Existence and uniqueness of solutions to dynamic models with occasionally binding constraints.

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Abstract: We present the first necessary and sufficient conditions for there to be a unique perfect-foresight solution to an otherwise linear dynamic model with occasionally binding constraints, given a fixed terminal condition. We derive further results on the existence of a solution in the presence of such terminal conditions. These results give determinacy conditions for models with occasionally binding constraints, much as Blanchard and Kahn (1980) did for linear models. In an application, we show that widely used New Keynesian models with endogenous states possess multiple perfect foresight equilibrium paths when there is a zero lower bound on nominal interest rates, even when agents believe that the central bank will eventually attain its long-run, positive inflation target. This illustrates that a credible long-run inflation target does not render the Taylor principle sufficient for determinacy in the presence of the zero lower bound. However, we show that price level targeting does restore determinacy providing agents believe that inflation will eventually be positive.

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https://github.com/tholden/dynareOBC/raw/master/TheoryPaper.pdf

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1. Introduction

Models with occasionally binding constraints are ubiquitous in modern macroeconomics, yet the profession has few theoretical tools for understanding the types of multiplicity they support. This is particularly relevant for the conduct of monetary policy, as the theoretical results on determinacy that justify the Taylor principle do not apply in models with occasionally binding constraints (OBCs), such as the zero lower bound (ZLB) on nominal interest rates. Since hitting the ZLB is associated with particularly poor economic outcomes, large welfare gains are available if equilibria featuring jumps to the ZLB can be ruled out.

In this paper, we develop theoretical tools for understanding the behaviour of otherwise linear models with occasionally binding constraints. We will provide the first necessary and sufficient conditions for there always to be a unique perfect foresight solution, returning to a given steady-state, in an otherwise linear model with occasionally binding constraints. These generalise the seminal results of Blanchard & Kahn (1980) for the linear case. Moreover, we show that these conditions are not satisfied by standard New Keynesian (NK) models.

Furthermore, we will provide both necessary conditions and sufficient conditions for there to always exist a perfect foresight solution, returning to a given steady-state, to an otherwise linear model with occasionally binding constraints. We also give existence conditions that are conditional on the economy's initial state. When no solution exists in some states, as we show to be the case in standard NK models, this implies that the model must converge to some alternative steady-state, if it has a solution at all. We note that while in the fully linear case, rational expectations and perfect-foresight solutions coincide, in the otherwise linear case considered here, this will not be the case. However, since under mild assumptions there are weakly more solutions under rational expectations than under perfect foresight,[2] our results imply lower bounds on the number of solutions under rational expectations.

As was observed by Benhabib, Schmitt-Grohé & Uribe (2001a; 2001b), in the presence of OBCs, there are often multiple steady-states. For example, a model with a ZLB on nominal interest rates and Taylor rule monetary policy when away from the bound will have an additional deflationary steady-state in which nominal interest rates are zero. Such multiple steady-states can sustain multiple equilibria and sunspots if agents are switching beliefs about the point to which the economy would converge without future uncertainty.[3]

However, the central banks of most major economies have announced (positive) inflation targets. Thus, convergence to a deflationary steady-state would represent a spectacular failure to hit the target. As argued by Christiano and Eichenbaum (2012), a central bank may rule out the deflationary equilibria in practice by switching to a money growth rule following severe

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[2] This is proven in Appendix E, online.

[3] The consequences of indeterminacy of this kind has been explored by Schmitt-Grohé & Uribe (2012), Mertens & Ravn (2014) and Aruoba, Cuba-Borda & Schorfheide (2014), amongst others.

deflation, along the lines of Christiano & Rostagno (2001). Furthermore, Richter & Throckmorton (2015) and Gavin et al. (2015) present evidence that the deflationary equilibrium is unstable[4] under rational expectations if shocks are large enough, making it much harder for agents to coordinate upon it. This suggests agents should believe that inflation will eventually return to the vicinity of its target, and they ought to place zero probability on paths converging to deflation. Such beliefs appear to be in line with the empirical evidence of Gürkaynak, Levin & Swanson (2010). If agents' beliefs satisfy these restrictions, then the kind of multiplicity discussed in the previous paragraph is ruled out. It is an important question, then, whether there are still multiple equilibria even when all agents believe that in the long-run, the economy will return to a particular steady-state. It is on such equilibria that we focus in this paper.

To understand why multiplicity is still possible even with the terminal condition fixed, suppose that somehow the model's agents knew that from next period onwards, the economy would be away from the bound. Then, in an otherwise linear model, expectations of next period's outcomes would be linear in today's variables. However, substituting out these expectations does not leave a linear system in today's variables, due to the occasionally binding constraint. For some models, this non-linear system will have two solutions, with one featuring a slack constraint, and the other having a binding constraint. Thus, even though the rule for forming expectations is pinned down, there may still be multiple possible outcomes today. Without the assumption that next period the economy is away from the bound, the scope for multiplicity is even richer, and there may be infinitely many solutions.

In an application, we show that multiplicity of perfect-foresight paths is the rule in otherwise linear New Keynesian models with endogenous state variables (e.g. price dispersion) and a ZLB. This means that even when agents' long-run expectations are pinned down, there is still multiplicity of equilibria, implying the Taylor principle is not sufficient for determinacy in the presence of occasionally binding constrains. Indeed, in these models, there are conditions under which the economy has one return path that never hits the ZLB, and another that does, so there may be multiplicity even when away from the bound. However, we show that under a price-targeting regime, there is a unique equilibrium path even when we impose the ZLB. Thus, if the standard arguments for the Taylor principle convince policy makers, then, given they face the ZLB, they ought to consider adopting a price level target.

We also show that for standard NK models with endogenous state variables, there is a positive probability of ending up in a state of the world (i.e. with certain state variables and shock realisations) in which there is no perfect foresight path returning to the "good" steady-state. Hence, if we suppose that in the stochastic model, agents deal with uncertainty by integrating over the space of possible future shock sequences, as in the original stochastic

[4] They show that policy function iteration is not stable near the deflationary equilibria.

extended path algorithm of Adjemian & Juillard (2013),[5] then such agents would always put positive probability on tending to the "bad" steady-state. Since the second steady-state is indeterminate in NK models, this implies global indeterminacy by a backwards induction argument. Once again though, price level targeting would be sufficient to restore determinacy.

The most relevant prior work for ours is that of Brendon, Paustian & Yates (2013; 2016), henceforth abbreviated to BPY. Like us, these authors examined perfect foresight equilibria of NK models with terminal conditions. In BPY (2013), the authors show analytically that in a very simple NK model, featuring a response to the growth rate in the Taylor rule, there are multiple perfect-foresight equilibria when all agents believe that with probability one, in one period's time, they will escape the bound and return to the neighbourhood of the "good" steady-state. Furthermore, in BPY (2013; 2016), the authors show numerically that in some select other models, there are multiple perfect-foresight equilibria when the economy begins at the steady-state, and all agents believe that the economy will jump to the bound, remain there for some number of periods, before leaving it endogenously, after which they believe they will never hit the bound again.

Relative to these authors, we will provide far more general theoretical results, and these will permit numerical analysis that is both more robust and less restrictive. This robustness and generality will prove crucial in showing multiplicity even in simple NK models, with entirely standard Taylor rules. For example, whereas BPY (2016) write that price-dispersion "does not have a strong enough impact on equilibrium allocations for the sort of propagation that we need", we show that the presence of price dispersion is sufficient for multiplicity. Likewise, whereas BPY (2013; 2016) find a much weaker role for multiplicity when the monetary rule does not include a response to the growth rate of output, our findings of multiplicity will not be at all dependent on such a response, implying very different policy prescriptions.

Further relevant papers are discussed in Section 6, in the course of examining our results. The rest of our paper is structured as follows. In the following section, Section 2, we present the key representation result which enables us to examine existence and uniqueness in models with OBCs and terminal conditions via examining the properties of linear complementarity problems. Section 3 introduces a simple example New Keynesian model with multiplicity, and provides further intuition. Next, Section 4 presents our main results on existence and uniqueness. We then apply these results to further NK models in Section 5. Section 7 concludes. All files needed for the replication of this paper's numerical results are included in the author's DynareOBC toolkit, which implements an algorithm for simulating models with occasionally binding constraints that we discuss in a companion paper (Holden 2016), as well as checking the existence and uniqueness conditions that we will discuss here.

[5] Strictly, this is not fully rational, as it is equivalent to assuming that agents act as if the uncertainty in all future periods would be resolved next period. However, in practice this appears to be a close approximation to full rationality, as demonstrated by Holden (2016). The authors of the original stochastic path method now have a more complicated version that is fully consistent with rationality (Adjemian & Juillard 2016).

2. Representation result

In this section, we present the representation result that establishes an equivalence between solutions of a DSGE model with occasionally binding constraints, and solutions of a linear complementarity problem. The key idea behind the result's proof is that an OBC provides a source of endogenous news about the future. If a shock hits, driving the economy to the bound in some future periods, then, in those future periods, the (lower) bounded variable will be higher that it would be otherwise.[6] Hence, any shock that causes the ZLB to be hit may be thought of as providing a source of endogenous news about future innovations to the monetary rule, of just the right magnitude needed to impose the ZLB. For example, if the size of a productivity shock is such that in the absence of the ZLB, nominal interest rates would be negative a year after the original shock, then, in the presence of the ZLB, the shock is providing endogenous news that nominal interest rates will be higher than otherwise in a year's time.

2.1. Problem set-ups

We start by defining the problem to be solved, and examining its relationship both to the problem without OBCs, and to a related problem with news shocks to the bounded variable.

In the absence of occasionally binding constraints, calculating an impulse response or performing a perfect foresight simulation exercise in a linear DSGE model is equivalent to solving the following problem:[7]

Problem 1 (Linear) Suppose that $x_0 \in \mathbb{R}^n$ is given. Find $x_t \in \mathbb{R}^n$ for $t \in \mathbb{N}^+$ such that $x_t \to \mu$ as $t \to \infty$, and such that for all $t \in \mathbb{N}^+$:

$$(A + B + C)\mu = Ax_{t-1} + Bx_t + Cx_{t+1}, \tag{1}$$

Throughout this paper, we will refer to equilibrium conditions such as equation (1) as "the model", conflating them with the optimisation problem(s) which gave rise to them. We make the following assumption in all the following:

Assumption 1 For any given $x_0 \in \mathbb{R}^n$, Problem 1 (Linear) has a unique solution, which (without loss of generality) takes the form $x_t = (I - F)\mu + Fx_{t-1}$, for $t \in \mathbb{N}^+$, where $0 = A + BF + CFF$, and where the eigenvalues of F are strictly inside the unit circle.

[6] The idea of imposing an OBC by adding news shocks is also present in Holden (2010), Hebden et al. (2011), Holden & Paetz (2012) and Bodenstein et al. (2013). Laséen & Svensson (2011) use a similar technique to impose a path of nominal interest rates, in a non-ZLB context. None of these papers formally establish our representation result. News shocks were introduced by Beaudry & Portier (2006).

[7] The absence of shocks and expectations here is without loss of generality. For suppose $(\hat{A} + \hat{B} + \hat{C})\hat{\mu} = \hat{A}\hat{x}_{t-1} + \hat{B}\hat{x}_t + \hat{C}\mathbb{E}_t\hat{x}_{t+1} + \widehat{D}\varepsilon_t$, with $\hat{x}_t \to \hat{\mu}$ as $t \to \infty$, and that $\varepsilon_t = 0$ for $t > 1$, as in an impulse response or perfect foresight simulation exercise. Then, if we define $x_t := \begin{bmatrix} \hat{x}_t \\ \varepsilon_{t+1} \end{bmatrix}$, $\mu := \begin{bmatrix} \hat{\mu} \\ 0 \end{bmatrix}$, $A := \begin{bmatrix} \hat{A} & \widehat{D} \\ 0 & 0 \end{bmatrix}$, $B := \begin{bmatrix} \hat{B} & 0 \\ 0 & I \end{bmatrix}$, $C := \begin{bmatrix} \hat{C} & 0 \\ 0 & 0 \end{bmatrix}$, then we are left with a problem in the form of Problem 1 (Linear), with the extended initial condition $x_0 = \begin{bmatrix} \hat{x}_0 \\ \varepsilon_1 \end{bmatrix}$, and the extended terminal condition $x_t \to \mu$ as $t \to \infty$. Expectations disappear as there is no uncertainty after period 0.

Conditions (A′) and (B) from Sims's (2002) generalisation of the standard Blanchard-Kahn (1980) conditions are necessary and sufficient for Assumption 1 to hold. Further, to avoid dealing specially with the knife-edge case of exact unit eigenvalues in the part of the model that is solved forward, here we rule it out with the subsequent assumption, which is, in any case, a necessary condition for perturbation to produce a consistent approximation to a non-linear model, and which is also necessary for the linear model to have a unique steady-state:

Assumption 2 $\det(A + B + C) \neq 0$.

We are interested in models featuring occasionally binding constraints. We will concentrate on models featuring a single ZLB type constraint in their first equation, which does not bind in steady-state, and which we treat as defining the first element of x_t. Generalising from this special case to models with one or more fully general bounds is straightforward, and is discussed in Appendix A. First, let us write $x_{1,t}$, $I_{1,\cdot}$, $A_{1,\cdot}$, $B_{1,\cdot}$, $C_{1,\cdot}$ for the first row of x_t, I, A, B, C (respectively) and $x_{-1,t}$, $I_{-1,\cdot}$, $A_{-1,\cdot}$, $B_{-1,\cdot}$, $C_{-1,\cdot}$ for the remainders. Likewise, we write $I_{\cdot,1}$ for the first column of I, and so on. Then, we are interested in:

Problem 2 (OBC) Suppose that $x_0 \in \mathbb{R}^n$ is given. Find $T \in \mathbb{N}$ and $x_t \in \mathbb{R}^n$ for $t \in \mathbb{N}^+$ such that $x_t \to \mu$ as $t \to \infty$, such that for all $t \in \mathbb{N}^+$:

$$x_{1,t} = \max\{0, I_{1,\cdot}\mu + A_{1,\cdot}(x_{t-1} - \mu) + (B_{1,\cdot} + I_{1,\cdot})(x_t - \mu) + C_{1,\cdot}(x_{t+1} - \mu)\},$$
$$(A_{-1,\cdot} + B_{-1,\cdot} + C_{-1,\cdot})\mu = A_{-1,\cdot}x_{t-1} + B_{-1,\cdot}x_t + C_{-1,\cdot}x_{t+1},$$

and such that $x_{1,t} > 0$ for $t > T$.

given:

Assumption 3 $\mu_1 > 0$.

Were it not for the max, this problem would be identical to Problem 1 (Linear), providing that Assumption 3 holds, as the existence of a $T \in \mathbb{N}$ such that $x_{1,t} > 0$ for $t > T$ is guaranteed by the fact that $x_{1,t} \to \mu_1$ as $t \to \infty$. In this problem, we are implicitly ruling out any solutions which get permanently stuck at an alternative steady-state, by assuming that the terminal condition remains as before. In the monetary policy context, this amounts to assuming that the central banks' long-run inflation target is credible. As we are assuming there is no uncertainty, the path of the endogenous variables will not necessarily match up with the path of their expectation in a richer model in which there was uncertainty, due to the non-linearity.

In many models, the occasionally binding constraint comes from the KKT conditions of an optimisation problem, which take the form $z_t \geq 0$, $\lambda_t \geq 0$ and $z_t\lambda_t = 0$. These may be converted into the max/min form since they are equivalent to the single equation $0 = \min\{z_t, \lambda_t\}$, which holds if and only if $z_t = \max\{0, z_t - \lambda_t\}$, an equation in the form of Problem 2 (OBC). Additionally, in Appendix D, we give an alternative procedure for converting KKT conditions into the form of Problem 2 (OBC), based on finding a "shadow" value of the constrained variable.

We will analyse Problem 2 (OBC) with the help of solutions to the auxiliary problem:

Problem 3 (News) Suppose that $T \in \mathbb{N}$, $x_0 \in \mathbb{R}^n$ and $y_0 \in \mathbb{R}^T$ is given. Find $x_t \in \mathbb{R}^n, y_t \in \mathbb{R}^T$ for $t \in \mathbb{N}^+$ such that $x_t \to \mu$, $y_t \to 0$, as $t \to \infty$, and such that for all $t \in \mathbb{N}^+$:

$$(A + B + C)\mu = Ax_{t-1} + Bx_t + Cx_{t+1} + I_{\cdot,1}y_{1,t-1},$$
$$y_{T,t} = 0, \qquad \forall i \in \{1, \dots, T-1\},\ y_{i,t} = y_{i+1,t-1}.$$

This may be thought of as a version of Problem 1 (Linear) with news shocks up to horizon T added to the first equation. By construction, the value of $y_{i,t}$ gives the shock that in period t is expected to arrive in i periods. Hence, as all information is known in period 0, $y_{t,0}$ gives the shock that will hit in period t, i.e. $y_{1,t-1} = y_{t,0}$ for $t \le T$, and $y_{1,t-1} = 0$ for $t > T$.

2.2. Relationships between the problems

A straightforward backwards induction argument (given in Appendix H.1, online) gives the following helpful result:

Lemma 1 There is a unique solution to Problem 3 (News) that is linear in x_0 and y_0.

For future reference, let $x_t^{(3,k)}$ be the solution to Problem 3 (News) when $x_0 = \mu$, $y_0 = I_{\cdot,k}$ (i.e. a vector which is all zeros apart from a 1 in position k). Then, by linearity, for arbitrary y_0 the solution to Problem 3 (News) when $x_0 = \mu$ is given by:

$$x_t - \mu = \sum_{k=1}^{T} y_{k,0}\left(x_t^{(3,k)} - \mu\right).$$

Now, let $M \in \mathbb{R}^{T\times T}$ satisfy:

$$M_{t,k} = x_{1,t}^{(3,k)} - \mu_1, \qquad \forall t,k \in \{1,..,T\}, \tag{2}$$

i.e. M horizontally stacks the (column-vector) relative impulse responses of the first variable to the news shocks, with the first column giving the response to a contemporaneous shock, the second column giving the response to a shock anticipated by one period, and so on. Then, this result implies that for arbitrary x_0 and y_0, the path of the first variable in the solution to Problem 3 (News) is given by:

$$(x_{1,1:T})' = q + My_0, \tag{3}$$

where $q := \left(x_{1,1:T}^{(1)}\right)'$ and where $x_t^{(1)}$ is the unique solution to Problem 1 (Linear), for the given x_0, i.e. q is the path of the first variable in the absence of news shocks or bounds. [8] Since M is not a function of either x_0 or y_0, equation (3) gives a highly convenient representation of the solution to Problem 3 (News).

Now let $x_t^{(2)}$ be a solution to Problem 2 (OBC) given some x_0. Since $x_t^{(2)} \to \mu$ as $t \to \infty$, there exists $T' \in \mathbb{N}$ such that for all $t > T'$, $x_{1,t}^{(2)} > 0$. We assume without loss of generality that $T' \le T$. We seek to relate the solution to Problem 2 (OBC) with the one to Problem 3 (News) for an appropriate choice of y_0. First, for all $t \in \mathbb{N}^+$, let:

$$\hat{e}_t := -\left[I_{1,\cdot}\mu + A_{1,\cdot}\left(x_{t-1}^{(2)} - \mu\right) + \left(B_{1,\cdot} + I_{1,\cdot}\right)\left(x_t^{(2)} - \mu\right) + C_{1,\cdot}\left(x_{t+1}^{(2)} - \mu\right)\right],$$

[8] This representation was also exploited by Holden (2010) and Holden and Paetz (2012).

$$e_t := \begin{cases} \hat{e}_t & \text{if } x_{1,t}^{(2)} = 0 \\ 0 & \text{if } x_{1,t}^{(2)} > 0 \end{cases}, \tag{4}$$

i.e. e_t is the shock that would need to hit the first equation for the positivity constraint on $x_{1,t}^{(2)}$ to be enforced. Note that by the definition of Problem 2 (OBC), $e_t \geq 0$ and $x_{1,t}^{(2)} e_t = 0$, for all $t \in \mathbb{N}^+$. Then, again from a backwards induction argument, (in Appendix H.2, online) we have:

Lemma 2 For any solution, $\left(T, x_t^{(2)}\right)$ to Problem 2 (OBC):

1) With $e_{1:T}$ as defined in equation (4), $e_{1:T} \geq 0$, $x_{1,1:T}^{(2)} \geq 0$ and $x_{1,1:T}^{(2)} \circ e_{1:T} = 0$, where $\circ$ denotes the Hadamard (entry-wise) product.
2) $x_t^{(2)}$ is also the unique solution to Problem 3 (News) with $x_0 = x_0^{(2)}$ and $y_0 = e_{1:T}'$.
3) If $x_t^{(2)}$ solves Problem 3 (News) with $x_0 = x_0^{(2)}$ and with some y_0, then $y_0 = e_{1:T}'$.

To use the easy solution to Problem 3 (News) to assist us in solving Problem 2 (OBC) just requires one more result. In particular, we show in Appendix H.3, online, that if $y_0 \in \mathbb{R}^T$ is such that $y_0 \geq 0$, $x_{1,1:T}^{(3)} \circ y_0' = 0$ and $x_{1,t}^{(3)} \geq 0$ for all $t \in \mathbb{N}$, where $x_t^{(3)}$ is the unique solution to Problem 3 (News) when started at x_0, y_0, then $x_t^{(3)}$ must also be a solution to Problem 2 (OBC). Together with Lemma 1, Lemma 2, and our representation of the solution of Problem 3 (News) from equation (3), this completes the proof of the following key theorem:

Theorem 1 The following hold:

1) Let $x_t^{(3)}$ be the unique solution to Problem 3 (News) given $T \in \mathbb{N}^+$, $x_0 \in \mathbb{R}^n$ and $y_0 \in \mathbb{R}^T$. Then $\left(T, x_t^{(3)}\right)$ is a solution to Problem 2 (OBC) given x_0 if and only if $y_0 \geq 0$, $y_0 \circ (q + My_0) = 0$, $q + My_0 \geq 0$ and $x_{1,t}^{(3)} \geq 0$ for all $t > T$.
2) Let $\left(T, x_t^{(2)}\right)$ be any solution to Problem 2 (OBC) given x_0. Then there exists a unique $y_0 \in \mathbb{R}^T$ such that $y_0 \geq 0$, $y_0 \circ (q + My_0) = 0$, $q + My_0 \geq 0$, and such that $x_t^{(2)}$ is the unique solution to Problem 3 (News) given T, x_0 and y_0.

2.3. The linear complementarity representation

Theorem 1 establishes that we may solve Problem 2 (OBC) by conjecturing a (sufficiently high) value for T and then solving the following problem:

Problem 4 (LCP) Suppose $T \in \mathbb{N}^+$, $q \in \mathbb{R}^T$ and $M \in \mathbb{R}^{T \times T}$ are given. Find $y \in \mathbb{R}^T$ such that $y \geq 0$, $y \circ (q + My) = 0$ and $q + My \geq 0$. We call this the **linear complementarity problem (LCP)** (q, M). (Cottle 2009)

These problems have been extensively studied, and so we can import results on the properties of LCPs to derive results on the properties of models with OBCs.

All the results in the literature on LCPs rest on properties of the matrix M, thus we would like to establish if the structure of our particular M implies it has any special properties. Unfortunately, we prove the following result in Appendix H.4, online, which means that M's origin implies no particular properties:

Proposition 1 For any $T \in \mathbb{N}^+$ and $\mathcal{M} \in \mathbb{R}^{T \times T}$, there exists a model in the form of Problem 2 (OBC) with a number of state variables given by a quadratic in T, such that $M = \mathcal{M}$ for that model, where M is defined as in equation (2), and such that for all $\mathcal{q} \in \mathbb{R}^T$, there exists an initial state for which $q = \mathcal{q}$, where q is the path of the bounded variable when constraints are ignored. (Holden 2016)

3. Intuition building results and examples

3.1. LCPs of size 1

When $T = 1$, it is particularly easy to characterise the properties of LCPs. This amounts to considering the behaviour of an economy in which everyone believes there will be at most one period at the bound. In this case, y gives the "shock" to the bounded equation necessary to impose the bound, and M gives the contemporaneous response of the bounded variable to an unanticipated shock: i.e. in a ZLB context, M gives the initial jump in nominal interest rates following a standard monetary policy shock.

First, suppose that M (a scalar as $T = 1$ for now) is positive. Then, if $q > 0$, for any $y \geq 0$, $q + My > 0$, so by the complementary slackness condition, in fact $y = 0$. Conversely, if $q \leq 0$, then there is a unique y satisfying the complementary slackness condition given by $y = -\frac{q}{M} \geq 0$. Thus, with $M > 0$, there is always a unique solution to the $T = 1$ LCP. With $M = 0$, $q + My = q$, so a solution to the LCP exists if and only if $q \geq 0$. It will be unique providing $q > 0$ (by the complementary slackness condition), but when $q = 0$, any $y \geq 0$ gives a solution. Finally, suppose that $M < 0$. Then, if $q > 0$, there are precisely two solutions. The "standard" solution has $y = 0$, but there is an additional solution featuring a jump to the bound in which $y = -\frac{q}{M} > 0$. If $q = 0$, then there is a unique solution ($y = 0$) and if $q < 0$, then with $y \geq 0$, $q + My < 0$, so there is no solution at all. Hence, the $T = 1$ LCP already provides examples of cases of uniqueness, non-existence and multiplicity.

3.2. The simple Brendon, Paustian & Yates (BPY) (2013) model

Brendon, Paustian & Yates (2013), henceforth BPY, provide a simple New Keynesian model that we can use to illustrate and better understand these cases. Its equations follow:

$$x_{i,t} = \max\{0, 1 - \beta + \alpha_{\Delta y}(x_{y,t} - x_{y,t-1}) + \alpha_\pi x_{\pi,t}\},$$
$$x_{y,t} = \mathbb{E}_t x_{y,t+1} - \frac{1}{\sigma}(x_{i,t} + \beta - 1 - \mathbb{E}_t x_{\pi,t+1}),$$
$$x_{\pi,t} = \beta \mathbb{E}_t x_{\pi,t+1} + \gamma x_{y,t},$$

where $x_{i,t}$ is the nominal interest rate, $x_{y,t}$ is the deviation of output from steady-state, $x_{\pi,t}$ is the deviation of inflation from steady-state, and $\beta \in (0,1)$, $\gamma, \sigma, \alpha_{\Delta y} \in (0, \infty)$, $\alpha_\pi \in (1, \infty)$ are parameters. The model's only departure from the textbook three equation NK model is the presence of an output growth rate term in the Taylor rule. This introduces an endogenous state variable in a tractable manner. In Appendix H.5, online, we prove the following:

Proposition 2 The BPY model is in the form of Problem 2 (OBC), and satisfies Assumptions 1, 2 and 3. With $T = 1$, $M < 0$ ($M = 0$) if and only if $\alpha_{\Delta y} > \sigma\alpha_{\pi}$ ($\alpha_{\Delta y} = \sigma\alpha_{\pi}$).

