital regulations are not effective in achieving second-best allocations unless they are combined with liquidity requirements.

Our results indicate that neither capital nor liquidity ratios alone are perfect predictors of potential instability; a better-capitalized banking system may end up conducting larger fire sales. Under partial regulation, for instance, although the capital ratios are higher than under complete regulation, the liquidity shock causes a larger disruption to financial markets.

Similarly, a more-liquid banking system may experience greater financial instability; banks are more liquid in the unregulated competitive equilibrium compared with partial regulation, but the shock leads to more distortions in the former.

We end this section by comparing bank size across three different regimes in the following proposition, and we discuss the implications of this result for simple leverage ratio regulation.

Proposition 6. *Under the log-quadratic functional form assumptions, bank balance sheet sizes across different regimes compare as follows:*

$$n(1+b) = n^{**}(1+b^{**}) > n^{*}(1+b^{*})$$

Proposition 6 shows that the bank size in the competitive equilibrium is equal to the socially optimal size. However, bank size is inefficiently small under partial regulation as there are both lower risky and liquid assets in this regime compared to the constrained optimum. Proposition 6 provides an interesting result on the regulation of leverage ratio, which can be defined as $\frac{e}{n(1+b)}$ in this setup. Proposition 6 shows that the optimal simple leverage ratio is the same under complete regulation and unregulated competitive equilibrium. Therefore, in the current setup, a leverage regulation applied in isolation would be ineffective.[20] However, the leverage regulation combined either with a liquidity ratio requirement or with a risk-weighted capital regulation would be sufficient to replicate the *constrained social optimum.*

4.2 Can regulating only liquidity be the solution?

In our model, fire sales are triggered by a restructuring shock in the bad state. Banks are solvent as long as they can cover this liquidity requirement because the return on the risky asset (R) is greater than the cost of restructuring needed to keep the investment alive (c). Therefore, one may wonder if the second-best allocations can be implemented using liquidity regulation alone, that is, without using capital requirements at all. The short answer is no. First, note that, in Lemma 3, we show that it is not optimal to avoid fire sales completely in the bad state by forcing banks to perfectly insure against the liquidity shock by setting $b = c$. Second, regulating only liquidity means that banks are free to choose their capital ratios. The questions then becomes whether banks choose the optimal capital ratio when the minimum liquidity requirement is set optimally.

Proposition 7. *Under the Efficiency, Elasticity, Regularity, and Technology assumptions, banks do not choose the constrained optimal risky investment level, n^{**}, if the regulator sets*

[20] Nevertheless, leverage ratio regulation might be an important method of addressing other market failures, such as risk shifting or informational asymmetries, which we do not study in this model.

the minimum liquidity ratio at the constrained optimal level, b^{**}; that is, $n_i(b^{**}) \neq n^{**}$.

Proposition 7 states that banks do not choose the second-best investment level when they are asked by the regulator to hold the second-best liquidity ratio. In fact, in the proof of the proposition we show that banks choose higher than the second-best level of risky investment; that is, $n_i(b^{**}) > n^{**}$, or equivalently, banks choose lower capital ratios compared with the second-best. Therefore, the second-best allocations cannot be implemented by regulating liquidity alone. Banks can take on the fire sale risk through both liquidity and capital channels. As a result, implementing the second-best requires restraining banks on both channels. Otherwise, banks use the unregulated channel to take more risk, undermining the regulator's intent to eliminate the inefficiency.

5 Extensions and further policy implications

5.1 More complex linear regulations and Pigouvian taxation

Our analysis so far indicates that we cannot implement the second-best allocations using a simple capital or liquidity regulation in isolation. However, it is worth asking whether we could implement the second-best using more complex rules that combine capital and liquidity regulations. In this section, we show that a linear combination of a capital and a liquidity regulation is sufficient to replicate the second-best allocations in a decentralized market.

Consider subjecting banks to the following linear rule $\tau_n n + \tau_b b \leq k$. We can write the Lagrangian of the banks' problem in this case as follows:

$$\mathcal{L}_i = \Gamma(n_i, b_i) - q(R-P)Q_i^s + \lambda(k - \tau_n n_i - \tau_b b_i)$$

The corresponding first-order conditions of this problem are as follows:

$$\frac{\partial \mathcal{L}_i}{\partial n_i} = \frac{\partial \Gamma}{\partial n_i} - q(R-P)\frac{\partial Q_i^s}{\partial n_i} - \lambda\tau_n = 0, \tag{19}$$

$$\frac{\partial \mathcal{L}_i}{\partial b_i} = \frac{\partial \Gamma}{\partial b_i} - q(R-P)\frac{\partial Q_i^s}{\partial b_i} - \lambda\tau_b = 0. \tag{20}$$

To see how this rule will work and how we find the optimal coefficients τ_n, τ_b, and k, compare the first-order conditions of the banks' problem above to those of the constrained planner's problem, given by (15). We can choose k and τ_n, τ_b such that $\lambda = 1$, without loss of generality. It is obvious that in this case if we define τ_n, τ_b as follows, our linear rule will implement the constrained optimal allocations:

$$\tau_n = q[(R-P)\frac{\partial Q^s}{\partial P}\frac{\partial P}{\partial n}] = q(R-P^{**})\frac{(c-b^{**})^2 n^{**}}{P^{**2}} > 0, \tag{21}$$

$$\tau_b = q[(R-P)\frac{\partial Q^s}{\partial P}\frac{\partial P}{\partial b}] = -q(R-P^{**})\frac{(c-b^{**})n^{**2}}{P^{**2}} < 0, \tag{22}$$

which implies that

$$k = \tau_n n^{**} + \tau_b b^{**} = -q(R-P)\frac{(c-b^{**})^2 n^{**}}{P^2} n^{**2} + q(R-P)\frac{(c-b^{**})n^{**2}}{P^2} b^{**}. \tag{23}$$

The optimal rule punishes banks for holding risky assets and rewards them for higher liquidity ratios. The rule is intuitive because in our model, more risky assets increase fire sales whereas more liquidity decreases fire sales. Therefore, the optimal rule indicates strategic substitution between risky assets and liquidity from the banks' perspective. Banks can satisfy the rule either by decreasing their risky investment or by increasing their liquidity ratios. In that regard, this linear rule provides more flexibility to banks compared to the joint implementation of capital and liquidity ratio requirements discussed earlier.

On a separate note, we can also implement the constrained efficient allocations using Pigouvian taxation instead of quantity-based rules. In this case, introducing two linear Pigouvian taxes, one for risky investment and one for the liquidity ratio, will be sufficient. The Pigouvian tax rates will then be equal to τ_n and τ_b, given by (21) and (22), respectively.

5.2 Implications for shadow banking

Central banks and regulatory institutions around the world mainly focus on regulating banks to improve financial stability. However, actions of nonbank financial institutions affect the stability of the system as well. Yet, some financial institutions are partially or totally exempt from bank regulations. We analyze how unregulated financial institutions react to bank regulation as well as what their reactions imply for financial stability. For that purpose, we introduce a new group of financial institutions that are identical to banks but not regulated.

We denote the choices of regulated institutions with $(\tilde{n}, \tilde{b})$ and those of unregulated institutions with (n, b). As before, n and $\tilde{n}$ are the amount of risky investment while b and $\tilde{b}$ denote the liquid asset per unit risky investment. Liquidity needs of regulated and unregulated institutions in bad times at $t = 1$ respectively are: $(c-\tilde{b})\tilde{n}$ and $(c-b)n$. The market clearing condition in the fire sale market is $(c-\tilde{b})\tilde{n}+(c-b)n = Py$. Thus, the fire sale price is a function of $\tilde{n}, \tilde{b}, n, b$. Below we analyze the response of unregulated institutions to bank regulation. In particular, we study the risky asset choice of an unregulated institution and see how it changes as the regulator limits the total risky investment $\tilde{n}$, in the regulated segment. An unregulated institution chooses n_i, b_i to maximize its expected profits, given by:

$$\Pi_i(n_i, b_i) = \Gamma(n_i, b_i) - q(R - P)Q_i^s(P, n_i, b_i), \tag{24}$$

where $Q_i^s(P, n_i, b_i) = (c - b_i)n_i/P$. Here, the atomistic institution takes the fire sale price $P(\tilde{n}, \tilde{b}, n, b)$ as given and we treat $\tilde{n}, \tilde{b}$ as parameters of the model because unregulated institutions take them as given. The regulator effectively determines the aggregate amount of $\tilde{n}$ using capital regulations and aggregate amount of $\tilde{b}$ using liquidity requirements. Therefore, the first-order condition of the unregulated institution with respect to n_i is

$$\frac{\partial \Pi(n_i, b_i)}{\partial n_i} = \frac{\partial \Gamma}{\partial n_i} - q(R - P)\frac{c - b_i}{P} = 0.$$

In order to see how optimal n_i changes with $\tilde{n}$, we need to evaluate the sign of the cross-partial derivative of the profit function:

$$\frac{\partial^2 \Pi(n_i, b_i)}{\partial \tilde{n} \partial n_i} = qR\frac{c - b_i}{P^2}\frac{\partial P}{\partial \tilde{n}} < 0.$$

Using the monotone comparative statics techniques outlined by Vives (2001), the negative sign of the cross-partial derivative (by Lemma 1 and 2) indicates that $n'(\tilde{n}) < 0$, that is, as regulation tightens risky investment level of banks, unregulated institutions respond by increasing their risky investment. Therefore, n_i and $\tilde{n}$ are strategic, yet imperfect substitutes from the unregulated institution's point of view.

Similarly, we can show that $b'(\tilde{b}) < 0$, that is, as the regulation require banks to increase their liquidity ratios, unregulated institutions respond by decreasing their liquidity ratios. Thus, unregulated institutions free ride on the liquidity of regulated institutions.

In similar ways, we can also show that $n'(\tilde{b}) > 0$, and $b'(\tilde{n}) > 0$. Regulations on $\tilde{n}$ and $\tilde{b}$ make the financial system more stable by increasing the fire sale price, which in turn create incentives for the unregulated institutions to invest more in risky assets and decrease their liquidity buffer. The behavior of unregulated institutions creates a counter force to the regulation.

To explain the intuition behind these results, we can consider another analogy from automotive safety regulations in the spirit of Peltzman (1975): Cars and motorcycles usually share the same roads. If we introduce speed restrictions on cars but not on motorcycles, roads will initially become safer, but this will create incentives for motorcycle riders to increase their driving intensity, creating a counter force to the regulation.

The effect analyzed in this section is similar to the one examined in international policy coordination literature such as Acharya (2003), Dell'Ariccia and Marquez (2006), and Kara

(2016). These papers show that bank regulations across countries are strategic substitutes, and hence, countries have an incentive to relax regulations when others tighten. Similarly, we show above that even in a given country bank capital and liquidity regulations have a public good property under fire sale externalities. If we regulate only some institutions, unregulated institutions that engage in similar investment behavior will free ride on the improved stability brought by disciplined institutions. Therefore, as argued by Farhi et al. (2009), efficient regulations should have a wide scope and apply to all relevant financial institutions.

5.3 A numerical example

In this section we explore the quantitative benefits of a liquidity requirement that supplements capital regulation. We calibrate our model period to be 2 years so that the total model length is 4 years. We let the expected return on the risky investment $R = 1.25$, which means that the annual return on risky investment is 5.7 percent ($1.057^4 = 1.25$). We let the probability of the bad state be $q = 0.25$ so that the crisis is expected to occur every 16 years (4/0.25). We choose the magnitude of the liquidity shock to be $c = 0.1$, which means that once the crisis hits, banks have to invest an additional 10 percent to keep the risky asset productive. Lastly, we choose the marginal operating cost parameter $\phi = 0.01$, a small number, and set the initial equity of banks to $e = 1$, without loss of generality. Table 1 provides the values of risky investment and liquidity ratios; prices of assets, fraction of assets sold, and total amount of fire sales in the bad state; and utility/profit measures in the competitive equilibrium, constrained planner's solution, and under partial regulation. We obtain these values using the closed-form solutions presented in the appendix B.

Table 1: Quantitative example

	Competitive equilibrium	Constrained efficient allocations	Partial regulation
Risky investment (n)	9.81	9.69	9.58
Liquidity ratio (b)	0.043	0.055	0.042
Bank size (n(1+b))	10.227	10.227	9.988
Price of risky asset (P)	0.688	0.815	0.695
Fraction of assets sold	0.08341	0.05504	0.08337
Total amount of fire sales	0.82	0.53	0.80
Total profits	1.0922	1.1018	1.0928
Bank profits	1.046	1.077	1.048
Traditional sector's profits	0.0462	0.0249	0.0447

The example shows that banks hold too many risky assets and too little liquidity in the competitive equilibrium compared to the constrained efficient allocations. Partial regulation restricts risky investment, but that comes at the expense of making banks even less liquid and reducing the risky investment below the complete regulation level. As a result, total profits increase only barely (0.05 percent), and the financial stability indicators (price of assets and the amount of fire sales) improve only marginally compared to the competitive equilibrium. The real improvement in welfare (0.88 percent) and financial stability (18 percent increase in asset prices and 34 percent decrease in the fraction of assets sold) comes when we add liquidity regulations to capital regulations and, hence, implement the constrained efficient allocations. We obtain similar results when we repeat this numerical example for a large range of parameters. Therefore, our quantitative analysis indicates that including liquidity requirements in addition to the capital regulations offers significant quantitative benefits in terms of welfare and financial stability.

The numerical example also shows that without transfers more regulation makes consumers worse off: The profits in the traditional sector decrease somewhat when we introduce capital regulations in isolation and then decreases significantly further when we add liquidity requirements to the regulatory toolkit. Regulation makes the traditional sector worse off by increasing the price of risky assets in the bad state. However, bank profits always increase with regulation and increase in absolute value more than the decrease in the traditional sector's profits. As a result, total profits increase with more regulation. We conclude that capital and liquidity regulations can be implemented in a Pareto-improving way by taxing banks and transferring resources to consumers.

