Does the Ross Recovery Theorem Work Empirically?

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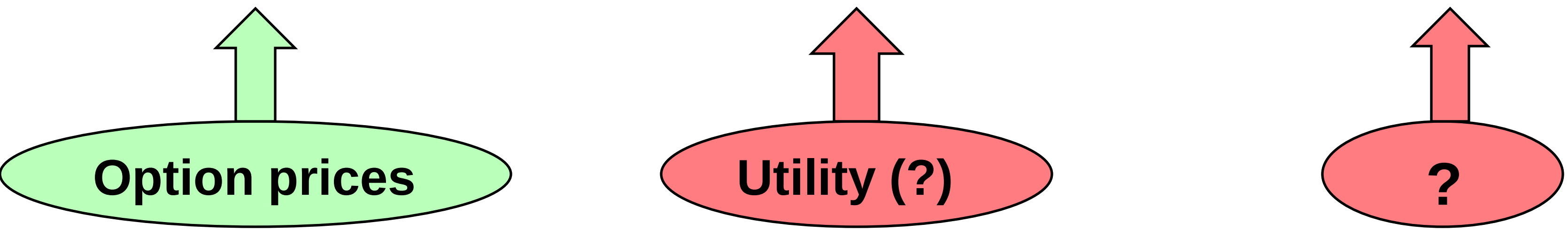
Motivation

- **Holy Grail: Forecast the future return of a stock or an index**
- **State prices (π) = pricing kernel (m) $\times$ physical prob (p)**

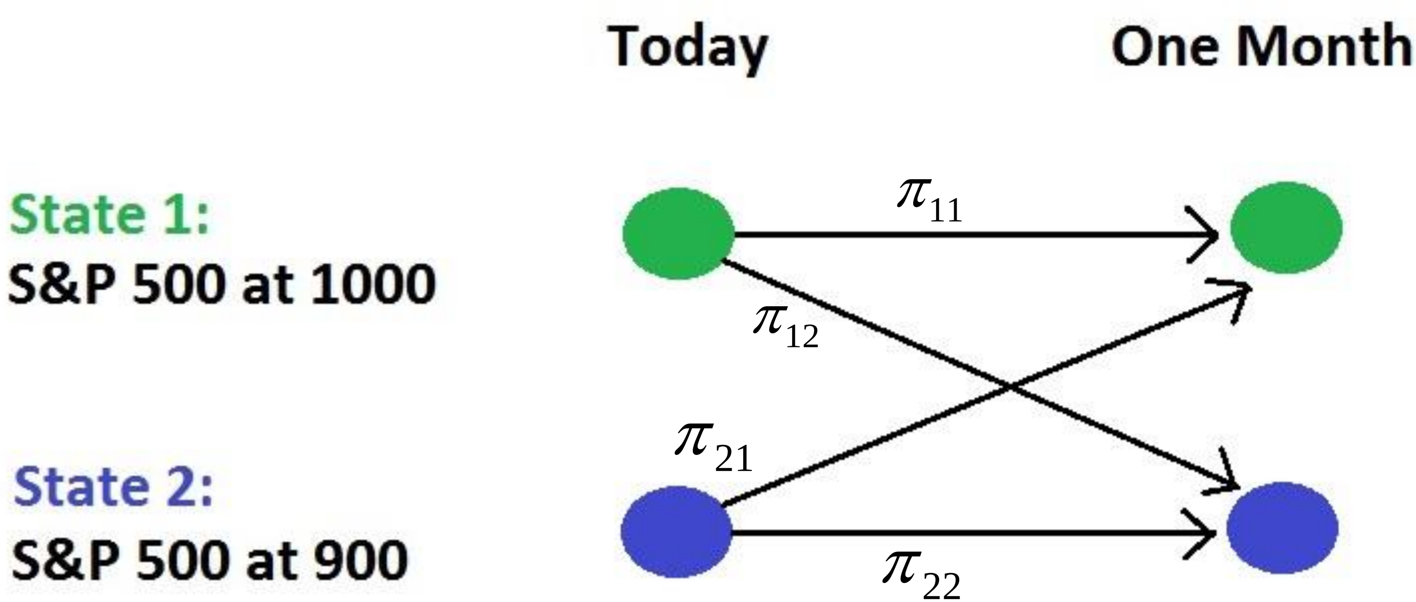


- **Normally: One quantity can be found from the other two**
- **Ross (2015): Determines the pricing kernel <u>AND</u> the physical probabilities from state prices**

C=f(S,t)

Ross Recovery needs Transition State Prices

- **Example: Two states (state 1 and state 2) and transition state prices π_{ij} of moving from state *i* to state *j* in one month**



How does Ross (2015) recover?

