Saving and Dissaving with Hyperbolic Discounting*

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Is the standard hyperbolic-discounting model capable of robust qualitative predictions for savings behavior? Despite results suggesting a negative answer, we provide a positive one. We give conditions under which all Markov equilibria display either saving at all wealth levels or dissaving at all wealth levels. Moreover, saving versus dissaving is determined by a simple condition comparing the interest rate to a threshold made up of impatience parameters only. Our robustness results illustrate a well-behaved side of the model and imply that qualitative behavior is determinate, dissipating indeterminacy concerns to the contrary (Krusell and Smith, 2003). We prove by construction that equilibria always exist and that multiplicity is present in some cases, highlighting that our robust predictions are not due to uniqueness. Similar results may be obtainable in related dynamic games, such as political economy models of public spending.

1 Introduction

This paper revisits consumption-saving decisions within the basic (quasi) hyperbolic-discounting model, as in Phelps and Pollak (1968) and Laibson (1997).[1] The horizon is infinite, time is discrete, and there is no uncertainty. Consumers save and borrow at a constant interest rate, and may face a limit to borrowing. Preferences over future consumption paths are not time consistent (Strotz, 1956). In particular, the entire future is heavily discounted against the present, while impatience is relatively modest between

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[1]As is well known, collective decision making models in the political economy literature, featuring alternating governments without commitment, can formally resemble the hyperbolic consumption-saving model that we study here, see e.g. Alesina and Tabellini (1990), Amador (2002), Azzimonti (2011). Another very related class of models are the intergenerational growth models, see e.g. Bernheim and Ray (1987).

neighboring future periods, inducing a present bias. Behavior is approached as a dynamic game across different "selves". Formally, a sequence of decision makers, one for each date, play against each other, deciding how much to consume and save given their current wealth, the state variable. We focus on (symmetric) Markov equilibria of this dynamic savings game, a common refinement in the literature. Our goal is to provide robust predictions for savings behavior across all Markov equilibria.

Early contributions by Phelps and Pollak (1968) and Laibson (1996) focused on linear equilibria, with constant saving rates, providing definite predictions for savings as a function of the interest rate and impatience parameters. However, there are two well-understood shortcomings with this approach. First, in the presence of binding borrowing limits, as in most recent contributions, linear equilibria do not exist. Second, even when linear equilibria exist, other nonlinear Markov equilibria may exist.[2] Unfortunately, studying Markov equilibria away from the linear case has been shown to be quite demanding, in part due to their non-smooth nature (Morris and Postlewaite, 1997; Harris and Laibson, 2002; Chatterjee and Eyigungor, 2016). Perhaps due to these challenges, little is known about the set of Markov equilibria and virtually no predictions have been offered for saving behavior (the stated object of study) that hold across all equilibria.

While clearly desirable, should we expect robust predictions of this sort to be possible? In an influential contribution, Krusell and Smith (2003) suggest a negative answer: they present a construction to argue that the standard hyperbolic-discounting model is ill behaved in the sense of displaying a continuum of Markov equilibria, each with a stable interior steady state, i.e. with savings above the steady state and dissavings below the steady state; importantly, the steady state can be placed anywhere. Most disturbingly, given our focus, this sort of indeterminacy implies that the model is incapable of robust qualitative predictions for savings behavior: for given parameters, at any given wealth level, the agent could be saving (for one set of equilibria) or dissaving (for another set of equilibria). The subsequent literature has echoed concern over these pathological-looking findings and invoked them in motivating new models, such as the continuous-time "instant gratification" model with return uncertainty introduced by Harris and Laibson (2013), the temptation with self-control model due to Gul and Pesendorfer (2004) or the dual-self model proposed by Fudenberg and Levine (2006).

Despite these apparent problems, we show that the standard hyperbolic-discounting

[2] Phelps and Pollak (1968, page 196) expressed a concern of this kind in their original paper: "In this analysis, we have confined ourselves to fixed-points described by a constant saving ratio over time. We are unsure whether or not there may exist saving sequences with non-constant ratios." We show in Section 5.2 that for some parameters there are other Markov equilibria, in addition to the linear ones Phelps-Pollak studied.

model is, in fact, capable of delivering predictions for savings behavior that hold across all Markov equilibria. These robustness results have been missed by the existing literature and help uncover a hitherto unappreciated well-behaved side of the model. We explain further below how to interpret Krusell and Smith (2003) in light of our findings but, in brief, our results imply that their constructions are not Markov equilibria in our standard hyperbolic-discounting model.

Our main result extends a well-known result from the time-consistent consumption-saving problem. Under exponential discounting a simple and intuitive condition determines whether the agent saves or dissaves: dissaving occurs if $R < \frac{1}{\delta}$, savings if $R > \frac{1}{\delta}$ and wealth is held constant if $R = \frac{1}{\delta}$ (where R is the gross interest rate and δ the discount factor). This fundamental result is engrained across a wide range of studies of intertemporal choice.

No comparable result exists for the standard hyperbolic-discounting model and our main contribution is to fill this gap. As long as utility is not too close to linear, we show that all Markov equilibria feature either saving for all wealth levels or dissaving for all wealth levels. Indeed, we show that, just as with exponential discounting, there exists a threshold interest rate R^* that is a function of impatience parameters only such that across *all* Markov equilibria agents save if $R > R^*$, dissave if $R < R^*$ and hold wealth constant if $R = R^*$. By implication, all Markov equilibria have the same qualitative predictions obtained in the earlier linear equilibrium analysis.

Robust predictions are possible by virtue of two fundamental principles uncovered by our analysis. For concreteness consider the case with $R > R^*$. Then our first principle states that as wealth rises the equilibrium policy function cannot reverse from dissaving to saving. Intuitively, agents have a natural inclination to save given the high interest rate. Additionally, any agent immediately below a region with saving has an extra motive to save, to enter the virtuous saving region, ensuring their successors save. This rules out reversals from dissaving to saving.

This No Reversal Principle still leaves open the possibility of dissaving in the upper tail of wealth levels. Our second principle rules this out. Otherwise, rich agents face the prospect of a very long spell of dissaving and we show that this deteriorates their utility to the point that they prefer to deviate towards holding wealth constant for one period. This particular deviation is critical to our analysis, as it has the advantage of being easy to compute with minimal information on an equilibrium path, without detailed knowledge of the savings policy function. We develop this insight into a recursive restriction on the sequence of utility values along an equilibrium outcome path that provides a simple, yet powerful, necessary condition. We exploit this tool to establish the aforementioned deterioration of utility along a dissaving spell to rule out dissaving in the upper tail.

The formal methodology which we develop in establishing these two principles may be useful in other related contexts. Of particular note is that our approach is non-local, in the sense that it does not invoke first-order conditions or rely on any smoothness properties of the equilibrium.

To put our robustness results in context, we also investigate the questions of existence and multiplicity of equilibria. First, we provide a general constructive proof of existence, building an equilibrium with dissaving when $R < R^*$ and saving when $R > R^*$, ensuring that our results are not vacuous in the sense of applying to a nonempty set of equilibria. Second, we show that multiple equilibria exist in some cases, highlighting that our robustness results are not due to uniqueness. We establish multiplicity by constructing equilibria explicitly.[3] Indeed, even for cases where a linear equilibrium exists we build alternative nonlinear equilibria, settling a question raised by Phelps and Pollak (1968). An important concern with models displaying multiple equilibria is that they may not offer meaningful predictions for behavior. Our robustness results obviate this concern by providing predictions across all Markov equilibria. Despite potential indeterminacy of equilibria, qualitative saving behavior is determinate.

Our robustness results shed light on the construction in Krusell and Smith (2003) and its proper interpretation. Their theorem constructs a continuum of saving functions with the property that wealth converges to a steady-state wealth level; in the consumption-saving context, this steady state can be placed anywhere.[4] Crucially, their constructions are only local in nature, in two ways: (i) the policy function is described only over an interval around the proposed steady state; and (ii) saving choices are shown to be optimal when confined to this very interval. By implication, they validly prove that these constructions represent equilibria for a modified game that restricts the agent to make choices within the endogenous interval that their result constructs around the proposed steady state.

Although the local nature of their construction is stated formally in their theorem, the result appear to have acquired a stronger interpretation elsewhere in their paper, as well as in the subsequent literature.[5] Indeed, the restriction to a local interval may appear in-

[3]We establish that the equilibrium is unique with constant wealth when the interest rate equals R^*. When R approaches R^* all our constructions converge to this constant wealth equilibrium.

[4]The standard consumption-saving model we focus on has a fixed interest rate. For generality, Krusell and Smith (2003) allow the saving technology to be weakly concave, treating the linear case as a special important case. The linear case is not only the standard in the hyperbolic-discounting literature but also popular in the political economy literature, e.g. Alesina and Tabellini (1990), Amador (2002), and Azzimonti (2011). We focus on the linear case, but discuss concave saving technologies in Section 5.3.

[5]Krusell and Smith (2003) set up their model and define equilibria without ad hoc upper bounds on

nocent on the surface, but, as we show, turns out to be critical. Ad hoc upper bounds on wealth accumulation lack any obvious economic motivation and have no place in the standard consumption-saving problem. Thus, the natural question which we pose is whether these local constructions can be extended globally, to all wealth levels, so that they represent equilibria for the original game. As a corollary to our results, we provide conditions for a negative answer.

Our main results provide conditions for robust predictions that, as we explained, were widely considered impossible. The sufficient conditions are relatively weak: for intermediate values of the interest rate only, we require a minimum amount of curvature in the utility function, between linear and logarithmic; this contains the empirical range of interest as well as the theoretical favorite benchmark case with log utility. It is natural to ask whether our sufficient conditions can be discarded to show that robust predictions are *always* possible. We provide a negative answer: for intermediate interest rates and near linear utility, we construct equilibria that reverse the natural sign of saving; indeed, equilibria may also be indeterminate. These results show that robust predictions cannot be taken for granted, highlighting the advantage of guaranteeing them under plausible conditions.

Finally, we discuss a few simple departures from the standard consumption-saving model. Our results adapt to a modified game where wealth choices are constrained to a discrete grid and an ad hoc upper bound—as is common in numerical work. This provides formal results to guide numerical work and highlights the non-local nature of our method. Second, we discuss how our results are affected when linear returns on wealth are replaced by concave saving technologies.

An important precursor to the discrete-time analysis we undertake here is the continuous-time one presented in Cao and Werning (2016). That paper studies a hyperbolic discounting model set in continuous time that builds on Harris and Laibson (2013). Continuous time allows for a constructive differential approach and delivers particularly sharp results. Indeed, some of those results were instrumental in providing appropriate conjectures for the discrete-time model. Despite these synergies, the analysis we undertook here in discrete time is very different and led us to develop quite different tools. Some differences in results remain, such as the greater extent of multiplicity that we find in the discrete time model.

wealth and the Introduction states that "Our main result is indeterminacy of Markov equilibrium savings rules: there is a continuum of such rules. These rules differ both in their stationary points and in their implied dynamics. [...] We construct these equilibria explicitly". Echoes of this interpretation appear throughout the subsequent literature.

2 A Savings Game

Preferences in period t are given by

$$u(c_t) + \beta\delta\left(u(c_{t+1}) + \delta u(c_{t+2}) + \delta^2 u(c_{t+3}) + \cdots\right),$$

for $u : [0,\infty) \to \{-\infty\} \cup \mathbb{R}$ increasing, concave and continuously differentiable over $c \in (0,\infty)$, with $\delta < 1$ and $\beta \le 1$. When $\beta = 1$ discounting is exponential and preferences are time consistent. When $\beta < 1$ preferences display a bias towards the present and are time inconsistent.[6]

Some of our analysis and results adopt an isoelastic utility function

$$u(c) = \frac{c^{1-\sigma}}{1-\sigma},$$

with $\sigma \neq 1$ and $\sigma > 0$; $u(c) = \log(c)$ when $\sigma = 1$. Isoelastic utility is a relatively standard assumption favored by the literature, which we adopt for convenience and comparability. The assumption is helpful to characterize equilibria on an unbounded state space. However, many of our results apply without isoelastic utility and all of our results hold with the weaker assumption that utility is isoelastic above some consumption level, since we only use the assumption to invoke homogeneity in the upper tail for wealth.

The budget constraint is

$$c_t + k_{t+1} = Rk_t,$$

for some gross interest rate $R > 1$.[7] The agent is also subject to a wealth limit

$$k_t \ge \underline{k},$$

for some $\underline{k} \ge 0$. Although we have not included labor income, this is without loss of generality since the present value of labor income can be lumped into wealth. Borrowing constraints against future income then translate into a strictly positive wealth limit, $\underline{k} > 0$, making this an especially relevant case.[8] The case $\underline{k} = 0$ represents the so-called 'natural

[6] We focus on the present-bias case, $\beta < 1$, which is also the main focus in the behavioral economics and political economy literatures. However, our methods and results likely extend straightforwardly to the future-bias case, $\beta > 1$, which may be relevant in some applications. For example, while some political economy settings map into a hyperbolic discounting with $\beta < 1$, others may imply $\beta > 1$.

[7] To simplify, in the text we assume $R > 1$, but discuss how our results carry over with $R \le 1$. See footnote 17.

[8] Let labor income be $y > 0$. Then the constraints are

$$c_t + k_{t+1} = y + Rk_t$$

and $k_{t+1} \ge \underline{k} \ge -\frac{1}{R-1}y$; the latter inequality is required to ensure that $c_t \ge 0$ is feasible. The change in borrowing limit' offering maximal liquidity.

Our results compare the gross interest rate R to a threshold

$$R^* \equiv 1 + \frac{1-\delta}{\beta\delta}.$$

Note that when $\beta = 1$ then $R^* = \frac{1}{\delta}$, and when $\beta < 1$ then $R^* \in (\frac{1}{\delta}, \frac{1}{\beta\delta})$.

Finally, following Phelps and Pollak (1968), we assume that the agent maximization problem for $\beta = 1$ is well defined. When utility is iso-elastic this is equivalent to the growth condition

$$\delta R^{1-\sigma} < 1.$$

Note that when $R > \frac{1}{\delta}$ this requires $\sigma > \underline{\underline{\sigma}}$ for some lower bound $\underline{\underline{\sigma}} \in (0,1)$ that depends on δ and R. Some of our results impose a separate lower bound $\underline{\sigma} \in (0,1)$. All our results apply for $\sigma \geq 1$, the empirically relevant range.

Markov Equilibria. A (symmetric) Markov equilibrium is a pair of functions $g : [\underline{k}, \infty) \to [\underline{k}, \infty)$ and $V : [\underline{k}, \infty) \to \mathbb{R}$ satisfying

$$V(k) = u(Rk - g(k)) + \delta V(g(k)) \qquad \forall k \geq \underline{k}, \tag{1}$$

$$u(Rk - g(k)) + \beta\delta V(g(k)) \geq u(Rk - k') + \beta\delta V(k') \qquad \forall k, k' \geq \underline{k}, \tag{2}$$

and the limiting condition $\delta^t V(g^t(k)) \to 0$ where $g^0 = k$ and $g^{t+1} = g(g^t)$.[9] Condition (2) ensures that the savings function g maximizes the utility of the current self with respect to savings k', taking as given how future consumption is affected by the chosen k', as summarized in the continuation value $V(k')$. Condition (1), together with the limit condition, ensures that $V(k)$ equals the discounted utility from consumption implied by the savings function, $V(k) = \sum_{t=0}^{\infty} \delta^t u(Rg^t(k) - g^{t+1}(k))$.

2.1 Prior State of Knowledge

Now that we have laid out the standard hyperbolic-discounting model we briefly review what was previously known about Markov equilibrium savings functions g. The upshot is that no comprehensive and robust predictions for behavior have been put forth.

variable $\hat{k}_t \equiv k_t + \frac{1}{R-1} y \geq 0$ gives $c_t + \hat{k}_{t+1} = R\hat{k}_t$ and $\hat{k}_t \geq \underline{\hat{k}}$. Note that this transformation requires $R > 1$, we treat the case with $R \leq 1$ separately.

[9] When $\underline{k} = 0$ and $u(0) = -\infty$ we modify the equilibrium definition slightly, replacing $V : [\underline{k}, \infty) \to \mathbb{R}$ with $V : [\underline{k}, \infty) \to \{-\infty\} \cup \mathbb{R}$, so that $V(0) = -\infty$, but requiring $V(k) > -\infty$ for $k > 0$. This ensures that all equilibria have $g(k) > 0$ for $k > 0$. This weak regularity requirement excludes pathologies such as $g(k) = 0$ and $V(k) = -\infty$ for all $k \geq 0$.

Linear Equilibria. As first shown by Phelps and Pollak (1968), linear equilibria exist in some cases. These have been extensively employed in the literature. In more detail, with isoelastic utility, whenever $R > R^*$ there exists a unique linear equilibrium with positive savings: $g(k) = \alpha k$ and $\alpha > 1$. However, it is not known whether there are other nonlinear equilibria (we provide an answer in Section 5.2), and if they do exist, whether all of them involve positive savings (our answer: Section 4). When $R < R^*$ and $\underline{k} = 0$ there exists a unique linear Markov equilibrium with dissavings: $g(k) = \alpha k$ and $\alpha < 1$. However, it has not been established whether other equilibria exist (our answer: Section 5.2), or whether they all involve dissaving (our answer: Section 4). Finally, the linear saving rule is no longer an equilibrium when $\underline{k} > 0$ and $R < R^*$.

Existence. Establishing the existence of Markov equilibria is important and non-trivial.[10] Progress has been made on this front. Given the linear equilibria mentioned above, existence is trivial when utility is isoelastic and either $R \geq R^*$ or $\underline{k} = 0$. Bernheim et al. (2015, Proposition 6) provide a proof of existence for $R \in (\frac{1}{\delta}, R^*)$ and $\underline{k} > 0$.[11] In Section 5.1 we provide a new result, showing existence for all $R \leq R^*$ and non-isoelastic utility functions. This ensures that our robustness results do not apply to an empty set of equilibria.

Indeterminacy? Krusell and Smith (2003) argue that for moderately high interest rates $R \in (R^*, \frac{1}{\beta\delta})$ Markov equilibria are indeterminate, with equilibria featuring dissavings at high wealth levels, instead of savings as in the linear equilibrium. In more detail, their result is as follows: pick any wealth $k^* \geq \underline{k}$, then they construct (g, V) (defined in the neighborhood of k^*) satisfying the two conditions presented above for a Markov equilibrium (with the maximization over k' in (2) also limited to this neighborhood). The savings function so constructed satisfies $g(k^*) = k^*$, $g(k) < k$ for $k > k^*$, and $g(k) > k$ for $k < k^*$. Thus, starting from any $k_0 \geq \underline{k}$ (in the neighborhood of k^*) the sequence $\{k_n\}$ defined by $k_{n+1} = g(k_n) = g^{n+1}(k_0)$ converges monotonically to k^*.

However, as noted in parenthesis above, these constructions are only local around a neighborhood of the proposed steady state k^*. There are, then, two possibilities. In the first, the constructions may characterize actual equilibria if there exists an extension of the savings function over the entire range of wealth $k \geq \underline{k}$. This appears to be the interpretation adopted by Krusell and Smith (2003) and the related literature. The second

[10] Note that the equilibrium of finite horizon versions of the game may not converge as the horizon is extended, in which case it does not provide existence nor selection of equilibria.

[11] Existence results are also available for extensions of the standard model. For example, Harris and Laibson (2001) prove existence and obtain smoothness properties by adding i.i.d. uncertainty in income.

possibility, however, is that the local constructs do not survive any attempt at a global extension and, thus, do not characterize a Markov equilibrium in the standard hyperbolic-discounting model. Our results provide conditions for this second possibility.

Robust Predictions? One prediction of the standard model that has been amply studied in the earlier literature concerns the emergence of discontinuities (see e.g. Morris and Postlewaite, 1997; Harris and Laibson, 2002). Most recently, Chatterjee and Eyigungor (2016) show that when $R = \frac{1}{\delta}$ Markov equilibria must feature discontinuities in the policy function. Fortunately, our method is uniquely suited to handle discontinuities and no special treatment is required. Much research has gone into showing that discontinuities may not survive extensions of the basic model that incorporate sufficient uncertainty.[12]

Although not their main focus, Chatterjee and Eyigungor (2016) also show that when $R = \frac{1}{\delta}$ *all* Markov equilibria must feature weak dissaving. Despite being special, this result is worth highlighting in our view because it provides a rare prediction for qualitative saving behavior across equilibria. The goal of our paper is to provide more extensive robust predictions of this kind. As we show, the economic logic for dissaving when $R \leq \frac{1}{\delta}$ is similar to that for $R = \frac{1}{\delta}$. These are relatively straightforward cases because, intuitively, at such low interest rates even a time-consistent consumer, $\beta = 1$, prefers to dissave; present bias, $\beta < 1$, only reinforces this conclusion. We will be mostly occupied with more challenging situations, involving intermediate interest rates such as $R \in (R^*, \frac{1}{\beta\delta})$ and $R \in (\frac{1}{\delta}, R^*)$. Indeed, Krusell and Smith (2003) suggest that robust qualitative prediction are not possible when $R \in (R^*, \frac{1}{\beta\delta})$: according to their constructions, at any wealth level the agent could save or dissave, depending on the equilibrium.

Finite Horizon. Following most of the literature, we focus on Markov equilibria of an infinite-horizon dynamic saving game. The resulting stationarity allows studying behavior without conditioning on the remaining time horizon. Finite horizon versions have also been studied and offer some advantages and disadvantages. On the one hand, with a finite horizon the equilibrium is essentially unique (up to a set for wealth of measure zero). By taking limits, this has been invoked as a selection device, especially in the absence of borrowing constraints where linear equilibria exist (Laibson, 1996; Krusell et al., 2010). On the other hand, when borrowing constraints are present the equilibrium does not necessarily converge as the horizon is extended, so it cannot always provide a selec-

[12] Harris and Laibson (2001, 2002) provide results and simulations to show that smooth equilibria arise for $R < \frac{1}{\delta}$ when uninsured income shocks are introduced. Chatterjee and Eyigungor (2016) also introduce actuarily fair lotteries over future wealth to convexify the value function V and produce a smooth and monotone responses for consumption and (expected) savings.

tion device for the infinite-horizon game. More importantly, the equilibrium is unique for any finite horizon, but little is known about it: clear predictions have not been formulated for savings, nor for consumption. Indeed, predictions for behavior appear more elusive than in the infinite-horizon setting.[13] Of course, in cases where the finite-horizon equilibrium *does* converge as the horizon is extended, our results provide relevant predictions for the equilibrium of the finite horizon game when the horizon is long enough.

3 Useful Preliminary Results

This section presents some preliminary results that turn out to be instrumental in proving our main results. These results are quite simple, yet novel and provide some independent insight. However, readers only interested in the main results can skip to the next section without loss of continuity.

