A Shadow Rate New Keynesian Model*

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Abstract

We propose a tractable and coherent framework that captures both conventional and unconventional monetary policies with the shadow fed funds rate. Empirically, we document the shadow rate's resemblance to an overall financial conditions index, various private interest rates, the Fed's balance sheet, and the Taylor rule. Theoretically, we demonstrate the impact of unconventional policies, such as QE and lending facilities, on the economy is identical to that of a negative shadow rate, making the latter a useful summary statistic for these policies. Our model generates the data-consistent result: a negative supply shock is always contractionary. It also salvages the New Keynesian model from the zero lower bound induced structural break.

Keywords: shadow rate, New Keynesian model, unconventional monetary policy, zero lower bound, QE, lending facilities

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1 Introduction

The zero lower bound (ZLB) poses a major issue for advanced economies and consequently economic research. It invalidates the traditional monetary policy tool because central banks are unable to further lower policy rates. Subsequently, central banks around the world have introduced unconventional policy tools such as large-scale asset purchases (or QE). How economic models accommodate the ZLB and unconventional monetary policy has become the new challenge for economic research. This paper proposes a novel New Keynesian (NK) model with the shadow rate to address this issue.

Policy makers and economists have documented empirically that the conventional and unconventional monetary policies work in a similar fashion. Yellen (2016), Reifschneider (2016) and Wu and Xia (2016) focus on the US economy, while Mouabbi and Sahuc (2017) study the Euro area. Powell (2013) assesses cross-border effects, and Blanchard (2016) investigates their transmission mechanisms. Belongia and Ireland (2017) establish a similar link between nominal GDP and monetary aggregates with or without the ZLB.

The goal of this paper is to echo the empirical findings and propose a coherent and tractable framework to summarize both the conventional and unconventional policies in the NK model. One prominent tool that empirical studies and policy analyses have widely adopted is the shadow rate of Wu and Xia (2016).[1] The shadow rate is the federal funds rate when the ZLB is not binding; otherwise, it is negative to account for unconventional policy tools. However, the question is how a negative interest rate can be mapped into an equilibrium concept. We warrant such a choice by (1) microfounding the negative shadow rate with unconventional policy tools: in a microfounded model, the central bank implements a conventional Taylor rule during normal times, and some major unconventional monetary policy when the ZLB binds. We then show unconventional monetary policy has identical

[1] For empirical studies, see, Basu and Bundick (2012), Nikolsko-Rzhevskyy et al. (2014), Aizenman et al. (2016), and Aastveit et al. (2017). For policy analyses, see Altig (2014) and Hakkio and Kahn (2014), and for media discussions, see The Wall Street Journal (2014), Bloomberg News (2016), Bloomberg Businessweek (2014), Forbes (2015), and Business Insider (2016).

effects on economic quantities as if the policy were implemented through a negative interest rate. Therefore, a negative shadow rate is not an actual policy instrument, but rather, it can be perceived as a summary statistic for unconventional monetary policy mapped into the interest rate domain. (2) We present new evidence to demonstrate the empirical relevance of the shadow rate, and (3) show that our model produces intuitive and sensible economic implications.

Our new shadow rate New Keynesian model (SRNKM) proposes the shadow rate as a tractable summary for all monetary policy tools, conventional or unconventional, allowing the model to remain linear without a ZLB-induced structural break. The shadow rate replaces the policy rate entering the IS curve. The ZLB on the Taylor rule is removed, which becomes a shadow rate Taylor rule. The Phillips curve stays the same. During normal times, this model is the same as the standard New Keynesian model. However, monetary policy remains active in our model when the ZLB prevails, which is not the case in the standard model.

We investigate new empirical evidence to establish the relevance of the shadow rate and validate our new model. First, at the ZLB, the shadow rate comoves almost perfectly with an overall financial conditions index and various private interest rates. They are the relevant indicators that drive agents' behavior and are the channel through which unconventional monetary policy propagates into the overall economy. Second, the shadow rate is highly correlated with the Fed's balance sheet, widely accepted as one summary for unconventional monetary policy, with the correlation being -0.94 throughout the QE phase. Third, the shadow rate follows the same Taylor rule as the fed funds rate did prior to the ZLB.

The standard NK model is associated with some distinctive modeling implications at the ZLB, some of which are counterfactual or puzzling. First, in such a model, a negative supply shock stimulates the economy at the ZLB. In contrast to this model implication, empirical evidence from Wieland (2015) and Garín et al. (2016) demonstrate the sign of output response is the same as the sign of the shock during normal times and at the ZLB. This conterfactual implication of the standard model is due to the lack of policy interventions at the ZLB. Our model restores the data-consistent implication by introducing unconventional monetary policy through the shadow rate. A related issue is the size of the government-spending multiplier. In a standard model without unconventional monetary policy, this multiplier is much larger at the ZLB. This larger multiplier also disappears in our model.

We then formalize the SRNKM with agents' optimization problems: at the ZLB, a negative shadow interest rate is not the actual borrowing or lending rate firms and households face, nor does the Fed set the shadow rate directly. But rather, the Fed monitors and targets the shadow rate, which is determined by the shadow rate Taylor rule, and achieves the target through some major unconventional policy.

The first example implements the negative shadow rate through QE programs. The central bank purchases bonds to lower their yields without changing the policy rate, which works by reducing the risk premium, motivated by the empirical research; see, for example, Hamilton and Wu (2012) and Gagnon et al. (2011). We demonstrate the QE purchases can be summarized equivalently by the shadow rate in our model, providing one microfoundation for the shadow rate IS curve. To achieve this equivalence, the model requires a linear relationship between log bond holdings by the Fed and the shadow rate. We verify this relationship in the data, with the correlation between these two variables being -0.92.

Second, we map lending facilities, which inject liquidity into the economy, into the shadow rate framework. The primary example of this policy is the Federal Reserve's Term Asset-Backed Securities Loan Facility. We model lending facilities by allowing the government to extend extra credit directly to the private sector; that is, the government can vary the loan-to-value ratio the borrowers face as a policy tool. The lending facilities are coupled with a tax policy on interest rate payments, which, according to Waller (2016) of the St. Louis Fed, is the nature of the recent negative interest rate policy in Europe and Japan. We then establish an equivalence between the shadow rate and the lending facilities – tax policy channel, which constitutes another microfoundation for the shadow rate IS curve.

Although much of the models are presented in the linearized form, the usefulness of the shadow rate goes beyond linearity. We demonstrate this point with the lending facilities – tax policy channel, where the equivalence is also established without linearization. Whether or not the model is linearized, the common theme is that the shadow rate serves as a summary statistic for various unconventional policy tools and does not introduce a structural break at the ZLB.

We extend our framework to accommodate the case where unconventional monetary policy is not fully active at the ZLB. We propose a simple smooth transition between normal times and the ZLB to capture the idea that zero is not a cutting edge, which is supported by the recent negative interest rate experience in Europe and Japan. The smooth transition allows us to further apply a Taylor series expansion to obtain a linear approximation for the extension where unconventional monetary policy is partially active. What is interesting is that it does not necessarily require unconventional monetary policy to be fully active for this extension and our main model to coincide. A weaker condition to arrive at our main model is the ZLB occurs occasionally, which is one popular assumption in the ZLB literature.

The shadow rate also salvages the NK model from issues arising from the structural break introduced by the ZLB, which imposes one of the biggest challenges for solving and estimating these models. Methods proposed in the literature to address this issue either produce economically uncompelling implications or are extremely computationally demanding. This challenge does not go away after the economy lifts off from the ZLB because research relies on historical data. Our SRNKM proposes a compelling solution to this challenge. It does not incur a structural break at the ZLB whether we work with a linear or non-linear model. Therefore, it restores the traditional solution and estimation methods' validity.

The rest of the paper after a brief literature review proceeds as follows. Section 2 proposes a three-equation linear SRNKM. Subsequently, Sections 3 and 4 map QE and lending facilities into this model theoretically. Section 5 extends our main model to accommodate partially active unconventional monetary policy. Section 6 discusses quantitative analyses, and Section 7 concludes.

Related literature: Our paper contributes to the DSGE literature on unconventional monetary policy. Cúrdia and Woodford (2011), Chen et al. (2012), and Gertler and Karadi (2013) study asset purchases, that is, QE. Gertler and Karadi (2011), Williamson (2012), and Del Negro et al. (2016) evaluate central banks' liquidity provision, along the lines of lending facilities. McKay et al. (2014), Del Negro et al. (2015), and Kulish et al. (2016) focus on forward guidance. We model QE and lending facilities directly. Our paper also speaks to forward guidance in the sense that the shadow rate reflects changes in medium- or long-term yields due to forward guidance. A direct mapping between the two is in Wu and Xia (2016).

Our paper differs from the existing literature in the follow respects. First, we use the shadow rate to provide one coherent framework for the ZLB period as well as for normal times, whereas models in the literature are specifically targeted for the ZLB. Consequently, our framework provides a natural extension to models researchers developed prior to the ZLB, because the shadow rate is the same as the fed funds rate when the ZLB is not binding.

Second, rather than focus on a specific policy tool, we use the shadow rate as a summary for all unconventional monetary policy measures. Third, the shadow rate is not subject to a structural break at the ZLB, which makes the model tractable and alleviates numerical and computational issues.

2 A shadow rate New Keynesian model (SRNKM)

In this and the next two sections, we propose a novel SRNKM, which captures both the conventional interest rate rule and unconventional policy tools in a coherent and tractable way. This section presents the three-equation linear version of the model, and Sections 3 - 4 then microfound this model with two major unconventional policy tools: QE and lending facilities. Subsection 2.1 sets up the standard New Keynesian model and demonstrates its lack of unconventional monetary policy. Subsection 2.2 introduces a linear SRNKM to address the issue with the standard model. Subsections 2.3 - 2.4 illustrate the shadow rate's empirical relevance. We then discuss our model's economic implications in 2.5, and computational advantages in 2.6.

2.1 Standard NK model and its lack of unconventional monetary policy

The standard linear NK model (e.g., see Galí (2008)) consists of

$$y_t = -\frac{1}{\sigma}(r_t - \mathbb{E}_t\pi_{t+1} - r) + \mathbb{E}_t y_{t+1}, \tag{2.1}$$

$$\pi_t = \beta\mathbb{E}_t\pi_{t+1} + \kappa(y_t - y_t^n), \tag{2.2}$$

$$s_t = \phi_s s_{t-1} + (1-\phi_s)\left[\phi_y(y_t - y_t^n) + \phi_\pi\pi_t + s\right], \tag{2.3}$$

$$r_t = \max(0,\, s_t), \tag{2.4}$$

where $\mathbb{E}$ is the expectation operator, lowercase letters are logs, letters without t subscripts are either coefficients or steady-state values, and all the coefficients are positive. y_t is output, and y_t^n is potential output or equilibrium output under flexible prices. π_t is inflation, and r_t is the policy rate. Equation (2.1) is the NK IS curve describing demand as a decreasing function of the real interest rate $r_t - \mathbb{E}_t\pi_{t+1}$, where σ is the reciprocal of the intertemporal elasticity of substitution. Equation (2.2) is the NK Phillips curve, characterizing the aggregate supply, where β is the discount factor, and κ depends on the degree of nominal rigidity and other preference parameters. The shadow fed funds rate s_t follows a Taylor (1993) rule in (2.3). The policy rate equals the shadow rate during normal times, and it is 0 when the shadow rate is negative.

In this framework, unconventional monetary policy plays no role at the ZLB. Once the ZLB hits, $s_t < 0$, the policy rate $r_t = 0$, and monetary policy is completely inactive in the system; see (2.1).

Unconventional policy tools, such as QE, target interest rates at longer maturities, and potentially with default and liquidity risks. This link has been confirmed by empirical research; for example, see Hamilton and Wu (2012), Bauer and Rudebusch (2014), and Swanson (2017). To fix ideas, it is useful to iterate the IS curve (2.1) forward to see an example of how other interest rates can affect agents' behavior when the policy rate is at its ZLB:

$$\begin{aligned} y_t &= -\frac{1}{\sigma}\sum_{i=1}^{n}\mathbb{E}_t(r_{t+i-1}-\pi_{t+i}-r)+\mathbb{E}_t y_{t+n} \\ &= -\frac{1}{\sigma}nr_{t,t+n}-\frac{1}{\sigma}\sum_{i=1}^{n}\mathbb{E}_t(-\pi_{t+i}-r)+\mathbb{E}_t y_{t+n}. \end{aligned} \tag{2.5}$$

Under the expectations hypothesis, $r_{t,t+n}$ is the long term interest rate from t to $t+n$. In this framework, agents make decisions based on the long-term real interest rate. Even if the short-term interest rate is constrained, the central bank can still stimulate the economy by moving the long-term interest rate. However, this is not the case in the standard NK model: the central bank cannot act now or in the near future, because of the ZLB constraint. Moreover, in the future when the ZLB is no longer an issue, agents do not expect the central bank to react in retrospect to a shock that happened at time t, because the Taylor rule (2.3) prescribes the central bank to respond only to concurrent shocks.

2.2 Linear SRNKM

In this section, we propose a novel NK model with the shadow fed funds rate to address the standard NK model's lack of unconventional monetary policy. The shadow rate extends the fed funds rate's role as the stance of monetary policy to the ZLB. When the ZLB constrains the risk-free rate r_t, unconventional monetary policy targets other interest rates, for example, $r_{t,t+n}$ in (2.5)'s terminology. The shadow rate can captures this policy intervention, because it is extracted from the entire yield curve.

The three-equation linear SRNKM is defined as follows:

Definition 1 *The shadow rate New Keynesian model consists of the shadow rate IS curve*

$$y_t = -\frac{1}{\sigma}(s_t - \mathbb{E}_t \pi_{t+1} - s) + \mathbb{E}_t y_{t+1}, \tag{2.6}$$

New Keynesian Phillips curve (2.2), and the shadow rate Taylor rule (2.3).

The benefit of using the shadow rate rather than introducing an additional monetary instrument just for the ZLB, such as a second interest rate or volume of money or assets, is that our model extends the standard NK model to incorporate unconventional monetary policy without introducing a structural break. During normal times when $s_t > 0$, the model is the same as the standard model. The difference is that monetary policy is still active at the ZLB, capturing central banks' efforts through unconventional policy tools.

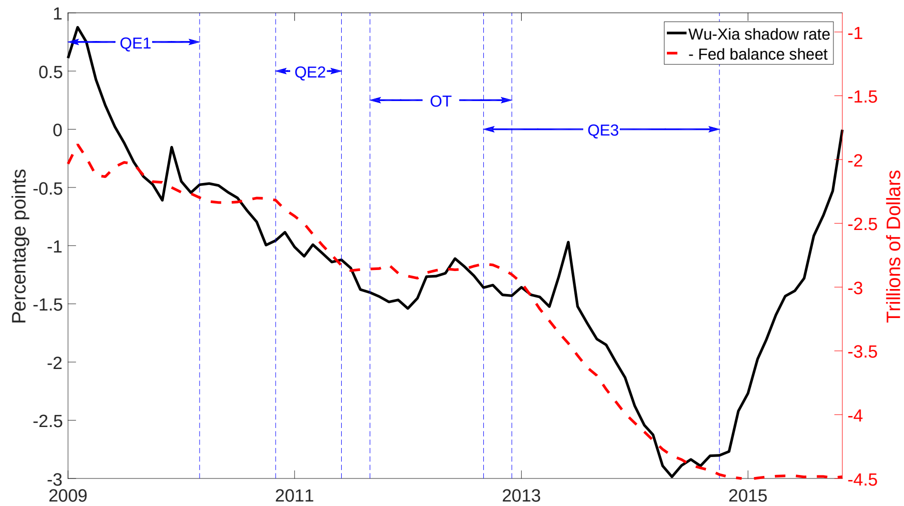
To further support the model, in Subsection 2.3, we document evidence that the shadow rate resembles the dynamics of overall financial conditions, various private interest rates, and the Fed's balance sheet. We formalize this link between the shadow rate and unconventional monetary policy theoretically in Sections 3 - 4 by microfounding (2.6) with two examples of unconventional policy tools. Note in our microfounded models, a negative shadow interest rate is not the actual borrowing or lending rate firms and households face. Rather, we propose to use it as a summary statistic for all the measures of the Fed's conventional and unconventional policies.