Assumptions:

- **Transition state prices π_{ij} are time-homogeneous and positive**
- **The pricing kernel can be written as:** $m_{ij} = \delta \times \frac{u'_j}{u'_i}$

Then:

- **Ross formulates an Eigenvalue problem via transition state prices π_{ij}**
- **That problem has only one unique solution where m_{ij} > 0**
- **That solution delivers pricing kernel m_{ij} <u>AND</u> physical probabilities p_{ij}**

Literature

- **Audrino, Huitema, and Ludwig (2015) apply the theorem empirically and develop a trading strategy based on the recovered moments**

- **Jensen, Lando and Pedersen (2017) develop a generalization of the recovery theorem that imposes additional structure. They test if their model is able to explain future returns with a regression analysis**

- **Borovicka, Hansen, and Scheinkman (2016) show that Ross recovers distorted probabilities: Ross recovery sets a crucial martingale component of the pricing kernel to unity**

- **Bakshi, Chabi-Yo, and Gao (2016) extract that missing component and show that it does not equal unity, but exhibits substantial variation**

The Proof of the Pudding is in the Eating

- **We use S&P 500 European put- and call option quotes:**
  January 1996 – August 2014 (OptionMetrics)

(1) **back out transition state prices from S&P 500 option quotes**

(2) **use Ross recovery to find the physical probabilities**

(3) **statistically test if future realized returns do indeed come from the recovered physical probabilities**

(4) **Ross recovery does NOT work in the data with this approach**

(5) **analyze why Ross recovery fails empirically**

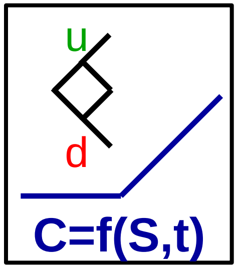
(1) Backing out Transition State Prices I

- **Problem: We do not know transition state prices for a ‘parallel universe state’, that is not the current state**

- **Assumption in Ross (2015): Transition state prices are time-homogeneous**
  - **Obtain spot state prices with one initial state and several maturities from option quotes (maturities up to one year)**
  - **Use them to obtain transition state prices for several initial states and one maturity (one month)**

