Pensions and Sovereign Default*

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Abstract

This paper studies the effect of public pension obligations on a government's decision to default. In the model, the government can renege on its pension promises but suffers a cost from losing the trust of households about future pensions. Large pension promises act as a commitment device for debt because they require the government to have regular access to credit markets. The government's decision to default is driven by its total obligations, not just its debt. This otherwise deterministic economy has an endogenous cycle in which the government has periods of high spending and increasing debt followed by periods of pension reform and debt reduction. The model successfully produces high debt in excess of 100% GDP without default and back-loaded pension cuts that match salient features of recent reforms in six EU nations.

1 Introduction

This paper proposes a tractable model that highlights the interaction between a government's decision to pay promised pensions and the decision to pay debt. Throughout the OECD, government debt obligations are large, but unfunded pensions are even larger. On average, debt is 90% of GDP while unfunded earned public pensions are 150-190% of GDP (Citigroup 2016, Gokhale 2009, Mink 2008, Disney 1999).

Unfunded pensions, like debt, are promises for future payments that are only backed by the guarantee of the government. They have a direct impact on government debt because they account for a large part of deficits in developed countries. For example, in Greece the pension deficit requires an annual transfer of 9-10% of GDP from the government budget (Blanchard 2015), while the country's total deficit averaged only 8.6% of GDP over the past 10 years. Thus, Greece's budget deficit can largely be thought of as a pension deficit.

The tight relationship between these two obligations was recently displayed during the EU debt crisis. Between 2010 and 2013, five EU member nations (Greece, Portugal, Ireland,

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Spain, and Cyprus) required bailouts to manage their sovereign debt and allay fears of default. For all five countries, these bailouts were conditional on spending reforms, most notably pension reforms. While many spending reductions were accepted, pension reforms became a major sticking point in these negotiations, particularly for Greece, demonstrating a strong unwillingness of these governments to deviate from the promised pension payments. During negotiations, Greece even offered to make equivalent cuts to other areas of spending rather than reduce pensions—an offer that was rejected. By specifically requiring pension reform, rather than simply spending reform, the ECB and IMF demonstrated their belief that the sustainability of public pensions has a strong impact on a country's likelihood of repaying debt.

In June of 2015, the choice between paying debt or paying pensions became a major issue in Greece. At the end of the month, the government had a \$1.7 billion interest payment to the IMF and a \$1.5 billion payment to social security funds, and did not have enough revenue to pay both. Up to the last day of that month, it was uncertain what the outcome would be. Would Greece renege on its pensions in order to pay the debt, or default on debt to pay the pensions? Or would additional bailout funds would be provided so that Greece could meet both obligations? In the end, no additional lending was provided and Greece defaulted on its debt payments in order to pay its pension funds, becoming the first developed nation to default on an IMF loan.

To capture this relationship, I develop a sovereign default model in which pension spending is promised in advance. Each period, the government has the choice to default on its debt and/or renege on the promised pension payments. Household labor supply decisions depend on their belief that the government will honor its pension promises. Specifically, households will work harder if they believe that higher labor income and pension contributions will result in them receiving a larger pension. If the government reneges on this promise, such as when Greece heavily cut pensions for top earners, households lose this incentive to work harder and free ride on the pension system. This mechanism is similar to the literature on the implicit social security tax (Murphy and Welch 1998, Disney 2004), where households have less desire to work when they do not believe their individual contributions will be fully repaid as pensions in the future.

The model successfully matches two key empirical facts: high debt and back-loaded pension reforms. Even with autarky as the only punishment for default, the government is able to sustain large amounts of debt without defaulting and chooses to accumulate large debt in equilibrium. This stands in contrast to the existing literature, where it is difficult to replicate the high debt to GDP ratios seen in the data. Compared to the 90% debt to GDP average for the OECD, government debt in the literature is typically only 5-20% GDP (e.g. Aguiar and Gopinath 2006 19%, Arellano 2008 6%, Arellano and Ramanarayanan 2012 16%, Cuadra and Sapriza 2008 7%, Yue 2010 10%).

The government can support large debt because high promised spending acts a commitment device. If current promised pension spending exceeds revenue, then the government can only pay the promised pensions if it is able to borrow from creditors. If the government chooses to default on debt, then it loses access to credit markets and will have to renege on pensions. These dynamics were observed in Argentina in 2001; after default the government passed the Zero Deficit Law which prohibited the government from spending more than revenue. To comply with the law, the government made large spending cuts, primarily to pensions, and began running a budget surplus.

If pension promises are made multiple periods in advance, then the government can sustain debt well in excess of 100% GDP without defaulting. The government enters each period with a schedule of promises, stating the promised spending not only for the current period but for several periods into the future. If any one of the promised spending amounts in the schedule exceeds tax revenue, then the government cannot fulfill this promised schedule unless it can borrow from creditors. This means that high future pension promises raise the value of continued market access, allowing the government to maintain large debt. Effectively, promises for future pensions allow the government to control its outside option of default.

The second success of the model is that it produces back-loaded pension cuts. This rationalizes the back-loaded fiscal consolidations documented in work by Romer (2012) and Devries, Guajardo, Leigh, and Pescatori (2011), and recent pension reforms in the EU. In late 2011, near the peak of the EU debt crisis, six countries (Spain, Ireland, UK, Italy, France, and the Netherlands) passed pension reforms, but the majority of the benefit cuts were placed many years into the future. In the model, pension spending is promised several periods in advance, which means that the government cannot alter current spending without losing households' trust. When spending and debt are both high, the government needs to borrow money to honor its current pension promises but creditors will only lend that money if the government promises lower future pensions. This creates a pattern of pension reform where spending is unchanged for several periods and then reduced for multiple periods. If the government is impatient, it chooses to make small initial cuts to future pensions followed by larger cuts, meaning that pension cuts are not only pushed several periods into the future but are also back-loaded.

In equilibrium, the decision to default is driven by total obligations, not just the level of debt. This result comes from the fact that the government cares more about its retirees than its creditors and therefore never chooses to renege on the pensions for retirees while still repaying its debt. This means that the debt is only repaid if the government is able to keep its pension promise, implying that the choice to repay debt depends on whether both the promised pensions and the debt can be paid.

The equilibrium features an endogenous cycle of debt and pension spending, despite the fact that there are no exogenous shocks. The model is deterministic, but due to the government's limited commitment there is no steady state value of debt or pensions. Promising high pension spending helps the government commit to repaying debt but cannot be maintained forever. The government will never choose to spend below revenue forever because it can simply default and spend its entire revenue on pensions in autarky. Therefore, within finite time, the economy enters a cycle in which the government switches between periods of high spending and increasing debt and periods of low spending and debt reduction.

The rest of the paper is organized as follows. Section 2 gives a brief review of the related literature on sovereign default. Sections 3-6 describe the model and its solution when spending promises are made one or more periods in advance. Section 7 shows the equilibrium choice of debt and the main parameters that drive the level of sustainable debt. Section 8 shows the pension cuts predicted by the model when the government has high debt and compares this with pension reforms for six EU nations during the European debt crisis. Section 9 concludes the paper and the appendix contains all proofs.

2 Literature Review

This paper contributes to a large literature on optimal debt and spending when governments have the ability to default (e.g. Bulow and Rogoff 1989, Cole and Kehoe 1998, Arellano 2008, Aguiar and Amador 2014). Building on the model of Eaton and Gersovitz 1981, these papers study the default choice of a government that cannot commit to repay creditors. A major puzzle in this literature is why governments do not default on large debt. For most models, the maximum sustainable debt is far below 90% GDP[1].

This paper demonstrates that spending obligations can be a quantitatively powerful commitment device for debt. The key alteration from the Eaton and Gersovitz model is that spending is promised N periods in advance. When $N = 0$, government spending is completely flexible and the model results are consistent with the literature. The model is deterministic and the only punishment for default is autarky, so the government will default on any positive amount of debt. When $N \geq 1$, spending is promised one or more periods in advance. Depending on the promised amount, the government may not be able to keep its spending promise if it goes into autarky. By choosing its promised spending in advance, the government is effectively controlling the value of its outside option to default.

The fact that defaulting on debt may force the government to renege on pensions is similar to the general reputation concept of Cole and Kehoe 1998. In their framework, defaulting would cause households to stop trusting the government's pension promises because it would reveal that the government's private type is low commitment. In comparison, households in this model do not care if the government defaults. They only stop trusting the government if it reneges on pensions. The connection to default comes from the fact that autarky may make it impossible for the government to meet its spending promises and force the government to renege.

Two recent papers study related issues. Aguiar and Amador (2014) analyze a government that cannot commit to not expropriating domestic capital and assume that the government must expropriate if it defaults. They use this model to examine how large debt may limit the growth of capital. In their calibration, steady state debt is 9% of GDP. Dovis, Golosov, and Shourideh (2016) study a similar economy with heterogeneous households where the government has an incentive to expropriate all household wealth and redistribute it evenly across all households. If the government defaults, they assume it must redistribute wealth. They characterize the dynamics of debt and spending in this model, but their focus is not on the level of sustainable debt.

These papers are related to this model in the sense that they all study a government that has debt obligations to foreign creditors and obligations to its citizens (e.g. not to expropriate). A crucial difference from those papers is that this model's government is not required to break its obligations to its citizens if it defaults. If promised spending is below revenue, then the government can default on its debt without reneging on its pension promises. It is only when promised spending exceeds revenue that default will force the government to renege. This produces very different behavior for spending and debt. For

[1] One paper that comes close is Chatterjee and Eyigungor 2012. Out of Argentina's 100% debt-to-GDP for 1993-2001, they are able to generate 70%. In order to get this high debt, they use long-term bonds, make the government considerably impatient with an annual discount factor of 0.82, and impose large output losses upon default.

example, in both of their models, default probability is increasing in debt. Because of this, a bailout in which a government is given access to more debt will only increase the risk of a default. In contrast, default risk in this model depends on the level of promised spending as well as debt. A bailout for a government that has high current promised spending may actually help avoid default, as long as the government promises lower future spending. This is studied in more detail in section 8.

3 Model

The model is a small, open economy. There is a single consumption good and labor is the only factor of production. Labor productivity is normalized to 1, so output simply equals total labor. The model has two types of agents: households and the government. Households make decisions about consumption and labor. The government makes decisions about debt and pensions, including whether to default on debt and whether to pay promised pensions. The government finances pension spending by taxing labor and borrowing from competitive, risk neutral foreign creditors.