2.3 Shadow rate: a summary for unconventional monetary policy

The new IS curve in (2.6) uses the shadow rate to summarize the effect of unconventional monetary policy on the economy. This section assesses the empirical validity of this assumption.

First, we compare the shadow rate with unconventional monetary policy directly, for which a popular measure is the Federal Reserve's balance sheet; see Figure 1. The Fed's assets (in red) grow from about \$2 trillion in 2009 to about \$4.5 trillion by the end of

Figure 1: Shadow rate and Fed's balance sheet



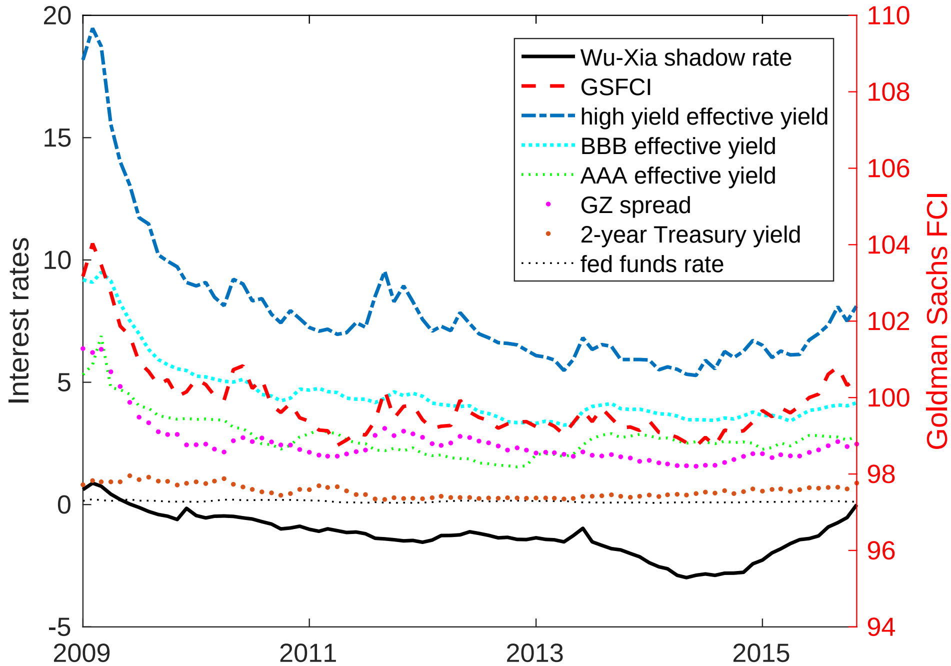
Notes: black solid line: the Wu-Xia shadow rate; red dashed line: the negative Fed's balance sheet. Left scale: interest rates in percentage points; right scale: negative Fed's balance sheet in trillions of dollars. QE1: the first round of QE from November 2008 to March 2010; QE2: the second round of QE from November 2010 to June 2011. OT: operation twist from September 2011 to December 2012. QE3: the third round of QE from September 2012 to October 2014. Data sources: Wu-Xia/Federal Reserve Bank of Atlanta and Federal Reserve Economic Data.

2014. The net expansion over this period reflects primarily QE. Meanwhile, the shadow rate drops from 0.6% to -3%. These two series are highly correlated with a correlation of -0.94 throughout the QE phase. The subsequent rise of the shadow rate is driven primarily by forward guidance, which is not reflected in the Fed's balance sheet.

Next, economic agents care about interest rates they face when they borrow or lend, or more broadly, the overall financial conditions, through which unconventional monetary policy transmits into the real economy. We investigate the comovement between the shadow rate and some key indicators. In Figure 2, the black line is Wu and Xia's (2016) shadow rate, and the red dashed line is the Goldman Sachs Financial Conditions Index (GSFCI) for the United States, which tracks broad financial markets including equity prices, the US dollar, Treasury yields, and credit spreads. The GSFCI and shadow rate have similar U-shaped dynamics, and their correlation is 0.8.

Moreover, various interest rates that private agents face also comove with the shadow

Figure 2: Shadow rate and private interest rates



Notes: black solid line: the Wu-Xia shadow rate; blue dash-dotted line: the BofA Merrill Lynch US High Yield Effective Yield; red dashed line: the Goldman Sachs Financial Conditions Index (right scale); cyan dotted line: the BofA Merrill Lynch US Corporate BBB Effective Yield; green dotted line: the BofA Merrill Lynch US Corporate AAA Effective Yield; magenta dots: the Gilchrist and Zakrajšek (2012) credit spread; brown dots: 2-year Treasury rate; black dotted line: the effective fed funds rate. Left scale: interest rates in percentage points; right scale: the Goldman Sachs Financial Conditions Index. The ZLB sample spans from January 2009 to November 2015. Data sources: Wu-Xia/Federal Reserve Bank of Atlanta, Federal Reserve Economic Data, and Simon Gilchrist's website.

rate. The blue dash-dotted line is the high yield effective yield, the cyan dotted line is the BBB effective yield, the green dotted line is the AAA effective yield. None of these corporate borrowing rates, whether investment grade or high yield, face the ZLB: they are at least 1.5% and display meaningful variation. They track the U-shaped dynamics of the shadow rate, and their correlations with the shadow rate are 0.8, 0.8, and 0.6, respectively.

We also compare it with some popular measures in the literature. First, we plot Gilchrist and Zakrajšek's (2012) credit spread in magenta dots, which proxies overall credit conditions, and its correlation with the Wu-Xia shadow rate is about 0.8. Swanson and Williams (2014) and Gertler and Karadi (2015) recommend using the 2-year Treasury yield as a proxy for monetary policy during the ZLB period. The shadow rate and the 2-year yield have a correlation of 0.6 at the ZLB, capturing similar information. The shadow rate has some advantages over the 2-year yield. First, after the QE2 was implemented, the 2-year yield was essentially zero for about two years, consequently, its correlations with private rates and GSFCI are lower than our shadow rate in general. Another advantage the shadow rate has over the 2-year rate is that the shadow rate moves in the same scale as these private rates, whereas the 2-year yield is moving in a smaller scale.

Overall, the resemblance between the shadow rate and the Fed's balance sheet validates its role as a summary for unconventional monetary policy. Its high correlations with the GSFCI and various interest rates support that the shadow rate IS curve in (2.6) describes how the private economy factors in the additional stimulus from unconventional policy tools.

2.4 Shadow rate Taylor rule

We have established the shadow rate as a tractable summary for unconventional monetary policy. Next, we assess whether the Taylor rule is a good description of the shadow rate dynamics by estimating (2.3) empirically via regressing the shadow rate on the output gap and inflation.[2] For the shadow rate, we take Wu and Xia's (2016) spliced series of the fed funds rate during normal times and shadow rate at the ZLB. The output gap is the difference between the log of GDP and the log of potential GDP, measured in 2009 chained dollars. Inflation is the log difference of the GDP Deflator, and the data are quarterly.

We run two regressions with different sample periods popular in the literature: the first regression uses the full sample from 1954Q1 to 2017Q1, and its results are in the top row of Figure 3. The second regression uses a subsample from 1985Q1 to 2017Q1, shown in the bottom row of Figure 3. The coefficient on inflation is 1.2 for the full sample and 1.4 for the subsample, consistent with the Taylor (1993) principle.

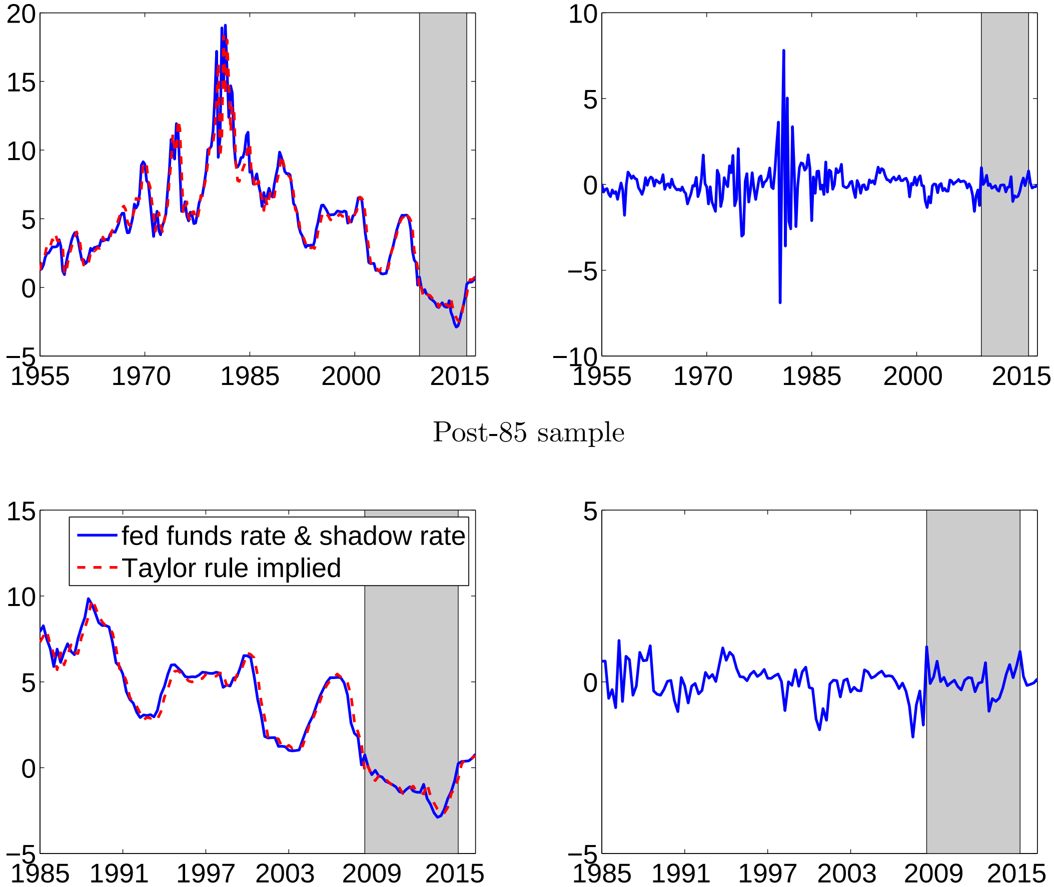
In the left column of Figure 3, we plot together the implemented monetary policy in blue and what the Taylor rule prescribes in red. The Taylor rule seems to be a good description of

[2] The specification in (2.3) follows the literature. Our results do not change if we use lagged inflation and output gap instead.

Figure 3: Taylor rule

Full sample

Post-85 sample



Notes: Left panel: blue line: observed fed funds rate and shadow rate; red dashed line: Taylor rule implied rate. Right panel: monetary policy shock. Shaded area: ZLB. Data are quarterly from 1954Q4 to 2017Q1 for the full sample, and 1985Q1 to 2017Q1 for the post-85 sample. Data sources: Wu-Xia/Federal Reserve Bank of Atlanta and Federal Reserve Economic Data.

what actually happens, including the ZLB period. We also plot the regression residual, which can be interpreted as the monetary policy shock, in the right column. Most prominently, in the top right panel, the size of the monetary policy shock was much larger during the 1980s when interest rates were high. On the contrary, the shock during the ZLB period had a similar size to the rest of the sample, regardless of which sample we use for the regression.

To see more formally whether a structural break exists, we test jointly whether the slope coefficients in the estimated Taylor rule have significantly changed at the ZLB: the F statistics for the full sample (post-85 sample) is 0.48 (1.42), which is smaller than the 5%

critical value, 2.64 (2.68). Therefore, we fail to reject the null of no structural break. This result is consistent with Wu and Xia's (2016) findings.

2.5 Economic implications

The standard NK model is associated with some counterfactual or puzzling modeling implications at the ZLB, because it does not allow any role for unconventional monetary policy. We focus on two such implications that are often discussed in the literature. First, a negative supply shock stimulates the economy, which is considered to be counterfactual.[3] Second, the government-spending multiplier is much larger than usual, and this finding is still under debate. We demonstrate qualitative implications in this section, and leave the discussion of quantitative implications to Section 6.

Both a transitory negative shock on productivity and a positive government-spending shock cause higher inflation. During normal times, in response to higher inflation, the interest rate increases more than one-for-one, implying a higher real interest rate, which in turn suppresses demand. This mechanism implies lower output in response to the negative supply shock, and a government multiplier less than 1.

The standard NK model suggests the opposite implications for both scenarios at the ZLB due to the lack of policy interventions. A constant policy rate in the standard NK model implies a lower real interest rate instead, which then stimulates private consumption, investment, and hence the overall economy. Therefore, the standard model implies a stimulative negative supply shock and larger government-spending multiplier.

In contrast to the implication of the standard NK model, empirical evidence from Wieland (2015) and Garín et al. (2016) demonstrate a similar response of output to a supply shock during normal times and at the ZLB. Our model with the shadow rate capturing unconventional monetary policy is able to generate this data-consistent implication. The shadow rate reacts positively to higher inflation through unconventional monetary policy, which is

[3] Christiano et al. (2015) point out this implication depends on whether the shock is temporary or permanent. We refer to models in the literature with a transitory shock as the "standard" model.

similar to how the conventional monetary policy works during normal times. Such a reaction increases the real rate private agents face, and implies a lower output in the model, which is consistent with the data. Moreover, the same model suggests the fiscal multiplier is the same as usual, contributing to the ongoing debate. This model implication is consistent with Braun et al. (2012), Mertens and Ravn (2014), Swanson and Williams (2014), and Wieland (2015).

2.6 Computational advantages

Besides the benefit of sensible economic implications, the SRNKM also salvages the NK model from the structural break introduced by the occasionally binding ZLB on the policy rate, which imposes one of the biggest challenges for solving and estimating these models.

One strand of research linearizes the equilibrium conditions without considering the ZLB, and then assumes the ZLB is driven by some exogenous shocks. This assumption greatly simplifies the solution, but it has several undesirable implications; for example, see Fernández-Villaverde et al. (2015) and Aruoba et al. (2016). Another strand of literature uses global projection methods to approximate agents' decision rules, such as Gust et al. (forthcoming), Fernández-Villaverde et al. (2015), and Aruoba et al. (2016). The non-linearity dramatically increases computing time and demands for more computing power.

This challenge does not go away after the economy lifts off from the ZLB. Instead, it becomes even more problematic as time goes on, because when the ZLB period is no longer at the end of the sample, researchers can not simply discard it. The central tension is how we treat the seven-year period of the ZLB.

Our shadow rate model proposes a compelling solution to this challenge. Our model does not incur a structural break at the ZLB as the standard model does; therefore, it restores traditional solution and estimation methods' validity.

3 Mapping QE into SRNKM

We have shown the relationship between the shadow rate and unconventional monetary policy empirically in Subsection 2.3. Next, we formalize this link. We microfound the SRNKM introduced in Section 2 using two major programs: QE in this section and lending facilities in Section 4.

3.1 Model of QE

The first policy tool is large-scale asset purchases (QE). QE programs target interest rates at longer maturities and/or with default and liquidity risks. When the ZLB binds for the fed funds rate, these interest rates are not constrained. We refer to the wedge between such an interest rate and the fed funds rate as the risk premium. The central bank purchases these bonds to lower their interest rates by reducing the risk premium.

Households maximize their utility:

$$\mathbb{E}_0 \sum_{t=0}^{\infty} \beta^t \left(\frac{C_t^{1-\sigma}}{1-\sigma} - \frac{L_t^{1+\eta}}{1+\eta} \right), \tag{3.1}$$

over consumption C_t and labor supply L_t, where η is the Frisch elasticity of labor supply, subject to the budget constraint:

$$C_t + \frac{B_t^H}{P_t} = \frac{R_{t-1}^B B_{t-1}^H}{P_t} + W_t L_t + T_t, \tag{3.2}$$

where B_{t-1}^H is the amount of nominal bonds households hold from $t-1$ to t, and the corresponding gross return on this nominal asset is R_{t-1}^B. P_t is the price level, W_t is the real wage, and T_t is net lump-sum transfers and profits. The first-order condition with respect to real bond holdings $\tilde{B}_t^H \equiv B_t^H / P_t$ is

$$C_t^{-\sigma} = \beta R_t^B \mathbb{E}_t \left(\frac{C_{t+1}^{-\sigma}}{\Pi_{t+1}} \right), \tag{3.3}$$

where $\Pi_{t+1} \equiv P_{t+1}/P_t$ is inflation from t to $t+1$.