Before turning to novel results we echo a well-known monotonicity property that follows from the supermodularity in (k, k') of the objective for any value function V.

Lemma 1. *For any Markov equilibrium (g, V), the savings function g is nondecreasing.*

By implication, a path $\{k_n\}_{n=0}^{\infty}$ satisfying $k_{n+1} = g(k_n)$ features either savings $k_{n+1} \geq k_n$ for all $n = 0, 1, \ldots$ or dissaving $k_{n+1} \leq k_n$ for all $n = 0, 1, \ldots$

3.1 Saving, Dissaving and Constant Wealth

The next three lemmas compare saving or dissaving, on the one hand, to holding wealth constant forever, on the other hand.

Welfare Comparisons. Define the value of holding wealth constant forever as

$$\bar{V}(k) \equiv \frac{u\left((R-1)\,k\right)}{1-\delta}.$$

The next two lemmas relate current saving behavior to the ranking of $V(k)$ versus $\bar{V}(k)$.

Lemma 2. *For any Markov equilibrium (g, V), if either:*

(a) $R \geq \frac{1}{\delta}$ and $g(k) < k$;

[13] With a finite horizon T, when $R > R^*$ it is natural to expect $g_T(k) < k$ for some values of k, and $g_T(k) > k$ for other values of k. Intuitively, a finite horizon pushes for dissaving, while the high interest rate $R > R^*$ pushes for saving. Except in linear equilibria, these two forces do not play out evenly across k. Indeed, this may happen even with $\beta = 1$. As for $R < R^*$, numerical explorations with $\beta < 1$ suggest equilibria exist with $g_T(k) > k$ for some values of k.

(b) $R \le \frac{1}{\delta}$ and $g(k) > k$;
then $V(k) < \bar{V}(k)$.

Strict dissaving is undesirable from the perspective of the exponential discounter when $R \ge \frac{1}{\delta}$, while strict saving is undesirable $R \le \frac{1}{\delta}$. To see why, consider case (a). First note that the monotonicity of g ensures that any initial dissaving $g(k_0) < k_0$ will perpetuate itself, $k_{n+1} = g(k_n) \le k_n$ for all $n = 0, 1, \ldots$ As is well known, when $R \ge \frac{1}{\delta}$ the optimum for an exponential discounter features saving, not dissaving; intuitively, holding wealth constant is not optimal, but still preferable to dissaving, $V(k) < \bar{V}(k)$. Case (b) is similar. Notably, Lemma 2 does not involve β in any way, since it follows directly from the definition of V and the monotonicity of g. The logic is purely mechanical and does not invoke optimality of the current self choice.[14]

Our next result does invoke optimality for the current self, showing that to entice positive savings the equilibrium must offer a greater value $V(k)$ than $\bar{V}(k)$.

Lemma 3. *For any Markov equilibrium (g, V), if $g(k) \ge k$ then $V(k) \ge \bar{V}(k)$; moreover, if $g(k) > k$ then $V(k) > \bar{V}(k)$.*

Unlike Lemma 2, this result does not compare R to $\frac{1}{\delta}$ and now both optimizing behavior and $\beta < 1$ are critical. The logic goes as follows. If the current self prefers to save rather than hold wealth constant for a period, then the previous self (who now discounts exponentially according to V) agrees. After all, the current self takes on lower consumption to get higher continuation utility, but given $\beta < 1$ is harsher at evaluating this tradeoff. This then implies $V(k) > \bar{V}(k)$.

Best Responding to Constant Wealth. Next, consider the best response of the current self to a hypothetical (non-equilibrium) situation where future selves hold wealth constant forever. Given k, the current self considers the objective over k' given by

$$\varphi(k, k') \equiv u(Rk - k') + \beta\delta\bar{V}(k'). \tag{3}$$

The function $\varphi(k, \cdot)$ is strictly concave, with a unique interior maximum $k' = k^*$ that varies continuously with k (by the Theorem of the Maximum). If $R > R^*$ then $\frac{\partial}{\partial k'}\varphi(k, k) > 0$, so $k^* > k$; the reverse is true for $R < R^*$.

Lemma 4. *Let φ be given by (3). For any fixed k, the function $\varphi(k, \cdot)$ is strictly concave and has an interior maximum $k^* > 0$. Moreover,*

(a) if $R > R^*$, then $k^* > k$ and $\varphi(k, \cdot)$ is strictly increasing for $k' \le k^*$;

[14] Indeed, it does not invoke optimizing behavior of future selves, except through the monotonicity of g.

(b) if $R < R^*$, then $k^* < k$ and $\varphi(k, \cdot)$ is strictly decreasing for $k' \geq k^*$;

(c) the maximum k^* is a continuous function of k.

By implication, when $R > R^*$ any amount of strict dissavings $k' < k$ is dominated by holding wealth constant $k' = k$, while some small amount of positive saving $k' > k$ dominates $k' = k$; the reverse is true when $R < R^*$. Based on this lemma, we say that when $R > R^*$ ($R < R^*$) there is a "natural inclination" to save (dissave).

3.2 A Recursive Necessary Conditions

Next, we introduce a tool that plays a crucial part in our method. For any Markov equilibrium (g, V) and initial wealth $k_0 \geq \underline{k}$ consider the path for wealth $\{k_n\}_{n=0}^{\infty}$ generated by $k_{n+1} = g(k_n)$ with associated value path $\{V_n\}$ given by $V_n = V(k_n) = \sum_{m=0}^{\infty} \delta^m u(Rk_{n+m} - k_{n+m+1})$. The next lemma provides a simple necessary condition for such paths.

Lemma 5. *If $\{k_n, V_n\}_{n=0}^{\infty}$ is generated by a Markov equilibrium (g, V) then for $n = 0, 1, \ldots$*

$$V_n = u(Rk_n - k_{n+1}) + \delta V_{n+1},$$

$$(1 - \beta\delta)u(Rk_n - k_{n+1}) + \beta\delta(1 - \delta)V_{n+1} \geq u((R - 1)k_n).$$

When utility is iso-elastic these conditions become

$$v_n = u(R - x_n) + \delta v_{n+1} x_n^{1-\sigma}, \tag{4}$$

$$(1 - \beta\delta)u(R - x_n) + \beta\delta(1 - \delta)v_{n+1} x_n^{1-\sigma} \geq u(R - 1), \tag{5}$$

for $n = 0, 1, \ldots$, where $\{x_n, v_n\}$ is the normalized path given by $x_n = \frac{k_{n+1}}{k_n}$ and $v_n = \frac{V_n}{k_n^{1-\sigma}}$.

The first condition is an accounting identity for values. The second condition rules out a particular deviation whereby the agent holds wealth constant for one period. When current wealth is k_n the equilibrium has the agent choosing k_{n+1} with continuation value V_{n+1}. The agent may consider deviating to $\tilde{k}_{n+1} = k_n$, in which case the ensuing path simply postpones the original one, $\tilde{k}_{n+1+s} = k_{n+s}$ for $s = 0, 1, \ldots$ giving continuation value $\tilde{V}_{n+1} = V_n$. This particular deviation has two distinct advantages. First, the deviation is easy to compute precisely because it resets the original path and does not fall outside of its range. Iso-elastic utility further simplifies by exploiting homogeneity.[15] Second, since the deviation holds wealth constant for a period, neither saving nor dissaving, it turns out to be perfectly suited to determine saving versus dissaving.

[15] When $\sigma = 1$, i.e. log utility, $V_n = v_n + \frac{1}{1-\delta} \log k_n$ and expressions (4) and (5) are adjusted accordingly, for example, $v_n = \log(R - x_n) + \delta(v_{n+1} + \frac{1}{1-\delta} \log x_n)$.

An important virtue of the necessary conditions derived above is that they can be evaluated directly on equilibrium paths, without further knowledge of g and V. Although this limits informational demands, it still falls short, since our goal is to characterize equilibria by ruling out entire patterns of behavior, instead of particular paths. The solution is to minimize the information required further by conditioning only on qualitative behavior. To this end, define correspondences for saving and a correspondence for dissaving

$$\Gamma_s(v_{n+1}) \equiv \{v_n | \exists x_n \in [1, R] \text{ satisfying (4) and (5)}\},$$
$$\Gamma_d(v_{n+1}) \equiv \{v_n | \exists x_n \in [0, 1] \text{ satisfying (4) and (5)}\}.$$

Lemma 6. *If utility is isoelastic, $\{k_n\}$ is a path generated by a Markov equilibrium (g, V) and $\{v_n\}$ is the normalized value path, then*

(a) if $g(k_0) \geq k_0$ then $v_n \in \Gamma_s(v_{n+1})$ for all $n = 0, 1, \ldots$

(b) if $g(k_0) \leq k_0$ then $v_n \in \Gamma_d(v_{n+1})$ for all $n = 0, 1, \ldots$

These necessary conditions turn out to be extremely convenient because of their simple recursive nature, with the single state variable $\{v_n\}$.

4 Robust Predictions for Saving Behavior

This section contains our main results. We establish conditions for Markov equilibria to involve either global savings or dissavings, that is, saving at all wealth levels or dissaving at all wealth levels. Moreover, for given parameters, all Markov equilibria involve global saving or global dissaving. Finally, whether saving or dissaving arises is determined by a simple condition comparing the interest rate to impatience parameters, R versus R^*. This condition can be seen as generalizing the well-known comparison of R versus $\frac{1}{\delta}$ for the standard time-consistent model. Taken together, our results imply that the standard hyperbolic-discounting model is capable of meaningful and robust qualitative predictions for economic behavior.

The conditions that we require for our robustness results are plausible, imposing a minimum curvature on utility only when the interest rate is in an intermediate range. However, we show that these conditions cannot be fully dispensed with, by constructing equilibria with near linear utility that reverse the sign of saving. These constructions are novel and establish sufficient conditions for indeterminacy of equilibria in the standard hyperbolic-discounting model.

Before establishing these results we present a crucial intermediate characterization of independent interest.

4.1 No Reversal Principle

Combining Lemmas 1–4 gives our first main result. It states that when $R > R^*$ behavior cannot reverse from dissaving to saving from low to high wealth; in other words, poverty traps are ruled out. Conversely, when $R < R^*$ behavior cannot reverse from saving to dissaving as wealth rise; in other words, interior stable steady states are impossible. We call this the No Reversal Principle.

Theorem 1 (No Reversal Principle)**.** *Consider a Markov Equilibrium (g, V). For any $\tilde{k} \geq \underline{k}$ and any $k_0 \geq \tilde{k}$ define the sequence $\{k_n\}$ by $k_{n+1} = g(k_n)$ and let $k_\infty = \lim_{n\to\infty} k_n$. Then*

(a) if $R > R^*$ and $g(\tilde{k}) < \tilde{k}$ then $g(k) < k$ for all $k \in [\tilde{k}, \infty)$ and $g(k_\infty) = k_\infty < \tilde{k}$;

(b) if $R < R^*$ and $g(\tilde{k}) > \tilde{k}$ then $g(k) > k$ for all $k \in [\tilde{k}, \infty)$ and $k_\infty = \infty$.

Although this result is of intrinsic interest, its main purpose is to significantly reduce the set of Markov equilibria, paving the path towards our results on global savings and dissaving. Indeed, thanks to the No Reversal Principle all that remains to be done is to characterize equilibrium behavior in the upper tail of wealth.

Intuitively, when $R > R^*$ a reversal from dissaving to saving unravels because an agent with wealth slightly below the region with saving should dissave, but actually prefers to deviate and save to reach the virtuous saving region. Similarly, when $R < R^*$ a reversal from saving to dissaving must be separated by a steady state, but an agent at such a steady state prefers to deviate and strictly dissave, given that $R < R^*$. In both cases, the proposed equilibrium unravels because a region with savings is attractive and we cannot expect agents to behave against their natural inclination (saving if $R > R^*$, dissaving if $R < R^*$) if that inclination leads them to a desirable saving region.

The results for the limit $k_\infty = \lim_{n\to\infty} k_n$ may appear redundant, but they are not. For instance, in part (a) we first conclude that $g(k) < k$ for all $k \geq \tilde{k}$, but this does not rule out $k_\infty > \tilde{k}$ or imply that k_∞ is a steady state ($g(k_\infty) = k_\infty$) since g may be discontinuous at k_∞.

4.2 Robust Global Saving: $R > R^*$

We first consider the more challenging case where the interest rate is relatively high, $R > R^*$. We provide conditions that rule out dissavings at any wealth level, for any Markov equilibrium. Indeed, we go a step further and show that saving must be strict.

Theorem 2 (Global Savings)**.** *Consider $R > R^*$, then all Markov equilibria feature strict saving, $g(k) > k$ for all $k \in [\underline{k}, \infty)$, if either* *(a)* $R \geq \frac{1}{\beta\delta}$*; or*

(b) $R \in (R^*, \frac{1}{\beta\delta})$*, utility is isoelastic with* $\sigma \geq \underline{\sigma}$ *for some* $\underline{\sigma} \in [0, 1)$*.*

Recall that when $R > R^*$ and utility is isoelastic a linear equilibrium with positive savings always exists. Although we later show that other equilibria may exist (see Section 5.2), Theorem 2 ensures that the linear equilibrium with positive saving is qualitatively representative of all Markov equilibria. In other words, it guarantees that saving is a robust prediction. Part (a) does not require any assumption on the utility function; the isoelastic assumption in part (b) can be relaxed, only requiring isoelastic utility above some large consumption level.

The economic logic behind Theorem 2 is as follows. Case (a) is relatively intuitive, since when $R \geq \frac{1}{\beta\delta}$ the incentive to avoid dissaving is overwhelming. Indeed, even an agent that discounts exponentially with the product discount factor $\beta\delta$ prefers to save. Intuitively, hyperbolic-discounting agents are more inclined to save than such heavy $\beta\delta$ exponential discounters.

This argument fails and things become more subtle when $R \in (R^*, \frac{1}{\beta\delta})$, so how is it that dissaving can be ruled out in this case? After all, if future selves dissave then the current self may find dissaving optimal and this may discourage savings despite a relatively high interest rate. This strategic complementarity is precisely behind the local construction in Krusell and Smith (2003). Their result can be interpreted as showing that sustaining dissavings in this manner is possible locally, for a modified game that restricts wealth choices to a sufficiently small interval. In contrast, Theorem 2 shows that, as long as utility is not too close to being linear, this is impossible for the standard hyperbolic-discounting model, which places no ad hoc upper bounds on wealth. In this sense, these local constructions cannot be extended to form part of an equilibrium in the standard model. Below we provide intuition as to why dissavings cannot be sustained in the upper tail for wealth.

By the No Reversal Principle, if dissaving happens at any wealth level, then it happens at all higher wealth levels. This implies that very rich agents face a very prolonged spell of declining wealth. Theorem 2 shows that when $\sigma \geq \underline{\sigma}$ the prospect of such a long spell of dissaving is so dire that some sufficiently rich agent must prefer to deviate and hold wealth unchanged for a period, postponing the undesirable dissaving path by one period. This rules out dissaving. We postpone a discussion of the role for the condition $\sigma \geq \underline{\sigma}$ to Section 4.5, where we explore what happens for $\sigma < \underline{\sigma}$.

Importantly, our corollary, that the local construction in Krusell and Smith (2003) cannot be extended over $[\underline{k}, \infty)$, is stronger than dismissing the particular extension which sets $g(k) = k^*$ for all $k \geq k^*$, where k^* is the proposed steady state (the local construction had $g(k) = k^*$ for $k \in [k^*, \bar{\bar{k}}]$ for some $\bar{\bar{k}} < \infty$). This extension can be easily dismissed, since the best-response to it has $k' \in (k^*, k)$ for sufficiently large k.[16] Although it is easy to rule out this particular extension, one might hope that there are others that survive. Theorem 2 lays such hopes to rest, ruling out any equilibrium with weak dissaving in the standard hyperbolic-discounting model.

We next provide a full proof of weak saving for part (a) of Theorem 2 and a sketch of the main argument of weak saving for part (b). Once weak saving is established we show that this implies strict saving. The full proof is contained in Appendix C.

Proof of Theorem 2 part (a). We have $R \geq \frac{1}{\beta\delta}$. Consider any Markov equilibrium. Condition (2) evaluated at $k' = k$ gives

$$u(Rk - g(k)) + \beta\delta V(g(k)) \geq u(Rk - k) + \beta\delta V(k).$$

Combining this with condition (1) and rearranging gives

$$(1 - \beta\delta)u(Rk - g(k)) + \beta\delta(1 - \delta)V(g(k)) \geq u(Rk - k).$$

Now, towards a contradiction, assume $g(k) < k$ at some k. By monotonicity (Lemma 1) this gives $g(g(k)) \leq g(k)$ and Lemma 2 part (a) then implies $V(g(k)) \leq \bar{V}(g(k))$, so

$$(1 - \beta\delta)u(Rk - g(k)) + \beta\delta(1 - \delta)\bar{V}(g(k)) \geq u(Rk - k). \tag{6}$$

This inequality will be shown to lead to a contradiction. To see this, use Lemma 4 part (a) with discount parameters $\hat{\beta} = 1$ and $\hat{\delta} = \beta\delta$, so that $\hat{R}^* = \frac{1}{\beta\delta}$ to obtain that for all $k' < k$

$$u(Rk - k') + \frac{\beta\delta}{1 - \beta\delta}u((R - 1)k') < \frac{1}{1 - \beta\delta}u((R - 1)k)$$

Setting $k' = g(k)$ and rearranging then contradicts (6). Thus, $g(k) \geq k$ for all $k \geq \underline{k}$.

Sketch of Proof for Theorem 2 part (b). Towards a contradiction, consider a Markov equilibrium with $g(\hat{k}) < \hat{k}$ at some $\hat{k} > \underline{k}$. By the No Reversal Principle (Theorem 1) then $g(k) < k$ for all $k \geq \hat{k}$. We derive a contradiction by showing that some sufficiently rich agent must wish to deviate.

Lemma 7 in Appendix E shows that there exists $\underline{\sigma} \in (0, 1)$ such that for each $\sigma \geq \underline{\sigma}$: (i)

[16] To see this consider maximizing $\hat{\varphi}(k, k') \equiv u(Rk - k') + \beta\delta(u(Rk' - k^*) + \frac{1}{1-\delta}u((R - 1)k^*)$. Then the optimum k' restricted to $k' \geq k^*$ is strictly greater than k^* for high k. This follows because $\frac{\partial}{\partial k'}\hat{\varphi}(k, k^*) = -u'(Rk - k^*) + \beta\delta R u((R - 1)k^*)$ so that as $k \to \infty$ we ensure $\frac{\partial}{\partial k'}\hat{\varphi}(k, k^*) > 0$ as long as $\lim_{c \to \infty} u'(c) = 0$.

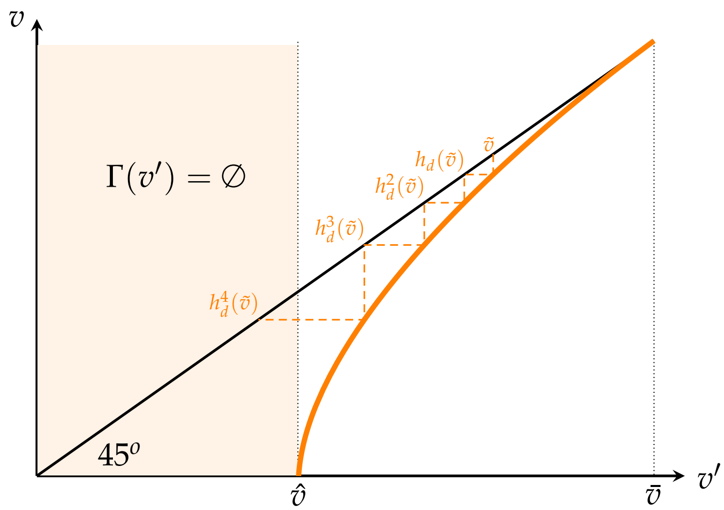

Figure 1: Equilibria with dissaving requires $v_n \leq h_d(v_{n+1})$ leading to $v_n < \hat{v}$, a contradiction (parameters: $\sigma = 0.5$, $\beta = 0.9$, $\delta = 0.97$ and $R = 1.04$).

for all $v' < \bar{v}$, $v \in \Gamma_d(v')$ implies $v < v'$; (ii) there exists $\hat{v} > \frac{u(0)}{1-\delta}$ such that $\Gamma_d(v) = \varnothing$ if and only if $v < \underline{v}$; and (iii) there exists $\underline{x} > 0$ such that for all $v' \in [\hat{v}, \bar{v}]$ if (5) holds then $x > \underline{x}$. For $v' \in [\hat{v}, \bar{v}]$ let

$$h_d(v') \equiv \sup \Gamma_d(v'). \tag{7}$$

Then h_d is continuous, strictly increasing with $h_d(v') < v'$ for all $v' < \bar{v}$ and $h_d(\bar{v}) = \bar{v}$. Figure 1 illustrates h_d. A key implication is that for any $\tilde{v} < \bar{v}$ we have $h_d^n(\tilde{v})$ decreasing in n and eventually strictly below $\hat{v}$.

To fix ideas and see how this property is useful, we first suppose there is some $\tilde{k} \geq \hat{k}$ with the property that we can always find a path that goes through $\tilde{k}$ after N periods, for any N: there exists $k_0 \geq \hat{k}$ such that $\{k_n\}$ given by $k_n = g^n(k_0)$ has $k_N = \tilde{k}$. An equilibrium requires $\hat{v} \leq v_{n-1} \leq h_d(v_n)$ for all $n = 1, 2, \ldots$ Lemma 2 implies $v_N = \tilde{v} < \bar{v}$. Then, since h_d is monotone and $h_d(v') < v'$ for all $v' \in [\hat{v}, \bar{v})$, it follows that $\hat{v} \leq v_1 \leq h_d^{N-1}(v_N) = h_d^{N-1}(\tilde{v})$. However, this is not possible since $h_d^{N-1}(\tilde{v}) < \hat{v}$ for some high enough N.

A long enough sequence going through any $\tilde{k}$ is guaranteed when g is continuous, so that g is invertible. However, one cannot rule out discontinuities. Fortunately, the same conclusions hold if we can establish that there exists some $\tilde{v} < \bar{v}$ such that for any N we can find a decreasing path $\{k_n\}$ starting at some $k_0 > \hat{k}$ with the property that $v_N \leq \tilde{v}$. We establish this weaker property for any equilibrium (g, V), allowing discontinuities, as follows.