The timing of the models is as follows. At the start of the period, households choose how much to work and consume. After working, some households exogenously retire. The government then makes decisions about debt and pension spending. Each period the government has the ability to default on bonds and/or renege on promised pensions. Defaulting means that the government does not repay any of the bonds. Reneging means the government pays pensions that are different from the promised amounts.

3.1 Households

The economy has a continuum of working households. At the end of each period, δ^r portion of workers retire and δ^b portion of new workers are born. Retired households die at rate δ^d. The birth, retirement, and death rates are all exogenous. Households have utility over labor and consumption that is given by $u(c, \ell) = \log c - \ell$ and have a discount factor of β.

Working households cannot save or borrow. This is consistent with a large literature showing that households struggle to save for their own retirement. The majority of households reach retirement with virtually no personal savings (Rhee and Boivie 2015, Morrissey 2016), and before the creation of public pension systems most American workers continued working until they died or became too injured to work (Fischer 1978). Since workers cannot save for retirement on their own, the government taxes workers and pays pensions to retirees.

Each period, working households must choose how much to work ℓ and consume c. Households pay taxes $\tau\ell$ as pension contributions and are promised pension replacement rate π if they retire at the end of the period. The replacement rate is the pension as a fraction of previous labor income. If the household chooses labor ℓ and then retires at the end of the period, the government promises to pay them pension $\pi\ell$ next period and every period until they die. Since the probability of retiring is δ^r and retirees die at rate δ^d, the utility value of promised pension $\pi\ell$ is

$$\delta^r \sum_{j=0}^{\infty} \beta^{j+1} \left(1-\delta^d\right)^j u\left(\pi \ell, 0\right) = \theta \log \pi \ell$$

where $\theta \equiv \frac{\beta \delta^r}{1-\beta\left(1-\delta^d\right)}$ is a measure of the benefit of receiving a pension until death.

Workers know that the government may renege on the promised pensions. Each worker believes that if the government reneges, then it will choose to pay a pension that is based on aggregate variables, not the worker's individual labor. For example, the government may have promised a replacement rate of 0.5, meaning that an individual worker's pension will be half of their labor income. This means that the worker can increase or decrease her pension by choosing to work more or less. However, if the government reneges on the promised pensions, then she believes that her pension will be determined by current tax revenue, debt, output, etc.

Since there are a continuum of workers, each individual cannot affect these aggregate variables. This means that the worker only believes she can affect her pension if the government does not renege. Let q be the household's belief that the government will not renege on pensions next period. The household's problem is

$$\begin{aligned} \max_{\ell} \quad & \log c - \ell + q\theta \log \pi \ell \\ \text{s.t.} \quad & c = \left(1-\tau\right)\ell \end{aligned}$$

which is solved by

$$\ell = 1 + q\theta \tag{1}$$

$$c = \left(1-\tau\right)\ell. \tag{2}$$

Note that the household does not believe it will receive zero pension income if the government reneges; it simply believes it will receive a pension that is independent of the individual household's decisions. This independence is why there is not a $(1-q)$ term in the optimization for the event that the government reneges. As we will see later, if the government reneges then it chooses to pay strictly positive pensions to all retirees. This means that there is never a scenario where households have negative infinite utility.

Looking at equation (1), we see that labor is increasing in q. When workers trust the government to keep its promise, there is an extra incentive to work because working more increases promised pensions. In the case of $q = 1$, pension contributions are viewed as forced savings. Workers are forced to save τ portion of their income and receive benefits that are proportional to their contributions. However, when $q = 0$ pension contributions are viewed purely as a tax on labor. Because workers don't believe that their individual contributions will have any effect on their future pensions, they have no incentive to work harder to provide more contributions. In this case, workers simply maximize their current period payoff of $\log c - \ell$ by choosing $\ell = 1$. While the workers still care about their future pensions, they act myopically because they don't believe that their decisions will affect their pensions.

3.2 Government

The government is infinitely lived and benevolent. Let $g = 1 + \delta^b - \delta^r$ be the growth rate of the workforce. To simplify notation, I express the payoffs and variables in per worker units.

The government enters period t with an amount of bonds per worker $\tilde{b}_t$ and a schedule of promised replacement rates for the next N periods, $\pi_t, ..., \pi_{t+N-1}$. Negative $\tilde{b}_t$ means the government is in debt. $N = 0$ corresponds to the case where the government makes no promises and simply chooses how much to spend on pensions at the end of each period. Positive N means that the government has already made promises about pension spending for some periods, but has the freedom to choose the promised replacement rates after that.

Each period, after households have chosen how much to work, the government chooses the pension amount to give to new retirees. It also decides whether to default on bonds and chooses the promised replacement rate for N periods in the future. Suppose the government never reneges and never defaults. Let $\bar{\ell} = 1 + \theta$ and $\bar{c} = (1 - \tau)\,\bar{\ell}$ be the labor and consumption when households completely trust the government. Given a path of promised replacement rates $(\pi_s)_{s=0}^{\infty}$, the payoff to the government at time t from never reneging and never defaulting is

$$\begin{aligned} \tilde{V}_t &= \sum_{s=t}^{\infty} \beta^{s-t} g^{s-t} \left(\log \bar{c} - \bar{\ell} + \theta \log \pi_s \bar{\ell}\right) \\ &= \frac{1}{1-\hat{\beta}} \left(\log \bar{c} - \bar{\ell}\right) + \theta \sum_{s=t}^{\infty} \hat{\beta}^{s-t} \log \pi_s \bar{\ell} \end{aligned} \tag{3}$$

where $\hat{\beta} = \beta g$.

The growth of the workforce from time t to s is g^{s-t} and the government discounts period s with factor β^{s-t}. Because the government never reneges, we know all workers will choose the same labor $\bar{\ell}$ and consumption $\bar{c}$ and get utility $\log \bar{c} - \bar{\ell}$. After households have worked, δ^r portion will retire. These newly retired workers will receive pension $\pi_s \bar{\ell}$ until they die. The utility of providing pension $\pi_s \bar{\ell}$ to δ^r retirees is simply $\theta \log \pi_s \bar{\ell}$.

Lemma 1. *The government strictly prefers households trust its promises. Specifically,* $\log \bar{c} - \bar{\ell} + \theta \log \pi \bar{\ell} > \log(1-\tau) - 1 + \theta \log \pi$ *for all* $\pi > 0$.

For simplicity, I assume that the government pays the full cost of the pensions for new retirees in the period that they retire. One interpretation of this is that once the workers retire and the government decides to give them a specific pension amount, it sets aside the money necessary to pay these pensions until the new retirees die. This money is placed in a trust that pays retirees the same amount every period they are alive, and cannot be touched by future governments. A second interpretation is that the government buys an annuity for each worker from foreign credit markets that pays the worker the same amount every period until they die. Let $\phi \pi \bar{\ell}$ be the cost of providing pension $\pi \bar{\ell}$ to δ^r new retirees. The only assumption for costs is $\phi > 0$.

If the government does not default or renege in period t, then its budget constraint is

$$\phi \pi_t \bar{\ell} - \tilde{b}_t = \tau \bar{\ell} - \frac{g}{R} \tilde{b}_{t+1}. \tag{4}$$

All terms in the budget constraint are in per worker units. The LHS has $\phi\pi_t\bar{\ell}$ as the pension spending and $-\tilde{b}_t$ as the cost of paying bonds. On the other side of the budget constraint, we have tax revenue $\tau\bar{\ell}$ and revenue from selling new bonds $-\frac{g}{R}\tilde{b}_{t+1}$. Because $\tilde{b}_{t+1}$ is bonds per worker, the effective interest rate is R/g.

To simplify the government's payoff, normalize bonds and promised pension spending by the maximum tax revenue $\tau\bar{\ell}$

$$\begin{aligned} b_t &= \tilde{b}_t/\tau\bar{\ell} \\ p_t &= \phi\pi\bar{\ell}/\tau\bar{\ell}. \end{aligned}$$

The budget constraint can be rewritten as

$$p_t - b_t = 1 - \frac{g}{R}b_{t+1}. \tag{5}$$

The government's payoff from never reneging or defaulting can be simplified to

$$V_t = \sum_{s=t}^{\infty} \hat{\beta}^{s-t} \log p_t \tag{6}$$

where the full government payoff is $\tilde{V}_t = \kappa + \theta V_t$ for $\kappa = \frac{1}{1-\hat{\beta}}\left(\log\bar{c} - \bar{\ell} + \theta\log\left(\tau\bar{\ell}/\phi\right)\right)$. I assume there is a lower bound on bonds that prevents Ponzi-schemes, but otherwise does not bind in equilibrium.

3.3 Equilibrium

The government starts period 0 with initial normalized bonds b_0 and promised spending $p_0, ..., p_{N-1}$. If the government does not renege or default for the first N periods, then future bonds are

$$b_t = \left(\frac{R}{g}\right)^t b_0 + \sum_{s=0}^{t-1}\left(\frac{R}{g}\right)^{t-s}(1-p_s), \forall t \leq N$$

where 1 represents the normalized revenue.

My assumptions for the model parameters and initial conditions are the following. I assume $b_t < 0\ \forall t \leq N$ meaning that the government starts in debt and will continue to be in debt for the first N periods if it does not default or renege. The parameter assumptions are that $\delta^r > 0$, $R/g > 1$, and $\hat{\beta}(R/g) = \beta R \leq 1$. The first condition simply states that workers actually retire, otherwise it makes no sense to discuss pensions. The second condition states that the effective interest rate for bonds per worker is greater than 1. Given a standard R of 1.02, this is true for all EU nations. If this were not satisfied then the government could choose to never pay back its debt and simply let debt per worker shrink to 0. This would mean that there is no amount of debt for which the government ever chooses to default, making concerns about sovereign default nonsensical. The last condition is a standard assumption for the discount factor. Having $\beta R > 1$ would imply that the government is a net saver, which clearly does not match the data for EU nations.

I focus on subgame perfect equilibria in which there is a punishment equilibrium in subgames where government defaults or reneges. The details of the subgames after a default or renege are given in the next section. The main concept is that defaulting will lead to autarky, while reneging will lead to distrust ($q = 0$ forever). Let $\underline{V}(p_0, ..., p_{N-1})$ be the payoff if the government deviates by defaulting and/or reneging when promised spending is $p_0, ..., p_{N-1}$.