Hence, by Theorem 1, when all agents believe the bound will be escaped after at most one period, if $\alpha_{\Delta y} < \sigma\alpha_{\pi}$, the model has a unique solution for all q, i.e. no matter what the nominal interest rate would be that period were no ZLB. If $\alpha_{\Delta y} = \sigma\alpha_{\pi}$, then the model has a unique solution whenever $q > 0$, infinitely many solutions when $q = 0$, and no solutions leaving the ZLB after one period when $q < 0$. Finally, if $\alpha_{\Delta y} > \sigma\alpha_{\pi}$ then the model has two solutions when $q > 0$, one solution when $q = 0$ and no solution escaping the ZLB next period when $q < 0$.

The mechanism here is as follows. The stronger the response to the growth rate, the more persistent is output, as the monetary rule implies additional stimulus if output was high last period. Suppose then that there was an unexpected positive shock to nominal interest rates. Then, due to the persistence, this would lower not just output and inflation today, but also output and inflation next period. With low expected inflation, real interest rates are high, giving consumers an additional reason to save, and thus further lowering output and inflation this period and next. With sufficiently high $\alpha_{\Delta y}$, this additional amplification is so strong that nominal interest rates fall this period, despite the positive shock, explaining why M may be negative.[9] Now, consider varying the magnitude of the original shock. For a sufficiently large shock, interest rates would hit zero. At this point, there is no observable evidence that a shock has arrived at all, since the ZLB implies that given the values of output and inflation, nominal interest rates should be zero even without a shock. Such a jump to the ZLB must then be a self-fulfilling prophecy. Agents expect low inflation, so they save, which, thanks to the monetary rule, implies low output tomorrow, rationalising the expectations of low inflation.

3.3. LCPs of size 2 and further intuition

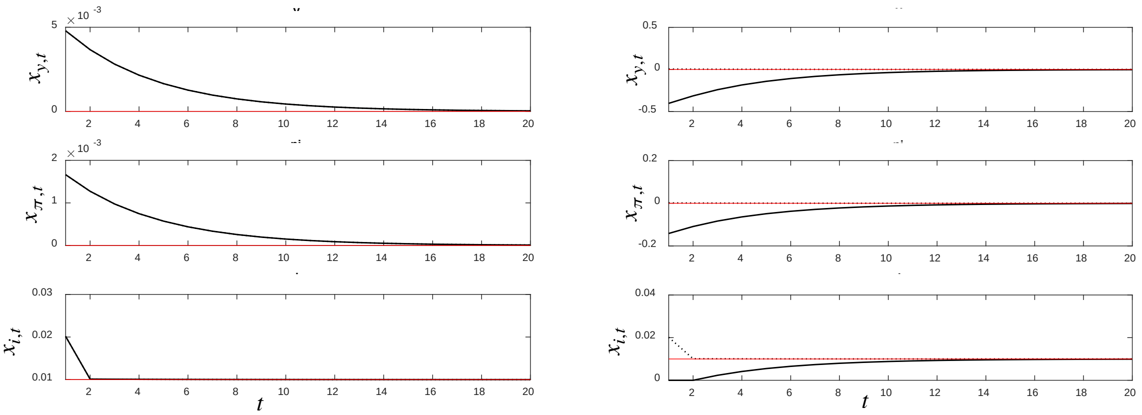
It is crucial for this mechanism that output is persistent enough that positive shocks to the monetary rule lower inflation expectations enough that nominal interest rates fall. While there may be other persistence mechanisms that could sustain this, it is arguable that any producing this result would be somewhat pathological. However, for multiplicity, we do not need that positive shocks to the bounded variable reduce its level. Instead, we only require that positive news shocks at several different horizons jointly lower the bounded variable by enough. We will illustrate this by considering the $T = 2$ special case, where we can again easily derive results from first principles.

[9] Note that this cannot happen without a response to growth rates, or some other endogenous state. For, without state variables, the period after the shock's arrival, inflation will be at steady state. Thus, in the period of the shock, real interest rates move one for one with nominal interest rates. Were the positive shock to the nominal interest rate to produce a fall in its level, then the Euler equation would imply high consumption today, also implying high inflation today via the Phillips curve. But, with consumption, inflation, and the shock all positive, the nominal interest rate must be above steady-state, contradicting our assumption that it had fallen.

Recall that a solution $\begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$ to the LCP $\left(\begin{bmatrix} q_1 \\ q_2 \end{bmatrix}, \begin{bmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{bmatrix}\right)$ satisfies $y_1 \geq 0$, $y_2 \geq 0$, $q_1 + M_{11}y_1 + M_{12}y_2 \geq 0$, $q_2 + M_{21}y_1 + M_{22}y_2 \geq 0$, $y_1(q_1 + M_{11}y_1 + M_{12}y_2) = 0$, and $y_2(q_2 + M_{21}y_1 + M_{22}y_2) = 0$. With two quadratics, there are up to four solutions, given by:

1) $y_1 = y_2 = 0$. Exists if $q_1 \geq 0$ and $q_2 \geq 0$.
2) $y_1 = -\frac{q_1}{M_{11}}$, $y_2 = 0$. Exists if $\frac{q_1}{M_{11}} \leq 0$ and $M_{11}q_2 \geq M_{21}q_1$.
3) $y_1 = 0$, $y_2 = -\frac{q_2}{M_{22}}$. Exists if $\frac{q_2}{M_{22}} \leq 0$ and $M_{22}q_1 \geq M_{12}q_2$.
4) $y_1 = \frac{M_{12}q_2 - M_{22}q_1}{M_{11}M_{22} - M_{12}M_{21}}$, $y_2 = \frac{M_{21}q_1 - M_{11}q_2}{M_{11}M_{22} - M_{12}M_{21}}$. Exists if $y_1 \geq 0$ and $y_2 \geq 0$.

So, there are multiple equilibria for at least some $q_1, q_2 \geq 0$ if and only if $M_{11} \leq 0$, $M_{22} \leq 0$ or $M_{11}M_{22} - M_{12}M_{21} \leq 0$.[10] Thus, for there to be solutions that jump to the bound, it is sufficient that $M_{12}M_{21}$ is large enough; we do not need positive shocks to have negative effects. Many different mechanisms can bring this about, as we will see when we consider further examples in Section 5.



Minimum $\|y\|_\infty$ solution **Minimum $\|q + My\|_\infty$ solution**

Figure 1: Alternative solutions following a magnitude 1 impulse to ε_t in the model of Section 3.4

3.4. An example of multiplicity

We finish this section with an example of multiplicity in the BPY (2013) model. This serves to illustrate the potential economic consequences of multiplicity in NK models. We present impulse responses to a shock to the Euler equation under two different solutions. With the shock added to the Euler equation, it now takes the form:

$$x_{y,t} = \mathbb{E}_t x_{y,t+1} - \frac{1}{\sigma}\left(x_{i,t} + \beta - 1 - \mathbb{E}_t x_{\pi,t+1} - (0.01)\varepsilon_t\right).$$

The rest of the BPY model's equations remain as they were given in Section 3.2. We take the parameterisation $\sigma = 1$, $\beta = 0.99$, $\gamma = \frac{(1-0.85)(1-\beta(0.85))}{0.85}(2 + \sigma)$, following BPY, and we additionally set $\alpha_\pi = 1.5$ and $\alpha_{\Delta y} = 1.6$, to ensure we are in the region with multiple solutions. In Figure 1, we show two alternative solutions to the impulse response to a magnitude 1 shock to ε_t. The solid line in the left plot gives the solution which minimises $\|y\|_\infty$. This solution never hits the bound, and is moderately expansionary. The solid line in the right plot gives the

[10] The last condition follows from 4) as were it the case that $M_{11}M_{22} > M_{12}M_{21}$, we would need $M_{12}q_2 \geq M_{22}q_1$ and $M_{21}q_1 \geq M_{11}q_2$ (to ensure $y_1 \geq 0$ and $y_2 \geq 0$). However, multiplying the last two inequalities gives $M_{12}M_{21}q_1q_2 \geq M_{11}M_{22}q_1q_2$, a contradiction.

solution which minimises $\|q + My\|_\infty$. (The dotted line there repeats the left plot, for comparison.) This solution stays at the bound for two periods, and is strongly contractionary, with a magnitude around 100 times larger than the other solution.

4. Existence and uniqueness results

We now turn to our main theoretical results on existence and uniqueness of perfect foresight solutions to models which are linear apart from an occasionally binding constraint. Further results are contained in the appendices, with Appendix D relating our results to the properties of models solvable via dynamic programming. We conclude this section with a practical guide to checking the existence and uniqueness conditions.

4.1. Relevant matrix properties

We start by giving definitions of the matrix properties that are required for the statement of our key existence and uniqueness results for $T > 2$. The properties of the solutions to the OBC model are determined by which of these matrix properties M possesses.[11]

Definition 1 (Principal sub-matrix, Principal minor) For a matrix $M \in \mathbb{R}^{T\times T}$, the **principal sub-matrices** of M are the matrices $\left[M_{i,j}\right]_{i,j=k_1,\dots,k_S}$, where $S, k_1, \dots, k_S \in \{1, \dots, T\}, k_1 < k_2 < \dots < k_S$, i.e. the **principal sub-matrices** of M are formed by deleting the same rows and columns. The **principal minors** of M are the determinants of M's principal sub-matrices.

Definition 2 (P(0)-matrix) A matrix $M \in \mathbb{R}^{T\times T}$ is called a **P-matrix (P_0-matrix)** if the principal minors of M are all strictly (weakly) positive. *Note: for symmetric M, M is a $P(_0)$-matrix if and only if it is positive (semi-)definite.*

Definition 3 (General positive (semi-)definite) A matrix $M \in \mathbb{R}^{T\times T}$ is called **general positive (semi-)definite** if $M + M'$ is positive (semi-)definite (p.(s.)d.).

For intuition on the relevance of these properties, recall that the definition of a linear complementarity problem (Problem 4 (LCP)) contained the complementary slackness type condition, $y \circ (q + My) = 0$. Equivalently then, $0 = y'(q + My) = y'q + y'My$. Now, if there is no multiplicity, $y'q$ is likely to be negative as the bound usually binds when q (the path in the absence of the bound) is negative. Thus, for the equation to be satisfied, $y'My = \frac{1}{2}y'(M + M')y$ should be positive, which certainly holds when M is general positive definite. More generally, y will usually have many zeros, since y is zero whenever the model is away from the bound. The remaining non-zero elements of y select a principal sub-matrix of M, which will be a P-matrix if M is a P-matrix. Since being a P-matrix is an alternative generalisation of positive-definiteness to non-symmetric matrices, this turns out to be sufficient for there to be a unique

[11] In each case, we give the definitions in a constructive form which makes clear both how the property might be verified computationally, and the links between definitions. These are not necessarily in the form which is standard in the original literature, however. For the original definitions, and the proofs of equivalence between the ones below and the originals, see Cottle, Pang & Stone (2009a) and Xu (1993).

solution to the original equation. In the $T = 1$ or $T = 2$ special case, being a P-matrix coincides with the conditions we found in Section 3.

We give two further definitions, which again help ensure that My can be made positive:

Definition 4 (S(0)-matrix) A matrix $M \in \mathbb{R}^{T \times T}$ is called an **S-matrix** (**S0-matrix)** if there exists $y \in \mathbb{R}^T$ such that $y > 0$ and $My \gg 0$ ($My \geq 0$). [12]

Definition 5 ((Strictly) Semi-monotone) A matrix $M \in \mathbb{R}^{T \times T}$ is called (**strictly) semi-monotone** if each of its principal sub-matrices is an **S0-matrix** (**S-matrix)**.

In the $T = 1$ case, being an S-matrix and being strictly semi-monotone coincide with positivity of M, so these conditions may again be interpreted as generalisations of the positivity we required for existence with $T = 1$. Additionally, in Appendix B we go on to definite (non-)degenerate, (row/column) sufficient matrices, and (strictly) copositive matrices, which are of secondary importance, and we note some relationships between the various classes.

A common "intuition" is that in models without state variables, M must be both a P matrix, and an S matrix. In fact, this is not true. Indeed, there are even purely static models for which M is in neither of these classes, as we prove the following result in Appendix H.6, online.

Proposition 3 There is a purely static model for which $M_{1:\infty,1:\infty} = -I_{\infty\times\infty}$, which is neither a P-matrix, nor an S-matrix, for any T.

4.2. Uniqueness results

We will now present fully general necessary and sufficient conditions for an LCP to have a unique solution. These imply equally general necessary and sufficient conditions for a DSGE model with occasionally binding constraints to have a unique solution, thanks to Theorem 1. Ideally, we would like the solution to exist and be unique for any possible path the bounded variable might have taken in the future were there no OBC, i.e. for any possible q. To see this, note that under a perfect foresight exercise we are ignoring the fact that shocks might hit the economy in future. More properly, we ought to take future uncertainty into account. One way to do this would be to follow the original stochastic extended path approach of Adjemian & Juillard (2013) by drawing lots of samples of future shocks for periods $1, \dots, S$, and averaging over these draws. [13] However, in a linear model with shocks with unbounded support, providing at least one shock has an impact on a given variable, the distribution of future paths of that variable has positive support over the entirety of $\mathbb{R}^S$. Thus, we would like M to be such that for any q, the linear complementarity problem (q, M) has a unique solution. For clarity,

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[12] These conditions may be rewritten as $\sup\{\varsigma \in \mathbb{R} | \exists y \geq 0 \text{ s.t. } \forall t \in \{1, \dots, T\}, (My)_t \geq \varsigma \wedge y_t \leq 1\} > 0$, and $\sup\left\{\sum_{t=1}^{T} y_t \middle| y \geq 0, My \geq 0 \wedge \forall t \in \{1, \dots, T\}, y_t \leq 1\right\} > 0$, respectively. As linear programming problems, these may be solved in time polynomial in T using the methods of e.g. Roos, Terlaky, and Vial (2006). Alternatively, by Ville's Theorem of the Alternative (Cottle, Pang & Stone 2009b), M is not an S0-matrix if and only if $-M'$ is an S-matrix.

[13] See Footnote 5 for caveats to this procedure.

we remind the reader that the matrix M is independent of the initial state, so existence and/or uniqueness for all possible initial states does not require considering more than one M matrix.

Theorem 2 The LCP (q, M) has a unique solution for all $q \in \mathbb{R}^T$, if and only if M is a P-matrix. If M is not a P-matrix, then for some q the LCP (q, M) has multiple solutions. (Samelson, Thrall & Wesler 1958; Cottle, Pang & Stone 2009a)

This theorem is the equivalent for models with OBCs of the key theorem of Blanchard & Kahn (1980). By testing whether our matrix M is a P-matrix we can immediately determine if the model possesses a unique solution no matter what the initial state is, and no matter what shocks (if any) are predicted to hit the model in future, for a fixed T. Since if M is a P-matrix, so too are all its principal sub-matrices, if M is not a P-matrix for some T, then we know that with larger T it would also not be a P-matrix. Thus, if for some T, M is not a P-matrix, then we know that the model does not have a unique solution, even for arbitrarily large T.

Since checking whether a matrix is a P-matrix can be onerous in practice, we also present both easier to verify necessary conditions, and easier to verify sufficient conditions. The following corollary gives necessary conditions for uniqueness:

Corollary 1 If for all $q \in \mathbb{R}^T$, the LCP (q, M) has a unique solution, then:

1. All of the principal sub-matrices of M are P-matrices, S-matrices and strictly semi-monotone. (Cottle, Pang & Stone 2009a)
2. M has a strictly positive diagonal. (Immediate from definition.)
3. All of the eigenvalues of M have complex arguments in the interval $\left(-\pi + \frac{\pi}{T}, \pi - \frac{\pi}{T}\right)$. (Fang 1989)

Theorem 2 also implies easily verified sufficient conditions for uniqueness:

Corollary 2 For an arbitrary matrix A, denote the spectral radius of A by $\rho(A)$, and its largest and smallest singular values by $\sigma_{\max}(A)$ and $\sigma_{\min}(A)$, respectively. Let $|A|$ be the matrix with $|A|_{ij} = |A_{ij}|$ for all i, j. Then, for any matrix $M \in \mathbb{R}^{T\times T}$, if there exist diagonal matrices $D_1, D_2 \in \mathbb{R}^{T\times T}$ with positive diagonals, such that $W := D_1 M D_2$ satisfies one of the following conditions, then for all $q \in \mathbb{R}^T$, the LCP (q, M) has a unique solution:

1. W is general positive definite. (Cottle, Pang & Stone 2009a)
2. W has a positive diagonal, and $\langle W\rangle^{-1}$ is a nonnegative matrix, where $\langle W\rangle$ is the matrix with $\langle W\rangle_{ij} = -|W_{ij}|$ for $i \neq j$ and $\langle W\rangle_{ii} = |W_{ii}|$. (Bai & Evans 1997)
3. $\rho(|I - W|) < 1$. (Li & Wu 2016)
4. $(I + W)'(I + W) - \sigma_{\max}(|I - W|)^2 I$ is positive definite. (Li & Wu 2016)
5. $\sigma_{\max}(|I - W|) < \sigma_{\min}(I + W)$. (Li & Wu 2016)
6. $\sigma_{\min}\left((I - W)^{-1}(I + W)\right) > 1$. (Li & Wu 2016)
7. $\sigma_{\max}\left((I + W)^{-1}(I - W)\right) < 1$. (Li & Wu 2016)
8. $\rho\left(\left|(I + W)^{-1}(I - W)\right|\right) < 1$. (Li & Wu 2016)

In our experience, whenever M is a P-matrix, it will usually satisfy one of these conditions when D_1 and D_2 are chosen so that all rows and columns of $|W|$ have maximum equal to 1, using the algorithm of Ruiz (2001).

Since some classes of models almost never possess a unique solution when at the zero lower bound, we might reasonably require a lesser condition, namely that at least when the solution to the model without a bound is a solution to the model with the bound, then it ought to be the unique solution. This is equivalent to requiring that when q is non-negative, the LCP (q, M) has a unique solution. Conditions for this are given in the following proposition:

Proposition 4 The LCP (q, M) has a unique solution for all $q \in \mathbb{R}^T$ with $q \gg 0$ ($q \geq 0$) if and only if M is (strictly) semi-monotone. (Cottle, Pang & Stone 2009a)

Hence, by verifying that M is semi-monotone, we can reassure ourselves that introducing the bound will not change the solution away from the bound. When this condition is violated, even when the economy is a long way from the bound, there may be solutions which jump to the bound. Again, since principal sub-matrices of (strictly) semi-monotone are (strictly) semi-monotone, a failure of (strict) semi-monotonicity for some T implies a failure for all larger T.

Where there are multiple solutions, we might like to be able to select one via some objective function. This is tractable when either the number of solutions is finite, or the solution set is convex. Conditions for this are given in the Appendix C.

4.3. Existence results

We now turn to conditions for existence of a solution to a model with occasionally binding constraints. The following further definition will be helpful:

Definition 6 (Feasible LCP) Suppose $q \in \mathbb{R}^T$, $M \in \mathbb{R}^{T \times T}$ are given. The LCP (q, M) is called **feasible** if there exists $y \in \mathbb{R}^T$ such that $y \geq 0$ and $q + My \geq 0$.

By construction, if an LCP (q, M) has a solution, then it is feasible, so being feasible is necessary for existence. Checking feasibility is straightforward for any particular (q, M), as to find a feasible solution we just need to solve a standard linear programming problem.

Note that if the LCP (q, M) is not feasible, then for any $\hat{q} \leq q$, if $y \geq 0$, then $\hat{q} + My \leq q + My < 0$ since (q, M) is not feasible, so the LCP $(\hat{q}, M)$ is also not feasible. Consequently, if there are any q for which the LCP is non-feasible, then there is a positive measure of such q. Thus, in a model in which q is uncertain, if there are some q for which the model has no solution satisfying the terminal condition, even with arbitrarily large T, then the model will have no solution satisfying the terminal condition with positive probability. Hence it is not consistent with rationality for agents to believe that our terminal condition is satisfied with certainty, so they would have to place positive probability on getting stuck in an alternative steady-state.

The next proposition gives an easily verified necessary condition for the global existence of a solution to a model with occasionally binding constraints, given some fixed horizon T:

Proposition 5 The LCP (q, M) is feasible for all $q \in \mathbb{R}^T$ if and only if M is an S-matrix. Hence, if the LCP (q, M) has a solution for all $q \in \mathbb{R}^T$, then M is an S-matrix.
(Cottle, Pang & Stone 2009a)

Of course, it may be the case that the M matrix is only an S-matrix when T is very large, so we must be careful in using this condition to imply non-existence of a solution. Furthermore, it may be the case that although there exists some $y \in \mathbb{R}^T$ with $y \geq 0$ such that $M_{1:T,1:T}y \gg 0$ (indexing the M matrix by its size for clarity), for any such y, $\inf_{t \in \mathbb{N}^+} M_{t,1:T}y < 0$, so for some $q^* > 0$ and $q \in \mathbb{R}^{\mathbb{N}^+}$ with $q_t \to q^*$ as $t \to \infty$, the infinite LCP $(q, M_{1:\infty,1:\infty})$ is not feasible under the additional restriction that $y_t = 0$ for $t > T$. Strictly, it is this infinite LCP which we ought to be solving, subject to the additional constraint that y has only finitely many non-zero elements, as implied by our terminal condition. From Proposition 5, we immediately have the following result on feasibility of the infinite problem:

Corollary 3 The infinite LCP $(q, M_{1:\infty,1:\infty})$ is feasible for all $q^* > 0$ and $q \in \mathbb{R}^{\mathbb{N}^+}$ with $q_t \to q^*$ as $t \to \infty$ if and only if $\varsigma := \sup_{\substack{y \in [0,1]^{\mathbb{N}^+} \\ \exists T \in \mathbb{N} \text{ s.t. } \forall t > T, y_t = 0}} \inf_{t \in \mathbb{N}^+} M_{t,1:\infty}y > 0$.

Consequently, if $\varsigma > 0$ then for every $q \in \mathbb{R}^{\mathbb{N}^+}$, for sufficiently large T, the finite problem $(q_{1:T}, M_{1:T,1:T})$ will be feasible, which is a sufficient condition for solvability. To evaluate this limit, we first need to derive constructive bounds on the M matrix for large T. We do this in Appendix H.8, online, where we prove that the rows and columns of M are converging to zero (with constructive bounds), and that the k^{th} diagonal of the M matrix is converging to the value $d_{1,k}$, to be defined (again with constructive bounds), where the principal diagonal is index zero, and indices increase as one moves up and to the right.

To explain the origins of $d_{1,k}$ we note the following lemma proved in Appendix H.7, online:

Lemma 3 The (time-reversed) difference equation $A\hat{d}_{k+1} + B\hat{d}_k + C\hat{d}_{k-1} = 0$ for all $k \in \mathbb{N}^+$ has a unique solution satisfying the terminal condition $\hat{d}_k \to 0$ as $k \to \infty$, given by $\hat{d}_k = H\hat{d}_{k-1}$, for all $k \in \mathbb{N}^+$, for some H with eigenvalues in the unit circle.

Then, we define $d_0 := -(AH + B + CF)^{-1}I_{\cdot,1}$, $d_k = Hd_{k-1}$, for all $k \in \mathbb{N}^+$, and $d_{-t} = Fd_{-(t-1)}$, for all $t \in \mathbb{N}^+$, so d_k follows the time reversed difference equation for positive indices, and the original difference equation for negative indices. This is opposite to what one might expect as time increases but diagonal indices decrease, as one descends the rows of M.

Using the resulting bounds on M, we can bound ς:

Proposition 6 There exists $\underline{\varsigma}_T, \overline{\varsigma}_T \geq 0$, computable in time polynomial in T, such that $\underline{\varsigma}_T \leq \varsigma \leq \overline{\varsigma}_T$, and $|\underline{\varsigma}_T - \overline{\varsigma}_T| \to 0$ as $T \to \infty$. Hence, if $\underline{\varsigma}_T > 0$ then the infinite LCP $(q, M_{1:\infty,1:\infty})$ is feasible for all $q \in \mathbb{R}^{\mathbb{N}^+}$ with $q_t \to q^* > 0$ as $t \to \infty$, and if $\overline{\varsigma}_T = 0$ then there is some some $q^* > 0$ and $q \in \mathbb{R}^{\mathbb{N}^+}$ with $q_t \to q^*$ as $t \to \infty$ such that the infinite LCP $(q, M_{1:\infty,1:\infty})$ has no solution.

This condition (proven in Appendix H.8) gives a simple test for feasibility with any sufficiently large T. It also provides a test giving strong numerical evidence of non-existence, since if $\overline{\varsigma}_T = 0 +$ numerical error, then $\varsigma = 0$ is likely.

We now turn to sufficient conditions for existence of a solution for finite T.

Proposition 7 The LCP (q, M) is solvable if it is feasible and, either:

1. M is row-sufficient, or,
2. M is copositive and for all non-singular principal sub-matrices W of M, all non-negative columns of W^{-1} possess a non-zero diagonal element.

(Cottle, Pang & Stone 2009a; Väliaho 1986)

If either condition 1 or condition 2 of Proposition 7 is satisfied, then to check existence for any particular q, we only need to solve a linear programming problem. As this will be faster than solving the particular LCP, this may be helpful in practice. Moreover:

Proposition 8 The LCP (q, M) is solvable for all $q \in \mathbb{R}^T$, if at least one of the following conditions holds: (Cottle, Pang & Stone 2009a)

1. M is an S-matrix, and either condition 1 or 2 of Proposition 7 is satisfied.
2. M is copositive and non-degenerate.
3. M is a P-, a strictly copositive or strictly semi-monotone matrix.