6 Discussion of assumptions

In this section, we show that our results are robust to some changes in the modeling environment. First, we provide a more general model with pledgeability constraints and derive the parameter region that give rise to the basic setup where the constraint does not bind in the initial period but it does in the bad state of interim period because of debt overhang. Second, we show that the aggregate nature of the liquidity shock is not material for the mechanism or the conclusions of the model. We allow the liquidity shock to be idiosyncratic rather than aggregate and show that this setup is isomorphic to the aggregate shock case with a smaller liquidity shock. Last, we discuss relaxing the convex operational cost assumption and its implications for our results.

6.1 Debt overhang and limited commitment

We assume that banks can only pledge a fraction $\alpha \in \{\alpha_H, \alpha_L\}$ of their revenues in the final period to lenders. The fraction that can be pledged is higher in good times than bad times, that is, $\alpha_H > \alpha_L$. $1-\alpha$ can be interpreted as a haircut for banks' assets. Haircuts tend to rise during times of financial distress, as documented by Shin (2008) and Gorton and Metrick (2012). We set $\alpha_H = 1$ without loss of generality and assume that $\alpha_L < 1$.

Banks' problem at $t = 0$ in the decentralized equilibrium is:

$$\max_{n_i, b_i, \gamma_i, l_{i0}, l_{i1}, r_i} \quad (1-q)[(R+b_i)n_i - r_i l_{i0}] + q\max\{R\gamma_i n_i - r_i l_{i0}, 0\} - e - \Phi((1+b_i)n_i), \quad (25)$$

subject to

$$l_{i0} \leq (1-q)(R+b_i)n_i + \alpha_L q R \gamma_i n_i \qquad \textit{Collateral constraint at t=0} \qquad (26)$$

$$l_{i1b} \leq \max\{\alpha_L \gamma_i R n_i - r_i l_{i0}, 0\} \qquad \textit{Collateral constraint at t=1, bad state} \qquad (27)$$

$$(1-q) r_i l_{i0} + q\min\{r_i l_{i0}, \alpha_L R \gamma_i n_i\} \geq l_{i0} \qquad \textit{PC of lenders at t=0} \qquad (28)$$

$$e + l_{i0} \geq n_i + b_i n_i \qquad \textit{BC of banks at t=0} \qquad (29)$$

$$l_{i1b} + P(1-\gamma_i)n_i + b_i n_i - c n_i \geq 0 \qquad \textit{BC of banks in the bad state a t=1} \qquad (30)$$

where $0 \leq \gamma_i \leq 1$ is the fraction of assets that a bank chooses to carry forward after receiving the liquidity shock in the bad state at $t = 1$.

The collateral constraint of banks at $t = 0$, depicted in (26), puts an upper bound on the leverage banks can take. Banks can borrow from consumers by pledging their future cash flows and the right side of the constraint (26) represents the expected pledgeable cash flow. In case of a good state at $t = 1$, no action is required: banks do not need any additional financing and they will wait until $t = 2$ to collect project returns. In case of a bad shock, however, banks prefer to borrow more if they have debt capacity. The collateral constraint depicted in (27) determines if banks have any debt capacity left given their existing debt. If their existing debt, $r_i l_{i0}$, exceeds their debt capacity $\alpha_L \gamma_i R n_i$ the right side of constraint equals zero, which implies that $l_{i1b} = 0$. In other words, if a bank enters the interim period with high amount of debt, it cannot raise additional funds due to a debt-overhang problem. Note that once the bad state is realized, banks' debt capacity is lower compared with $t = 0$. The participation constraint of consumers at $t = 0$ is shown in (28) which states that expected payoff from lending to banks should be at least as much as the amount lent l_{i0}. Banks' budget constraints at $t = 0$ and $t = 1$ are provided in (29) and (30). In a bad state at $t = 1$ a bank has three potential sources of liquidity to cover restructuring cost of risky assets: liquidity hoarded from the initial period, $b_i n_i$, amount raised by fire sales, $P(1-\gamma_i)n_i$, and additional borrowing l_{i1b}.

We would like to focus on an equilibrium where the collateral constraint does not bind at the initial period, but it does in the bad state at $t = 1$ due to a debt-overhang problem. Banks raise $l_{i0} = (1 + b_i)n_i - e$ amount of long-term debt from lenders at $t = 0$ and promise them a non-contingent payment of $r_i l_{i0}$ at $t = 2$, where $r_i \geq 1$ is the associated gross interest rate. A debt-overhang problem arises when the pledgeable return of a bank is, at most, enough to honor the existing debt even if the fire sale can be completely avoided—that is, if $r_i l_{i0} \geq \alpha_L R n_i$. In such a case, the bank cannot raise any additional funds in the bad state, that is, $l_{i1b} = 0$. Therefore, we call $r_i l_{i0} \geq \alpha_L R n_i$ as the debt-overhang condition, and under this condition the collateral constraint at $t = 1$, given by (27), binds. Using $l_{i0} = (1 + b_i)n_i - e$ we can rewrite the collateral constraint at $t = 0$, given by (26), and the debt-overhang condition at $t = 1$ as follows:

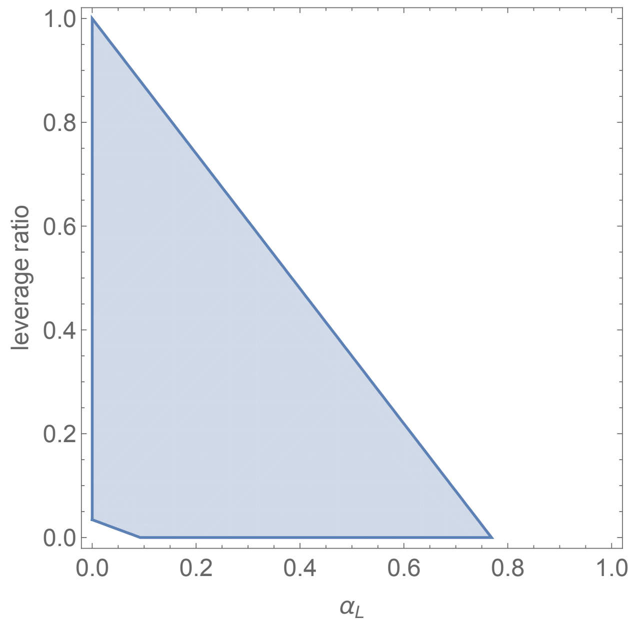
$$(1 + b_i)n_i - e \leq (1 - q)(R + b_i)n_i + \alpha_L q R \gamma_i n_i \qquad \textit{Collateral constraint at t=0} \tag{31}$$

$$r_i(1 + b_i)n_i - r_i e \geq \alpha_L R n_i \qquad \textit{Debt-overhang condition at t=1} \tag{32}$$

We are interested in a parameter region in which conditions (31) and (32) are both satisfied. It is clear that when α_L or e is high, the first collateral constraint (31) is more likely to be satisfied whereas the debt-overhang condition (32) is less likely to be so. Thus, there is an intermediate range of α_L and e such that while the first constraint is satisfied, the debt overhang problem arises in equilibrium at $t = 1$. With high equity, e, banks need less outside funds at $t = 0$ thus the first constraint is less likely to bind. When α_L is high enough, the debt capacity of banks are high and accordingly the collateral constraint at $t = 0$ would not bind. Banks enter $t = 1$ with a given debt of $r_i l_{i0}$. Therefore if α_L is low, or if $r_i l_{i0}$ is high, the debt capacity at $t = 1$ can be already saturated with the existing debt. Because lower e implies higher initial debt, l_{i0}, it also leads to debt-overhang problem at $t = 1$. Thus α_L and e jointly determine whether a debt-overhang problem will arise at $t = 1$, and we focus on the parameter region where it does arise. Note that the composition of the liability side has no effect on the fire sale externality, and hence, on the results in our setup. Therefore, instead of the level of equity e we can look at the simple leverage ratio $k_i = e/(n_i + b_i n_i)$. Using the closed-form solutions in the appendix we can determine the range for leverage ratio, k_i, and fraction of pledgeable revenue in the bad state α_L that leads to debt-overhang problem. As we can see by the shaded region in Figure 3, debt-overhang problem arises for a reasonably large parameter space without violating the collateral constraint at $t = 0$.

When the parameters e and α_L lie in the shaded region in Figure 3 and the debt-overhang

Figure 3: Parameter region that leads to debt-overhang problem



arises, the bank will be in default in the bad state.[21] Hence it has to pay initial lenders a positive interest rate, $r_i > 1$, in good times to compensate. In an equilibrium where fire sales takes place in the bad state, the interest on initial bank debt has to be such that the participation constraint (PC) of initial lenders binds:

$$(1-q)r_i l_{i0} + \alpha_L q R \gamma_i n_i = l_{i0} \qquad PC \tag{33}$$

For the PC condition of consumers to be satisfied, r_i has to be such that

$$r_i \geq \frac{l_{i0} - \alpha_L q R \gamma_i n_i}{(1-q) l_{i0}} \equiv r_i^*. \tag{34}$$

To obtain this constraint we rearrange the PC condition (33) and note that $\min\{\alpha_L R \gamma_i n_i, r_i l_{i0}\} = \alpha_L R \gamma_i n_i$. In order to maximize profits, banks set $r_i = r_i^*$. Note that, profit of banks in the bad state in this case is $R\gamma_i n_i - \alpha_L R \gamma_i n_i = (1-\alpha_L) R \gamma_i n_i$. Now, we can substitute the optimal r_i^* back into the banks' objective function (25) and simplify to obtain:

$$\max_{n_i, b_i, \gamma_i, l_{i0}} \quad (1-q)(R+b_i)n_i + qR\gamma_i n_i - (1+b_i)n_i - \Phi((1+b_i)n_i),$$

[21] If a bank is in default after fire sales, we assume that it is required by law to manage the remaining assets until final period and deliver returns to consumers.

subject to

$$e + l_{i0} \geq n_i + b_i n_i \qquad \textit{BC of banks at t=0} \tag{35}$$

$$P(1-\gamma_i)n_i + b_i n_i - c n_i \geq 0 \qquad \textit{BC of banks in the bad state a t=1} \tag{36}$$

Thus, in the parameter space in which banks are not collateral constrained at t=0 but a debt-overhang problem arises at $t = 1$, banks' optimization problem that we obtain is identical to the one presented in our benchmark model.

6.2 Idiosyncratic liquidity shocks

In the basic model, the liquidity shock is aggregate in nature, as in Lorenzoni (2008). In this section, we show that the aggregate nature of the liquidity shock is without loss of generality and that our results do not change if we allow idiosyncratic liquidity shocks. In this more general setup, liquidity shocks hit only a fraction of the banks. Thus, banks are ex-post heterogeneous in terms of their liquidity needs. Banks that receive the liquidity shock need funding while others are left with excess liquidity. Banks with excess liquidity can use these resources to buy the risky assets from the distressed banks, potentially at fire sale prices.[22] Therefore, in this variant of the model, banks hoard liquidity also for a strategic purpose: They can use their liquid assets to buy risky assets at fire sale prices. This function of liquidity is also present in the models of Acharya, Shin, and Yorulmazer (2011), Allen and Gale (2004b), Allen and Gale (2004a), and Gorton and Huang (2004). The amount of risky assets that can be bought with the liquid holdings of a shock-free bank is $b_i n_i / P$.

First, we analyze the case conditionally on the liquidity shock but without knowing which banks receive the shock. We assume that, conditional on being in the bad state, the probability of being hit with a liquidity shock is λ for each bank. Hence, by the law of large numbers, a fraction λ of banks is hit by the liquidity shock in the bad state. The expected profit of a bank before the realization of which banks receive the shock, conditional on the bad state, is $\lambda R \gamma_i n_i + (1-\lambda)(n_i + \frac{b_i n_i}{P})R$. The first term, $\lambda R \gamma_i n_i$, is the return from remaining risky assets after fire sales multiplied by the probability of receiving the liquidity shock, λ. The amount of remaining risky assets after fire sales is denoted by γ_i for bank i, as in the benchmark model. The second term captures the returns from risky investment in the case without the liquidity shock, including the returns from risky asset bought using hoarded liquidity. We substitute for γ_i and rewrite the expected profit conditional on the

[22] In principle, it is possible that the amount of excess liquidity in the banking system exceeds the liquidity needs of the shock-receiving banks. At the end of this subsection we explain why this situation does not arise in equilibrium.

bad state, as follows:

$$\begin{aligned}\Pi_{i|bad} &= \lambda R\left(1-\frac{c-b_i}{P}\right)n_i+(1-\lambda)n_iR+(1-\lambda)\frac{b_in_i}{P}R,\\ &= \lambda Rn_i-\frac{\lambda Rcn_i}{P}+\lambda R\frac{b_in_i}{P}+(1-\lambda)Rn_i+\frac{b_in_i}{P}R-\lambda\frac{b_in_i}{P}R.\end{aligned}$$

Further simplification yields:

$$\Pi_{i|bad}=R\left(1-\frac{c\lambda-b_i}{P}\right)n_i=R\tilde{\gamma_i}n_i,$$

where $\tilde{\gamma_i}=1-\frac{c\lambda-b_i}{P}$. This $\tilde{\gamma_i}$ is similar to γ_i in the basic setup; the only difference is that the size of the liquidity shock, c, is replaced with $c\lambda$ in the numerator of the definition. In this setup, when we set $\lambda=1$, we are back to our benchmark case. Thus, allowing λ to be between zero and one provides a more general model. In order to write the expected profits of banks at $t=0$ in this more general setup, we simply note that the economy ends up in the bad state only with probability q and obtains the returns derived earlier, while good times arise with probability $1-q$ and feature returns that are the same as in the benchmark case:

$$\Pi_i=(1-q)(R+b_i)n_i+qR\left(1-\frac{c\lambda-b_i}{P}\right)n_i-D(n_i(1+b_i)).$$

Compared with the benchmark case, the only difference in banks' expected profit at $t=0$ is that c is replaced with $c\lambda$. For completeness, we conclude by writing the demand and supply functions in this more-general case. The aggregate liquidity need in the bad state is $\lambda(c-b)n$, and the liquidity supply is $(1-\lambda)bn+PQ^d(P)$. Equating demand and supply yields $\lambda(c-b)n=(1-\lambda)bn+PQ^d(P)$, and simplifying reduces this market-clearing condition to $(\lambda c-b)n=PQ^d(P)$. Compared with the market-clearing condition in the original setup, the only difference is, again, that c has been replaced with λc. Thus, in this new setup, if we relabel $\lambda c=\tilde{c}$, we are back to our original setup where c is replaced with $\tilde{c}$.