First, Theorem 1 part (a) ensures that for any $k_0 \geq \hat{k}$ the sequence $\{k_n\}$ generated by $k_{n+1} = g(k_n)$ eventually goes strictly below $\hat{k}$. Second, because $g(k) < k$ for all $k \geq \hat{k}$, there must exist a crossing point $\tilde{k} \geq \hat{k}$ with the property that if $k \geq \tilde{k}$ and $g(k) \leq \tilde{k}$ then $\frac{g(k)}{k} \leq \bar{x}$ for some $\bar{x} < 1$. This implies that if a path crosses $\tilde{k}$ at N, so that $k_{N+1} \leq \tilde{k} < k_N$, then $v_N \leq \tilde{v} < \bar{v}$. This follows because $v_N = u(R - x_N) + \delta v_{N+1} x_N^{1-\sigma} \leq u(R - x_N) + \delta \bar{v} x_N^{1-\sigma} \leq u(R - \bar{x}) + \delta \bar{v} \bar{x}^{1-\sigma} \equiv \tilde{v}$; the first inequality due to Lemma 2, the second by the monotonicity of $u(R - x) + \delta \bar{v} x^{1-\sigma}$ over $x < 1$ when $R > 1/\delta$ (Lemma 4 with $\beta = 1$). Finally, since $\frac{k_{n+1}}{k_n} \geq \underline{x} > 0$, as mentioned earlier, we can find a large enough $k_0 > \tilde{k}$ such that the sequence crosses $\tilde{k}$ in exactly N periods, for any N. This gives the desired properties and concludes the sketch of the proof.

4.3 Robust Global Dissaving: $R < R^*$

We now turn to the reverse case and assume the interest rate is low, $R < R^*$. We provide conditions that ensure dissaving at all wealth levels, across all Markov equilibria.[17] Thanks to the No Reversal Principle (Theorem 1), all that remains is to rule out saving in the upper tail, for high wealth.

Theorem 3 (Global Dissavings)**.** *Consider $R \in (0, R^*)$, then all Markov equilibria feature weak dissaving, $g(k) \leq k$ for all $k \in [\underline{k}, \infty)$ if either*

(a) $R \leq \frac{1}{\delta}$;

(b) $R \in (\frac{1}{\delta}, R^*)$, utility is isoelastic with $\sigma \geq \underline{\sigma}$ for some $\underline{\sigma} \in (0, 1)$.

Below we provide a sketch of the proof for part (b). Part (a) follows immediately given our lemmas, by noting that $g(k) > k$ generates a contradiction between Lemma 2 ($V(k) < \bar{V}(k)$) and Lemma 3 ($V(k) > \bar{V}(k)$). Intuitively, this case is relatively straightforward, since even a time-consistent decision maker prefers to dissave when $R \leq \frac{1}{\delta}$; then, as one would expect, $\beta < 1$ only reinforces this conclusion.

However, a different and subtler line of reasoning is required to prove case (b), when $R \in (\frac{1}{\delta}, R^*)$. In this intermediate region of interest rates, dissaving emerges from the time–inconsistency problem. Recall that we must rule out saving in the upper tail of wealth. Intuitively, because the interest rate is low, saving can only be sustained by the expectation of reaching a rewarding region with very high utility. We show that such expectations are unfounded, much like a Ponzi scheme.[18]

[17] We have assumed that $R > 1$ to simplify the exposition. When $R \leq 1$ we cannot apply the transformation that sets $y = 0$ and $\underline{k} \geq 0$ without loss of generality, as in footnote 8. However, Theorem 3 part (a) easily follows with $R \leq 1$, because Lemmas 2 and 3 carry over. Indeed, Theorem 1 and Lemmas 1–4 all carry over.

[18] We can relax the isoelastic assumption in part (b), only requiring isoelastic utility above some large consumption level.

Note that Theorem 3 establishes weak dissaving, but stops short of claiming strict dissaving. There is good reason for this, as it is known that there may exist equilibria with interior steady states.[19]

Sketch of Proof for Theorem 3 part (b). We proceed by contradiction: assume $g(\hat{k}) > \hat{k}$ for some $\hat{k} \geq \underline{k}$. By the No Reversal Principle (Theorem 1 Part b) then $g(k) > k$ for all $k \geq \hat{k}$. The sequence $\{k_n\}$ generated by $k_0 = \hat{k}$ and $k_{n+1} = g^{n+1}(\hat{k}) = g(k_n)$ is strictly increasing and $\lim_{n\to\infty} k_n = \infty$. We derive a contradiction using the necessary conditions in Lemma 5. Define $h_s(v') \equiv \sup \Gamma_s(v')$. Along an equilibrium path where $k_{n+1} \geq k_n$ we must have $v_n \leq h_s(v_{n+1})$. For $\sigma \geq \underline{\sigma}$ for some $\underline{\sigma} \in (0,1)$ we show that (i) h_s is continuous and strictly increasing in v', (ii) $h_s(\bar{v}) = \bar{v}$, and (iii) $h_s(v') < v'$ for all $v' > \bar{v}$.

Let v^{FB} denote the value for the optimization problem associated with $\beta = 1$. We then have $v_n \leq v^{FB}$ and since h_s is increasing $v_n \leq h_s(v_{n+1}) \leq h_s(v^{FB})$. Iterating on these inequalities using h_s one obtains $v_0 \leq h_s^n(v^{FB})$, which must hold for any $n = 0, 1, \ldots$ Since $h_s^n(v^{FB}) \to \bar{v}$ as $n \to \infty$ this implies $v_0 \leq \bar{v}$. However, $k_1 > k_0$ implies $v_0 > \bar{v}$ by Lemma 2, a contradiction. This concludes the sketch of the proof.

4.4 A Loose End: $R = R^*$

In terms of the parameter space, $R = R^*$ represents a knife-edged (zero measure) scenario. However, in a general equilibrium setting it may represent an important focal point. Theorems 2 and 3 suggest that (and $\sigma \geq \underline{\sigma}$) the unique equilibrium with $R = R^*$ holds wealth constant, $g(k) = k$. Formally, however, this does not follow from our results, which are stated with strict inequalities $R > R^*$ and $R < R^*$. Indeed, the case $R = R^*$ is subtle and requires special treatment. As it turns out, the proofs of Theorems 2 and 3 apply to $R = R^*$ unchanged if we could invoke the No Reversal Principle. However, the delicate issue is that the proof of the latter exploited $R \neq R^*$ to get $k^*(k) \neq k$ from Lemma 4 and generate strict deviations towards regions with savings. Nevertheless, we are able to modify the arguments and prove the desired result.

Theorem 4. *Suppose utility is isoelastic and $R = R^*$. Then there exists $\underline{\sigma} \in (0,1)$ such that for all $\sigma \geq \underline{\sigma}$ the unique Markov equilibrium (g, V) holds wealth constant: $g(k) = k$ for all $k \geq \underline{k}$.*

[19] See, for example, (Chatterjee and Eyigungor, 2016, Theorem 5). Cao and Werning (2016) study a continuous-time version model with similar equilibria. In continuous time, one can show that the Markov equilibrium is unique near $\underline{k}$. One can also show that there is at most one continuous equilibrium and provide conditions for its existence.

4.5 Near Linear Utility and Indeterminate Savings

Theorems 2 and 3 parts (a) provide robust predictions for any utility function, while parts (b) require the utility function to be iso-elastic and sufficiently concave, imposing a lower bound $\underline{\sigma} \in (0,1)$. To gauge the latter, note that the Intertemporal Elasticity of Substitution equals $\frac{1}{\sigma}$ and it is standard to favor modest values for this elasticity, below 1, implying $\sigma > 1$. However, logarithmic utility $\sigma = 1$ is an important theoretical benchmark for its simplicity (see for example, Barro, 1999, Krusell et al., 2002, Azzimonti, 2011 and Halac and Yared, 2014). Our results apply in all these empirical and theoretical cases of interest and also holds for some interval of σ below one. Indeed, comparing parts (a) and (b) in the two theorems suggests that $\underline{\sigma} \downarrow 0$ as $R \uparrow \frac{1}{\beta\delta}$ or $R \downarrow \frac{1}{\delta}$.

Can one strengthen our theorems to remove the lower bound on σ altogether, so that $\underline{\sigma} = 0$? Or is it the case that saving behavior can be reversed for σ near zero? Next we provide an answer, demonstrating the second possibility by construction.

Dissaving with $R \in [R^*, \frac{1}{\beta\delta})$. Our construction is slightly simpler for $\underline{k} = 0$ and goes as follows. Suppose a steady-state solution $v' = v < \bar{v}$ and $x^* < 1$ exists to conditions (4) and (5) (this is ruled out by $\sigma \geq \underline{\sigma}$, but may be possible otherwise); suppose further that $u'(\frac{R}{(x^*)^2} - 1) \geq \beta\delta R u'(R - x^*)$. Then we set

$$g(k) = (x^*)^{n+1}\hat{k} \qquad \forall k \in [(x^*)^n\hat{k}, (x^*)^{n-1}\hat{k}) \tag{8}$$

for all $n = \ldots, -2, -1, 0, 1, 2, \ldots$ for any $\hat{k} > 0$. We can verify this is an equilibrium: condition (5) with equality ensures indifference across jumps, so that an agent with $k = (x^*)^n\hat{k}$ is indifferent to $k' = k$ or $k' = x^*k$; condition $u'(\frac{R}{(x^*)^2} - 1) \geq \beta\delta R u'(R - x^*)$ is a Kuhn-Tucker condition that ensures the corner $k' = x^*k$ is preferable to any interior choice $k' \in ((x^*)^n\hat{k}, (x^*)^{n-1}\hat{k})$. (For example, all conditions are met when $\sigma = 0.27$, $\beta = 0.8$, $\delta = 0.97$ and $R = 1.04$.)

Omitting details, a similar non-homogeneous construction is possible for $\underline{k} > 0$; we require a slightly stronger Kuhn-Tucker condition $u'(\frac{R}{(x^*)^2} - 1) \geq \beta\delta R u'(R - 1)$. Naturally, for high k the construction for $\underline{k} > 0$ approaches that for $\underline{k} = 0$.

Why does low σ make dissaving possible? Recall that a linear equilibrium with strict saving always exists. When σ is low, consumers are more responsive to incentives, heightening strategic complementarities, opening the door to wider multiplicity. Intuitively, strong enough strategic complementarities allow for equilibria that overturn the natural inclination to save with $R > R^*$. The issue is subtle, however, since this discussion could also suggest multiple linear equilibria, perhaps with dissaving. Yet, the linear equilibrium is always unique and features saving. Nonlinear policies g appear to play up strategic complementarities.

Saving with $R \in (\frac{1}{\delta}, R^*]$. Once again, suppose there exists a steady-state solution to conditions (4) and (5) with equality but this time with $x^* > 1$ (this is ruled out by $\sigma \geq \underline{\sigma}$, but possible in some cases otherwise); suppose further that the Kuhn-Tucker condition $u'(R-1) \geq \beta\delta R u'(R-x^*)$ holds at this x^*. Then for any $\hat{k} \in (\underline{k}, x^*\underline{k}]$ set $g(k) = \hat{k}$ for $k \in [\underline{k}, \hat{k})$ and

$$g(k) = (x^*)^{n+1}\hat{k} \qquad \forall k \in [(x^*)^n\hat{k}, (x^*)^{n+1}\hat{k}), \tag{9}$$

for all $n \in \{0, 1, 2, \dots\}$. Once again, this is guaranteed to be an equilibrium thanks to the indifference condition (5) and the Kuhn-Tucker requirement. (For example, all the conditions are met for $\sigma = 0.25$, $\beta = 0.7$, $\delta = 0.97$ and $R = 1.04$.)

When $R \in (R^*, \frac{1}{\beta\delta})$ one can also generate equilibria with savings to the left of an arbitrary steady state k^* and dissavings to the right of it. This can be done by combining our dissaving construction, to the right of a steady state, with the saving construction in Krusell and Smith (2003), to the left of a steady state. However, when $R \in (\frac{1}{\delta}, R^*)$ our Non Reversal Principle rules such a possibility out.

The stepwise nature of both our global constructions and its verification are inspired by the local constructs in Krusell and Smith (2003). However, there are some important differences. First and foremost, our constructions are global and do represent equilibria for the standard hyperbolic-discounting model. Second, our first construction features dissaving along a step function, with wealth asymptoting towards the steady state. This is in contrast to their local construction which defined a flat policy function to the right of the steady state, with wealth reaching the steady state in a single step. Our second construction, the one with saving, has no parallel in Krusell and Smith (2003) since they did not consider $R \in (\frac{1}{\delta}, R^*)$ nor saving in the upper tail for wealth. Indeed, as we explain in Section 5.3, our construction requires a global analysis and would be impossible over any bounded interval . Finally, a crucial difference is that we require σ near zero to build equilibria, while they do not impose any assumptions on the utility function. This highlights the nonlocal nature of our construction and is precisely why we are able to rule out such constructions, as well as others, when $\sigma \geq \underline{\sigma} \in (0, 1)$ in Theorems 2 and 3.

5 Further Results: Existence, Multiplicity, and Extensions

We now present a few additional results for the standard hyperbolic-discounting model. In particular, we provide existence and multiplicity results. We also discuss a few extensions of this model, adapting our results to discrete grids, ad hoc upper bounds on wealth, and concave saving technologies.

5.1 Equilibrium Existence

Our robust characterization of equilibria would be vacuous if it applied to an empty set of equilibria. Fortunately, we now establish that this is never the case: Markov equilibria always exist in the standard hyperbolic-discounting model.[20]

Theorem 5. *Assume isoelastic utility, then a Markov equilibrium always exists. When $\underline{k} > 0$ and $R < R^*$ the same conclusion holds without the isoelastic utility assumption.*

Proof. Under isoelastic utility, when $R \geq R^*$ then $g(k) = \alpha k \geq k$ is an equilibrium; when $R < R^*$ and $\underline{k} = 0$ then $g(k) = \alpha k < k$ is an equilibrium. When $R < R^*$ and $\underline{k} > 0$ Appendix G establishes the existence of an equilibrium by construction with dissaving $g(k) \leq k$. □

It may be tempting to take existence of equilibria for granted. However, as is well understood, existence is not obvious for games with a continuum of actions; especially not for Markov equilibria in pure strategies in dynamic games. Indeed, proving existence in our model for the nontrivial case ($R < R^*$ and $\underline{k} > 0$) turns out to be quite involved. Rather than appealing to a fixed point theorem, which we found no obvious way of invoking, our argument is constructive. This has the virtue of suggesting its computation via the constructive algorithm that we spell out.

Bernheim et al. (2015, Proposition 6) provide an existence result for $R \in (\frac{1}{\delta}, R^*)$ and isoelastic utility using a constructive proof. Our proof shares some features with theirs, but is different in a crucial way which allows for $R < R^*$ and non-isoelastic utility. In a nutshell our proof proceeds as follows. We are looking for a dissaving equilibrium. Thus, we momentarily impose $k' \leq k$ in the agent maximization, forcing dissaving. We then seek to sweep from low to high wealth to construct an equilibrium, in a self generating manner. To start, for a small enough interval near the lower bound on wealth, we set

[20]With isoelastic utility, the only difficult case involves $R < R^*$ and $\underline{k} > 0$, since linear equilibria exist in the other cases. In this case, however, we do not require isoelastic utility. We omit existence proofs in the other cases without isoelastic utility, but we conjecture that a similar constructive proof is possible if we assume $u(c)$ is isoelastic for high and low c but not for intermediate c.

$g(k) = \underline{k}$ and compute the associated value functions $V(k) = u(Rk - \underline{k}) + \beta\frac{\delta}{1-\delta}u((R - 1)\underline{k})$ and $W(k) = u(Rk - g(k)) + \beta\delta V(k)$ over this small interval. It turns out that $W(k) > \bar{W}(k) = u((R-1)k) + \beta\delta\bar{V}(k)$ and a few other useful properties hold. We then show that whenever these properties hold we can extend these functions (g, V, W) to the right over a small enough interval, while preserving these same properties.[21] Iterating on this algorithm, and restarting it when necessary, then produces a candidate equilibrium over $[\underline{k}, \infty)$. We finally verify that this candidate is indeed an equilibrium.

5.2 Multiplicity of Equilibria

In this paper we focus on Markov equilibria. This refinement reduces the set of equilibria, but does not necessarily lead to uniqueness. Indeed, we now argue that multiple equilibria are present in our model for cases that satisfy the conditions of our theorems. It is, thus, especially noteworthy that robust qualitative predictions for saving behavior are possible despite such multiplicity. In contrast, Cao and Werning (2016) studied a continuous-time model and established uniqueness when $R > R^*$, highlighting a difference between these two formulations.

Saving Multiplicity with $R \in (R^*, \frac{1}{\beta\delta})$. Recall that when utility is isoelastic there is a linear equilibrium with saving. Under the conditions of Theorem 2, all equilibria involve saving. Do our results hold because the equilibrium is unique and equal to the linear one? A similar question was raised by Phelps and Pollak (1968), but no answer has been given to date. We provide a negative answer.

Our construction is identical to the one spelled out in (9), except that there we assumed $R < R^*$. Here, thanks to $R > R^*$, we can be sure that there exists a steady-state solution $v' = v > \bar{v}$ and $x^* > 1$ to conditions (4) and (5) with the latter set as an equality. Then if x^* is not too large so that the Kuhn-Tucker condition $u'(R-1) \geq \beta\delta R u'(R - x^*)$ holds (which is impossible if $R > \frac{1}{\beta\delta}$), then (9) is an equilibrium, as before. Since $\hat{k} \in (\underline{k}, x^*\underline{k}]$ is arbitrary this establishes indeterminacy, although all these equilibria feature the same rate of growth for wealth after one period.

In addition, when $\sigma > \underline{\sigma}$ as in Theorem 4 we can show that x^* goes to 1 when $R \downarrow R^*$. Because $\beta\delta R < 1$, this implies that the Kuhn-Tucker condition holds and our constructions are equilibria. These equilibria, however, converge to $g(k) = k$ as $R \downarrow R^*$, which is

[21] Our algorithm is designed to stop as soon as $W(k)$ reaches $\bar{W}(k)$, since $W(k) > \bar{W}(k)$ is a property that is required to be preserved. This is the key difference with the proof strategy in Bernheim et al. (2015, Proposition 6), which goes beyond this point, but then tries to prove there exists an earlier point with $W(k) = \bar{W}(k)$; this last step in their strategy requires $R > \frac{1}{\delta}$ in a crucial way.

consonant with our uniqueness result for $R = R^*$.

Dissaving Multiplicity with $R < R^*$**.** Multiplicity is possible but more limited for the case $\underline{k} > 0$. First, although there may be multiple equilibria, there cannot be indeterminacy. Second, near $\underline{k}$ all equilibrium policy functions coincide. In this sense, for low k there is uniqueness.

When $\underline{k} = 0$ a linear equilibrium exists with dissaving. We build alternative dissaving equilibria. Assume isoelastic utility.

First, we can again use our previous construction, the dissaving equilibrium in (8). This time, because $R < R^*$ we can ensure a steady-state solution with $x^* < 1$. With $\sigma > \underline{\sigma}$ as in Theorem 4 then x^* goes to 1 when $R \uparrow R^*$ and the Kuhn-Tucker condition is satisfied and that these constructions are equilibria.

Second, we can provide another construction in some cases. With a strictly positive lower bound on wealth there may exist equilibria with interior steady states. When this is the case (normalizing so that $\underline{k} = 1$) there is a policy function $\hat{g}$ over $k \in [1, \infty)$ with $\hat{g}(1) = 1$, $\hat{g}(\hat{x}) = \hat{x} > 1$, $\hat{g}(k) < k$ for all $k \in (1, \hat{x})$. Now for any $\hat{k} > 0$ we set

$$g(k) = \hat{x}^n \hat{k} \hat{g}\left(\frac{k}{\hat{x}^n \hat{k}}\right) \qquad \text{for } k \in [\hat{x}^n \hat{k}, \hat{x}^{n+1} \hat{k}),$$

for all $n \in \{\ldots, -2, -1, 0, 1, 2, \ldots\}$. It can be verified that this constitutes an equilibrium. Since $\hat{k} > 0$ was arbitrary this proves indeterminacy. These equilibria are in addition to the linear one.

5.3 Extensions: Grids, Ad Hoc Bounds and Concave Technologies

We have adopted standard assumptions: individuals face linear returns on their wealth, with limits to borrowing but no limits to saving. These assumptions, which can be taken as defining what is meant by a standard consumption-savings problem, are also natural. In particular, note that, even if aggregate technology is concave or total wealth and resources are bounded, general-equilibrium models confront consumers with linear budget sets at given interest rates without upper bounds on individual wealth choices (see Barro, 1999, for such a growth model with hyperbolic consumers). The important point is that non-linear technology or upper bounds on economy-wide resources do not necessitate imposing these constraints on individuals.[22]

[22] In other contexts a nonlinear savings technology at the individual level may be relevant. For example, suppose a firm is owned by a single owner and cannot borrow. Then capital in the firm is constrained by the savings of its "entrepreneur". This problem has received attention in the Development Economics

Nevertheless, exploring how our results are affected or not by these assumptions sheds light on economic mechanisms at play and helps compare our results to the literature. In addition, for practical reasons, computational work is forced to impose an ad hoc upper bound on wealth. It also typically constrains wealth choices to lie on a discrete grid. Thus, investigating the effects of these constraints can help interpret numerical findings.

Discrete Grids. We start by showing that our results apply to a modification of the saving game that forces wealth choices to live on a discrete grid instead of the interval $[\underline{k}, \infty)$. Grids are common in numerical work, so this extension is of direct relevance. In addition, showing that our results apply to grid-restricted game serves to highlight the nonlocal and non-smooth nature of our method, and draw a stark contrast to approaches based on first-order Euler equations.

Consider a discrete grid $\{k_n\}$ of wealth levels satisfying $\underline{k} = \kappa_0 < \kappa_1 < \cdots$ and $\kappa_n \to \infty$. A finite grid is covered by combining the present analysis with the next one on ad hoc upper bounds.

We first note that Lemma 1–3 apply immediately when the agent choice is restricted to any grid. Secondly, Theorem 1–3 also follow as long as we can invoke the main implication of Lemma 4 parts (a) and (b) adapted to the grid: agents best responding to constant wealth must prefer a grid point different than their own. Using Lemma 4, this can be ensured if the grid is sufficiently dense, so that $\kappa_{n+1} < k^*(\kappa_n)$ when $R > R^*$ and $k^*(\kappa_n) < \kappa_{n-1}$ when $R < R^*$, or equivalently

$$R > R^* \quad \text{then} \quad |\kappa_{n+1} - \kappa_n| < |k^*(\kappa_n) - \kappa_n|\,, \tag{10a}$$

$$R > R^* \quad \text{then} \quad |\kappa_n - \kappa_{n-1}| < |k^*(\kappa_n) - \kappa_n|\,. \tag{10b}$$

When utility is isoelastic, $k^* = \alpha^* k$, so both conditions are equivalent to the simple logarithmic grid spacing condition $|\log(\kappa_{n+1}) - \log(\kappa_n)| < |\log \alpha^*|$. We summarize this result in the next theorem.