If condition 1, 2 or 3 of Proposition 8 is satisfied, then the LCP will always have a solution. Therefore, for any path of the bounded variable in the absence of the bound, we will also be able to solve the model when the bound is imposed. Monetary policy makers should ideally choose a policy rule that produces a model that satisfies one of these three conditions, since otherwise there is a positive probability that only solutions converging to the "bad" steady-state will exist for some values of state variables and shock realisations.

Finally, in the special case of nonnegative M matrices we can derive conditions for existence that are both necessary and sufficient:

Proposition 9 If M is a nonnegative matrix, then the LCP (q, M) is solvable for all $q \in \mathbb{R}^T$ if and only if M has a positive diagonal. (Cottle, Pang & Stone 2009a)

4.4. Checking the existence and uniqueness conditions in practice

This section has presented many results, but the practical details of what one should test and in what order may still be unclear. Luckily, a lot of the decisions are automated by the author's DynareOBC toolkit, but we present a suggested testing procedure here in any case. This also serves to give an overview of our results and their limitations.

For checking feasibility and existence, the most powerful result is Proposition 6. If the lower bound from Proposition 6 is positive, for all sufficiently high T, the LCP is always feasible. If further conditions are satisfied for a given T, (see Proposition 7 and Proposition 8) then this guarantees existence for that particular T. However, since the additional conditions are sufficient and not necessary, in practice it may not be worth checking them, as we have never encountered a problem without a solution that was nonetheless feasible. Finding a T for which Proposition 6 produces a positive lower bound on ς requires a bit of trial and error. T will need to be big enough that the asymptotic approximation is accurate, which usually requires T to be bigger than the time it takes for the model's dynamics to die out. However, if T is too large, then DynareOBC's conservative approach to handling numerical error means that it can be difficult to reject $\underline{\varsigma} = 0$. Usually though, an intermediary value for T can be found at which we can establish $\underline{\varsigma} > 0$, even with a conservative approach to numerical error.

For checking non-existence, Proposition 6 can still be useful, though in this case, it does not provide definitive proof of non-feasibility, due to inescapable numerical inaccuracies. For a particular T, we may test if M is not an S-matrix in time polynomial in T by solving a simple linear programming problem. If M is not an S-matrix, then by Proposition 5, there are some q for which there is no solution which finally escapes the bound after at most T periods. With T larger than the time it takes for the model's dynamics to die out, this provides further evidence of non-existence for arbitrarily large T. In any case, given that only having a solution that stays at the bound for 250 years is arguably as bad as having no solution at all, for medium scale models, we suggest to just check if M is an S-matrix with $T = 1000$.

For checking uniqueness vs multiplicity, it is important to remember that while we can prove uniqueness for a given finite T by proving that the M matrix is a P-matrix, once we have found one T for which M is not a P-matrix (so there are multiple solutions, by Theorem 2), we know the same is true for all higher T. If we wish to prove that there is a unique solution up to some horizon T, then the best approach is to begin by testing the sufficient conditions from Corollary 2, with our suggested D_1 and D_2. If none of these conditions pass, then it is likely that M is not a P-matrix. In any case, checking that an M which fails the conditions of Corollary 2 is a P-matrix for very large T may not be computationally feasible, though finding a counter-example usually is.

If we wish to establish multiplicity, then Corollary 1 provides a guide. It is trivial to check if M has any nonpositive elements on its diagonal, in which case it cannot be a P-matrix. We can also check whether $d_{0,1} := -I_{1,\cdot}(AH + B + CF)^{-1}I_{\cdot,1} \leq 0$, in which case for sufficiently large T, M cannot be a P-matrix, as $d_{0,1}$ is the limit of the diagonal of M. It is also trivial to check the eigenvalue condition given in Corollary 1, and that M is an S-matrix. If none of these checks established that M is not a P-matrix, then a search for a principal sub-matrix with negative determinant is the obvious next step. It is sensible to begin by checking the contiguous principal sub-matrices.[14] These correspond to a single spell at the ZLB which is natural given that impulse responses in DSGE models tend to be single peaked. This is so reliable a diagnostic (and so fast) that DynareOBC reports it automatically for all models. Continuing, one could then check all the 2×2 principal sub-matrices, then the 3×3 ones, and

[14] Some care must be taken though as checking the signs of determinants of large matrices is numerically unreliable.

so on. With T around the half-life of the model's dynamics, usually one of these tests will quickly produce the required counter-example. A similar search strategy can be used to rule out semi-monotonicity, implying multiplicity when away from the bound, by Proposition 4.

Given the computational challenge of verifying whether M is a P-matrix, without Corollary 2, it may be tempting to wonder if our results really enable one to accomplish anything that could not have been accomplished by a naïve brute force approach. For example, it has been suggested that given T and an initial state, one could check for multiple equilibria by considering all of the 2^T possible combinations of periods at which the model could be at the bound and testing if each guess is consistent with the model, following, for example, the solution algorithms of Fair and Taylor (1983) or Guerrieri & Iacoviello (2012). Since there are 2^T principal sub-matrices of M, it might seem likely that this will be computationally very similar to checking if M is a P-matrix. However, our uniqueness results are not conditional on q or the initial state, rather they give conditions under which there is a unique solution for any possible path that the economy would take in the absence of the bound. Thus, while the brute force approach may tell you about uniqueness given an initial state in a reasonable amount of time, using our results, in a comparable amount of time you will learn whether there are multiple solutions for any possible q. A brute force approach to checking for all possible initial conditions would require one to solve a linear programming problem for each pair of possible sets of periods at the bound, of which there are $2^{2T-1} - 2^{T-1}$.[15] This is far more computationally demanding than our approach, and becomes intractable for even very small T. Additionally, our approach is numerically more robust, allows the easy management of the effects of numerical error to avoid false positives and false negatives, and requires less work in each step. Finally, we stress that in most cases, thanks to Corollary 1 and Corollary 2, no such search of the sub-matrices of M is required under our approach, and a proof or counter-example may be produced in time polynomial in T, just as it may be when checking for existence with our results.

5. Applications to New Keynesian models

Since one of the great successes of the original Blanchard & Kahn (1980) result has been the development of the Taylor principle, and since the zero lower bound remains of great interest to policy makers, we now seek to apply our theoretical results to New Keynesian models with a ZLB. We stress though that our results are likely to have application to many other classes of model, including, for example, models with financial frictions, or models with overlapping generations and borrowing constrains.

[15] Given the periods in the constrained regime, the economy's path is linear in the initial state. Excepting knife edge cases of rank deficiency, any multiplicity must involve two paths each at the bound in a different set of periods. Consequently, a brute force approach to finding multiplicity unconditional on the initial state is to guess two different sets of periods at which the economy is at the bound, then solve a linear programming problem to find out if there is a value of the initial state for which the regimes on each path agree with their respective guesses.

In the first subsection here, we re-examine the simple Brendon, Paustian & Yates (BPY) (2013) model in light of our results, before going on to consider a variant of it with price targeting, which we show to produce determinacy. In the BPY (2013) model, multiplicity and non-existence stem from a response to growth rates in the Taylor rule. However, we do not want to give the impression that multiplicity and non-existence are only caused by such a response, or that they are only a problem in carefully constructed theoretical examples. Thus, in subsection 5.2, we show that a standard NK model with positive steady-state inflation and a ZLB possesses multiple equilibria in some states, and no solutions in others, even with an entirely standard Taylor rule. We also show that here too price level targeting is sufficient to restore determinacy. Finally, we show that these conclusions also carry through to the posterior-modes of the Smets & Wouters (2003; 2007) models.

5.1. The simple Brendon, Paustian & Yates (2013) (BPY) model

Recall that in Section 3.2, we showed that if $\alpha_{\Delta y} > \sigma\alpha_\pi$ in the BPY (2013) model, then with $T = 1$, $M < 0$. When $T > 1$, this implies that M is neither P_0, general positive semi-definite, semi-monotone, co-positive, nor sufficient, since the top-left 1×1 principal sub-matrix of M is the same as when $T = 1$. Thus, if anything, when $T > 1$, the parameter region in which there are multiple solutions (when away from the bound or at it) is larger. However, numerical experiments suggest that this parameter region in fact remains the same as T increases, which is unsurprising given the weak persistence of this model. Thus, if we want more interesting results with higher T, we need to consider a model with a stronger persistence mechanism.

One obvious possibility is to consider models with either persistence in the interest rate, or persistence in the "shadow" rate that would hold were it not for the ZLB. In Appendix G, online, we find that persistence in the shadow interest rate, introduced in a standard way, does not appear to change the determinacy region providing T is large enough. However, note though that we may also introduce persistence in shadow interest rates by setting:

$$x_{d,t} = (1-\rho)(1-\beta) + \left(\alpha_{\Delta y}\left(x_{y,t} - x_{y,t-1}\right) + \alpha_\pi x_{\pi,t}\right) + \rho x_{d,t-1},$$

where $x_{i,t} = \max\{0, x_{d,t}\}$. If the second bracketed term was multiplied by $(1-\rho)$, then this would be entirely standard, however as written here, in the limit as $\rho \to 1$, this tends to:

$$x_{d,t} = 1 - \beta + \alpha_{\Delta y} x_{y,t} + \alpha_\pi x_{p,t}$$

where $x_{p,t}$ is the price level, so $x_{\pi,t} = x_{p,t} - x_{p,t-1}$. This is a level targeting rule, with nominal GDP targeting as a special case with $\alpha_{\Delta y} = \alpha_\pi$. Note that the omission of the $(1-\rho)$ coefficient on $\alpha_{\Delta y}$ and α_π is akin to having a "true" response to output growth of $\frac{\alpha_{\Delta y}}{1-\rho}$ and a "true" response to inflation of $\frac{\alpha_\pi}{1-\rho}$, so in the limit as $\rho \to 1$, we effectively have an infinitely strong response to these quantities. It turns out that this is sufficient to produce determinacy for all $\alpha_{\Delta y}, \alpha_\pi \in (0, \infty)$. In particular, given the model:

$$x_{i,t} = \max\{0, 1 - \beta + \alpha_{\Delta y} x_{y,t} + \alpha_\pi x_{p,t}\},$$
$$x_{y,t} = \mathbb{E}_t x_{y,t+1} - \frac{1}{\sigma}\left(x_{i,t} + \beta - 1 - \mathbb{E}_t x_{p,t+1} + x_{p,t}\right),$$
$$x_{p,t} - x_{p,t-1} = \beta \mathbb{E}_t x_{p,t+1} - \beta x_{p,t} + \gamma x_{y,t},$$

we prove in Appendix H.9, online, that the following proposition holds:

Proposition 10 The BPY model with price targeting is in the form of Problem 2 (OBC), and satisfies Assumptions 1, 2 and 3. With $T = 1$, $M > 0$ for all $\alpha_\pi \in (0, \infty)$, $\alpha_{\Delta y} \in [0, \infty)$.

Furthermore, with $\sigma = 1$, $\beta = 0.99$, $\gamma = \frac{(1-0.85)(1-\beta(0.85))}{0.85}(2+\sigma)$, as before, and $\alpha_{\Delta y} = 1$, $\alpha_\pi = 1$, if we check our lower bound on ς with $T = 20$, we find that $\varsigma > 0.042$. Hence, this model is always feasible for any sufficiently large T. Given that $d_0 > 0$ for this model, and that for $T = 1000$, M is a P-matrix by our sufficient conditions from Corollary 2, this is strongly suggestive of the existence of a unique solution for any q and for arbitrarily large T.

5.2. The linearized Fernández-Villaverde et al. (2015) model

The discussion of the BPY (2013) model might lead one to believe that multiplicity and non-existence is solely a consequence of overly aggressive monetary responses to output growth, and overly weak monetary responses to inflation. However, it turns out that in basic NK models with positive inflation in steady-state, and hence price dispersion, even without any monetary response to output growth, and even with extremely aggressive monetary responses to inflation, there are still multiple equilibria in some states of the world (i.e. for some q), and no solutions in others. Price level targeting again fixes these problems though.

We show these results in the Fernández-Villaverde et al. (2015) model, which is a basic non-linear New Keynesian model without capital or price indexation of non-resetting firms, but featuring (non-valued) government spending and steady-state inflation (and hence price-dispersion). We refer the reader to the original paper for the model's equations. After substitutions, the model has four non-linear equations which are functions of gross inflation, labour supply, price dispersion and an auxiliary variable introduced from the firms' price-setting first order condition. Of these variables, only price dispersion enters with a lag. We linearize the model around its steady-state, and then reintroduce the "max" operator which linearization removed from the Taylor rule. [16] All parameters are set to the values given in Fernández-Villaverde et al. (2015). There is no response to output growth in the Taylor rule, so any multiplicity cannot be a consequence of the mechanism highlighted by BPY (2013).

For this model, numerical calculations reveal that with $T \leq 14$, M is a P-matrix. However, with $T \geq 15$, M is not a P matrix, and thus there are certainly some states of the world (some q) in which the model has multiple solutions. Furthermore, with $T = 1000$, our upper bound on ς from Proposition 6 implies that $\varsigma \leq 0 +$ numerical error, suggesting that M is not an S-matrix for arbitrarily large T, by Corollary 3. If this is correct, then even for arbitrarily large T, there are some q for which no solution exists.

[16] Before linearization, we transform the model's variables so that the transformed variables take values on the entire real line. I.e. we work with the logarithms of labour supply, price dispersion and the auxiliary variable. For inflation, we note that inflation is always less than $\theta^{\frac{1}{1-\varepsilon}}$ (in the notation of Fernández-Villaverde et al. (2015)). Thus, we work with a logit transformation of inflation over $\theta^{\frac{1}{1-\varepsilon}}$.

To make the mechanism behind these results clear, we will compare the Fernández-Villaverde et al. (2015) model to an altered version of it with full indexation to steady-state inflation of prices that are not set optimally. To a first order approximation, the model with full indexation never has any price dispersion, and thus has no endogenous state variables. It is thus a purely forwards looking model, and so it is perhaps unsurprising that it should have a unique equilibrium given a terminal condition, even in the presence of the ZLB.

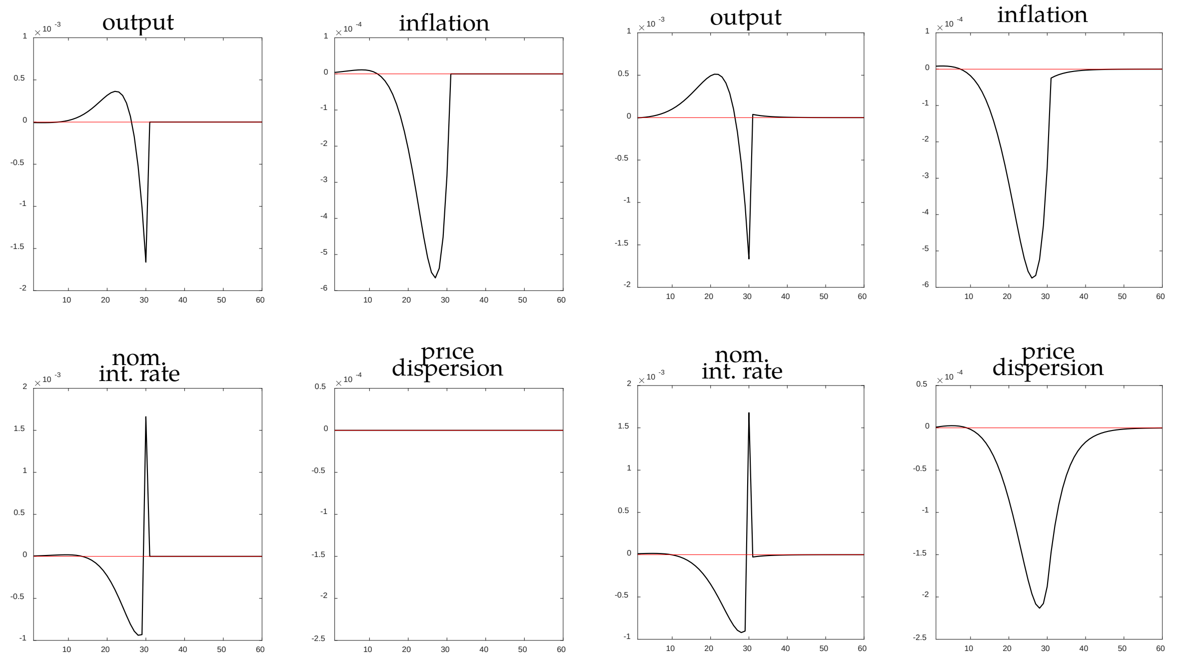


Figure 2: Impulse responses to a shock announced in period 1, but hitting in period 30, in basic New Keynesian models with (left) and without (right) indexation to steady-state inflation.
All variables are in logarithms. In both cases, the model and parameters are taken from Fernández-Villaverde et al. (2015), the only change being the addition of complete price indexation to steady-state inflation for non-updating firms in the left hand plots.

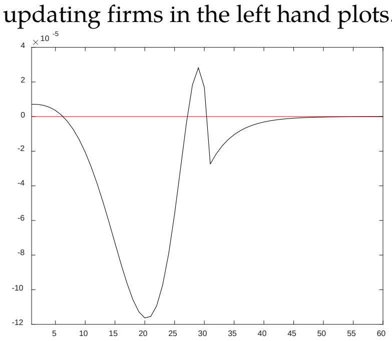


Figure 3: Difference between the IRFs of nominal interest rates from the two models shown in Figure 2.
Negative values imply that nominal interest rates are lower in the model without indexation.

In Figure 2 we plot the impulse responses of first order approximations to both models to a shock to nominal interest rates that is announced in period one but that does not hit until period thirty. For both models, the shape is similar, however, in the model without indexation, the presence of price dispersion reduces inflation both before and after the shock hits. This is because the predicted fall in inflation compresses the price distribution, reducing dispersion, and thus reducing the number of firms making large adjustments. The fall in price dispersion also increases output, due to lower efficiency losses from miss-pricing. However, the effect on interest rates is dominated by the negative inflation effect, as the Taylor-rule coefficient on output cannot be too high if there is to be determinacy.[17] For reference, the difference between the IRFs of nominal interest rates in each model is plotted in Figure 3, making clear that interest rates are on average lower following the shock in the model without indexation.

Remarkably, this small difference in the impulse responses between models is enough that the linearized model without indexation has multiple equilibria given a ZLB, but the linearized model with full indexation is determinate. This illustrates just how fragile is the uniqueness in the linearized purely forward-looking model. Informally, what is needed for multiplicity is that the impulse responses to positive news shocks to interest rates are sufficiently negative for a sufficiently high amount of time that a linear combination of them could be negative in every period in which a shock arrives. Here, price dispersion is providing the required additional reduction to nominal interest rates following a news shock.

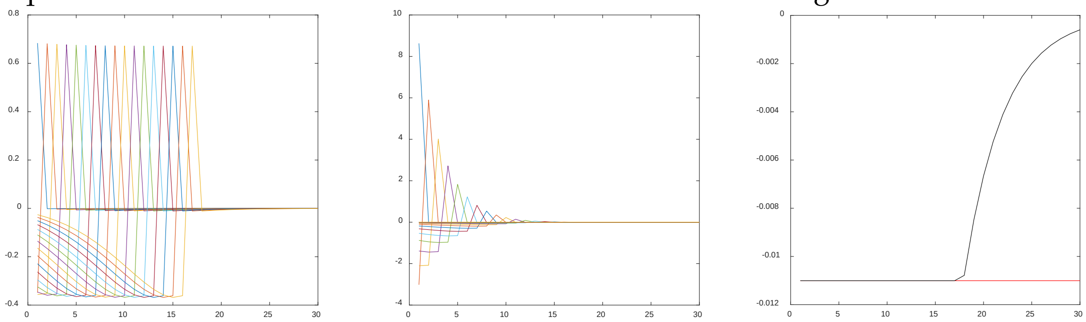


Figure 4: Construction of multiple equilibria in the Fernández-Villaverde et al. (2015) model.
The left plot shows the IRFs to news shocks arriving zero to sixteen quarters after becoming known. The middle plot shows the same IRFs scaled appropriately. The right plot shows the sum of the scaled IRFs shown in the central figure, where the red line gives the ZLB's location, relative to steady-state.

We illustrate how multiplicity emerges in the model without indexation by showing, in Figure 4, the construction of an additional equilibrium which jumps to the ZLB for seventeen quarters.[18] If the economy is to be at the bound for seventeen quarters, then for those seventeen quarters, the nominal interest rate must be higher than it would be according to the Taylor rule, meaning that we need to consider seventeen endogenous news shocks, at horizons from zero to sixteen quarters into the future. The impulse responses to unit shocks of this kind are shown in the leftmost plot. Each impulse response has broadly the same shape as the one shown for nominal interest rates in the right of Figure 2. The central figure plots the same impulse responses again, but now each line is scaled by a constant so that their sum gives the line shown in black in the rightmost plot. In this rightmost plot, the red line gives the ZLB's location, relative to steady-state, thus the combined impulse response spends seventeen

[17] One might think the situation would be different if the response to output was high enough that the rise in output after the shock produced a rise in interest rates. However, as observed by Ascari and Ropele (2009), the determinacy region is smaller in the presence of price dispersion than would be suggested by the Taylor criterion. Numerical experiments suggest that in all the determinate region, interest rates are below steady-state following the shock.
[18] Seventeen quarters was the minimum span for which an equilibrium of this form could be found.

quarters at the ZLB before returning to steady-state. Since there are only "news shocks" in the periods in which the economy is at the ZLB, this gives a perfect foresight rational expectations equilibrium which makes a self-fulfilling jump to the ZLB.

The situation is quite different under price level targeting. In particular, if we replace inflation in the monetary rule with the price level relative to its linear trend, which evolves according to:

$$x_{p,t} = x_{p,t-1} + x_{\pi,t} - x_{\pi}, \tag{5}$$

then with $T = 200$, the lower bound from Proposition 6 implies that $\varsigma > 0.003$, and hence that for all sufficiently large T, M is an S-matrix (by Corollary 3), so there is always a feasible solution. Furthermore, even with $T = 1000$, M is a P-matrix by our sufficient conditions from Corollary 2. This is strongly suggestive of uniqueness even for arbitrarily large T, given the reasonably short-lived dynamics of the model.

5.3. The Smets & Wouters (2003; 2007) models

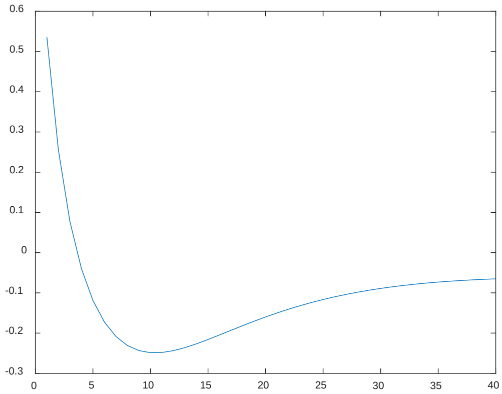
Smets & Wouters (2003) and Smets & Wouters (2007) are the canonical medium-scale linear DSGE models, featuring assorted shocks, habits, price and wage indexation, capital (with adjustment costs), (costly) variable utilisation and general monetary policy reaction functions. The former model is estimated on Euro area data, while the latter is estimated on US data. The latter model also contains trend growth (permitting its estimation on non-detrended data), and a slightly more general aggregator across industries. However, overall, they are quite similar models, and any differences in their behaviour chiefly stems from differences in the estimated parameters. Since both models are incredibly well known in the literature, we omit their equations here, referring the reader to the original papers for further details.

To assess the likelihood of multiple equilibria at or away from the zero lower bound, we augment each model with a ZLB on nominal interest rates, and evaluate the properties of each model's M matrix at the estimated posterior-modes from the original papers. To minimise the deviation from the original papers, we do not introduce an auxiliary variable for shadow nominal interest rates, so the monetary rules take the form of $x_{r,t} = \max\{0, (1-\rho_r)(\cdots) + \rho_r x_{r,t-1} + \cdots\}$, in both cases. Our results would be essentially identical with a shadow nominal interest rate though.

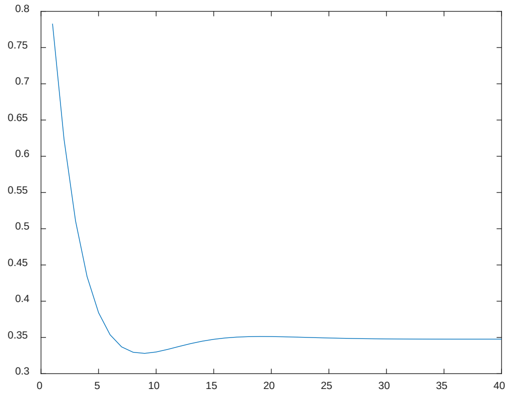
If the diagonal of the M matrix ever goes negative, then the M matrix cannot be semi-monotone, or P_0, and hence the model will sometimes have multiple solutions even when away from the zero lower bound (i.e. for some positive q), by Proposition 4. In Figure 5, we plot the diagonal of the M matrix for each model in turn,[19] i.e. the impact on nominal interest rates in period t of news in period 1 that a positive, magnitude one shock will hit nominal interest rates in period t. Immediately, we see that while in the US model, these impacts remain positive at all horizons, in the Euro area model, these impacts turn negative after just a few

[19] The MOD files for the Smets & Wouters (2003) and (2007) models were derived, respectively, from the Macro Model Database (Wieland et al. 2012) and files provided by Johannes Pfeifer here: http://goo.gl/CP53x5.

periods, and remain so at least up to period 40. Therefore, in the ZLB augmented Smets & Wouters (2003) model, there is not always a unique equilibrium. Furthermore, if a run of future shocks was drawn from a distribution with unbounded support, then the value of these shocks was revealed to the model's agents (as in the stochastic extended path), then there would be a positive probability that the model without the ZLB would always feature positive interest rates, but that the model with the ZLB could hit zero.