Linearizing the QE Euler equation and imposing the goods market-clearing condition $Y_t = C_t$ yield the QE IS curve:

$$y_t = -\frac{1}{\sigma}\left(r_t^B - \mathbb{E}_t\pi_{t+1} - r^B\right) + \mathbb{E}_t y_{t+1}. \tag{3.4}$$

The QE IS curve differs from the standard IS curve (2.1) in that the return on bonds rather than the fed funds rate is the relevant interest rate households face.

Define

$$rp_t \equiv r_t^B - r_t, \tag{3.5}$$

where the policy rate r_t follows the Taylor rule during normal times as in (2.3) and (2.4). We refer to the wedge between the two rates rp_t as the risk premium.

We provide some intuition for where the risk premium could potentially come from. The first possible source could be the term premium. According to (2.5), the long term interest rate is $r_{t,t+n} = \frac{1}{n}\mathbb{E}_t(r_{t+i-1})$, and the expectations hypothesis holds. This is inconsistent with the empirical literature on bond prices, where a time-varying term premium is a well-established fact.[4] Equation (3.5) is a tractable shortcut to introduce the term premium: let the long term bond $r^B_{t,t+n} \equiv \frac{1}{n}\mathbb{E}_t(r^B_{t+i-1}) = \frac{1}{n}\mathbb{E}_t(r_{t+i-1}) + \frac{1}{n}\mathbb{E}_t(rp_{t+i-1})$, where the first term is the average future short term interest rate, and the second term is the term premim. Alternatively, we can interpret the risk premium as the corporate spread, see the model in Appendix A.1. Moreover, a similar term has been interpreted as the convenience yield by Krishnamurthy and Vissing-Jorgensen (2012) and Del Negro et al. (2017). Bernanke et al. (1999) introduce a risk premium due to costly state verification.

Empirical research, for example, Gagnon et al. (2011), Krishnamurthy and Vissing-

[4] For time-varying term premium in the term structure literature, see Wright (2011), Bauer, Rudebusch and Wu (2012, 2014), and Creal and Wu (forthcoming, 2017).

Jorgensen (2011), and Hamilton and Wu (2012), finds a larger amount of bonds the central bank holds through QE operations are associated with a lower risk premium, which suggests rp_t is a decreasing function of the total purchase of bonds by the central bank b_t^{CB}:[5]

$$rp_t'(b_t^{CB}) < 0. \tag{3.6}$$

This negative relationship, without additional assumptions about functional forms, then suggests the following in the linear model:

$$rp_t(b_t^{CB}) = rp - \varsigma(b_t^{CB} - b^{CB}), \tag{3.7}$$

where $\varsigma > 0$.

During normal times, $b_t^{CB} = b^{CB}$, and $rp_t(b^{CB}) = rp$. The assumption of a constant risk premium can be relaxed to allow some stochastic shocks, and this extension does not change our results. An implementation with the shock specified similar to Smets and Wouters (2007) and Campbell et al. (2016) is in Appendix A.2. When the ZLB binds $r_t = 0$, the central bank implements QE to increase its bond holdings b_t^{CB} in order to provide further stimulus. The total supply of bonds is held by households and the central bank: $B_t = B_t^H + B_t^{CB}$, where B_t can be time-varying and subject to exogenous shocks.

The linearized QE model consists of the new IS curve (3.4) and the risk premium channel of bond purchase (3.5) and (3.7), and together with the usual Phillips curve (2.2) and policy rule (2.3) and (2.4).

[5]The international economics literature also establishes a similar relationship between bond quantity and risk premium; see Uribe and Yue (2006) and Nason and Rogers (2006), for example, with the former motivating it by some cost associated with financial intermediaries who facilitate bond transactions.

3.2 Shadow rate equivalence for QE

Monetary policy enters the IS curve (3.4) through the return on the bond

$$r_t^B = r_t + rp - \varsigma(b_t^{CB} - b^{CB}). \tag{3.8}$$

During normal times, $b_t^{CB} = b^{CB}$, $r_t^B = r_t + rp$, and monetary policy operates through the usual Taylor rule on r_t, which is equal to the shadow rate s_t. At the ZLB, the policy rate no longer moves, $r_t = 0$, and the overall effect of monetary policy is $r_t^B = rp - \varsigma(b_t^{CB} - b^{CB})$. If

$$b_t^{CB} = b^{CB} - \frac{s_t}{\varsigma}, \tag{3.9}$$

then

$$r_t^B = s_t + rp \tag{3.10}$$

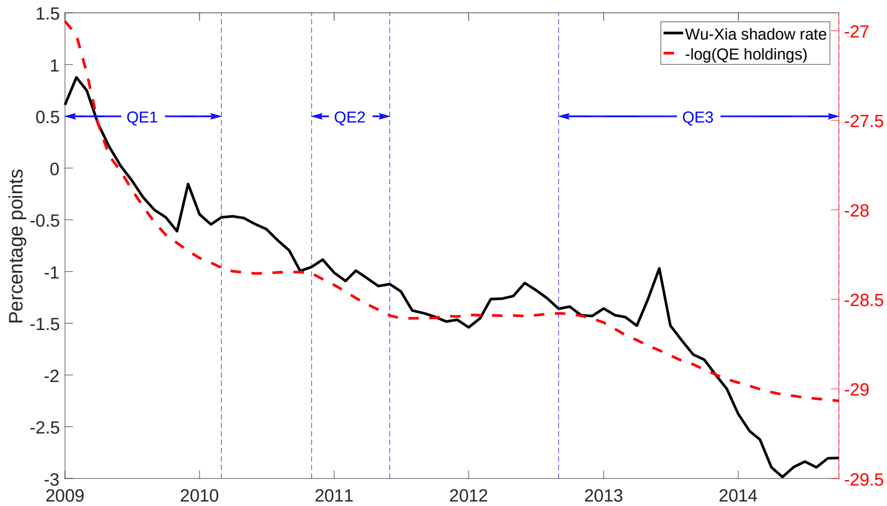
captures both the conventional monetary policy during normal times and unconventional policy at the ZLB. Although the return on the bonds in (3.8) deviates from the conventional policy rate r_t with a time-varying wedge, the difference between the return on the bond in (3.10) and s_t is a constant. This leads to the following proposition.

Proposition 1 *The shadow rate New Keynesian model represented by the shadow rate IS curve (2.6), New Keynesian Phillips curve (2.2), and shadow rate Taylor rule (2.3) is equivalent to a model where monetary policy is implemented by the conventional Taylor rule during normal times and QE at the ZLB that changes the risk premium through (3.7) if*

$$\begin{cases} r_t = s_t,\ b_t^{CB} = b^{CB} & \text{for } s_t \geq 0 \\ r_t = 0,\ b_t^{CB} \text{ follows (3.9)} & \text{for } s_t < 0. \end{cases}$$

Proof: See Appendix B.

Figure 4: QE and shadow rate



Notes: black solid line: Wu and Xia's (2016) shadow rate; red dashed line: the negative of the log of the Fed's asset holdings through QEs (including Treasury securities, Federal agency debt securities, and mortgage-backed securities). Left scale: interest rates in percentage points; right scale: negative of log asset holdings. Data sources: Wu-Xia/Federal Reserve Bank of Atlanta and Federal Reserve Statistical Release H.4.1.

Proposition 1 establishes QE as one microfoundation for (2.6). Note that at the ZLB, a negative shadow interest rate is not the actual borrowing or lending rate firms and households face. The Fed monitors and targets the shadow rate, which is determined by the shadow rate Taylor rule. The Fed does not set the shadow rate directly, but rather, it achieves the target through purchasing and selling bonds according to Proposition 1. An analogy during normal times is that the Fed reaches the federal funds rate target through open market operations.

An extension from government bonds to corporate bonds is in Appendix A.1. The equivalence holds regardless of who issues bonds, as long as the relationships between the risk premium, bond holdings, and the shadow rate in Proposition 1 hold.

3.3 Quantifying the assumption in Proposition 1

Proposition 1 assumes a linear relationship between b_t^{CB} and s_t with a negative correlation at the ZLB in (3.9). Figure 4 verifies this relationship in the data, where the shadow rate is in black and the negative of the log of the Fed's asset holdings through QE purchases is in red, including Treasury securities, Federal agency debt securities, and mortgage-backed securities. They comove with a high correlation of 0.92 from QE1 to QE3.

The data also inform us about the coefficient ς and the effects of QE programs on the shadow rate. We regress the shadow rate s_t on log asset holdings of the Fed b_t^{CB}, and the slope coefficient is -1.83, which means when the Fed increases its bond holdings by 1%, the shadow rate decreases by 0.0183%. QE1 increases the Fed's holdings on Treasuries, Federal agency debt securities, and mortgage-backed securities from 490 billion to 2 trillion, mapping into about a 2.5% decrease in the shadow rate. This number is larger than the actual change in the shadow rate, and the difference can be explained by unwinding lending facilities. QE3 is another larger operation, changing the Fed's asset holdings from 2.6 trillion to 4.2 trillion. Although QE3 is as big an operation as QE1 in the dollar amount, the percentage change of QE3 is much smaller. Our model implies a 0.9% decrease in the shadow rate. The difference between this number and the actual change can be explained by the expansionary forward guidance at the time.

4 Mapping lending facilities into SRNKM

In this section, we map lending facilities into the SRNKM introduced in Section 2. These facilities inject liquidity into the economy by extending loans to the private sector. One prominent example is the Federal Reserve's Term Asset-Backed Securities Loan Facility, which has been assessed by, for example, Ashcraft et al. (2010) and Del Negro et al. (2016). Similar policies implemented by other central banks include the Eurosystem's valuation haircuts and UK's credit controls.

4.1 Model of lending facilities

We extend the standard model characterized by (2.1) - (2.4) in the following respects. First, we replace the standard wholesale firms with entrepreneurs, who produce intermediate goods using capital and labor and then sell them in a competitive market to the retailers. Entrepreneurs maximize their lifetime utility. They have a lower discount factor and are less patient than households. As a result, they borrow from households using capital as collateral up to a constant loan-to-value ratio allowed by the households. Second, we allow the government to have two additional policy tools at the ZLB. It can loosen entrepreneurs' borrowing constraint by directly lending to them through lending facilities, effectively making the loan-to-value ratio higher and time varying. Another policy the government employs at the ZLB is a tax on the interest rate income of households and a subsidy to entrepreneurs. Taxing interest rate income can be motivated by the recent phenomenon of negative interest rates in Europe and Japan, according to Waller (2016) of the St. Louis Fed. The pre-tax/subsidy private interest rate imposes a constant markup over the policy rate $R_t^B = R_t RP$, which is a simplified version of the setup in Section 3.

Entrepreneurs (denoted by a superscript E) produce intermediate goods Y_t^E according to a Cobb-Douglas function,

$$Y_t^E = AK_{t-1}^{\alpha}(L_t)^{1-\alpha}, \tag{4.1}$$

where A is technology, K_{t-1} is physical capital used at period t and determined at $t-1$, and α is capital share of production. Capital accumulates following the law of motion: $K_t = I_t + (1-\delta)K_{t-1}$, where δ is the depreciation rate and I_t is investment. Entrepreneurs sell intermediate goods to retailers at the price P_t^E, and retailers then mark up the price by $X_t \equiv P_t/P_t^E$.

Entrepreneurs choose consumption C_t^E, investment on capital stock I_t, and labor input L_t to maximize their utility

$$\mathbb{E}_0 \sum_{t=0}^{\infty} \gamma^t \log C_t^E, \tag{4.2}$$

where their discount factor γ is smaller than households' β. They borrow against capital stock with the constraint

$$\tilde{B}_t \leq M_t \mathbb{E}_t \left(\frac{K_t \Pi_{t+1}}{R_t^B} \right), \tag{4.3}$$

issue real corporate bonds $\tilde{B}_t$, and take the loan-to-value ratio M_t as given. The entrepreneurs' budget constraint is

$$\frac{Y_t^E}{X_t} + \tilde{B}_t = \frac{R_{t-1}^B \tilde{B}_{t-1}}{\mathcal{T}_{t-1} \Pi_t} + W_t L_t + I_t + C_t^E, \tag{4.4}$$

where the tax/transfer schedule $\mathcal{T}_{t-1}$ is posted at $t-1$ and levied at t. The first-order conditions are labor demand and the consumption Euler equation:

$$W_t = \frac{(1-\alpha) A K_{t-1}^{\alpha} L_t^{-\alpha}}{X_t}, \tag{4.5}$$

$$\frac{1}{C_t^E} \left(1 - \frac{M_t \mathbb{E}_t \Pi_{t+1}}{R_t^B} \right) = \gamma \mathbb{E}_t \left[\frac{1}{C_{t+1}^E} \left(\frac{\alpha Y_{t+1}^E}{X_{t+1} K_t} - \frac{M_t}{\mathcal{T}_t} + 1 - \delta \right) \right]. \tag{4.6}$$

Households maximize their utility (3.1) subject to the budget constraint:

$$C_t + \tilde{B}_t^H = \frac{R_{t-1}^B \tilde{B}_{t-1}^H}{\mathcal{T}_{t-1} \Pi_t} + W_t L_t + T_t. \tag{4.7}$$

Hence, their consumption Euler equation is

$$C_t^{-\sigma} = \beta R_t^B \mathbb{E}_t \left(\frac{C_{t+1}^{-\sigma}}{\Pi_{t+1} \mathcal{T}_t} \right), \tag{4.8}$$

and labor supply satisfies

$$W_t = C_t^{\sigma} L_t^{\eta}. \tag{4.9}$$

Households are willing to lend entrepreneurs $\tilde{B}_t^H$ with a constant loan-to-value ratio M:

$$\tilde{B}_t^H \leq M \mathbb{E}_t \left(\frac{K_t \Pi_{t+1}}{R_t^B} \right). \tag{4.10}$$

During normal times, $\tilde{B}_t = \tilde{B}_t^H$ and $M_t = M$. At the ZLB, the central bank can provide extra credit $\tilde{B}_t^{CB}$ to firms through lending facilities allowing $M_t > M$. The total credit firms obtain equals households' bond holdings plus the central bank's bond holdings $\tilde{B}_t = \tilde{B}_t^H + \tilde{B}_t^{CB}$.

The monopolistically competitive final goods producers, who face Calvo-stickiness, behave the same as in the benchmark model. The government still implements the Taylor rule during normal times. The goods market clears if

$$Y_t = C_t + C_t^E + I_t. \tag{4.11}$$

4.2 Shadow rate equivalence for lending facilities

To boost the economy, the central bank can implement either conventional or unconventional monetary policy. During normal times, it lowers R_t. At the ZLB, it implements unconventional policy tools by increasing M_t and increasing tax/transfer $\mathcal{T}_t$. Unconventional policy tools stimulate the economy through the following channels. First, a higher loan-to-value ratio allows entrepreneurs to secure more loans. Second, the tax benefit for entrepreneurs' interest rate payment effectively lowers their borrowing cost, encouraging them to borrow, consume, invest, and produce more. All together, these channels boost the aggregate demand, and help the economy eventually escape the "liquidity trap".

Not only do conventional and unconventional monetary policies work through similar channels, but they are also observationally equivalent in our model, because they always appear in pairs in equilibrium conditions. In households' consumption Euler equation (4.8) and households' and entrepreneurs' budget constraints (4.7) and (4.4), government policy appears in the form $R_t/\mathcal{T}_t$. In the entrepreneurs' borrowing constraint (4.3) and first-order condition (4.6), it appears in the form R_t/M_t. Hence, lowering R_t or increasing M_t and $\mathcal{T}_t$ can reduce $R_t/\mathcal{T}_t$ and R_t/M_t. Moreover, $M_t/\mathcal{T}_t$ enters entrepreneurs' Euler equation (4.6), and moving both proportionally keeps this ratio constant.