It would be possible to have no fire sales in the bad state in this setup if the liquid assets in the hands of shock-free banks were in excess of the liquidity need of shock-receiving banks, so that the risky assets were traded within the banking system without needing to sell to the traditional sector. Although this case is possible in principle, it is never observed in equilibrium because it is not optimal for banks to hoard sufficient liquidity for this case to arise. Comparing the demand for liquidity with the supply of liquidity in the case of the liquidity shock, it is clear that the fire sales arise if and only if λcn is greater than bn. In other words, fire sales are observed in equilibrium as long as $\tilde{c}>b$. Given that $\tilde{c}$ is a parameter, the ex-ante liquidity choice of banks determines whether fire sales occur. As we know from the benchmark case, banks optimally set $b_i < c$. Because this is true for any parameter value, it is true for $\tilde{c}$ as well. The intuition is the same: Holding liquidity is costly if the shock does not materialize. Thus, for banks to hoard liquidity, there must be some additional return to holding liquidity in case of the liquidity shock. This additional return is only possible if the fire sale price is less than R, which is only possible if there are fire sales. In other words, if there will not be any fire sales in the bad state—that is, if $P = R$—then there is no benefit to holding liquidity. But this contradicts the assumption of sufficient liquidity in the banking system.

6.3 Operational costs of a bank

We assume a convex operational cost similar to the ones imposed by Van den Heuvel (2008) and Acharya (2003, 2009) because it ensures the existence of an equilibrium. However, the form of this function is not essential for our key results, as long as an equilibrium exists. To be more specific, Lemmas 2, 3 and Proposition 7 do not require any specific functional form and Proposition 4 is robust to some alternative modeling choices such as concave cost functions, like the square root or natural logarithm functions.[23]

In our model, the net interest rate on bank deposits is zero. Without an additional cost (such as an operational cost) banks can borrow more from depositors and park these funds as liquid assets (cash) in their portfolios. In that way they could freely insure against the fire sale risk. We believe that such a scenario is not realistic. First, banks do face costs to attract deposits. Second, unlimited amount of funds from depositors at zero cost would cancel out the opportunity cost of holding liquid assets, namely the cost of bygone profits from other investments. A constant balance sheet size would emphasize this opportunity cost mechanism, as it does in many papers in the literature. However, with a fixed balance sheet size, the choice between risky assets and liquid assets boils down to a mere portfolio allocation problem. A setup with a single choice variable does not allow the type of interactions we study. Thus, by employing a flexible balance sheet size, we avoid two extreme assumptions; namely, that banks have an unlimited amount of funds at their disposal, and that bank balance sheet size is inflexible.

Furthermore, whether bank size matters for the inefficiencies banks create is also discussed in the context of the recent financial crisis, as well as how bank regulation might affect bank size. Regulatory rules might affect bank profitability, which may lead banks to resize their operations. To speak to these discussions, a flexible balance sheet size is important because it also allows us to study the optimal size of banks' balance sheets. Our result in Proposition 6 emphasizes that the composition of a bank's balance sheet matters more than its size, and

[23]These results are not included here but are available upon request from the authors.

that regulation does not necessarily imply a reduction in balance sheet size.

7 Conclusion

In this paper, we investigate the optimal design of bank regulation and the interaction between capital and liquidity requirements. Our model is characterized by fire sale externalities, because atomistic banks do not take into account the effect of their initial portfolio choices on the fire sale price. Existence of these fire sale externalities creates an inefficiency. In the unregulated competitive equilibrium, banks overinvest in the risky asset and underinvest in the liquid asset compared to a constrained planner's allocations. We investigate whether the constrained efficient allocations can be implemented using quantity-based capital and liquidity regulations, as in the Basel Accords. The regulation required is macroprudential because it addresses the instability in the banking system by targeting aggregate capital and liquidity ratios.

Our results indicate that the pre-Basel III regulatory framework, with its reliance only on capital requirements, was ineffective in addressing systemic instability caused by fire sales. Capital requirements can lead to less severe fire sales by forcing banks to reduce risky assets—however, we show that banks respond to stricter capital requirements by decreasing their liquidity ratios. Anticipating this response, the regulator preemptively sets capital ratios at high levels. Ultimately, this interplay between banks and the regulator leads to lower levels of risky assets and liquidity compared to the second-best allocations. Macroprudential liquidity requirements that complement capital regulations, as in Basel III, can implement the second best, improve financial stability and allow for a higher level of investment in risky assets.

It is important to highlight that our results cannot be interpreted as indicating that the actual capital regulation requirements were too high in a particular country (such as the U.S.) in the pre-crisis period, which corresponds to pre-Basel III framework, and that now they should be relaxed. Our results only say that if capital regulations were set optimally from a welfare maximizing point in the absence of liquidity regulation, they would be set at higher levels compared to the second-best environment in which the regulator is also endowed with the liquidity regulation tool. Our model is not meant to be quantitative and hence does not speak to whether actual capital ratios in practice either under Basel I/II or Basel III are too low or too high. However, many studies, most famously Admati et al. (2010), have argued that current minimum capital requirements are too low.

The message of this paper goes beyond bank regulation. Our results imply that capital ratios are not a perfect predictor of the stability of the banking system or any individual bank under a potential distress scenario. Without sufficient liquidity buffers, banks' capital can easily erode with fire sale losses. Under fire sale externalities, then, a well-capitalized banking system may experience greater losses than a less-capitalized banking system with strong liquidity buffers. Thus, capital ratios alone cannot be barometers of soundness of individual banks or a banking system.

The Basel III liquidity ratio LCR currently applies to only large banks in the U.S. In contrast, our results suggests that liquidity regulations should apply even to small banks because in our model all banks are small by definition, as we consider atomistic banks that engage in fire sales markets and take asset prices as given. Answering the question of whether liquidity regulations should be applied differently to large and small banks, like the question of whether they should be applied differently to well-capitalized and poorly-capitalized banks, is beyond the scope of our current model. We leave these interesting theoretical and policy questions to future research.

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Bank regulation under fire sale externalities

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Internet Appendix

A Proofs omitted in the main text

Lemma 1. *The fire sale price of risky asset, $P(n,b)$, is decreasing in n and increasing in b. The fraction of risky assets sold, $1-\gamma(n,b)$, is increasing in n and decreasing in b.*

Proof. Part 1: $P(n,b)$, is decreasing in n and increasing in b.
The asset market clearing condition in the bad state at $t=1$ is given as

$$Q^s(P) = \frac{c-b}{P}n = Q^d(P),$$

which can be written as

$$(c-b)n = PQ^d(P). \tag{A.1}$$

First, take the partial derivative of both sides of this last equation with respect to n:

$$c-b = \frac{\partial P}{\partial n}Q^d(P) + P\frac{\partial Q^d(P)}{\partial P}\frac{\partial P}{\partial n} = \frac{\partial P}{\partial n}\left\{Q^d(P) + P\frac{\partial Q^d(P)}{\partial P}\right\} = \frac{\partial P}{\partial n}Q^d(P)\left\{1+\epsilon^d\right\},$$

where

$$\epsilon^d = \frac{\partial Q^d(P)}{\partial P}\frac{P}{Q^d},$$

is the price elasticity of the traditional sector's demand function. Rearranging the last equation gives

$$\frac{\partial P}{\partial n} = \frac{c-b}{Q^d(P)(1+\epsilon^d)} < 0$$

since $1+\epsilon^d < 0$ by the *Elasticity* assumption, and $c-b>0$ by assumption here because we are examining the case with fire sales. We later show in Lemma 2 and 3 that $c-b>0$ indeed holds in equilibrium.

Now take the partial derivative of both sides of (A.1) with respect to b:

$$-n = \frac{\partial P}{\partial b}Q^d(P) + P\frac{\partial Q^d(P)}{\partial P}\frac{\partial P}{\partial b} = \frac{\partial P}{\partial b}\left\{Q^d(P) + P\frac{\partial Q^d(P)}{\partial P}\right\} = \frac{\partial P}{\partial b}Q^d(P)\left\{1+\epsilon^d\right\}.$$

Rearranging the last equation gives

$$\frac{\partial P}{\partial b} = -\frac{n}{Q^d(P)(1+\epsilon^d)} > 0.$$

because $1+\epsilon^d < 0$ by *Elasticity* assumption.

Part 2: $1-\gamma(n,b)$, is increasing in n and decreasing in b.
Using (4) we can write banks' asset sales in equilibrium as $1-\gamma(n,b) = (c-b)/P(n,b)$. Note that

$$\frac{\partial(1-\gamma)}{\partial n} = \frac{\partial(1-\gamma)}{\partial P}\frac{\partial P}{\partial n} > 0,$$

because $\partial(1-\gamma)/\partial P = -c/P^2 < 0$ from (4) and by Lemma 1 we have that $\partial P/\partial n < 0$. Similarly, we can obtain

$$\frac{\partial(1-\gamma)}{\partial b} = -\frac{1}{P} + \frac{\partial(1-\gamma)}{\partial P}\frac{\partial P}{\partial b} < 0,$$

since $\partial(1-\gamma)/\partial P < 0$ as shown above, and by Lemma 1 we have that $\partial P/\partial b > 0$. □

Lemma 2. *Under the Efficiency and Technology assumptions, banks always take fire sale risk in equilibrium; that is, $b_i < c$ for all banks.*

Proof. It is straightforward to show that banks never carry excess liquidity in equilibrium, that is, $b_i > c$. This is because when $b_i > c$ the liquid assets in excess of the shock, $(b_i - c)n$, have no use even in the bad state; the expected return on liquid assets is equal to one and dominated by the expected return on the risky asset, $R - cq$, by the *Technology* assumption.

To prove $b_i < c$, we start with the full insurance case, that is $b_i = c$, and move ε amount of investment from liquid asset to risky asset, and show that this reallocation is profitable. Deviating bank get exposed to fire sale risk as a result of this reallocation. First, we rewrite expected profit in terms of the total amount of liquid assets at the bank, defined as $B_i \equiv b_i n_i$, rather than the liquidity ratio, b:

$$\begin{aligned}
\Pi_i &= (1-q)(R+b_i)n_i + qR\left(1 - \frac{c-b_i}{P}\right)n_i - D(n_i + n_i b_i), \\
&= (1-q)(Rn_i + B_i) + qR\left(n_i - \frac{cn_i - B_i}{P}\right) - D(n_i + B_i),
\end{aligned}$$

In case of perfect insurance the size of liquidity hoarded at the initial period is equal to the size of the liquidity need in the bad state, that is, $B_i = cn_i$. Expected profit in the full insurance case boils down to $\Pi_{fi} = Rn_i + (1-q)B_i - D(n_i + B_i)$. Now, moving some amount of funds in the initial period from liquid assets to the risky investment, yields $B_{i,new} = B_i - \varepsilon$ and $n_{i,new} = n_i + \varepsilon$. Let us denote the fire sale price after the reallocation by P_ε. Expected profit is as follows after the reallocation of funds

$$\begin{aligned}
\Pi_{i,new} &= \Pi_{i,fi} + \varepsilon(R - 1 + q) - qR\frac{(1+c)\varepsilon}{P_\varepsilon}, \\
&= \Pi_{i,fi} + \varepsilon(R - 1 + q) - qR\frac{(1+c)\varepsilon}{P_\varepsilon},
\end{aligned}$$

where $P_\varepsilon = F'(0)$. The following equation provides the condition for the deviating bank to profit from this reallocation of funds away from full insurance:

$$\varepsilon(R-1+q) - qR\frac{(1+c)\varepsilon}{P_\varepsilon} \geq 0. \tag{A.2}$$

Using *Efficiency* assumption, $P_\varepsilon = F'(0) > \nu$ implies that this condition is satisfied and deviation is profitable.

□

Proposition 1. *Under the Efficiency, Elasticity, Regularity, and Technology assumptions, the competitive equilibrium price of assets is given by*

$$P = \frac{qR(1+c)}{R-1+q}. \tag{A.3}$$

The equilibrium price, P, is increasing in the probability of the liquidity shock, q, and the size of the shock, c, but decreasing in the return on the risky assets, R.