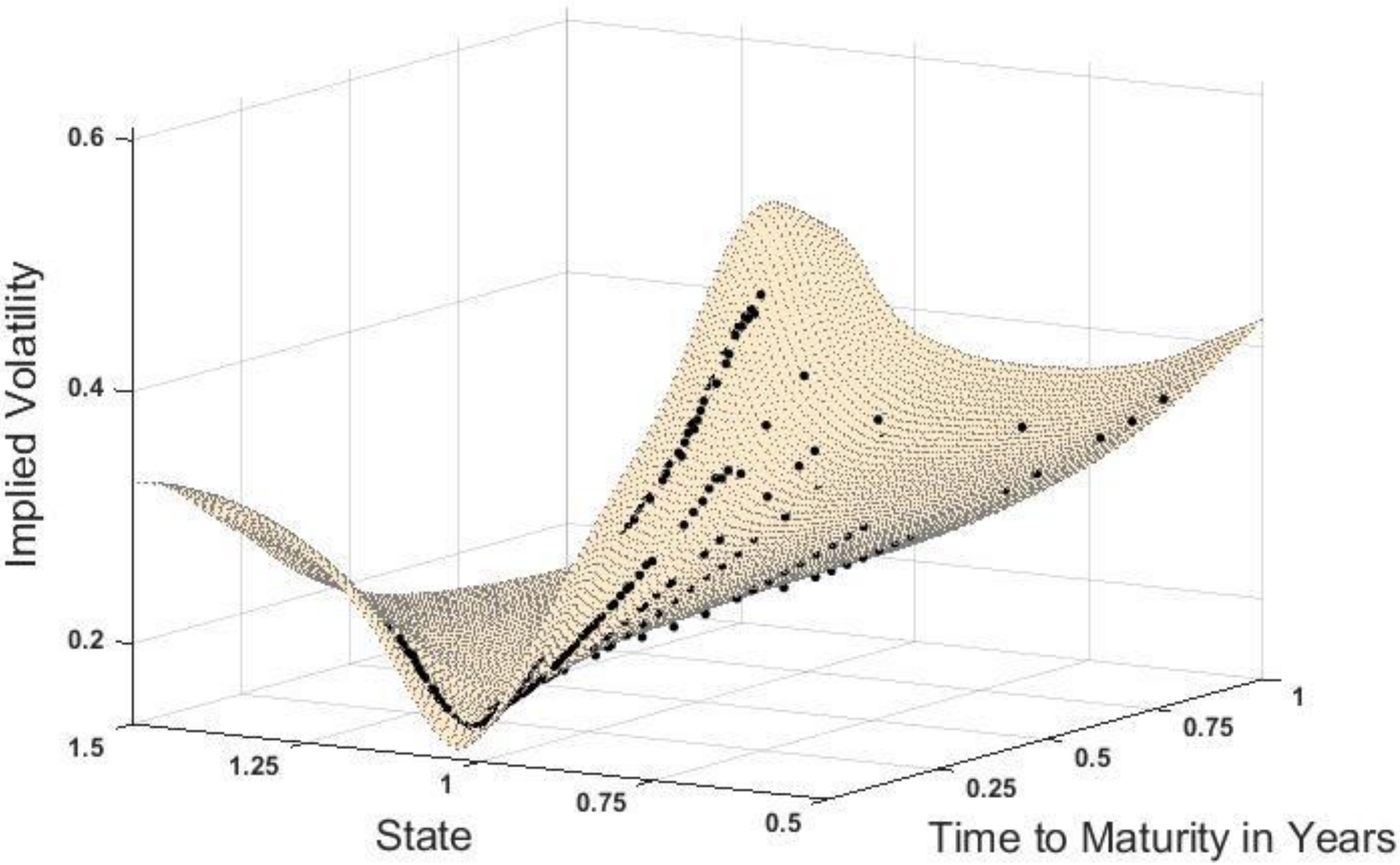
➢ **Get π_{ij} from shifting spot state prices:**