The equilibrium allocation given initial condition $\{b_0, p_0, ..., p_{N-1}\}$ is then defined as the allocation $(b_t, p_t)_{t=0}^{\infty}$ that maximizes V_0 subject to (i) the budget constraint (5) and (ii) the participation constraint

$$V_t \geq \underline{V}(p_t, ..., p_{t+N-1}), \forall t. \tag{7}$$

This allocation will be the path of play for the SPE. All subgames that are not played in equilibrium will be summarized by the punishment equilibria, i.e. the continuation equilibria if the government deviates by defaulting and/or reneging.

3.4 Punishment Payoffs

I assume that the punishment for defaulting is autarky and the punishment for reneging is that households distrust the government. If the government deviates by defaulting on bonds and reneging on pensions, then its payoff is

$$\begin{aligned}\tilde{V}^{d,r} &= \log \bar{c} - \bar{\ell} + \theta \log\left(\tau\bar{\ell}/\phi\right) \\ &+ \frac{\hat{\beta}}{1-\hat{\beta}}\left[\log(1-\tau) - 1 + \theta\log(\tau/\phi)\right].\end{aligned}$$

Households have already chosen labor $\bar{\ell}$ before the government reneges, so initial workers get utility $\log \bar{c} - \bar{\ell}$. Since the government is in autarky, it simply spends all of its revenue on pensions, meaning the initial group of new retirees get pension $\tau\bar{\ell}/\phi$. After the first period, households reduce their labor to 1 because they don't trust the government. This means that all future workers get utility $\log(1-\tau) - 1$ and future retirees get pension τ/ϕ.

Subtracting κ and dividing by θ, gives

$$V^{d,r} = \frac{\hat{\beta}}{1-\hat{\beta}}\left[1 - \left(1 + \frac{1}{\theta}\right)\log \bar{\ell}\right] \tag{8}$$

since $\theta = \bar{\ell} - 1$.

Suppose the government chooses to default on bonds but still pay the promised pensions. This subgame must be an equilibrium. So, if the government defaults in period t and there is a future period s where the government reneges, then all households choose labor 1 for period s and all future periods. This means that there is no punishment for reneging in period $s-1$ because future labor is already low, making reneging in period $s-1$ an optimal choice. This argument can be repeated to conclude that it is optimal for the government to renege in period t. Therefore, in the equilibrium for this subgame, the government either reneges immediately or never reneges. This means that the government only chooses to default without reneging if it is optimal to never renege.

Since the government is in autarky, the highest spending that the government can choose for future periods is $p=1$, meaning that it promises to spend 100% of its revenue. Because the government never reneges in this subgame, households always choose labor $\bar{l}$. The payoff is then

$$\tilde{V}^d\left(p_0,...,p_{N-1}\right) = \kappa+\theta\sum_{j=0}^{N-1}\hat{\beta}^j\log p_j$$

which is normalized by subtracting κ and dividing by θ to give

$$V^d\left(p_0,...,p_{N-1}\right) = \sum_{j=0}^{N-1}\hat{\beta}^j\log p_j. \tag{9}$$

The deviation payoff is then defined as

$$\underline{V}\left(p_0,...,p_{N-1}\right) = \begin{cases}\max\left\{V^{d,r},V^d\left(p_0,...,p_{N-1}\right)\right\} & \text{if } p_0,..,p_{N-1}<1\\ V^{d,r} & \text{otherwise.}\end{cases} \tag{10}$$

$V^d\left(p_0,...,p_{N-1}\right)$ is the payoff when the government defaults and never reneges. Since the maximum p the government can pay in autarky is 1, this deviation is not possible if $p_i>1$ for any $0\leq i\leq N-1$. In order to make the SPE well defined, V^d is not allowed as a possible deviation when $p_i=1$ for some $0\leq i\leq N-1$, i.e. there is a promised spending that is 100% of revenue. This ensures that the choice set of the government is closed. Otherwise, there would be $p>1$ arbitrarily close to 1 that satisfy (7) but $p=1$ might not.

There is a third possible deviation that the government could make, which is reneging on pensions and not defaulting on bonds. Once the government reneges on pensions, spending is completely flexible. Because the only punishment for default is autarky and the model is deterministic, the government cannot support any debt when spending is completely flexible. Therefore, unless bonds are strictly positive, a government which reneges will find it optimal to also immediately default. In the next section I show that the government always chooses weakly negative bonds, which means that the constraint not to renege without defaulting never binds. Because of this, there is no need for a third participation constraint.

4 Recursive Formulation

The equilibrium allocation given $\{b_0,p_0,..,p_{N-1}\}$ is the solution to the recursive problem

$$\begin{aligned} V\left(b,p_0,..,p_{N-1}\right) &= \max_{p_N}\log p_0+\hat{\beta}V\left(b',p_1,..,p_N\right)\\ b' &= \left(R/g\right)\left(1-p_0+b\right)\\ V\left(b,p_0,..,p_{N-1}\right) &\geq \underline{V}\left(p_0,...,p_{N-1}\right). \end{aligned} \tag{11}$$

Because the value function appears in the participation constraint, there may be multiple solutions to the functional equation. The solution that corresponds to the equilibrium is the highest value solution to the recursive problem. This is shown formally in the Appendix. The Appendix also shows how this recursive problem can be transformed to a simple two state variable recursive problem, with one continuous variable for future bonds and one integer variable that takes values from 0 to $N-1$. This makes calculating the value function computationally easy. A solution to the recursive problem exists if and only if the initial conditions satisfy $V(b_0, p_0, .., p_{N-1}) \geq \underline{V}(p_0, ..., p_{N-1})$. In this case, the participation constraint (11) can moved one period forward

$$V(b', p_1, .., p_N) \quad \geq \quad \underline{V}(p_1, ..., p_N).$$

The solution to this recursive problem will have a government which is always weakly in debt. Formally, for $i = 0, .., N$ let

$$b^{(i)} \quad = \quad \left(\frac{R}{g}\right)^i b_0 + \sum_{j=0}^{i-1} \left(\frac{R}{g}\right)^{i-j} (1 - p_j)$$

be the future debt if the government complies with its promised spending. In choosing p_N, the government is also choosing $b^{(N+1)}$. We have assumed that the initial bonds are non-positive and the initial promised spending is such that $b^{(N)} < 0$, so by Proposition 1 the government will always choose $b^{(N+1)} \leq 0$. The fact that $b^{(N+1)} \leq 0$ implies that the next period choice of $b^{(N+2)}$ will also be non-positive, which means all future bonds will be non-positive.

Proposition 1. *If $b^{(N)} \leq 0$, then optimal $b^{(N+1)} \leq 0$.*

Proof. See Appendix. □

The lower bounds on government bonds are characterized by $\underline{b}, b^*$, which are the lowest bonds such that the government does not default and renege or just default. The government always has the option to immediately default on its debt and renege on its promised spending, and $\underline{b}$ gives the lowest b such that this option is not chosen. If promised spending does not exceed revenue, the government also has the option to immediately default, pay the promised pensions and then choose future promised pensions. In order not to default, the continuation value after the first N periods for not defaulting or reneging must be at least as great as the continuation value after N periods for defaulting, which is simply 0. When the government has the ability to default and still pay the promised pensions, b^* is the lowest possible bonds the government can have after meetings its promised spending such that it does not want to default. Since $\theta > 0$, $V^{d,r} < 0$ which means $\underline{b} < b^*$.

$$\underline{b} \quad = \quad \min\left\{b : \exists (p_0, .., p_{N-1}) \text{ s.t. } V(b, p_0, ..., p_{N-1}) \geq V^{d,r}\right\} \tag{12}$$

$$b^* \quad = \quad \min\left\{b : \exists (p_0, .., p_{N-1}) \text{ s.t. } V(b, p_0, ..., p_{N-1}) \geq 0\right\} \tag{13}$$

The next section examines the dynamics of bonds as well as spending for this government under the simplest form of the model, when the government only has one period of promised spending.

5 One Period of Obligations

This section gives the equilibrium when the government enters each period with a promised amount for current spending, $N = 1$. To provide a benchmark for comparison, I first summarize the results when $N = 0$. This government has completely flexible spending. It enters each period with no promises for spending and can choose to spend any amount it desires, so long as its bonds for the next period do not violate the participation constraint. It is easy to show in this case that the government cannot support any debt.

Since initial bonds are negative, the government immediately defaults and enters autarky. The equilibrium bonds and spending are 0 and 1 for each period. This matches the typical result in the literature that when spending is completely flexible and autarky is the only punishment for default, the government cannot support large debt. This comes from the fact that aggregate shocks are not large enough to make autarky a severe punishment. Since this model is deterministic, the government can support exactly 0 debt when spending is flexible.

When $N = 1$, however, autarky can become a more significant punishment. If the government enters the period with a spending promise that exceeds its revenue (i.e. $p \geq 1$), then it can only keep this promise if it is able to borrow. Since there is a cost to reneging on pensions, there is an additional cost of defaulting when spending is high because defaulting implies the government will also have to renege. Because of this, high promised spending can act as a commitment device for debt. Governments with large promised pensions must remain in creditor's good graces, otherwise they will have to renege on their own citizens.

Figure 1 shows the regions of the state space (b, p) where the government chooses to default or default and renege. Starting from the bottom right, as bonds decrease and promised spending increases, the government eventually enters a region where it is optimal to default, which is shown in blue. This makes intuitive sense. As obligations rise, the government will eventually break one of its promises. Given the choice between defaulting on foreign creditors or reneging on its own citizens, the government chooses to default.

In the blue region, promised spending is below revenue, so the government can default and still pay the promised pensions. As promised spending increases past 1, the government enters a region where it is no longer optimal to default, shown by the white space between the red and blue regions. Here, bonds are negative enough that the government would like to default, but defaulting would mean that the government cannot pay the promised pensions. In order to avoid reneging on pensions, the government chooses not default on its debt. This is the commitment power of high promised spending.

Continuing to move towards the top left of the graph, we see that this commitment power has limits. As bonds decrease and promised spending increases, the government eventually reaches the region shown in red where it chooses to default on debt and renege on pensions. In this region, obligations have become so large that the government cannot borrow enough to pay them both. The government cannot pay the promised pensions if it defaults, so it must renege. Since it is forced to renege, the government also chooses to default because it has negative bonds.