Theorem 6. *Suppose wealth choices are constrained to lie on a discrete grid $\{\kappa_n\}$ satisfying $\kappa_n \to \infty$ and (10). Then Theorems 1–3 apply to this restricted game.*

A discrete grid poses obvious difficulties for the existence of pure strategy equilibria, so Theorem 5 is unlikely to extend unmodified. However, this may not be a major prac-

literature, but constitutes a marked departure from the standard consumption-saving problem. Another interesting departure is when households face a progressive wealth tax. Then the relevant after-tax "technology" is strictly concave, even if the the before-tax return is constant.

tical concern for numerical work. Our results may extend to allow for mixed strategy equilibria.

Ad Hoc Upper Bound on Wealth. Consider now the imposition of an ad hoc upper bound $k \leq \bar{k}$ that constrains the agent maximization problem. One can verify that Theorems 1 and Lemmas 1–4 are unaffected. By assumption $g(\bar{k}) \leq \bar{k}$, so $g(k) \leq k$ for all $k \in [\underline{k}, \bar{k}]$ when $R < R^*$. Thus, the conclusion in Theorem 3 is strengthened, applying without any conditions on the utility function and highlighting the No Reversal Principle as a fundamental result of its own.[23]

Theorem 2 part (a) with $R \geq \frac{1}{\beta\delta}$ also applies without change. In contrast, Theorem 2 part (b) with $R \in (R^*, \frac{1}{\beta\delta})$ required wealth to be unbounded above. Indeed, ad hoc upper bounds clearly creates the potential for dissavings since the Krusell and Smith (2003) local constructions are valid equilibria when choices are restricted to small enough intervals. Although we cannot expect the same conclusions, we offer a variant with essentially the same practical implications. The proof for this result is provided in Appendix C.

Theorem 7. *Suppose wealth choices are constrained by an ad hoc upper bound $\bar{k}$. Then in any Markov equilibria,*

(a) if $R < R^*$ then $g(k) \leq k$ for all $k \in [\underline{k}, \bar{k}]$;

(b) if $R \geq \frac{1}{\beta\delta}$ then $g(k) > k$ for all $k \in [\underline{k}, \bar{k})$;

(c) if $R \in (R^*, \frac{1}{\beta\delta})$ and utility is isoelastic with $\sigma \geq \underline{\sigma}$ for some $\underline{\sigma} \in [0, 1)$, then $g(k) > k$ for all $k \in [\underline{k}, \zeta\bar{k}]$ where $\zeta < 1$ is independent of $\bar{k}$.

In practice, we are often interested in behavior over some bounded interval of wealth; suppose we set the ad hoc bound high enough relative to such an interval (as it should, if the its imposition was purely for numerical reasons); then the model robustly predicts agents in the interval of interest will strictly save. In this sense, the economic thrust of Theorem 2 is unchanged.

Concave Returns. Consider a saving game that replaces the linear returns Rk in the budget constraint with a weakly concave return function $f : [\underline{k}, \infty) \to [\underline{k}, \infty)$.[24] To adapt our results we assume f is affine above some $\bar{k} > 0$.

All our results then carry over without modification as long as the necessary comparisons between the marginal return $f'(k)$ (taking the place of R) and the corresponding

[23] Thus, the savings equilibrium with $R < R^*$ with σ near zero constructed in Section 4.5 cannot exist and so they would never be encountered numerically.

[24] An upper bound on wealth can be seen as a case with concave returns f that is piecewise linear with return 0 above $\bar{k}$.

threshold in each case (i.e. R^*, $\frac{1}{\delta}$ or $\frac{1}{\delta\beta}$) holds uniformly, for all k. For example, if $f'(k) \geq \frac{1}{\beta\delta}$ for all k then we must have strict saving regardless of the degree of concavity in f. Thus, concavity per se has no effect on our results.

What about nonuniform cases where $f'(k) < R^*$ for high k but $f'(k) > R^*$ for low k? As it turns out, robust predictions for dissaving remain: one can say that wherever $f'(k) < R^*$ then $g(k) \leq k$.[25] Likewise, wherever $f'(k) \geq \frac{1}{\beta\delta}$ then $g(k) > k$. However, when $R \in (R^*, \frac{1}{\beta\delta})$ Theorem 3 part (b) made use of a high marginal return throughout the upper tail of wealth. Thus, we cannot claim that saving prevails pointwise wherever $f'(k) \in (R^*, \frac{1}{\beta\delta})$. This opens the door for indeterminacy with saving and dissaving. Saying more formally is challenging, but based on numerical explorations this appears to be the case. In particular, some of the local constructions in Krusell and Smith (2003) may extend globally. However, in general, not all local constructions can be extended and the set of global steady-states remains a strict subset of the set of local steady-states.[26,27]

6 Conclusions

We have revisited the standard hyperbolic-discounting model. Our main finding is that under plausible conditions, as long as the utility function is not too close to being linear, the set of Markov equilibria has the well-behaved property of predicting saving or dissaving behavior globally, that is, for all wealth levels. Moreover, the condition for saving versus dissaving behavior is a simple one that compares the interest rate to discounting parameters: wealth rises if $R > R^*$ and falls if $R < R^*$.

Our results uncovering robust predictions under plausible conditions should not be taken for granted. First, our findings run counter to interpretations of previous results suggesting equilibrium indeterminacy. Second, we establish that robust predictions are not always possible; for some parameters and near linear utility we construct (global)

[25] This result holds without assuming that f becomes linear at the top if, instead, $\lim_{k\to\infty} f'(k) < 1/\delta$, as is common in growth models.

[26] We thank Tony Smith for showing us numerical results consistent with this pattern for a discrete-time model that allows for mixed strategies over a grid (based on work in Krusell and Smith, 2008). We obtained similar patterns numerically within the continuous-time deterministic model from Cao and Werning (2016). In our exploration, as we parametrize the concavity of the saving technology and take the limit towards linear technology, the set of global steady-states shrinks to an empty set, while the set of local steady-states expands.

[27] Greater concavity may cut two ways: it may increase the fraction of local steady state constructions that can be validly extended to become global equilibria, but at the same time shrink the interval of local steady-state wealth levels. Indeed, if the saving technology has a kink (an extreme form of concavity) with return below R^* for all $k > k^*$ and return above $\frac{1}{\beta\delta} > R^*$ for all $k < k^*$, then k^* is the unique steady-state and it must be globally stable, with saving below k^* and dissaving above k^*.

equilibria that reverse the natural sign of saving. Third, it may seem natural to seek robust predictions for consumption, in addition to wealth, but this goal appears elusive. In particular, in the time-consistent $\beta = 1$ case consumption falls over time if $R < \frac{1}{\delta}$ and rises if $R > \frac{1}{\delta}$.[28] Unfortunately, when $\beta < 1$ this is no longer guaranteed.[29,30] Our results do indicate, however, that non-monotonicity of consumption is limited in the sense that the present value of consumption $k_t = \sum_{s=0}^{\infty} R^{-s} c_{t+s}$ must be monotone.

Throughout this paper we have purposefully stayed within the confines of the most standard hyperbolic-discounting model. It is an open question whether our methods can adapt to extensions, such as the introduction of uncertainty. However, in our view, it is important to establish basic properties within the standard model, before turning to such extensions. Indeed, few sharp analytical results are available with uncertainty even for the time-consistent benchmark with $\beta = 1$, which is why the incomplete markets literature often resorts to numerical simulations. Basic results without uncertainty, however, provide an important benchmark to guide economic intuition into these less tractable extensions.

Finally, although we have studied the behavioral problem of a hyperbolic-discounting consumer, our analysis should be helpful in other time-inconsistent dynamic games, such as in political economy models of public debt.

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[28] This follows directly from the Euler equation $u'(c_t) = \delta R u'(c_{t+1})$ whenever the borrowing constraint does not bind.

[29] Numerical explorations indicate that the equilibrium path for consumption $c_t = Rk_t - g(k_t)$ may be non-monotone over time. When our results hold, the path for k_t is monotone, so a non-monotone path for c_t obtains because consumption $Rk - g(k)$ may not be everywhere increasing in k.

[30] The Euler equation no longer holds, but instead, following Harris and Laibson (2001), at points of differentiability $k_{t+1} > 0$ one obtains $u'(c_t) = \delta[\beta + (1-\beta)\frac{1}{R} g'(k_{t+1})] R\, u'(c_{t+1})$. Thus, if $g'(k_{t+1})$ is not constant then $\{c_t\}$ may not be monotone.

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Appendix

A Proofs from Section 3

The following lemmas apply to any Markov equilibrium (g, V).

A.1 Proof of Lemma 2

We first show that if $R \geq \frac{1}{\delta}$ and $g(k) < k$, then $V(k) < \bar{V}(k)$. We then show that if $R \leq \frac{1}{\delta}$ and $g(k) > k$ then $V(k) < \bar{V}(k)$.

When $R \geq \frac{1}{\delta}$, we show that $\bar{V}(k)$ is the value function of the dynamic optimization problem

$$\max_{\{k_n\}} \sum_{n=0}^{\infty} \delta^n u(Rk_n - k_{n+1})$$

subject to $k_0 = k$ and $k_{n+1} \leq k_n$. We show that $\bar{V}$ is the solution of the Bellman equation associated to the problem:

$$W(k) = \max_{k' \leq k} \{u(Rk - k') + \delta W(k')\}.$$

Consider the maximization problem

$$\max_{k' \leq k} \{u(Rk - k') + \frac{\delta}{1-\delta} u\left((R-1)\, k'\right)\}$$

where $\frac{u((R-1)k')}{1-\delta} = \bar{V}(k')$. The first-order condition for k' is

$$-u'(Rk - k') + \frac{R-1}{\frac{1}{\delta} - 1} u'((R-1)\, k') \geq 0,$$

with equality if $k' < k$. Since $R \geq \frac{1}{\delta}$, the inequality holds at $k' = k$, and is violated for any $k' < k$. Therefore, $k' = k$ is the unique solution of this maximization problem and $W(k) = \bar{V}(k)$. Note that this optimum is unique. Since $g(k) < k$ and g is monotone, we have $g^n(k)$ a decreasing sequence, so this is feasible but not optimal; it follows that $V(k) < \bar{V}(k)$.

The proof for $R \leq \frac{1}{\delta}$ is symmetric.

A.2 Proof of Lemma 3

From the definition of equilibrium,

$$u(Rk - g(k)) + \beta\delta V(g(k)) \geq u(Rk - k) + \beta\delta V(k),$$

Rearranging,

$$\beta\left(u(Rk - g(k)) + \delta V(g(k))\right) + (1-\beta)u(Rk - g(k)) \\ \geq \beta\left(u(Rk-k) + \delta V(k)\right) + (1-\beta)u(Rk-k),$$

or equivalently,

$$u(Rk - g(k)) + \delta V(g(k)) \geq u(Rk-k) + \delta V(k) + \frac{1-\beta}{\beta}(u(Rk-k) - u(Rk - g(k))).$$

Now if $g(k) \geq k$ and $\beta \leq 1$, the last term in the right side is positive. Therefore,

$$u(Rk - g(k)) + \delta V(g(k)) \geq u(Rk-k) + \delta V(k),$$

The left side equals $V(k)$, so that

$$V(k) \geq u(Rk-k) + \delta V(k).$$

This implies $V(k) \geq \frac{1}{1-\delta}u(Rk-k) = \bar{V}(k)$. The inequality is strict if $g(k) > k$.

A.3 Proof of Lemma 5

The first condition follows directly from (1) at $k = k_n$ given that $V_n = V(k_n)$, $k_n = g(k_n)$ and $V_{n+1} = V(k_{n+1})$. The second condition follows from inequality (2) at $k = k_n$ by setting $k' = k_n$, given that $V_n = V(k_n)$, $k_{n+1} = g(k_n)$ and $V_{n+1} = V(k_{n+1})$, then using (1) and rearranging.

With isoelastic utility we can divide these expressions by $k_n^{1-\sigma}$ and rearrange to obtain (4) and (5).

B Proof of No Reversal Principle: Theorem 1

Part (a): $R > R^*$ Assume by contradiction that there exists $\hat{k} > \tilde{k}$ such that $g(\hat{k}) \geq \hat{k}$. By hypothesis we have that

$$\check{k} \equiv \inf\left\{k \geq \tilde{k} : g(k) \geq k\right\} \leq \hat{k} < \infty. \tag{11}$$

There are two cases to consider, both leading to a contradiction.

Case 1: First suppose $g(\check{k}) \geq \check{k}$. This implies that $\tilde{k} < \check{k}$ and, hence, that $g(k) < k$ for all $k \in (\tilde{k}, \check{k})$. Since g is a Markov equilibrium savings function it must satisfy (2) for all k, implying

$$u(Rk - g(k)) + \beta\delta V(g(k)) \geq u(Rk - \check{k}) + \beta\delta V(\check{k}).$$

By Lemma 3, $V(\check{k}) \geq \bar{V}(\check{k})$. This implies

$$u(Rk - \check{k}) + \beta\delta V(\check{k}) \geq u(Rk - \check{k}) + \beta\delta \bar{V}(\check{k}).$$

Combining the two inequalities gives for all $k \geq \underline{k}$

$$u(Rk - g(k)) + \beta\delta V(g(k)) \geq u(Rk - \check{k}) + \beta\delta \bar{V}(\check{k}). \tag{12}$$

Now consider $k \in (\tilde{k}, \check{k})$, so that $g(k) < k < \check{k}$. By monotonicity of g, then $g(g(k)) \leq g(k)$. If this weak inequality holds with equality then $V(g(k)) = \bar{V}(g(k))$, otherwise, if the inequality is strict, by Lemma 2 part (a) we have $V(g(k)) < \bar{V}(g(k))$. Combining, we conclude that $V(g(k)) \leq \bar{V}(g(k))$ and using (12) then gives that for all $k \in (\tilde{k}, \check{k})$

$$u(Rk - g(k)) + \beta\delta \bar{V}(g(k)) \geq u(Rk - \check{k}) + \beta\delta \bar{V}(\check{k}). \tag{13}$$

We shall derive a contradiction with this inequality.

Indeed, since $g(k) < k$, Lemma 4 part (a) shows that

$$u(Rk - g(k)) + \beta\delta \bar{V}(g(k)) < u(Rk - k) + \beta\delta \bar{V}(k).$$

By Lemma 4 part (a), $k^*(\check{k}) > \check{k}$. In addition, because $k^*(k)$ is continuous in k (part c), there exists $k < \check{k}$ sufficiently close to $\check{k}$ such that $k^*(k) > \check{k}$. Also by Lemma 4 part (a), for such k,

$$u(Rk - k) + \beta\delta \bar{V}(k) < u(Rk - \check{k}) + \beta\delta \bar{V}(\check{k}).$$

Together the last two inequalities contradict inequality (13).

This proves that $g(\check{k}) \geq \check{k}$ is not possible.

Case 2: Now suppose $g(\check{k}) < \check{k}$. We shall find that this case also leads to a contradiction.

Since $g(\check{k}) < \check{k}$, the definition of $\check{k}$ implies that there exists a sequence $\{l_n\}$ with $l_n > \check{k}$, $\lim_{n\to\infty} l_n = \check{k}$ satisfying $g(l_n) \geq l_n$. Since g is a Markov equilibrium savings function it must satisfy (2) for $k = \check{k}$, implying

$$u(R\check{k} - g(\check{k})) + \beta\delta V(g(\check{k})) \geq u(R\check{k} - l_n) + \beta\delta V(l_n).$$

By Lemma 3, since $g(l_n) \geq l_n$ we have $V(l_n) \geq \bar{V}(l_n)$ and thus

$$u(R\check{k} - l_n) + \beta\delta V(l_n) \geq u(R\check{k} - l_n) + \beta\delta \bar{V}(l_n).$$

Combining gives

$$u(R\check{k} - g(\check{k})) + \beta\delta V(g(\check{k})) \geq u(R\check{k} - l_n) + \beta\delta \bar{V}(l_n).$$

Also by Lemma 2 (for $R > \frac{1}{\delta}$), $V(g(\check{k})) \le \bar{V}(g(\check{k}))$, implying

$$u(R\check{k} - g(\check{k})) + \beta\delta\bar{V}(g(\check{k})) \ge u(R\check{k} - l_n) + \beta\delta\bar{V}(l_n).$$

Now taking the limit as $n \to \infty$, this implies

$$u(R\check{k} - g(\check{k})) + \beta\delta\bar{V}(g(\check{k})) \ge u(R\check{k} - \check{k}) + \beta\delta\bar{V}(\check{k}).$$

However, since $g(\check{k}) < \check{k}$, this contradicts the conclusion of Lemma 4 part (a) (for $k = \check{k}$ and $k' = g(\check{k})$). This contradiction implies that $g(\check{k}) < \check{k}$ is not possible.

Since either Case 1 or Case 2 must hold and both lead to a contradiction we conclude that the hypothesis that there exists $\hat{k}$ such that $g(\hat{k}) \ge \hat{k}$ must be false. Therefore, we have shown that $g(k) < k$ for all $k \ge \tilde{k}$.

Now starting from $k_0 \ge \tilde{k}$, because $k_1 = g(k_0) < k_0$, and g is monotone. The resulting sequence $\{k_n\}$ is decreasing and bounded below by $\underline{k}$. Therefore the limit $\lim_{n\to\infty} k_n = k_\infty \ge \underline{k}$ exists. We show that k_∞ must be a steady-state.

Indeed, because k_n is decreasing, $k_\infty \le k_n$. Again, by the monotonicity of g, $g(k_\infty) \le g(k_n) = k_{n+1}$. Taking the limit $n \to \infty$, we obtain $g(k_\infty) \le k_\infty$. To show that $g(k_\infty) = k_\infty$, we just need to rule out $g(k_\infty) < k_\infty$. We show this by contradiction. Assume $g(k_\infty) < k_\infty$. Because of (2) at $k = k_\infty$,

$$u(Rk_\infty - g(k_\infty)) + \beta\delta V(g(k_\infty)) \ge u(Rk_\infty - k_{n+1}) + \beta\delta V(k_{n+1}).$$

By Lemma 2, $V(g(k_\infty)) \le \bar{V}(g(k_\infty))$. Therefore

$$u(Rk_\infty - g(k_\infty)) + \beta\delta\bar{V}(g(k_\infty)) \ge u(Rk_\infty - k_{n+1}) + \beta\delta V(k_{n+1}).$$

Now as $n \to \infty$, $V(k_{n+1}) = \sum_{m=0}^{\infty} \delta^m u(Rk_{n+1+m} - k_{n+2+m}) \to \bar{V}(k_\infty)$ and $k_{n+1} \to k_\infty$, the inequality above implies

$$u(Rk_\infty - g(k_\infty)) + \beta\delta\bar{V}(g(k_\infty)) \ge u(Rk_\infty - k_\infty) + \beta\delta\bar{V}(k_\infty).$$

which contradicts the conclusion in part (a) of Lemma 4 (for $k = k_\infty$ and $k' = g(k_\infty) < k_\infty$). Therefore by contradiction we must have $g(k_\infty) = k_\infty$ as desired.

Lastly, since $g(k) < k$ for all $k \ge \tilde{k}$. It follows that $k_\infty < \tilde{k}$.

Part (b): $R < R^*$ We proceed again by contradiction and assume that there exists $\hat{k} > \tilde{k}$ such that $g(\hat{k}) \le \hat{k}$. Therefore,

$$\check{k} = \inf\{k \ge \tilde{k} : g(k) \le k\} \le \hat{k} < \infty. \tag{14}$$

By definition, $g(k) > k$ for all $k \in (\tilde{k}, \check{k})$. We first show that $\check{k}$ is a steady-state, so that $g(\check{k}) = \check{k}$. If $g(\check{k}) > \check{k}$, then by the definition of $\check{k}$, there exists $k \in (\check{k}, g(\check{k}))$ such that $g(k) \le k$. However this implies

$$g(k) \le k < g(\check{k}),$$

which contradicts the monotonicity of g shown in Lemma 1. Therefore $g(\check{k}) \le \check{k}$. Also by the definition of $\check{k}$,

$$k < g(k) \le g(\check{k}) \le \check{k}$$

for all $k \in (\tilde{k}, \check{k})$. Taking the limit $k \to \check{k}$ from the left, we obtain $g(\check{k}) = \check{k}$.

Since g is a Markov equilibrium savings function it must satisfy (2) for $k = \check{k}$, implying

$$u(R\check{k} - g(\check{k})) + \beta\delta V(g(\check{k})) \ge u(R\check{k} - k) + \beta\delta V(k),$$

for all $k \in (\tilde{k}, \check{k})$. Since $g(\check{k}) = \check{k}$ which implies $V(g(\check{k})) = \bar{V}(\check{k})$ and $g(k) > k$ which implies $V(k) > \bar{V}(k)$ by Lemma 3, we obtain

$$u(R\check{k} - \check{k}) + \beta\delta\bar{V}(\check{k}) > u(R\check{k} - k) + \beta\delta\bar{V}(k).$$

This contradicts the conclusion of Lemma 4 part (b) for $k < \check{k}$ and sufficiently close to $\check{k}$ such that $k^*(\check{k}) < k < \check{k}$.

We conclude that the hypothesis that there exists $\hat{k}$ such that $g(\hat{k}) \le \hat{k}$ must be false. Therefore, we have shown that $g(k) > k$ for all $k \ge \underline{\tilde{k}}$.

Now starting from $k_0 \ge \tilde{k}$, the resulting sequence $\{k_n\}$ is strictly increasing. We show that $\lim_{n\to\infty} k_n = \infty$. Assume the contrary: $\lim_{n\to\infty} k_n = k_\infty < \infty$. Because $k_\infty > k_0 \ge \tilde{k}$, and the result established above, $g(k_\infty) > k_\infty$. By Lemma 3, we conclude that $V(k_\infty) > \bar{V}(k_\infty)$. Now

$$V(k_{n+1}) = \sum_{m=0}^{\infty} \delta^m u(Rk_{n+1+m} - k_{n+2+m}) \to \bar{V}(k_\infty),$$

because $k_n \to k_\infty$. Since (g, V) is a Markov equilibrium condition (2) must hold for $k = k_n$, implying

$$u(Rk_n - k_{n+1}) + \beta\delta V(k_{n+1}) \ge u(Rk_n - k_\infty) + \beta\delta V(k_\infty).$$

Taking the limit $n \to \infty$ on both sides gives $\bar{V}(k_\infty) \ge V(k_\infty)$, a contradiction. Therefore, we have shown that $\lim_{n\to\infty} k_n = \infty$.