Proof. The first-order conditions of the banks' problem (7) with respect to n_i and b_i respectively are:

$$(1-q)(R+b_i) + qR\gamma_i = D^{'}(n_i(1+b_i))(1+b_i), \tag{A.4}$$

$$(1-q)n_i + qR\frac{1}{P}n_i = D^{'}(n_i(1+b_i))n_i, \tag{A.5}$$

where $\gamma_i = 1-(c-b_i)/P$ as obtained in the previous section. Combining the two equations gives:

$$(1-q)R + (1-q)b_i + qR + qR(\frac{b_i - c}{P}) = (1-q) + (1-q)b_i + \frac{qR}{P} + \frac{qR}{P}b_i.$$

In this last equation, the terms that involve the liquidity ratio, b_i, on both sides cancel out each other, and hence we can solve for P, the competitive equilibrium price of assets. It is straightforward to obtain the sign of the derivative of the equilibrium price with respect to model parameters, R, c, q. □

Proposition 2. *Under the log-quadratic functional form assumptions, the comparative statics for the competitive equilibrium risky investment level, n, and liquidity ratio, b, are as follows:*

1. *The risky investment level (n) is increasing in the return on the risky asset (R) and decreasing in the size of the liquidity shock (c), probability of the bad state (q), and the marginal cost parameter (ϕ).*

2. *The liquidity ratio (b) is increasing in the return on the risky asset (R), size of the liquidity shock (c), and the probability of the bad state (q), and decreasing in the marginal cost parameter (ϕ).*

Proof. The derivatives below use the following closed-form solution for the competitive equilibrium risky investment level and liquidity ratio as obtained in Section B.1:

$$n = \frac{[R-1-qc][R-1+q+2\phi R(1+c)]}{(R-1+q)(1+c)^2 2\phi}, \qquad b = \frac{cq - \frac{2\phi R}{\tau+1}}{q + \frac{2\phi R}{\tau+1}}.$$

In most derivatives below, we use the *Technology* assumption ($R - 1 - qc > 0$) to obtain the sign. The derivatives for the risky investment level and their signs can be obtained as follows after some algebraic manipulation:

$$\begin{aligned}
\frac{\partial n}{\partial R} &= \frac{(R-1+q)^2 + 2\phi(1+c)[(R+q-1)^2 + (1-q)q(1+c)]}{(R-1+q)^2(1+c)^2 2\phi} > 0. \\
\frac{\partial n}{\partial c} &= \frac{-[2(R-1) + q(1-c) + 2\phi R(1+c)]}{2\phi(1+c)^3} < 0. \\
\frac{\partial n}{\partial q} &= \frac{-c(R-1+q)^2 - 2\phi R(1+c)(R-1)(1+c)}{(R-1+q)^2(1+c)^2 2\phi} < 0. \\
\frac{\partial n}{\partial \phi} &= \frac{-(R-1-qc)(R-1+q)}{2(R-1+q)(1+c)^2\phi^2} < 0.
\end{aligned}$$

Similarly, the derivatives for the liquidity ratio and their signs can be obtained as follows:

$$\frac{\partial b}{\partial R} = \frac{\frac{2\phi(1-q)}{(\tau+1)^2}}{[\frac{2\phi R}{\tau+1} + q]^2} > 0. \qquad \frac{\partial b}{\partial c} = \frac{q^2}{[\frac{2\phi R}{\tau+1} + q]^2} > 0.$$

$$\frac{\partial b}{\partial q} = \frac{\frac{2\phi R}{(\tau+1)^2}}{[\frac{2\phi R}{\tau+1} + q]^2} > 0. \qquad \frac{\partial b}{\partial \phi} = \frac{-\frac{2R}{\tau+1}q(1+c)}{[\frac{2\phi R}{\tau+1} + q]^2} < 0.$$

□

Lemma 3. *Under the risk neutrality, Efficiency, and Technology assumptions, it is optimal for the constrained planner to take fire sale risk; that is, the constrained optimal liquidity ratio satisfies $b < c$.*

Proof. In principle, it is possible to completely insure against the fire sale risk. Under full insurance, similar to some interpretation of narrow banking (Freixas and Rochet, 2008, Chapter 7.2.2), the banks are able to cover the liquidity need even in the worst scenario by using their liquid holdings. However, we show that full insurance is not optimal and the constrained social planner takes some fire sale risk, by setting the aggregate liquidity ratio less than the liquidity need in the bad state, that is, by setting $b < c$.

To show this, we start with the full insurance case, that is $b = c$, and move ε amount of investment from liquid asset to risky asset, and show that this reallocation provides a Pareto improvement. Banks get exposed to fire sale risk as a result of this reallocation. As we show in Lemma 2 this reallocation does not hurt banks as long fire-sale price is not dramatically different from the fundamental value of risky assets, i.e. $R \geq F'(0) > \nu$ as given by the

Efficiency assumption. Consumer are better off with this reallocation because the expected profit of traditional sector they own is now positive, $q[F(y_\varepsilon) - y_\varepsilon P_\varepsilon] > 0$ while it was zero under the full insurance case.

□

Proposition 3. *Under the risk neutrality, Efficiency, Elasticity, and Technology assumptions, the competitive equilibrium is constrained inefficient. Furthermore, under the log-quadratic functional form assumptions, competitive equilibrium allocations compare to the constrained efficient allocations as follows:*

1. *Risky investment levels: $n > n^{**}$*
2. *Liquidity ratios: $b < b^{**}$*

Proof. To prove that the competitive equilibrium is constrained inefficient we compare the equations that define the competitive equilibrium and constrained planner's allocations, and show that the additional terms in the constrained planner's problem are strictly different than zero. The first-order conditions of the competitive equilibrium are defined in Section 3.2, shown in equation (9), and the ones for constrained planner's case is derived in Section 3.3, shown in equations (15) and (B.11).

$$\frac{\partial \Gamma}{\partial x_i} - q(R-P)\frac{\partial Q_i^s}{\partial x_i} = 0, \quad \forall x_i \in \{n_i, b_i\} \tag{A.6}$$

$$\frac{\partial \Gamma}{\partial x} - q(R-P)\frac{\partial Q^s}{\partial x} - q(R-P)\frac{\partial Q^s}{\partial P}\frac{\partial P}{\partial x} = 0, \quad \forall x \in \{n, b\} \tag{A.7}$$

where $Q_i^s(P, n_i, b_i) = \frac{c-b_i}{P} n_i$.

Using (5), we obtain that $\frac{\partial Q^s}{\partial P} = -(c-b)n/P^2 < 0$, that is, the supply of assets is downward-sloping for banks. Therefore, the extra term in constrained planner's problem is negative for risky investment because in Lemma 1 we have shown that $\partial P/\partial n < 0$. This term captures the extra units of fire sales by other banks, caused by each banks' additional investment in the risky asset. Similarly, when comparing the first-order conditions with respect to the liquidity ratio, the extra term in constrained planner's problem is positive because we have shown in Lemma 1 that $\partial P/\partial b > 0$. This term captures the public good property of liquidity: The liquid asset held by banks not only insures them against the fire sale risk but also constitutes a positive externality on other banks via greater fire sale prices.

We defer the proof of the second part of this proposition to Lemmas 5 and 6, which are under the proof of Proposition 5 below. □

Proposition 4. *Let the operational cost of a bank be given by $\Phi(x) = \phi x^2$. Then, banks decrease their liquidity ratio as the regulator tightens capital requirements; that is, $b'(n) \geq 0$ for any concave technology function for the traditional sector, $F(\cdot)$, that satisfies the Elasticity and Regularity assumptions along with either:*
(i) $F'(0) = R$, or (ii) $F'(0) \leq R$ and $R < \frac{F'(F' + yF'')}{F' + 2yF''}$ for all $y \geq 0$.

Proof. We are studying the partial regulation case, in which banks are free to choose their liquidity ratio b_i but the regulator limits their choice of n_i. Therefore, we can write banks' expected profit function at $t = 0$ as follows

$$\max_{b_i} \Pi_i(b_i; n) = (1-q)\{R + b_i\}n + qR\gamma_i n - D(n(1+b_i)). \tag{A.8}$$

Here, we can treat n like a parameter of the model because banks take it as given. The regulator, in a sense, determines the aggregate amount of n. Therefore, the first-order conditions of the banks' problem above is

$$\begin{aligned}\frac{\partial \Pi(b_i; n)}{\partial b_i} &= (1-q)n + qRn\frac{\partial \gamma}{\partial b_i} - n - 2\phi n^2(1+b_i) = 0 \\ &= (1-q)n + qRn\frac{1}{P} - n - 2\phi n^2(1+b_i) = 0,\end{aligned}$$

which can be simplified as

$$q\left(\frac{R}{P} - 1\right) = 2\phi n(1+b_i). \tag{A.9}$$

Note that we can obtain the derivative of the equilibrium price with respect to the regulatory parameter, n, as follows:

$$\frac{\partial P}{\partial n} = \frac{F_2(c - b_i)}{F_1 + yF_2}, \tag{A.10}$$

where $F_k \equiv \frac{d^k F(y)}{dy^k}$ for $k = 1, 2$, and y shows the quantity of assets sold to the traditional sector in fire sales.

Banks' profit function exhibits increasing differences in b_i and n if its cross derivative is positive. Increasing differences means that $b'(n) > 0$, that is, the optimal choice of b_i in banks' problem is increasing with the regulatory parameter, n. We can obtain the cross derivative of banks' expected profit as

$$\frac{\partial^2 \Pi(n, b_i)}{\partial b_i \partial n} = (1-q) + qR\left(\frac{1}{P} - \frac{n}{P^2}\frac{\partial P}{\partial n}\right) - 1 - 4\phi n(1+b_i).$$

Substituting for $\phi n(1+b_i)$ from the banks' first-order condition (A.9) and using the expression for $\partial P/\partial n$, given by (A.10), and we can simplify the cross derivative of banks' expected profit as follows

$$
\begin{aligned}
\frac{\partial^2\Pi(n,b_i)}{\partial b_i \partial n} &= (1-q)+qR\left(\frac{1}{P}-\frac{n}{P^2}\frac{F_2(c-b_i)}{F_1+yF_2}\right)-1-2q\left(\frac{R}{P}-1\right), \\
&= -q+qR\left(\frac{1}{P}-\frac{n(c-b_i)}{Py}\frac{1}{P}\frac{yF_2}{F_1+yF_2}\right)-\frac{2qR}{P}+2q, \\
&= q+qR\left(\frac{1}{P}-\frac{n(c-b_i)}{Py}\frac{1}{P}\frac{yF_2}{F_1+yF_2}\right)-\frac{2qR}{P},
\end{aligned}
$$

where in the second line we manipulated the second term within the parentheses by multiplying and dividing by y. Now, use of the equality of $y = n(c-b_i)/P$ in equilibrium and finally substitute $P = F_1$ to get:

$$
\begin{aligned}
\frac{\partial^2\Pi(n,b_i)}{\partial b_i \partial n} &= q+qR\left(\frac{1}{P}-\frac{1}{P}\frac{yF_2}{F_1+yF_2}\right)-\frac{2qR}{P} = q-qR\left(\frac{1}{P}+\frac{1}{P}\frac{yF_2}{F_1+yF_2}\right) \\
&= q\left\{1-R\frac{[F_1+2yF_2]}{F_1(F_1+yF_2)}\right\}.
\end{aligned}
$$

Increasing differences hold if

$$
\frac{\partial^2\Pi(b_i;n)}{\partial b_i \partial n} > 0 \Leftrightarrow R < \frac{F_1(F_1+yF_2)}{F_1+2yF_2} \equiv \kappa. \tag{A.11}
$$

Therefore, if we assume that the traditional sector's technology F satisfies (A.11), we are done. If we do not make this assumption, we can instead assume that $F_1(0) = R$ and show that (A.11) holds for all $y > 0$. Note that when y is equal to zero κ is equal to F_1 by definition, and we have that $F_1(0) = R$ by assumption. Therefore, in order to show that $\kappa > R$ for all $y > 0$, all we need to show is that κ is increasing in y. Below we show that the derivative of κ with respect to y is indeed positive:

$$
\begin{aligned}
\frac{d\kappa}{dy} &= \frac{[F_2(F_1+yF_2)+F_1(F_2+F_2+yF_3)][F_1+2yF_2]}{(F_1+2yF_2)^2}, \\
&= \frac{[3F_1F_2+yF_2^2+F_1F_3y][F_1+yF_2+yF_2]-[F_1(F_1+yF_2)][F_2+2F_2+2yF_3]}{(F_1+2yF_2)^2}
\end{aligned} \tag{A.12}
$$

Because the denominator of the derivative is positive we focus on the numerator to obtain the sign of the derivative. The numerator of (A.12) can be simplified as follows:

$$
\begin{aligned}
\frac{d\kappa}{dy}\times(F_1+2yF_2)^2 &= y(F_2^2-F_1F_3)(F_1+yF_2)+yF_2[3F_1F_2+yF_2^2+F_1F_3y], \\
&= y(F_2^2-F_1F_3)F_1+yF_2[yF_2^2-yF_1F_3+3F_1F_2+yF_2^2+yF_1F_3], \\
&= y(F_2^2-F_1F_3)F_1+yF_2[3F_1F_2+2yF_2^2].
\end{aligned}
$$

Divide both sides with y to simplify further:

$$\begin{aligned}
\frac{d\kappa}{dy} \times \frac{(F_1 + 2yF_2)^2}{y} &= F_1F_2^2 - F_1^2F_3 + 3F_1F_2^2 + 2yF_2^3 \\
&= 4F_1F_2 - F_1^2F_3 + 2yF_2^3 \\
&= 2F_1F_2 - F_1^2F_3 + 2F_1F_2 + 2yF_2^3 \\
&= F_1(2F_2^2 - F_1F_3) + 2F_2^2(F_1 + yF_2) > 0.
\end{aligned}$$

$2F_2^2 - F_1F_3$ is positive due to the *Regularity* assumption, and $F_1 + yF_2$ is positive due to the *Elasticity* assumption. □

Proposition 5. *Under the log-quadratic functional form assumptions, risky investment levels, liquidity ratios, and financial stability measures under competitive equilibrium* (n, b, P, $1-\gamma$, $(1-\gamma)n$), *partial regulation equilibrium* (n^*, b^*, P^*, $1-\gamma^*$, $(1-\gamma^*)n^*$), *and complete regulation equilibrium* (n^{**}, b^{**}, P^{**}, $1-\gamma^{**}$, $(1-\gamma^{**})n^{**}$) *compare as follows:*

1. *Risky investment levels:* $n > n^{**} > n^*$
2. *Liquidity ratios:* $b^{**} > b > b^*$
3. *Financial stability measures*
   (a) *Price of assets in the bad state:* $P^{**} > P^* > P$
   (b) *Fraction of assets sold:* $1-\gamma > 1-\gamma^* > 1-\gamma^{**}$
   (c) *Total fire sales:* $(1-\gamma)n > (1-\gamma^*)n^* > (1-\gamma^{**})n^{**}$

Proof. Proof of this proposition is established through a series of lemmas below. □

Lemma 4. $P^{**} > P^* > P$

Proof. **Part 1:** $P^* > P$. First, note that we obtain the competitive equilibrium price of assets in Proposition 1 as:

$$P = \frac{qR(1+c)}{R-1+q} = \frac{\beta}{R\sigma},$$

using the definitions of σ, β from (B.28) and (B.32). Now, take the cubic equation given by (B.33) and divide it by $R\sigma$ to obtain:

$$R\{2\phi\sigma P^{*3} + (\sigma qR - 2\phi\beta)P^* - q\beta\} + 2\phi\beta P^{*2} - 2\phi(1+c)P^{*3} = 0$$

$$R\left[\frac{2\phi}{R}P^{*3} + \left(q - \frac{2\phi\beta}{\sigma R}\right)P^* - \frac{q\beta}{\sigma R}\right] + \frac{2\phi\beta}{\sigma R}P^{*2} - \frac{2\phi(1+c)}{\sigma R}P^{*3} = 0$$