The boundary of the default and renege region is given by (b, p) where $b' < \underline{b}$. Similarly, the boundary for the default region is given by $b' < b^*$. Since $b' = (R/g)(b - p + 1)$, this means that the government's decision to deviate is driven by total obligations $b - p$, not just

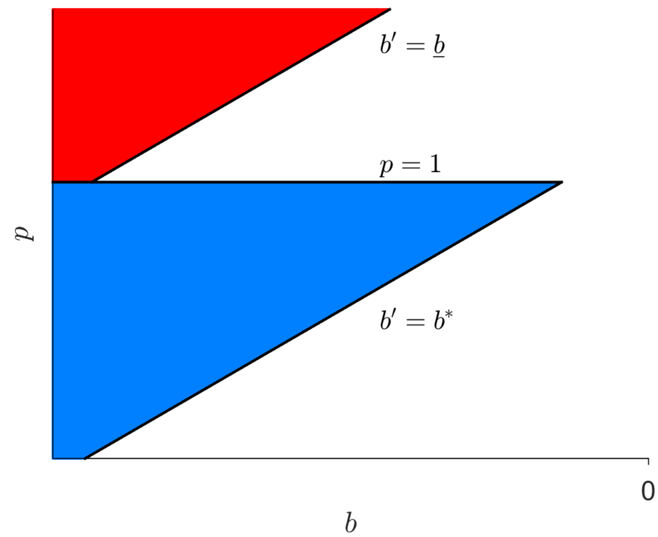

Figure 1: Default and Renege Regions

The blue region is defined by $V(b,p) < V^d(p)$ (default region). The red region is defined by $V(b,p) < V^{d,r}$ (default and renege region).

its current bonds. The government's level of promised spending determines if it has a bond limit of $\underline{b}$ or b^* (i.e. whether it is committed to repaying debt) and total obligations $b - p$ then dictate if the government deviates. The main intuition is that the government cares more about its citizens than the foreign creditors. Therefore, it only pays its debt if it is also able to pay the promised pensions, which means its decision to default depends on total obligations rather than just the level of bonds.

The commitment power of promising high spending cannot last forever. Eventually the government must spend less than 1, otherwise its bonds will fall below $\underline{b}$. Conversely, the government will never choose to spend less than 1 forever because it could simply default, pay the current low promised spending and then spend 1 in all future periods. The equilibrium must have a cycle of debt and spending with some periods of high spending and increasing debt followed by periods of low spending and debt reduction.

Definition. The path of bonds and spending $\{b_t, p_t\}_{t=0}^{\infty}$ enters cycle $(b^i, p^i)_{i=0}^{m-1}$ if there exists finite k such that $(b_{k+i}, p_{k+i}) = (b^i, p^i)$ for all $0 \leq i \leq m-1$ and $(b_t, p_t) = (b_{t+m}, p_{t+m})$ for all $t \geq k$.

Theorem 1. *There exists a cycle such that for any* $\{b_0, p_0\}$ *where an equilibrium exists,* $(b_t, p_t)_{t=0}^{\infty}$ *enters this cycle.*

Proof. See Appendix. □

This cycle is $(b^i, p^i)_{i=0}^{m-1}$ where m, p^0 solve

$$
\begin{aligned}
\max_{m,p^0} \quad & \sum_{i=0}^{m-1} \hat{\beta}^i \log p^i \\
\text{s.t.} \quad & \\
p^i &= \max\left\{(\beta R)^i p^0, 1\right\} \ \forall i < m-1 \qquad (14) \\
p^{m-1} &= (\beta R)^{m-1} p^0 \qquad (15) \\
\left(1 - \left(\frac{R}{g}\right)^m\right) b^* &= \sum_{i=0}^{m-1} \left(\frac{R}{g}\right)^{m-i} \left(1 - p^i\right).
\end{aligned}
$$

The value of b^* can be found by setting the above maximization equal to 0. The cycle $(b^i, p^i)_{i=0}^{m-1}$ is given by (14), (15), $b^0 = b^*$ and $b^{i+1} = \frac{R}{g}\left(b^i + 1 - p^i\right)$.

Figures 2 and 3 show the equilibrium cycle for a specific parametrization. Changing the model parameters will alter b^*, p^0, m but the qualitative properties of the cycle will be unchanged. Starting with the point in the top right, the cycle begins at (b^*, p^0). Spending is high, so bonds become more negative each period. Spending falls by a factor of βR until it eventually hits 1. From there, spending remains at 1 for several periods because there is a commitment benefit of promising high spending.

Eventually the government must cut spending. The longer the government waits to cut spending, the larger the cut will need to be. Once the government promises spending below 1, it loses the commitment power of high promised spending. Because of this, creditors will only lend to the government if the drop in spending is sufficiently large that the government will not want to default next period. After this period of low spending, bonds increase enough that the government can begin the cycle again with high spending. The appeal of restarting the cycle with high spending is the reason why the government does not default at the last point in the cycle.

6 Multiple Periods of Obligations

The case of $N = 1$ captures most of the intuition of the model, but the success in matching the motivating facts comes when $N > 1$. In reality, governments often make promises many years in advance. However, they also frequently alter past promises after the fact. Because of this, N should not be taken literally as the longest horizon promise a government can make. Instead, it should be thought of as a property of the households that measures how many years in advance the government must warn households of a pension change in order not to lose their trust. The less forgiving households are, the greater N will be.

When N is greater than 1, the government enters each period not just with a promised amount for current spending, but with a schedule of promised spending for the current period and the $N - 1$ succeeding periods. If any of the promised spending amounts are not less than 1, then the government will not be able to keep its promise if it is in autarky. As in the $N = 1$ case, when the government cannot keep its spending promise in autarky, it has greater commitment not to default on debt. Because of this, the government intentionally chooses a schedule of spending promises that is never entirely affordable.

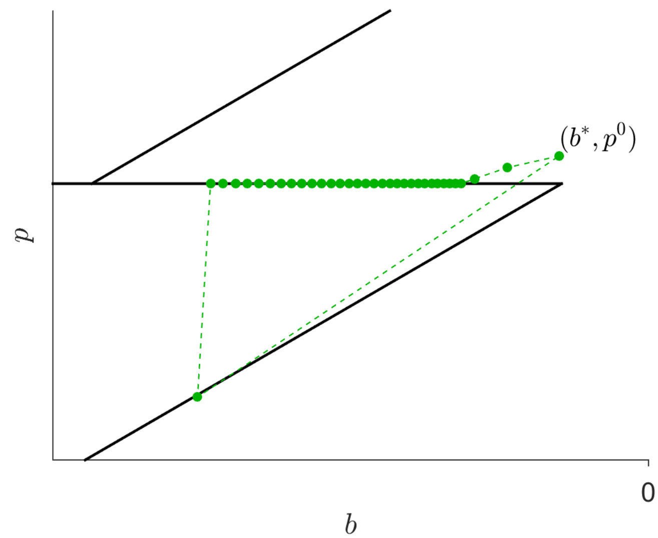

Figure 2: Bond and Spending Cycle

Black lines represent $p = 1$, $b' = b^*$, $b' = \underline{b}$. Green line with markers is equilibrium cycle.

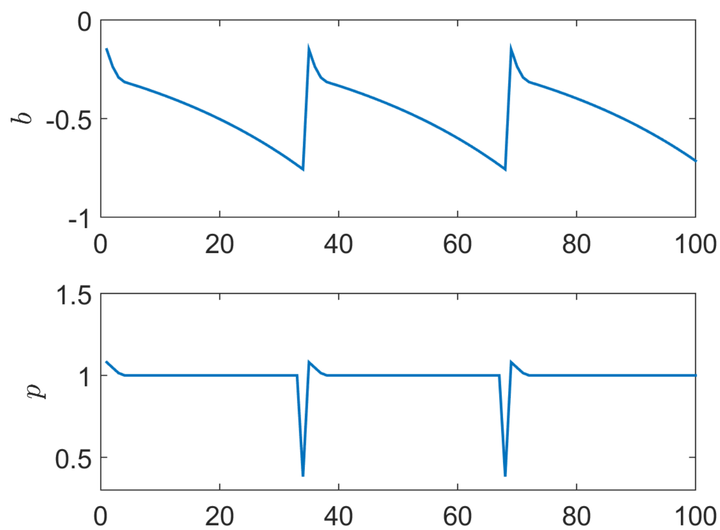

Figure 3: Bond and Spending Cycle, $N = 1$

Proposition 2. *For $N > 1$, if $p_1, ..., p_{N-1} < 1$, then $p_N(b, p_0, p_1, ..., p_{N-1}) \geq 1 \; \forall b$.*

Proof. See Appendix. $\square$

By doing this, the government ensures that it never has the option to default without reneging. This reduces the incentive problem of the government and allows it to take on larger amounts of debt. When $N = 1$, this is not possible. Eventually, promised spending must be below 1, which means the government has the ability to default without reneging. When $N > 1$, the government has the ability to choose low spending in some periods without ever having an entire schedule of promised spending that is less than 1.

This strategy of always choosing a schedule of promises that is never entirely affordable means that there will be a cycle of spending. Spending must be low in some periods to avoid debt exploding to infinity and must be high in other periods to maintain commitment power.

Theorem 2. *There exists a cycle such that for any $\{b_0, p_0, ..., p_{N-1}\}$ where an equilibrium allocation exists, $(b_t, p_t)_{t=0}^{\infty}$ enters this cycle.*

Proof. See Appendix. $\square$

This cycle $(b^i, p^i)_{i=0}^{N-1}$ is exactly N periods long. The spending cycle $(p^i)_{i=0}^{N-1}$ solves

$$\begin{aligned}
p^i &= \min\left\{(\beta R)^i p^0, \exp\left(\left(1-\hat{\beta}\right)V^{d,r}\right)\right\} \; \forall i < N-1 \\
p^{N-1} &= 1 \\
\left(1-\left(\frac{R}{g}\right)^N\right)b^0 &= \sum_{i=0}^{N-1}\left(\frac{R}{g}\right)^{N-i}(1-p_i).
\end{aligned}$$

If $\beta R < 1$, then $b^0 = \underline{b}$, otherwise $b^0 = b^{(N)}$ i.e. the bonds after the initial promised spending $(p_0, ..., p_{N-1})$. The bond cycle $(b^i)_{i=0}^{N-1}$ is given by $b^{i+1} = \frac{R}{g}(b^i + 1 - p^i)$. The lower bound $\underline{b}$ can be found by setting $\sum_{i=0}^{N-1} \hat{\beta}^i \log p^i$ equal to $V^{d,r}\left(1-\hat{\beta}^N\right)$. Figure 4 shows the cycle of spending for $N = 4$ for $\beta R < 1$. In comparison to the cycle with $N = 1$, bonds are much more negative and the cuts to spending are smaller and spread out over several periods rather than being concentrated in one period.