The following proposition formalizes the equivalence between conventional and unconventional policies, both of which can be captured by the shadow rate.

Proposition 2 *If*

$$\begin{cases} R_t = S_t,\ \mathcal{T}_t = 1,\ M_t = M & \text{for } S_t \geq 1 \\ R_t = 1,\ \mathcal{T}_t = M_t/M = 1/S_t & \text{for } S_t < 1, \end{cases}$$

then the shadow rate S_t *is the sufficient statistic for all conventional and unconventional policies.*

Proof: See Appendix B.

Note the equivalence in Proposition 2 is achieved without linearization, which extends the usefulness of the shadow rate beyond linearized models. Whether the model is linearized or not, the shadow rate is a summary for all policy instruments and does not introduce a structural break at the ZLB.

The equivalence can also be extended to the linear model, which describes the equilibrium allocation $\{c_t, c_t^E, y_t, k_t, i_t, \tilde{b}_t\}_{t=0}^{\infty}$ and prices and policies $\{x_t, \pi_t, r_t, s_t, m_t, \tau_t\}_{t=0}^{\infty}$, and consists of (2.3), (2.4), policy rules for changing m_t and τ_t, and

$$c_t = -\frac{1}{\sigma}(r_t - \tau_t - \mathbb{E}_t\pi_{t+1} - r) + \mathbb{E}_t c_{t+1}, \tag{4.12}$$

$$C^E c_t^E = \alpha\frac{Y}{X}(y_t - x_t) + \tilde{B}\tilde{b}_t - R^B\tilde{B}(r_{t-1} + \tilde{b}_{t-1} - \tau_{t-1} - \pi_{t-1} + rp) - I i_t + \Lambda_1, \tag{4.13}$$

$$\tilde{b}_t = \mathbb{E}_t(k_t + \pi_{t+1} + m_t - r_t - rp), \tag{4.14}$$

$$\begin{aligned} 0 = &\left(1 - \frac{M}{R^B}\right)(c_t^E - \mathbb{E}_t c_{t+1}^E) + \frac{\gamma\alpha Y}{XK}\mathbb{E}_t(y_{t+1} - x_{t+1} - k_t) \\ &+ \frac{M}{R^B}\mathbb{E}_t(\pi_{t+1} - r_t + m_t - rp) + \gamma M(\tau_t - m_t) + \Lambda_2, \end{aligned} \tag{4.15}$$

$$y_t = \frac{\alpha(1+\eta)}{\alpha+\eta}k_{t-1} - \frac{1-\alpha}{\alpha+\eta}(x_t + \sigma c_t) + \frac{1+\eta}{\alpha+\eta}a + \frac{1-\alpha}{\alpha+\eta}\log(1-\alpha), \tag{4.16}$$

$$k_t = (1-\delta)k_{t-1} + \delta i_t - \delta\log\delta, \tag{4.17}$$

$$\pi_t = \beta\mathbb{E}_t\pi_{t+1} - \lambda\left(x_t - x\right), \tag{4.18}$$

$$y_t = \frac{C}{Y}c_t + \frac{C^E}{Y}c_t^E + \left(1 - \frac{C}{Y} - \frac{C^E}{Y}\right)i_t, \tag{4.19}$$

where Λ_1 and Λ_2 are functions of steady-state values, defined in Appendix A.1. (4.12) linearizes the households' consumption Euler equation (4.8), and it differs from the standard

Euler equation (2.1) mainly because of the tax. (4.13) is from the entrepreneurs' budget constraint (4.4) and labor demand first-order condition (4.5). (4.14) is the linear expression for the borrowing constraint (4.3) when it is binding. (4.15) linearizes the entrepreneurs' consumption Euler equation (4.6). (4.16) combines the production function (4.1) and labor supply first-order condition (4.9). (4.17) is the linearized capital accumulation. (4.18) is the NK Phillips curve expressed with the price markup, which is equivalent to (2.2), and $\lambda = \kappa/(\sigma+\eta)$. (4.19) is the linearized version of the goods market-clearing condition (4.11).

Finally, the following proposition builds the equivalence between the shadow rate and the lending facility – tax policy in the linear model:

Proposition 3 *The shadow rate New Keynesian model represented by the shadow rate IS curve*

$$c_t \quad = \quad -\frac{1}{\sigma}(s_t - \mathbb{E}_t\pi_{t+1} - s) + \mathbb{E}_t c_{t+1}, \tag{4.20}$$

the shadow rate Taylor rule (2.3), together with (4.16) - (4.19) and

$$C^E c_t^E = \alpha\frac{Y}{X}(y_t - x_t) + \tilde{B}\tilde{b}_t - R^B\tilde{B}(s_{t-1} + rp + \tilde{b}_{t-1} - \pi_{t-1}) - Ii_t + \Lambda_1, \tag{4.21}$$

$$\tilde{b}_t = \mathbb{E}_t(k_t + \pi_{t+1} + m - s_t - rp), \tag{4.22}$$

$$\begin{aligned} 0 = &\left(1 - \frac{M}{R^B}\right)(c_t^E - \mathbb{E}_t c_{t+1}^E) + \frac{\gamma\alpha Y}{XK}\mathbb{E}_t(y_{t+1} - x_{t+1} - k_t) \\ &+\frac{M}{R^B}\mathbb{E}_t(\pi_{t+1} - s_t - rp + m) - \gamma M m + \Lambda_2 \end{aligned} \tag{4.23}$$

is equivalent to the model summarized by (2.3) - (2.4) and (4.12) - (4.19), where monetary policy is implemented by the conventional Taylor rule during normal times and lending facility – tax policy at the ZLB if

$$\begin{cases} r_t = s_t,\ \tau_t = 0,\ m_t = m & \textit{for } s_t \geq 0 \\ r_t = 0,\ \tau_t = m_t - m = -s_t & \textit{for } s_t < 0. \end{cases}$$

Proof: See Appendix B.

Hence, Proposition 3 establishes the lending facility – tax policy channel as another microfoundation for (2.6), because (4.20) is (2.6) without imposing the market-clearing condition.

5 Partially active monetary policy

So far, our discussion on the SRNKM in Definition 1 is based on the argument that unconventional monetary policy is fully active, that is, the central bank follows the same Taylor rule with and without the ZLB. The difference is how they implement the policy. What if monetary policy is partially active at the ZLB? This section extends our model to accommodate this possibility, and illustrates the relevance of the SRNKM in this environment.

Definition 2 *The SRNKM with partially active monetary policy consists of the IS curve*

$$y_t = -\frac{1}{\sigma}(\mathcal{S}_t - \mathbb{E}_t\pi_{t+1} - \mathcal{S}) + \mathbb{E}_t y_{t+1}, \tag{5.1}$$

New Keynesian Phillips curve (2.2), and the shadow rate Taylor rule (2.3), where

$$\begin{cases} \mathcal{S}_t = s_t \ for \ s_t \geq 0 \\ \mathcal{S}_t = \lambda s_t \ for \ s_t < 0, \end{cases} \tag{5.2}$$

and $0 \leq \lambda \leq 1$ *represents how active monetary policy is at the ZLB.*

We approximate (5.2) with a smooth transition between the two regimes

$$\mathcal{S}_t = \left(\frac{exp(\varphi s_t)}{1 + exp(\varphi s_t)}(1 - \lambda) + \lambda \right) s_t, \tag{5.3}$$

where the auxiliary parameter $\varphi > 0$ controls the transition speed. This smooth transition has several intuitive implications. First, if $\lambda < 1$, s_t only needs to be close to zero, and does not have to be 0 for monetary policy to be less effective. Second, a small positive s_t and a small negative s_t are similar in terms of how active monetary policy is. Third, this formulation is consistent with the recent experience of negative interest rates in Europe and Japan in the sense that the ZLB does not provide a cutting edge at zero.

The mathematical convenience of the smooth function in (5.3) allows us to further linearize the model.

Proposition 4 *The SRNKM defined in Definition 2 can be approximated by the linear system of (2.2), (2.3), (5.1) and*

$$\mathcal{S}_t = s_t - (1-\omega)(1-\lambda)s_t + \varphi s\omega(1-\omega)(1-\lambda)(s_t - s), \tag{5.4}$$

where ω is the fraction of normal times.

Proof: See Appendix B.

To gain some intuition, for $s_t \approx s$, (5.4) becomes

$$\mathcal{S}_t = [\omega + (1-\omega)\lambda]s_t. \tag{5.5}$$

This relationship is rather intuitive: for a fraction ω of the time, we are out of the ZLB, and the loading is 1; for a fraction $1-\omega$ of the time, we are at the ZLB, and the loading is λ. The condition for determinacy becomes $(\omega + (1-\omega)\lambda)\phi_\pi > 1$.

Next, we study under what condition the model in Definition 1 can approximate the model in Definition 2.

Proposition 5 *The sufficient condition for the model in Definition 2 to be approximated by* *the model in Definition 1 is*

$$(1-\omega)(1-\lambda)=0 \tag{5.6}$$

Proof: See Appendix B.

The trivial case is when monetary policy is fully active, $\lambda = 1$, Definition 2 becomes Definition 1. This coincidence is by definition.

What is more interesting about Proposition 5 is that we do not necessarily need the strong assumption that unconventional monetary policy is fully active for our main model in Definition 1 to be valid. But rather, an alternative assumption $\omega = 1$ that also works is popular in the ZLB literature: the ZLB only lasts finite periods, whereas the subsequent normal times is infinite. Researchers use this assumption to guarantee determinacy, and/or solve the model with the ZLB backwards; for example, see Guerrieri and Iacoviello (2015) and Fernández-Villaverde et al. (2015). Proposition 5 states that our main model in Definition 1 is a first order approximation to NK models with the ZLB constraint under a popular assumption.

6 Quantitative analyses

The mechanism for how the SRNKM works has been demonstrated qualitatively in Section 2. In this section, we study the quantitative implications of this model. We first explain the quantitative model and solution method. Next, we discuss the consequence of an inflation shock at the ZLB and relate it to the supply shock and government-spending shock discussed in Subsection 2.5.

6.1 Model and methodology

Shadow rate vs. standard model We analyze contrasts between our shadow rate model and the standard model. We term any model that does not have unconventional monetary policy the standard model. Although the standard model we introduce in this section has many more ingredients than the standard three-equation NK model, they share similar qualitative implications that are discussed in Subsection 2.5. In a standard model, $r_t = 0$ enters the Euler equation, budget constraint, borrowing constraint, and so on at the ZLB. By contrast, the shadow model has unconventional monetary policy. It replaces r_t with the negative shadow rate s_t at the ZLB.

Quantitative model Many components are from Iacoviello's (2005) full model, including five sectors, of which two are households. Both types of households work, consume, and hold housing stocks. They differ in their discount factors. Patient households have a higher discount factor and save. Impatient households have a lower discount factor and borrow from patient households using their existing housing as collateral. Entrepreneurs also have a lower discount factor than patient households, and hence borrow from them with collateral as well. Entrepreneurs consume, invest, and hold houses. They use housing, capital, and labor as inputs to produce identical intermediate goods and sell them in a competitive market to retailers. Retailers are monopolistically competitive. They differentiate intermediate goods into final goods, and set prices with Calvo-type stickiness. The central bank implements a Taylor rule.

We have shown in Sections 3 and 4 how the negative shadow rate can be implemented through various unconventional policy tools. These unconventional tools set our model apart from Iacoviello's (2005). First, we use a time-varying risk premium to capture QE discussed in Section 3. Second, we allow the loan-to-value ratio to be time-varying to model lending facilities. Additionally, lenders' (borrowers') bond returns (payments) are subject to a time-varying tax (subsidy) at the ZLB. These two policies together constitute the channel discussed in Section 4. We also differ from his model by allowing the government to adjust the aggregate demand through changing its expenditure, so that we can study the government-spending multiplier. The detailed model setup is in Appendix C.1. Many parameter values are taken from Iacoviello (2005) and Fernández-Villaverde et al. (2015), and more calibration details are in Appendix C.2,

Solution method For our shadow rate model capturing unconventional monetary policy, we work with a linear model where only the shadow rate enters the model representing all possible channels for monetary policy. In this case, the constraint of the ZLB for the policy rate does not impose any non-linearity in our model. Full details of the linear model are in Appendix C.4.1. After we solve the model, we use the results from Propositions 1 - 3 to demonstrate how the negative shadow rate can be implemented with underlying unconventional policy tools discussed in Sections 3 and 4. The details are in Appendix C.4.2 - Appendix C.4.3. As a comparison, we also analyze the standard model with the ZLB constraint (see details in Appendix C.4.4) with the method of Guerrieri and Iacoviello (2015).

ZLB environment To create a ZLB environment, we follow Christiano et al. (2011), Fernández-Villaverde et al. (2015), and many other to impose a series of positive preference shocks on the economy.[6] The shocks last from period 1 to 15. The total shock size is 5% implying a size of 5/15% per period. They cause people to save more, push the nominal policy rate r_t to zero at period 8, and keep it there until about period 20. The impulse responses to this sequence of shocks are in Appendix D.[7]

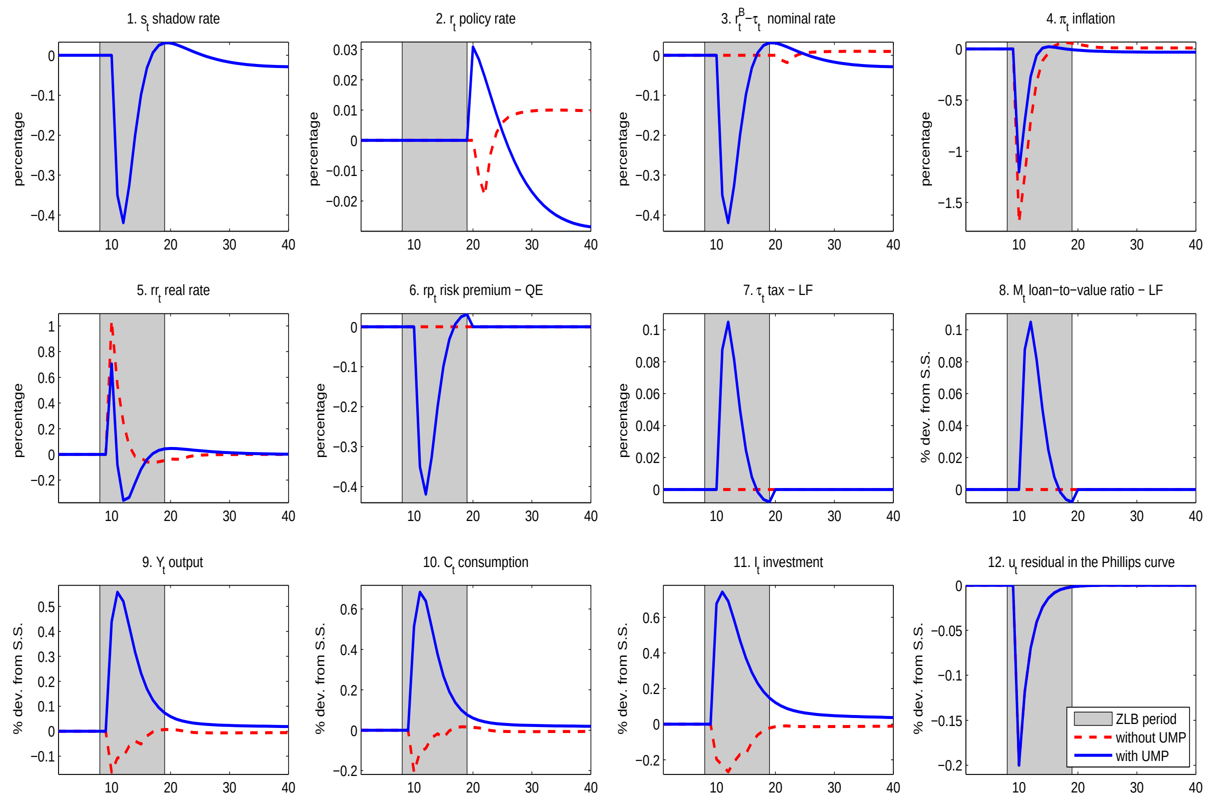
6.2 Negative inflation shock at the ZLB

One of the major concerns about the ZLB is deflation. Once the economy encounters a deflationary spiral, the problem will exacerbate: a decrease in price leads to lower production, which in turn contributes to a lower wage and demand. Lower demand further decreases the

[6] Schorfheide et al. (2014) and Creal and Wu (2016) introduce preference shocks to study risk premium.