$$\pi_j^{(t+1\,month)} = \sum_i \pi_i^{(t)} \pi_{ij}$$



(1) Backing out Transition State Prices II

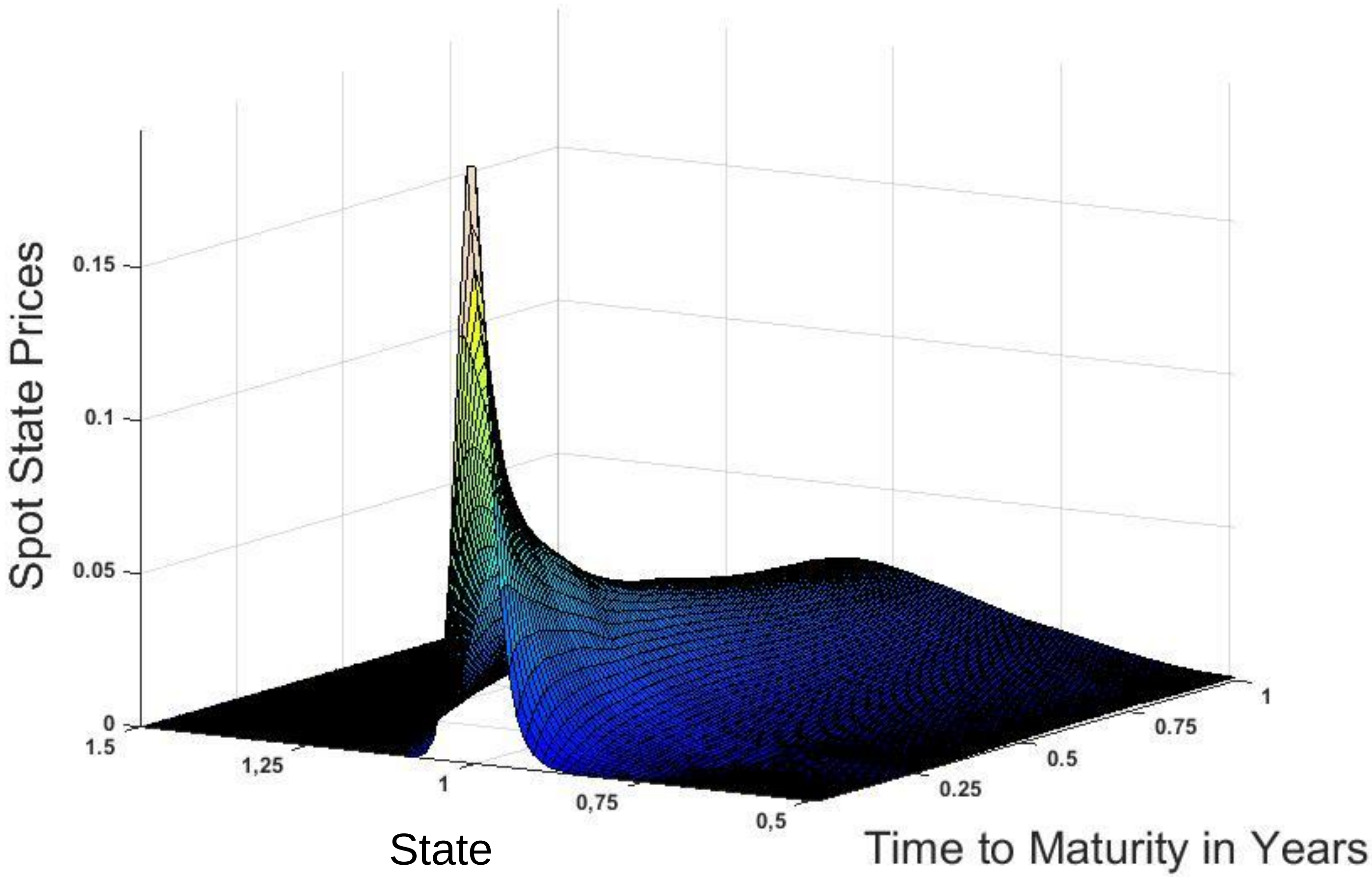
- **Obtain smooth implied volatilities (February 17, 2010)**

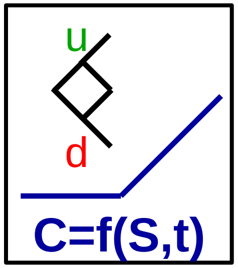


C=f(S,t)