7 Sustainable Debt

The main quantitative test of the model is to see if the government chooses to accumulate large amounts of debt and can support this debt without defaulting. In the model, b represents bonds normalized by tax revenue. Debt to GDP is then simply $-b\tau$. The value of τ is based on the average EU revenue/GDP for 2003-2013. The discount factor and world interest rate β, R are set to standard values of 0.95 and 1.02. The retirement rate and the death rate are calculated using EU average retirement ages and life expectancy at 65 for 2003-2013.

Table 2 shows the average debt to GDP ratio for the equilibrium cycle for different values of N and g. Two things are immediately apparent. First, when $N > 1$ the government is

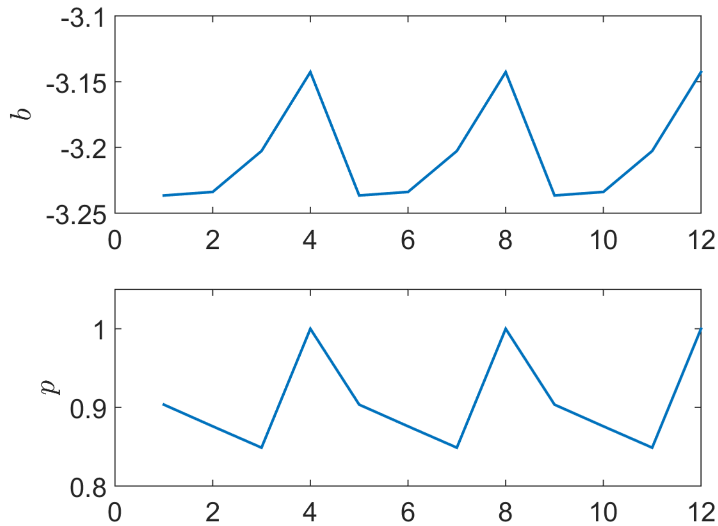

Figure 4: Bond and Spending Cycle, $N = 4$

able to sustain a large amount of debt without defaulting. The benchmark for comparison is always the $N = 0$ case in which the government cannot support any debt. When $N = 1$, the government has some commitment power from high promised spending, but is limited by the fact that it will eventually have the ability to default without reneging once spending is cut.

Table 1: Parameter Values

β	0.95	
R	1.02	
τ	0.4	Revenue/GDP
δ^r	2.3%	1/(Retirement Age-18)
δ^d	4.6%	1/(65+Life Expectancy-Retirement Age)

With $N > 1$, the government is able to always maintain a schedule of promised pensions that is unaffordable in autarky. Therefore default cannot occur without reneging. The punishment of losing household trust is large enough that the government can commit to repay debt in excess of 100% GDP. As N increases, so does the equilibrium debt.

In the limit as $N \to \infty$, the equilibrium approaches the efficient outcome. The equilibrium cycle becomes a single point (i.e. a steady state) and is efficient in the sense that the government is always given the maximum punishment if it defaults, which is autarky plus the loss of household trust, regardless of promised spending. While this results in higher equilibrium debt than the case of finite N, the vast majority of the commitment power is captured at $N = 2$. So, even when promises are only made a few periods in advance, the government is able to come close to the efficient level of debt by always choosing a schedule of promised spending that is not entirely affordable. Figure 5 shows the evolution of debt

Table 2: Mean Cycle Debt (% GDP)

	N				
g	0	1	2	4	∞
0.99	0	18	120	124	130
1	0	26	184	190	196
1.01	0	42	370	384	396

for finite N and how this compares to the efficient case.

The second clear result is that the growth rate of the workforce significantly impacts the amount of debt that can be sustained. Increasing g increases debt by raising $\hat{\beta}$ and lowering R/g. A higher growth rate means that the government puts more weight on future periods because it will have more citizens in those periods, so the effective discount factor $\hat{\beta}$ increases. At the same time, higher g also means that future debt is spread over more workers and more revenue, decreasing the effective interest rate on bonds normalized by revenue R/g.

The increased patience of the government makes the punishment of defaulting and reneging more painful and the lower effective interest rate makes large debt more manageable. While the government's discount factor has increased, its patience relative to the market $\hat{\beta}(R/g) = \beta R$ has not changed. This means the increase in g has not changed the government's desire to accumulate debt. This combination of effects is why g tends to be the most important variable in determining equilibrium debt once $N > 1$.

The importance of g explains why demographic changes in the EU are a large concern for debt sustainability. Using the average retirement age and birth rate, g is 1 for Greece in 1980 and approximately 0.99 for both Greece and the EU over 2003-2013. This small change in the workforce growth rate cuts the equilibrium debt to GDP by roughly a third for all N values. This means that countries must either default or make massive reductions to their future debt in order move to the new equilibrium cycle.

8 Pension Reform

Along with supporting high debt, the model produces a reasonable response of spending when debt and promised pensions have become too high. Suppose the government's obligations mean that debt will be at or near the maximum sustainable level, i.e. $(b, p_0, .., p_{N-1})$ imply that $b^{(N)}$ will be at or near $\underline{b}$. In a typical model of sovereign debt, concern about debt sustainability would put downward pressure on current spending. However, in this model the government does not want to renege on current promised spending. Instead, the government honors the promised spending and choose to cut future pension promises.

This strategy of only cutting pensions N periods in the future creates a clear pattern in the government's spending response. Figure 6 shows the response for $N = 4$ when $p_0, .., p_3 = 1$. The dashed line separates periods where spending has already been promised from those where it has not. Even though the government already knows at $t = 0$ that its spending will push debt to the maximum sustainable level, it does not deviate from the already promised pension spending for periods $0 - 3$.

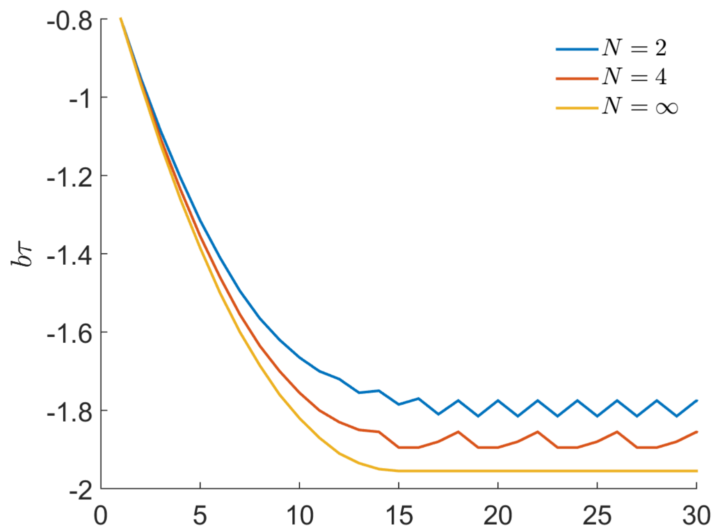

Figure 5: Evolution of Debt/GDP

Instead, the government promises lower pensions for period $t = 4$. In the next period, the government continues to honor the promised spending and chooses to promise low pensions for $t = 5$. If $\beta R < 1$ then the government will choose to back-load the cuts to spending. This means that spending cuts are not only pushed N periods into the future, but also become bigger over time. Thus, for an impatient government the response of spending is to have no change from the promised amounts for $N - 1$ periods, followed by increasingly large cuts.

The model government's response to high debt can be compared to recent pension reforms in the EU. In late 2011, when the EU debt crisis was near its peak, six countries passed pension reforms. The wide variation in pension systems makes comparison difficult, however, virtually all pension systems feature a full retirement age. Workers who retire before this age receive a reduced pension while those that retire later receive increased pensions. An increase in the full retirement age is equivalent to a reduction in the promised pension, as a worker who would of previously received full pension benefits now only receives early retirement benefits and a worker who would have received a late retirement bonus now only gets her normal pension. Since comparisons across time and countries in the full retirement age are easy to make, I use this as my measure of benefit cuts.

Figure 7 shows the change in full male retirement age for the six reforms[2]. The general pattern matches the prediction of the model quite well. For each country, the reform has no effect on the retirement age for several years, after which increasingly large changes are made. Based on the reforms, the appropriate value of N for each country is between 3 and 7.

[2]Change in retirement age for Italy and Spain are adjusted to account for exemptions. Italian workers with 36 years of contributions were exempt from changes for the first 5 years. Spanish workers with 35 years of contributions were exempt from changes for first 3 years.

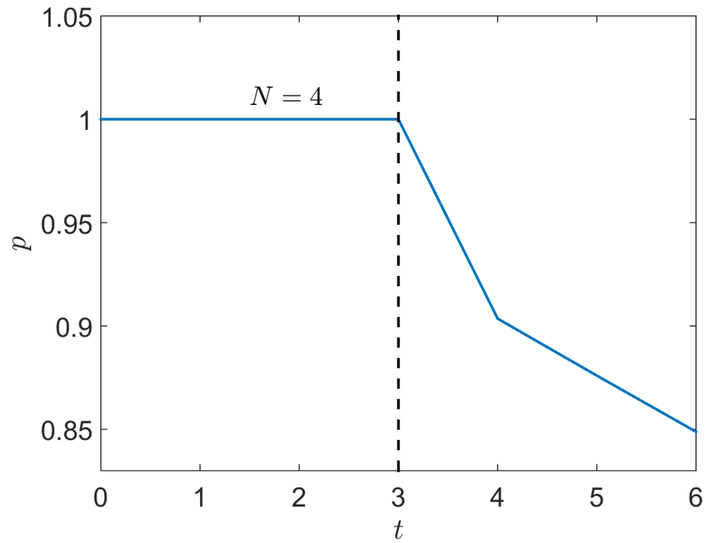

Figure 6: Back-loaded Spending Cuts

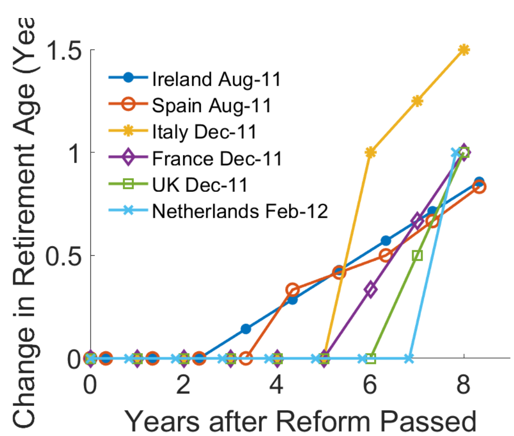

Figure 7: Male Full Retirement Age Changes

9 Conclusion

This paper develops a tractable model for studying the interaction of government debt and pensions. The equilibrium of this model can be found analytically and the full dynamics are given by the solution to a simple two variable recursive problem. The model provides several key theoretical insights into the role of pensions in sovereign default: (i) large promised pensions act as a commitment device for debt, (ii) default is driven by total obligations rather than just debt, (iii) even in a deterministic setting the government chooses to cycle between periods of high spending and debt accumulation and periods of low spending and fiscal consolidation. When promised spending exceeds revenue, the government must maintain access to credit markets in order to keep its promise to its citizens. This makes autarky a significant punishment and implies that the decision to default is driven by whether the government can pay its debt and still honor its promised pensions. If spending is chosen multiple periods in advance, then the government intentionally chooses a schedule of promised spending that is not entirely affordable in order to keep its outside option value of defaulting low. This means that the government voluntarily alternates between high and low spending so that is debt does not grow beyond the maximum sustainable level and its schedule of pension spending cannot be met in autarky.