C Proofs for Robust Saving: Theorem 2 and Theorem 7

This appendix groups the proof of Theorem 2 and its related extension Theorem 7.

C.1 Theorem 2

We first prove weak savings, that $g(k) \le k$ for all $k \ge \underline{k}$. Having established weak savings, we prove strict savings.

Weak Savings under part (a): $R \ge \frac{1}{\beta\delta}$**.** The proof is contained in the main body.

Weak Savings for part (b): $R \in (R^*, \frac{1}{\beta\delta})$. Assume $\sigma > \underline{\sigma} \in (0,1)$ so that Lemma 7 applies. Thus, there exists $\hat{v} > \frac{u(0)}{1-\delta}$ such that $\Gamma_d(v') = \emptyset$ if and only if $v' < \hat{v}$. Also, there exists $\underline{x} \in (0,1)$ such that if $v' \in [\hat{v}, \bar{v})$ then for all x satisfying (5) we have $x \geq \underline{x}$. Finally, we have that h_d is continuous, increasing, and $h_d(v') < v'$ over $[\hat{v}, \bar{v})$.

We now proceed by contradiction: suppose there exists $\check{k} \geq \underline{k}$ such that $g(\check{k}) < \check{k}$. Theorem 1 part (a) then implies $g(k) < k$ for all $k \geq \check{k}$. Now pick any $l^* > \bar{k}$ and pick any $\tilde{k} \in (g(l^*), l^*)$ and define

$$\bar{x} \equiv \max\left\{\frac{\tilde{k}}{l^*}, \frac{g(l^*)}{\tilde{k}}\right\} < 1.$$

Then it follows that $k \geq \tilde{k}$ and $g(k) \leq \tilde{k}$ implies $\frac{g(k)}{k} \leq \bar{x}$.

Define

$$\chi(x) \equiv u(R - x) + \delta\bar{v}x^{1-\sigma}. \tag{15}$$

Since $R > \frac{1}{\delta}$, $\chi(x)$ is strictly increasing over $x \in [0,1]$. Set $\tilde{v} \equiv \chi(\bar{x}) < \chi(1) = \bar{v}$.

Now given $\tilde{v}$, let N be the lowest value such that $h_d^{(N)}(\tilde{v}) < \hat{v}$. Such a finite N exists by virtue of the fact that we guaranteed h_d is continuous and $h_d(v') < v'$ for all $v' \in [\hat{v}, \bar{v})$.

Given N, $\underline{x}$ and $\tilde{k}$, choose $k_0 = \underline{x}^{-(N+1)}\tilde{k} > \tilde{k}$ and consider the sequence $\{k_n\}$ generated by the savings function $k_{n+1} = g(k_n)$. Let $\{x_n, v_n\}$ denote the resulting normalized sequence as in Section 3.2. By Lemma 2, $v_n \leq \bar{v}$ for all $n = 0, 1, \ldots$ By the definition of Γ_d we must have $v_n \in \Gamma_d(v_{n+1})$ for all $n = 0, 1, \ldots$ This is equivalent to requiring $\hat{v} \leq v_{n+1}$ and $v_n \leq h_d(v_{n+1})$ for all $n = 0, 1, \ldots$

Since $g(k) < k$ for all $k \geq \check{k}$ and g is monotone, the sequence $\{k_n\}$ is strictly decreasing. By Theorem 1 Part a, $\lim_{n\to\infty} k_n < \check{k}$. Thus, there exists a crossing point at n^* such that $k_{n^*} \geq \tilde{k}$ and $k_{n^*+1} \leq \tilde{k}$. By our choice of k_0 and the fact that $x_n > \underline{x}$, we guaranteed that $n^* \geq N + 1$. From the definition of $\tilde{k}$, we have $x_{n^*} \leq \bar{x}$ and hence

$$\begin{aligned} v_{n^*} &= u(R - x_{n^*}) + \delta v_{n^*+1} x_{n^*}^{1-\sigma} \\ &\leq u(R - x_{n^*}) + \delta\bar{v}x_{n^*}^{1-\sigma} = \chi(x_{n^*}) \leq \chi(\bar{x}) = \tilde{v}. \end{aligned}$$

An equilibrium requires $\hat{v} \leq v_{n^*-m} \leq h_d^{(m)}(v_{n^*}) \leq h_d^{(m)}(\tilde{v})$ for $m = 1, \ldots, n^* - 1$. However, $\hat{v} \leq h_d^{(m)}(\tilde{v})$ for $m = N \leq n^* - 1$ is impossible since $h_d^{(N)}(\tilde{v}) < \hat{v}$.

Therefore, by contradiction, we obtain weak saving in all Markov equilibrium.

Weak Savings Implies Strict Savings. Having established that $g(k) \geq k$ for all k, we now show how strict savings follows, so that $g(k) > k$ for all k. Assume towards a contradiction that $g(\check{k}) = \check{k}$ for some $\check{k} \geq \underline{k}$. By the Markov equilibrium condition (2) this implies

$$u((R-1)\check{k}) + \beta\delta V(\check{k}) \geq u(R\check{k} - k) + \beta\delta V(k)$$

for all $k \geq \underline{k}$. Since $V(\check{k}) = \bar{V}(\check{k})$ this implies

$$u((R-1)\check{k}) + \beta\delta\bar{V}(\check{k}) \geq u(R\check{k} - k) + \beta\delta V(k)$$

for all $k \geq \underline{k}$. Since we have shown that $g(k) \geq k$ for all $k \geq \underline{k}$, Lemma 3 implies $V(k) \geq \bar{V}(k)$. Thus,

$$u((R-1)\breve{k}) + \beta\delta\bar{V}(\breve{k}) \geq u(R\breve{k} - k) + \beta\delta\bar{V}(k)$$

for all $k \geq \underline{k}$. However, this contradicts the conclusion of Lemma 4 part (a) (for $k = \breve{k}$ and $k' > \breve{k}$). Thus, it follows that $g(k) > k$ for all $k \geq \underline{k}$.

C.2 Theorem 7

The proof for weak dissavings in Theorem 2 relies crucially on the assumption that the wealth domain is unbounded above (since $\tilde{v}$,N and therefore K depend on endogenous equilibrium object such as g, V) and is not directly applicable to the environment with an ah-hoc upper bound on wealth. Now we offer an alternative proof that shows weak savings in the model with an ad-hoc upper bound: $g(k) \geq k$ for all $k \leq \zeta\bar{k}$ for some $\zeta < 1$ independent of $\bar{k}$. Having established weak dissavings, we can proceed as in the proof of Theorem 2 to establish strict savings for $k \leq \zeta\bar{k}$.

Proof for Weak Savings. Assume $\sigma > \underline{\sigma}$ such that Lemmas 7, 8, and 9 apply. Let $\gamma > 1$ and $\tilde{v} < \bar{v}$ as defined in Lemma 8 which depend only on model primitives σ, β, δ, R and are independent of $\bar{k}$. Given $\tilde{v}$, let N be such that $h_d^{(N)}(\tilde{v}) < \hat{v}$. N exists and is finite because h_d is continuous and $h_d(v') < v'$ for all $v' < \bar{v}$ as shown in Lemma 7. Let $\underline{x} < 1$ be defined as in Lemma 7ii. In addition, $N, \underline{x}$ depends only on model primitives because h_d, $\tilde{v}$, $\hat{v}$, $\bar{v}$ do. Define ζ as

$$\zeta = \frac{1}{\gamma}\underline{x}^{N+1} < 1,$$

which itself depends only on model primitives.

We show by contradiction that $g(k) \geq k$ for all $k \leq \zeta\bar{k}$. Assume the contrary: there exists $\breve{k} \in [\underline{k}, \zeta\bar{k}]$ such that $g(\breve{k}) < \breve{k}$. Theorem 1 (Part a) then implies that $g(k) < k$ for all $k \geq \breve{k}$. When $\underline{k} > 0$, the equilibrium definition immediately implies $g(\underline{k}) \geq \underline{k}$. When $\underline{k} = 0$, Lemma 9 shows that there exists $k > 0$ such that $g(k) \geq k$. In either case, we can find $\underline{\underline{k}} > 0$ such that $g(\underline{\underline{k}}) \geq \underline{\underline{k}}$. Because $g(k) < k$ for all $k \geq \breve{k}$, it must be that $\underline{\underline{k}} < \breve{k}$.

Now let $k_0 = \bar{k}$ and consider the sequence $\{k_n\}$ generated by the savings function $k_{n+1} = g(k_n)$. As shown in Theorem 1 Part a, the limit of this sequence, k^s, must be a steady-state and $k^s < \breve{k}$. By monotonicity of g, $k^s \geq \underline{\underline{k}} > 0$. Let $\tilde{k} = \gamma k^s$. By Lemma 8, for any k such that $k \geq \tilde{k}$ and $g(k) \leq \tilde{k}$ then $v(k) \leq \tilde{v}$.

Construct the sequences $\{x_n\}$ and $\{v_n\}$ as in Section 3.2. By Lemma 2, $v_n \leq \bar{v}$ for all $n = 0, 1, \ldots$ By the definition of Γ_d we must have $v_n \in \Gamma_d(v_{n+1})$. Lemma 7ii ensures that $x_n > \underline{x}$, for all $n = 0, 1, \ldots$

Because $k_0 = \bar{k} > \tilde{k}$ and $\lim_{n\to\infty} k_n = k^s < \tilde{k}$, there exists a crossing point n^* such that $k_{n^*} \geq \tilde{k}$ and $k_{n^*+1} \leq \tilde{k}$. Now since $k_0 = \bar{k}$ and

$$\tilde{k} \geq k_{n^*+1} = k_0 \prod_{i=0}^{n^*} x_i > k_0 \underline{x}^{n^*+1} = \bar{k}\underline{x}^{n^*+1},$$

we have

$$\underline{x}^{-n^*-1}\tilde{k} > \bar{k} > \underline{x}^{-(N+1)}\gamma k^s = \underline{x}^{-(N+1)}\tilde{k}.$$

It follows that $n^* \geq N+1$. From the definition of $\tilde{k}$, we have $v_{n^*} \leq \tilde{v}$. With this property, we obtain a contradiction as in the previous proof for unbounded domain. Therefore, $g(k) \geq k$ for all $k \leq \zeta\bar{k}$.

D Proof of Robust Dissaving: Theorem 3

Assume that $g(k) > k$ at some $k \geq \underline{k}$. We show that this must lead to a contradiction.

Part (a): $R \leq \frac{1}{\delta}$ The proof is already included in the main body of the paper.

Part (b): $\frac{1}{\delta} < R < R^*$ Consider the correspondence Γ_s defined in Subsection 4.3. For any equilibrium path with $k_{n+1} \geq k_n$ we must have $v_n \in \Gamma_s(v_{n+1})$. Recall that $\bar{v} \equiv \frac{u(R-1)}{1-\delta}$. One can show that the function

$$F_s(v') \equiv \max_{x\in[1,R]} (1-\beta\delta)u(R-x) + \beta\delta(1-\delta)v'x^{1-\sigma}.$$

is strictly increasing in v' and that $F_s(\bar{v}) = u(R-1)$. It follows that the set of $x \in [1, R]$ for which (5) holds is non empty if and only if $v' \geq \bar{v}$. Thus, $\Gamma_s(v') \neq \emptyset$ if and only if $v' \geq \bar{v}$. For all $v' \geq \bar{v}$, we already defined

$$h_s(v') \equiv \sup \Gamma_s(v'). \tag{16}$$

Lemma 10 shows that there exists $\underline{\sigma} < 1$ such that for $\sigma > \underline{\sigma}$, $h_s(v') < v'$ for all $v' > \bar{v}$. Assume $\sigma > \underline{\sigma}$, we will use this function h_s to obtain a contradiction. Suppose $g(\hat{k}) > \hat{k}$ at some $\hat{k} \geq \underline{k}$. Then by the No Reversal Principle, $g(k) > k$ for all $k \geq \hat{k}$.

Consider the sequence of wealth $\{k_n\}_{n=1}^{\infty}$ generated by $k_{n+1} = g(k_n)$ and $k_0 = \hat{k}$. By Lemma 1, $\{k_n\}$ is non-decreasing and by Theorem 1 Part b, $\lim_{n\to\infty} k_n = \infty$. We show that this leads to a contradiction. Let $v_n = \frac{V(k_n)}{k_n^{1-\sigma}}$. Because the sequence features strict saving and $k_1 > k_0$, by Lemma 3, we know that $v_0 > \bar{v}$. Let v^{FB} denote the (normalized) value of the optimum for the exponential consumer with commitment, i.e.

$$v^{FB}k^{1-\sigma} = \max_{\{l_n\},l_0=k} \sum_{n=0}^{\infty} \delta^n u(Rl_n - l_{n+1})$$

and $v^{FB} < \infty$ since $\delta R^{1-\sigma} < 1$. We then have

$$v_n \leq v^{FB}$$

and since h_s is an increasing function this gives

$$v_n \le h_s(v_{n+1}) \le h_s(v^{FB}).$$

Continuing in this way one obtains

$$v_0 \le h_s^n(v^{FB}),$$

for any $n = 0, 1, \dots$ Taking the limit as $n \to \infty$ gives $h_s^n(v^{FB}) \to \bar{v}$, so we conclude that $v_0 \le \bar{v}$. A contradiction.

Therefore we have shown that $g(k) \le k$ for all $k \ge \underline{k}$.

E Supporting Lemmas

E.1 Lemmas Supporting Proofs of Theorem 2 and Theorem 7

Lemma 7. *Suppose $R \in [R^*, \frac{1}{\beta\delta})$ and let Γ_d and h_d be defined as in Section 4.2. Then there exists $\underline{\sigma} < 1$ such that for all $\sigma > \underline{\sigma}$, the following properties hold.*

(i) There exists $\hat{v} \in (\frac{u(0)}{1-\delta}, \bar{v})$ such that $\Gamma_d(v') = \emptyset$ if and only if $v' < \hat{v}$.

(ii) There exists $\underline{x} > 0$ such that for all $v' \in [\hat{v}, \bar{v}]$, if (5) holds then $x > \underline{x}$.

(iii) $h_d(v')$ is continuous, strictly increasing in v' and $h_d(\bar{v}) = \bar{v}$ and $h_d(v') < v'$ for all $v' < \bar{v}$.

Proof. Define $\underline{\sigma}_1$ as the unique $\sigma \in (0, 1)$ such that $(1 - \beta\delta)u(R) + \beta\delta u(0) = u(R - 1)$ or equivalently

$$(1 - \beta\delta)R^{1-\sigma} = (R - 1)^{1-\sigma}, \tag{17}$$

which exists because $R < \frac{1}{\beta\delta}$. When $\sigma > \underline{\sigma}_1$ then the $(1 - \beta\delta)u(R) + \beta\delta u(0) < u(R - 1)$.

Similarly, define $\underline{\sigma}_2$ as the unique $\sigma \in (0, 1)$ such that

$$(1 - \beta\delta)\sigma(2 - \sigma) = 1 - \beta. \tag{18}$$

When $\sigma > \underline{\sigma}_2$ then $(1 - \beta\delta)\sigma(2 - \sigma) > 1 - \beta$.

Now define $\underline{\sigma} = \max\{\underline{\sigma}_1, \underline{\sigma}_2\}$. We show that (i), (ii), and (iii) hold.

Proof of Part (i). Define

$$F(v', x) \equiv (1 - \beta\delta)u(R - x) + \beta\delta(1 - \delta)v'x^{1-\sigma},$$

$$\mathcal{F}(v') \equiv \left\{x \in [0, 1] : F(v', x) \ge u\left(R - 1\right)\right\},$$

and

$$f(v') = \max_{x \in [0,1]} F(v', x).$$

Then $f(v')$ is strictly increasing in v'. In addition $f(\bar{v}) > F(\bar{v}, 1) = u(R-1)$ and $\lim_{v' \downarrow \frac{u(0)}{1-\delta}} f(v') = (1-\beta\delta)u(R)$. Because $\sigma > \underline{\sigma}_1$ we have $(1-\beta\delta)u(R) < u(R-1)$. Therefore, $\lim_{v' \downarrow \frac{u(0)}{1-\delta}} f(v') < u(R-1)$. Consequently, there exists a unique $\hat{v} \in (\frac{u(0)}{1-\delta}, \bar{v})$ such that $f(v') = u(R-1)$ and $f(v') < u(R-1)$, so that $\mathcal{F}(v') = \emptyset$ and $\Gamma_d(v') = \emptyset$ for $v' < \hat{v}$.

Proof of Part (ii). Because $\sigma > \underline{\sigma}_1$:

$$1 - \beta\delta < \left(\frac{R-1}{R}\right)^{1-\sigma}.$$

Since

$$(1-\beta\delta)u(R-x) + \beta\delta(1-\delta)v' x^{1-\sigma} \geq u(R-1)$$

and $v' \leq \bar{v}$,we have

$$(1-\beta\delta)u(R-x) + \beta\delta(1-\delta)\bar{v} x^{1-\sigma} \geq u(R-1). \tag{19}$$

As $x \to 0$, the left hand side converges to

$$(1-\beta\delta)u(R),$$

when $\sigma < 1$ and to $-\infty$ when $\sigma > 1$, both are strictly less than $u(R-1)$ given $\sigma > \underline{\sigma}_1$. Therefore, by continuity, there exists $\underline{x} > 0$ such that (19) holds with the strict reversed inequality when $x \leq \underline{x}$. For such $\underline{x}$, (19) implies $x > \underline{x}$.

Proof of Part (iii). Define

$$H(v', x) \equiv u\left(R - x\right) + \delta v' x^{1-\sigma}.$$

Using H,we can write the function h_d as

$$h_d(v') = \max_{x \in \mathcal{F}(v')} H\left(v', x\right).$$

By Berge's Maximum Theorem, h_d is continuous.

It is easy to show that F are strictly concave in x. Therefore $\mathcal{F}(v')$ is an interval, i.e. $\mathcal{F}(v') = [x_1(v'), x_2(v')]$ where $0 \leq x_1(v') \leq x_2(v') \leq 1$. Since $F(v'_1, x) \leq F(v'_2, x)$ for any $v'_1 < v'_2$, we have $\mathcal{F}(v'_1) \subset \mathcal{F}(v'_2)$.

In addition $H(v'_1, x) \leq H(v'_2, x)$ for any $v'_1 < v'_2$, therefore $h_d(v'_1) \leq h_d(v'_2)$. It is also easy to rule out equality ($h_d(v'_1) = h_d(v'_2)$ only if $v'_1 = 0$ however in this case $v'_2 > v'_1 = 0$ leading to $h_d(v'_1) = u(R) < h_d(v'_2)$, a contradiction). Therefore $h_d(v'_1) < h_d(v'_2)$.

Now at $v' = \bar{v}$, we have, $F\left(\bar{v}, 1\right) = u(R-1)$. Also

$$\frac{\partial F}{\partial x} \equiv -(1-\beta\delta)u'(R-x) + \beta\delta(1-\delta)v'(1-\sigma)x^{-\sigma}.$$

Using that $\bar{v} = \frac{u(R-1)}{1-\delta}$ and $u'(R-1) = (1-\sigma)u(R-1)(R-1)$ we obtain

$$\frac{\partial F}{\partial x}(\bar{v}, 1) = u'(R-1)(\beta\delta R - 1).$$

Since $\beta\delta R < 1$, $\frac{\partial F}{\partial x} < 0$, we have $\mathcal{F}(\bar{v}) = [x_1, 1]$ where $0 \le x_1 < 1$. Now

$$\frac{\partial H}{\partial x} = -u'(R-x) + \delta v'(1-\sigma)x^{-\sigma}.$$

Since $\bar{v} = \frac{u(R-1)}{1-\delta}$,

$$\frac{\partial H}{\partial x}(\bar{v}, 1) = u'(R-1)\left(\frac{\delta R - 1}{1-\delta}\right).$$

Because $R > \frac{1}{\delta}$, $\frac{\partial H}{\partial x}(\bar{v}, 1) > 0$. Moreover, H is concave in x, therefore H is strictly increasing for $x \in [x_1, 1]$. So $h_d(\bar{v}) = H(\bar{v}, 1) = \bar{v}$.

Now we show that $h_d(v') < v'$ for all $v' < \bar{v}$. We rewrite function $h_d(.)$ as

$$h_d(v') = \max_{x \in [0,1]} u(R-x) + \delta v' x^{1-\sigma} \tag{20}$$

subject to

$$(1-\beta\delta)u(R-x) + \beta\delta(1-\delta)v' x^{1-\sigma} \ge u(R-1). \tag{21}$$

Let $\hat{x}$ be the solution to this optimization problem. We show that $h_d(v') < v'$ separately in two cases.

Case 1: $\sigma < 1$. We will show that for σ sufficiently close to 1, and if $\hat{x} < 1$, constraint (21) holds with equality, i.e.

$$(1-\beta\delta)u(R-\hat{x}) + \beta\delta(1-\delta)v'\hat{x}^{1-\sigma} = u(R-1).$$

We show that $h_d(v') < v'$ when $v' \neq \bar{v}$ in this case. Indeed, we have

$$h_d(v') = u(R-\hat{x}) + \delta v' \hat{x}^{1-\sigma}$$

and

$$(1-\beta\delta)u(R-\hat{x}) + \beta\delta(1-\delta)v'\hat{x}^{1-\sigma} = u(R-1).$$

Showing that $h_d(v') \le v'$ is equivalent to showing

$$v' \ge \frac{u(R-\hat{x})}{1-\delta\hat{x}^{1-\sigma}}.$$

From the second equation,

$$v' = \frac{u(R-1) - (1-\beta\delta)u(R-\hat{x})}{\beta\delta(1-\delta)\hat{x}^{1-\sigma}}.$$

Therefore, we need to show

$$u(R-1) \geq u(R-\hat{x}) \left(\frac{\beta\delta(1-\delta)\hat{x}^{1-\sigma}}{1-\delta\hat{x}^{1-\sigma}} + 1 - \beta\delta \right).$$

Let

$$\begin{aligned} \phi(x) &\equiv u(R-x) \left(\frac{\beta\delta(1-\delta)x^{1-\sigma}}{1-\delta x^{1-\sigma}} + 1 - \beta\delta \right) \\ &= u(R-x) \left(\frac{\beta(1-\delta)}{1-\delta x^{1-\sigma}} + 1 - \beta \right). \end{aligned} \tag{22}$$

In Lemma 7a below, we show that for $\sigma \in (\underline{\sigma}_2, 1)$, , $\phi'(x) > 0$ for $x < 1$. Therefore $\phi(\hat{x}) < \phi(1) = u(R-1)$ for all $\hat{x} < 1$. This implies $h_d(v') \leq v'$. In addition, we obtain an equality if and only if $\hat{x} = 1$ and $v' = \frac{u(R-1)}{1-\delta} = \bar{v}$. Otherwise $h_d(v') < v'$.