Note that $(1+c)/\sigma = P$, and substitute this into the equation above and manipulate:

$$R\left[\frac{2\phi}{R}P^{*3} + (q - 2\phi P)P^* - qP\right] + 2\phi PP^{*2} - \frac{2\phi}{R}PP^{*3} = 0$$

$$R\left(\frac{2\phi}{R}P^{*2} + q\right)P^* - \left(2\phi RP^* + qR - 2\phi P^{*2} + \frac{2\phi}{R}P^{*3}\right)P = 0$$

From this last equivalence we can obtain the price ratios in these two cases as:

$$\frac{P}{P^*} = \frac{2\phi P^{*2} + qR}{\frac{2\phi}{R} P^{*3} - 2\phi P^{*2} + 2\phi R P^* + qR} = \frac{2\phi R P^{*2} + qR^2}{2\phi P^{*3} - 2\phi R P^{*2} + 2\phi R^2 P^* + qR^2}, \tag{A.13}$$

In order to show that $P < P^*$, we need to show that the numerator of this ratio is less then its denominator, that is

$$\begin{aligned} 2\phi R P^{*2} + qR^2 &< 2\phi P^{*3} - 2\phi R P^{*2} + 2\phi R^2 P^* + qR^2 \\ 4\phi R P^{*2} &< 2\phi P^*(P^{*2} + R^2) \\ 0 &< (R - P^*)^2 \end{aligned}$$

The last inequality holds because we must have $P^* < R$ in equilibrium. $P^* < R$ holds in equilibrium for the following reason: Assumption *Efficiency* states that $P^* \leq R$, yet the equality cannot arise in equilibrium as $P^* = R$ implies $P = R$ as well due to (A.13). However, given the solution for P in Proposition 1, $P < R$ holds due to the *Technology* assumption, $R - cq - 1 > 0$. Thus, we must have $P^* < R$.

Part 2: $P^{**} > P^*$. First, note that

$$R - 1 - qc = R - 1 + q - q - qc = R - 1 + q - q(1 + c) = qR\sigma - q(1 + c) = q(\sigma R - 1 - c),$$

where σ, β are defined by (B.28) and (B.32). Using this equivalence we can write the polynomial equation that gives P^{**}, equation (B.17), as

$$\begin{aligned} (R - 1 - qc)P^{**2} + q\beta P^{**} - qR\beta = 0 \\ q(\sigma R - 1 - c)P^{**2} + q\beta P^{**} - qR\beta = 0 \\ \frac{\sigma R - 1 - c}{R} P^{**2} + \frac{\beta}{R} P^{**} = \beta \end{aligned}$$

Now substitute β using the last equation above into the cubic equation that gives P^*, equation (B.33):

$$\begin{aligned} 2\phi(\sigma R - 1 - c)P^{*3} + 2\phi\beta P^{*2} + R(\sigma q R - 2\phi\beta)P^* - qR\beta = 0 \\ 2\phi(\sigma R - 1 - c)P^{*3} + 2\phi\beta P^{*2} + R\left(\sigma q R - 2\phi\frac{\sigma R - 1 - c}{R} P^{**2} - 2\phi\frac{\beta}{R} P^{**}\right) P^* - qR\beta = 0 \\ 2\phi(\sigma R - 1 - c)P^{*3} + 2\phi\beta P^{*2} + \sigma q R^2 P^* - 2\phi(\sigma R - 1 - c)P^{**2}P^* - 2\phi\beta P^{**}P^* - qR\beta = 0 \\ (2\phi\sigma R - 1 - c)P^*(P^{*2} - P^{**2}) + 2\phi\beta P^*(P^* - P^{**}) + qR(\sigma R P^* - \beta) = 0 \end{aligned} \tag{A.14}$$

Note that the first two terms in (A.14) must have the same sign, which will be equal to the inverse of the sign of the last term, $qR(\sigma R P^* - \beta)$. Therefore, in order to show that $P^* - P^{**} < 0$, we need to show that $qR(\sigma R P^* - \beta) > 0$. We can write this last terms as

$$qR(\sigma R P^* - \beta) = qR^2\sigma P^* - q(1 + c)R^2 > 0 \Leftrightarrow \sigma P^* - 1 - c > 0.$$

Note that by Part 1, we know that $P < P^*$. Hence, if $\sigma P - 1 - c \geq 0$ then we must necessarily have $\sigma P^* - 1 - c > 0$. Using the closed-form solution of the competitive equilibrium, given by (12), we can show that:

$$\sigma P - 1 - c = \frac{R - 1 + q}{qR} \frac{qR(1+c)}{R - 1 + q} - 1 - c = 0$$

Therefore, we must have $\sigma P^* - 1 - c > 0$, which implies that $P^{**} > P^*$ in order for equation (A.14) to hold. □

Lemma 5. $b^{**} > b > b^*$

Proof. **Part 1:** $b^{**} > b$. Note that the closed-form solutions for the liquidity ratios in these two cases were obtained in equations (B.4) and (B.18) as:

$$b = \frac{cq(\tau + 1) - 2\phi R}{q(\tau + 1) + 2\phi R}, \qquad b^{**} = \frac{cq(\tau^{**} + 1)^2 - 2\phi R}{q(\tau^{**} + 1)^2 + 2\phi R}.$$

Comparing the liquidity ratios under competitive equilibrium (b) and under the constrained planner's solution (b^{**}), we see that they have the same following functional form:

$$f(x) = \frac{cqx - 2\phi R}{qx + 2\phi R}. \tag{A.15}$$

The only difference is $x = \tau + 1$ in the competitive case versus $x = (\tau^{**} + 1)^2$ in the constrained planner's problem. First, note that

$$f'(x) = \frac{cq(qx + 2\phi R) - (cqx - 2\phi R)q}{(qx + 2\phi R)^2} = \frac{2\phi Rq(1 + c)}{(qx + 2\phi R)^2} > 0. \tag{A.16}$$

Therefore, in order to show that $b^{**} > b$, all we need to show is that $(\tau^{**} + 1)^2 > \tau + 1$, which can be written equivalently as:

$$\frac{R^2}{P^{**2}} > \frac{R}{P} \Leftrightarrow P^{**2} < RP.$$

Now, substitute P^{**2} from the solution to the constrained planner's problem, given by (B.17) and the competitive equilibrium price, P, from (12) to write this inequality as:

$$\begin{aligned}
\frac{q\beta(R - P^{**})}{R - 1 - qc} &< R\frac{qR(1+c)}{R - 1 + q} \\
R - P^{**} &< R\frac{R - 1 - qc}{R - 1 + q} \\
R\left(1 - \frac{R - 1 - qc}{R - 1 + q}\right) &< P^{**} \\
\frac{Rq(1+c)}{R - 1 + q} = P &< P^{**}.
\end{aligned}$$

The last inequality holds by Lemma 4. Therefore, $(\tau^{**}+1)^2 > \tau + 1$, which implies that $b^{**} > b$.

Part 2: $b > b^*$. Note that the closed-form solutions for the liquidity ratios in these two cases were obtained in equations (B.4) and (B.21)as:

$$b = \frac{cq(\tau+1) - 2\phi R}{q(\tau+1) + 2\phi R}, \qquad b^{**} = \frac{cq(\tau^*+1) - 2\phi R}{q(\tau^*+1) + 2\phi R}.$$

Comparing the liquidity ratios under competitive equilibrium (b) and under the partial regulation case (b^*), we see that they have the same functional form, $f(x)$, given above by (A.15). The only difference is $x = \tau + 1$ in the competitive case versus $x = \tau^* + 1$ in the partial case. We have shown above, by (A.16), that $f'(x) > 0$. Therefore, in order to show that $b > b^*$, all we need to show is that $\tau > \tau^*$. Note that because $\tau^* = R/P^* - 1$ and $\tau = R/P - 1$, and $P^* > P$ by Lemma 4, we have that $\tau > \tau^*$. This completes the proof. □

Lemma 6. $n > n^{**} > n^*$

Proof. **Part 1:** $n > n^{**}$. Using the closed-form solution for the competitive equilibrium, (B.5), and for the constrained planner's problem, (B.19), the difference in risky investment levels across these two cases can be written as

$$\begin{aligned} n - n^{**} &= \frac{\tau}{\tau+1}\frac{q(\tau+1) + 2\phi R}{2\phi(1+c)} - \frac{\tau^{**}}{\tau^{**}+1}\frac{q(\tau^{**}+1)^2 + 2\phi R}{2\phi(1+c)} \\ &= \frac{1}{2\phi(1+c)}\left\{\frac{\tau[q(\tau+1) + 2\phi R]}{(\tau+1)} - \frac{\tau^{**}[q(\tau^{**}+1)^2 + 2\phi R]}{(\tau^{**}+1)}\right\} \\ &= \frac{1}{2\phi(1+c)}\left\{q\tau + 2\phi R\frac{\tau}{(\tau+1)} - q\tau^{**}(\tau^{**}+1) - 2\phi R\frac{\tau^{**}}{(\tau^{**}+1)}\right\} \\ &= \frac{1}{2\phi(1+c)}\left\{q\tau - q\tau^{**}(\tau^{**}+1) + 2\phi R\frac{\tau}{(\tau+1)} - 2\phi R\frac{\tau^{**}}{(\tau^{**}+1)}\right\} \end{aligned}$$

First, note that $\tau = R/P - 1 > \tau^{**} = R/P^{**} - 1$ by Lemma 4, and this implies that $2\phi R\frac{\tau}{(\tau+1)} - 2\phi R\frac{\tau^{**}}{(\tau^{**}+1)}$ is positive. Therefore, $n - n^{**}$ is positive if $q\tau - q\tau^{**}(\tau^{**}+1) \geq 0$. Next, we show that this inequality indeed holds. From (B.16) we have $R - 1 - qc = \frac{qR(R-P^{**})(1+c)}{P^2}$, which implies that:

$$\tau = \frac{R - 1 - qc}{q(1+c)} = \frac{R(R - P^{**})}{P^{**2}} = \frac{R}{P^{**}}\left(\frac{R}{P^{**}} - 1\right) = \tau^{**}(\tau^{**}+1),$$

where we use that $\tau = R/P - 1$ and $P = \frac{qR(1+c)}{R-1+q}$, as given by 12.

Part 2: $n^{**} > n^*$. For the second part of this lemma, we use the fact that $P^{**} > P^*$ as shown by Lemma 4. Using the closed-form solution for n^{**} from (B.19) and n^* from (B.22), we can write the difference in risky investment levels across these two cases as:

$$\begin{aligned} n^{**} - n^* &= \frac{\tau^{**}}{\tau^{**}+1}\frac{q(\tau^{**}+1)^2 + 2\phi R}{2\phi(1+c)} - \frac{\tau^*}{\tau^*+1}\frac{q(\tau^*+1)+2\phi R}{2\phi(1+c)}, \\ &= \frac{1}{2\phi(1+c)}\left\{\frac{\tau^{**}}{\tau^{**}+1}[q(\tau^{**}+1)^2+2\phi R] - \frac{\tau^*}{\tau^*+1}[q(\tau^*+1)+2\phi R]\right\}, \\ &= \frac{\Theta}{2\phi(1+c)(\tau^*+1)(\tau^{**}+1)} \end{aligned} \tag{A.17}$$

where

$$\begin{aligned} \Theta &\equiv q(\tau^{**}+1)(\tau^*+1)[\tau^{**}(\tau^{**}+1)-\tau^*] + 2\phi R[\tau^{**}(\tau^*+1)-\tau^*(\tau^{**}+1)] \\ &= q(\tau^{**}+1)(\tau^*+1)[\tau-\tau^*] + 2\phi R[\tau^{**}-\tau^*], \end{aligned} \tag{A.18}$$

where we use the equivalence, $\tau = \tau^{**}(\tau^{**}+1)$, obtained in Part 1 above. Since the denominator of (A.17) is positive, in order to prove that $n^{**} - n^* > 0$, it suffices to show that $\Theta > 0$. In order to show that this inequality holds, first, we would like to write $2\phi R$ that shows up Θ in terms of τ's. For that start from the cubic equation that gives the partial equilibrium price as obtained by (B.33):

$$\begin{aligned} 0 &= \frac{2\phi}{q}(R-1-qc)P^{*3} + 2\phi R(1+c)P^{*2} + R^2(\sigma q - 2\phi(1+c))P^* - (1+c)qR^2, \\ 0 &= \frac{2\phi}{q}q(1+c)\tau P^{*3} + 2\phi R(1+c)P^{*2} + R(R-1+q-2\phi R(1+c))P^* - (1+c)qR^2, \\ 0 &= 2\phi(1+c)\tau P^{*3} + 2\phi R(1+c)P^{*2} + R\left(\frac{qR(1+c)}{P} - 2\phi R(1+c)\right)P^* - (1+c)qR^2, \\ 0 &= 2\phi(1+c)\tau P^{*3} + 2\phi R(1+c)P^{*2} + \frac{qR^2(1+c)}{P}P^* - 2\phi R^2(1+c)P^* - (1+c)qR^2, \\ 0 &= 2\phi(1+c)\tau\frac{R^3}{(\tau^*+1)^3} + 2\phi R(1+c)\frac{R^2}{(\tau^*+1)^2} + \frac{qR^2(1+c)}{P}\frac{R}{\tau^*+1} - 2\phi R^2(1+c)\frac{R}{\tau^*+1} - (1+c)qR^2, \\ 0 &= 2\phi(1+c)\tau\frac{R^3}{(\tau^*+1)^3} + 2\phi(1+c)\frac{R^3}{(\tau^*+1)^2} + qR^2(1+c)\frac{\tau+1}{R}\frac{R}{\tau^*+1} - 2\phi(1+c)\frac{R^3}{\tau^*+1} - (1+c)qR^2, \\ 0 &= \frac{2\phi(1+c)R^3}{(\tau^*+1)^3}[\tau+\tau^*+1-(\tau^*+1)^2] + qR^2(1+c)\left[\frac{\tau+1}{\tau^*+1}-1\right], \\ 0 &= \frac{2\phi R}{(\tau^*+1)^2}[\tau-\tau^*(\tau^*+1)] - q(\tau^*-\tau), \end{aligned}$$