The Smets & Wouters **(2003)** model



The Smets & Wouters **(2007)** model

Figure 5: The diagonals of the M matrices for the Smets & Wouters (2003; 2007) models

It remains for us to assess whether M is a $\mathrm{P}_{(0)}$-matrix or (strictly) semi-monotone for the Smets & Wouters (2007) model. Numerical calculations reveal that for $T < 9$, M is a P-matrix, and hence is strictly semi-monotone. However, with $T \geq 9$, the top-left 9×9 sub-matrix of M has negative determinant and is not an S or $\mathrm{S}_{(0)}$ matrix. Thus, for $T \geq 9$, M is not a $\mathrm{P}_{(0)}$-matrix or (strictly) semi-monotone, and hence this model also has multiple equilibria, even when away from the bound. While placing a larger coefficient on inflation in the Taylor rule can make the Euro area picture more like the US one, with a positive diagonal to the M matrix, even with incredibly large coefficients, M remains a non-P-matrix for both models. This is driven by the fact that both the real and nominal rigidities in the model help reduce the average value of the impulse response to a positive news shock to the monetary rule. Following such a shock's arrival, they help ensure that the fall in output is persistent. Prior to its arrival, consumption habits and capital or investment adjustment costs help produce a larger anticipatory recession. Hence, in both the Euro area and the US, we ought to take seriously the possibility that the existence of the ZLB produces non-uniqueness.

As an example of such non-uniqueness, in Figure 6 we plot two different solutions following the most likely combination of shocks to the Smets & Wouters (2007) model that would produce negative interest rates for a year in the absence of a ZLB.[20] In both cases, the dotted line shows the response in the absence of the ZLB. Particularly notable is the flip in sign, since

[20] We find the vector w that minimises $w'w$ subject to $\bar{r} + Zw \leq 0$, where $\bar{r}$ is the steady-state interest rate, and columns of Z give four periods of the impulse response of interest rates to a given shocks. This gives the following shock magnitudes: productivity, 3.56%; risk premium, −2.70%; government, −1.63%; investment, −4.43%; monetary, −2.81%; price mark-up, −3.19%; wage mark-up, −4.14%.

the shocks most likely to take the model to the ZLB for a year are expansionary ones reducing prices (i.e. positive productivity and negative mark-up shocks). Section 6.3 shows an example of multiplicity in the Smets & Wouters (2003) model, and discusses the economic relevance of such multiplicity.

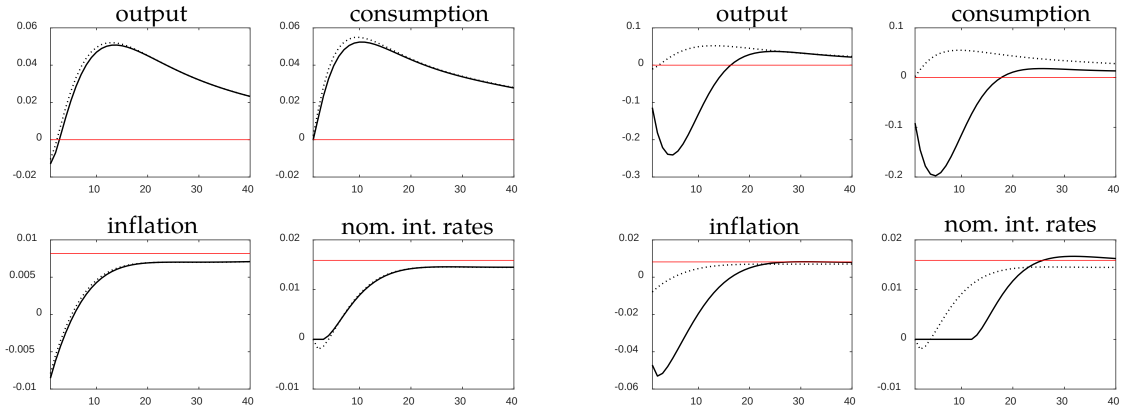


Figure 6: Two alternative solutions following a combination of shocks to the Smets & Wouters (2007) model
All variables are in logarithms. The precise combination of shocks is detailed in footnote 20.

In addition, it turns out that for neither model is M an S-matrix even with $T = 1000$, and thus for both models there are some $q \in \mathbb{R}^{1000}$ for which no solution exists. This is strongly suggestive of non-existence for some q even for arbitrarily large T. This is reinforced by the fact that for the Smets & Wouters (2007) model, with $T = 1000$, Proposition 6 gives that $\varsigma \leq 0 +$ numerical error.

Alternatively, suppose we replace the monetary rule in both models by:

$$x_{r,t} = \max\{0, (1 - \rho_r)(x_{y,t} + x_{p,t}) + \rho_r x_{r,t-1}\}$$

where ρ_r is as in the respective original model, where the price level $x_{p,t}$ again evolves per equation (5), and where $x_{y,t}$ is output relative to its linear trend. Then, for both models, with $T = 1000$, M was a P-matrix by our sufficient conditions from Corollary 2. Furthermore, from Proposition 6, with $T = 1000$, for the Euro area model we have that $\varsigma > 3 \times 10^{-7}$ and for the US model we have that $\varsigma > 0.002$, so Corollary 3 implies that a solution always exists to both models for sufficiently large T. As one would expect, this result is also robust to departures from equal, unit, coefficients. Thus, price level targeting again appears to be sufficient for determinacy in the presence of the ZLB.

6. Additional discussion

Before concluding, we present additional discussion of the significance of our results in the context of the wider literature. We begin by further discussing the uniqueness results, before turning to the existence ones. We then discuss whether the multiplicity we find could be a factor in explaining real world outcomes. We end the section with further discussion of our case for price level targeting.

6.1. Uniqueness and multiplicity

We have presented necessary and sufficient conditions for uniqueness in otherwise linear models with terminal conditions. Some caveats are in order though.

Bodenstein (2010) showed that linearization can exclude equilibria, and Braun, Körber & Waki (2012) show that there may be multiple perfect-foresight solutions to a non-linear NK model with ZLB, converging to the non-deflationary steady-state. However, it turns out that the linearized version of their model has a unique equilibrium, even when the ZLB is imposed. Thus, the multiplicity we find is strictly in addition to the multiplicity found by those authors. While the theoretical and computational methods used by Braun, Körber & Waki (2012) have the great advantage that they can cope with fully non-linear models, it appears that they cannot cope with endogenous state variables, which limits their applicability. By producing tools for analysing otherwise linear models including state variables, our tools and results provide a complement to those of Braun, Körber & Waki (2012). For evidence of the continued relevance of our results in a non-linear setting, note that the multiplicity found in a simple linearized model in BPY (2013) is also found in the equivalent non-linear model in BPY (2016).

Of course, ideally, we would have liked to analyse models with other nonlinearities apart from the occasionally binding constraint(s). However, we maintain that studying multiplicity in otherwise linear models is still an important exercise. Firstly, macroeconomists have long relied on existence and uniqueness results based on linearization of models without occasionally binding constraints, even though this may produce spurious uniqueness in some circumstances. Secondly, it is nearly impossible to find all perfect foresight solutions in general non-linear models, since this is equivalent to finding all the solutions to a huge system of non-linear equations, when even finding all the solutions to large systems of quadratic equations is computationally intractable. At least if we have the full set of solutions to the otherwise linear model, we may use homotopy continuation methods to map these solutions into solutions of the non-linear model. Furthermore, finding all solutions under uncertainty is at least as difficult in general, as the policy function is also defined by a large system of non-linear equations. Thirdly, Christiano and Eichenbaum (2012) argue that e-learnability considerations render the additional equilibria of Braun, Körber & Waki (2012) mere "mathematical curiosities", suggesting that the equilibria that exist in the linearized model are of independent interest, whatever one's view on this debate. Finally, our main results for NK models imply non-uniqueness, so concerns of spurious uniqueness under linearization will not be relevant in these cases.

From the preceding discussion, we see that our choice to focus on otherwise-linear models under perfect-foresight, with fixed terminal conditions, has biased our results in favour of uniqueness for three distinct reasons. Firstly, because there are potentially more solutions under rational expectations than under perfect-foresight, as we prove in Appendix E, online; secondly, because there are potentially other solutions returning to alternate steady-states; and thirdly, because the original fully non-linear model may possess yet more solutions. This means our results on the multiplicity of solutions to New Keynesian models are all the more surprising, and that it is all the more likely that multiplicity of equilibria is an important factor in explaining actual economies' spells at the ZLB.

Duarte (2016) considers how a central bank might ensure determinacy in a simple continuous time new Keynesian model. Like us, he finds that the Taylor principle is not sufficient in the presence of the ZLB. He shows that determinacy may be produced by using a rule that holds interest rates at zero for a history dependent amount of time, before switching to a $\max\{0, \dots\}$ Taylor rule. While we do not allow for such switches in central bank behaviour, we do find an important role for history dependence, through price targeting.

Jones (2015) also presents a uniqueness result for models with occasionally binding constraints. He shows that if one knows the set of periods at which the constraint binds, then under standard assumptions, there is a unique path. However, there is no reason there should be a unique set of periods at which the constraint binds, consistent with the model. The multiplicity for models with occasionally binding constraints precisely stems from there being multiple sets of periods at which the model could be at the bound. Our results are not conditional on knowing in advance the periods at which the constraint binds.

Finally, uniqueness results have also been derived in the Markov switching literature, see e.g. Davig & Leeper (2007) and Farmer, Waggoner & Zha (2010; 2011), though the assumed exogeneity of the switching in these papers limits their application to endogenous OBCs such as the ZLB. Determinacy results with endogenous switching were derived by Marx & Barthelemy (2013), but they only apply to forward looking models that are sufficiently close to ones with exogenous switching, and there is no reason e.g. a standard NK model with a ZLB should have this property. Our results do not have this limitation.

6.2. Existence and non-existence

We also produced conditions for the existence of any perfect-foresight solution to an otherwise linear model with a terminal condition. These results provide new intuition for the prior literature on existence under rational expectations, which has found that NK models with a ZLB might have no solution at all if the variance of shocks is too high. For example, Mendes (2011) derived analytic results on existence as a function of the variance of a demand shock, and Basu & Bundick (2015) showed the potential quantitative relevance of such results. Furthermore, conditions for the existence of an equilibrium in a simple NK model with discretionary monetary policy are derived in close form for a model with a two-state Markov shock by Nakata & Schmidt (2014). They show that the economy must spend a small amount of time in the bad state for the equilibrium to exist, which again links existence to variance.

While our results are not directly related to the variance of shocks, as we work under perfect foresight, they are nonetheless linked. We showed that whether a perfect foresight solution exists depends on the perfect-foresight path taken by nominal interest rates in the absence of the bound. Throughout, we assumed that this path was arbitrary, as there is always some information about future shocks that could be revealed today to produce a given path. However, in a model with a small number of shocks, all of bounded support, and no information about future shocks, clearly not all paths are possible for nominal interest rates in the absence of the bound. The more shocks are added (e.g. news shocks), and the wider their support, the greater will be the support of the space of possible paths for nominal interest rates in the absence of the ZLB, and hence, the more likely will be non-existence of a solution for a positive measure of paths, helping to explain the prior results.

There has also been some prior work by Richter & Throckmorton (2015) and Gavin et al. (2015; Appendix B) that has related a kind of eductive stability (the convergence of policy function iteration) to other properties of the model. Non-convergence of policy function iteration is suggestive of non-existence, though not definitive evidence. While the procedure of the cited authors has the advantage of working with the fully non-linear model under rational expectations, this limitation means that it cannot directly address the question of existence. By contrast, our results are theoretical and directly address existence. Thus, both procedures should be viewed as complementary; while ours definitively answers the question of existence in the slightly limited world of perfect foresight, otherwise linear models, the Richter & Throckmorton results give answers on stability in a richer setting.

Another approach to establishing the existence of an equilibrium is to produce it to satisfactory accuracy, by solving the model in some way. Under perfect foresight, the procedure outlined in this paper's companion is a possibility (Holden 2016), and the method of Guerrieri & Iacoviello (2012) is a prominent alternative. Under rational expectations, policy function iteration methods have been used by Fernández-Villaverde et al. (2015) and Richter & Throckmorton (2015), amongst others. However, this approach cannot establish non-existence or prove uniqueness. As such it is of little use to the policy maker who wants policy guidance to ensure existence and/or uniqueness. Furthermore, if the problem is solved globally, one cannot in general rule out that there is not an area of non-existence outside of the grid on which the model was solved. Similarly, if the model is solved under perfect foresight for a given initial state, then the fact that a solution exists for that initial point gives no guarantees that a solution should exist for other initial points. Thus, there is an essential role for more general results on global existence, as we have produced here.

6.3. Economic significance of multiplicity at the ZLB

There are two reasons why one might be sceptical about the economic significance of the multiple equilibria caused by the presence of the ZLB that we find. Firstly, as with any non-fundamental equilibrium, the coordination of beliefs needed to sustain the equilibrium may be difficult. Secondly, as we have seen, self-fulfilling jumps to the ZLB may feature implausibly large falls in output and inflation. This reflects the implausibly large response to news about future policy innovations, a problem that has been termed the "forward guidance puzzle" in the literature (Carlstrom, Fuerst & Paustian 2015; Del Negro, Giannoni & Patterson 2015).[21] However, if the economy is already in a recession, then both problems are substantially ameliorated. If interest rates are already low, then it does not seem too great a stretch to suggest that a drop in confidence may lead people to expect to hit the ZLB. Even more plausibly, if the economy is already at the ZLB, then small changes in confidence could easily select an equilibrium featuring a longer spell at the ZLB than in the equilibrium with the shortest time there. Indeed, there is no good reason people should coordinate on the equilibrium with the shortest time at the ZLB. Moreover, with interest rates already low, the size of the required self-fulfilling news shock is much smaller, meaning that the additional drop in output and inflation caused by a jump to the ZLB will be much more moderate.

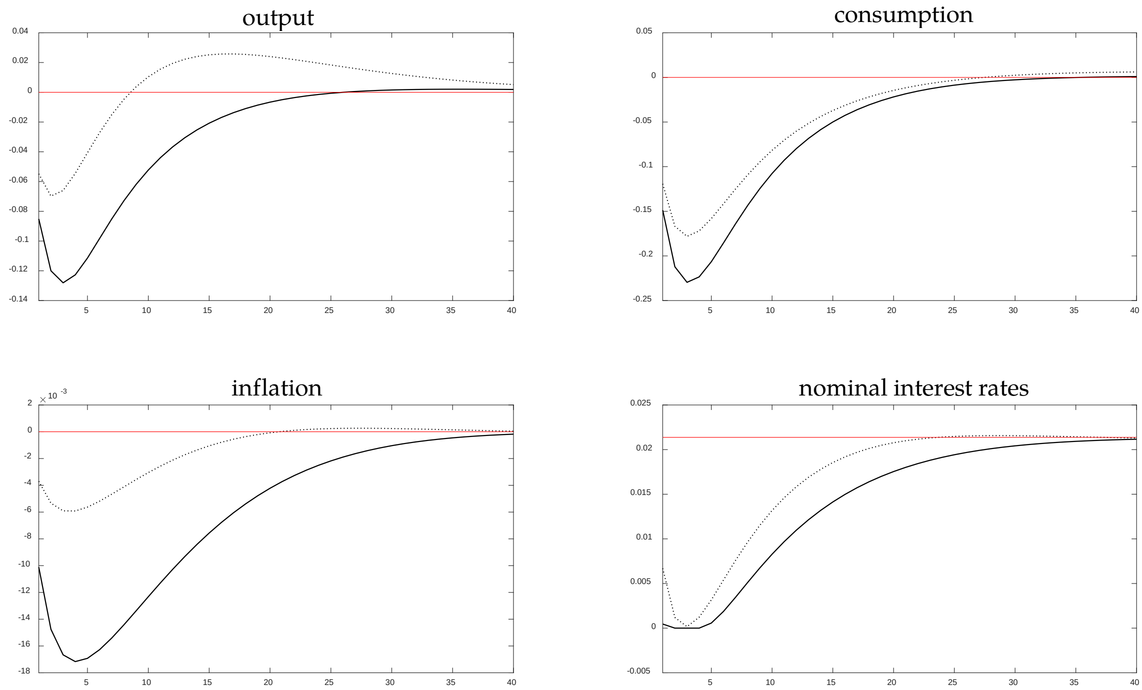


Figure 7: Two solutions following a preference shock in the Smets & Wouters (2003) model.
All variables are in logarithms. The dotted line is a solution which does not hit the bound. The solid line is an alternative solution which does hit the bound.

As an illustration, in Figure 7 we plot the impulse response to a large magnitude preference shock (scaling utility), in the Smets & Wouters (2003) model.[22] The shock is not quite large

[21] McKay, Nakamura & Steinsson (2016) point out that these implausibly large responses to news are muted in models with heterogeneous agents, and give a simple "discounted Euler" approximation that produces similar results to a full heterogeneous agent model. While including a discounted Euler equation makes it harder to generate multiplicity (e.g. reducing the parameter space with multiplicity in the BPY (2013) model), when there is multiplicity, the resulting responses are much larger, as the weaker response to news means the required endogenous news shocks need to be much greater in order to drive the model to the bound.
[22] The shock is 22.5 standard deviations. While this is implausibly large, the economy could be driven to the bound with a run of much smaller shocks. It is also worth recalling that the model was estimated on the great moderation period, and so the estimated standard deviations may be too low. Finally, recent evidence (Cúrdia, del Negro & Greenwald 2014) suggests that the shocks in DSGE models should be fat tailed, making large shocks more likely.

enough to send the economy to the ZLB[23] in the standard solution, shown with a dotted line. However, there is an alternative solution in which the economy jumps to the bound one period after the initial shock, remaining there for three periods. While the alternative solution features larger drops in output and inflation, the falls are broadly in line with the magnitude of the crisis, with Eurozone GDP and consumption now being about 20% below a pre-crisis log-linear trend, and the largest drop in Eurozone consumption inflation from 2008q3 to 2008q4 being around 1%.[24] Considering this, we view it as plausible that multiplicity of equilibria was a significant component of the explanation for the great recession.

6.4. Price level targeting

Our results suggest that given belief in an eventual return to inflation, a determinate equilibrium may be produced in standard NK models if the central bank switches to targeting the price level, rather than the inflation rate. As the previous figure made clear, the welfare benefits to this could be substantial. There is of course a large literature advocating price level targeting already. Vestin (2006) made an important early contribution by showing that its history dependence mimics the optimal rule, a conclusion reinforced by Giannoni (2010). Eggertsson & Woodford (2003) showed the particular desirability of price level targeting in the presence of the ZLB, since it produces inflation after the bound is escaped. A later contribution by Nakov (2008) showed that this result survived taking a fully global solution, and Coibion, Gorodnichenko & Wieland (2012) showed that it still holds in a richer model. More recently, Basu & Bundick (2015) have argued that a response to the price level avoids the kinds of equilibrium non-existence problems stressed by Mendes (2011), while also solving the contractionary bias caused by the ZLB. Our argument is distinct from these; we showed that in the presence of the ZLB, inflation targeting rules are indeterminate, even conditional on an eventual return to inflation, whereas price level targeting rules produce determinacy, in the sense of the existence of a unique path returning to the inflationary steady state.

Our results are also distinct from those of Adão, Correia & Teles (2011) who showed that if the central bank is not constrained to respect the ZLB out of equilibrium, and if the central bank uses a rule that responds to the right hand side of the Euler equation, then a globally unique equilibrium may be produced, even without ruling out explosive beliefs about prices. Their rule has the flavour of a (future) price-targeting rule, due to the presence of future prices in the right-hand side of the Euler equation. Here though, we are assuming that the central bank must satisfy the ZLB even out of equilibrium, which makes it harder to produce

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[23] Since the Smets & Wouters (2003) model does not include trend growth, it is impossible to produce a steady-state value for nominal interest rates that is consistent with both the model and the data. We choose to follow the data, setting the steady-state of nominal interest rates to its mean level over the same sample period used by Smets & Wouters (2003), using data from the same source (Fagan, Henry & Mestre 2005).

[24] Data was again from the area-wide model database (Fagan, Henry & Mestre 2005).

uniqueness. Additionally, we do not require that the central bank can choose a knife-edge value for its response to the (future) price-level, or that it knows the precise form of agents' utility functions, both of which are apparently required by the rule of Adão, Correia & Teles and which may be difficult in practice. However, in line with the New Keynesian literature, we maintain the standard assumption that explosive paths for inflation are ruled out, an assumption which the knife-edge rules of Adão, Correia & Teles do not require.[25]

Somewhat contrary to our results, Armenter (2016) shows that in a simple otherwise linear NK model, if the central bank pursues Markov (discretionary) policy subject to an objective targeting inflation, nominal GDP or the price level, then the presence of a ZLB produces additional equilibria quite generally. This difference between our results and those of Armenter (2016) is driven both by the fact that we rule out getting stuck in the neighbourhood of the deflationary steady-state by assumption, and since we assume commitment to a rule.

7. Conclusion

This paper provides the first general theoretical results on existence and uniqueness for otherwise linear models with occasionally binding constraints, given terminal conditions. As such, it may be viewed as doing for models with occasionally binding constraints what Blanchard & Kahn (1980) did for linear models.

We provided necessary and sufficient conditions for the existence of a unique equilibrium, as well as such conditions for uniqueness when away from the bound, all conditional on the economy returning to a specific steady-state. We also provide a toolkit ("DynareOBC") which verifies these conditions. In our application to New Keynesian models, we showed that these conditions were violated in entirely standard models, rather than just being a consequence of policy rules responding to growth rates. In the presence of multiplicity, there is the potential for additional endogenous volatility from sunspots, so the welfare benefits of avoiding multiplicity may be substantial. Moreover, we saw that the additional equilibria may feature huge drops in output, providing further reasons for their avoidance.

Luckily, our results give policy makers a solution. Providing agents believe in an eventual return to inflation, pursuing a price level targeting rule will produce a determinate equilibrium in standard NK models, even in the presence of the ZLB. Consequently, if one believes the arguments for the Taylor principle in the absence of the ZLB, then one should advocate price level targeting if the ZLB constraint is inescapable.

In addition, we provided conditions for existence of any solution that converges to the "good" steady-state, conditions which again may be verified using our DynareOBC toolkit. We showed that under inflation targeting, standard NK models again failed to satisfy these conditions over all the space of state variables and shocks. Whereas the literature started by

[25] Note that the unstable solutions under price level targeting feature exponential growth in the logarithm of the price level, which also implies explosions in inflation rates.

Benhabib, Schmitt-Grohé & Uribe (2001a; 2001b) showed that the existence of a "bad" steady-state may imply additional volatility if agents long-run beliefs are not pinned down by the inflation target, here we showed that under inflation targeting, there was positive probability of arriving in a state from which there was no way for the economy to converge to the "good" steady-state. This in turn implies that agents should not place prior certainty on converging to the "good" steady-state, thus rationalising the beliefs required to get the kind of global multiplicity at the zero lower bound that these and other authors have focussed on.

Appendix A: Generalizations

It is straightforward to generalise the results of this paper to less restrictive otherwise linear models with occasionally binding constraints.

Firstly, if the constraint is on a variable other than $x_{1,t}$, or in another equation than the first, then all of the results go through as before, just by relabelling and rearranging. Furthermore, if the constraint takes the form of $z_{1,t} = \max\{z_{2,t}, z_{3,t}\}$, where $z_{1,t}$, $z_{2,t}$ and $z_{3,t}$ are linear expressions in the contemporaneous values, lags and leads of x_t, then, assuming without loss of generality that $z_{3,\cdot} > z_{2,\cdot}$ in steady-state, we have that $z_{1,t} - z_{2,t} = \max\{0, z_{3,t} - z_{2,t}\}$. Hence, adding a new auxiliary variable $x_{n+1,t}$, with the associated equation $x_{n+1,t} = z_{1,t} - z_{2,t}$, and replacing the constrained equation with $x_{n+1,t} = \max\{0, z_{3,t} - z_{2,t}\}$, we have a new equation in the form covered by our results. Moreover, if rather than a max we have a min, we just use the fact that if $z_{1,t} = \min\{z_{2,t}, z_{3,t}\}$, then $-z_{1,t} = \max\{-z_{2,t}, -z_{3,t}\}$, which is covered by the generalisation just established. The easiest encoding of the complementary slackness conditions, $z_t \geq 0$, $\lambda_t \geq 0$ and $z_t \lambda_t = 0$, is $0 = \min\{z_t, \lambda_t\}$, which is of this form.

To deal with multiple occasionally binding constraints, we use the representation from Holden and Paetz (2012). Suppose there are c constrained variables in the model. For $a \in \{1, \dots, c\}$, let $q^{(a)}$ be the path of the a^{th} constrained variable in the absence of all constraints. For $a, b \in \{1, \dots, c\}$, let $M^{(a,b)}$ be the matrix created by horizontally stacking the column vector relative impulse responses of the a^{th} constrained variable to magnitude 1 news shocks at horizon $0, \dots, T-1$ to the equation defining the b^{th} constrained variables. For example, if $c = 1$ so there is a single constraint, then we would have that $M^{(1,1)} = M$ as defined in equation (2). Finally, let:

$$q := \begin{bmatrix} q^{(1)} \\ \vdots \\ q^{(c)} \end{bmatrix}, \qquad M := \begin{bmatrix} M^{(1,1)} & \cdots & M^{(1,c)} \\ \vdots & \ddots & \vdots \\ M^{(c,1)} & \cdots & M^{(c,c)} \end{bmatrix},$$

and let y be a solution to the LCP (q, M). Then the vertically stacked paths of the constrained variables in a solution which satisfies these constraints is given by $q + My$, and Theorem 1 goes through as before.

Appendix B: Additional matrix properties and their relationships

Definition 7 ((Non-)Degenerate matrix) A matrix $M \in \mathbb{R}^{T \times T}$ is called a **non-degenerate matrix** if the principal minors of M are all non-zero. M is called a **degenerate matrix** if it is not a non-degenerate matrix.

Definition 8 (Sufficient matrices) Let $M \in \mathbb{R}^{T \times T}$. M is called **column sufficient** if M is a P_0-matrix, and for each principal sub-matrix $W := [M_{i,j}]_{i,j=k_1,\dots,k_S}$ of M, with zero determinant, and for each proper principal sub-matrix $[W_{i,j}]_{i,j=l_1,\dots,l_R}$ of W ($R < S$), with zero determinant, the columns of $[W_{i,j}]_{\substack{i=1,\dots,S \\ j=l_1,\dots,l_R}}$ do not form a basis for the column space of W.[26] M is called **row sufficient** if M' is column sufficient. M is called **sufficient** if it is column sufficient and row sufficient.