I demonstrate that under a simple calibration of the model this commitment device is quantitatively powerful and produces spending responses to high debt that match the relevant features of recent EU pension reforms. If spending is promised multiple periods in advance, then the government can support well over 100% debt to GDP without defaulting. When the government needs to reduce debt, it chooses to honor its current schedule of promised spending and promise lower future spending. This creates a clear pattern of spending reform, where no change is made for several periods followed by increasingly large cuts, which is verified empirically by recent pension reforms. The model allows for flexible choice in the demographic variables (e.g. birth, retirement, and death rates do not need to produce a stationary distribution of workers and retirees) and shows that the growth rate of the workforce has a significant impact on the level of sustainable debt. This implies that the falling birth rates and increasing retirement rates in the EU and most developed nations will be a major hurdle for the sustainability of the current pension systems and debt.

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Appendix

This appendix contains proofs of the main propositions.

Recursive Problem Solves Equilibrium

An equilibrium is a solution to

$$\begin{aligned} V_0^* &= \sup_{\{b_t,p_t\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \hat{\beta}^t \log p_t \\ b_{t+1} &= (R/g)(1-p_t+b_t) \\ V_t^* &\geq \underline{V}(p_t,...,p_{t+N-1}) \ \ \forall t \geq 0 \\ b_t &\geq -M \ \forall \text{t} \geq 0 \\ (b_0,p_0,...,p_{N-1}) & \quad \text{given} \end{aligned}$$

where $-M$ is a lower bound on bonds to prevent Ponzi-schemes that is assumed not to bind in equilibrium. This has the following dual problem for bonds

$$\begin{aligned} b_0^* &= \inf_{\{V_t,p_t\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} (g/R)^t (p_t-1) \\ V_{t+1} &= (V_t - \log p_t)/\hat{\beta} \\ V_t &\geq \underline{V}(p_t,...,p_{t+N-1}) \ \ \forall t \geq 0 \\ (V_0,p_0,...,p_{N-1}) & \quad \text{given.} \end{aligned}$$

Given initial conditions $(V_0, p_0, .., p_{N-1})$ where $V_0 \geq \underline{V}(p_0,...,p_{N-1})$, the solution to the dual problem is given by the solution to the recursive problem

$$\begin{aligned} B(V,p_0,..,p_{N-1}) &= \min_{p_N} p_0 - 1 + (g/R)\, B(V',p_1,...,p_N) \\ \log p_0 + \hat{\beta} V' &\geq V \\ V' &\geq \underline{V}(p_1,...,p_N)\,. \end{aligned}$$

This comes from the fact that the choice set is always non-empty and V is bounded below. Since $(g/R) < 1$, the functional equation has a unique solution.

The final step is simply to show that solution to the recursive dual problem solves the functional equation for the original problem. Let $\hat{V}(b, p_0, .., .p_{N-1})$ be defined such that $B\left(\hat{V}(b,p_0,...,p_{N-1}), p_0,...,p_{N-1}\right) = b$, i.e. the inverse of the bond functional equation. Let

$$\begin{aligned} F(b,p_0,..,p_{N-1}) &= \max_{p_N} \log p_0 + \hat{\beta}\hat{V}(b',p_1,..,p_N) \\ b' &= (R/g)(1-p_0+b) \\ V(b',p_1,..,p_N) &\geq \underline{V}(p_1,...,p_N)\,. \end{aligned}$$

From the dual recursive problem, we know there exists a choice of p_N such that $\log p_0 + \hat{\beta}\hat{V}(b', p_1, .., p_N) = \hat{V}(b, p_0, ..., p_{N-1})$. Suppose that $F(b, p_0, .., p_{N-1}) > \hat{V}(b, p_0, .., p_{N-1})$. This means that for some $\varepsilon > 0$, $F(b - \varepsilon, p_0, .., p_{N-1}) \geq \hat{V}(b, p_0, .., p_{N-1})$. This contradicts $B\left(\hat{V}(b, p_0, .., p_{N-1}), p_0, ..., p_{N-1}\right)$ being the minimum amount of bonds necessary to generate value $\hat{V}(b, p_0, .., p_{N-1})$. Therefore, this cannot be possible and we must have $F(b, p_0, .., p_{N-1}) = \hat{V}(b, p_0, .., p_{N-1})$.

Rewriting Recursive Problem

If the government does not renege or default, it's pension spending and future bonds are given for the first N periods. So the government chooses future promises $\{p_t, b_{t+1}\}_{t=N}^{\infty}$ in order to maximize its payoff after the first N periods. We define a value function $W(b, k)$ that gives the payoff of the government's optimal plan after the first N periods.

The deviation payoff changes if promised pension spending is less than 1 for N sequential periods. Let k_t be the number of sequential periods that promised spending has been below 1 before period t, or $N-1$ if this number is greater than $N-1$. Let $i_t = \max\{i : p_i \geq 1, t - N \leq i \leq t-1\}$ be the number of periods since spending was at least 1.

$$k_t = \begin{cases} N & \text{if } p_{t-N}, .., p_{t-1} < 1 \\ N - i_t & \text{Otherwise.} \end{cases}$$

The value function is then

$$\begin{aligned} W(b, k) &= \max_{p, b'} \log p + \hat{\beta} W(b', k') \\ b' &= (R/g)(1 - p + b) \\ k' &= \begin{cases} \min(N, k+1) & \text{if } p < 1 \\ 0 & \text{otherwise} \end{cases} \\ W(b', k') &\geq \underline{W}(k') \end{aligned}$$

where k acts as a counter, keeping track of how many periods spending has been below 1. The deviation payoff is given by

$$\underline{W}(k) = \begin{cases} 0 & \text{if } k = N \\ V^{d,r} & \text{otherwise.} \end{cases}$$

This is similar to $\underline{V}$, with $V^{d,r}$ as the payoff from defaulting and reneging, and 0 as the payoff from just defaulting. The payoff from just defaulting is 0 because we are writing the payoffs looking N periods into the future. If the government defaults without reneging at time t, then we know its payoff in period $t + N$ is simply 0 because it chooses all future spending to be exactly 1.

Now, I will show how this value function can be derived from the original value function

$$
\begin{aligned}
V\left(b, p_0, .., p_{N-1}\right) &= \max_{p_N} \log p_0 + \hat{\beta} V\left(b', p_1, .., p_N\right) \\
b' &= (R/g)\left(1 - p_0 + b\right) \\
V\left(b', p_1, .., p_N\right) &\geq \underline{V}\left(p_1, ..., p_N\right).
\end{aligned}
$$

We guess $V\left(b, p_0, ..., p_{N-1}\right) = \sum_{i=0}^{N-1} \hat{\beta}^i \log p_i + \hat{\beta}^N \left(W\left(b^{(N)}, k\right) - 1\left\{W\left(b^{(N)}, k\right) < \alpha\right\}\infty\right)$, where $b^{(N)} = \left(\frac{R}{g}\right)^N b + \sum_{i=0}^{N-1}\left(\frac{R}{g}\right)^{N-i}\left(1 - p_i\right)$, $k = \max\left(N - \max\left\{i : p_{i-1} \geq 1\right\}, N-1\right)$ and $\alpha = \max_{0 \leq i \leq N-1}\left\{\hat{\beta}^{-i} V^{d,r} - \hat{\beta}^{-N} \sum_{j=N-i}^{N-1} \hat{\beta}^j \log p_j\right\}$. Plugging this into the recursive formulation gives

$$
\begin{aligned}
V\left(b, p_0, .., p_{N-1}\right) &= \log p_0 + \sum_{i=1}^{N-1} \hat{\beta}^i \log p_i + \\
&\quad \hat{\beta}^N \left[\max_{p_N} \log p_N + \hat{\beta}\left(W\left(b^{(N+1)}, k'\right) - 1\left\{W\left(b^{(N+1)}, k'\right) < \alpha'\right\}\infty\right)\right] \\
b^{(N+1)} &= \left(\frac{R}{g}\right)^{N+1} b + \sum_{i=0}^{N}\left(\frac{R}{g}\right)^{N-i}\left(1 - p_i\right) \\
k' &= \begin{cases} N & \text{if } p_1, .., p_N < 1 \\ N - \max\left\{i : p_i \geq 1\right\} & \text{Otherwise.} \end{cases} \\
\alpha' &= \max_{0 \leq i \leq N-1}\left\{\hat{\beta}^{-i} V^{d,r} - \hat{\beta}^{-N} \sum_{j=N-i}^{N-1} \hat{\beta}^j \log p_{j+1}\right\}
\end{aligned}
$$

which is equal to $\sum_{i=0}^{N-1} \hat{\beta}^i \log p_i + \hat{\beta}^N \left(W\left(b^{(N)}, k\right) - 1\left\{W\left(b^{(N)}, k\right) < \alpha\right\}\infty\right)$ using the definitions of $b^{(N)}, k, \alpha$ given above. So it satisfies the recursive equation. This means that the policy function $p_N\left(b, p_0, ..., p_{N-1}\right)$ for V is the same as the policy function $p\left(b^{(N)}, k\right)$ for W. The bounds on bonds can be written as

$$
\begin{aligned}
\underline{b} &= \min\left\{b : W\left(b, 0\right) \geq V^{d,r}\right\} \\
b^* &= \min\left\{b : W\left(b, N\right) \geq 0\right\}.
\end{aligned}
$$

Sometimes it is useful i the proofs to split the problem of $W\left(b, k\right)$ into a choice of the best $p < 1$ and the best $p \geq 1$. Let $W^L\left(b, k\right)$ represent the recursive problem when the government chooses to spend low and let $p^L\left(b, k\right)$ represent the government's optimal choice of spending when it chooses to spend low.

$$
\begin{aligned}
W^L\left(b, k\right) &= \max_{p^L, b'} \log p^L + \hat{\beta} W\left(b', k\right) \\
p^L &\leq 1 \\
b' &= (R/g)\left(1 - p^L + b\right) \\
k' &= \min\left(N, k+1\right) \\
W\left(b', k'\right) &\geq \underline{W}\left(b', k'\right).
\end{aligned}
$$

Let $W^H(b)$ and $p^H(b)$ represent the recursive problem when the government chooses to spend high. We know $k' = 0$ regardless of the current value of k, so k does not matter.

$$\begin{aligned} W^H(b) &= \max_{p^H, b'} \log p^H + \hat{\beta} W(b', 0) \\ p^H &\geq 1 \\ b' &= (R/g)\left(1 - p^H + b\right) \\ W(b', 0) &\geq V^{d,r}. \end{aligned}$$

We then have that $W(b,k) = \max\left\{W^L(b,k), W^H(b)\right\}$.