where in the first line we use definition of σ, given by (B.28), to write $\sigma R - 1 - c = (R-1-qc)/q$, while using $\tau = R/P - 1 = (R-1-qc)/[q(1+c)]$ in the second line. In the third line we replaced $R-1+q$ with $\frac{qR(1+c)}{P}$ using equation (12) for price in competitive equilibrium and later we use $P^* = R/(\tau^*+1)$ to replace P^*. From the last equation above we can obtain:

$$2\phi R = \frac{q(\tau^*+1)^2(\tau^*-\tau)}{\tau-\tau^*(\tau^*+1)} = \frac{q(\tau^*+1)^2(\tau^*-\tau)}{\tau^{**}(\tau^{**}+1)-\tau^*(\tau^*+1)},$$

where we use the equivalence, $\tau = \tau^{**}(\tau^{**}+1)$, again. Now we plug this expression for $2\phi R$ back into (A.18) and show below that $\Theta > 0$ holds:

$$\begin{aligned}
q(\tau^{**}+1)(\tau^{*}+1)[\tau-\tau^{*}] &> 2\phi R[\tau^{*}-\tau^{**}] = \frac{q(\tau^{*}+1)^2(\tau^{*}-\tau)}{\tau^{**}(\tau^{**}+1)-\tau^{*}(\tau^{*}+1)}[\tau^{*}-\tau^{**}] \\
\tau^{**}+1 &> \frac{(\tau^{*}+1)(-1)(\tau^{*}-\tau^{**})}{\tau^{**}(\tau^{**}+1)-\tau^{*}(\tau^{*}+1)} \\
\tau^{**}+1 &> \frac{(\tau^{*}+1)(\tau^{*}-\tau^{**})}{\tau^{*}(\tau^{*}+1)-\tau^{**}(\tau^{**}+1)}
\end{aligned}$$

This inequality can be simplified further as follows:

$$\begin{aligned}
(\tau^{**}+1)\tau^{*}(\tau^{*}+1)-\tau^{**}(\tau^{**}+1)^2 &> (\tau^{*}+1)(\tau^{*}-\tau^{**}) \\
(\tau^{*}+1)[\tau^{*}(\tau^{**}+1)-(\tau^{*}-\tau^{**})] &> \tau^{**}(\tau^{**}+1)^2 \\
(\tau^{*}+1)\tau^{**}(\tau^{*}+1) &> \tau^{**}(\tau^{**}+1)^2 \\
(\tau^{*}+1)^2 &> (\tau^{**}+1)^2.
\end{aligned}$$

This inequality is true because $P^{**} > P^{*}$, as shown by Lemma 4, which implies that $\tau^{*} > \tau^{**}$, using the definitions $\tau^{*} = R/P^{*} - 1$ and $\tau^{**} = R/P^{**} - 1$. □

Lemma 7. $1-\gamma > 1-\gamma^{*} > 1-\gamma^{**}$

Proof.

$$1-\gamma = \frac{c-b}{P} \text{ together with } b^{**} > b^{*} \text{ and } P^{**} > P^{*} \implies 1-\gamma^{*} > 1-\gamma^{**}$$

To obtain $(1-\gamma) > (1-\gamma^{*})$, we can equivalently show that $\frac{\frac{c-b}{P}}{\frac{c-b^{*}}{P^{*}}} > 1$.

Using equations (B.4) and (B.21) for b and b^{*} respectively, $b = \frac{qc(\tau+1)-2\phi R}{2\phi R+q(\tau+1)} \implies c-b = \frac{2\phi R(1+c)}{2\phi R+q(\tau+1)}$, and similarly we can derive $c-b^{*} = \frac{2\phi R(1+c)}{2\phi R+q(\tau^{*}+1)}$. Writing τ and τ^{*} in terms of P and P^{*} we get the following,

$$\frac{\frac{c-b}{P}}{\frac{c-b^{*}}{P^{*}}} = \frac{c-b}{c-b^{*}}\frac{P}{P^{*}} = \frac{2\phi P^{*}+q}{2\phi P+q}\frac{P}{P^{*}}\frac{P^{*}}{P} > 1.$$

The last inequality holds because $P^{*} > P$ by Lemma 4. □

Lemma 8. $(1-\gamma)n > (1-\gamma^{*})n^{*} > (1-\gamma^{**})n^{**}$

Proof. Given that the demand function for risky assets in the interim period is downward sloping (continuous and differentiable as well), the prices disclose the amount of fire sales. Hence, we can use the results in Lemma 4 to prove this lemma:

$$(1-\gamma)n = \frac{R}{P} - 1 \text{ and } P^{**} > P^{*} > P \implies (1-\gamma^{**})n^{**} < (1-\gamma^{*})n^{*} < (1-\gamma)n.$$

$\square$

Proposition 6. *Under the log-quadratic functional form assumptions, bank balance sheet sizes across different regimes compare as follows:*

$$n(1+b) = n^{**}(1+b^{**}) > n^{*}(1+b^{*})$$

Proof. Using the closed-form solutions in Sections B.1 and B.2, we can write the bank size under the competitive equilibrium and constrained planner's problem as follows:

$$n(1+b) \;=\; \frac{\tau}{\tau+1}\frac{2\phi R + q(\tau+1)}{2\phi(1+c)}\frac{q(\tau+1)(1+c)}{2\phi R + q(\tau+1)} = \frac{\tau}{\tau+1}\frac{q(\tau+1)}{2\phi} = \frac{q\tau}{2\phi}.$$

$$n^{**}(1+b^{**}) \;=\; \frac{\tau^{**}}{\tau^{**}+1}\frac{2\phi R + q(\tau^{**}+1)^2}{2\phi(1+c)}\frac{q(\tau^{**}+1)^2(1+c)}{2\phi R + q(\tau^{**}+1)^2} = \frac{q\tau^{**}(\tau^{**}+1)}{2\phi}.$$

Above we use equations (B.5) and (B.4) for the balance sheet size in competitive equilibrium and equations (B.18) and (B.19) for the constrained planner's case. Note that in Part I of Lemma 6 we show that $\tau = \tau^{**}(\tau^{**}+1)$. Thus, comparing the equations above we conclude $n(1+b) = n^{**}(1+b^{**})$.

Lastly, $b^{**} > b > b^{*}$, as shown in Lemma 5, and $n > n^{**} > n^{*}$, as shown in Lemma 6, together imply that $n(1+b) > n^{*}(1+b^{*})$, that is, the bank balance sheet size is the smallest under partial regulation. $\square$

Proposition 7. *Under the Efficiency, Elasticity, Regularity, and Technology assumptions, banks do not choose the constrained optimal risky investment level, n^{**}, if the regulator sets the minimum liquidity ratio at the constrained optimal level, b^{**}; that is, $n_i(b^{**}) \neq n^{**}$.*

Proof. We first study bank behavior under liquidity regulation alone. In this case, the regulator chooses the optimal liquidity ratio, b, at $t = 0$ to maximize the net expected social welfare but allows banks to freely choose their risky investment level, n_i. Consider the problem of a bank first: For a given regulatory liquidity ratio, b, a bank chooses the level of risky investment, n_i, to maximize its expected profits:

$$\max_{n_i} \Pi_i(n_i; b) = \max_{n_i} \; (R + b - qc)n_i - D(n_i(1+b)) - q(R-P)Q_i^s(P, n_i, b) \tag{A.19}$$

The first-order condition of the banks' problem (A.19) with respect to n_i is

$$R + b - qc - D'(n_i(1+b))(1+b) - q(R-P)\frac{\partial Q_i^s}{\partial n_i} = 0, \tag{A.20}$$

where $Q_i^s(P, n_i, b) = (1-\gamma)n_i = \frac{c-b}{P}n_i$.

We then compare this first-order condition with the corresponding one of the constrained planner's problem, given by (B.12). These two first-order conditions are written explicitly below for comparison:

$$
\begin{aligned}
\Psi &\equiv (1-q)(R+b)+qR\Big\{\gamma+\frac{\partial\gamma}{\partial n}n\Big\}+q\left\{F'((1-\gamma)n)\left(1-\gamma-\frac{\partial\gamma}{\partial n}n\right)-c+b\right\}-D^{'}(\cdot)(1+b)=0,\\
\Upsilon &\equiv (1-q)(R+b)+qR\gamma_i-D^{'}(\cdot)(1+b)=0.
\end{aligned}
$$

The constrained planner's first-order condition, Ψ, includes extra terms because the planner internalizes the effects of fire sale externalities. These extra terms are:

$$
Z=qR\frac{\partial\gamma}{\partial n}n+q\left\{F'((1-\gamma)n)\left(1-\gamma-\frac{\partial\gamma}{\partial n}n\right)-c+b\right\}
$$

Hence, we can write $\Psi=\Upsilon+Z$. We first show that the sum of these extra terms is negative:

$$
\begin{aligned}
Z &= qR\frac{\partial\gamma}{\partial n}n+q\left\{F'((1-\gamma)n)\left(1-\gamma-\frac{\partial\gamma}{\partial n}n\right)-c+b\right\}\\
&= qR\frac{\partial\gamma}{\partial n}n+q\left\{P\left(\frac{c-b}{P}-\frac{\partial\gamma}{\partial n}n\right)-c+b\right\}\\
&= qR\frac{\partial\gamma}{\partial n}n+q\left\{c-b-\frac{\partial\gamma}{\partial n}nP-c+b\right\}\\
&= qR\frac{\partial\gamma}{\partial n}n-qP\frac{\partial\gamma}{\partial n}=q\frac{\partial\gamma}{\partial n}n(R-P)<0,
\end{aligned}
$$

where we use that in equilibrium $F'((1-\gamma)n^{**})=P^{**}$. The sign of Z is negative because $R>P^{**}$ by the *Efficiency* assumption, and $\partial\gamma/\partial n<0$ by Lemma 1.

$Z<0$ implies that banks' first-order condition, Υ, evaluated at the constrained efficient allocations, n^{**}, b^{**} is positive, that is $\Upsilon(n^{**}, b^{**})>0$. On the contrary, we have $\Upsilon(n(b^{**}), b^{**})=0$ by definition of optimality. Furthermore, we can show that Υ is decreasing in n for a given b, that is:

$$
\frac{\partial\Upsilon}{\partial n}=qR\frac{\partial\gamma}{\partial n}-D^{''}(\cdot)(1+b)^2<0,
$$

because $D^{''}(\cdot)>0$ by assumption and $\partial\gamma/\partial n<0$ by Lemma 1. Therefore, we must have $n(b^{**})>n^{**}$. □

B Closed-form solutions

B.1 A closed-form solution for the competitive equilibrium

Suppose that the operational cost of a bank is given by $\Phi(x) = \phi x^2$ and let the traditional sector's technology function be given by $F = Rln(1+y)$. Firms in traditional sector choose how much assets, y, to buy from banks in the bad state at $t = 1$ to maximize their profits, $F(y) - Py$, where P is the price of assets. The first-order condition of this problem yields (inverse) demand function of firms in traditional sector for risky assets:

$$P = F'(y) = \frac{R}{1+y} \text{ and hence } y = F'^{-1}(P) = \frac{R}{P} - 1 \equiv Q^d(P). \tag{B.1}$$

We solve for the competitive equilibrium price, P, in the main text, as shown by (12). Now, use this solution in the demand side function and define the total amount of assets purchased by the traditional sector, τ, in terms of the exogenous variables as follows:

$$y = \frac{R}{P} - 1 = \frac{R-1+q}{q(1+c)} - 1 \equiv \tau. \tag{B.2}$$

We obtain the total supply of asset by banks as $(1-\gamma)n$ by (5) in Section 3.1.2. Hence, the market clearing condition, $(1-\gamma)n = \tau$, yields:

$$(c-b)n = P\tau \implies n = \frac{P\tau}{c-b}. \tag{B.3}$$

This equation gives the investment level, n, as a function of the liquidity ratio, b. We can solve for the latter from the first-order conditions of banks' problem in the decentralized case, given by (A.4-A.5), as derived in the proof of Proposition 1 below. Using $\frac{R}{P} = \tau + 1$ from (B.2) and the functional form of the operational cost, $\Phi'(n(1+b)) = 2\phi n(1+b)$, in the first-order condition with respect to b, given by (A.5) yields:

$$\begin{aligned} 1 - q + q(\tau+1) &= 1 + 2\phi n(1+b), \\ 1 + q\tau &= 1 + 2\phi \frac{P\tau}{c-b}(1+b), \end{aligned}$$

where in the second line we use $n = P\tau/(c-b)$ from (B.3). Substituting $R/(\tau+1)$ for P from (B.2) yields

$$c - b = \frac{2\phi}{q}\frac{R(1+b)}{\tau+1}.$$

Finally, rearrange to obtain the liquidity ratio in the competitive equilibrium as

$$b = \frac{cq(\tau+1) - 2\phi R}{q(\tau+1) + 2\phi R}. \tag{B.4}$$

To obtain the risky investment level in the competitive equilibrium substitute this expression for b in (B.3):

$$n = \frac{\tau}{\tau+1}\frac{q(\tau+1)+2\phi R}{2\phi(1+c)} \tag{B.5}$$

B.2 A closed-form solution for the constrained planner's problem

Lemma 3 allows us to focus on the $b < c$ case when analyzing the constrained planner's problem which simplifies as follows:

$$\max_{n,b,y} \Gamma(n,b) - q\{(R-P)Q^s(P,n,b)\} - (1-q)T_2, \tag{B.6}$$

$$\text{subject to } y = Q^s(P,n,b),$$

$$F'(y) = P,$$

$$(1-q)T_2 + 3\omega + q[F(y) - Py] \geq U_{i,CE}, \tag{B.7}$$

where $Q^s = \frac{c-b}{P}n$. The transfers must be such that consumers receive in expectation what they lose from the change in the amount of fire sales and price of assets:

$$(1-q)T_2 = q[F(y^{CE}) - P^{CE}y^{CE} - F(y) + Py]. \tag{B.8}$$

In other words, the constraint on consumers' expected utility binds. Substituting this value for transfers back into the planner's problem helps us to get rid of the constraint on consumers' expected utility:

$$\max_{n,b,y} \Gamma(n,b) - q\{(R-P)Q^s(P,n,b)\} + q[F(y) - Py] - q[F(y^{CE}) - P^{CE}y^{CE}], \tag{B.9}$$

$$\text{subject to } y = Q^s(P,n,b),$$

$$F'(y) = P.$$

Corresponding first-order conditions with respect to $x \in \{n, b\}$ are, respectively,

$$\frac{\partial \Gamma}{\partial x} - q\left[(R-P)\left(\frac{\partial Q^s}{\partial x} + \frac{\partial Q^s}{\partial P}\frac{\partial P}{\partial x}\right) - Q^s\frac{\partial P}{\partial x} - (F'(y) - P)\left(\frac{\partial y}{\partial x} + \frac{\partial y}{\partial P}\frac{\partial P}{\partial x}\right) + y\frac{\partial P}{\partial x}\right] = 0. \tag{B.10}$$