[7] Our results in Sections 6.2 - 6.4 are robust to alternative shocks to create the ZLB environment, for example, inflation shocks.

Figure 5: Negative inflation shock



Notes: We hit the economy with two types of shocks. First, a series of positive preference shocks occurs in periods 1 - 15, and the total shock size is 5%. Second, a negative inflation shock happens in period 10 with a size of 0.2%. We difference out the effect of preference shocks, and only plot the additional effect of the inflation shock. The red lines represent the case in which, when the policy rate is bounded by zero, no unconventional monetary policy is implemented. The blue lines represent the case in which unconventional monetary policy is implemented through a negative shadow rate. Plots 1-7 are in percentage, among which plots 1-6 are annualized. Plots 8-12 are measured in percentage deviation from the steady state. The shadow rate in plot 1 is an overall measure of monetary policy. In normal times, it is implemented through the policy rate in plot 2. At the ZLB, it can be implemented through QE in plot 6 (per Section 3) or the lending facilities (LF) in plots 7 and 8 (per Section 4). The shaded area marks periods 8-19.

price. In this section, we investigate the effect of unconventional monetary policy in fighting deflation at the ZLB through the lens of our SRNKM.

In addition to the positive preference shocks to create the ZLB environment, we introduce a negative inflation shock of the size 0.2% at period 10. To investigate its marginal impact on the economy, we plot in Figure 5 the difference between the total effects of both shocks and the effects of only preference shocks. The red lines capture the standard model without unconventional monetary policy, while the blue lines represent the shadow rate model when unconventional monetary policy is present.

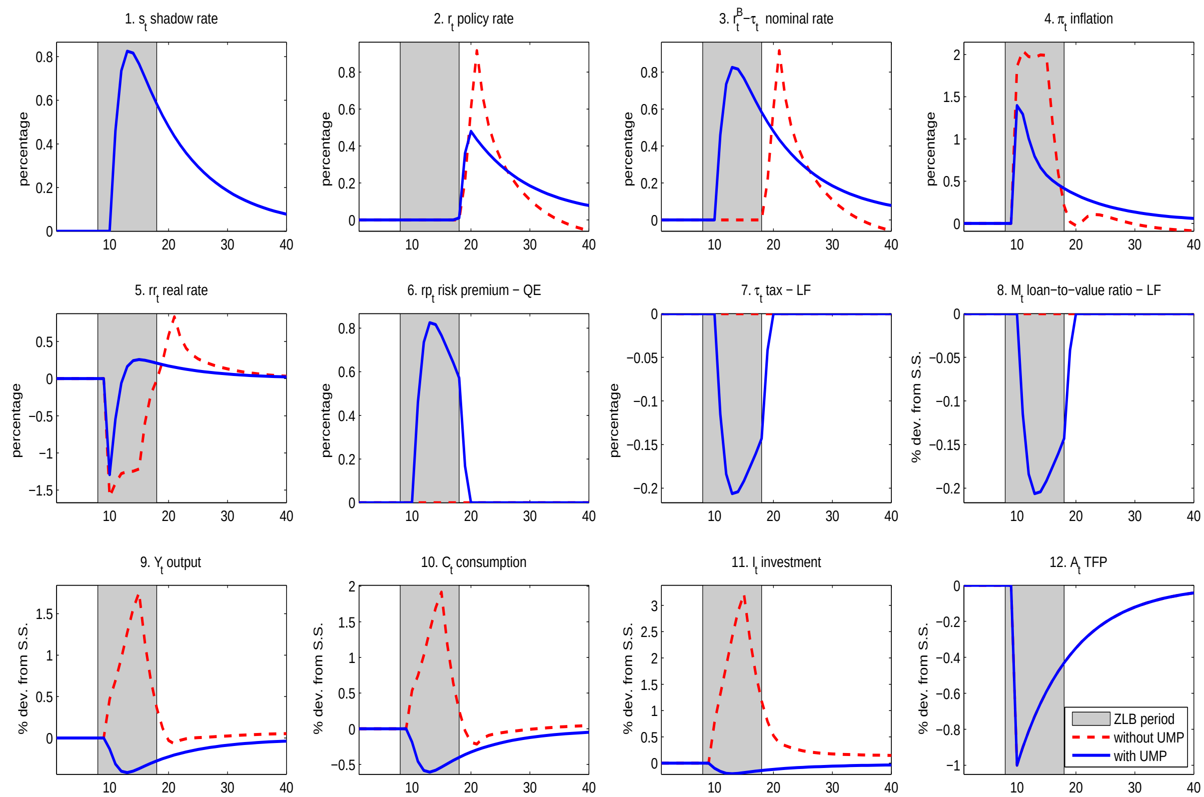
Inflation and hence inflation expectation decrease less in the shadow rate model (in blue) compared to the standard model (in red); that is, the responsiveness of unconventional monetary policy alleviates some of the deflationary concern. The policy rate does not respond in either case when the ZLB is binding. However, the access to unconventional monetary policy allows the shadow rate to further drop by 0.4% at the maximum. The post-tax nominal return on bonds $r_t^B - \tau_t$ (in plot 3) defined in Section 3 or 4 moves one-for-one with the shadow rate.

A lower shadow rate can be implemented either through a QE channel in Section 3 (blue line in plot 6) or a lending facility – fiscal policy in Section 4 (blue lines in plots 7 and 8). For example, the 0.4% drop in the shadow rate can be obtained by the same amount of change in the risk premium through bond purchases. The amount of purchases, however, is state-dependent, and it amounts to 130, 530, 660, or 1080 billion dollars if the pre-purchase level of the central bank's bond holdings were at the pre-QE1, pre-QE2, pre-QE3, or post-QE3 level, respectively. Alternatively, the loan-to-value ratio and tax rate go up by 0.1%. Translating these numbers into the annual rate, $0.1\% \times 4 = 0.4\%$, can explain the same amount of change in the shadow rate.

With unconventional monetary policy, a lower nominal rate and higher inflation expectation imply a less and more transitory increase in the real rate $rr_t \equiv r_t^B - \tau_t - \mathbb{E}_t \pi_{t+1}$, which incentivise firms to invest and households to consume more. For example, output increases by 0.6% in the shadow rate model, whereas it decreases by 0.2% in the standard model.

The differences in responses to the inflation shock provide the basic mechanism to explain the economic implications discussed in Subsection 2.5, which we will turn to next.

Figure 6: Negative TFP shock



Notes: We hit the economy with two types of shocks. First, a series of positive preference shocks occurs in periods 1-15, and the total shock size is 5%. Second, a negative TFP shock happens in period 10 with a size of 1%. We difference out the effect of preference shocks and only plot the additional effect of the TFP shock. The red lines represent the case in which, when the policy rate is bounded by zero, no unconventional monetary policy is implemented. The blue lines represent the case in which unconventional monetary policy is implemented through a negative shadow rate. Plots 1-7 are in percentage, among which plots 1-6 are annualized. Plots 8-12 are measured in percentage deviation from the steady state. The shadow rate in plot 1 is an overall measure of monetary policy. In normal times, it is implemented through the policy rate in plot 2. At the ZLB, it can be implemented through QE in plot 6 (per Section 3) or the lending facilities (LF) in plots 7 and 8 (per Section 4). The shaded area marks periods 8-18.

6.3 Negative supply shock at the ZLB

In this section, we quantify the effects of a negative supply shock discussed in Subsection 2.5. We add a -1% TFP innovation at period 10 and plot in Figure 6 its marginal contributions.

The standard model shows a negative supply shock increases output at the ZLB (red line in plot 9), which contradicts the empirical findings; for example, see Wieland (2015) and Garín et al. (2016). By contrast, our shadow rate model (blue line) produces a negative impact of such a shock. This result is data-consistent. The same contrast can be further extended to other real variables: consumption and investment. More specifically, with the presence of unconventional monetary policy, output decreases by 0.4%, consumption by 0.6%, and investment by 0.2%.

The differences in impulse responses reflect whether monetary policy is active. This works through the same mechanism as explained in Subsection 6.2. The differences are the directions and magnitudes. In the case with active unconventional monetary policy, the shadow rate increases by 0.8%, which can be implemented by selling 180, 750, 940, or 1530 billion dollars of bonds if the pre-sale holdings were at the pre-QE1, pre-QE2, pre-QE3, or post-QE3 level, respectively. Alternatively, the government can decrease the loan-to-value ratio and tax rate by 0.2%.

6.4 Government-spending multiplier at the ZLB

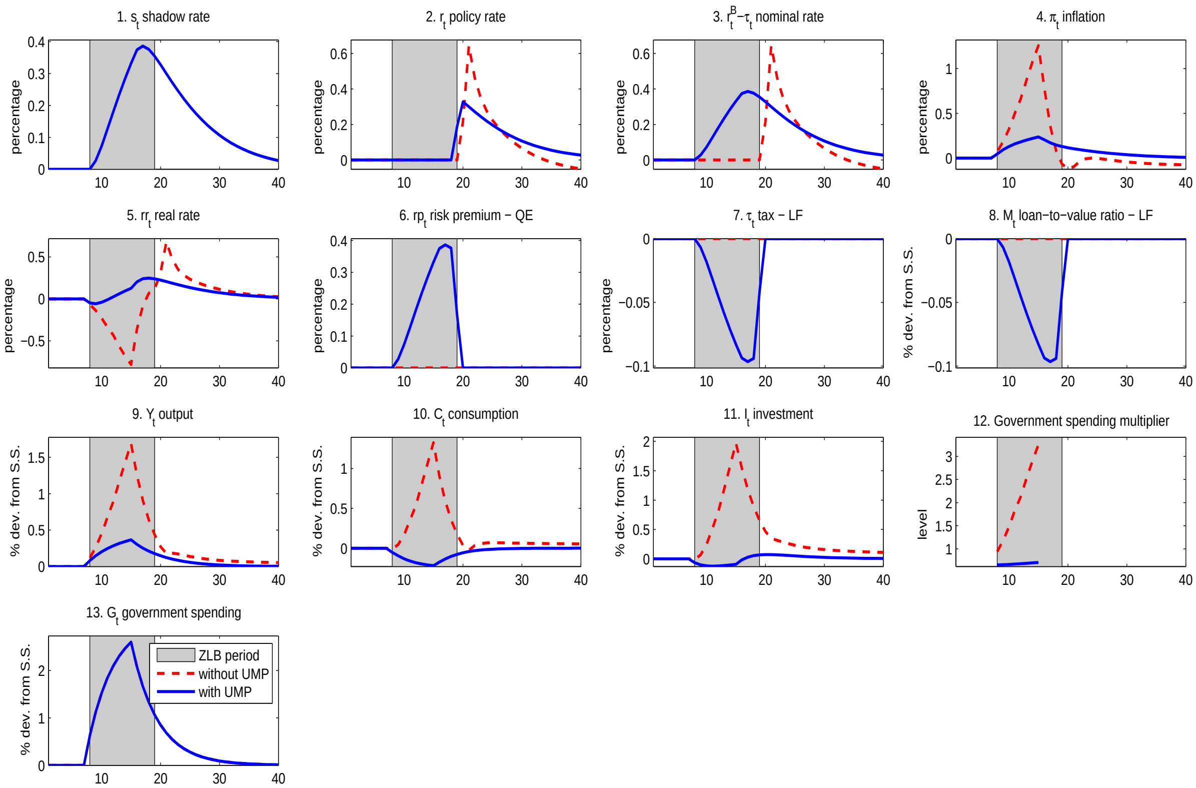
This section quantifies the government-spending multiplier discussed in Subsection 2.5. We introduce a sequence of government spending shocks from period 8 to 15 with a total (per-period) size of 5% (0.625%), and plot their impact in Figure 7.

The government-spending multiplier in the standard model (red line in plot 12) is mostly above 1 and peaks at 3.2 when the policy rate is bounded at zero and the central bank takes no additional measures to smooth the economy. By contrast, the number is less than 0.7 in the shadow rate model (in blue) when the central bank monitors and adjusts the shadow rate through implementing unconventional monetary policy.

Positive government shocks push up the aggregate demand, which leads to a rising pressure on inflation. This again lands itself as another application of the mechanism explained in Subsection 6.2. Higher inflation without policy intervention boosts the private economy, yielding a multiplier greater than 1.

By contrast, in our shadow rate model, the shadow rate increases by 0.4% in response to

Figure 7: Positive government-spending shocks



Notes: We hit the economy with two types of shocks. First, a series of positive preference shocks occurs in periods 1 - 15, and the total shock size is 5%. Second, government-spending shocks occur from periods 8-15 with a total size of 5%. We difference out the effect of preference shocks, and only plot the additional effect of the government-spending shock. The red lines represent the case in which, when the policy rate is bounded by zero, no unconventional monetary policy is implemented. The blue lines represent the case in which unconventional monetary policy is implemented through a negative shadow rate. Plots 1-7 are in percentage, among which plots 1-6 are annualized. Plots 8-12 are measured in percentage deviation from the steady state. The shadow rate in plot 1 is an overall measure of monetary policy. In normal times, it is implemented through the policy rate in plot 2. At the ZLB, it can be implemented through QE in plot 6 (per Section 3) or the lending facilities (LF) in plots 7 and 8 (per Section 4). The shaded area marks periods 8-19.

such a shock, crowding out private consumption by 0.2% and investment by 0.1%. Although output still increases by 0.4%, its change is less than the shocks themselves, producing a smaller multiplier. The change in the shadow rate in our model can be implemented through increasing the risk premium by 0.4%, which requires to sell 100, 390, 490, or 800 billion dollars of bonds if the pre-sale holdings were at the pre-QE1, pre-QE2, pre-QE3, or post-QE3 level, respectively. An alternative policy is to reduce the loan-to-value ratio and tax rate by 0.1%.

7 Conclusion

We have built a New Keynesian model with the shadow rate, which coherently captures the conventional interest rate rule in normal times, and unconventional monetary policy at the ZLB. The model is the same as the standard New Keynesian model when the policy rate is above zero. When the policy rate is binding at zero, however, unlike the standard model with an inactive monetary policy, the central bank in our model continues to monitor and adjust the shadow rate following the shadow rate Taylor rule. A negative shadow rate prescribed by this Taylor rule can be implemented, for example, by QE and/or lending facilities. Our model restores the data-consistent result that a negative supply shock is always contractionary. Relatedly, the unusually large government-spending multiplier in the standard New Keynesian model at the ZLB also disappears. In addition to incorporating unconventional policy tools in a sensible and tractable way, our model does not incur a structural break at the ZLB whether we work with a linear or non-linear model. Hence, it restores existing solution and estimation methods' validity.

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Appendix A Alternative specifications for QE

Appendix A.1 QE with corporate bonds

Entrepreneurs Bonds are issued by entrepreneurs (denoted by a superscript E) instead of the government. They produce intermediate good Y_t^E according to a Cobb-Douglas function,

$$Y_t^E = AK_{t-1}^{\alpha}(L_t)^{1-\alpha}, \tag{A.1}$$

where A is technology, K_{t-1} is physical capital used at period t and determined at $t-1$, L_t is labor, and α is the capital share of production. Capital accumulates following the law of motion: $K_t = I_t + (1-\delta)K_{t-1}$, where δ is the depreciation rate and I_t is investment. Entrepreneurs sell the intermediate goods to retailers at price P_t^E, and the markup for the retailers is $X_t \equiv P_t/P_t^E$.