Now we show that if $\sigma < 1$ but $\sigma > \underline{\sigma}_1$ and $x_2(v') < 1$, constraint (21) binds at $x = \hat{x}$. Indeed, we rewrite the constraint as

$$u(R-x) + \frac{\beta\delta(1-\delta)}{1-\beta\delta} v' x^{1-\sigma} \geq \frac{1}{1-\beta\delta} u(R-1).$$

Both the left side of this constraint and the objective function in (20) are single-peaked. Since $\frac{\beta\delta(1-\delta)}{1-\beta\delta} < \delta$, the peak of the left side of this constraint lies to the left of the peak of the objective function.

The constraint is then equivalent to $x \in [x_1(v'), x_2(v')]$ where the constraint holds with equality at $x_1(v')$ and if $x_2(v') \leq 1$ it also holds with equality at $x_2(v')$, otherwise $x_2(v') = 1$. Lemma 7b below shows that for $\sigma \in (\underline{\sigma}_1, 1)$, if the peak of the objective function $x^* \leq 1$ then

$$u(R-x^*) + \frac{\beta\delta(1-\delta)}{1-\beta\delta} v'(x^*)^{1-\sigma} < \frac{1}{1-\beta\delta} u(R-1).$$

Thus, $x^* > x_2(v')$. Therefore, the objective function is strictly increasing in $x \in [x_1(v'), x_2(v')]$, which then implies that $\hat{x} = x_2(v') < 1$. If $x^* > 1$, then $x_2(v') < x^*$ and the same conclusion holds.

We have shown that for $\sigma \in (\underline{\sigma}, 1)$ and if $\hat{x} < 1$, then $h_d(v') < v'$. The remaining possibility is that $x_2(v') = \hat{x} = 1$. Because $\hat{v} \leq v' < \bar{v}$, we show that $x_2(v') = \hat{x} < 1$, so this case does not apply. Indeed, when $x = 1$, since $v' < \bar{v}$

$$(1-\beta\delta)u(R-1) + \beta\delta(1-\delta)v' < u(R-1).$$

In addition $v' \geq \hat{v}$ implies that there exists $x < 1$ satisfying

$$(1-\beta\delta)u(R-x) + \beta\delta(1-\delta)v' x^{1-\sigma} \geq u(R-1).$$

Therefore $x_2(v') < 1$.

Case 2: $\sigma > 1$. There are three possibilities Case 2i, Case 2ii, and Case 2iii.

Case 2i: $\hat{x} < x_2(v') \leq 1$. Then $\hat{x}$ is a local maximum of the objective function, (20), determining h_d.The F.O.C. in $\hat{x}$ implies

$$v' = \frac{(R-\hat{x})^{-\sigma}}{\delta(1-\sigma)\,(\hat{x})^{-\sigma}}.$$

So

$$h_d(v') = u(R-\hat{x}) + \delta\frac{(R-\hat{x})^{-\sigma}}{\delta(1-\sigma)}\hat{x}.$$

Therefore

$$h_d(v') < v'$$

if and only if

$$(R-\hat{x})^{1-\sigma} + (R-\hat{x})^{-\sigma}\,\hat{x} > \frac{(R-\hat{x})^{-\sigma}}{\delta}$$

or equivalently, $R\,(\hat{x})^{-\sigma} > \frac{1}{\delta}$. This is true since $R > R^* > \frac{1}{\delta}$ and $\hat{x} < 1$.

Case 2ii: $\hat{x} = x_2(v') < 1$. If $\delta\hat{x}^{1-\sigma} \geq 1$, given that $v' < 0$ and $u < 0$,

$$h_d(v') = u(R-\hat{x}) + \delta v'\hat{x}^{1-\sigma} < v'.$$

Now if $\delta\hat{x}^{1-\sigma} \leq 1$, Lemma 7a shows that ϕ is defined in (22) is strictly increasing in x when $\delta x^{1-\sigma} \leq 1$. Therefore $\phi(\hat{x}) < \phi(1)$. As shown for the case $\sigma < 1$, this implies $h_d(v') < v'$.

Case 2iii: $\hat{x} = x_2(v') = 1$. As shown above for $\sigma < 1$, this case does not arise when $v' < \bar{v}$. .

Lemma 7a. *Suppose $R \geq R^*$. Let ϕ be defined by (22). When either i) $\sigma \in (\underline{\sigma}_2, 1)$ or ii) $\sigma > 1$ and $\delta x^{1-\sigma} \leq 1$, then $\phi'(x) > 0$ for all $x \in [0,1)$.*

Differentiating $\phi(x)$, we get

$$\begin{aligned}
\phi'(x) &= -u'(R-x)\left(\frac{\beta(1-\delta)}{1-\delta x^{1-\sigma}} + 1 - \beta\right) + u(R-x)\frac{\beta(1-\delta)\delta(1-\sigma)x^{-\sigma}}{(1-\delta x^{1-\sigma})^2} \\
&= u'(R-x)\left(\frac{\beta(1-\delta)\delta x^{-\sigma}}{(1-\delta x^{1-\sigma})^2}(R-x) - \frac{\beta(1-\delta)}{1-\delta x^{1-\sigma}} - (1-\beta)\right) \\
&= u'(R-x)\frac{x^{-\sigma}}{(1-\delta x^{1-\sigma})^2}\psi(x),
\end{aligned}$$

where

$$\psi(x) \equiv \beta(1-\delta)\delta(R-x) - \beta(1-\delta)x^{\sigma}\left(1-\delta x^{1-\sigma}\right) - (1-\beta)\left(1-\delta x^{1-\sigma}\right)^{2} x^{\sigma} \tag{23}$$
$$= \beta(1-\delta)\delta R - \beta(1-\delta)x^{\sigma} - (1-\beta)\left(1-\delta x^{1-\sigma}\right)^{2} x^{\sigma}$$
$$= \beta(1-\delta)\delta R - (1-\beta\delta)x^{\sigma} + 2(1-\beta)\delta x - (1-\beta)\delta^{2}x^{2-\sigma}.$$

Showing $\phi'(x) > 0$ is equivalent to showing $\psi(x) > 0$, which we do below.

At $x = 1$,

$$\psi(1) = (1-\delta)\beta\delta\left(R - 1 - \frac{1-\delta}{\beta\delta}\right) = (1-\delta)\beta\delta(R-R^{*}) \geq 0$$

given that $R \geq R^{*}$.

First consider the case $\sigma < 1$. The derivative $\psi'(x)$ is given by

$$\psi'(x) = -\left(1-\beta\delta\right)\sigma x^{\sigma-1} + \left(1-\beta\right)2\delta - (1-\beta)\delta^{2}(2-\sigma)x^{1-\sigma}. \tag{24}$$

By the Cauchy-Schwarz inequality,

$$\psi'(x) \leq (1-\beta)2\delta - \sqrt{(1-\beta\delta)(1-\beta)\delta^{2}\sigma(2-\sigma)} < 0,$$

where the last inequality comes from (18).Therefore $\psi(x) > \psi(1) \geq 0$ for all $x \in [0,1]$.

Now consider the case $\sigma > 1$. We first show that ψ is concave in x as long as $\delta x^{1-\sigma} \leq 1$. Indeed,

$$\psi''(x) = -(1-\beta\delta)x^{\sigma-2}\sigma(\sigma-1) + (1-\beta)\delta^{2}(2-\sigma)x^{-\sigma}(\sigma-1).$$

If $\sigma \geq 2$ then $\psi''(x) < 0$ since both terms on the right side are negative. If $1 < \sigma < 2$, then

$$\psi''(x) = x^{\sigma-2}(\sigma-1)\left(-(1-\beta\delta)\sigma + (1-\beta)\delta^{2}x^{2-2\sigma}(2-\sigma)\right)$$
$$\leq x^{\sigma-2}(\sigma-1)\left(-(1-\beta\delta)\sigma + (1-\beta)(2-\sigma)\right) < 0,$$

where the first inequality comes from $\delta x^{1-\sigma} \leq 1$ and the second inequality comes from $\sigma > 2-\sigma$ and $1-\beta\delta > 1-\beta$.

Because $\psi(x)$ is strictly concave in x, in order to show that $\psi(x) > 0$, we just need to show that $\psi(x) \geq 0$ at the two extremes: when $x = 1$ or when $\delta x^{1-\sigma} = 1$.

When $x = 1$, $\psi(1) \geq 0$ is shown above. When $\delta x^{1-\sigma} = 1$,

$$\psi(x) = \beta(1-\delta)\delta R - (1-\beta\delta)x^{\sigma} + 2(1-\beta)\delta x - (1-\beta)\delta^{2}x^{2-\sigma}$$
$$= \beta(1-\delta)\delta R - (1-\beta\delta)x^{\sigma} + 2(1-\beta)\delta x - (1-\beta)\delta x$$
$$= \beta(1-\delta)\delta R - (1-\beta\delta)\delta x + 2(1-\beta)\delta x - (1-\beta)\delta x$$
$$= \beta(1-\delta)\delta(R-x) > 0.$$

Lemma 7b. *Suppose $R \geq R^*$. and $\sigma < 1$. Let*

$$x^* = \arg\max_{x \geq 0} u(R - x) + \delta v' x^{1-\sigma}.$$

For all $\sigma > \underline{\sigma}_1$, if $x^* \leq 1$ then

$$(1 - \beta\delta)u(R - x^*) + \beta\delta(1 - \delta)v' (x^*)^{1-\sigma} < u(R - 1). \tag{25}$$

Indeed,

$$(R - x^*)^{-\sigma} = \delta v'(1 - \sigma) (x^*)^{-\sigma}$$

Therefore

$$v' = \frac{(R - x^*)^{-\sigma}}{\delta(1 - \sigma) (x^*)^{-\sigma}}.$$

Plugging this into the LHS of (25), we obtain

$$\begin{aligned}
&(1 - \beta\delta)u(R - x^*) + \beta\delta(1 - \delta)v' (x^*)^{1-\sigma} \\
&= (1 - \beta\delta)u(R - x^*) + \beta\delta(1 - \delta)\frac{(R - x^*)^{-\sigma}}{\delta(1 - \sigma) (x^*)^{-\sigma}} (x^*)^{1-\sigma} \\
&= (1 - \beta\delta)\frac{(R - x^*)^{1-\sigma}}{1 - \sigma} + \beta(1 - \delta)\frac{(R - x^*)^{-\sigma}}{(x^*)^{-\sigma}} \frac{(x^*)^{1-\sigma}}{1 - \sigma}.
\end{aligned}$$

When $\sigma < 1$, the desired inequality (25) is then equivalent to

$$(1 - \beta\delta) (R - x^*) + \beta(1 - \delta)x^* < (R - x^*)^{\sigma} (R - 1)^{1-\sigma},$$

or

$$(1 - \beta\delta)R < (1 - \beta) x^* + (R - x^*)^{\sigma} (R - 1)^{1-\sigma}.$$

The right hand side is concave, therefore we just need to show that the inequality holds at $x^* = 1$ and $x^* = 0$. At $x^* = 1$, it is equivalent to

$$(1 - \beta\delta)R < (1 - \beta) + R - 1$$

or $\frac{1}{\delta} < R$ which is satisfied because $R \geq R^* > \frac{1}{\delta}$.

At $x^* = 0$, the inequality is equivalent to

$$1 - \beta\delta < \left(1 - \frac{1}{R}\right)^{1-\sigma},$$

which holds because $\sigma > \underline{\sigma}_1$.

Lemma 8. *Assume isoelastic utility with $\sigma > \underline{\sigma}$ defined in Lemma 7 and $R > R^*$. If $k^s > 0$ is a* *steady state, then there exists $\tilde{k} > k^s$ and $\tilde{v} < \bar{v}$ such that if $k \geq \tilde{k}$ and $g(k) \leq \tilde{k}$ then*

$$\frac{V(k)}{k^{1-\sigma}} \leq \tilde{v}.$$

In addition $\gamma = \frac{\tilde{k}}{k^s} > 1$ and $\tilde{v}$ depends only on model primitives: σ, β, δ, R.

Proof. Let $k^*(\cdot)$ be defined in Lemma 4 as the optimal response to constant wealth. Because $R > R^*$, we have $k^*(k^s) > k^s$ and let $\tilde{\tilde{k}}$ and $\tilde{k}$ denote $\frac{k^s+k^*(k^s)}{2}$ and $k^*(k^s)$ respectively. Then by Lemma 4, we have, for all $k \in [\tilde{\tilde{k}}, \tilde{k}]$:

$$\begin{aligned} u(Rk^s - k) + \beta\delta\bar{V}(k) &\geq u(Rk^s - \tilde{\tilde{k}}) + \beta\delta\bar{V}(\tilde{\tilde{k}}) \\ &= u(Rk^s - k^s) + \beta\delta\bar{V}(k^s) + \Delta, \end{aligned}$$

where $\Delta = u(Rk^s - \tilde{\tilde{k}}) + \beta\delta\bar{V}(\tilde{\tilde{k}}) - u(Rk^s - k^s) - \beta\delta\bar{V}(k^s) > 0$. For any k in the same interval, because an agent with wealth k^s prefers k^s to k, we have:

$$u(Rk^s - k^s) + \beta\delta\bar{V}(k^s) \geq u(Rk^s - k) + \beta\delta V(k).$$

Combining this with the previous inequality, we arrive at:

$$u(Rk^s - k) + \beta\delta\bar{V}(k) \geq \Delta + u(Rk^s - k) + \beta\delta V(k),$$

or

$$\frac{V(k)}{k^{1-\sigma}} \leq \bar{v} - \frac{\Delta}{\beta\delta k^{1-\sigma}} \leq \bar{v} - \frac{\Delta}{\beta\delta\tilde{k}^{1-\sigma}}.$$

Now let $\tilde{v}$ be defined by

$$\tilde{v} = \max\left\{\bar{v} - \frac{\Delta}{\beta\delta\tilde{k}^{1-\sigma}}, \chi\left(\frac{\tilde{\tilde{k}}}{\bar{\tilde{k}}}\right)\right\},$$

where χ is defined in (15). It follows immediately that $\tilde{v} < \bar{v}$. We show that $\tilde{k}$ and $\tilde{v}$ satisfy the desired property. Indeed, if $k \geq \tilde{k}$, and $g(k) \leq \tilde{k}$, there are two possibilities:

i. $g(k) \in [\tilde{\tilde{k}}, \tilde{k}]$: in this case, $\frac{V(k)}{k^{1-\sigma}} \leq \bar{v} - \frac{\Delta}{\beta\delta k^{1-\sigma}} \leq \tilde{v}$ as shown above.

ii. $g(k) \leq \tilde{\tilde{k}}$: in this case, $\frac{g(k)}{k} \leq \frac{\tilde{\tilde{k}}}{\bar{\tilde{k}}}$. Therefore

$$\begin{aligned} \frac{V(k)}{k^{1-\sigma}} &= \frac{u(Rk - g(k)) + \beta\delta V(g(k))}{k^{1-\sigma}} \\ &\leq \frac{u(Rk - g(k)) + \beta\delta\bar{V}(g(k))}{k^{1-\sigma}} \\ &= \chi\left(\frac{g(k)}{k}\right) \leq \chi\left(\frac{\tilde{\tilde{k}}}{\bar{\tilde{k}}}\right) \leq \tilde{v}, \end{aligned}$$

where the first inequality comes from Lemma 2 and $g(k) \leq k$ and the second inequality comes from monotonicity of χ.

Lastly, by homogeneity, $\frac{\tilde{k}}{k^s}$ and $\tilde{v}$ only depend on model primitives. □

Lemma 9. *Assume isoelastic utility with $\sigma > \underline{\sigma}$ defined in Lemma 7, $R > R^*$, and $\underline{k} = 0$. Then in any Markov equilibrium, $g(k) \geq k$ for some $k > 0$.*

Proof. We show the result by contradiction. Assume the contrary: $g(k) < k$ for all $k > 0$. Starting from any wealth level $k_0 > 0$ consider the sequence $\{k_n\}_{n=0}^{\infty}$ generated by the savings function g. We have $k_{n+1} = g(k_n) < k_n$ for all $n \geq 0$. Let

$$v_n = \frac{V(k_n)}{k_n^{1-\sigma}}.$$

By Lemma 2, $V(k_n) < \bar{V}(k_n)$, which implies $v_n \leq \bar{v}$.

The sequence $\{k_n\}$ is strictly decreasing. It is also bounded from below by 0, so it must converge to some $\tilde{k} \geq 0$. If $\tilde{k} > 0$, as shown in Theorem 1 Part a, it must be that $g(\tilde{k}) = \tilde{k}$, which is ruled out by the contradiction assumption that $g(k) < k$ for all $k > 0$. Consequently, we have:

$$\lim_{n\to\infty} k_n = 0. \tag{26}$$

Now, we show that

$$\lim_{n\to\infty} \frac{k_{n+1}}{k_n} = 1. \tag{27}$$

Since $\frac{k_{n+1}}{k_n} < 1$ for all $n \geq 0$, to show (27), we just need to show that for any $\gamma > 0$, there exists n_γ such that $\frac{k_{n+1}}{k_n} \geq 1 - \gamma$ for all $n > n_\gamma$. Indeed, let $\tilde{v} = \chi(1-\gamma) < \bar{v}$ where χ is defined in (15). By Lemma 7, there exists n_γ such that $h_d^{(n)}(\tilde{v}) < \hat{v}$ for all $n \geq n_\gamma$. We show by contradiction that $\frac{k_{n+1}}{k_n} \geq 1 - \gamma$ for all $n > n_\gamma$. Assume by contradiction that $\frac{k_{n^*+1}}{k_{n^*}} < 1 - \gamma$ for some $n^* > n_\gamma$. By Lemma 2, $v_{n^*+1} \leq \bar{v}$, hence

$$\begin{aligned} v_{n^*} &= u(R - x_{n^*}) + \delta v_{n^*+1} x_{n^*}^{1-\sigma} \\ &\leq u(R - x_{n^*}) + \delta \bar{v} x_{n^*}^{1-\sigma} \leq \chi(1-\gamma) = \tilde{v}. \end{aligned}$$

From the definition of h_d, we have:

$$v_{n^*} \leq h_d(v_{n^*+1}) \leq \tilde{v}.$$

Therefore $v_{n^*-n_\gamma} \leq h_d^{(n_\gamma)}(\tilde{v}) < \hat{v}$. Which implies $\Gamma_d(v_{n^*-n_\gamma}) = \emptyset$, a contradiction, since $v_{n^*-n_\gamma-1} \in \Gamma_d(v_{n^*-n_\gamma})$. So we obtain (27) by contradiction.

Similarly, we can show that

$$\lim_{n\to\infty} \frac{V(k_n)}{k_n^{1-\sigma}} = \bar{v}. \tag{28}$$

Armed with (26), (27), and (28), we can derive the ultimate contradiction. Let $\bar{\varphi}(x) \equiv u(R-x) + \beta\delta x^{1-\sigma}\bar{v}$. Because $R > R^*$, $\bar{\varphi}'(1) = u'(R-1)\left(1 - \frac{\beta\delta(R-1)}{1-\delta}\right) > 0$. Therefore, there exist $0 < \epsilon_1 < \epsilon_2$ such that

$$\bar{\varphi}(x) > \bar{\varphi}(1) \; \forall x \in [1+\epsilon_1, 1+\epsilon_2]\,. \tag{29}$$

Because of (27), there exists $N > 0$ such that

$$0 < \log(k_n) - \log(k_{n+1}) < \log(1+\epsilon_2) - \log(1+\epsilon_1) \; \forall n \geq N. \tag{30}$$

Given (30) and (26), for each $n \geq N$, there exists $m(n) > n$ such that

$$\log(1+\epsilon_2) \geq \log(k_n) - \log(k_{m(n)}) \geq \log(1+\epsilon_1),$$

or equivalently

$$1+\epsilon_2 \geq \frac{k_n}{k_{m(n)}} \geq 1+\epsilon_1.$$

From the optimality of $k_{m(n)+1}$ given $k_{m(n)}$,

$$u(Rk_{m(n)} - k_{m(n)+1}) + \beta\delta V(k_{m(n)+1}) \geq u(Rk_{m(n)} - k_n) + \beta\delta V(k_n).$$

Dividing both sides by $k_{m(n)}^{1-\sigma}$ and rearranging, the last inequality leads to

$$\begin{aligned} u\left(R - \frac{k_{m(n)+1}}{k_{m(n)}}\right) + \beta\delta \frac{V(k_{m(n)+1})}{k_{m(n)+1}^{1-\sigma}} \left(\frac{k_{m(n)+1}}{k_{m(n)}}\right)^{1-\sigma} \\ \geq u\left(R - \frac{k_n}{k_{m(n)}}\right) + \beta\delta \frac{V(k_n)}{k_n^{1-\sigma}} \left(\frac{k_n}{k_{m(n)}}\right)^{1-\sigma}. \end{aligned} \tag{31}$$

We can extract a subsequence n_p such that $\frac{k_{n_p}}{k_{m(n_p)}}$ converges to some $x^* \in [1+\epsilon_1, 1+\epsilon_2]$. Applying (31) for $n = n_p$ and take the limit $p \to \infty$, using (27) and (28), we arrive at

$$\bar{\varphi}(1) = u(R-1) + \beta\delta\bar{v} \geq u(R - x^*) + \beta\delta\bar{v}(x^*)^{1-\sigma} = \bar{\varphi}(x^*),$$

which contradicts (29). We obtain the desired contradiction. □

E.2 Lemmas Supporting Proof of Theorem 3

Lemma 10. *Suppose $R \in (\frac{1}{\delta}, R^*]$. Let h_s be defined by (16). Then*
(i) $h_s(v')$ is continuous, strictly increasing in v';
(ii) $h_s(\bar{v}) = \bar{v}$; and
(iii) if $\sigma > \underline{\sigma}_2$, defined in (18), then $h_s(v') < v'$ for all $\bar{v} < v'$.

Proof. Proof of parts (i) and (ii): The proof of this part is similar to the proof of part (iii) in

Lemma 7. In particular, let H and F be defined as in that proof and

$$\mathcal{F}_s(v') = \left\{x \in [1,R] : F(v',x) \geq u\left(R-1\right)\right\}.$$

Given these definitions, we can write the function h_s as

$$h_s(v') = \max_{x \in \mathcal{F}_s(v')} H\left(v',x\right).$$

By Berge's Maximum Theorem, h_s is continuous.

Since F is strictly concave in x, $\mathcal{F}_s(v')$ is an interval, i.e. $\mathcal{F}_s(v') = [x_1(v'), x_2(v')]$ where $1 \leq x_1(v') \leq x_2(v') \leq R$. Since $F(v'_1, x) \leq F(v'_2, x)$ for any $v'_1 < v'_2$, we have $\mathcal{F}_s(v'_1) \subset \mathcal{F}_s(v'_2)$. In addition $H(v'_1, x) \leq H(v'_2, x)$ for any $v'_1 < v'_2$, therefore $h_s(v'_1) \leq h_s(v'_2)$. It is also easy to show that this inequality must be strict.