Proof of Lemma 1

This lemma is equivalent to $\log \bar{\ell} + \theta \log \bar{\ell} > \bar{\ell} - 1$. Since $\bar{\ell} = 1 + \theta$, this becomes $\bar{\ell} \log \bar{\ell} > \bar{\ell} - 1$ which is always true because $\bar{\ell} > 1$.

Proof of Proposition 1

The following claim is helpful in the proof of proposition 1.

Claim 1. For $W(b, N-1)$, if $b \leq \frac{g}{R} b^*$ and $p(b, N-1) < 1$ then $b'(b, N-1) = b^*$.

Proof. This is shown by contradiction. Suppose $\exists b \leq \frac{g}{R} b^*$ such that $p(b, N-1) < 1$ and $b'(b, N-1) \neq b^*$. Then $k' = N$ and $W(b', N) \geq 0$. This means that $b' > b^*$ and $W(b', N) > 0$. Since no participation constraints bind, $p' = \beta R p < 1$. Further, until a participation constraint binds, all spending will be less than 1 by the same argument. If a participation constraint never binds, then $W(b', N) < 0$ because spending is always below 1. If a participation constraint does bind, the continuation value will be at most 0. This means that $W(b', N) < 0$ since spending is below 1 until the participation constraint binds and the continuation value once the constraint binds is non-positive. In both cases, $W(b', N) < 0$. This contradicts $W(b', N) > 0$. □

The proof of proposition 1 is shown by contradiction. First, the optimal p_N for $V(b, p_0, ..., p_{N-1})$ is the optimal p for $W\left(b^{(N)}, k\right)$ where k is the counter for how many periods spending has been below 1. This means that the optimal $b^{(N)}$ for $V(b, p_0, ..., p_{N-1})$ is the optimal b' for $W\left(b^{(N)}, k\right)$.

Let $b'(b,k)$ and $p(b,k)$ be the policy functions for $W(b,k)$. Suppose there exists (b,k) with $b \leq 0$ such that optimal $b'(b,k) > 0$. This means that $p(b,k) < 1$ because $b' > b$. We know that for any positive bonds b', $W(b', k') > 0$ for all k'. This because the government can choose to spend $1 + b'$ for one period and then spend 1 forever after that, giving $W(b', k') \geq \log(1 + b') > 0$.

Since $W(b', k') > 0$, no participation constraints bind which means $p' = \beta R p < 1$ from the Euler equation. This implies $b'' > 0$. Since $b'' > 0$, the argument can be repeated to conclude that $p'' < 1$ and $b^{(3)} > 0$. Repeat the argument to see that $p^{(i)} < 1$ for all $i \geq 1$. This means that $W(b', k') < 0$ because all future spending is below 1. This contradicts $W(b', k') > 0$.

As a corollary, I also show that if $\beta R < 1$, then $b'(b,k) < 0$ for $b \le 0$. This corollary is used in the proof of theorem 2. To prove this, we just need to show that $b'(b,k) \ne 0$ for $b \le 0$ when $\beta R < 1$. In the paragraph below, I show that $\beta R < 1$ implies $W(0,k) > 0$ for all k. If $b'(b,k) = 0$, then $p(b,k) \le 1$ and no incentive constraints bind. From the Euler equation, $p' = \beta R p < 1$ and $b'' > 0$. Since $b'' > 0$, the argument can be repeated to conclude that $p'' < 1$ and $b^{(3)} > 0$. Repeat the argument to see that $p^{(i)} < 1$ for all $i \ge 1$. This means that $W(b', k') < 0$ because all future spending is below 1. This contradicts $W(b', k') = W(0, k') > 0$.

In this paragraph, I show that $W(0,k) > 0$ for all k when $\beta R < 1$. We know that $W(0,k) \ge 0$, since the government can always choose to spend 1 in each period. Consider the following spending path. The government spends p in the first period, $1 + (1-p)\frac{R}{g}$ in the second period, and then 1 for all remaining periods. The spending in the second period ensures that bonds are 0 after the first two periods. At $p = 1$, the payoff of this plan is 0 and the derivative w.r.t. p is $p^{-1} - \beta R \left(1 + (1-p)\frac{R}{g}\right)^{-1} = 1 - \beta R > 0$. Since the derivative is positive, the government can do strictly better by choosing $p > 1$ and will generate a strictly positive payoff. Since this plan is feasible, $W(0,k)$ must provide at least as high a payoff, so $W(0,k) > 0$.

Proof of Theorem 1

The optimal spending promise p' for $V(b,p)$ is the optimal p for $W\left(\frac{R}{g}(b + 1 - p), k\right)$ where $k = 1\{p < 1\}$. The proof is split into two steps.

The first step is to prove that the equilibrium path for the state $\left(b^{(i)}, k^{(i)}\right)_{i=0}^{\infty}$ must hit $(b^*, 1)$ at some point (i.e. the participation constraint for defaulting without reneging must eventually bind). Obviously, if this participation constraint ever binds, then the state must be $(b^*, 1)$ at some point. If the participation constraint for defaulting and reneging ever binds, then in the next period the current bond value is $\underline{b}$ and the counter k is 0. We know $W(\underline{b}, 0) = V^{d,r}$ and the continuation value must be at least $V^{d,r}$, so spending must be below 1. By claim 6, this means $b' = b^*$ since current bonds are $\underline{b} \le b^* \le \frac{g}{R} b^*$ and spending is below 1. This means that the next period state is $(b^*, 1)$.

Suppose a participation constraint never binds. In this case, the Euler equation states that $p^{(i)} = (\beta R)^i p$ if $(\beta R)^i p > 1$ and $p^{(i)} = (\beta R)^i p$ or 1 if $(\beta R)^i p < 1$, where p is the initial choice of spending. Since initial bonds are negative, spending must be below 1 at some point, otherwise bonds will grow to $-\infty$. Once spending is below 1 in one period, we know future spending will always be at most 1. This means that the continuation value is at most 0. Since spending is below 1, the continuation value must be at least 0, otherwise it violates the default without reneging participation constraint. Since spending is below 1 and the continuation value is 0, the next state must be $(b^*, 1)$.

The second step is to show how this creates a cycle. Suppose the state has reached $(b^*, 1)$. We know bonds are always non-positive, so $b^* \le 0$. We therefore consider two cases. First, suppose $b^* < 0$. In this case, we can repeat the above argument to show that the state must eventually return to $(b^*, 1)$. Further, we know that the next state cannot be $(b^*, 1)$. If the next state was $(b^*, 1)$ then spending would have to be below 1, because bonds are negative. This would means that $W(b^*, 1) = \log p + \hat{\beta} W(b^*, 1) < 0$ which violates the definition of b^*.

So, in this case, we have a multi-period cycle that starts at $(b^*, 1)$ and eventually returns. This means that equilibrium bonds and spending will also have a multi-period cycle.

For the second case, suppose $b^* = 0$ which occurs if $\beta R = 1$. Then bonds must either stay at 0 forever or eventually drop below 0. If bonds fall below 0 then the above argument can be repeated to show that the state must eventually return to $(b^*, 1)$, creating a cycle. If bonds stay forever at 0 then p must always be 1, creating a steady-state (b, p) of $(0, 1)$ which is a one period cycle.

Proof of Proposition 2

This proposition is equivalent to $p(b, N-1) \geq 1$ for all b for the simplified recursive problem $W(b, k)$.

The first step is to prove the proposition for $b > \frac{g}{R}b^*$. Suppose there exists $b > \frac{g}{R}b^*$ such that $p(b, N-1) < 1$. Then $b'(b, N-1) > b^*$, which means the continuation value is strictly positive. Since current spending is below 1, we know that future spending $p^{(i)}$ will be either 1 or $(\beta R)^i p$ until a participation constraint binds. Once a participation constraint binds, the continuation value will be at most 0. This means that $W(b'(b, N-1), N) \leq 0$, which contradicts $b'(b, N-1) > b^*$. Therefore, for all $b > \frac{g}{R}b^*$ it must be that $p(b, N-1) \geq 1$.

The second step is the prove the proposition for $b \leq \frac{g}{R}b^*$. Consider the following function

$$
\begin{aligned}
\hat{W}(b) &= \max_{m.p} \sum_{i=0}^{m-1} \hat{\beta}^i \log p_i + \hat{\beta}^m V^{d,r} \\
p_i &= \begin{cases} 1 & \text{if } \mod(i, N) = N-1 \\ (\beta R)^i p & \text{otherwise} \end{cases} \\
V^{d,r} &\leq \sum_{i=j}^{m-1} \hat{\beta}^{i-j} \log p_i + \hat{\beta}^{m-j} V^{d,r} \; \forall 0 \leq j \leq m-1 \\
\underline{b} - \left(\frac{R}{g}\right)^m b &= \sum_{i=0}^{m-1} \left(\frac{R}{g}\right)^{m-i} (1 - p_i).
\end{aligned}
$$

Since $b \leq \frac{g}{R}b^*$, we know from claim 1 that $W^L(b, N-1) = \log\left(1 + b - \frac{g}{R}b^*\right)$. We also know that $W^H(b, N-1) \geq \hat{\beta} W\left(\frac{R}{g}b, 0\right) \geq \hat{\beta}\hat{W}\left(\frac{R}{g}b\right)$. It can be shown with some algebra that $\hat{\beta}\hat{W}\left(\frac{R}{g}b\right) \geq \log\left(1 + b - \frac{g}{R}b^*\right)$. Therefore, $W^L(b, N-1) \leq W^H(b, N-1)$.