Definition 9 ((Strictly) Copositive) A matrix $M \in \mathbb{R}^{T \times T}$ is called **(strictly) copositive** if $M + M'$ is (strictly) semi-monotone.[27]

Cottle, Pang & Stone (2009a) note the following relationships between these classes (amongst others):

Lemma 4 The following hold:

1) All general positive semi-definite matrices are copositive and sufficient.
2) P_0 includes skew-symmetric, general p.s.d., sufficient and P matrices.
3) All P_0-matrices, and all copositive matrices are semi-monotone, and all P-matrices, and all strictly copositive matrices are strictly semi-monotone.

Appendix C: Other properties of the solution set

Conditions for having a finite or convex set of solutions are given in the following propositions:

Proposition 11 The LCP (q, M) has a finite (possibly zero) number of solutions for all $q \in \mathbb{R}^T$ if and only if M is non-degenerate. (Cottle, Pang & Stone 2009a)

Proposition 12 The LCP (q, M) has a convex (possibly empty) set of solutions for all $q \in \mathbb{R}^T$ if and only if M is column sufficient. (Cottle, Pang & Stone 2009a)

Appendix D: Results from dynamic programming

Alternative existence and uniqueness results for the infinite T problem can be established via dynamic programming methods, under the assumption that Problem 2 (OBC) comes from the first order conditions solution of a social planner problem. These have the advantage that

[26] This may be checked via the singular value decomposition.

[27] Väliaho (1986) contains an alternative characterisation which avoids solving any linear programming problems.

their conditions are potentially much easier to evaluate, though they also have somewhat limited applicability. We focus here on uniqueness results, since these are of greater interest.

Suppose that the social planner in some economy solves the following problem:

Problem 5 (Linear-Quadratic) Suppose $\mu \in \mathbb{R}^n$, $\Psi^{(0)} \in \mathbb{R}^{c\times 1}$ and $\Psi^{(1)} \in \mathbb{R}^{c\times 2n}$ are given, where $c \in \mathbb{N}$. Define $\tilde{\Gamma}: \mathbb{R}^n \to \mathbb{P}(\mathbb{R}^n)$ (where $\mathbb{P}$ denotes the power-set operator) by:

$$\tilde{\Gamma}(x) = \left\{ z \in \mathbb{R}^n \middle| 0 \le \Psi^{(0)} + \Psi^{(1)} \begin{bmatrix} x-\mu \\ z-\mu \end{bmatrix} \right\}, \tag{6}$$

for all $x \in \mathbb{R}^n$. *(Note: $\tilde{\Gamma}(x)$ will give the set of feasible values for next period's state if the current state is x. Equality constraints may be included by including an identical lower bound and upper bound.)* Define:

$$\widetilde{X} := \left\{ x \in \mathbb{R}^n \middle| \tilde{\Gamma}(x) \ne \emptyset \right\}, \tag{7}$$

and suppose without loss of generality that for all $x \in \mathbb{R}^n$, $\tilde{\Gamma}(x) \cap \widetilde{X} = \tilde{\Gamma}(x)$. *(Note: this means that the linear inequalities bounding $\widetilde{X}$ are already included in those in the definition of $\tilde{\Gamma}(x)$. It is without loss of generality as the planner will never choose an $\tilde{x} \in \tilde{\Gamma}(x)$ such that $\tilde{\Gamma}(\tilde{x}) = \emptyset$.)* Further define $\tilde{\mathcal{F}}: \widetilde{X} \times \widetilde{X} \to \mathbb{R}$ by:

$$\tilde{\mathcal{F}}(x,z) = u^{(0)} + u^{(1)} \begin{bmatrix} x-\mu \\ z-\mu \end{bmatrix} + \frac{1}{2} \begin{bmatrix} x-\mu \\ z-\mu \end{bmatrix}' \tilde{u}^{(2)} \begin{bmatrix} x-\mu \\ z-\mu \end{bmatrix}, \tag{8}$$

for all $x, z \in \widetilde{X}$, where $u^{(0)} \in \mathbb{R}$, $u^{(1)} \in \mathbb{R}^{1\times 2n}$ and $\tilde{u}^{(2)} = \tilde{u}^{(2)\prime} \in \mathbb{R}^{2n\times 2n}$ are given. Finally, suppose $x_0 \in \widetilde{X}$ is given and $\beta \in (0,1)$, and choose $x_1, x_2, \ldots$ to maximise:

$$\liminf_{T\to\infty} \sum_{t=1}^{T} \beta^{t-1} \tilde{\mathcal{F}}(x_{t-1}, x_t) \tag{9}$$

subject to the constraints that for all $t \in \mathbb{N}^+$, $x_t \in \tilde{\Gamma}(x_{t-1})$.

To ensure the problem is well behaved, we make the following assumption:

Assumption 4 $\tilde{u}^{(2)}$ is negative-definite.

In Appendix H.10, online, we establish the following (unsurprising) result:

Proposition 13 If either $\widetilde{X}$ is compact, or, $\tilde{\Gamma}(x)$ is compact valued and $x \in \tilde{\Gamma}(x)$ for all $x \in \widetilde{X}$, then for all $x_0 \in \widetilde{X}$, there is a unique path $(x_t)_{t=0}^{\infty}$ which solves Problem 5 (Linear-Quadratic).

We wish to use this result to establish the uniqueness of the solution to the first order conditions. The Lagrangian for our problem is given by:

$$\sum_{t=1}^{\infty} \beta^{t-1} \left[\tilde{\mathcal{F}}(x_{t-1}, x_t) + \lambda_{\Psi,t}' \left[\Psi^{(0)} + \Psi^{(1)} \begin{bmatrix} x_{t-1}-\mu \\ x_t-\mu \end{bmatrix} \right] \right], \tag{10}$$

for some KKT-multipliers $\lambda_t \in \mathbb{R}^c$ for all $t \in \mathbb{N}^+$. Taking the first order conditions leads to the following necessary KKT conditions, for all $t \in \mathbb{N}^+$:

$$0 = u_{\cdot,2}^{(1)} + \begin{bmatrix} x_{t-1}-\mu \\ x_t-\mu \end{bmatrix}' \tilde{u}_{\cdot,2}^{(2)} + \lambda_t' \Psi_{\cdot,2}^{(1)} + \beta \left[u_{\cdot,1}^{(1)} + \begin{bmatrix} x_t-\mu \\ x_{t+1}-\mu \end{bmatrix}' \tilde{u}_{\cdot,1}^{(2)} + \lambda_{t+1}' \Psi_{\cdot,1}^{(1)} \right], \tag{11}$$

$$0 \le \Psi^{(0)} + \Psi^{(1)} \begin{bmatrix} x_{t-1}-\mu \\ x_t-\mu \end{bmatrix}, \qquad 0 \le \lambda_t, \qquad 0 = \lambda_t \circ \left[\Psi^{(0)} + \Psi^{(1)} \begin{bmatrix} x_{t-1}-\mu \\ x_t-\mu \end{bmatrix} \right], \tag{12}$$

where subscripts 1 and 2 refer to blocks of rows or columns of length n. Additionally, for μ to be the steady-state of x_t and $\bar{\lambda}$ to be the steady-state of λ_t, we require:

$$0 = u^{(1)}_{\cdot,2} + \bar{\lambda}'\Psi^{(1)}_{\cdot,2} + \beta\big[u^{(1)}_{\cdot,1} + \bar{\lambda}'\Psi^{(1)}_{\cdot,1}\big], \tag{13}$$

$$0 \le \Psi^{(0)}, \qquad 0 \le \bar{\lambda}, \qquad 0 = \bar{\lambda} \circ \Psi^{(0)}. \tag{14}$$

In Appendix H.11, online, we prove the following result:

Proposition 14 Suppose that for all $t \in \mathbb{N}$, $(x_t)_{t=1}^{\infty}$ and $(\lambda_t)_{t=1}^{\infty}$ satisfy the KKT conditions given in equations (11) and (12), and that as $t \to \infty$, $x_t \to \mu$ and $\lambda_t \to \bar{\lambda}$, where μ and λ satisfy the steady-state KKT conditions given in equations (13) and (14). Then $(x_t)_{t=1}^{\infty}$ solves Problem 5 (Linear-Quadratic). If, further, either condition of Proposition 13 is satisfied, then $(x_t)_{t=1}^{\infty}$ is the unique solution to Problem 5 (Linear-Quadratic), and there can be no other solutions to the KKT conditions given in equations (11) and (12) satisfying $x_t \to \mu$ and $\lambda_t \to \bar{\lambda}$ as $t \to \infty$.

Now, it is possible to convert the KKT conditions given in equations (11) and (12) into a problem in the form of the multiple-bound generalisation of Problem 2 (OBC) quite generally. To see this, first note that we may rewrite equation (11) as:

$$\begin{aligned}0 = u^{(1)\prime}_{\cdot,2} + \tilde{u}^{(2)}_{2,1}(x_{t-1} - \mu) + \tilde{u}^{(2)}_{2,2}(x_t - \mu) + \Psi^{(1)\prime}_{\cdot,2}\lambda_t \\ + \beta\big[u^{(1)\prime}_{\cdot,1} + \tilde{u}^{(2)}_{1,1}(x_t - \mu) + \tilde{u}^{(2)}_{1,2}(x_{t+1} - \mu) + \Psi^{(1)\prime}_{\cdot,1}\lambda_{t+1}\big].\end{aligned}$$

Now, $\tilde{u}^{(2)}_{2,2} + \beta u^{(2)}_{1,1}$ is negative definite, hence we may define $\mho := \Psi^{(1)}_{\cdot,2}\big[\tilde{u}^{(2)}_{2,2} + \beta\tilde{u}^{(2)}_{1,1}\big]^{-1}$, so:

$$\begin{aligned}&\Psi^{(0)} + \Psi^{(1)}\begin{bmatrix} x_{t-1} - \mu \\ x_t - \mu \end{bmatrix} \\ &= \Psi^{(0)} + \big(\Psi^{(1)}_{\cdot,1} - \mho\tilde{u}^{(2)}_{2,1}\big)(x_{t-1} - \mu) - \mho\Big[u^{(1)\prime}_{\cdot,2} + \beta\big[u^{(1)\prime}_{\cdot,1} + \tilde{u}^{(2)}_{1,2}(x_{t+1} - \mu) + \Psi^{(1)\prime}_{\cdot,1}\lambda_{t+1}\big]\Big] \\ &- \Psi^{(1)}_{\cdot,2}\big[\tilde{u}^{(2)}_{2,2} + \beta\tilde{u}^{(2)}_{1,1}\big]^{-1}\Psi^{(1)\prime}_{\cdot,2}\lambda_t.\end{aligned} \tag{15}$$

Moreover, equation (12) implies that if the k^{th} element of $\Psi^{(0)} + \Psi^{(1)}\begin{bmatrix} x_{t-1} - \mu \\ x_t - \mu \end{bmatrix}$ is positive, then the k^{th} element of λ_t is zero, so:

$$\Psi^{(0)} + \Psi^{(1)}\begin{bmatrix} x_{t-1} - \mu \\ x_t - \mu \end{bmatrix} = \max\{0, z_t\}, \tag{16}$$

where:

$$\begin{aligned}z_t := \Psi^{(0)} + \big(\Psi^{(1)}_{\cdot,1} - \mho\tilde{u}^{(2)}_{2,1}\big)(x_{t-1} - \mu) - \mho\Big[u^{(1)\prime}_{\cdot,2} + \beta\big[u^{(1)\prime}_{\cdot,1} + \tilde{u}^{(2)}_{1,2}(x_{t+1} - \mu) + \Psi^{(1)\prime}_{\cdot,1}\lambda_{t+1}\big]\Big] \\ - \Big[\Psi^{(1)}_{\cdot,2}\big[\tilde{u}^{(2)}_{2,2} + \beta\tilde{u}^{(2)}_{1,1}\big]^{-1}\Psi^{(1)\prime}_{\cdot,2} + \mathcal{W}\Big]\lambda_t,\end{aligned}$$

and $\mathcal{W} \in \mathbb{R}^{c \times c}$ is an arbitrary, positive diagonal matrix. A natural choice is:

$$\mathcal{W} := -\operatorname{diag}^{-1}\operatorname{diag}\Big[\Psi^{(1)}_{\cdot,2}\big[\tilde{u}^{(2)}_{2,2} + \beta\tilde{u}^{(2)}_{1,1}\big]^{-1}\Psi^{(1)\prime}_{\cdot,2}\Big],$$

providing this is positive (it is nonnegative at least as $\tilde{u}^{(2)}_{2,2} + \beta\tilde{u}^{(2)}_{1,1}$ is negative definite), where the diag operator maps matrices to a vector containing their diagonal, and diag^{-1} maps vectors to a matrix with the given vector on the diagonal, and zeros elsewhere.

We claim that we may replace equation (12) with equation (16) without changing the model. We have already shown that equation (12) implies equation (16), so we just have to prove the converse. We continue to suppose equation (11) holds, and thus, so too does equation (15). Then, from subtracting equation (15) from equation (16), we have that:

$$\mathcal{W}\lambda_t = \max\{-z_t, 0\}.$$

Hence, as $\mathcal{W}$ is a positive diagonal matrix, and the right-hand side is nonnegative, $\lambda_t \geq 0$. Furthermore, the kth element of λ_t is non-negative if and only if the kth element of z_t is non-positive (as $\mathcal{W}$ is a positive diagonal matrix), which in turn holds if and only if the kth element of $\Psi^{(0)} + \Psi^{(1)} \begin{bmatrix} x_{t-1} - \mu \\ x_t - \mu \end{bmatrix}$ is equal to zero, by equation (16). Thus equation (12) is satisfied.

Combined with our previous results, this gives the following proposition:

Proposition 15 Suppose we are given a problem in the form of Problem 5 (Linear-Quadratic). Then, the KKT conditions of that problem may be placed into the form of the multiple-bound generalisation of Problem 2 (OBC). Let (q_{x_0}, M) be the infinite LCP corresponding to this representation, given initial state $x_0 \in \widetilde{X}$. Then, if y is a solution to the LCP, $q_{x_0} + My$ gives the stacked paths of the bounded variables in a solution to Problem 5 (Linear-Quadratic). If, further, either condition of Proposition 13 is satisfied, then this LCP has a unique solution for all $x_0 \in \widetilde{X}$, which gives the unique solution to Problem 5 (Linear-Quadratic), and, for sufficiently large T^*, the finite LCP $\left(q_{x_0}^{(T^*)}, M^{(T^*)}\right)$ has a unique solution $y^{(T^*)}$ for all $x_0 \in \widetilde{X}$, where $q_{x_0}^{(T^*)} + M^{(T^*)} y^{(T^*)}$ gives the first T^* periods of the stacked paths of the bounded variables in a solution to Problem 5 (Linear-Quadratic).

This proposition provides some evidence that the LCP will have a unique solution when it is generated from a dynamic programming problem with a unique solution. In Appendix F, online, we derive similar results for models with more general constraints and objective functions. The proof of this proposition also showed an alternative method for converting KKT conditions into equations of the form handled by our methods.

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Appendix E: Relationship between multiplicity under perfect-foresight, and multiplicity under rational expectations

By augmenting the state-space appropriately, the first order conditions of a general, non-linear, rational expectations, DSGE model may always be placed in the form:

$$0 = \mathbb{E}_t \hat{f}(\hat{x}_{t-1}, \hat{x}_t, \hat{x}_{t+1}, \sigma\varepsilon_t),$$

for all $t \in \mathbb{Z}$, where $\sigma \in [0,1]$, $\hat{f}: (\mathbb{R}^{\hat{n}})^3 \times \mathbb{R}^m \to \mathbb{R}^{\hat{n}}$, and where for all $t \in \mathbb{Z}$, $\hat{x}_t \in \mathbb{R}^{\hat{n}}$, $\varepsilon_t \in \mathbb{R}^m$, $\mathbb{E}_{t-1}\varepsilon_t = 0$, and $\mathbb{E}_t\hat{x}_t = \hat{x}_t$. Since f is arbitrary, without loss of generality we may further assume that $\varepsilon_t \sim \text{NIID}(0, I)$. We further assume:

Assumption 5 $\hat{f}$ is everywhere continuous.

The continuity of $\hat{f}$ does rule out some models, but all models in which the only source of non-differentiability is a max or min operator (like those studied in this paper and its computational companion (Holden 2016)) will have a continuous $\hat{f}$.

Now, by further augmenting the state space, we can then find a continuous function $f: (\mathbb{R}^n)^3 \times \mathbb{R}^m \to \mathbb{R}^n$ such that for all $t \in \mathbb{Z}$:

$$0 = f(x_{t-1}, x_t, \mathbb{E}_t x_{t+1}, \sigma\varepsilon_t),$$

where for all $t \in \mathbb{Z}$, $x_t \in \mathbb{R}^n$ and $\mathbb{E}_t x_t = x_t$.[28] A solution to this model is given by a policy function. Given f is continuous, it is natural to restrict attention to continuous policy functions.[29] Furthermore, given the model's transversality conditions, we are usually only interested in stationary, Markov solutions, so the policy function will not be a function of t or of lags of the state. Additionally, in this paper we are only interested in solutions in which the deterministic model converges to some particular steady-state μ. Thus, we make the following assumption:

Assumption 6 The policy function is given by a continuous function: $g: [0,1] \times \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$, such that for all $(\sigma, x, e) \in [0,1] \times \mathbb{R}^n \times \mathbb{R}^m$:

$$0 = f(x, g(\sigma, x, e), \mathbb{E}_\varepsilon g(\sigma, g(\sigma, x, e), \sigma\varepsilon), e),$$

[28] For example, we may use the equations: $\hat{x}_t^{\circ} = \hat{x}_{t-1}$, $\hat{\varepsilon}_t = \varepsilon_t$, $z_t = \hat{f}(\hat{x}_{t-1}^{\circ}, \hat{x}_{t-1}, \hat{x}_t, \sigma\hat{\varepsilon}_{t-1})$, $0 = \mathbb{E}_t z_{t+1}$, with $x_t \coloneqq [\hat{x}_t' \quad \hat{x}_t^{\circ\prime} \quad \hat{\varepsilon}_t' \quad z_t']'$.

[29] Note also that in standard dynamic programming applications, the policy function will be continuous. See e.g. Theorem 9.8 of Stokey, Lucas, and Prescott (1989).

where $\varepsilon \sim \mathrm{N}(0, I)$ and $\mathbb{E}_\varepsilon$ denotes an expectation with respect to ε. Furthermore, for all $x_0 \in \mathbb{R}^n$, the recurrence $x_t = g(0, x_{t-1}, 0)$ satisfies $x_t \to \mu$ as $t \to \infty$.

To produce a lower bound on the number of policy functions satisfying Assumption 6, we need two further assumptions. The first assumption just gives the existence of the "time iteration" (a.k.a. "policy function iteration") operator $\mathcal{T}$, and ensures that it has a fixed point.

Assumption 7 Let $\mathcal{G}$ denote the space of all continuous functions $[0,1] \times \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$. We assume there exists a function $\mathcal{T}: \mathcal{G} \to \mathcal{G}$ such that for all $(g, \sigma, x, e) \in \mathcal{G} \times [0,1] \times \mathbb{R}^n \times \mathbb{R}^m$:

$$0 = f\left(x, \mathcal{T}(g)(\sigma, x, e), \mathbb{E}_\varepsilon g\left(\sigma, \mathcal{T}(g)(\sigma, x, e), \sigma\varepsilon\right), e\right).$$

We further assume that if there exists some $(g, \sigma) \in \mathcal{G} \times [0,1]$ such that for all $(x, e) \in \mathbb{R}^n \times \mathbb{R}^m$:

$$0 = f\left(x, g(\sigma, x, e), \mathbb{E}_\varepsilon g\left(\sigma, g(\sigma, x, e), \sigma\varepsilon\right), e\right),$$

then for all $(x, e) \in \mathbb{R}^n \times \mathbb{R}^m$, $\mathcal{T}(g)(\sigma, x, e) = g(\sigma, x, e)$.

The second assumption ensures that time iteration always converges when started from a solution to the model with no uncertainty after the current period. This is a weak assumption since the policy functions under uncertainty are invariably close to the policy function in the absence of uncertainty.

Assumption 8 Let $h: \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$ be a continuous function giving a solution to the model in which there is no future uncertainty, i.e. for all $(x, e) \in \mathbb{R}^n \times \mathbb{R}^m$:

$$0 = f(x, h(x, e), h(h(x, e), 0), e).$$

Further, define $g_{h,0} \in \mathcal{G}$ by $g_{h,0}(\sigma, x, e) = h(x, e)$ for all $(\sigma, x, e) \in [0,1] \times \mathbb{R}^n \times \mathbb{R}^m$, and define $g_{h,k} \in \mathcal{G}$ inductively by $g_{h,k+1} = \mathcal{T}(g_{h,k})$ for all $k \in \mathbb{N}$. Then there exists some $g_{h,\infty} \in \mathcal{G}$ such that $g_{h,\infty} = \mathcal{T}(g_{h,\infty})$ and for all $(\sigma, x, e) \in [0,1] \times \mathbb{R}^n \times \mathbb{R}^m$, $g_{h,k}(\sigma, x, e) \to g_{h,\infty}(\sigma, x, e)$ as $k \to \infty$.

Note, by construction, if h is as in Assumption 8, then for all $(x, e) \in \mathbb{R}^n \times \mathbb{R}^m$:

$$0 = f(x, g_{h,0}(0, x, e), \mathbb{E}_\varepsilon g_{h,0}(0, g_{h,0}(0, x, e), 0\varepsilon), e).$$

Hence, by Assumption 7, for all $k \in \mathbb{N}$, all $x(x, e) \in \mathbb{R}^n \times \mathbb{R}^m$, $g_{h,k}(0, x, e) = g_{h,0}(0, x, e)$. Consequently, for all $(x, e) \in \mathbb{R}^n \times \mathbb{R}^m$, $g_{h,\infty}(0, x, e) = g_{h,0}(0, x, e) = h(x, e)$.

Now suppose that h_1 and h_2 were as in Assumption 8, and that there exists $(x, e) \in \mathbb{R}^n \times \mathbb{R}^m$, such that $h_1(x, e) \neq h_2(x, e)$. Then, by the continuity of $g_{h_1,\infty}$ and $g_{h_2,\infty}$, there is some $\mathcal{S} \subseteq [0,1] \times \mathbb{R}^n \times \mathbb{R}^m$ of positive measure, with $(0, x, e) \in \mathcal{S}$, such that for all $(\sigma, x, e) \in \mathcal{S}$, $g_{h_1,\infty}(\sigma, x, e) \neq g_{h_2,\infty}(\sigma, x, e)$. Hence, the rational expectations policy functions differ, at least for small σ. Thus, if Assumption 7 and Assumption 8 are satisfied, there are at least as many policy functions satisfying Assumption 6 as there are solutions to the model in which there is no future uncertainty.

Appendix F: Results from and for general dynamic programming problems

Here we consider non-linear dynamic programming problems with general objective functions. Consider then the following generalisation of Problem 5 (Linear-Quadratic):

Problem 6 (Non-linear) Suppose $\Gamma: \mathbb{R}^n \to \mathbb{P}(\mathbb{R}^n)$ is a given compact, convex valued continuous function. Define $X := \{x \in \mathbb{R}^n | \Gamma(x) \neq \emptyset\}$, and suppose without loss of generality that for all $x \in \mathbb{R}^n$, $\Gamma(x) \cap X = \Gamma(x)$. Further suppose that $\mathcal{F}: X \times X \to \mathbb{R}$ is a given twice continuously differentiable, concave function, and that $x_0 \in X$ and $\beta \in (0,1)$ are given. Choose $x_1, x_2, \ldots$ to maximise:

$$\liminf_{T \to \infty} \sum_{t=1}^{T} \beta^{t-1} \mathcal{F}(x_{t-1}, x_t),$$

subject to the constraints that for all $t \in \mathbb{N}^+$, $x_t \in \Gamma(x_{t-1})$.

For tractability, we make the following additional assumption, which enables us to uniformly approximate Γ by a finite number of inequalities:

Assumption 9 X is compact.

Then, by Theorem 4.8 of Stokey, Lucas, and Prescott (1989), there is a unique solution to Problem 6 (Non-linear) for any x_0. We further assume the following to ensure that there is a natural point to approximate around:[30]

Assumption 10 There exists $\mu \in X$ such that for any given $x_0 \in X$, in the solution to Problem 6 (Non-linear) with that x_0, as $t \to \infty$, $x_t \to \mu$.

Having defined μ, we can let $\tilde{\mathcal{F}}$ be a second order Taylor approximation to $\mathcal{F}$ around μ, which will take the form of equation (8). Assumption 4 will be satisfied for this approximation thanks to the concavity of $\mathcal{F}$. To apply the previous results, we also then need to approximate the constraints.

Suppose first that the graph of Γ is convex, i.e. the set $\{(x, z) | x \in X, z \in \Gamma(x)\}$ is convex. Since it is also compact, by Assumption 9, for any $\epsilon > 0$, there exists $c \in \mathbb{N}$, $\Psi^{(0)} \in \mathbb{R}^{c \times 1}$ and $\Psi^{(1)} \in \mathbb{R}^{c \times 2n}$ such that with $\tilde{\Gamma}$ defined as in equation (6) and $\tilde{X}$ defined as in equation (7):

1) $\mu \in \tilde{X} \subseteq X$,
2) for all $x \in X$, there exists $\tilde{x} \in \tilde{X}$ such that $\|x - \tilde{x}\|_2 < \epsilon$,
3) for all $x \in \tilde{X}$, $\tilde{\Gamma}(x) \subseteq \Gamma(x)$,
4) for all $x \in \tilde{X}$, and for all $z \in \Gamma(x)$, there exists $\tilde{z} \in \tilde{\Gamma}(x)$ such that $\|z - \tilde{z}\|_2 < \epsilon$.

(This follows from standard properties of convex sets.) Then, by our previous results, the following proposition is immediate:

[30] If X is convex, then the existence of a fixed point of the policy function is a consequence of Brouwer's Fixed Point Theorem, but there is no reason the fixed point guaranteed by Brouwer's Theorem should be even locally attractive.