Now at $v' = \bar{v}$, we have, $F\left(\bar{v}, 1\right) = u(R-1)$. Consider the derivative $\frac{\partial F}{\partial x}\left(\bar{v}, x\right)$

$$\frac{\partial F}{\partial x} = -(1-\beta\delta)u'(R-x) + \beta\delta(1-\delta)v'(1-\sigma)x^{-\sigma}.$$

Since $\bar{v} = \frac{u(R-1)}{1-\delta}$,

$$\frac{\partial F}{\partial x}\left(\bar{v}, 1\right) = u'\left(R-1\right)\left(\beta\delta R - 1\right).$$

Because $\beta\delta R < 1$, $\frac{\partial F}{\partial x} < 0$, therefore $\mathcal{F}_s(\bar{v}) = \{1\}$. So $h_s(\bar{v}) = H(\bar{v}, 1) = \bar{v}$. For $v' < \bar{v}$, $\mathcal{F}_s(v') = \emptyset$.

Proof of part (iii): Let

$$v = h_s(v')$$

and let $\hat{x} \in [1,R]$ be the solution to the constrained optimization problem that defines h_s. Then

$$v = u(R-\hat{x}) + \delta\hat{x}^{1-\sigma}v'$$

and

$$(1-\beta\delta)u(R-\hat{x}) + \beta\delta(1-\delta)v'\hat{x}^{1-\sigma} \geq u(R-1).$$

Therefore

$$\beta(1-\delta)v \geq u(R-1) - (1-\beta)u(R-\hat{x}). \tag{32}$$

Since $v' = \frac{v-u(R-\hat{x})}{\delta\hat{x}^{1-\sigma}}$, showing $h_s(v') < v'$ is equivalent to showing

$$\frac{v-u(R-\hat{x})}{\delta\hat{x}^{1-\sigma}} > v$$

or

$$v > \frac{u(R-\hat{x})}{1-\delta\hat{x}^{1-\sigma}}.$$

By (32), this is obtained if

$$\frac{u(R-1)-(1-\beta)u(R-\hat{x})}{\beta(1-\delta)} > \frac{u(R-\hat{x})}{1-\delta\hat{x}^{1-\sigma}}$$

for all $\hat{x} \in (1, R]$ (because $v' > \bar{v}$ and $\hat{x} > 1$). After rearranging, we rewrite this inequality as

$$u(R-1) \geq u(R-\hat{x})\left(\frac{\beta(1-\delta)}{1-\delta\hat{x}^{1-\sigma}} + 1 - \beta\right).$$

The right hand side is $\phi(\hat{x})$, where ϕ is defined in (22). By definition $\phi(1) = u(R-1)$. It is sufficient to show that ϕ is strictly decreasing over $[1, R]$.

Indeed,

$$\phi'(x) = u'(R-\hat{x})\frac{x^{-\sigma}}{(1-\delta x^{1-\sigma})^2}\psi(x)$$

where ψ is defined in (23). As shown in Lemma 10a below, $\psi(x) < 0$ for $x > 1$ if $\sigma > 1$ or $\sigma \in (\underline{\sigma}_2, 1)$. □

Lemma 10a. *Let $\psi(x)$ be defined as in (23), then $\psi(x) < 0$ for all $x \in (1, R]$ if either i) $\sigma > 1$ or ii) $\sigma \in (\underline{\sigma}_2, 1)$.*

Proof. In Case i) or Case ii), we show that ψ is strictly decreasing over $[1, R]$. Therefore $\psi(x) < \psi(1) = (1-\delta)\beta\delta(R - R^*) \leq 0$ from the expression for ψ in (23).

From the expression for $\psi'(x)$ given in (24), $\psi'(x) < 0$ if and only if

$$-(1-\beta\delta)\,\sigma y + (1-\beta)\,2\delta - (1-\beta)\delta^2(2-\sigma)\frac{1}{y} < 0,$$

where $y = x^{\sigma-1}$. If $\sigma > 1$, $y \geq 1$. The derivative of the last expression in y is strictly negative for $y \geq 1$,

$$-(1-\beta\delta)\sigma + (1-\beta)\delta^2(2-\sigma)\frac{1}{y^2} < 0,$$

since

$$1-\beta\delta > (1-\beta)\delta^2$$

and $\sigma \geq 2 - \sigma$. Now at $y = 1$, the value of the expression is

$$\begin{aligned}
&-(1-\beta\delta)\,\sigma + (1-\beta)\,2\delta - (1-\beta)\delta^2(2-\sigma)\\
&= -(1-\beta\delta-\delta^2(1-\beta))\sigma + (1-\beta)2\delta(1-\delta)\\
&< -(1-\beta\delta-\delta^2(1-\beta)) + (1-\beta)2\delta(1-\delta)\\
&= -(1-\delta)(1-\delta+\beta\delta) < 0.
\end{aligned}$$

So $\psi'(x) < 0$ for all $x \geq 1$.

When $\sigma < 1$, then

$$\begin{aligned}\psi'(x) &= -(1-\beta\delta)\,\sigma x^{\sigma-1} + (1-\beta)\,2\delta - (1-\beta)\delta^2(2-\sigma)x^{1-\sigma} \\ &\leq -2\sqrt{(1-\beta\delta)\,\sigma(1-\beta)\delta^2(2-\sigma)} + (1-\beta)2\delta,\end{aligned}$$

by the Cauchy–Schwarz inequality. The last expression is strictly negative when (18) holds with strict inequality. □

F Case $R = R^*$: Proof of Theorem 4

Choose $\underline{\sigma} < 1$ such that for all $\sigma > \underline{\sigma}$, Lemma 7 and Lemma 10 apply: $h_d(v') < v'$ for all $v' < \bar{v}$ and $h_s(v') < v'$ for all $v' > \bar{v}$. We prove the uniqueness result in two steps. First, we extend the conclusions of the No Reversal Principle stated in Theorem 1 to $R = R^*$ with $\sigma \geq \underline{\sigma}$. Then, we use the principle to rule out $g(k) > k$ and $g(k) < k$ at any k.

First, observe that with $R = R^*$ we have a version of Lemma 4 with $k^*(k) = k$, so that

$$\varphi(k,k) > \varphi(k,k') \tag{33}$$

for all $k' \neq k$.

No Reversal Principle Part (a). We first show that there cannot be reversal from strict dissavings to weak savings. We proceed as in the proof of Theorem 1 part (a): assume by contradiction that there exists $\tilde{k} < \hat{k}$ and $g(\tilde{k}) < \tilde{k}$ and $g(\hat{k}) \geq \hat{k}$. Define $\check{k}$ as in (11). There are two cases.

Case 1: $g(\check{k}) \geq \check{k}$. In this case $g(k) < k$ for all $k \in (\tilde{k}, \check{k})$. We first show by contradiction that $\lim_{k \uparrow \check{k}} g(k) = \check{k}$. Otherwise, there exists a sequence $l_n \uparrow \check{k}$ such that $\lim_{n\to\infty} g(l_n) = k' < k$. Because of (2), we have

$$u(Rl_n - g(l_n)) + \beta\delta V(g(l_n)) \geq u(Rl_n - \check{k}) + \beta\delta V(\check{k}).$$

By Lemma 2, $V(g(l_n)) \leq \bar{V}(g(l_n))$ and by Lemma 3, $V(\check{k}) \geq \bar{V}(\check{k})$. Therefore, the last inequality implies:

$$u(Rl_n - g(l_n)) + \beta\delta \bar{V}(g(l_n)) \geq u(Rl_n - \check{k}) + \beta\delta \bar{V}(\check{k}).$$

Taking the limit $n \to \infty$, we arrive at

$$u(R\check{k} - k') + \beta\delta \bar{V}(k') \geq u(R\check{k} - \check{k}) + \beta\delta \bar{V}(\check{k}),$$

which contradicts (33). Therefore, by contradiction we have shown that $\lim_{k \uparrow \check{k}} g(k) = \check{k}$.

Now, pick $k^* \in (g(\tilde{k}), \tilde{k})$ and define $\tilde{x} < 1$ and $\tilde{v} < \bar{v}$ as in the proof of Theorem 2: $\tilde{x} \equiv \max\left\{\frac{k^*}{\tilde{k}}, \frac{g(\tilde{k})}{k^*}\right\} < 1$ and $\tilde{v} = \chi(\tilde{x}) < \bar{v}$. Then for k sufficiently close to $\check{k}$ such that

$g(k) > k^*$ (guaranteed, since $\lim_{k\uparrow\breve{k}} g(k) = \breve{k} > k^*$), the sequence $\{g^{(n)}(k)\}$ crosses k^* at some n^*: $k_{n^*+1} \leq k^* \leq k_{n^*}$. This implies that $v_{n^*} \leq \tilde{v}$, just as in the proof of Theorem 2. Since $\sigma > \underline{\sigma}$, recall that we have $h_d(v') < v'$ for all $v' < \bar{v}$. Thus, it follows that $v_1 < \tilde{v}$, or equivalently $\frac{V(g(k))}{(g(k))^{1-\sigma}} < \tilde{v}$.

Starting from the equilibrium condition

$$u(Rk - g(k)) + \beta\delta V(g(k)) \geq u(Rk - \breve{k}) + \beta\delta V(\breve{k})$$

and using that $V(\breve{k}) \geq \bar{V}(\breve{k})$ and $\tilde{v}\,(g(k))^{1-\sigma} > V(g(k))$ we obtain

$$u(Rk - g(k)) + \beta\delta\tilde{v}(g(k))^{1-\sigma} > u(Rk - \breve{k}) + \beta\delta\bar{v}(\breve{k})^{1-\sigma}.$$

Taking the limit $k \uparrow \breve{k}$,

$$u(R\breve{k} - \breve{k}) + \beta\delta\tilde{v}(\breve{k})^{1-\sigma} > u(R\breve{k} - \breve{k}) + \beta\delta\bar{v}(\breve{k})^{1-\sigma},$$

implying $\tilde{v} > \bar{v}$, which contradicts the property that $\tilde{v} < \bar{v}$.

Case 2: $g(\breve{k}) < \breve{k}$**.** The proof of Theorem 1 part (a) for this case applies here as well without change.

Therefore, combining both cases, we obtain that $g(k) < k$ for all $k \geq \tilde{k}$ as in the Theorem 1 part (a).

The proof that for any $k_0 \geq \tilde{k}$ we have $\lim_{n\to\infty} k_n = k_\infty$ is a steady state and $k_\infty < \tilde{k}$ is the same as that for Theorem 1 part (a), it applies here for $R = R^*$ without change.

No Reversal Principle Part (b). We now show that there cannot be reversal from strict savings to weak dissavings. We again proceed as in the proof of Theorem 1 part (b): assume by contradiction that $\tilde{k} < \hat{k}$ and $g(\tilde{k}) > \tilde{k}$ and $g(\hat{k}) \leq \hat{k}$. Define $\breve{k}$ as in (14). Then $\breve{k}$ must be a steady-state; the proof of this fact is unchanged from the proof of Theorem 1 part (b).

Consider the sequence $k_0 = \tilde{k}$ and $k_{n+1} = g(k_n) \geq k_n$. Then $\lim_{n\to\infty} k_n \leq \breve{k}$. Because $g(k) > k$ for all $k \in [\tilde{k}, \breve{k})$, we must have $\lim_{n\to\infty} k_n = \breve{k}$. There are two cases.

Case 1. $k_n = \breve{k}$ **for high enough** n**.** Let n^* be such that $k_{n^*} < k_{n^*+1} = \breve{k}$. This implies $v_{n^*} \leq h_s(v_{n^*+1}) = h_s(\bar{v}) = \bar{v}$. This contradicts Lemma 3, which implies $v_{n^*} > \bar{v}$.

Case 2: $k_n < \breve{k}$ **for all** $n = 0, 1, \ldots$ We have that $\{k_n\}$ is a strictly increasing sequence. Recall that because $\sigma \geq \underline{\sigma}$ we have $h_s(v') < v'$ for all $v' > \bar{v}$ and h_s is increasing. Thus,

$$v_0 \leq h_s^{(n)}(v^n) \leq h_s^{(n)}(v^{FB}),$$

for all n, where $v^{FB} < \infty$ as defined in the proof of Theorem 3. Taking the limit $n \to \infty$, we obtain $v_0 \leq \bar{v}$. This then contradicts Lemma 3, which implies $v_0 > \bar{v}$.

Therefore, combining both cases, we obtain that $g(k) > k$ for all $k \geq \tilde{k}$ as in the No Reversal result part (b).

The proof that for any $k_0 \geq \tilde{k}$ we have $\lim_{n\to\infty} k_n = \infty$ is the same as that for Theorem 1 part (b), it applies here for $R = R^*$ without change.

Uniqueness. Armed with these extensions of the No Reversal Principle, we now prove our uniqueness result. Assume an equilibrium (g, V) and assume towards a contradiction that $g(\tilde{k}) \neq \tilde{k}$ for some $\tilde{k} \geq \underline{k}$.

If $g(\tilde{k}) < \tilde{k}$ at some $\tilde{k} > \underline{k}$, then by the No Reversal Principle part (a) we have $g(k) < k$ for all $k \geq \tilde{k}$. We can then proceed exactly as in the proof of Theorem 2 to obtain a contradiction, the proof applies without change to $R = R^*$.

If $g(\tilde{k}) > \tilde{k}$ at some $\tilde{k} \geq \underline{k}$, then by the No Reversal Principle part (b) we have $g(k) > k$ for all $k > \tilde{k}$. We can then proceed exactly as in the proof of Theorem 3 to obtain a contradiction, the proof applies without change to $R = R^*$.

Therefore, by contradiction, we have established that $g(k) = k$ for all $k \geq \underline{k}$.

G Proof of Existence: Theorem 5

We consider the nontrivial case with $R < R^*$ and $\underline{k} > 0$. Our proof adapts the continuous-time analysis in Cao and Werning (2016) to construct a Markov equilibrium.

Setting the Stage: Self-Preserving Properties. For each $k^* > \underline{k}$, define $\mathcal{B}(k^*)$ to be the set of policy and value functions (g, V, W) defined over $[\underline{k}, k^*)$ with the following properties:

P1. $g(\underline{k}) = \underline{k}$ and $\underline{k} \leq g(k) < k$ for $\underline{k} < k < k^*$;

P2. g and V are upper semi-continuous (u.s.c) and W is continuous;

P3. g, V and W satisfy

$$V(k) = u(Rk - g(k)) + \delta V(g(k)) \qquad \forall k \in [\underline{k}, k^*) \tag{34}$$

$$W(k) = u(Rk - g(k)) + \beta\delta V(g(k)) \geq u(Rk - k') + \beta\delta V(k') \qquad \forall k \in [\underline{k}, k^*), k' \in [\underline{k}, k]; \tag{35}$$

P4. W satisfies

$$W(k) > \bar{W}(k) \equiv \left(1 + \frac{\beta\delta}{1-\delta}\right) u((R-1)k) \qquad k \in (\underline{k}, k^*). \tag{36}$$

Proof Sketch and Roadmap. Let us first lay out the main structure of the arguments, without all the detailed calculations.

We first construct functions satisfying P1-P4 on some small enough interval around the lower bound, producing $(g_0, V_0, W_0) \in \mathcal{B}(k_0^*)$ for some $k_0^* > \underline{k}$.

Second, we provide an algorithm that takes any $(g, V, W) \in \mathcal{B}(k^*)$ with

$$\lim_{k \uparrow k^*} W(k) > \bar{W}(k^*)$$

and constructs an extension $(\tilde{g}, \tilde{V}, \tilde{W}) \in \mathcal{B}(\tilde{k}^*)$ where $\tilde{k}^* > k^*$, with $(\tilde{g}, \tilde{V}, \tilde{W})$ coinciding with (g, V, W) over $k \in [\underline{k}, k^*)$. The extension works by guaranteeing that over a small enough extension interval $k \in [k^*, \tilde{k}^*)$ the optimum $\tilde{g}(k)$ falls below k^*, and we satisfy P1–P4. This requires various detailed calculations and bounds, to take a small enough step.

Next we construct a candidate equilibrium by iterating on this extension. Starting from $(g_0, V_0, W_0) \in \mathcal{B}(k_0^*)$ we can extend the functions over a sequence of expanding of intervals. The end result of this iterative process is what we call our extension algorithm There are two cases to consider. In the first case, the algorithm can be applied indefinitely and the intervals expand without bound covering all of $[\underline{k}, \infty)$. This provides a candidate equilibrium directly.

In the second case, the algorithm must stop after a finite number of iterations, if we reach a solution with $\lim_{k \uparrow k^*} W(k^*) = \bar{W}(k^*)$, or it continues indefinitely but the intervals converge. Under either possibility we are left with a bounded interval $[\underline{k}, k^*)$ for some $k^* < \infty$. Crucially, we show that $\lim_{k \uparrow k^*} W(k) = \bar{W}(k^*)$. We then restart the algorithm *as if* k^* were the lower bound, instead of $\underline{k}$. Let $\underline{k}_0^* = \underline{k}$ and $\underline{k}_1^* = k^*$.

Once we replay our algorithm starting from $\underline{k}_1^*$, it may produce again a bounded interval $[\underline{k}_1^*, \underline{k}_2^*)$; it it does, we restart the extension algorithm using $\underline{k}_2^*$ as the lower bound. Repeating this as many times as necessary, or indefinitely, we can cover all of $[\underline{k}, \infty)$, i.e. this procedure cannot get stuck.[31] This produces our candidate equilibrium (g, V).

Finally, we verify that our candidate is, indeed, an equilibrium. Fortunately, because of P1-P4, there is only one condition left to verify: $g(k) \in \arg\max_{k' \geq \underline{k}} \{u(Rk - k') + \beta\delta V(k')\}$. Recall that for $k \in (\underline{k}, k^*)$ P3 imposed the weaker condition: $g(k) \in \arg\max_{k' \in [\underline{k}, k]} \{u(Rk - k') + \beta\delta V(k')\}$. To show that our construction satisfies the stronger condition, our proof uses that, given single-crossing, local incentive compatibility implies global incentive compatibility. By local incentive compatibility we mean here that no agent k prefers to imitate choices $g(\tilde{k})$ made by neighbors $\tilde{k}$ on both sides of k.

In the interior of the intervals generated by our algorithm, local incentive compatibility is guaranteed by $g(k) < k$ (property P1). At points $\{\underline{k}_n^*\}$ where the algorithm is restarted local incentive compatibility relies, among other things, crucially on the fact that our construction ensures that $\lim_{k \uparrow \underline{k}_n^*} W(k) = \bar{W}(\underline{k}_n^*)$. An agent at $\underline{k}_n^*$ is postulated to set $g(\underline{k}_n^*) = \underline{k}_n^*$ and obtain $W(\underline{k}_n^*) = \bar{W}(\underline{k}_n^*)$, just as if $\underline{k}_n^*$ acted as a binding lower bound. The fact that $\lim_{k \uparrow \underline{k}_n^*} W(k) = \bar{W}(\underline{k}_n^*)$ guarantees that this agent has no incentive to deviate to a nearby $k' < \underline{k}_n^*$; likewise, agents right below $\underline{k}_n^*$ have no incentive to deviate upwards to $\underline{k}_n^*$. If instead $\lim_{k \uparrow \underline{k}_n^*} W(k) \neq \bar{W}(\underline{k}_n^*)$ one of these two local incentives to deviate would be present, violating an equilibrium condition.

Having verified this that all the equilibrium conditions are met, this completes the construction of a Markov equilibrium.

[31] Under isoelastic utility we can exploit homogeneity and simply extend the construction past k^* by using blown up replicas of the single solution found over $[\underline{k}, k^*)$, but our argument does not require isoelastic utility.

Construction of Initial Function satisfying P1-P4. First, let us construct $(g_0, V_0, W_0) \in \mathcal{B}(k_0^*)$ for some $k_0^* > \underline{k}$. Set $g_0(k) = \underline{k}$ for all $k \geq \underline{k}$ and

$$V_0(k) = u(Rk - \underline{k}) + \frac{\delta}{1-\delta} u((R-1)\underline{k}),$$
$$W_0(k) = u(Rk - \underline{k}) + \frac{\beta\delta}{1-\delta} u((R-1)\underline{k}).$$

It is immediate that P1 and P2 are satisfied. By concavity of u, for k such that the first-order condition for $k' = \underline{k}$

$$u'(Rk - \underline{k}) \geq \beta\delta R u'((R-1)\underline{k})$$

holds P3 is satisfied. Because $R < R^* < \frac{1}{\beta\delta}$, the first order condition is satisfied for $k \in [\underline{k}, k_0^*)$ for k_0^* sufficiently close to $\underline{k}$. Also, $W_0(\underline{k}) = \bar{W}(\underline{k})$ and

$$W_0'(\underline{k}) = Ru'((R-1)\underline{k}) > \bar{W}'(\underline{k}) = \left(1 + \frac{\beta\delta}{1-\delta}\right)(R-1)u'((R-1)\underline{k})$$

since $R < R^*$. Therefore, for k_0^* sufficiently close to $\underline{k}$ then P4 is satisfied. We chose k_0^* such that the aforementioned conditions are satisfied and (g_0, V_0, W_0) as the restriction of the corresponding functions defined above to $[\underline{k}, k_0^*)$.

Building the Extension Algorithm. Assume $(g, V, W) \in \mathcal{B}(\tilde{k})$. Then (35) implies $W(k)$ is increasing in k. Therefore, $\lim_{k \uparrow \tilde{k}} W(\tilde{k})$ exists and because of (36), $\lim_{k \uparrow \tilde{k}} W(k) \geq \bar{W}(k)$. We show that if this inequality is strict then we can always extend the system to a larger interval $[\underline{k}, \tilde{\tilde{k}})$, i.e., an element in $\mathcal{B}(\tilde{\tilde{k}})$ as follows. Let

$$\Delta = \inf \left\{ W(k) - \bar{W}(k) : \frac{\underline{k} + k_0^*}{2} \leq k < \tilde{k} \right\}.$$

It follows that $\Delta > 0$ because $\lim_{k \uparrow \tilde{k}} W(k) > \bar{W}(\tilde{k})$, P4 holds and both W and $\bar{W}$ are continuous.

We chose $\epsilon < \min \left\{ \frac{\tilde{k}}{2}, \tilde{k} - \frac{\underline{k} + k_0^*}{2} \right\}$ sufficiently small such that:

$$(1 - \delta(1-\beta))u((R-1)k) - u(Rk - k') + \delta(1-\beta)u((R-1)k') > -(1-\delta)\frac{\Delta}{4}$$

for all $\tilde{k} - \epsilon \leq k' \leq k < \tilde{k}$.