Entrepreneurs choose consumption C_t^E, investment on capital stock I_t, and labor input L_t to maximize their utility

$$\mathbb{E}_0 \sum_{t=0}^{\infty} \gamma^t \log C_t^E, \tag{A.2}$$

where the entrepreneurs' discount factor γ is smaller than households' β. Their borrowing constraint is

$$\tilde{B}_t \leq M\mathbb{E}_t\left(\frac{K_t \Pi_{t+1}}{R_t^B}\right), \tag{A.3}$$

where $\tilde{B}_t$ is the amount of real corporate bonds issued by the entrepreneurs at t, and the gross return on this asset from t to $t+1$ is R_t^B. $\Pi_{t+1} \equiv P_{t+1}/P_t$ is inflation. M is the loan-to-value ratio. The entrepreneurs' budget constraint is

$$\frac{Y_t^E}{X_t} + \tilde{B}_t = \frac{R_{t-1}^B \tilde{B}_{t-1}}{\Pi_t} + W_t L_t + I_t + C_t^E, \tag{A.4}$$

where W_t is the real wage. The first-order conditions are labor demand and the consumption Euler equation:

$$W_t = \frac{(1-\alpha)AK_{t-1}^{\alpha}L_t^{-\alpha}}{X_t}, \tag{A.5}$$

$$\frac{1}{C_t^E}\left(1 - \frac{M\mathbb{E}_t\Pi_{t+1}}{R_t^B}\right) = \gamma\mathbb{E}_t\left[\frac{1}{C_{t+1}^E}\left(\frac{\alpha Y_{t+1}^E}{X_{t+1}K_t} - M + 1 - \delta\right)\right]. \tag{A.6}$$

Households and government The households' problem is the same as in Section 3.1. The central bank is also the same as in Section 3.1: it follows the Taylor rule (2.3) and (2.4) during normal times, and purchases bonds to lower risk premium at the ZLB according to (3.5) and (3.7). The goods market-clearing condition is $Y_t = C_t + C_t^E + I_t$.

Equilibrium The linear system describing the equilibrium allocation $\{c_t, c_t^E, y_t, k_t, i_t, \tilde{b}_t, b_t^{CB}\}_{t=0}^{\infty}$ and prices $\{x_t, \pi_t, w_t, r_t^B, r_t, rp_t, s_t\}_{t=0}^{\infty}$ consists of (2.3), (2.4), (3.5), (3.7), a policy rule for central bank purchases at the ZLB, and

$$c_t = -\frac{1}{\sigma}(r_t^B - \mathbb{E}_t\pi_{t+1} - r^B) + \mathbb{E}_t c_{t+1}, \tag{A.7}$$

$$C^E c_t^E = \alpha\frac{Y}{X}(y_t - x_t) + \tilde{B}\tilde{b}_t - R^B\tilde{B}(r_{t-1}^B + \tilde{b}_{t-1} - \pi_{t-1}) - Ii_t + \Lambda_1, \tag{A.8}$$

$$\tilde{b}_t = \mathbb{E}_t(k_t + \pi_{t+1} + m - r_t^B), \tag{A.9}$$

$$0 = \left(1 - \frac{M}{R^B}\right)(c_t^E - \mathbb{E}_t c_{t+1}^E) + \frac{\gamma\alpha Y}{XK}\mathbb{E}_t(y_{t+1} - x_{t+1} - k_t) + \frac{M}{R^B}\mathbb{E}_t(\pi_{t+1} - r_t^B) + \tilde{\Lambda}_2, \tag{A.10}$$

$$y_t = \frac{\alpha(1+\eta)}{\alpha+\eta}k_{t-1} - \frac{1-\alpha}{\alpha+\eta}(x_t + \sigma c_t) + \frac{1+\eta}{\alpha+\eta}a + \frac{1-\alpha}{\alpha+\eta}\log(1-\alpha), \tag{A.11}$$

$$k_t = (1-\delta)k_{t-1} + \delta i_t - \delta\log\delta, \tag{A.12}$$

$$\pi_t = \beta\mathbb{E}_t\pi_{t+1} - \lambda\left(x_t - x\right), \tag{A.13}$$

$$y_t = \frac{C}{Y}c_t + \frac{C^E}{Y}c_t^E + \left(1 - \frac{C}{Y} - \frac{C^E}{Y}\right)i_t, \tag{A.14}$$

where $\Lambda_1 = C^E\log C^E - \alpha\frac{Y}{X}\log\frac{Y}{X} - \tilde{B}\log\tilde{B} + R^B\tilde{B}\log R^B\tilde{B} + I\log I$, $\tilde{\Lambda}_2 = -\frac{\gamma\alpha Y}{XK}\log\frac{Y}{XK} + \frac{M}{R^B}\log R^B$. The Λ_2 in (4.15) is $\Lambda_2 = \tilde{\Lambda}_2 - \left(\frac{1}{R^B} - \gamma\right)M\log M$.

Equivalence Therefore, Proposition 1 becomes

Corollary 1 *The shadow rate New Keynesian model represented by the shadow rate IS curve*

$$c_t = -\frac{1}{\sigma}(s_t - \mathbb{E}_t\pi_{t+1} - s) + \mathbb{E}_t c_{t+1}, \tag{A.15}$$

the shadow rate Taylor rule (2.3), together with (A.11) - (A.14) and

$$C^E c_t^E = \alpha\frac{Y}{X}(y_t - x_t) + \tilde{B}\tilde{b}_t - R^B\tilde{B}(s_{t-1} + rp + \tilde{b}_{t-1} - \pi_{t-1}) - Ii_t + \Lambda_1, \tag{A.16}$$

$$\tilde{b}_t = \mathbb{E}_t(k_t + \pi_{t+1} + m - s_t - rp), \tag{A.17}$$

$$0 = \left(1 - \frac{M}{R^B}\right)(c_t^E - \mathbb{E}_t c_{t+1}^E) + \frac{\gamma\alpha Y}{XK}\mathbb{E}_t(y_{t+1} - x_{t+1} - k_t) + \frac{M}{R^B}\mathbb{E}_t(\pi_{t+1} - s_t - rp + m) + \Lambda_2 \tag{A.18}$$

is equivalent to the model summarized by (2.3), (2.4), (3.5), (3.7), and (A.7) - (A.14), where monetary policy is implemented by the conventional Taylor rule during normal times and QE that changes risk premium if

$$\begin{cases} r_t = s_t,\ b_t^{CB} = b^{CB} & \text{for } s_t \geq 0 \\ r_t = 0,\ b_t^{CB} \text{ follows (3.9)} & \text{for } s_t < 0. \end{cases}$$

Appendix A.2 Time-varying risk premium

We add an exogenous premium shock, similar to Smets and Wouters (2007): (3.7) becomes

$$rp_t(b_t^{CB}) = rp - \varsigma(b_t^{CB} - b^{CB}) + \epsilon_{b,t}, \tag{A.19}$$

where $\epsilon_{b,t}$ is a white noise. With this extension, the risk premium is time-varying during normal times when $b_t^{CB} = b^{CB}$. Under the conditions imposed in Proposition 1, $r_t^B = s_t + rp + \epsilon_{b,t}$. Imposing the market-clearing condition, the shadow rate IS curve in (2.6) becomes

$$y_t = -\frac{1}{\sigma}(s_t - \mathbb{E}_t\pi_{t+1} - s + \epsilon_{b,t}) + \mathbb{E}_t y_{t+1}. \tag{A.20}$$

Other equilibrium conditions remain the same, including the New Keynesian Phillips curve (2.2) and shadow rate Taylor rule (2.3). Therefore, Proposition 1 becomes

Corollary 2 *The shadow rate New Keynesian model represented by the shadow rate IS curve (A.20), New Keynesian Phillips curve (2.2), and shadow rate Taylor rule (2.3) is equivalent to a model where monetary policy is implemented by the conventional Taylor rule during normal times and QE at the ZLB that changes the risk premium through (A.19) if*

$$\begin{cases} r_t = s_t,\ b_t^{CB} = b^{CB} & \text{for } s_t \geq 0 \\ r_t = 0,\ b_t^{CB} \text{ follows (3.9)} & \text{for } s_t < 0. \end{cases}$$

Appendix B Proof of Propositions

Proof for Proposition 1 To prove the equivalence, the key is to show the QE IS curve in (3.4) is the same as the shadow rate IS curve (2.6).

During normal times $b_t^{CB} = b^{CB}, r_t^B = r_t + rp, r_t = s_t$, the QE Euler equation (3.4) becomes

$$\begin{aligned} y_t &= -\frac{1}{\sigma}\left(r_t + rp - \mathbb{E}_t\pi_{t+1} - r^B\right) + \mathbb{E}_t y_{t+1} \\ &= -\frac{1}{\sigma}\left(s_t - \mathbb{E}_t\pi_{t+1} - s\right) + \mathbb{E}_t y_{t+1}. \end{aligned} \tag{B.1}$$

At the ZLB $r_t = 0, b_t^{CB} = b^{CB} - \frac{s_t}{\varsigma}$, use the unconventional monetary policy in (3.7), and (3.4) becomes

$$\begin{aligned} y_t &= -\frac{1}{\sigma}\left(rp - \varsigma(b_t^{CB} - b^{CB}) - \mathbb{E}_t\pi_{t+1} - r^B\right) + \mathbb{E}_t y_{t+1} \\ &= -\frac{1}{\sigma}\left(s_t - \mathbb{E}_t\pi_{t+1} - s\right) + \mathbb{E}_t y_{t+1}. \end{aligned} \tag{B.2}$$

Therefore, in both cases, we have shown (3.4) and (2.6) are equivalent. ∎

Proof for Proposition 2 During normal times, $R_t = S_t$, $\mathcal{T}_t = 1$, and $M_t = M$ imply

$$R_t/\mathcal{T}_t = S_t, R_t/M_t = S_t/M, M_t/\mathcal{T}_t = M.$$

At the ZLB, $\mathcal{T}_t = M_t/M = 1/S_t$, and $R_t = 1$ imply

$$R_t/\mathcal{T}_t = S_t, R_t/M_t = S_t/M, M_t/\mathcal{T}_t = M.$$

Therefore, government policies, whether conventional or conventional, can be summarized by the single variable S_t. ∎

Proof for Proposition 3 $r_t - \tau_t$ enters (4.12) and (4.13), and Proposition 2 have shown $r_t - \tau_t = log(R_t/\mathcal{T}_t) = s_t$. $r_t - m_t$ enters (4.14) and (4.15), and Proposition 2 have shown $r_t - m_t = log(R_t/M_t) = s_t - m$. $\tau_t - m_t$ enters (4.15), and Proposition 2 have shown $m_t - \tau_t = log(M_t/\mathcal{T}_t) = m$. Therefore, equations (4.12) - (4.15) can be expressed equivalently with the shadow rate as in (4.20) - (4.23). ∎

Proof for Proposition 4 Log-linearize $\frac{exp(\varphi s_t)}{1+exp(\varphi s_t)}$ in (5.3):

$$\frac{exp(\varphi s_t)}{1+exp(\varphi s_t)} = \frac{exp(\varphi s)}{1+exp(\varphi s)} + \frac{exp(\varphi s)}{1+exp(\varphi s)}\varphi\hat{s}_t - \frac{exp(2\varphi s)}{(1+exp(\varphi s))^2}\varphi\hat{s}_t, \tag{B.3}$$

where $\hat{s}_t = s_t - s$. Using (5.3) to compute SS

$$\frac{exp(\varphi s)}{1 + exp(\varphi s)} = \frac{\frac{\mathcal{S}}{s} - \lambda}{1 - \lambda} \tag{B.4}$$

Suppose normal times takes ω fraction, and ZLB takes $1 - \omega$ fraction, then

$$\mathcal{S}/s = \omega + (1 - \omega)\lambda. \tag{B.5}$$

Apply this to (B.4), $\frac{exp(\varphi s)}{1+exp(\varphi s)} = \omega$. Apply the approximation in (B.3) to (5.3).

$$\mathcal{S}_t = ([1 + \varphi(1 - \omega)\hat{s}_t]\, \omega(1 - \lambda) + \lambda)\, s_t. \tag{B.6}$$

Furthermore,

$$\begin{aligned}
\hat{\mathcal{S}}_t + \mathcal{S} &= ([1 + \varphi(1 - \omega)\hat{s}_t]\, \omega(1 - \lambda) + \lambda)\, (\hat{s}_t + s) \\
&= ([1 + \varphi(1 - \omega)\hat{s}_t]\, \omega(1 - \lambda) + \lambda)\, \hat{s}_t + ([1 + \varphi(1 - \omega)\hat{s}_t]\, \omega(1 - \lambda) + \lambda)\, s \\
&= \omega(1 - \lambda)\hat{s}_t + \lambda\hat{s}_t + \omega(1 - \lambda)s + \varphi s(1 - \omega)\omega(1 - \lambda)\hat{s}_t + \lambda s
\end{aligned}$$

Cancel steady states using (B.5), we obtain

$$\hat{\mathcal{S}}_t = [\omega(1 - \lambda) + \lambda + \varphi s(1 - \omega)\omega(1 - \lambda)]\hat{s}_t, \tag{B.7}$$

which is equivalent to (5.4). ■

Proof for Proposition 5 Impose the condition in (5.6), (5.4) becomes

$$\mathcal{S}_t = s_t. \tag{B.8}$$

■

Appendix C Quantitative model

Appendix C.1 Setup

Appendix C.1.1 Patient households

Patient households (denoted with a superscript P) maximize their lifetime utility:

$$\mathbb{E}_0 \sum_{t=0}^{\infty} \left(\prod_{i=1}^{t} \beta_i \right) \left[\log C_t^P + j \log H_t^P - (L_t^P)^{1+\eta}/(1 + \eta) + \chi_{\mathcal{M}} \log(\mathcal{M}_t^P / P_t) \right],$$

where β_t is the discount factor fluctuating around mean β and following the process $\beta_t/\beta = (\beta_{t-1}/\beta)^{\rho_\beta}\, \varepsilon_{\beta,t}$. C_t^P is consumption, j indicates the marginal utility of housing, H_t^P is the holdings of housing, L_t^P is hours of work, and $\mathcal{M}_t^P/P_t$ is the real money balance.

Assume households lend in nominal terms at time $t-1$ with the amount of loan B_{t-1}^P, and receive $R_{t-1}^B/\mathcal{T}_{t-1}$ at time t. The bond return R_t^B is higher than the policy rate R_t by a risk premium RP_t and $R_t^B = R_t RP_t$. The gross tax rate on bond return $\mathcal{T}_{t-1}$ is assumed to be known $t-1$. The budget constraint of households follows:

$$C_t^P + Q_t \Delta H_t^P + \frac{B_t^P}{P_t} = \frac{R_{t-1}^B B_{t-1}^P}{\mathcal{T}_{t-1} P_t} + W_t^P L_t^P + \mathcal{D}_t + T_t^P - \frac{\Delta \mathcal{M}_t^P}{P_t} - \mathcal{T}_t^P, \tag{C.1}$$

where Δ is the first difference operator. Q_t denotes the real housing price, W_t^P is the real wage, and

$\Pi_t \equiv P_t/P_{t-1}$ is the gross inflation rate. $\mathcal{D}_t$ is the lump-sump profits received from the retailer, T_t^P is the central bank transfer, and $\mathcal{T}_t^P$ is the lump-sum tax.