Proposition 16 Suppose we are given a problem in the form of Problem 6 (Non-linear) (and which satisfies Assumption 9 and Assumption 10). If the graph of Γ is convex, then we can construct a problem in the form of the multiple-bound generalisation of Problem 2 (OBC) which encodes a local approximation to the original dynamic programming problem around $x_t = \mu$. Furthermore, the LCP corresponding to this approximation will have a unique solution for all $x_0 \in \widetilde{X}$. Moreover, the approximation is consistent for quadratic objectives in the sense that as the number of inequalities used to approximate Γ goes to infinity, the approximate value function converges uniformly to the true value function.

Unfortunately, if the graph of Γ is non-convex, then we will not be able to derive similar results. To see the best we could do along similar proof lines, here we merely sketch the construction of an approximation to the graph of Γ in this case. We will need to assume that there exists $z \in \operatorname{int} \Gamma(x)$ for all $x \in X$, which precludes the existence of equality constraints.[31] We first approximate the graph of Γ by a polytope (i.e. n dimensional polygon) contained in the graph of Γ such that all points in the graph of Γ are within $\frac{\epsilon}{2}$ of a point in the polytope. Then, providing ϵ is sufficiently small, for each simplicial surface element of the polytope, indexed by $k \in \{1, \dots, c\}$, we can find a quadratic function $q_k : X \times X \to \mathbb{R}$ with:

$$q_k = \Psi_k^{(0)} + \Psi_{k,\cdot}^{(1)} \begin{bmatrix} x - \mu \\ z - \mu \end{bmatrix} + \begin{bmatrix} x - \mu \\ z - \mu \end{bmatrix}' \Psi_k^{(2)} \begin{bmatrix} x - \mu \\ z - \mu \end{bmatrix}$$

for all $x, z \in X$ and such that q_k is zero at the corners of the simplicial surface element, such that q_k is nonpositive on its surface, such that $\Psi_k^{(2)}$ is symmetric positive definite, and such that all points in the polytope are within $\frac{\epsilon}{2}$ of a point in the set:

$$\{(x, z) \in X \times X | \forall k \in \{1, \dots, S\}, 0 \le q_k(x, z)\}.$$

This gives a set of quadratic constraints that approximate Γ. If we then define:

$$\tilde{u}^{(2)} := u^{(2)} + \sum_{k=1}^{c} \bar{\lambda}'_{\Psi,k} \Psi_k^{(2)},$$

where $u^{(2)}$ is the Hessian of $\mathcal{F}$, then the Lagrangian in equation (10) is the same as what would be obtained from taking a second order Taylor approximation to the Lagrangian of the problem of maximising our non-linear objective subject to the approximate quadratic constraints, suggesting it may perform acceptably well for x near μ, along similar lines to the results of Levine, Pearlman, and Pierse (2008) and Benigno & Woodford (2012). However, existence of a unique solution to the original problem cannot be used to establish even the existence of a solution of the approximated problem, since only linear approximations to the quadratic constraints would be imposed by our algorithm, giving a reduced choice set (as the quadratic terms are positive definite).

[31] This is often not too much of a restriction, since equality constraints may be substituted out.

Appendix G: The BPY model with shadow interest rate persistence

Following BPY (2013), we introduce persistence in the shadow interest rate by replacing the previous Taylor rule with $x_{i,t} = \max\{0, x_{d,t}\}$, where $x_{d,t}$, the shadow nominal interest rate is given by:

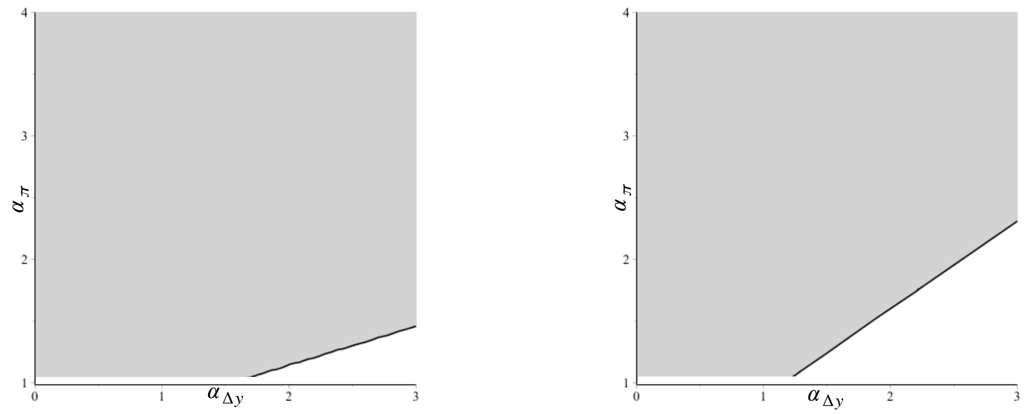
$$x_{d,t} = (1-\rho)\left(1 - \beta + \alpha_{\Delta y}\left(x_{y,t} - x_{y,t-1}\right) + \alpha_\pi x_{\pi,t}\right) + \rho x_{d,t-1}.$$

It is easy to verify that this may be put in the form of Problem 2 (OBC), and that with $\beta \in (0,1)$, $\gamma, \sigma, \alpha_{\Delta y} \in (0, \infty)$, $\alpha_\pi \in (1, \infty)$, $\rho \in (-1,1)$, Assumption 2 is satisfied. For our numerical exercise, we again set $\sigma = 1$, $\beta = 0.99$, $\gamma = \frac{(1-0.85)(1-\beta(0.85))}{0.85}(2+\sigma)$, $\rho = 0.5$, following BPY.

In Figure 8, we plot the regions in $(\alpha_{\Delta y}, \alpha_\pi)$ space in which M is a P-matrix (P_0-matrix) when $T = 2$ or $T = 4$. As may be seen, in the smaller T case, the P-matrix region is much larger. This relationship appears to continue to hold for both larger and smaller T, with the equivalent $T = 1$ plot being almost entirely shaded, and the large T plot tending to the equivalent plot from the model without monetary policy persistence. Intuitively, the persistence in the shadow nominal interest rate dampens the immediate response of nominal interest rates to inflation and output growth, making it harder to induce a ZLB episode over short-horizons.

Further evidence that the long-horizon behaviour is the same as in the model without persistence is provided by the fact that with $\alpha_\pi = 1.5$ and $\alpha_{\Delta y} = 1.05$, [32] then M is a P-matrix with $T = 20$. Moreover, from Proposition 6 with $T = 50$, we have that $\varsigma > 6.385 \times 10^{-8}$, so M is an S-matrix for all sufficiently large T, by Corollary 3.

On the other hand, with $\alpha_\pi = 1.5$ and $\alpha_{\Delta y} = 1.51$, then with $T = 200$, M is not an S-matrix,[33] meaning that for all sufficiently large T, M is not a P-matrix, so there are sometimes multiple solutions. Additionally, from Proposition 6 with $T = 200$, $\varsigma \le 0 +$ numerical error, meaning that it is likely that the model does not always possess a solution, no matter how high is T.



$T = 2$ **$T = 4$**

Figure 8: Regions in which M is a P-matrix (shaded grey) or a P_0-matrix (shaded grey, plus the black line), when $T = 2$ (left) or $T = 4$ (right).

[32] Results for larger $\alpha_{\Delta y}$ were impossible due to numerical errors.

[33] This was verified a second way by checking that $-M'$ was an S0-matrix, as discussed in footnote 12.

Appendix H: Proofs

Appendix H.1: Proof of Lemma 1

Since $y_{1,t-1} = 0$ for $t > T$, and using Assumption 1, $(x_{T+1} - \mu) = F(x_T - \mu)$, so with $t = T$, defining $s_{T+1} := 0$, $(x_{t+1} - \mu) = s_{t+1} + F(x_t - \mu)$. Proceeding now by backwards induction on t, note that $0 = A(x_{t-1} - \mu) + B(x_t - \mu) + CF(x_t - \mu) + Cs_{t+1} + I_{\cdot,1}y_{t,0}$, so:

$$\begin{aligned}(x_t - \mu) &= -(B + CF)^{-1}[A(x_{t-1} - \mu) + Cs_{t+1} + I_{\cdot,1}y_{t,0}] \\ &= F(x_{t-1} - \mu) - (B + CF)^{-1}(Cs_{t+1} + I_{\cdot,1}y_{t,0}),\end{aligned}$$

i.e., if we define: $s_t := -(B + CF)^{-1}(Cs_{t+1} + I_{\cdot,1}y_{t,0})$, then $(x_t - \mu) = s_t + F(x_{t-1} - \mu)$. By induction then, this holds for all $t \in \{1, \dots, T\}$, as required.[34]

Appendix H.2: Proof of Lemma 2

From the definition of Problem 2 (OBC), we also have that for all $t \in \mathbb{N}^+$, $0 = A\big(x_{t-1}^{(2)} - \mu\big) + B\big(x_t^{(2)} - \mu\big) + C\big(x_{t+1}^{(2)} - \mu\big) + I_{\cdot,1}e_t$. Furthermore, if $t > T$, then $t > T'$, and hence $e_t = 0$. Hence, by Assumption 1, $\big(x_{T+1}^{(2)} - \mu\big) = F\big(x_T^{(2)} - \mu\big)$. Thus, much as before, with $t = T$, defining $\tilde{s}_{T+1} := 0$, $\big(x_{t+1}^{(2)} - \mu\big) = \tilde{s}_{t+1} + F\big(x_t^{(2)} - \mu\big)$. Consequently, $0 = A\big(x_{t-1}^{(2)} - \mu\big) + B\big(x_t^{(2)} - \mu\big) + CF\big(x_t^{(2)} - \mu\big) + C\tilde{s}_{t+1} + I_{\cdot,1}e_t$, so $\big(x_t^{(2)} - \mu\big) = F\big(x_{t-1}^{(2)} - \mu\big) - (B + CF)^{-1}(C\tilde{s}_{t+1} + I_{\cdot,1}e_t)$, i.e., if we define: $\tilde{s}_t := -(B + CF)^{-1}(C\tilde{s}_{t+1} + I_{\cdot,1}e_t)$, then $\big(x_t^{(2)} - \mu\big) = \tilde{s}_t + F\big(x_{t-1}^{(2)} - \mu\big)$. As before, by induction this must hold for all $t \in \{1, \dots, T\}$. By comparing the definitions of s_t and $\tilde{s}_t$, and the laws of motion of x_t under both problems, we then immediately have that if Problem 3 (News) is started with $x_0 = x_0^{(2)}$ and $y_0 = e'_{1:T}$, then $x_t^{(2)}$ solves Problem 3 (News). Conversely, if $x_t^{(2)}$ solves Problem 3 (News) for some y_0, then from the laws of motion of x_t under both problems it must be the case that $\tilde{s}_t = s_t$ for all $t \in \mathbb{N}$, and hence from the definitions of s_t and $\tilde{s}_t$, we have that $y_0 = e'_{1:T}$. This establishes the result.

Appendix H.3: Proof of Theorem 1

Suppose that $y_0 \in \mathbb{R}^T$ is such that $y_0 \geq 0$, $x_{1,1:T}^{(3)} \circ y'_0 = 0$ and $x_{1,t}^{(3)} \geq 0$ for all $t \in \mathbb{N}$, where $x_t^{(3)}$ is the unique solution to Problem 3 (News) when started at x_0, y_0. We would like to prove that in this case $x_t^{(3)}$ must also be a solution to Problem 2 (OBC). I.e., we must prove that for all $t \in \mathbb{N}^+$:

$$\begin{aligned}x_{1,t}^{(3)} &= \max\big\{0, I_{1,\cdot}\mu + A_{1,\cdot}\big(x_{t-1}^{(3)} - \mu\big) + (B_{1,\cdot} + I_{1,\cdot})\big(x_t^{(3)} - \mu\big) + C_{1,\cdot}\big(x_{t+1}^{(3)} - \mu\big)\big\}, \qquad (17) \\ (A_{-1,\cdot} + B_{-1,\cdot} + C_{-1,\cdot})\mu &= A_{-1,\cdot}x_{t-1}^{(3)} + B_{-1,\cdot}x_t^{(3)} + C_{-1,\cdot}x_{t+1}^{(3)}.\end{aligned}$$

By the definition of Problem 3 (News), the latter equation must hold with equality. Hence, we just need to prove that equation (17) holds for all $t \in \mathbb{N}^+$. So, let $t \in \mathbb{N}^+$. Now, if $x_{1,t}^{(3)} > 0$, then $y_{t,0} = 0$, by the complementary slackness type condition ($x_{1,1:T}^{(3)} \circ y'_0 = 0$).

[34] This representation of the solution to Problem 3 (News) was inspired by that of Anderson (2015).

Thus, from the definition of Problem 3 (News):

$$\begin{aligned}x_{1,t}^{(3)} &= I_{1,\cdot}\mu + A_{1,\cdot}(x_{t-1}^{(3)}-\mu) + (B_{1,\cdot}+I_{1,\cdot})(x_t^{(3)}-\mu) + C_{1,\cdot}(x_{t+1}^{(3)}-\mu)\\ &= \max\{0, I_{1,\cdot}\mu + A_{1,\cdot}(x_{t-1}^{(3)}-\mu) + (B_{1,\cdot}+I_{1,\cdot})(x_t^{(3)}-\mu) + C_{1,\cdot}(x_{t+1}^{(3)}-\mu)\},\end{aligned}$$

as required. The only remaining case is that $x_{1,t}^{(3)} = 0$ (since $x_{1,t}^{(3)} \ge 0$ for all $t \in \mathbb{N}$, by assumption), which implies that:

$$\begin{aligned}x_{1,t}^{(3)} = 0 &= A_{1,\cdot}(x_{t-1}-\mu) + B_{1,\cdot}(x_t-\mu) + C_{1,\cdot}(x_{t+1}-\mu) + y_{t,0}\\ &= I_{1,\cdot}\mu + A_{1,\cdot}(x_{t-1}-\mu) + (B_{1,\cdot}+I_{1,\cdot})(x_t-\mu) + C_{1,\cdot}(x_{t+1}-\mu) + y_{t,0},\end{aligned}$$

by the definition of Problem 3 (News). Thus:

$$I_{1,\cdot}\mu + A_{1,\cdot}(x_{t-1}-\mu) + (B_{1,\cdot}+I_{1,\cdot})(x_t-\mu) + C_{1,\cdot}(x_{t+1}-\mu) = -y_{t,0} \le 0.$$

Consequently, equation (17) holds in this case too, completing the proof.

Appendix H.4: Proof of Proposition 1

Let $\mathcal{M} \in \mathbb{R}^{T\times T}$. Consider a model with the following equations:

$$\begin{gathered}a_t = \max\{0, b_t\},\\ a_t = 1 + \sum_{j=1}^{T}\sum_{k=1}^{T} \mathcal{M}_{j,k}(c_{j-1,k-1,t} - c_{j,k,t}) + d_{0,t},\\ c_{0,0,t} = a_t - b_t,\\ c_{0,k,t} = \mathbb{E}_t c_{0,k-1,t+1}, \qquad \forall k \in \{1,\dots,T\},\\ c_{j,k,t} = c_{j-1,k,t-1}, \qquad \forall j \in \{1,\dots,T\}, k \in \{0,\dots,T\},\\ d_{k,t} = d_{k+1,t-1}, \qquad \forall k \in \{0,\dots,T-1\},\\ d_{T,t} = 0\end{gathered}$$

with steady-state $a_\cdot = b_\cdot = 1$, $c_{j,k,\cdot} = 0$, $d_{k,\cdot} = 0$ for all $j,k, \in \{0,\dots,T\}$. Defining:

$$x_t := [a_t \quad b_t \quad (\operatorname{vec} c_{\cdot,\cdot,t})' \quad d_{\cdot,t}']'$$

and dropping expectations, this model is then in the form of Problem 2 (OBC).

Now consider the model's Problem 3 (News) type equivalent, in which for $t \in \mathbb{N}^+$:

$$a_t = \begin{cases} b_t + y_{t,0} & \text{if } t \le T\\ b_t & \text{if } t > T'\end{cases}$$

where $y_{\cdot,\cdot}$ is defined as in Problem 3 (News). Thus, if $c_{j,k,0} = 0$ and $d_{k,0} = 0$ for all $j,k \in \{0,\dots,T\}$, then for all $t \in \mathbb{N}^+$, $j,k \in \{0,\dots,T\}$:

$$c_{0,k,t} = \begin{cases} y_{t+k,0} & \text{if } t+k \le T\\ 0 & \text{if } t+k > 0'\end{cases}$$

$$c_{j,k,t} = \begin{cases} c_{0,k,t-j} & \text{if } t-j > 0\\ 0 & \text{if } t-j \le 0\end{cases} = \begin{cases} y_{t+k-j,0} & \text{if } t-j>0, t+k-j \le T\\ 0 & \text{otherwise}\end{cases}.$$

Hence, for all $t \in \mathbb{N}^+$, $j,k \in \{1,\dots,T\}$:

$$c_{j-1,k-1,t} - c_{j,k,t} = \begin{cases} y_{t+k-j,0} & \text{if } t-j = 0, t+k-j \le T\\ 0 & \text{otherwise}\end{cases} = \begin{cases} y_{k,0} & \text{if } t = j\\ 0 & \text{otherwise}\end{cases}.$$

Therefore, for all $t \in \{1,\dots,T\}$:

$$a_t - 1 = \sum_{k=1}^{T} \mathcal{M}_{t,k} y_{k,0}.$$

Consequently, if $y_{k,0} = I_{\cdot,l}$ for some $l \in \{1,\dots,T\}$, then $a_t - 1 = \mathcal{M}_{t,l}$ (i.e. the relative impulse response to a news-shock at horizon l) is the l^{th} column of $\mathcal{M}$.

Finally, note that in the model's Problem 1 (Linear) equivalent, if $\mathcal{c}_{j,k,0} = 0$ for all $j,k \in \{0,\dots,T\}$, then for all $t \in \mathbb{N}^+$, $\mathcal{a}_t = \mathcal{b}_t = \mathcal{d}_{0,t} = \mathcal{d}_{t,0}$. Hence, if $\mathcal{d}_{\cdot,0} = \mathcal{q}$ for some $\mathcal{q} \in \mathbb{R}^T$, then $q = \mathcal{q}$ for this model.

Appendix H.5: Proof of Proposition 2

Defining $x_t = [x_{i,t} \quad x_{y,t} \quad x_{\pi,t}]'$, the BPY model is in the form of Problem 2 (OBC), with:

$$A := \begin{bmatrix} 0 & -\alpha_{\Delta y} & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \qquad B := \begin{bmatrix} -1 & \alpha_{\Delta y} & \alpha_\pi \\ -\dfrac{1}{\sigma} & -1 & 0 \\ 0 & \gamma & -1 \end{bmatrix}, \qquad C := \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & \dfrac{1}{\sigma} \\ 0 & 0 & \beta \end{bmatrix}.$$

Assumption 2 is satisfied for this model as:

$$\det(A + B + C) = \det \begin{bmatrix} -1 & 0 & \alpha_\pi \\ -\dfrac{1}{\sigma} & 0 & \dfrac{1}{\sigma} \\ 0 & \gamma & -1 \end{bmatrix} \neq 0$$

as $\alpha_\pi \neq 1$ and $\gamma \neq 0$. Let $f := F_{2,2}$, where F is as in Assumption 1. Then:

$$F = \begin{bmatrix} 0 & \alpha_{\Delta y}(f-1) + \alpha_\pi \dfrac{\gamma f}{1-\beta f} & 0 \\ 0 & f & 0 \\ 0 & \dfrac{\gamma f}{1-\beta f} & 0 \end{bmatrix}.$$

Hence:

$$f = f^2 - \frac{1}{\sigma}\left(\alpha_{\Delta y}(f-1) + \alpha_\pi \frac{\gamma f}{1-\beta f} - \frac{\gamma f^2}{1-\beta f}\right),$$

i.e.:

$$\beta\sigma f^3 - \Big(\big(\alpha_{\Delta y} + \sigma\big)\beta + \gamma + \sigma\Big) f^2 + \Big((1+\beta)\alpha_{\Delta y} + \gamma\alpha_\pi + \sigma\Big) f - \alpha_{\Delta y} = 0. \tag{18}$$

When $f \leq 0$, the left-hand side is negative, and when $f = 1$, the left-hand side equals $(\alpha_\pi - 1)\gamma > 0$ (by assumption on α_π), hence equation (18) has either one or three solutions in $(0,1)$, and no solutions in $(-\infty,0]$. We wish to prove there is a unique solution in $(-1,1)$. First note that when $\alpha_\pi = 1$, the discriminant of the polynomial is:

$$\Big((1-\beta)\big(\alpha_{\Delta y} - \sigma\big) - \gamma\Big)^2 \Big(\big(\beta\alpha_{\Delta y}\big)^2 + 2\beta(\gamma - \sigma)\alpha_{\Delta y} + (\gamma + \sigma)^2\Big).$$

The first multiplicand is positive. The second is minimised when $\sigma = \beta\alpha_{\Delta y} - \gamma$, at the value $4\beta\gamma\alpha_{\Delta y} > 0$, hence this multiplicand is positive too. Consequently, at least for small α_π, there are three real solutions for f, so there may be multiple solutions in $(0,1)$.

Suppose for a contradiction that there were at least three solutions to equation (18) in $(0,1)$ (double counting repeated roots), even for arbitrary large $\beta \in (0,1)$. Let $f_1, f_2, f_3 \in (0,1)$ be the three roots. Then, by Vieta's formulas:

$$3 > f_1 + f_2 + f_3 = \frac{(\alpha_{\Delta y} + \sigma)\beta + \gamma + \sigma}{\beta\sigma},$$

$$3 > f_1 f_2 + f_1 f_3 + f_2 f_3 = \frac{(1+\beta)\alpha_{\Delta y} + \gamma\alpha_\pi + \sigma}{\beta\sigma},$$

$$1 > f_1 f_2 f_3 = \frac{\alpha_{\Delta y}}{\beta\sigma},$$

so:

$$
\begin{gathered}
(2\beta-1)\sigma > \beta\alpha_{\Delta y} + \gamma > \gamma > 0 \\
\beta > \frac{1}{2}, \qquad (2\beta-1)\sigma > \gamma, \\
\beta\sigma > \beta\alpha_{\Delta y} + \gamma + \sigma(1-\beta), \\
2\beta\sigma > (1+\beta)\alpha_{\Delta y} + \gamma\alpha_{\pi} + \sigma(1-\beta), \\
\beta\sigma > \alpha_{\Delta y}.
\end{gathered}
$$

Also, the first derivative of equation (18) must be positive at $f = 1$, so:

$$(1-\beta)\big(\alpha_{\Delta y} - \sigma\big) + (\alpha_{\pi} - 2)\gamma > 0.$$

Combining these inequalities gives the bounds:

$$
\begin{gathered}
0 < \alpha_{\Delta y} < 2\sigma - \frac{\gamma+\sigma}{\beta}, \\
2 + \frac{(1-\beta)\big(\sigma - \alpha_{\Delta y}\big)}{\gamma} < \alpha_{\pi} < \frac{(3\beta-1)\sigma - (1+\beta)\alpha_{\Delta y}}{\gamma}.
\end{gathered}
$$

Furthermore, if there are multiple solutions to equation (18), then the discriminant of its first derivative must be nonnegative, i.e.:

$$\Big((\alpha_{\Delta y} + \sigma)\beta + \gamma + \sigma\Big)^2 - 3\beta\sigma\Big((1+\beta)\alpha_{\Delta y} + \gamma\alpha_{\pi} + \sigma\Big) \geq 0.$$

Therefore, we have the following bounds on α_{π}:

$$2 + \frac{(1-\beta)\big(\sigma - \alpha_{\Delta y}\big)}{\gamma} < \alpha_{\pi} \leq \frac{\Big((\alpha_{\Delta y} + \sigma)\beta + \gamma + \sigma\Big)^2 - 3\beta\sigma\Big((1+\beta)\alpha_{\Delta y} + \sigma\Big)}{3\beta\sigma\gamma}$$

since,

$$
\begin{aligned}
&\frac{(3\beta-1)\sigma - (1+\beta)\alpha_{\Delta y}}{\gamma} - \frac{\Big((\alpha_{\Delta y} + \sigma)\beta + \gamma + \sigma\Big)^2 - 3\beta\sigma\Big((1+\beta)\alpha_{\Delta y} + \sigma\Big)}{3\beta\sigma\gamma} \\
&\qquad = \frac{\Big((2\sigma - \alpha_{\Delta y})\beta - \gamma - \sigma\Big)\Big((4\sigma + \alpha_{\Delta y})\beta + \gamma + \sigma\Big)}{3\beta\gamma\sigma} > 0
\end{aligned}
$$

as $\alpha_{\Delta y} < 2\sigma - \frac{\gamma+\sigma}{\beta}$. Consequently, there exists $\lambda, \mu, \kappa \in [0,1]$ such that:

$$
\begin{gathered}
\alpha_{\pi} = (1-\lambda)\left[2 + \frac{(1-\beta)\big(\sigma - \alpha_{\Delta y}\big)}{\gamma}\right] + \lambda\left[\frac{\Big((\alpha_{\Delta y} + \sigma)\beta + \gamma + \sigma\Big)^2 - 3\beta\sigma\Big((1+\beta)\alpha_{\Delta y} + \sigma\Big)}{3\beta\sigma\gamma}\right], \\
\alpha_{\Delta y} = (1-\mu)[0] + \mu\left[2\sigma - \frac{\gamma+\sigma}{\beta}\right], \\
\gamma = (1-\kappa)[0] + \kappa\big[(2\beta-1)\sigma\big]
\end{gathered}
$$

These simultaneous equations have unique solutions for α_{π}, $\alpha_{\Delta y}$ and γ in terms of λ, μ and κ. Substituting these solutions into the discriminant of equation (18) gives a polynomial in $\lambda, \mu, \kappa, \beta, \sigma$. As such, an exact global maximum of the discriminant may be found subject to the constraints $\lambda, \mu, \kappa \in [0,1]$, $\beta \in \big[\frac{1}{2}, 1\big]$, $\sigma \in [0, \infty)$, by using an exact compact polynomial optimisation solver, such as that in the Maple computer algebra package. Doing this gives a maximum of 0 when $\beta \in \big\{\frac{1}{2}, 1\big\}$, $\kappa = 1$ and $\sigma = 0$. But of course, we actually require that $\beta \in \big(\frac{1}{2}, 1\big)$, $\kappa < 1$, $\sigma > 0$. Thus, by continuity, the discriminant is negative over the entire domain.