Now for k, k' such that $\tilde{k} - \epsilon \le k' \le k < \tilde{k}$, using (34) and (35), we have

$$
\begin{aligned}
& W(k) - \big(u(Rk-k') + \beta\delta V(k')\big) \\
& = W(k) - \big(u(Rk-k') + \beta\delta(u(Rk'-g(k')) + \delta V(g(k')))\big) \\
& = W(k) - \delta W(k') - \big(u(Rk-k') - \delta(1-\beta)u(Rk'-g(k'))\big) \\
& \ge (1-\delta)W(k) - \big(u(Rk-k') - \delta(1-\beta)u(Rk'-k')\big)\,, \qquad (37)
\end{aligned}
$$

where the last inequality is due to W being monotone, $k' \le k$ and $g(k') \le k'$. Let $\eta(k,k')$ denote the last expression:

$$\eta(k,k') \equiv (1-\delta)W(k) - \big(u(Rk-k') - \delta(1-\beta)u(Rk'-k')\big)\,.$$

We write the last inequality as

$$
\begin{aligned}
& W(k) - \big(u(Rk-k') + \beta\delta V(k')\big) \\
& \ge \eta(k,k') \\
& = (1-\delta)(W(k) - \bar{W}(k)) \\
& + \big\{(1-\delta(1-\beta))u((R-1)k) - u(Rk-k') + \delta(1-\beta)u((R-1)k')\big\} \\
& > (1-\delta)\frac{\Delta}{2} - (1-\delta)\frac{\Delta}{4} = (1-\delta)\frac{\Delta}{4}. \qquad (38)
\end{aligned}
$$

We define $\tilde{W}$ over $[\underline{k}, \infty)$ as:

$$\tilde{W}(k) = \max_{k' \in [\underline{k}, \tilde{k}-\epsilon]} \{u(Rk-k') + \beta\delta V(k')\}.$$

It is easy to see that for $k_1 < k_2$,

$$\min_{k' \in [\underline{k}, \tilde{k}-\epsilon]} \big\{u(Rk_1 - k') - u(Rk_2 - k')\big\} + \tilde{W}(k_2) \le \tilde{W}(k_1) \le \tilde{W}(k_2).$$

Therefore $\tilde{W}$ is continuous. Because of (38), $W(k) = \tilde{W}(k)$ for all $k \in (\tilde{k} - \epsilon, \tilde{k})$. So, $\lim_{k \uparrow \tilde{k}} W(k) = \tilde{W}(\tilde{k})$.

We define the following function $\Lambda_1(\tilde{k}, \Delta)$ as the solution x to the following equation (we set $\Lambda_1 = \infty$ if a solution does not exist):

$$u(Rx - \tilde{k}) = u((R-1)\tilde{k}) + (1-\delta)\Delta.$$

It follows immediately that $\Lambda_1(\tilde{k}, \Delta) > \tilde{k}$. We use Λ_1 to define $\tilde{\tilde{k}}_1 > \tilde{k}$ as:

$$\tilde{\tilde{k}}_1 = \Lambda_1(\tilde{k}, \Delta).$$

Let $\tilde{\eta}$ denote the same expression for η with W being replaced by $\tilde{W}$. Then for $k \in [\tilde{k}, \tilde{\tilde{k}}_1]$:

$$\begin{aligned}\tilde{\eta}(k, \tilde{k}) &= (1-\delta)\tilde{W}(k) - \left(u(Rk - k') - \delta(1-\beta)u(Rk' - k')\right) \\ &\geq (1-\delta)\tilde{W}(\tilde{k}) - \left(u(Rk - k') - \delta(1-\beta)u(Rk' - k')\right) \\ &\geq (1-\delta)\tilde{W}(\tilde{k}) - \left(u(Rk - \tilde{k}) - \delta(1-\beta)u(R\tilde{k} - \tilde{k})\right), \\ &\geq (1-\delta)\tilde{W}(\tilde{k}) - \left(u(R\tilde{\tilde{k}}_1 - \tilde{k}) - \delta(1-\beta)u(R\tilde{k} - \tilde{k})\right),\end{aligned}$$

where the first inequality comes from the monotonicity of $\tilde{W}$ and the second inequality comes from $k' \geq \tilde{k}$, and the last inequality comes from $k \leq \tilde{\tilde{k}}_1$. From the definition of Δ, we have $\tilde{W}(\tilde{k}) \geq \bar{W}(\tilde{k}) + \Delta$. Therefore,

$$\tilde{\eta}(k, \tilde{k}) \geq (1-\delta)\Delta - \left(u(R\tilde{\tilde{k}}_1 - \tilde{k}) - u((R-1)\tilde{k})\right) = 0,$$

by the definition of $\tilde{\tilde{k}}_1$.

Similarly, we define the following function $\Lambda_2(\tilde{k}, \Delta)$ as the solution x to the following equation (we set $\Lambda_2 = \infty$ if a solution does not exist):

$$u(Rx - \frac{\tilde{k}}{2}) - u(R\tilde{k} - \frac{\tilde{k}}{2}) = (1-\delta)\frac{\Delta}{4}.$$

It follows immediately that $\Lambda_2(\tilde{k}, \Delta) > \tilde{k}$. We use Λ_2 to define $\tilde{\tilde{k}}_2 > \tilde{k}$ as:

$$\tilde{\tilde{k}}_2 = \Lambda_2(\tilde{k}, \Delta).$$

Because u is increasing and concave and $\epsilon < \frac{\tilde{k}}{2}$, for all $k \in [\tilde{k}, \tilde{\tilde{k}}]$ and $k' \in [\tilde{k} - \epsilon, \tilde{k}]$:

$$u(Rk - k') - u(R\tilde{k} - k') < u(R\tilde{\tilde{k}}_2 - \frac{\tilde{k}}{2}) - u(R\tilde{k} - \frac{\tilde{k}}{2}) = (1-\delta)\frac{\Delta}{4}.$$

Let

$$\tilde{\tilde{k}} = \sup\left\{\tilde{k} \leq k \leq \min\{\tilde{\tilde{k}}_1, \tilde{\tilde{k}}_2\} : \tilde{W}(k') > \bar{W}(k') \forall k' \in [\tilde{k}, k]\right\},$$

Because $\tilde{W}(\tilde{k}) = \lim_{k \uparrow \tilde{k}} W(k) > \bar{W}(\tilde{k})$ and $\tilde{W}$ is continuous, we have $\tilde{\tilde{k}} > \tilde{k}$, and

$$\tilde{W}(k) > \bar{W}(k) \forall k \in [\tilde{k}, \tilde{\tilde{k}}). \tag{39}$$

Notice also that if $\tilde{\tilde{k}} < \min\{\tilde{\tilde{k}}_1, \tilde{\tilde{k}}_2\}$ then $\tilde{W}(\tilde{\tilde{k}}) = \bar{W}(\tilde{\tilde{k}})$.

Now set $W(k) = \tilde{W}(k)$ for $k \in [\tilde{k}, \tilde{\tilde{k}})$ and

$$G(k) = \arg\max_{k' \in [\underline{k}, \tilde{k} - \epsilon]} u(Rk - k') + \beta\delta V(k')$$

and

$$g(k) = \max G(k)$$

(because V is u.s.c G is non-empty and compact, and consequently g is uniquely defined) and

$$V(k) = u(Rk - g(k)) + \delta V(g(k)).$$

Verifying the Extension Satisfies P1-P4. We now show that P1-P4 are satisfied for (g, V, W) over $[\underline{k}, \tilde{k})$.

It is immediate that P1 is satisfied. To show P2, we show that g is u.s.c., that is, if $k_n \to k$, and $\lim_{n\to\infty} g(k_n) = l$ then $l \leq g(k)$. To do so, we first show that $l \in G(k)$. Indeed, for each n, and $k' \in [\underline{k}, \tilde{k} - \epsilon]$:

$$u(Rk_n - g(k_n)) + \beta\delta V(g(k_n)) \geq u(Rg(k_n) - k') + \beta\delta V(k').$$

Taking the limit $n \to \infty$, and using u.s.c of V over $[\underline{k}, \tilde{k})$, we have

$$u(Rk - l) + \beta\delta V(l) \geq u(Rl - k') + \beta\delta V(k').$$

Therefore, $l \in G(k)$, and consequently, from the definition of g, $l \leq g(k)$ as desired. Thus g is u.s.c. In addition, W is continuous because $\tilde{W}$ is continuous. Now, using (34) and (35), we have

$$V(k) = \frac{1}{\beta}W(k) - \left(\frac{1}{\beta} - 1\right) u(Rk - g(k)).$$

Because W is continuous and g is u.s.c, V is also u.s.c. Thus, P2 is satisfied.

We now show that g, V, W satisfy P3 over $[\underline{k}, \tilde{\tilde{k}})$. Indeed, for $k \in [\tilde{k}, \tilde{\tilde{k}})$ and $\tilde{k} \leq k' \leq k$, again using (34) and (35) as in (37), we obtain:

$$W(k) - \big(u(Rk - k') + \beta\delta V(k')\big) \geq \eta(k, k') \geq \eta(k, \tilde{k}) > 0.$$

For $k' \in [\tilde{k} - \epsilon, \tilde{k}]$, using (38), we have:

$$\begin{aligned}
&W(k) - \big(u(Rk - k') + \beta\delta V(k')\big) \\
&\geq W(\tilde{k}) - \big(u(R\tilde{k} - k') + \beta\delta V(k')\big) - \big(u(Rk - k') - u(R\tilde{k} - k')\big) \\
&\geq (1 - \delta)\frac{\Delta}{4} - (1 - \delta)\frac{\Delta}{4} = 0.
\end{aligned}$$

For $k' \in [\underline{k}, \tilde{k} - \epsilon]$, $W(k) \geq (u(Rk - k') + \beta\delta V(k'))$ directly from the definition of W.

P4 is also satisfied because $W(k) = \tilde{W}(k) > \bar{W}(k)$ for $k < \tilde{\tilde{k}}$ and (39). Therefore, we have constructed a system with the desired properties P1-P4 over $[\underline{k}, \tilde{\tilde{k}})$.

Constructing the Candidate Equilibrium. Now, starting from the initial system in $\mathcal{B}(k_0^*)$, we construct the sequence of extensions for g, V, W over $[\underline{k}, k_n^*)$ with $k_0^* < k_1^* < \ldots$ as follows.

Given k_n^*, and the system g, V, W defined over $[\underline{k}, k_n^*)$ we calculate

$$\Delta_n = \inf\left\{W(k) - \bar{W}(k) : \frac{\underline{k} + k_0^*}{2} \le k < k_n^*\right\}.$$

If $\Delta_n = 0$ then we stop and set $k^* = k_n^*$. Because of P4, we have

$$\lim_{k \uparrow k^*} W(k) = \bar{W}(k^*). \tag{40}$$

If $\Delta_n > 0$, then we set $\tilde{k} = k_n^*$ and use the Extension Algorithm to extend the system to $[\underline{k}, \tilde{\tilde{k}})$ where $\tilde{\tilde{k}}$ is defined as in the Extension Algorithm.

If we never stop, i.e. $\Delta_n > 0$ for all $n > 0$, then there are two possibilities: i. $\lim_{n\to\infty} k_n^* = +\infty$ or ii. $\lim_{n\to\infty} k_n^* = k^* < +\infty$. In this latter case, we show by contradiction that (40) holds. Otherwise,

$$\Delta^* = \inf\left\{W(k) - \bar{W}(k) : \frac{\underline{k} + k_0^*}{2} \le k < k^*\right\} > 0.$$

From the definition of Δ^* and Δ_n , it is easy to see that $\Delta_n \ge \Delta^*$ for all $n \ge 0$. Because $\Delta_{n+1} > 0$, from the Extension Algorithm, we have:

$$k_{n+1}^* = \min\left\{\Lambda_1(k_n^*, \Delta_n), \Lambda_2(k_n^*, \Delta_n)\right\}.$$

For $i \in \{1, 2\}$, let

$$d_i = \min_{\tilde{k} \in [\underline{k}, k^*]} \left\{\Lambda_i(\tilde{k}, \Delta^*) - \tilde{k}\right\} > 0.$$

Then

$$k_{n+1}^* - k_n^* \ge \min\{d_1, d_2\} > 0$$

for all $n \ge 0$. Therefore $\lim_{n\to\infty} k_n^* = \infty$. A contradiction.

To sum up, there are two cases and we construct a candidate equilibrium in each case as follows.

Case 1: We have $\lim_{n\to\infty} k_n^* = \infty$ and thus have constructed (g, V, W) satisfying P1-P4 over $[\underline{k}, \infty)$. We define the candidate equilibrium as (g, V).

Case 2: We reach a finite $k^* < \infty$ and $\lim_{k\to k^*} W(k) = \bar{W}(k^*)$. Now we reset $\underline{k} = k^*$ and restart the Extension Algorithm again.[32] For each $\underline{k}$ let $\Phi(\underline{k})$ denote k^* generated by the successive applications of the Extension Algorithm described above. Starting from $\underline{k}_0^* = \underline{k}$, consider the sequence $\{\underline{k}_n^*\}_{n=0}^{\infty}$ generated by Φ (the sequence has finite elements if

[32]Under isoelastic utility, we can simply use homogeneity to replicate the system. Indeed, let $x^* = \frac{k^*}{\underline{k}}$, we define the system over $\left[\underline{k}(x^*)^n, \underline{k}(x^*)^{n+1}\right)$ for $n = 1, 2, \ldots$ as:

$$\begin{aligned}&(g(k), V(k), W(k))\\ &= \left((x^*)^n g\left(k(x^*)^{-n}\right), (x^*)^{(1-\sigma)n} V\left(k(x^*)^{-n}\right), (x^*)^{(1-\sigma)n} W\left(k(x^*)^{-n}\right)\right).\end{aligned}$$

$\underline{k}_n^* = \infty$ for some n). We show by contradiction that $\lim_{n\to\infty} \underline{k}_n^* = \infty$. Assume the contrary:

$$\lim_{n\to\infty} \underline{k}_n^* = k^\infty < \infty.$$

Let $\check{k}(\underline{k})$ be defined by (we set $\check{k}(\underline{k}) = \infty$ if it does not exist):

$$u'(R\check{k} - \underline{k}) = \beta\delta R u'((R-1)\underline{k}).$$

It is easy to show that $\check{k}(\underline{k}) > \underline{k}$, because $R < R^* < \frac{1}{\beta\delta}$, and is continuous in $\underline{k}$. Therefore, there exists $\underline{l} < k^\infty$ such that $\check{k}(\underline{l}) > k^\infty$. There exits N such that $\underline{k}_n^* > \underline{l}$ for all $n \geq N$. Therefore $\check{k}(\underline{k}_n^*) > k^\infty > \underline{k}_{n+1}^* = \Phi(\underline{k}_n^*)$ for all $n \geq N$. By the Extension Algorithm, it means that $W_0^n(\underline{k}_{n+1}^*) = \bar{W}(\underline{k}_{n+1}^*)$, where W_0^n defined over $[\underline{k}_n^*, \infty)$ by:

$$W_0^n(k) = u(Rk - \underline{k}_n^*) + \frac{\beta\delta}{1-\delta} u((R-1)\underline{k}_n^*).$$

By the Mean Value Theorem, there exist $\hat{k}_n, \hat{\hat{k}}_{n+1} \in [\underline{k}_n^*, \underline{k}_{n+1}^*]$ such that

$$\begin{aligned}
& Ru'(R\hat{k}_n - \underline{k}_n^*) = \frac{d}{dk} W_0^n(\hat{k}_n) \\
&= \frac{W_0^n(\underline{k}_{n+1}^*) - W_0^n(\underline{k}_n^*)}{\underline{k}_{n+1}^* - \underline{k}_n^*} \\
&= \frac{\bar{W}(\underline{k}_{n+1}^*) - \bar{W}(\underline{k}_n^*)}{\underline{k}_{n+1}^* - \underline{k}_n^*} \\
&= \frac{d}{dk} \bar{W}(\hat{\hat{k}}_n) = (1 + \frac{\beta\delta}{1-\delta})(R-1)u'((R-1)\hat{\hat{k}}_n).
\end{aligned} \tag{41}$$

Taking the limit $n \to \infty$ and using the Squeeze Theorem, we have $\hat{k}_n, \hat{\hat{k}}_n \to k^\infty$. Therefore, (41) implies

$$Ru'((R-1)k^\infty) = (1 + \frac{\beta\delta}{1-\delta})(R-1)u'((R-1)k^\infty)$$

or $R = R^*$, which contradicts the assumption that $R < R^*$.

Therefore, $\lim_{n\to\infty} \underline{k}_n^* = \infty$ and our extensions cover the whole half interval $[\underline{k}, \infty)$.

Adapting this proof, we can also show that there exists a finite number (or zero) of steady-states in each finite interval.

Verifying Candidate is an Equilibrium. We verify that the candidate equilibrium above is indeed a Markov equilibrium. That is, given $k \geq \underline{k}$, for any $k' \geq \underline{k}$,

$$W(k) = u(Rk - g(k)) + \beta\delta V(g(k)) \geq u(Rk - k') + \beta\delta V(k'). \tag{42}$$

First, we notice that in Case 2, around a steady-state $\underline{k}^*$, there exists an interval $[\underline{k}^* - \epsilon^*, \underline{k}^* + \epsilon^*]$ such that $\underline{k}^*$ prefers $g(\underline{k}^*) = \underline{k}^*$ to any other $g(k')$ for $k' \in [\underline{k}^* - \epsilon^*, \underline{k}^* + \epsilon^*]$.

Indeed, for $k' \in [\underline{k}^* - \epsilon^*, \underline{k}^*)$, and $k' < k'' < \underline{k}^*$, we have:

$$W(k'') \geq u(Rk'' - g(k')) + \beta\delta V(g(k')).$$

Taking the limit $k'' \uparrow k^*$ and using the continuity of W, we obtain:

$$W(\underline{k}^*) = \bar{W}(\underline{k}^*) \geq u(R\underline{k}^* - g(k')) + \beta\delta V(g(k')).$$

Now for $k' \in (\underline{k}^*, \underline{k}^* + \epsilon^*]$, our construction implies that $g(k') = \underline{k}^*$ for ϵ^* sufficient small. Therefore, because $R < R^* < \frac{1}{\beta\delta}$, direct calculations imply that:

$$\begin{aligned}
W(\underline{k}^*) &= \bar{W}(\underline{k}^*) \\
&\geq u(R\underline{k}^* - k') + \beta\delta u(Rk' - \underline{k}^*) + \frac{\beta\delta^2}{1-\delta} u((R-1)\underline{k}^*) \\
&= u(R\underline{k}^* - k') + \beta\delta V(k').
\end{aligned}$$

In addition, we show that any $k' \in [\underline{k}^* - \epsilon^*, \underline{k}^*)$ prefers $g(k')$ to $\underline{k}^*$. Because $R < R^*$, by Lemma 4 and by choosing ϵ^* small

$$\bar{W}(k') = u(Rk' - k') + \beta\delta\bar{V}(k') \geq u(Rk' - \underline{k}^*) + \beta\delta\bar{V}(\underline{k}^*).$$

Therefore,

$$W(k') > \bar{W}(k') \geq u(Rk' - \underline{k}^*) + \beta\delta\bar{V}(\underline{k}^*) = u(Rk' - \underline{k}^*) + \beta\delta V(\underline{k}^*).$$

Now we are in a position to show (42).

If $k' > k$, we create a chain

$$k = k_0 < k_1 < \cdots < k_N = k'$$

such that k_n prefers $g(k_n)$ to $g(k_{n+1})$:

$$u(Rk_n - g(k_n)) + \beta\delta V(g(k_n)) \geq u(Rk_n - g(k_{n+1})) + \beta\delta V(g(k_{n+1})).$$

By single-crossing, k_0 prefers $g(k_n)$ to $g(k_{n+1})$ for $n = 0, 1, \ldots, N-1$. Therefore, k_0 prefers $g(k_0)$ to $g(k_N)$, that is k prefers $g(k)$ to $g(k')$. In addition, because $g(k') \leq k'$ and P3 in the definition of $\mathcal{B}$, k' prefers $g(k')$ to k'. So, again by single-crossing k prefers $g(k')$ to k'. Thus k prefers $g(k)$ to k', that is (42) holds.

To create the chain, we consider two cases:

Case A: If there is no steady-state between, or at, k and k'. Then because g is u.s.c. there exists $\epsilon > 0$ such that $g(l) < l - \epsilon$ for any $l \in [k, k']$. We choose N such that $\frac{k'-k}{N} < \epsilon$ and

$$k_n = k + n\frac{k'-k}{N}.$$

So $g(k_n) \leq g(k_{n+1}) < k_n$. By P3, k_n prefers $g(k_n)$ to $g(k_{n+1})$.

Case B: If there are steady states between k and k' (as shown above there can only be a finite number of them):

$$k \leq \underline{k}_r^* < \cdots < \underline{k}_s^* \leq k'.$$

We initiate the chain with points $\{\underline{k}_m^* - \epsilon_m^*, \underline{k}_m^*, \underline{k}_m^* + \epsilon_m^*\}_{m=r}^s$, where ϵ_m^* defined for each steady-state $\underline{k}_m^*$ as above. For resulting intervals that do not contain a steady-state, such as $[\underline{k}_m^* + \epsilon_m^*, \underline{k}_{m+1}^* - \epsilon_{m+1}^*]$, we use the chain construction in Case A to add points in between.

If $k' \leq k$. Again there are two cases. First, if there is no steady-state between k and k', then (42) is a direct application of P3. Second, if there are steady-states:

$$\underline{k}_r^* \leq k' < \underline{k}_{r+1}^* < \cdots < \underline{k}_s^* \leq k.$$

From P3 , we have

$$W(k'') \geq u(Rk'' - k') + \beta\delta V(k')$$

for $k' < k'' < \underline{k}_{r+1}^*$. Taking the limit $k'' \uparrow \underline{k}_{r+1}^*$ and using the continuity of W, we obtain that $\underline{k}_{r+1}^*$ prefers $\underline{k}_{r+1}^*$ to k'. Therefore by single-crossing, $\underline{k}_{r+2}^*$ prefers $\underline{k}_{r+1}^*$ to k'. In addition, $\underline{k}_{r+2}^*$ prefers $\underline{k}_{r+2}^*$ to $\underline{k}_{r+1}^*$. Therefore, $\underline{k}_{r+2}^*$ prefers $\underline{k}_{r+2}^*$ to k'. Keep iterating, we obtain k prefers $\underline{k}_s^*$ to k'. Lastly, from P3, k prefers $g(k)$ to $\underline{k}_s^*$. Thus k prefers $g(k)$ to k' as desired.