The first-order conditions for consumption, labor supply, and housing demand are

$$\frac{1}{C_t^P} = \mathbb{E}_t\left(\frac{\beta_{t+1}R_t^B}{\mathcal{T}_t\Pi_{t+1}C_{t+1}^P}\right) \tag{C.2}$$

$$W_t^P = (L_t^P)^\eta C_t^P \tag{C.3}$$

$$\frac{Q_t}{C_t^P} = \frac{j}{H_t^P} + \mathbb{E}_t\left(\frac{\beta_{t+1}Q_{t+1}}{C_{t+1}^P}\right). \tag{C.4}$$

Appendix C.1.2 Impatient households

Impatient households (denoted with a superscript I) have a lower discount factor β^I than the patient ones, which guarantees the borrowing constraint for the impatient households binds in equilibrium. They choose consumption C_t^I, housing service H_t^I, labor supply L_t^I, and the real money balance $\Delta\mathcal{M}_t^I/P_t$ to maximize lifetime utility given by

$$\mathbb{E}_0\sum_{t=0}^{\infty}(\beta^I)^t\left[\log C_t^I + j\log H_t^I - (L_t^I)^{1+\eta}/(1+\eta) + \chi_{\mathcal{M}}\log(\mathcal{M}_t^I/P_t)\right].$$

The budget constraint and borrowing constraint are

$$C_t^I + Q_t\Delta H_t^I + \frac{R_{t-1}^B B_{t-1}^I}{\mathcal{T}_{t-1}P_t} = \frac{B_t^I}{P_t} + W_t^I L_t^I + T_t^I - \frac{\Delta\mathcal{M}_t^I}{P_t} - \mathcal{T}_t^I \tag{C.5}$$

$$B_t^I/P_t \le M_t^I\mathbb{E}_t(Q_{t+1}H_t^I\Pi_{t+1}/R_t^B), \tag{C.6}$$

where B_t^I is the nominal loan borrowed by the impatient households, W_t^I is the real wage, T_t^I is the central bank transfer, $\mathcal{T}_t^I$ is the lump-sum tax. M_t^I is the loan-to-value ratio faced by the impatient households, where $M_t^I = M^I$ during normal times when the patient household is the sole lender. The first-order conditions for labor supply and housing service can be summarized as follows:

$$W_t^I = (L_t^I)^\eta C_t^I \tag{C.7}$$

$$\frac{Q_t}{C_t^I} = \frac{j}{H_t^I} + \mathbb{E}_t\left[\beta^I\frac{Q_{t+1}}{C_{t+1}^I}\left(1-\frac{M_t^I}{\mathcal{T}_t}\right) + \frac{M_t^I Q_{t+1}\Pi_{t+1}}{C_t^I R_t^B}\right]. \tag{C.8}$$

Appendix C.1.3 Entrepreneurs

Entrepreneurs (denoted by superscript E) produce intermediate good Y_t^E according to a Cobb-Douglas function:

$$Y_t^E = A_t K_{t-1}^\mu (H_{t-1}^E)^\nu (L_t^P)^{\alpha(1-\mu-\nu)}(L_t^I)^{(1-\alpha)(1-\mu-\nu)}, \tag{C.9}$$

where the technology A_t has a random shock $A_t/A = (A_{t-1}/A)^{\rho_a}\,\varepsilon_{a,t}$ and A is normalized to be 1. Both the housing input H_{t-1}^E and physical capital K_{t-1} used for the period t production are determined at time $t-1$. Capital accumulates following the law of motion: $K_t = I_t + (1-\delta)K_{t-1}$, where δ is the depreciation rate and I_t is investment. Capital installation entails an adjustment cost: $\xi_{K,t} = \psi(I_t/K_{t-1}-\delta)^2K_{t-1}/(2\delta)$. Entrepreneurs sell the intermediate goods to retailers at price P_t^E. The markup for the retailers is $X_t \equiv P_t/P_t^E$.

Entrepreneurs choose consumption C_t^E, investment on capital stock I_t, housing service H_t^E, and labor input L_t^P and L_t^I to maximize their utility $\mathbb{E}_0\sum_{t=0}^{\infty}\gamma^t\log C_t^E$, where the entrepreneurs' discount factor γ is smaller than β. The borrowing constraint entrepreneurs face is

$$B_t^E/P_t \le M_t^E\mathbb{E}_t(Q_{t+1}H_t^E\Pi_{t+1}/R_t^B), \tag{C.10}$$

where $M_t^E = M^E$ when the patient household is the sole lender during normal times. The budget constraint is

$$\frac{Y_t^E}{X_t} + \frac{B_t^E}{P_t} = C_t^E + Q_t \Delta H_t^E + \frac{R_{t-1}^B B_{t-1}^E}{\mathcal{T}_{t-1} P_t} + W_t^P L_t^P + W_t^I L_t^I + I_t + \xi_{K,t}. \tag{C.11}$$

The first-order conditions can be expressed in four equations:

$$\frac{Q_t}{C_t^E} = \mathbb{E}_t \left\{ \frac{\gamma}{C_{t+1}^E} \left[\frac{\nu Y_{t+1}^E}{X_{t+1} H_t^E} + \left(1 - \frac{M_t^E}{\mathcal{T}_t}\right) Q_{t+1} \right] + \frac{1}{C_t^E} \frac{M_t^E Q_{t+1} \Pi_{t+1}}{R_t^B} \right\} \tag{C.12}$$

$$W_t^P = \frac{\alpha(1-\mu-\nu) Y_t^E}{X_t L_t^P} \tag{C.13}$$

$$W_t^I = \frac{(1-\alpha)(1-\mu-\nu) Y_t^E}{X_t L_t^I} \tag{C.14}$$

$$\frac{1}{C_t^E} \left[1 + \frac{\psi}{\delta} \left(\frac{I_t}{K_{t-1}} - \delta \right) \right] = \gamma \mathbb{E}_t \left\{ \frac{1}{C_{t+1}^E} \left[\frac{\mu Y_{t+1}^E}{X_{t+1} K_t} + (1-\delta) - \frac{\psi}{2\delta} \left(\delta - \frac{I_{t+1}}{K_t} \right) \left(2 - \delta + \frac{I_{t+1}}{K_t} \right) \right] \right\}. \tag{C.15}$$

Appendix C.1.4 Retailers

A continuum of retailers of mass 1, indexed by z, buy intermediate goods Y_t^E from entrepreneurs at P_t^E in a competitive market, differentiate one unit of goods at no cost into one unit of retail goods $Y_t(z)$, and sell it at the price $P_t(z)$. Final goods Y_t are from a CES aggregation of the differentiated goods produced by retailers, $Y_t = (\int_0^1 Y_t(z)^{1-\frac{1}{\epsilon}} dz)^{\frac{\epsilon}{\epsilon-1}}$, the aggregate price index is $P_t = (\int_0^1 P_t(z)^{1-\epsilon} dz)^{\frac{1}{1-\epsilon}}$, and the individual demand curve is $Y(z) = \left(\frac{P_t(z)}{P_t}\right)^{-\epsilon} Y_t$, where ϵ is the elasticity of substitution for the CES aggregation.

They face Calvo-stickiness: the sales price can be updated every period with a probability of $1-\theta$. When retailers can optimize the price, they reset it at $P_t^*(z)$; otherwise, the price is partially indexed to the past and steady state inflation; that is,

$$P_t(z) = \begin{cases} P_{t-1}(z) \left(\frac{P_{t-1}}{P_{t-2}}\right)^{\xi_p} \Pi^{1-\xi_p}, \\ P_t^*(z) \end{cases} \tag{C.16}$$

where Π is the steady-state inflation.

The optimal price $P_t^*(z)$ set by retailers that can change price at time t solves:

$$\max_{P_t^*(z)} \sum_{k=0}^{\infty} \theta^k \mathbb{E}_t \left[\Lambda_{t,k} (P_t / P_{t+k}) \left(P_t^*(z) \left(\frac{P_{t+k-1}}{P_{t-1}} \right)^{\xi_p} \Pi^{(1-\xi_p)k} Y_{t+k|t}(z) - P_{t+k}^E Y_{t+k|t}(z) \right) \right],$$

where $\Lambda_{t,k} \equiv \beta^k (C_t^P / C_{t+k}^P)$ is the patient households' real stochastic discount factor between t and $t+k$, and subject to

$$Y_{t+k|t}(z) = \left(\frac{P_t^*(z) \left(\frac{P_{t+k-1}}{P_{t-1}}\right)^{\xi_p} \Pi^{(1-\xi_p)k}}{P_{t+k}} \right)^{-\epsilon} Y_{t+k}.$$

The first-order condition for the retailer's problem takes the form

$$\sum_{k=0}^{\infty} \theta^k \mathbb{E}_t \left[\Lambda_{t,k} P_t \left(\frac{(\epsilon-1) P_t^*(z) (P_{t+k-1}/P_{t-1})^{\xi_p} \Pi^{(1-\xi_p)k}}{P_{t+k}} - \frac{\epsilon}{X_{t+k}} \right) Y_{t+k|t}(z) \right] = 0. \tag{C.17}$$

The aggregate price level evolves as follows:

$$P_t = \left\{ \theta \left[P_{t-1} \left(\frac{P_{t-1}}{P_{t-2}} \right)^{\xi_p} \Pi^{1-\xi_p} \right] + (1-\theta)(P_t^*)^{1-\epsilon} \right\}^{1/(1-\epsilon)}. \tag{C.18}$$

Appendix C.1.5 Government

Corollary 3 *Propositions 1 - 3 imply the central bank's operations satisfy*

$$
\begin{aligned}
&R_t RP_t/\mathcal{T}_t = S_t RP/\mathcal{T}\\
&R_t RP_t/M_t^I = S_t RP/M^I\\
&R_t RP_t/M_t^E = S_t RP/M^E\\
&M_t^I/\mathcal{T}_t = M^I/\mathcal{T}\\
&M_t^E/\mathcal{T}_t = M^E/\mathcal{T},
\end{aligned}
$$

where the shadow rate follows a Taylor rule:[8]

$$
\frac{S_t}{R} = \left(\frac{S_{t-1}}{R}\right)^{\phi_s} \left[(\Pi_{t-1}/\Pi)^{\phi_\pi}(Y_{t-1}/Y)^{\phi_y}\right]^{1-\phi_s}, \tag{C.19}
$$

and R, Π, and Y are steady-state policy rate, inflation, and output, respectively.

The central bank prints money $\Delta\mathcal{M}_t^{CB}$ to finance bond purchases B_t^{CB} and transfers. All money supplied by the central bank is eventually held by the households.

$$
\Delta\mathcal{M}_t^{CB}/P_t = \Delta\mathcal{M}_t^P/P_t + \Delta\mathcal{M}_t^I/P_t = T_t^P + T_t^I + T_t^G + B_t^{CB}/P_t - B_{t-1}^{CB}R_{t-1}^B/\mathcal{T}_{t-1}P_t, \tag{C.20}
$$

where T_t^G is the real transfer from the central bank to the government. The central bank holds bonds due to QE B_t^{QE} per (3.7) and lending facilities according to

$$
B_t^{LF} = (M_t^I - M^I)\mathbb{E}_t(Q_{t+1}H_t^I\Pi_{t+1}/R_t^B) + (M_t^E - M^E)\mathbb{E}_t(Q_{t+1}H_t^E\Pi_{t+1}/R_t^B), \tag{C.21}
$$

and $B_t^{CB} = B_t^{QE} + B_t^{LF}$.

The fiscal authority collects lump-sum tax from households to finance government spending:

$$
G_t = T_t^G + \mathcal{T}_t^P + \mathcal{T}_t^I, \tag{C.22}
$$

where G_t is government spending, and follows the process:

$$
\frac{G_t}{G} = \left(\frac{G_{t-1}}{G}\right)^{\rho_g} \varepsilon_{g,t}, \tag{C.23}
$$

where $\varepsilon_{g,t}$ is the government-spending shock, and $\mathcal{T}_t^P(\mathcal{T}_t^I)$ is a lump-sum tax to patient (impatient) households. The share of the lump-sum tax of each sector is determined by its wage share, respectively:

$$
\mathcal{T}_t^P = \alpha(G_t - T_t^G) \tag{C.24}
$$

$$
\mathcal{T}_t^I = (1-\alpha)(G_t - T_t^G). \tag{C.25}
$$

Appendix C.1.6 Equilibrium

The equilibrium consists of an allocation $\{H_t^E, H_t^P, H_t^I, L_t^E, L_t^P, L_t^I, Y_t, C_t^E, C_t^P, C_t^I, I_t, K_t, B_t^E, B_t^P, B_t^I, B_t^{CB}, G_t, \Delta\mathcal{M}_t^{CB}, \Delta\mathcal{M}_t^P, \Delta\mathcal{M}_t^I, T_t^P, T_t^I, T_t^G, \mathcal{T}_t^P, \mathcal{T}_t^I\}_{t=0}^{\infty}$, and a sequence of prices $\{W_t^P, W_t^I, S_t, P_t, P_t^*, X_t, Q_t\}_{t=0}^{\infty}$, that solves the household and firm problems and market-clearing conditions: $H_t^E + H_t^P + H_t^I = H$, $C_t^E + C_t^P + C_t^I + I_t + G_t = Y_t$, $B_t^P + B_t^{CB} = B_t^E + B_t^I$, (C.20), (C.22).

Table C.1: Calibrated parameters in the quantitative model

para	description	source	value
β	discount factor of patient households	Iacoviello (2005)	0.99
β^I	discount factor of impatient households	Iacoviello (2005)	0.95
γ	discount factor of entrepreneurs	Iacoviello (2005)	0.98
j	steady-state weight on housing services	Iacoviello (2005)	0.1
η	labor supply aversion	Iacoviello (2005)	0.01
μ	capital share in production	Iacoviello (2005)	0.3
ν	housing share in production	Iacoviello (2005)	0.03
δ	capital depreciation rate	Iacoviello (2005)	0.03
X	steady state gross markup	Iacoviello (2005)	1.05
θ	probability that cannot re-optimize	Iacoviello (2005)	0.75
α	patient households' wage share	Iacoviello (2005)	0.64
M^E	loan-to-value ratio for entrepreneurs	Iacoviello (2005)	0.89
M^I	loan-to-value ratio for impatient households	Iacoviello (2005)	0.55
ϕ_s	interest rate persistence	Iacoviello (2005)	0.73
ϕ_y	interest rate response to output	Iacoviello (2005)	0.27
ϕ_π	interest rate response to inflation	Iacoviello (2005)	0.13
$\frac{G}{Y}$	steady-state government-spending-to-output ratio	Fernández-Villaverde et al. (2015)	0.20
ρ_a	autocorrelation of technology shock	Fernández-Villaverde et al. (2015)	0.90
ρ_g	autocorrelation of government-spending shock	Fernández-Villaverde et al. (2015)	0.80
ρ_β	autocorrelation of discount rate shock	Fernández-Villaverde et al. (2015)	0.80
σ_a	standard deviation of technology shock	Fernández-Villaverde et al. (2015)	0.0025
σ_g	standard deviation of government-spending shock	Fernández-Villaverde et al. (2015)	0.0025
σ_β	standard deviation of discount rate shock	Fernández-Villaverde et al. (2015)	0.0025
ξ_p	price indexation	Smets and Wouters (2007)	0.24
Π	steady-state inflation	2% annual inflation	1.005
$\frac{\tilde{B}^{CB}}{\tilde{B}^E+\tilde{B}^I}$	steady-state central bank bond holdings ratio	Fed's asset holdings	0.02
$\mathcal{T}$	steady-state tax (subsidy) on interest rate income (payment)	no tax in normal times	1
rp	steady-state risk premium	3.6% risk premium annually	1.009

Appendix C.2 Calibration

Table C.1 presents the calibrated parameters. Many of them are from Iacoviello (2005), Fernández-Villaverde et al. (2015), and Smets and Wouters (2007). For other parameters, we match the following empirical moments. The steady-state gross inflation is set to 1.005, which implies a 2% annual inflation rate. The steady-state central bank's bond holdings ratio matches the average ratio of Fed' total assets and all sectors' debt securities and loans in the U.S. during 2003 -2007, which is 2%. The steady-state tax on the gross interest rate income is set to 1 to imply zero tax on net interest rate income during normal times. The net quarterly risk premium is set to 0.9% to match the 3.6% average historical annual risk premium.