This gives the required contradiction to our assumption of three roots to the polynomial, establishing that Assumption 1 holds for this model.

Now, when $T = 1$, M is equal to the top left element of the matrix $-(B + CF)^{-1}$, i.e.:

$$M = \frac{\beta\sigma f^2 - \big((1+\beta)\sigma + \gamma\big)f + \sigma}{\beta\sigma f^2 - \Big((1+\beta)\sigma + \gamma + \beta\alpha_{\Delta y}\Big)f + \sigma + \alpha_{\Delta y} + \gamma\alpha_\pi}.$$

Now, multiplying the denominator by f gives:

$$\begin{aligned}
&\beta\sigma f^3 - \Big((1+\beta)\sigma + \gamma + \beta\alpha_{\Delta y}\Big)f^2 + (\sigma + \alpha_{\Delta y} + \gamma\alpha_\pi)f \\
&\qquad = \Big[\beta\sigma f^3 - \Big((\alpha_{\Delta y} + \sigma)\beta + \gamma + \sigma\Big)f^2 + \Big((1+\beta)\alpha_{\Delta y} + \gamma\alpha_\pi + \sigma\Big)f - \alpha_{\Delta y}\Big] \\
&\qquad - \big[\beta\alpha_{\Delta y} f - \alpha_{\Delta y}\big] = (1-\beta f)\alpha_{\Delta y} > 0,
\end{aligned}$$

by equation (18). Hence, the sign of M is that of $\beta\sigma f^2 - \big((1+\beta)\sigma + \gamma\big)f + \sigma$. I.e., M is negative if and only if:

$$\begin{aligned}
&\frac{\big((1+\beta)\sigma + \gamma\big) - \sqrt{\big((1+\beta)\sigma + \gamma\big)^2 - 4\beta\sigma^2}}{2\beta\sigma} < f \\
&\qquad < \frac{\big((1+\beta)\sigma + \gamma\big) + \sqrt{\big((1+\beta)\sigma + \gamma\big)^2 - 4\beta\sigma^2}}{2\beta\sigma}.
\end{aligned}$$

The upper limit is greater than 1, so only the lower is relevant. To translate this bound on f into a bound on $\alpha_{\Delta y}$, we first need to establish that f is monotonic in $\alpha_{\Delta y}$.

Totally differentiating equation (18) gives:

$$\Big[3\beta\sigma f^2 - 2\Big((\alpha_{\Delta y} + \sigma)\beta + \gamma + \sigma\Big)f + \Big((1+\beta)\alpha_{\Delta y} + \gamma\alpha_\pi + \sigma\Big)\Big]\frac{df}{d\alpha_{\Delta y}} = (1-\beta f)(1-f) > 0.$$

Thus, the sign of $\frac{df}{d\alpha_{\Delta y}}$ is equal to that of:

$$3\beta\sigma f^2 - 2\Big((\alpha_{\Delta y} + \sigma)\beta + \gamma + \sigma\Big)f + \Big((1+\beta)\alpha_{\Delta y} + \gamma\alpha_\pi + \sigma\Big).$$

Note, however, that this expression is just the derivative of the left-hand side of equation (18) with respect to f.

To establish the sign of $\frac{df}{d\alpha_{\Delta y}}$, we consider two cases. First, suppose that equation (18) has three real solutions. Then, the unique solution to equation (18) in (0,1) is its lowest solution. Hence, this solution must be below the first local maximum of the left-hand side of equation (19). Consequently, at the $f \in (0,1)$, which solves equation (18), $3\beta\sigma f^2 - 2\Big((\alpha_{\Delta y} + \sigma)\beta + \gamma + \sigma\Big)f + \Big((1+\beta)\alpha_{\Delta y} + \gamma\alpha_\pi + \sigma\Big) > 0$. Alternatively, suppose that equation (18) has a unique real solution. Then the left-hand side of this equation cannot change sign in between its local maximum and its local minimum (if it has any). Thus, at the $f \in (0,1)$ at which it changes sign, we must have that $3\beta\sigma f^2 - 2\Big((\alpha_{\Delta y} + \sigma)\beta + \gamma + \sigma\Big)f + \Big((1+\beta)\alpha_{\Delta y} + \gamma\alpha_\pi + \sigma\Big) > 0$. Therefore, in either case $\frac{df}{d\alpha_{\Delta y}} > 0$, meaning that f is monotonic increasing in $\alpha_{\Delta y}$.

Consequently, to find the critical $(f, \alpha_{\Delta y})$ at which M changes sign, it is sufficient to find the lowest solution with respect to both f and $\alpha_{\Delta y}$ of the pair of equations:

$$\begin{gathered}
\beta\sigma f^2 - \big((1+\beta)\sigma + \gamma\big)f + \sigma = 0, \\
\beta\sigma f^3 - \Big((\alpha_{\Delta y} + \sigma)\beta + \gamma + \sigma\Big)f^2 + \Big((1+\beta)\alpha_{\Delta y} + \gamma\alpha_\pi + \sigma\Big)f - \alpha_{\Delta y} = 0.
\end{gathered}$$

The former implies that:

$$\beta\sigma f^3 - \big((1+\beta)\sigma + \gamma\big)f^2 + \sigma f = 0,$$

so, by the latter:

$$\alpha_{\Delta y}\beta f^2 - \Big((1+\beta)\alpha_{\Delta y} + \gamma\alpha_\pi\Big)f + \alpha_{\Delta y} = 0.$$

If $\alpha_{\Delta y} = \sigma\alpha_\pi$, then this equation holds if and only if:

$$\sigma\beta f^2 - \big((1+\beta)\sigma + \gamma\big)f + \sigma = 0.$$

Therefore, the critical $(f, \alpha_{\Delta y})$ at which M changes sign are given by:

$$\alpha_{\Delta y} = \sigma\alpha_\pi,$$

$$f = \frac{\big((1+\beta)\sigma + \gamma\big) - \sqrt{\big((1+\beta)\sigma + \gamma\big)^2 - 4\beta\sigma^2}}{2\beta\sigma}.$$

Thus, M is negative if and only if $\alpha_{\Delta y} > \sigma\alpha_\pi$, and M is zero if and only if $\alpha_{\Delta y} = \sigma\alpha_\pi$.

Appendix H.6: Proof of Proposition 3

Consider the model:

$$a_t = \max\{0, b_t\}, \qquad a_t = 1 - c_t, \qquad c_t = a_t - b_t.$$

The model has steady-state $a = b = 1$, $c = 0$. Furthermore, in the model's Problem 3 (News) type equivalent, in which for $t \in \mathbb{N}^+$:

$$a_t = \begin{cases} b_t + y_{t,0} & \text{if } t \le T \\ b_t & \text{if } t > T \end{cases}$$

where $y_{\cdot,\cdot}$ is defined as in Problem 3 (News), we have that:

$$c_t = \begin{cases} y_{t,0} & \text{if } t \le T \\ 0 & \text{if } t > T \end{cases}$$

so:

$$b_t = \begin{cases} 1 - 2y_{t,0} & \text{if } t \le T \\ 1 & \text{if } t > T \end{cases}$$

implying:

$$a_t = \begin{cases} 1 - y_{t,0} & \text{if } t \le T \\ 1 & \text{if } t > T \end{cases}.$$

thus, $M = -I$ for this model.

Appendix H.7: Proof of Lemma 3

First, define $G := -C(B + CF)^{-1}$, and note that if L is the lag (right-shift) operator, the model from Problem 1 (Linear) can be written as:

$$L^{-1}(ALL + BL + C)(x - \mu) = 0.$$

Furthermore, by the definitions of F and G:

$$(L - G)(B + CF)(I - FL) = ALL + BL + C,$$

so, the stability of the model from Problem 1 (Linear) is determined by the solutions for $z \in \mathbb{C}$ of the polynomial:

$$0 = \det\big(Az^2 + Bz + C\big) = \det(Iz - G)\det(B + CF)\det(I - Fz).$$

Now by Assumption 1, all of the roots of $\det(I - Fz)$ are strictly outside of the unit circle, and all of the other roots of $\det\big(Az^2 + Bz + C\big)$ are weakly inside the unit circle (else there would be indeterminacy), thus, all of the roots of $\det(Iz - G)$ are weakly inside the unit circle.

Therefore, if we write $\rho_{\mathcal{M}}$ for the spectral radius of some matrix $\mathcal{M}$, then, by this discussion and Assumption 2, $\rho_G < 1$.

Now consider the time reversed model:

$$L\big(AL^{-1}L^{-1} + BL^{-1} + C\big)d = 0,$$

subject to the terminal condition that $d_k \to 0$ as $k \to \infty$. Now, let $z \in \mathbb{C}$, $z \neq 0$ be a solution to:

$$0 = \det\big(Az^2 + Bz + C\big),$$

and define $\tilde{z} = z^{-1}$, so:

$$0 = \det\big(A + B\tilde{z} + C\tilde{z}^2\big) = z^{-2}\det\big(Az^2 + Bz + C\big) = \det(I - G\tilde{z})\det(B + CF)\det(I\tilde{z} - F).$$

By Assumption 1, all the roots of $\det(I\tilde{z} - F)$ are inside the unit circle, thus they cannot contribute to the dynamics of the time reversed process, else the terminal condition would be violated. Thus, the time reversed model has a unique solution satisfying the terminal condition with a transition matrix with the same eigenvalues as G. Consequently, this solution can be calculated via standard methods for solving linear DSGE models, and it will be given by $d_k = Hd_{k-1}$, for all $k > 0$, where $H = -(B + AH)^{-1}C$, and $\phi_H = \phi_G < 1$, by Assumption 2.

Appendix H.8: Proof of Proposition 6

Let $s_t^*, x_t^* \in \mathbb{R}^{n\times\mathbb{N}^+}$ be such that for any $y \in \mathbb{R}^{\mathbb{N}^+}$, the k^{th} columns of $s_t^* y$ and $x_t^* y$ give the value of s_t and x_t following a magnitude 1 news shock at horizon k, i.e. when $x_0 = \mu$ and y_0 is the k^{th} row of $I_{\mathbb{N}^+\times\mathbb{N}^+}$. Then:

$$\begin{aligned} s_t^* &= -(B + CF)^{-1}\big[I_{\cdot,1}I_{t,1:\infty} + GI_{\cdot,1}I_{t+1,1:\infty} + G^2 I_{\cdot,1}I_{t+2,1:\infty} + \cdots\big] \\ &= -(B + CF)^{-1}\sum_{k=0}^{\infty}(GL)^k I_{\cdot,1}I_{t,1:\infty} \\ &= -(B + CF)^{-1}(I - GL)^{-1}I_{\cdot,1}I_{t,1:\infty}, \end{aligned}$$

where the infinite sums are well defined as $\rho_G < 1$, and where $I_{t,1:\infty} \in \mathbb{R}^{1\times\mathbb{N}^+}$ is a row vector with zeros everywhere except position t where there is a 1. Thus:

$$s_t^* = \begin{bmatrix} 0_{n\times(t-1)} & s_1^* \end{bmatrix} = L^{t-1}s_1^*.$$

Furthermore,

$$(x_t^* - \mu^*) = \sum_{j=1}^{t} F^{t-j}s_k^* = \sum_{j=1}^{t} F^{t-j}L^{j-1}s_1^*,$$

i.e.:

$$(x_t^* - \mu^*)_{\cdot,k} = \sum_{j=1}^{t} F^{t-j}s_{1,\cdot,k+1-j}^* = -\sum_{j=1}^{\min\{t,k\}} F^{t-j}(B + CF)^{-1}G^{k-j}I_{\cdot,1},$$

and so, the k^{th} offset diagonal of M ($k \in \mathbb{Z}$) is given by the first row of the k^{th} column of:

$$L^{-t}(x_t^* - \mu^*) = L^{-1}\sum_{j=1}^{t}\big(FL^{-1}\big)^{t-j}s_1^* = L^{-1}\sum_{j=0}^{t-1}\big(FL^{-1}\big)^{j}s_1^*,$$

where we abuse notation slightly by allowing L^{-1} to give a result with indices in $\mathbb{Z}$ rather than $\mathbb{N}^+$, with padding by zeros. Consequently, for all $k \in \mathbb{N}^+$, $M_{t,k} = \mathrm{O}\big(t^n\rho_F^t\big)$, as $t \to \infty$, for all $t \in \mathbb{N}^+$, $M_{t,k} = \mathrm{O}\big(t^n\rho_G^k\big)$, as $k \to \infty$, and for all $k \in \mathbb{Z}$, $M_{t,t+k} - \lim_{\tau\to\infty} M_{\tau,\tau+k} = \mathrm{O}\big(t^{n-1}(\rho_F\rho_G)^t\big)$, as $t \to \infty$.

Hence,

$$\sup_{y\in[0,1]^{\mathbb{N}^+}} \inf_{t\in\mathbb{N}^+} M_{t,1:\infty} y$$

exists and is well defined, and so:

$$\varsigma = \sup_{\substack{y\in[0,1]^{\mathbb{N}^+}\\ \exists T\in\mathbb{N} \text{ s.t. } \forall t>T, y_t=0}} \inf_{t\in\mathbb{N}^+} M_{t,1:\infty} y = \sup_{y\in[0,1]^{\mathbb{N}^+}} \inf_{t\in\mathbb{N}^+} M_{t,1:\infty} y,$$

since every point in $[0,1]^{\mathbb{N}^+}$ is a limit (under the supremum norm) of a sequence of points in the set:

$$\left\{y \in [0,1]^{\mathbb{N}^+} \middle| \exists T \in \mathbb{N} \text{ s.t. } \forall t > T, y_t = 0\right\}.$$

Thus, we just need to provide conditions under which $\sup_{y\in[0,1]^{\mathbb{N}^+}} \inf_{t\in\mathbb{N}^+} M_{t,1:\infty} y > 0$.

To produce such conditions, we need constructive bounds on M, even if they have slightly worse convergence rates. For any matrix, $\mathcal{M} \in \mathbb{R}^{n\times n}$ with $\rho_{\mathcal{M}} < 1$, and any $\phi \in (\rho_{\mathcal{M}}, 1)$, let:

$$\mathcal{C}_{\mathcal{M},\phi} := \sup_{k\in\mathbb{N}} \left\| \left(\mathcal{M}\phi^{-1}\right)^k \right\|_2.$$

Furthermore, for any matrix, $\mathcal{M} \in \mathbb{R}^{n\times n}$ with $\rho_{\mathcal{M}} < 1$, and any $\epsilon > 0$, let:

$$\rho_{\mathcal{M},\epsilon} := \max\{|z| | z \in \mathbb{C}, \sigma_{\min}(\mathcal{M} - zI) = \epsilon\},$$

where $\sigma_{\min}(\mathcal{M} - zI)$ is the minimum singular value of $\mathcal{M} - zI$, and let $\epsilon^*(\mathcal{M}) \in (0, \infty]$ solve:

$$\rho_{\mathcal{M},\epsilon} = 1.$$

(This has a solution in $(0, \infty]$ by continuity as $\rho_{\mathcal{M}} < 1$.) Then, by Theorem 16.2 of Trefethen and Embree (2005), for any $K \in \mathbb{N}$ and $k > K$:

$$\left\| \left(\mathcal{M}\phi^{-1}\right)^k \right\|_2 \le \left\| \left(\mathcal{M}\phi^{-1}\right)^K \right\|_2 \left\| \left(\mathcal{M}\phi^{-1}\right)^{k-K} \right\|_2 \le \frac{\left\| \left(\mathcal{M}\phi^{-1}\right)^K \right\|_2}{\epsilon^*(\mathcal{M}\phi^{-1})}.$$

Now, $\left\| \left(\mathcal{M}\phi^{-1}\right)^K \right\|_2 \to 0$ as $K \to \infty$, hence, there exists some $K \in \mathbb{N}$ such that:

$$\sup_{k=0,\dots,K} \left\| \left(\mathcal{M}\phi^{-1}\right)^k \right\|_2 \ge \frac{\left\| \left(\mathcal{M}\phi^{-1}\right)^K \right\|_2}{\epsilon^*(\mathcal{M}\phi^{-1})} \ge \sup_{k>K} \left\| \left(\mathcal{M}\phi^{-1}\right)^k \right\|_2,$$

meaning $\mathcal{C}_{\mathcal{M},\phi} = \sup_{k=0,\dots,K} \left\| \left(\mathcal{M}\phi^{-1}\right)^k \right\|_2$. The quantity $\rho_{\mathcal{M},\epsilon}$ (and hence $\epsilon^*(\mathcal{M})$) may be efficiently computed using the methods described by Wright and Trefethen (2001), and implemented in their EigTool toolkit[35]. Thus, $\mathcal{C}_{\mathcal{M},\phi}$ may be calculated in finitely many operations by iterating over $K \in \mathbb{N}$ until a K is found which satisfies:

$$\sup_{k=0,\dots,K} \left\| \left(\mathcal{M}\phi^{-1}\right)^k \right\|_2 \ge \frac{\left\| \left(\mathcal{M}\phi^{-1}\right)^K \right\|_2}{\epsilon^*(\mathcal{M}\phi^{-1})}.$$

From the definition of $\mathcal{C}_{\mathcal{M},\phi}$, we have that for any $k \in \mathbb{N}$ and any $\phi \in (\rho_{\mathcal{M}}, 1)$:

$$\left\|\mathcal{M}^k\right\|_2 \le \mathcal{C}_{\mathcal{M},\phi}\phi^k.$$

Now, fix $\phi_F \in (\rho_F, 1)$ and $\phi_G \in (\rho_G, 1)$,[36] and define:

$$\mathcal{D}_{\phi_F,\phi_G} := \mathcal{C}_{F,\phi_F}\mathcal{C}_{G,\phi_F}\left\|(B + CF)^{-1}\right\|_2,$$

[35] This toolkit is available from https://github.com/eigtool/eigtool, and is included in DynareOBC.

[36] In practice, we try a grid of values, as it is problem dependent whether high ϕ_F and low $\mathcal{K}(\mathcal{M}\phi^{-1})$ is preferable to low ϕ_F and high $\mathcal{K}(\mathcal{M}\phi^{-1})$.

then, for all $t, k \in \mathbb{N}^+$:

$$\left|M_{t,k}\right| = \left|(x_t^* - \mu^*)_{1,k}\right| \le \left\|(x_t^* - \mu^*)_{\cdot,k}\right\|_2 \le \sum_{j=1}^{\min\{t,k\}} \left\|F^{t-j}\right\|_2 \left\|(B+CF)^{-1}\right\|_2 \left\|G^{k-j}\right\|_2$$
$$\le \mathcal{D}_{\phi_F,\phi_G} \sum_{j=1}^{\min\{t,k\}} \phi_F^{t-j}\phi_G^{k-j} = \mathcal{D}_{\phi_F,\phi_G}\phi_F^t\phi_G^k \frac{(\phi_F\phi_G)^{-\min\{t,k\}} - 1}{1 - \phi_F\phi_G}.$$

Additionally, for all $t \in \mathbb{N}^+, k \in \mathbb{Z}$:

$$\begin{aligned}\left|M_{t,t+k} - \lim_{\tau\to\infty} M_{\tau,\tau+k}\right| &= \left|\left(L^{-t}(x_t^* - \mu^*)\right)_{1,k} - \left(\lim_{\tau\to\infty} L^{-t}(x_t^* - \mu^*)\right)_{1,k}\right| \\ &\le \left\|\left(L^{-1}\sum_{j=0}^{t-1}\left(FL^{-1}\right)^j s_1^* - L^{-1}\sum_{j=0}^{\infty}\left(FL^{-1}\right)^j s_1^*\right)_{\cdot,k}\right\|_2 \\ &= \left\|\left(\sum_{j=\max\{t,-k\}}^{\infty} F^j s_{1,\cdot,j+k+1}^*\right)_{\cdot,0}\right\|_2 \\ &= \left\|\sum_{j=\max\{t,-k\}}^{\infty} F^j(B+CF)^{-1}G^{j+k}I_{\cdot,1}\right\|_2 \\ &\le \sum_{j=\max\{t,-k\}}^{\infty} \left\|F^j\right\|_2 \left\|(B+CF)^{-1}\right\|_2 \left\|G^{j+k}\right\|_2 \\ &\le \mathcal{D}_{\phi_F,\phi_G} \sum_{j=\max\{t,-k\}}^{\infty} \phi_F^j\phi_G^{j+k} = \mathcal{D}_{\phi_F,\phi_G} \frac{\phi_F^{\max\{t,-k\}}\phi_G^{\max\{0,t+k\}}}{1-\phi_F\phi_G},\end{aligned}$$

so, for all $t, k \in \mathbb{N}^+$:

$$\left|M_{t,k} - \lim_{\tau\to\infty} M_{\tau,\tau+k-t}\right| \le \mathcal{D}_{\phi_F,\phi_G} \frac{\phi_F^t\phi_G^k}{1-\phi_F\phi_G}.$$

To evaluate $\lim_{\tau\to\infty} M_{\tau,\tau+k-t}$, note that this limit is the top element from the $(k-t)^{\text{th}}$ column of:

$$\begin{aligned} d &:= \lim_{\tau\to\infty} L^{-\tau}(x_\tau^* - \mu^*) = L^{-1}\left(I - FL^{-1}\right)^{-1} s_1^* \\ &= -\left(I - FL^{-1}\right)^{-1}(B+CF)^{-1}(I-GL)^{-1}I_{\cdot,1}I_{0,-\infty:\infty},\end{aligned}$$

where $I_{0,-\infty:\infty} \in \mathbb{R}^{1\times\mathbb{Z}}$ is zero everywhere apart from index 0 where it equals 1. Hence, by the definitions of F and G:

$$AL^{-1}d + Bd + CLd = -I_{\cdot,1}I_{0,-\infty:\infty}.$$

In other words, if we write d_k in place of $d_{\cdot,k}$ for convenience, then, for all $k \in \mathbb{Z}$:

$$Ad_{k+1} + Bd_k + Cd_{k-1} = -\begin{cases} I_{\cdot,1} & \text{if } k = 0 \\ 0 & \text{otherwise} \end{cases}$$

I.e. the homogeneous part of the difference equation for d_{-t} is identical to that of $x_t - \mu$. The time reversal here is intuitive since we are indexing diagonals such that indices increase as we move up and to the right in M, but time is increasing as we move down in M.

Exploiting the possibility of reversing time is the key to easy evaluating d_k. First, note that for $k < 0$, it must be the case that $d_k = Fd_{k+1}$, since the shock has already "occurred" (remember, that we are going forwards in "time" when we reduce k). Likewise, since $d_k \to 0$ as $k \to \infty$, as we have already proved that the first row of M converges to zero, by Lemma 3, it must be the case that $d_k = Hd_{k-1}$, for all $k > 0$, where $H = -(B + AH)^{-1}C$, and $\phi_H < 1$.

It just remains to determine the value of d_0. By the previous results, this must satisfy:

$$-I_{\cdot,1} = Ad_1 + Bd_0 + Cd_{-1} = (AH + B + CF)d_0.$$

Hence:

$$d_0 = -(AH + B + CF)^{-1} I_{\cdot,1}.$$

This gives a readily computed solution for the limits of the diagonals of M. Lastly, note that:

$$|d_{-t,1}| \le \|d_{-t}\|_2 = \|F^t d_0\|_2 \le \|F^t\|_2 \|d_0\|_2 \le \mathcal{C}_{F,\phi_F} \phi_F^t \|d_0\|_2,$$

and:

$$|d_{t,1}| \le \|d_t\|_2 = \|H^t d_0\|_2 \le \|H^t\|_2 \|d_0\|_2 \le \mathcal{C}_{H,\phi_H} \phi_H^t \|d_0\|_2.$$

We will use these results in producing our bounds on ς.