Proof of Theorem 2

The proof of Theorem 2 is split into several claims. These claims are all for $N > 1$.

Claim 2. Once a participation constraint binds, spending is at most 1 forever.

Proof. Let (b, k) be the current state when a participation constraint binds. From proposition 2, we know that if $k = N-1$ then $p(b, k) \geq 1$. Because the participation constraint binds, we know $p(b, k) = 1$, $b'(b, k) = \underline{b}$ and $k = 0$. Therefore, if a participation constraint binds at $k = N-1$, then the next period state will be $(\underline{b}, 0)$, where the participation constraint also binds. So WLOG assume $k < N-1$.

Because the participation constraint binds and $k < N-1$, $W(b,k) = V^{d,r}$. Since the continuation value is always at least $V^{d,r}$, $W(b,k) = V^{d,r}$ implies $p(b,k) < 1$. Until another participation constraint binds, we know that $p^{(i)} = (\beta R)^i p$ or 1 from the Euler equation. So spending is weakly less than 1 until another participation constraint binds. Once another participation constraint binds, we can repeat the above argument to conclude that current spending is at most 1 and will not exceed 1 until another participation constraint binds. This means that spending will always be weakly less than 1. □

Claim 3. For all k, if $p^L(b,k) < 1$ then $p^L(b,k)$ is increasing in b.

Proof. Given k, let $F\left(b, p^L\right) = \log p^L + \hat{\beta} W\left(\frac{R}{g}\left(b + 1 - p^L\right), k'\right)$ where $k' = \max(N, k+1)$. $F_1\left(b, p^L\right) = \beta R W'\left(\frac{R}{g}\left(b + 1 - p^L\right), k'\right)$ which is increasing in p^L. By Topkis' Theorem, p^L is increasing in b. □

Claim 4. If a participation constraint binds, then spending is below 1 until $k = N-1$. Formally, if $W(b,k) = V^{d,r}$ then $p^{(i)} < 1$ for all $i < N-1-k$.

Proof. Suppose $W(b,k) = V^{d,r}$ and $p^{(i)} \geq 1$ for some $i < N-1-k$. We know $\sum_{i=0}^{\infty} \hat{\beta}^i \log p^{(i)} = V^{d,r}$. Consider the following alternative spending path $\tilde{p}^{(0)} = 1 + \left(1 - \frac{g}{R}\right) \underline{b}$ and $\tilde{p}^{(i+1)} = p^{(i)}$. First, I show that $\log\left(1 + \left(1 - \frac{g}{R}\right)\underline{b}\right) > \left(1 - \hat{\beta}\right) V^{d,r}$. Consider the unconstrained version of the government's problem, where defaulting without reneging is never allowed.

$$
\begin{aligned}
\tilde{W}(b) &= \max_{p,b'} \log p + \hat{\beta} \tilde{W}(b') \\
b' &= (R/g)(1 - p + b) \\
\tilde{W}(b') &\geq V^{d,r}.
\end{aligned}
$$

The lowest possible b for this unconstrained problem is $\tilde{\underline{b}}$ where $\frac{1}{1-\hat{\beta}} \log\left(1 + \left(1 - \frac{g}{R}\right)\tilde{\underline{b}}\right) = V^{d,r}$, i.e. the government spends the same amount each period has utility exactly equal to the outside option. From proposition 2, we know that spending for $W(b,k)$ can never be held at a fixed level forever, so $\tilde{W}(b) > W(b,k)$ for all k and $\tilde{\underline{b}} < \underline{b}$. This means $\log\left(1 + \left(1 - \frac{g}{R}\right)\underline{b}\right) > \left(1 - \hat{\beta}\right) V^{d,r}$.

The value of this alternative spending path is $\log\left(1 + \left(1 - \frac{g}{R}\right)\underline{b}\right) + \hat{\beta} \sum_{i=0}^{\infty} \hat{\beta}^i \log p^{(i)} = \log\left(1 + \left(1 - \frac{g}{R}\right)\underline{b}\right) + \hat{\beta} V^{d,r} > V^{d,r}$. This alternative spending path satisfies the participation constraints and creates a higher utility payoff for the government. Therefore, $p^{(i)}$ cannot be the optimal spending path. □

Claim 5. For $k < N-1$, let $\underline{p}(k) = p(b,k)$ where b is such that $W(b,k) = V^{d,r}$. Then $\underline{p}(k)$ is nonincreasing in k.

Proof. The proof is done by showing that $\underline{p}(k) \geq \underline{p}(k+1)$ for all $k < N-2$. Let b_k be such that $W(b_k, k) = V^{d,r}$. There are two cases. First, suppose $W(b'(b_k, k), k+1) = V^{d,r}$. Then $\underline{p}(k) = \exp\left(\left(1-\hat{\beta}\right) V^{d,r}\right)$. Since $W(b_{k+1}, k+1) = V^{d,r}$ and the continuation value must be at least $V^{d,r}$, $\underline{p}(k+1) = p(b_{k+1}, k+1) \leq \exp\left(\left(1-\hat{\beta}\right) V^{d,r}\right) = \underline{p}(k)$.

Second, suppose $W(b'(b_k, k), k+1) > V^{d,r}$. This means that $b'(b_k, k) > b_{k+1}$. Then $\underline{p}(k) = \frac{1}{\beta R} p(b'(b_k, k), k+1) \geq p(b'(b_k, k), k+1)$. We know $p(b'(b_k, k), k+1) \geq p^L(b'(b_k, k), k+1) \geq p^L(b_{k+1}, k+1) = \underline{p}(k+1)$ from claim 3. Therefore, $\underline{p}(k) \geq \underline{p}(k+1)$. □

Claim 6. If $W(b, k) = V^{d,r}$ then $b^{(N-k)} = \underline{b}$ and $k^{(N-k)} = 0$. This means that once a participation constraint binds, another participation constraint binds in $N-k$ periods.

Proof. Suppose $W(b, k) = V^{d,r}$. From claim 4, we know that spending will be below 1 for $N-1-k$ periods. From proposition 2, we know that spending will then be at least 1 in $N-k$ periods. This means that $k^{(N-k)} = 0$, i.e. k will be 0 in $N-k$ periods.

We now prove that $b^{(N-k)} = \underline{b}$ by contradiction. Suppose $b^{(N-k)} > \underline{b}$. Let $j = \max\{0 \leq i \leq N-1-k :$ which gives the last period where the participation constraint was binding. Since $b^{(N-k)} > \underline{b}$, no participation constraints bind in the choice of $b^{(N-k)}$ which means $(\beta R)^{N-k-j} W_1\left(b^{(N-k)}, 0\right) = W_1\left(b^{(j)}, k+j\right) = 1/p\left(b^{(j)}, k+j\right)$. Since $p^L(b, 0)$ is increasing in b, $W_1\left(b^{(N-k)}, 0\right) < 1/p^L(\underline{b}, 0)$. So $p\left(b^{(j)}, k+j\right) = p^L\left(b^{(j)}, k+j\right) > p^L(\underline{b}, 0)$. This means that $\underline{p}(k+j) > \underline{p}(0)$ which contradicts claim 5. □

These claims can be combined to conclude that once a participation constraint binds $W(b, k) = V^{d,r}$, the following is true: (i) $p^{(i)} < 1$ if $k^{(i)} < N-1$, (ii) $p^{(i)} = 1$ if $k^{(i)} = N-1$, and (iii) $b^{(i)} = \underline{b}$ if $k^{(i)} = 0$. The first piece comes from third and fourth claims. The second piece comes from the first claim and proposition 2. The third piece comes from the third claim. This means that once a constraint binds, the economy enters a cycle that starts at $(\underline{b}, 0)$ and returns to $(\underline{b}, 0)$ in N periods. The spending cycle $(p^i)_{i=0}^{N-1}$ must solve

$$
\begin{aligned}
p^i &= \min\left\{(\beta R)^i p^0, \exp\left(\left(1-\hat{\beta}\right) V^{d,r}\right)\right\} \ \forall i < N-1 \\
p^{N-1} &= 1 \\
\left(1 - \left(\frac{R}{g}\right)^N\right) \underline{b} &= \sum_{i=0}^{N-1} \left(\frac{R}{g}\right)^{N-i} (1 - p_i).
\end{aligned}
$$

where $W(\underline{b}, 0) = \frac{1}{1-\hat{\beta}^N} \sum_{i=0}^{N-1} \hat{\beta}^i \log p^i = V^{d,r}$. Given the value of $\underline{b}$, there is only one possible value of p^0 that satisfies the above equations, so this cycle is unique. The bond cycle $(b^i)_{i=0}^{N-1}$ is given by $b^0 = \underline{b}$ and $b^{i+1} = \frac{R}{g}(b^i + 1 - p^i)$.

For $\beta R < 1$, a participation constraint will eventually bind. This is shown by contradiction. Suppose a participation constraint never binds. Let p be the first chosen level of spending after the N periods of initially promised spending. The Euler equation states that $p^{(i)} = (\beta R)^i p$ if $(\beta R)^i p > 1$ and $p^{(i)} = (\beta R)^i p$ or 1 if $(\beta R)^i p < 1$. Since initial bonds are negative, spending must be below 1 at some point, otherwise bonds will grow to $-\infty$. Once spending is below 1 in one period, we know future spending will always be at most 1. Eventually, $(\beta R)^i p$ will become so low that it is never possible to choose $p^{(i)} = (\beta R)^i p$ without violating the participation constraints. This means that after some point, all spending must be 1. From the corollary for proposition 1, we know that bonds are always negative when $\beta R < 1$ (this is shown at the end of the proof of proposition 1). This means that it is not possible for all spending to be 1, since bonds will grow to $-\infty$.

For $\beta R = 1$, the optimal choice is to simply maintain debt at its current level because the government wants to have flat spending across time. In this case, the cycle will solve

$$\begin{aligned} p^i &= p^0 \ \forall i < N-1 \\ p^{N-1} &= 1 \\ \left(1 - \left(\frac{R}{g}\right)^N\right) b &= \sum_{i=0}^{N-1} \left(\frac{R}{g}\right)^{N-i} (1 - p_i) \end{aligned}$$

where b is the initial bonds.