Using an envelope argument, $F'(y) = P$, and the market clearing condition $y = Q^s$, we can simplify the optimality condition for welfare to:

$$\frac{\partial \Gamma}{\partial x} - q(R-P)\frac{\partial Q^s}{\partial x} - q(R-P)\frac{\partial Q^s}{\partial P}\frac{\partial P}{\partial x} = 0, \quad \forall x \in \{n, b\}. \tag{B.11}$$

Using the equilibrium condition $y = Q^s = (1-\gamma)n = \frac{(c-b)n}{P}$ and $\Gamma(n_i, b_i) \equiv (R + b_i - qc)n_i - D(n_i(1+b_i))$ we can write the the first-order conditions of the planner's problem with respect to n and b are respectively:

$$(1-q)(R+b) + qR\Big\{\gamma + \frac{\partial \gamma}{\partial n}n\Big\} + q\left\{F'((1-\gamma)n)\left(1 - \gamma - \frac{\partial \gamma}{\partial n}n\right) - c + b\right\} = D^{'}(n(1+b))(1+b), \tag{B.12}$$

$$(1-q)n + qR\frac{\partial \gamma}{\partial b}n + q\left\{F'((1-\gamma)n)\left(-\frac{\partial \gamma}{\partial b}\right)n + n\right\} = D^{'}(n(1+b))n, \tag{B.13}$$

where $\gamma = 1 + \frac{b-c}{P}$ from banks' problem in the bad state, as obtained in Section 3.1.2. Combining the two first-order conditions to obtain:

$$(1-q)(R+b) + qR\Big\{\gamma + \frac{\partial\gamma}{\partial n}n\Big\} + q\left\{F'((1-\gamma)n)\left(1-\gamma-\frac{\partial\gamma}{\partial n}n\right) - c + b\right\} = \\ \left[(1-q) + qR\frac{\partial\gamma}{\partial b} + qF'((1-\gamma)n)\left(-\frac{\partial\gamma}{\partial b}\right) + q\right](1+b). \quad \text{(B.14)}$$

From this point on we use the log-quadratic functional form assumptions in order to get closed form solutions to the planner's problem. First, note that using the functional form for traditional sector's demand, given by (B.1), in the market clearing condition (6) yields the price of assets in the bad state as a function of initial portfolio allocations:

$$E(P,n,b) = Q^d(P) - Q^s(P,n,b) = 0 \implies \frac{R-P}{P} = \frac{c-b}{P}n \implies P = R-(c-b)n. \quad \text{(B.15)}$$

Substituting $\frac{\partial\gamma}{\partial n} = -\frac{(c-b)^2}{P^2}$ and $\frac{\partial\gamma}{\partial b} = \frac{R}{P^2}$, and later $P = R-(c-b)n$ into (B.14) and noting that $F'((1-\gamma)n) = P$ yields:

$$(1-q)(R+b) + qR\Big\{1 - \frac{c-b}{P} - \frac{(c-b)^2}{P^2}n\Big\} + q\left\{P\left(\frac{c-b}{P} + \frac{(c-b)^2}{P^2}n\right) - c + b\right\} = \\ \left[(1-q) + qR\frac{R}{P^2} - qP\frac{R}{P^2} + q\right](1+b),$$

or equivalently:

$$(1-q)(R-1) + (1-q)(1+b) + qR - qR\Big\{\frac{(c-b)P + (c-b)^2 n}{P^2}\Big\} + q\left\{P\frac{(c-b)P + (c-b)^2 n}{P^2} - c + b\right\} = \\ (1-q)(1+b) + q\frac{R}{P^2}(R-P)(1+b) + q(1+b).$$

Note that $(c-b)P + (c-b)^2 n = (c-b)[R-(c-b)n] + (c-b)^2 n = R(c-b)$. Substitute this equilavance into the equation above and simplify:

$$R - 1 + q - qR\frac{R(c-b)}{P^2} + qP\frac{R(c-b)}{P^2} - qc + qb = q\frac{R}{P^2}(R-P)(1+b) + q + qb.$$

$$\begin{aligned}
R-1-qc &= \frac{qR}{P^2}\{(R-P)(1+b) + R(c-b) - P(c-b)\} \\
R-1-qc &= \frac{qR}{P^2}\{(R-[R-(c-b)n](1+b) + R(c-b) - [R-(c-b)n](c-b)\} \\
R-1-qc &= \frac{qR}{P^2}\{(R-R+(c-b)n(1+b) + R(c-b) - R(c-b) + (c-b)^2 n\} \\
R-1-qc &= \frac{qR}{P^2}\{(c-b)n(1+b) + (c-b)^2 n\}
\end{aligned}$$

Further simplification yields:

$$\begin{aligned} R - 1 - qc &= \frac{qR(c-b)n(1+c)}{P^2} \\ R - 1 - qc &= \frac{qR(R-P)(1+c)}{P^2}, \end{aligned} \tag{B.16}$$

where we substitute $P = R - (c-b)n$ in the second line using the market clearing condition (B.15), and $(c-b)n = R - P$ using the same condition again in the last line above. From (B.16) we obtain the following quadratic equation in P:

$$(R - 1 - qc)P^2 + qR(1+c)P - qR^2(1+c) = 0, \tag{B.17}$$

which we can solve for the price of assets under constrained planner's solution, P^{**}:

$$P^{**} = \frac{-qR(1+c) + \sqrt{q^2R^2(1+c)^2 + 4(R-1-qc)qR^2(1+c)}}{2(R-1-qc)}.$$

We can define $\tau^{**} \equiv R/P^{**} - 1$ similar to (B.2) to represent the total amount of assets sold under fire sales to the traditional sector in terms of the model parameters, and write risky investment as a function of the liquidity ratio as $n^{**} = P^{**}\tau^{**}/(c-b)$ using the market clearing condition, similar to (B.15).

We use these equations to solve for the constrained efficient portfolio allocations n^{**}, b^{**}. For that start from the first-order condition with respect to b given above by (B.13):

$$\begin{aligned} 1 - q + qR\frac{\partial\gamma}{\partial b} + q\left\{F'((1-\gamma)n)\left(-\frac{\partial\gamma}{\partial b}\right) + 1\right\} &= D'(n(1+b)), \\ 1 - q + qR\frac{R}{P^2} + q\left\{-P\frac{R}{P^2} + 1\right\} &= 1 + 2\phi n(1+b), \\ q\frac{R^2}{P^2} - q\frac{R}{P} &= 2\phi n(1+b). \end{aligned}$$

Writing all endogenous variables in terms of τ^* and simplifying yields

$$\begin{aligned} q(\tau^{**}+1)^2 - q(\tau^{**}+1) &= 2\phi\frac{P\tau^{**}}{c-b}(1+b), \\ q(\tau^{**}+1)(\tau^{**}+1-1) &= 2\phi\frac{R}{\tau^{**}+1}\frac{\tau^{**}}{c-b}(1+b), \\ q(\tau^{**}+1)^2\tau^{**}(c-b) &= 2\phi R\tau^{**}(1+b), \\ q(\tau^{**}+1)^2c - 2\phi R &= b\{2\phi R + q(\tau^{**}+1)^2\}, \end{aligned}$$

where we use $R/P^{**} = \tau^{**} + 1$ and $n^{**} = P^{**}\tau^{**}/(c-b)$. For future reference, using the second from the last number, we can obtain the liquidity shortage per risky asset in the constrained planner's solution as

$$c - b^{**} = \frac{2\phi R(1+b^{**})}{q(\tau^{**}+1)^2}.$$

We can obtain the closed-form solution for the constrained efficient liquidity ratio, b^{**}, by rearranging the last equation above, as

$$b^{**} = \frac{cq(\tau^{**}+1)^2 - 2\phi R}{q(\tau^{**}+1)^2 + 2\phi R}. \tag{B.18}$$

Finally, we can obtain the closed-form solution for the risky investment level by substituting b^{**} into $n^{**} = P^{**}\tau^{**}/(c-b)$ and using $P^{**} = R/(\tau^{**}+1)$

$$n^{**} \quad = \quad \frac{P\tau^{**}}{c-b} = \frac{R\tau^{**}}{\tau^{**}+1}\frac{q(\tau^{**}+1)^2 + 2\phi R}{2\phi R(1+c)} = \frac{\tau^{**}}{\tau^{**}+1}\frac{q(\tau^{**}+1)^2 + 2\phi R}{2\phi(1+c)}. \tag{B.19}$$

B.3 A closed-form solution for the partial regulation case

In the partial regulation case, we consider the problem of a planner who chooses the level of risky investment, n, at $t = 0$ in a Pareto efficient way but allows banks to freely choose their liquidity ratio, b_i. We first analyze banks' problem in this setting and then turn to the planner's problem. When the planner's optimal investment level is introduced as a regulatory upper bound on investment level, n, banks set $n_i = n$ and choose the liquidity ratio, b_i, to maximize their expected profits:

$$\max_{b_i} \Pi_i(b_i; n) = (1-q)\{R + b_i\}n + qR\gamma_i n - D(n(1+b_i)).$$

The first-order condition of the banks' problem (B.3) with respect to b_i is

$$1 - q + qR\frac{1}{P} = D^{'}(n(1+b_i)). \tag{B.20}$$

We use the log-quadratic functional form assumptions as in the closed-form solutions of the unregulated competitive equilibrium in Section B.1 and constrained planner's problem in Section B.2. We can also define $\tau^* \equiv R/P^* - 1$ similar to (B.2) to represent the total amount of assets sold under fire sales to the traditional sector in terms of the model parameters, and write risky investment as a function of the liquidity ratio as $n^* = P^*\tau^*/(c-b)$ using the market clearing condition, similar to (B.15). Now, use the functional-form for the operational cost in banks' first-order condition and manipulate

$$\begin{aligned}
1 - q + \frac{qR}{P} \quad &= \quad 1 + 2\phi n(1+b), \\
q\left(\frac{R}{P} - 1\right) \quad &= \quad 2\phi\frac{P\tau}{c-b}(1+b), \\
q\tau \quad &= \quad 2\phi\frac{R}{\tau+1}\frac{\tau}{c-b}(1+b),
\end{aligned}$$

where we first use $n = \frac{P\tau}{c-b}$ and then substitute $P = \frac{R}{\tau+1}$. From the last equation we can obtain an expression for the liquidity ratio in this case in terms of τ^* as follows

$$b^* = \frac{qc(\tau^*+1) - 2\phi R}{q(\tau^*+1) + 2\phi R}. \tag{B.21}$$

Using $n = \frac{P\tau}{c-b}$ and $P = \frac{R}{\tau+1}$ once more, we can obtain a similar expression for the risky investment level in this case in terms of τ^* as follows:

$$n^* = \frac{\tau^*}{\tau^*+1} \frac{q(\tau^*+1) + 2\phi R}{2\phi(1+c)}. \tag{B.22}$$

All that remains now is to obtain a closed-form solution for $\tau^* = R/P^* - 1$, and substitute that in (B.21) and (B.22) to obtain closed-form solutions for n^* and b^*. To obtain a closed-form solution for P^* we analyze the regulator's problem. The regulator takes into account that for any given n, the banks optimally choose their liquidity ratio $b(n)$, as shown by the response function (17).