First, fix $T \in \mathbb{N}^+$, and define a new matrix $\underline{M}^{(T)} \in \mathbb{R}^{\mathbb{N}^+ \times \mathbb{N}^+}$ by $\underline{M}^{(T)}_{1:T,1:T} = M_{1:T,1:T}$, and for all $t, k \in \mathbb{N}^+$, with $\min\{t,k\} > T$, $\underline{M}^{(T)}_{t,k} = d_{k-t,1} - \mathcal{D}_{\phi_F,\phi_G} \frac{\phi_F^t \phi_G^k}{1-\phi_F\phi_G}$, then:

$$\begin{aligned}
\varsigma &\ge \max_{\substack{y \in [0,1]^T \\ y_\infty \in [0,1]}} \inf_{t \in \mathbb{N}^+} M_{t,1:\infty} \begin{bmatrix} y \\ y_\infty 1_{\infty \times 1} \end{bmatrix} \ge \max_{\substack{y \in [0,1]^T \\ y_\infty \in [0,1]}} \inf_{t \in \mathbb{N}^+} \underline{M}^{(T)}_{t,1:\infty} \begin{bmatrix} y \\ y_\infty 1_{\infty \times 1} \end{bmatrix} \\
&= \max_{\substack{y \in [0,1]^T \\ y_\infty \in [0,1]}} \min \left\{ \begin{array}{c} \min_{t=1,\dots,T} \left[M_{t,1:T} y + \sum_{k=T+1}^{\infty} \left(d_{k-t,1} - \mathcal{D}_{\phi_F,\phi_G} \frac{\phi_F^t \phi_G^k}{1-\phi_F\phi_G} \right) y_\infty \right], \\ \inf_{t \in \mathbb{N}^+, t > T} \left[\sum_{k=1}^{T} \left(d_{k-t,1} - \mathcal{D}_{\phi_F,\phi_G} \frac{\phi_F^t \phi_G^k}{1-\phi_F\phi_G} \right) y_k + \sum_{k=T+1}^{\infty} \left(d_{k-t,1} - \mathcal{D}_{\phi_F,\phi_G} \frac{\phi_F^t \phi_G^k}{1-\phi_F\phi_G} \right) y_\infty \right] \end{array} \right\} \\
&\ge \max_{\substack{y \in [0,1]^T \\ y_\infty \in [0,1]}} \min \left\{ \begin{array}{c} \min_{t=1,\dots,T} \left[M_{t,1:T} y + \left((I-H)^{-1} d_{T+1-t} \right)_1 y_\infty - \mathcal{D}_{\phi_F,\phi_G} \frac{\phi_F^t \phi_G^{T+1}}{(1-\phi_F\phi_G)(1-\phi_G)} y_\infty \right], \\ \min_{t=T+1,\dots,2T} \left[\begin{array}{c} \sum_{k=1}^{T} \left(d_{-(t-k),1} - \mathcal{D}_{\phi_F,\phi_G} \frac{\phi_F^t \phi_G^k}{1-\phi_F\phi_G} \right) y_k + \left((I-F)^{-1} (d_{-1} - d_{-(t-T)}) \right)_1 y_\infty \\ + \left((I-H)^{-1} d_0 \right)_1 y_\infty - \mathcal{D}_{\phi_F,\phi_G} \frac{\phi_F^t \phi_G^{T+1}}{(1-\phi_F\phi_G)(1-\phi_G)} y_\infty \end{array} \right], \\ \inf_{t \in \mathbb{N}^+, t > 2T} \left[\begin{array}{c} \sum_{k=1}^{T} d_{-(t-k),1} y_k + \left((I-F)^{-1} (d_{-1} - d_{-(t-T)}) \right)_1 y_\infty \\ + \left((I-H)^{-1} d_0 \right)_1 y_\infty - \mathcal{D}_{\phi_F,\phi_G} \frac{\phi_F^{2T+1} \phi_G}{(1-\phi_F\phi_G)(1-\phi_G)} \end{array} \right] \end{array} \right\}.
\end{aligned}$$

Now, for $t \ge T$:

$$\begin{aligned}
\left| \left((I-F)^{-1} d_{-(t-T)} \right)_1 \right| &\le \left\| (I-F)^{-1} d_{-(t-T)} \right\|_2 \le \left\| (I-F)^{-1} \right\|_2 \left\| d_{-(t-T)} \right\|_2 \\
&\le \mathcal{C}_{F,\phi_F} \phi_F^{t-T} \left\| (I-F)^{-1} \right\|_2 \|d_0\|_2,
\end{aligned}$$

so:

$$\begin{aligned}
&\sum_{k=1}^{T} d_{-(t-k),1} y_k - \left((I-F)^{-1} d_{-(t-T)} \right)_1 y_\infty \\
&\qquad \ge -\sum_{k=1}^{T} \mathcal{C}_{F,\phi_F} \phi_F^{t-k} \|d_0\|_2 - \mathcal{C}_{F,\phi_F} \phi_F^{t-T} \left\| (I-F)^{-1} \right\|_2 \|d_0\|_2 y_\infty \\
&\qquad = -\mathcal{C}_{F,\phi_F} \frac{\phi_F^t (\phi_F^{-T} - 1)}{1-\phi_F} \|d_0\|_2 - \mathcal{C}_{F,\phi_F} \phi_F^{t-T} \left\| (I-F)^{-1} \right\|_2 \|d_0\|_2 y_\infty.
\end{aligned}$$

Thus $\varsigma \geq \underline{\varsigma}$, where:

$$\underline{\varsigma}_T := \max_{\substack{y\in[0,1]^T \\ y_\infty\in[0,1]}} \min \left\{ \begin{array}{c} \min_{t=1,\dots,T} \left[M_{t,1:T}y + \left((I-H)^{-1}d_{T+1-t}\right)_1 y_\infty - \mathcal{D}_{\phi_F,\phi_G} \frac{\phi_F^t \phi_G^{T+1}}{(1-\phi_F\phi_G)(1-\phi_G)} y_\infty \right], \\ \min_{t=T+1,\dots,2T} \left[\begin{array}{c} \sum_{k=1}^{T} \left(d_{-(t-k),1} - \mathcal{D}_{\phi_F,\phi_G} \frac{\phi_F^t \phi_G^k}{1-\phi_F\phi_G} \right) y_k + \left((I-F)^{-1}(d_{-1} - d_{-(t-T)})\right)_1 y_\infty \\ + \left((I-H)^{-1}d_0\right)_1 y_\infty - \mathcal{D}_{\phi_F,\phi_G} \frac{\phi_F^t \phi_G^{T+1}}{(1-\phi_F\phi_G)(1-\phi_G)} y_\infty \end{array} \right], \\ \left[\begin{array}{c} -\mathcal{C}_{F,\phi_F} \frac{\phi_F^{2T+1}(\phi_F^{-T}-1)}{1-\phi_F} \|d_0\|_2 - \mathcal{C}_{F,\phi_F}\phi_F^{T+1} \|(I-F)^{-1}\|_2 \|d_0\|_2 y_\infty + \left((I-F)^{-1}d_{-1}\right)_1 y_\infty \\ + \left((I-H)^{-1}d_0\right)_1 y_\infty - \mathcal{D}_{\phi_F,\phi_G} \frac{\phi_F^{2T+1}\phi_G}{(1-\phi_F\phi_G)(1-\phi_G)} \end{array} \right] \end{array} \right\}.$$

It is worth noting that as $T \to \infty$, the final minimand in this expression tends to:

$$\left((I-F)^{-1}d_{-1}\right)_1 y_\infty + \left((I-H)^{-1}d_0\right)_1 y_\infty,$$

i.e. a positive multiple of the doubly infinite sum of $d_{1,k}$ over all $k \in \mathbb{Z}$. If this expression is negative, then our lower bound on ς will be negative as well, and hence uninformative.

To construct an upper bound on ς, fix $T \in \mathbb{N}^+$, and define a new matrix $\overline{M}^{(T)} \in \mathbb{R}^{\mathbb{N}^+\times\mathbb{N}^+}$ by $\overline{M}^{(T)}_{1:T,1:T} = M_{1:T,1:T}$, and for all $t,k \in \mathbb{N}^+$, with $\min\{t,k\} > T$, $\overline{M}^{(T)}_{t,k} = |d_{k-t,1}| + \mathcal{D}_{\phi_F,\phi_G} \frac{\phi_F^t\phi_G^k}{1-\phi_F\phi_G}$. Then:

$$\begin{aligned} \varsigma &= \sup_{y\in[0,1]^{\mathbb{N}^+}} \inf_{t\in\mathbb{N}^+} M_{t,1:\infty}y \leq \sup_{y\in[0,1]^{\mathbb{N}^+}} \inf_{t\in\mathbb{N}^+} \overline{M}_{t,1:\infty}y \leq \sup_{y\in[0,1]^{\mathbb{N}^+}} \min_{t=1,\dots,T} \overline{M}_{t,1:\infty}y \\ &\leq \max_{y\in[0,1]^T} \min_{t=1,\dots,T} \overline{M}_{t,1:\infty} \begin{bmatrix} y \\ 1_{\infty\times 1} \end{bmatrix} \\ &\leq \max_{y\in[0,1]^T} \min_{t=1,\dots,T} \left[M_{t,1:T}y + \sum_{k=T+1}^{\infty} |d_{k-t,1}| + \sum_{k=T+1}^{\infty} \mathcal{D}_{\phi_F,\phi_G} \frac{\phi_F^t\phi_G^k}{1-\phi_F\phi_G} \right] \\ &\leq \max_{y\in[0,1]^T} \min_{t=1,\dots,T} \left[M_{t,1:T}y + \sum_{k=T+1-t}^{\infty} |d_{k,1}| + \mathcal{D}_{\phi_F,\phi_G} \frac{\phi_F^t\phi_G^{T+1}}{1-\phi_F\phi_G} \sum_{k=0}^{\infty} \phi_G^k \right] \\ &\leq \max_{y\in[0,1]^T} \min_{t=1,\dots,T} \left[M_{t,1:T}y + \mathcal{C}_{H,\phi_H}\|d_0\|_2 \phi_H^{T+1-t} \sum_{k=0}^{\infty} \phi_H^k + \mathcal{D}_{\phi_F,\phi_G} \frac{\phi_F^t\phi_G^{T+1}}{(1-\phi_F\phi_G)(1-\phi_G)} \right] \\ &= \overline{\varsigma}_T := \max_{y\in[0,1]^T} \min_{t=1,\dots,T} \left[M_{t,1:T}y + \frac{\mathcal{C}_{H,\phi_H}\|d_0\|_2\phi_H^{T+1-t}}{1-\phi_H} + \mathcal{D}_{\phi_F,\phi_G} \frac{\phi_F^t\phi_G^{T+1}}{(1-\phi_F\phi_G)(1-\phi_G)} \right]. \end{aligned}$$

Appendix H.9: Proof of Proposition 10

Defining $x_t = [x_{i,t} \quad x_{y,t} \quad x_{p,t}]'$, the price targeting model from Section 5.1 is in the form of Problem 2 (OBC), with:

$$A := \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \qquad B := \begin{bmatrix} -1 & \alpha_{\Delta y} & \alpha_\pi \\ -\frac{1}{\sigma} & -1 & -\frac{1}{\sigma} \\ 0 & \gamma & -1-\beta \end{bmatrix}, \qquad C := \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & \frac{1}{\sigma} \\ 0 & 0 & \beta \end{bmatrix}.$$

Assumption 2 is satisfied for this model as:

$$\det(A+B+C) = \det \begin{bmatrix} -1 & \alpha_{\Delta y} & \alpha_\pi \\ -\frac{1}{\sigma} & 0 & 0 \\ 0 & \gamma & -1 \end{bmatrix} \neq 0$$

as $\alpha_{\Delta y} \neq 0$ and $\alpha_\pi \neq 0$. Let $f := F_{3,3}$, where F is as in Assumption 1.

Then:

$$F = \begin{bmatrix} 0 & 0 & \dfrac{f(1-f)(\sigma\alpha_\pi - \alpha_{\Delta y})}{\alpha_{\Delta y} + (1-f)\sigma} \\ 0 & 0 & \dfrac{f(1-f-\alpha_\pi)}{\alpha_{\Delta y} + (1-f)\sigma} \\ 0 & 0 & f \end{bmatrix},$$

and so:

$$\beta\sigma f^3 - \Big((1+2\beta)\sigma + \beta\alpha_{\Delta y} + \gamma\Big)f^2 + \Big((2+\beta)\sigma + (1+\beta)\alpha_{\Delta y} + (1+\alpha_\pi)\gamma\Big)f - (\sigma + \alpha_{\Delta y}) = 0.$$

Now define:

$$\hat{\alpha}_{\Delta y} := \sigma + \alpha_{\Delta y}, \qquad \hat{\alpha}_\pi := 1 + \alpha_\pi$$

so:

$$\beta\sigma f^3 - \Big((\hat{\alpha}_{\Delta y} + \sigma)\beta + \gamma + \sigma\Big)f^2 + \Big((1+\beta)\hat{\alpha}_{\Delta y} + \gamma\hat{\alpha}_\pi + \sigma\Big)f - \hat{\alpha}_{\Delta y} = 0.$$

This is identical to the equation for f in Appendix H.5, apart from the fact that $\hat{\alpha}_{\Delta y}$ has replaced $\alpha_{\Delta y}$ and $\hat{\alpha}_\pi$ has replaced α_π. Hence, by the results of Appendix H.5, Assumption 1 holds for this model as well.

Finally, for this model, with $T = 1$, we have that:

$$M = \frac{(1-f)(1+(1-f)\beta)\sigma^2 + \Big((1+(1-f)\beta)\alpha_{\Delta y} + \big((1-f) + \alpha_\pi f\big)\gamma\Big)\sigma + (1-f)\gamma\alpha_{\Delta y}}{\Big((1-f)(1+(1-f)\beta)\sigma + (1+(1-f)\beta)\alpha_{\Delta y} + \big((1-f)+\alpha_\pi\big)\gamma\Big)(\sigma + \alpha_{\Delta y})} > 0.$$

Appendix H.10: Proof of Proposition 13

If $\widetilde{X}$ is compact, then Γ is compact valued. Furthermore, $\widetilde{X}$ is clearly convex, and Γ is continuous. Thus assumption 4.3 of Stokey, Lucas, and Prescott (1989) (henceforth: SLP) is satisfied. Since the continuous image of a compact set is compact, $\widetilde{\mathcal{F}}$ is bounded above and below, so assumption 4.4 of SLP is satisfied as well. Furthermore, $\widetilde{\mathcal{F}}$ is concave and Γ is convex, so assumptions 4.7 and 4.8 of SLP are satisfied too. Thus, by Theorem 4.6 of SLP, with $\mathcal{B}$ defined as in equation (19) and v^* defined as in equation (20), $\mathcal{B}$ has a unique fixed point which is continuous and equal to v^*. Moreover, by Theorem 4.8 of SLP, there is a unique policy function which attains the supremum in the definition of $\mathcal{B}(v^*) = v^*$.

Now suppose that $\widetilde{X}$ is possibly non-compact, but $\tilde{\Gamma}(x)$ is compact valued and $x \in \tilde{\Gamma}(x)$ for all $x \in \widetilde{X}$. We first note that for all $x, z \in \widetilde{X}$:

$$\widetilde{\mathcal{F}}(x,z) \le u^{(0)} - \frac{1}{2}u^{(1)}\tilde{u}^{(2)^{-1}}u^{(1)'},$$

thus, our objective function is bounded above without additional assumptions. For a lower bound, we assume that for all $x \in \widetilde{X}$, $x \in \tilde{\Gamma}(x)$, so holding the state fixed is always feasible. This is true in very many standard applications. Then, the value of setting $x_t = x_0$ for all $t \in \mathbb{N}^+$ provides a lower bound for our objective function.

More precisely, we define $\mathbb{V} := \{v | v: \widetilde{X} \to [-\infty, \infty)\}$ and $\underline{v}, \overline{v} \in \mathbb{V}$ by:

$$\underline{v}(x) = \frac{1}{1-\beta}\widetilde{\mathcal{F}}(x_0, x_0),$$

$$\overline{v}(x) = \frac{1}{1-\beta}\Big[u^{(0)} - \frac{1}{2}u^{(1)}\tilde{u}^{(2)^{-1}}u^{(1)'}\Big],$$

for all $x \in \widetilde{X}$.

Finally, define $\mathcal{B}\colon \mathbb{V} \to \mathbb{V}$ by:

$$\mathcal{B}(v)(x) = \sup_{z \in \tilde{\Gamma}(x)} \left[\tilde{\mathcal{F}}(x,z) + \beta v(z) \right] \tag{19}$$

for all $v \in \mathbb{V}$ and for all $x \in \widetilde{X}$. Then $\mathcal{B}(\underline{v}) \geq \underline{v}$ and $\mathcal{B}(\overline{v}) \leq \overline{v}$. Furthermore, if some sequence $(x_t)_{t=1}^{\infty}$ satisfies the constraint that for all $t \in \mathbb{N}^+$, $x_t \in \tilde{\Gamma}(x_{t-1})$, and the objective in (8) is finite for that sequence, then it must be the case that $\|x_t\|_\infty t\beta^{\frac{t}{2}} \to 0$ as $t \to \infty$ (by the comparison test), so:

$$\liminf_{t\to\infty} \beta^t \underline{v}(x_t) = 0.$$

Additionally, for any sequence $(x_t)_{t=1}^{\infty}$:

$$\limsup_{t\to\infty} \beta^t \overline{v}(x_t) = 0.$$

Thus, our dynamic programming problem satisfies the assumptions of Theorem 2.1 of Kamihigashi (2014), and so $\mathcal{B}$ has a unique fixed point in $[\underline{v}, \overline{v}]$ to which $\mathcal{B}^k(\underline{v})$ converges pointwise, monotonically, as $k \to \infty$, and which is equal to the function $v^*\colon \widetilde{X} \to \mathbb{R}$ defined by:

$$v^*(x_0) = \sup\left\{ \textstyle\sum_{t=1}^{\infty} \beta^{t-1} \tilde{\mathcal{F}}(x_{t-1}, x_t) \,\middle|\, \forall t \in \mathbb{N}^+, x_t \in \Gamma(x_{t-1}) \right\}, \tag{20}$$

for all $x_0 \in \widetilde{X}$.

Furthermore, if we define:

$$\mathbb{W} := \{ v \in V | v \text{ is continuous on } \widetilde{X}, v \text{ is concave on } \widetilde{X} \},$$

then as $\tilde{u}^{(2)}$ is negative-definite, $\underline{v} \in \mathbb{W}$. Additionally, under the assumption that $\tilde{\Gamma}(x)$ is compact valued, if $v \in \mathbb{W}$, then $\mathcal{B}(v) \in \mathbb{W}$, by the Theorem of the Maximum,[37] and, furthermore, there is a unique policy function which attains the supremum in the definition of $\mathcal{B}(v)$. Moreover, $v^* = \lim_{k\to\infty} \mathcal{B}^k(\underline{v})$ is concave and lower semi-continuous on $\widetilde{X}$.[38] We just need to prove that v^* is upper semi-continuous.[39] Suppose for a contradiction that it is not, so there exists $x^* \in \widetilde{X}$ such that:

$$\limsup_{x\to x^*} v^*(x) > \lim_{k\to\infty} v^*(x^*).$$

Then, there exists $\delta > 0$ such that for all $\epsilon > 0$, there exists $x_0^{(\epsilon)} \in \widetilde{X}$ with $\|x^* - x_0^{(\epsilon)}\|_\infty < \epsilon$ such that:

$$v^*\big(x_0^{(\epsilon)}\big) > \delta + v^*(x^*).$$

Now, by the definition of a supremum, for all $\epsilon > 0$, there exists $\big(x_t^{(\epsilon)}\big)_{t=1}^{\infty}$ such that for all $t \in \mathbb{N}^+$, $x_t^{(\epsilon)} \in \Gamma\big(x_{t-1}^{(\epsilon)}\big)$ and:

$$v^*\big(x_0^{(\epsilon)}\big) < \delta + \sum_{t=1}^{\infty} \beta^{t-1} \tilde{\mathcal{F}}\big(x_{t-1}^{(\epsilon)}, x_t^{(\epsilon)}\big).$$

Hence:

$$\sum_{t=1}^{\infty} \beta^{t-1} \tilde{\mathcal{F}}\big(x_{t-1}^{(\epsilon)}, x_t^{(\epsilon)}\big) > v^*\big(x_0^{(\epsilon)}\big) - \delta > v^*(x^*).$$

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[37] See e.g. Theorem 3.6 and following of Stokey, Lucas, and Prescott (1989).

[38] See e.g. Lemma 2.41 of Aliprantis and Border (2013).

[39] In the following, we broadly follow the proof of Lemma 3.3 of Kamihigashi and Roy (2003).

Now, let $\mathcal{S}_0 := \{x \in \tilde{X} \| \|x^* - x\|_\infty \le 1\}$, and for $t \in \mathbb{N}^+$, let $\mathcal{S}_t := \Gamma(\mathcal{S}_{t-1})$. Then, since we are assuming Γ is compact valued, for all $t \in \mathbb{N}$, $\mathcal{S}_t$ is compact by the continuity of Γ. Furthermore, for all $t \in \mathbb{N}$ and $\epsilon \in (0,1)$, $x_t^{(\epsilon)} \in \mathcal{S}_t$. Hence, $\prod_{t=0}^{\infty} \mathcal{S}_t$ is sequentially compact in the product topology. Thus, there exists a sequence $(\epsilon_k)_{k=1}^{\infty}$ with $\epsilon_k \to 0$ as $k \to \infty$ and such that $x_t^{(\epsilon_k)}$ converges for all $t \in \mathbb{N}$. Let $x_t := \lim_{k\to\infty} x_t^{(\epsilon_k)}$, and note that $x^* = x_0 \in \mathcal{S}_0 \subseteq \tilde{X}$, and that for all $t, k \in \mathbb{N}^+$, $x_t^{(\epsilon_k)} \in \Gamma\left(x_{t-1}^{(\epsilon_k)}\right)$, so by the continuity of Γ, $x_t \in \Gamma(x_{t-1})$ for all $t \in \mathbb{N}^+$. Thus, by Fatou's Lemma:

$$v^*(x^*) \ge \sum_{t=1}^{\infty} \beta^{t-1} \tilde{\mathcal{F}}(x_{t-1}, x_t) \ge \limsup_{k\to\infty} \sum_{t=1}^{\infty} \beta^{t-1} \tilde{\mathcal{F}}\left(x_{t-1}^{(\epsilon,k)}, x_t^{(\epsilon,k)}\right) > v^*(x^*),$$

which gives the required contradiction. Thus, v^* is continuous and concave, and there is a unique policy function attaining the supremum in the definition of $\mathcal{B}(v^*) = v^*$.

Appendix H.11: Proof of Proposition 14

Suppose that $(x_t)_{t=1}^{\infty}$, $(\lambda_t)_{t=1}^{\infty}$ satisfy the KKT conditions given in equations (11) and (12), and that $x_t \to \mu$ and $\lambda_t \to \bar{\lambda}$ as $t \to \infty$. Let $(z_t)_{t=0}^{\infty}$ satisfy $z_0 = x_0$ and $z_t \in \tilde{\Gamma}(z_{t-1})$ for all $t \in \mathbb{N}^+$. Then, by the KKT conditions and the concavity of:

$$(x_{t-1}, x_t) \mapsto \tilde{\mathcal{F}}(x_{t-1}, x_t) + \lambda_t' \left[\Psi^{(0)} + \Psi^{(1)} \begin{bmatrix} x_{t-1} - \mu \\ x_t - \mu \end{bmatrix}\right],$$

we have that for all $T \in \mathbb{N}^+$:[40]

$$\begin{aligned}
&\sum_{t=1}^{T} \beta^{t-1} \left[\tilde{\mathcal{F}}(x_{t-1}, x_t) - \tilde{\mathcal{F}}(z_{t-1}, z_t)\right] \\
&= \sum_{t=1}^{T} \beta^{t-1} \left[\tilde{\mathcal{F}}(x_{t-1}, x_t) + \lambda_t' \left[\Psi^{(0)} + \Psi^{(1)} \begin{bmatrix} x_{t-1} - \mu \\ x_t - \mu \end{bmatrix}\right] - \tilde{\mathcal{F}}(z_{t-1}, z_t)\right] \\
&\ge \sum_{t=1}^{T} \beta^{t-1} \Bigg[\tilde{\mathcal{F}}(x_{t-1}, x_t) + \lambda_t' \left[\Psi^{(0)} + \Psi^{(1)} \begin{bmatrix} x_{t-1} - \mu \\ x_t - \mu \end{bmatrix}\right] - \tilde{\mathcal{F}}(z_{t-1}, z_t) \\
&\qquad\qquad - \lambda_t' \left[\Psi^{(0)} + \Psi^{(1)} \begin{bmatrix} z_{t-1} - \mu \\ z_t - \mu \end{bmatrix}\right]\Bigg] \\
&\ge \sum_{t=1}^{T} \beta^{t-1} \Bigg[\left[u_{\cdot,2}^{(1)} + \begin{bmatrix} x_{t-1} - \mu \\ x_t - \mu \end{bmatrix}' \tilde{u}_{\cdot,2}^{(2)} + \lambda_t' \Psi_{\cdot,2}^{(1)}\right](x_t - z_t) \\
&\qquad\qquad + \left[u_{\cdot,1}^{(1)} + \begin{bmatrix} x_{t-1} - \mu \\ x_t - \mu \end{bmatrix}' \tilde{u}_{\cdot,1}^{(2)} + \lambda_t' \Psi_{\cdot,1}^{(1)}\right](x_{t-1} - z_{t-1})\Bigg] \\
&= \sum_{t=1}^{T} \beta^{t-1} \Bigg[\left[u_{\cdot,2}^{(1)} + \begin{bmatrix} x_{t-1} - \mu \\ x_t - \mu \end{bmatrix}' \tilde{u}_{\cdot,2}^{(2)} + \lambda_t' \Psi_{\cdot,2}^{(1)} + \beta\left[u_{\cdot,1}^{(1)} + \begin{bmatrix} x_t - \mu \\ x_{t+1} - \mu \end{bmatrix}' \tilde{u}_{\cdot,1}^{(2)} + \lambda_{t+1}' \Psi_{\cdot,1}^{(1)}\right]\right](x_t \\
&\qquad\qquad - z_t)\Bigg] + \beta^T \left[u_{\cdot,1}^{(1)} + \begin{bmatrix} x_T - \mu \\ x_{T+1} - \mu \end{bmatrix}' \tilde{u}_{\cdot,1}^{(2)} + \lambda_{T+1}' \Psi_{\cdot,1}^{(1)}\right](z_T - x_T) \\
&= \beta^T \left[u_{\cdot,1}^{(1)} + \begin{bmatrix} x_T - \mu \\ x_{T+1} - \mu \end{bmatrix}' \tilde{u}_{\cdot,1}^{(2)} + \lambda_{T+1}' \Psi_{\cdot,1}^{(1)}\right](z_T - x_T).
\end{aligned}$$

Thus:

$$\begin{aligned}
\sum_{t=1}^{\infty} \beta^{t-1} \left[\tilde{\mathcal{F}}(x_{t-1}, x_t) - \tilde{\mathcal{F}}(z_{t-1}, z_t)\right] &\ge \lim_{T\to\infty} \beta^T \left[u_{\cdot,1}^{(1)} + \begin{bmatrix} x_T - \mu \\ x_{T+1} - \mu \end{bmatrix}' \tilde{u}_{\cdot,1}^{(2)} + \lambda_{T+1}' \Psi_{\cdot,1}^{(1)}\right](z_T - x_T) \\
&= \lim_{T\to\infty} \beta^T \left[u_{\cdot,1}^{(1)} + \bar{\lambda}' \Psi_{\cdot,1}^{(1)}\right](z_T - \mu) = \lim_{T\to\infty} \beta^T \left[u_{\cdot,1}^{(1)} + \bar{\lambda}' \Psi_{\cdot,1}^{(1)}\right] z_T.
\end{aligned}$$

[40] Here, we broadly follow the proof of Theorem 4.15 of Stokey, Lucas, and Prescott (1989).

Now, suppose $\lim_{T\to\infty} \beta^T z_T \neq 0$, then since $\tilde{u}^{(2)}$ is negative definite:

$$\sum_{t=1}^{\infty} \beta^{t-1} \tilde{\mathcal{F}}(z_{t-1}, z_t) = -\infty,$$

so $(z_t)_{t=0}^{\infty}$ cannot be optimal. Hence, regardless of the value of $\lim_{T\to\infty} \beta^T z_T$:

$$\sum_{t=1}^{\infty} \beta^{t-1} \Big[\tilde{\mathcal{F}}(x_{t-1}, x_t) - \tilde{\mathcal{F}}(z_{t-1}, z_t) \Big] \geq 0,$$

which implies that $(x_t)_{t=1}^{\infty}$ solves Problem 5 (Linear-Quadratic).