Appendix C.3 Steady state

The patient households' Euler equation gives us the steady-state private borrowing rate, shadow rate, and the real private borrowing rate:

$$R^B = \Pi/\beta \tag{C.26}$$

$$S = R = R^B/RP \tag{C.27}$$

$$RR^B = 1/\beta. \tag{C.28}$$

Entrepreneurs' first-order conditions on housing, the borrowing constraint, and budget constraint give

[8]We follow Iacoviello (2005) to assume the Taylor rule depends on lagged output and inflation. Whether the variables are lagged or contemporaneous does not affect our results.

their real estate share, debt-to-output, and consumption-to-output ratio:

$$\frac{QH^E}{Y} = \frac{\gamma\nu}{X(1-\gamma^e)} \tag{C.29}$$

$$\frac{\tilde{B}^E}{Y} = \beta M^E \frac{QH^E}{Y} \tag{C.30}$$

$$\frac{C^E}{Y} = \left[\mu + \nu - \frac{\delta\gamma\mu}{1-\gamma(1-\delta)} - (1-\beta)M^E X \frac{QH^E}{Y}\right]\frac{1}{X}, \tag{C.31}$$

where $\gamma^e = (1-M^E)\gamma + M^E\beta$.

Impatient households' budget constraint, borrowing constraint, and first-order condition on housing give their real estate share, debt-to-output, and consumption-to-output ratio:

$$\frac{QH^I}{C^I} = \frac{j}{[1-\beta^I(1-M^I) - \frac{M^I}{RR^B}]} \tag{C.32}$$

$$\frac{\tilde{B}^I}{QH^I} = \frac{M^I\Pi}{R^B} \tag{C.33}$$

$$\frac{C^I}{Y} = \frac{s^I - \alpha\frac{G-T^G}{Y}}{1 + \frac{QH^I}{C^I}(RR^B - 1)\frac{\tilde{B}^I}{QH^I}}, \tag{C.34}$$

where $s^I = \frac{(1-\alpha)(1-\mu-\nu)}{X}$ is the income share of impatient households.

In the steady state, real balances, transfers to households and central bank bond holdings are constant, (C.20) implies

$$T^G = (\frac{R^B}{\Pi} - 1)\tilde{B}^{CB}. \tag{C.35}$$

The bond-market-clearing condition, patient households' budget constraint, and first-order condition with respect to housing imply

$$\frac{\tilde{B}^P}{Y} = \frac{\tilde{B}^E}{Y} + \frac{\tilde{B}^I}{Y} - \frac{\tilde{B}^{CB}}{Y} \tag{C.36}$$

$$\frac{C^P}{Y} = s^P - (1-\alpha)\frac{G-T^G}{Y} + (RR^B - 1)\frac{\tilde{B}^P}{Y} \tag{C.37}$$

$$\frac{QH^P}{C^P} = \frac{j}{1-\beta} \tag{C.38}$$

$$\frac{QH^P}{Y} = \frac{QH^P}{C^P}\frac{C^P}{Y}, \tag{C.39}$$

where

$$s^P = [\alpha(1-\mu-\nu) + X - 1]/X$$

is the income shares of patient households.

Housing shares of different sectors follows:

$$\frac{H^E}{H^P} = \frac{QH^E}{Y} / \frac{QH^P}{Y} \tag{C.40}$$

$$\frac{H^I}{H^P} = \frac{QH^I}{Y} / \frac{QH^E}{Y}. \tag{C.41}$$

The investment-output ratio follows:

$$\frac{I}{Y} = 1 - \frac{C^E}{Y} - \frac{C^I}{Y} - \frac{C^P}{Y} - \frac{G}{Y} \tag{C.42}$$

Appendix C.4 Log-linearized model

We present the linear model with the shadow rate first in Appendix C.4.1. Then, we map it into unconventional policy tools in Appendix C.4.2 - Appendix C.4.3. Appendix C.4.4 explains the implementation of the standard model without unconventional monetary policy.

Appendix C.4.1 Shadow rate representation

In this representation, we summarize all policy tools, conventional and unconventional, with a single variable S_t according to Corollary 3. Let hatted variables in lower case denote percentage changes from the steady state. The model can be expressed in the following blocks of equations:
1. Aggregate demand:

$$\widehat{y}_t = \frac{C^E}{Y}\widehat{c}^E_t + \frac{C^P}{Y}\widehat{c}^P_t + \frac{C^I}{Y}\widehat{c}^I_t + \frac{I}{Y}\widehat{i}_t + \frac{G}{Y}\widehat{g}_t \tag{C.43}$$

$$\widehat{c}^P_t = \mathbb{E}_t(\widehat{c}^P_{t+1} - \widehat{s}_t + \widehat{\pi}_{t+1} - \widehat{\beta}_{t+1}) \tag{C.44}$$

$$\widehat{i}_t - \widehat{k}_{t-1} = \gamma\left(\mathbb{E}_t\widehat{i}_{t+1} - \widehat{k}_t\right) + \frac{1-\gamma(1-\delta)}{\psi}\left[\mathbb{E}_t\left(\widehat{y}_{t+1} - \widehat{x}_{t+1}\right) - \widehat{k}_t\right] + \frac{1}{\psi}\left(\widehat{c}^E_t - \mathbb{E}_t\widehat{c}^E_{t+1}\right) \tag{C.45}$$

2. Housing/consumption margin:

$$\begin{aligned}\widehat{q}_t = {} & \gamma^e\mathbb{E}_t\widehat{q}_{t+1} + (1-\gamma^e)\left(\mathbb{E}_t\widehat{y}_{t+1} - \mathbb{E}_t\widehat{x}_{t+1} - \widehat{h}^E_t\right) + \left(1 - M^E\beta\right)\left(\widehat{c}^E_t - \mathbb{E}_t\widehat{c}^E_{t+1}\right) \\ & + M^E\beta\left(\mathbb{E}_t\widehat{\pi}_{t+1} - \widehat{s}_t\right)\end{aligned} \tag{C.46}$$

$$\widehat{q}_t = \gamma^h\mathbb{E}_t\widehat{q}_{t+1} - \left(1-\gamma^h\right)\widehat{h}^I_t + M^I\beta\left(\mathbb{E}_t\widehat{\pi}_{t+1} - \widehat{s}_t\right) + \left(1 - \frac{M^I\beta}{\mathcal{T}}\right)\widehat{c}^I_t - \beta^I\left(1 - M^I\right)\mathbb{E}_t\widehat{c}^I_{t+1} \tag{C.47}$$

$$\widehat{q}_t = \beta\mathbb{E}_t(\widehat{q}_{t+1} + \widehat{\beta}_{t+1}) + \left(\widehat{c}^P_t - \beta\mathbb{E}_t\widehat{c}^P_{t+1}\right) + (1-\beta)\frac{H^E}{H^P}\widehat{h}^E_t - (1-\beta)\frac{H^I}{H^P}\widehat{h}^I_t, \tag{C.48}$$

where

$$\gamma^h = M^I\beta + \left(1 - M^I\right)\beta^I$$

3. Borrowing constraints:

$$\widehat{\tilde{b}}^E_t = \mathbb{E}_t\widehat{q}_{t+1} - \left(\widehat{s}_t - \mathbb{E}_t\widehat{\pi}_{t+1}\right) + \widehat{h}^E_t \tag{C.49}$$

$$\widehat{\tilde{b}}^I_t = \mathbb{E}_t\widehat{q}_{t+1} - \left(\widehat{s}_t - \mathbb{E}_t\widehat{\pi}_{t+1}\right) + \widehat{h}^I_t \tag{C.50}$$

4. Aggregate supply:

$$\widehat{y}_t = \frac{1+\eta}{\eta+\nu+\mu}(\widehat{a}_t + \nu\widehat{h}^E_{t-1} + \mu\widehat{k}_{t-1}) - \frac{1-\nu-\mu}{\eta+\nu+\mu}(\widehat{x}_t + \alpha\widehat{c}^P_t + (1-\alpha)\widehat{c}^I_t) \tag{C.51}$$

$$\widehat{\pi}_t = \frac{\beta}{1+\beta\xi_p}\mathbb{E}_t\widehat{\pi}_{t+1} + \frac{\xi_p}{1+\beta\xi_p}\widehat{\pi}_{t-1} - \frac{1}{1+\beta\xi_p}\varkappa\widehat{x}_t + \widehat{e}_{\pi,t}, \tag{C.52}$$

where

$$\varkappa = (1-\theta)(1-\beta\theta)/\theta$$

5. Flows of funds/evolution of state variables:

$$\widehat{k}_t = \delta\widehat{i}_t + (1-\delta)\widehat{k}_{t-1} \tag{C.53}$$

$$\begin{aligned}\frac{\tilde{B}^E}{Y}\widehat{\tilde{b}}_t^E &= \frac{C^E}{Y}\widehat{c}_t^E + \frac{QH^E}{Y}(\widehat{h}_t^E - \widehat{h}_{t-1}^E) + \frac{I}{Y}\widehat{i}_t + RR^B\frac{\tilde{B}^E}{Y}(\widehat{s}_{t-1} - \widehat{\pi}_t + \widehat{\tilde{b}}_{t-1}^E) \\ &\quad -(1-s^P-s^I)(\widehat{y}_t - \widehat{x}_t)\end{aligned} \tag{C.54}$$

$$\begin{aligned}\frac{\tilde{B}^I}{Y}\widehat{\tilde{b}}_t^I &= \frac{C^I}{Y}\widehat{c}_t^I + \frac{QH^I}{Y}(\widehat{h}_t^I - \widehat{h}_{t-1}^I) + RR^e\frac{\tilde{B}^I}{Y}(\widehat{s}_{t-1} - \widehat{\pi}_t + \widehat{\tilde{b}}_{t-1}^I) \\ &\quad -s^I(\widehat{y}_t - \widehat{x}_t) - \frac{(1-\alpha)G}{Y}\widehat{g}_t\end{aligned} \tag{C.55}$$

6. Monetary policy rule and shock processes:

$$\widehat{s}_t = (1-\phi_s)[(1+\phi_\pi)\widehat{\pi}_{t-1} + \phi_y Y\widehat{y}_{t-1}] + \phi_s\widehat{s}_{t-1} \tag{C.56}$$

$$\widehat{a}_t = \rho_a\widehat{a}_{t-1} + \widehat{e}_{a,t} \tag{C.57}$$

$$\widehat{\beta}_t = \rho_\beta\widehat{\beta}_{t-1} + \widehat{e}_{\beta,t} \tag{C.58}$$

$$\widehat{g}_t = \rho_g\widehat{g}_{t-1} + \widehat{e}_{g,t}. \tag{C.59}$$

Appendix C.4.2 QE

During normal times, the central bank varies R_t, whereas at the ZLB, it lowers RP_t through purchasing bonds from impatient households and entrepreneurs to decrease the bond supply to patient households. In this case, we keep the following policy variables constant: $M_t^I = M^I$, $M_t^E = M^E$, and $\mathcal{T}_t = \mathcal{T}$.

Proposition 1 suggests the following implementation of monetary policy through $\widehat{r}_t$ and $\widehat{rp}_t$ to obtain the linear model presented in Appendix C.4.1

$$\begin{cases}\widehat{r}_t = \widehat{s}_t, \widehat{rp}_t = 0 & \text{for } s_t \geq 0 \\ \widehat{r}_t = -s, \widehat{rp}_t = \widehat{s}_t + s & \text{for } s_t < 0.\end{cases}$$

Appendix C.4.3 Lending facilities

In this case, we keep the risk premium constant $R_t^B = R_t RP$. At the ZLB, the government can increase the loan-to-value ratio so that impatient households and entrepreneurs can borrow more money for consumption and production, whereas the patient households still lend according to the borrowing constraints with constant loan-to-value ratios. Moreover, a tax is placed on interest rate income, which is then transferred to the borrowers.

Proposition 3 implies to implement $\widehat{r}_t, \widehat{m}_t^I, \widehat{m}_t^E, \widehat{\tau}_t$ according to

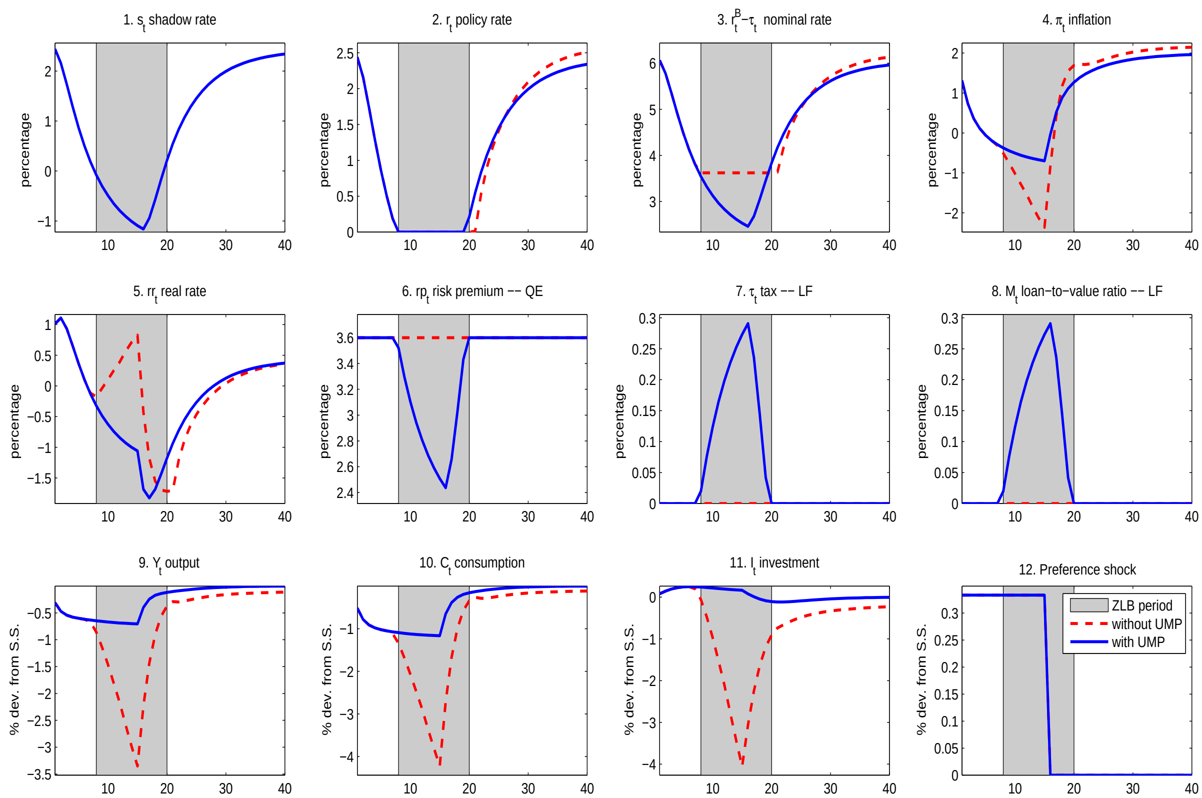
$$\begin{cases}\widehat{r}_t = \widehat{s}_t, \widehat{\tau}_t = \widehat{m}_t^I = \widehat{m}_t^E = 0 & \text{for } s_t \geq 0 \\ \widehat{r}_t = -s, \widehat{\tau}_t = \widehat{m}_t^I = \widehat{m}_t^E = -(\widehat{s}_t + s) & \text{for } s_t < 0.\end{cases}$$

Appendix C.4.4 No unconventional monetary policy

For the standard model without unconventional monetary policy, replace $\widehat{s}_t$ with $\widehat{r}_t$ in (C.43) - (C.55), and augment the monetary policy in (C.56) with (2.4).

Appendix D ZLB environment

Figure D.1: Positive preference shocks



Notes: We hit the economy with a series of positive preference shocks, which occurs in periods 1 - 15, and the total shock size is 5%. The red lines represent the case in which, when the policy rate is bounded by zero, no unconventional monetary policy is implemented. The blue lines represent the case in which unconventional monetary policy is implemented through a negative shadow rate. Plots 1-7 are levels in percentage, among which plots 1-6 are annualized. Plots 8-12 are measured in percentage deviation from the steady state. The shadow rate in plot 1 is an overall measure of monetary policy. In normal times, it is implemented through the policy rate in plot 2. At the ZLB, it can be implemented through QE in plot 6 (per Section 3) or the lending facilities (LF) in plots 7 and 8 (per Section 4). The shaded area marks the ZLB period from 8-20.