The planner takes this reaction function into account while choosing the risky investment level to maximize the expected bank profits subject to the constraint that consumers' utility after transfers is at least as high as in the competitive equilibrium:

$$\begin{aligned} \max_{n,y} \Gamma(n, b(n)) - q\{(R-P)Q^s(P, n, b(n)) - (1-q)T_2\}, & \\ \text{subject to } y = Q^s(P, n, b(n)), & \\ F'(y) = P, & \\ \frac{d\Pi_i(b_i; n)}{db_i}, = 0 & \\ (1-q)T_2 + 3\omega + q[F(y) - Py] \geq U_i^{CE}. & \end{aligned}$$

The transfers must be such that consumers receive in expectation what they lose from the change in the amount of fire sales and price of assets:

$$(1-q)T_2 = q[F(y^{CE}) - P^{CE}y^{CE} - F(y) + Py] \tag{B.23}$$

In other words, the constraint on consumers' expected utility binds. Substituting this value for transfers back into the planner's problem helps us to get rid of the constraint on consumers' expected utility:

$$\begin{aligned} \max_{n,y} \Gamma(n, b(n)) - q\{(R-P)Q^s(P, n, b(n))\} + q[F(y) - Py] - q[F(y^{CE}) - P^{CE}y^{CE}], & \quad \text{(B.24)} \\ \text{subject to } y = Q^s(P, n, b(n)), & \\ F'(y) = P. & \\ \frac{d\Pi_i(b_i; n)}{db_i}, = 0 & \end{aligned}$$

The optimal risky investment level in this case is determined by the following first-order condition of the planner's problem with respect to n:

$$\begin{aligned} \frac{\partial \Gamma}{\partial n} + \frac{\partial \Gamma}{\partial b} b'(n) - q \left[(R-P) \left(\frac{\partial Q^s}{\partial n} + \frac{\partial Q^s}{\partial b} b'(n) + \frac{\partial Q^s}{\partial P} \frac{dP}{dn} \right) - Q^s \frac{dP}{dn} \right] & \\ +q \left[(F'(y) - P) \left(\frac{\partial y}{\partial n} + \frac{\partial y}{\partial b} b'(n) + \frac{\partial y}{\partial P} \frac{dP}{dn} \right) + y \frac{dP}{dn} \right] & = 0, \end{aligned}$$

where

$$Q^s = \frac{c-b}{P}n \quad \text{and} \quad \frac{dP}{dn} = \frac{\partial P}{\partial n} + \frac{\partial P}{\partial b}b'(n).$$

Using an envelope argument, $F'(y) = P$, and the market clearing condition $y = Q^s$, we can simplify the optimality condition for welfare to:

$$\frac{\partial \Gamma}{\partial n} + \frac{\partial \Gamma}{\partial b}b'(n) - q(R-P)\left(\frac{\partial Q^s}{\partial n} + \frac{\partial Q^s}{\partial b}b'(n)\right) - q(R-P)\frac{\partial Q^s}{\partial P}\frac{dP}{dn} = 0. \quad \text{(B.25)}$$

Using the equilibrium condition $y = Q^s = (1-\gamma)n\frac{(c-b)n}{P}$ and $\Gamma(n, b_i) \equiv (R + b_i - qc)n - D(n(1+b_i))$ we can write the first-order condition as

$$(1-q)\{R + b(n) + nb^{'}(n)\} + qR\left\{\gamma + n\frac{d\gamma}{dn}\right\} + q[F'(\cdot)\left(1 - \gamma - \frac{d\gamma}{dn}n\right) - c + b(n) + nb^{'}(n)] = D^{'}(n(1+b))\{1 + b(n) + nb^{'}(n)\}. \text{(B.26)}$$

We use the log-quadratic functional form assumptions as in the closed-form solutions of the unregulated competitive equilibrium in Section B.1 and constrained planner's problem in Section B.2. First, note that substituting for P using (B.15) into γ, given by (4), we get

$$\gamma = 1 + \frac{b(n) - c}{P} = 1 + \frac{b(n) - c}{R + (b(n) - c)n},$$

Using this equivalence, we can obtain the total derivative of γ with respect to n as:

$$\begin{aligned} \frac{d\gamma}{dn} &= \frac{\partial \gamma}{\partial b}b'(n) + \frac{\partial \gamma}{\partial n} \\ &= \frac{P - (b(n) - c)n}{P^2}b'(n) - \frac{(b(n) - c)^2}{P^2} \\ &= \frac{b'(n)}{P} - \frac{nb'(n)(b(n) - c)}{P^2} - \frac{(b(n) - c)^2}{P^2}. \end{aligned} \quad \text{(B.27)}$$

Replacing $d\gamma/dn$ in the first-order condition (B.26) with (B.27) and rearranging yields

$$\begin{aligned} &(1-q)\{R + b(n)\} + qR\left(1 + \frac{b(n) - c}{P} - \frac{n(b(n) - c)^2}{P^2}\right) + nb^{'}(n)\left\{1 - q + \frac{qR}{P} - D'(\cdot) - \frac{qR(b(n) - c)n}{P^2}\right\} \\ &+q\left[\left(-\frac{b(n) - c}{P} - \frac{b'(n)n}{P} + \frac{n^2b'(n)(b(n) - c)}{P^2} + \frac{n(b(n) - c)^2}{P^2}\right)P + (b(n) - c) + nb'(n)\right] \\ &-D^{'}(\cdot)\{1 + b(n)\} = 0, \end{aligned}$$

where we replace $F'((1-\gamma)n) = P$ using the market clearing condition (B.15) in the second line. We have that $1 - q + qR/P - D'(\cdot) = 0$ from the banks' first-order condition (B.20).

Hence, the first-order condition above can further be simplified as follows:

$$R-1+q-\frac{qR^2(1+c)}{P^2}-\frac{qRn(b(n)-c)(1+b(n))}{P^2}-\frac{qRb'(n)n^2(b(n)-c)}{P^2}$$
$$+\frac{qn(b(n)-c)^2P}{P^2}+\frac{qb'(n)n^2(b(n)-c)P}{P^2}=0.$$

Divide the last equation by qR to obtain

$$\frac{R-1+q}{qR}-\frac{R(1+c)}{P^2}-\frac{n(b(n)-c)(1+b(n))}{P^2}-\frac{b'(n)n^2(b(n)-c)}{P^2}$$
$$+\frac{n(b(n)-c)^2P}{RP^2}+\frac{b'(n)n^2(b(n)-c)P}{RP^2}=0.$$

Let us define

$$\sigma \equiv \frac{R-1+q}{qR}. \tag{B.28}$$

Using this definition, we can write this first-order condition as

$$\frac{1}{P^2}\left\{\sigma P^2-R(1+c)-n(b(n)-c)(1+b(n))-b^{'}(n)(b(n)-c)n^2\right\}$$
$$+\frac{1}{P^2}\left\{\frac{(b-c)^2nP}{R}+b^{'}(n)\frac{n^2(b-c)P}{R}\right\}=0. \tag{B.29}$$

We focus on the terms inside the braces because in equilibrium price must be strictly positive. Using this term, we would like to write endogenous variables n and b in terms of the parameters of the model and P, and then, use these expression in the first-order conditions of the banks' problem (B.20) to obtain a closed-form solution for P. For that, first, below we obtain $1+b(n)$, $n(b(n)-c)$ and $b'(n)$ in terms of the parameters of the model and P starting from the banks' first-order condition (B.20):

$$\begin{aligned}(1-q)+q\frac{R}{P} &= 1+2\phi n(1+b), \qquad \text{(B.30)}\\ q(R-P) &= P2\phi n(1+b),\\ q(R-P) &= [R+(b-c)n]2\phi n(1+b),\\ -q(b-c)n &= 2\phi n(1+b)R+2\phi n(1+b)(b-c)n,\\ -(b-c)[q+2\phi n(1+b)] &= 2\phi(1+b)R,\end{aligned}$$

where we substitute for $P=R+(b-c)n$ using (B.15). Now, take the derivative of both sides with respect to n, and collect terms that involve $b^{'}(n)$:

$$\begin{aligned}-b'(n)[q+2\phi n(1+b)]-2\phi(b-c)[1+b+nb'(n)] &= 2\phi Rb'(n),\\ -b'(n)[q+2\phi n(1+b)]-2\phi(b-c)(1+b)-2\phi(b-c)nb'(n) &= 2\phi Rb'(n),\\ -b'(n)[2\phi R+2\phi n(b-c)+q+2\phi n(1+b)] &= 2\phi(b-c)(1+b),\\ -b'(n)[2\phi R+q+2\phi n(2b+1-c)] &= 2\phi(b-c)(1+b).\end{aligned}$$

From the last equation we obtain:

$$b'(n) = \frac{-2\phi(b-c)(1+b)}{2\phi R + q + 2\phi n(2b+1-c)}. \tag{B.31}$$

We further simplify $b'(n)$ in order to eliminate b from this expression. In order to do this simplification, note that first, from the market clearing condition at $t=1$, $P = R + (b-c)n$, as derived in (B.15), we can obtain that

$$b - c = -\frac{R-P}{n}.$$

Second, from the banks' first-order condition, given by (B.30), we can obtain that

$$1 + b \quad = \quad \frac{q}{2\phi n}\left(\frac{R}{P} - 1\right).$$

Use these values for $1+b$ and $b-c$ into (B.31) to write $b'(n)$ as a function of n, P and the parameters of the model as follows

$$\begin{aligned}
b'(n) &= \frac{-2\phi(-1)\frac{R-P}{n}\frac{q}{2\phi n}\left(\frac{R}{P}-1\right)}{2\phi R + q - 2\phi(R-P) + 2\phi\frac{q}{2\phi}\left(\frac{R}{P}-1\right)}, \\
&= \frac{\frac{q}{n^2 P}(R-P)^2}{\frac{1}{P}[2\phi RP + qP - 2\phi P(R-P) + q(R-P)]}, \\
&= \frac{q(R-P)^2}{n^2[2\phi RP + qP - 2\phi RP + 2\phi P^2 + qR - qP]}, \\
&= \frac{q(R-P)^2}{n^2[2\phi P^2 + qR]}.
\end{aligned}$$

Eventually, use the expressions obtained for $1+b(n)$, $n(b(n)-c)$ and $b'(n)$ above to rewrite the term inside the braces in (B.29) as:

$$\sigma P^2 - R(1+c) + (R-P)\frac{q(R-P)}{2\phi Pn} + \frac{q(R-P)^2}{n^2[2\phi P^2 + qR]}\frac{R-P}{n}n^2 + \frac{P(R-P)^2}{nR} - \frac{q(R-P)^2}{n^2[2\phi P^2 + qR]}\frac{R-P}{nR}Pn^2 = 0$$

$$\sigma P^2 - R(1+c) + \frac{q(R-P)^2}{n}\left[\frac{1}{2\phi P} + \frac{R-P}{2\phi P^2 + qR}\right] + \frac{(R-P)^2 P}{nR}\frac{2\phi P^2 + qP}{2\phi P^2 + qR} = 0$$

Note that the last equation takes the form of $A + B/n + C/n = 0$ where A, B, C group relevant terms. Therefore, we can obtain n in the form of $n = -B/A - C/A$, that is, from the last equation we can obtain n in terms of P and the parameters of the model:

$$n = \frac{q(R-P)^2\left[\frac{1}{2\phi P} + \frac{R-P}{2\phi P^2 + qR}\right]}{R(1+c) - \sigma P^2} + \frac{(R-P)^2\frac{(2\phi P^2 + qP)P}{(2\phi P^2 + qR)R}}{R(1+c) - \sigma P^2} \equiv \psi_1(P) + \psi_2(P).$$

We can similarly obtain an expression for b in terms of P and the parameters of the model using the equilibrium price function $P = R + (b-c)n$, which implies that

$$b = \frac{P-R}{n} + c = \frac{P-R+cn}{n} = \frac{P-R+c[\psi_1(P)+\psi_2(P)]}{\psi_1(P)+\psi_2(P)}.$$

Now, substitute these expressions for n and b back into the banks' first-order condition (B.30) in order to obtain a fixed-point equation that involves only P as an endogenous variable, from which we can solve for the equilibrium price P:

$$\begin{aligned}
2\phi n(1+b) &= -q + \frac{qR}{P}, \\
2\phi[\psi_1(P)+\psi_2(P)]\left[\frac{P-R+c[\psi_1(P)+\psi_2(P)]}{\psi_1(P)+\psi_2(P)} + 1\right] + q &= \frac{qR}{P} \\
2\phi P\{P-R+(1+c)[\psi_1(P)+\psi_2(P)]\} + qP &= qR \\
-2\phi P(R-P) + 2\phi P(1+c)[\psi_1(P)+\psi_2(P)]\} &= q(R-P) \\
2\phi(1+c)P[\psi_1(P)+\psi_2(P)] &= (2\phi P + q)(R-P) \\
2\phi(1+c)P(R-P)^2\left\{q\left[\frac{1}{2\phi P} + \frac{R-P}{2\phi P^2+qR}\right] + \frac{P(2\phi P^2+qP)}{R(2\phi P^2+qR)}\right\} &= [R(1+c)-\sigma P^2](R-P)(2\phi P+q) \\
2\phi(1+c)P(R-P)\left\{q\frac{R(2\phi PR+qR)}{(2\phi P^2+qR)R} + \frac{2\phi P^2(2\phi P^2+qP)}{2\phi PR(2\phi P^2+qR)}\right\} &= [R(1+c)-\sigma P^2](2\phi P+q)
\end{aligned}$$

Now, we sum the terms in side the braces on the left-hand side and multiply both sides with the common denominator of the left-hand side after summation and simplify further to get:

$$\begin{aligned}
2\phi(1+c)P(R-P)\{qR(2\phi PR+qR)+2\phi P^2(2\phi P^2+qP)\} &= [R(1+c)-\sigma P^2](2\phi P+q)2\phi PR(2\phi P^2+qR) \\
(1+c)(R-P)\{qR^2(2\phi P+q)+2\phi P^3(2\phi P+q)\} &= [R(1+c)-\sigma P^2](2\phi P+q)R(2\phi P^2+qR) \\
(1+c)(R-P)(2\phi P+q)(qR^2+2\phi P^3) &= [R(1+c)-\sigma P^2](2\phi P+q)R(2\phi P^2+qR) \\
(1+c)(R-P)(qR^2+2\phi P^3) &= [R(1+c)-\sigma P^2]R(2\phi P^2+qR) \\
(1+c)R(qR^2+2\phi P^3)-(1+c)P(qR^2+2\phi P^3) &= R^2(1+c)(2\phi P^2+qR)-\sigma P^2R(2\phi P^2+qR) \\
(1+c)R2\phi P^2-(1+c)qR^2-(1+c)2\phi P^3 &= R^2(1+c)2\phi P-\sigma R2\phi P^3-\sigma PRqR
\end{aligned}$$

We can rearrange this last equation to obtain a cubic equation in terms of the partial equilibrium price:

$$2\phi(\sigma R-1-c)P^3 + 2\phi R(1+c)P^2 + R^2(\sigma q - 2\phi(1+c))P - (1+c)qR^2 \quad = \quad 0.$$

Define

$$\beta \equiv R(1+c). \tag{B.32}$$

Replacing β for $R(1+c)$ we can also write the cubic equation for the partial regulation price as follows:

$$2\phi(\sigma R-1-c)P^{*3} + 2\phi\beta P^{*2} + R(\sigma qR - 2\phi\beta)P^* - qR\beta = 0 \tag{B.33}$$

It is easy to show that this cubic equation has only one real root and two complex conjugate roots. The only real root can easily be obtained using Vieta's substitution for cubic equations.