(1) Backing out Transition State Prices III

- **Obtain spot (!) state prices from implied volatilities (February 17, 2010)**





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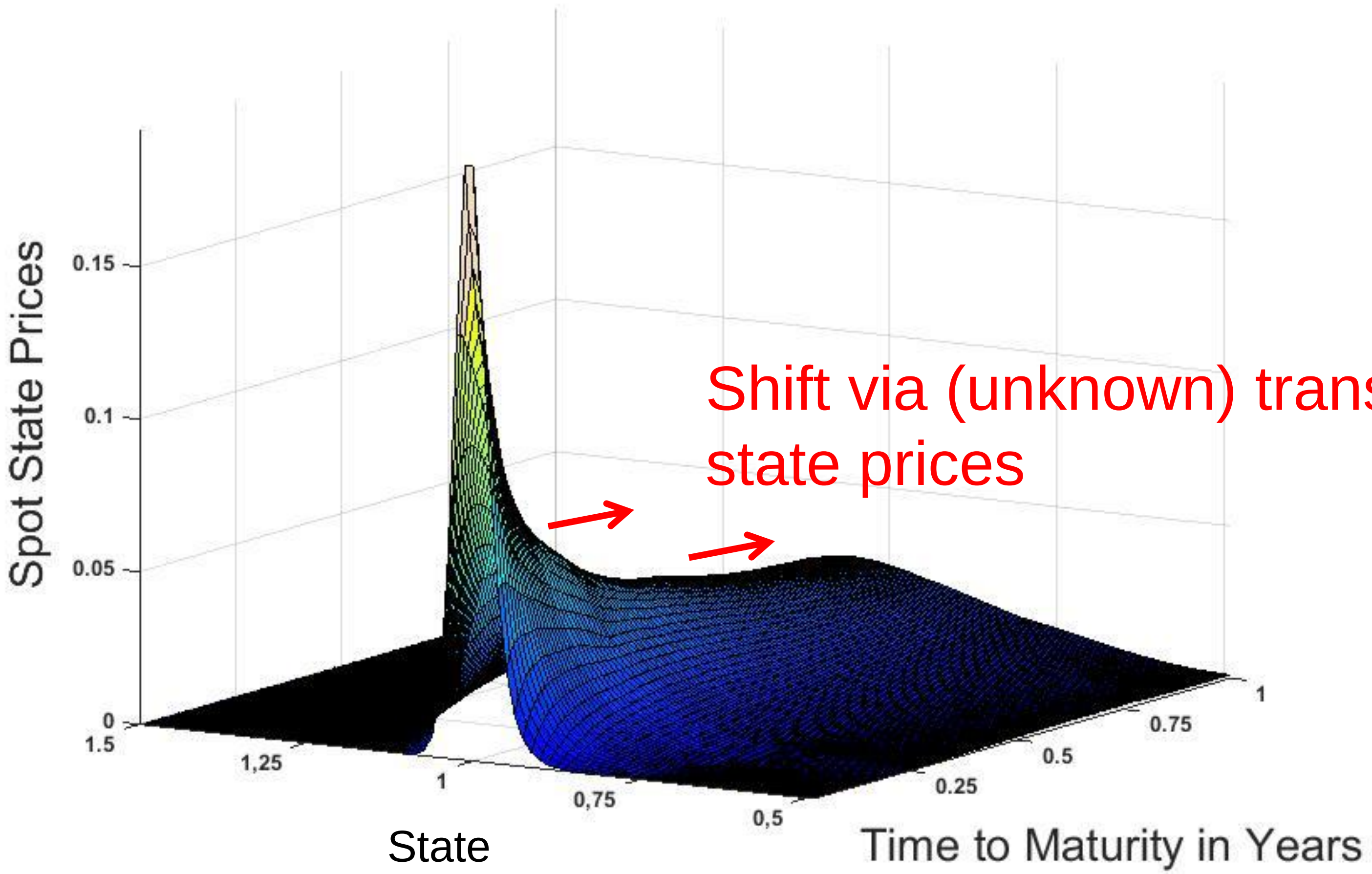


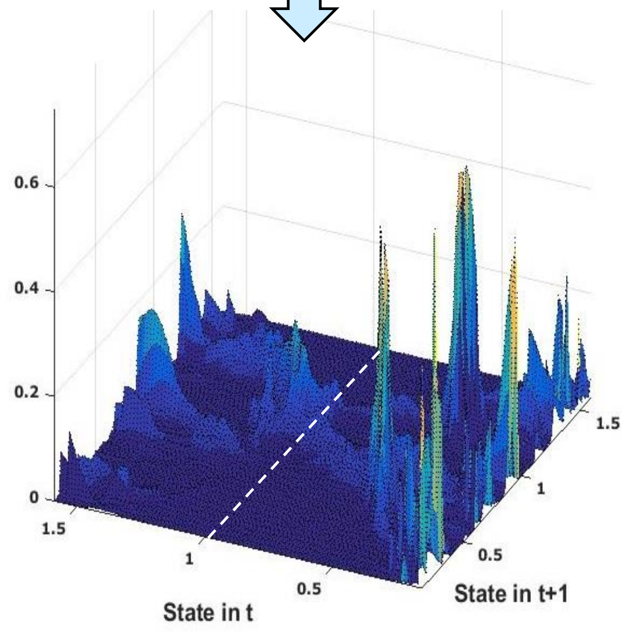
(1) Backing out Transition State Prices IV

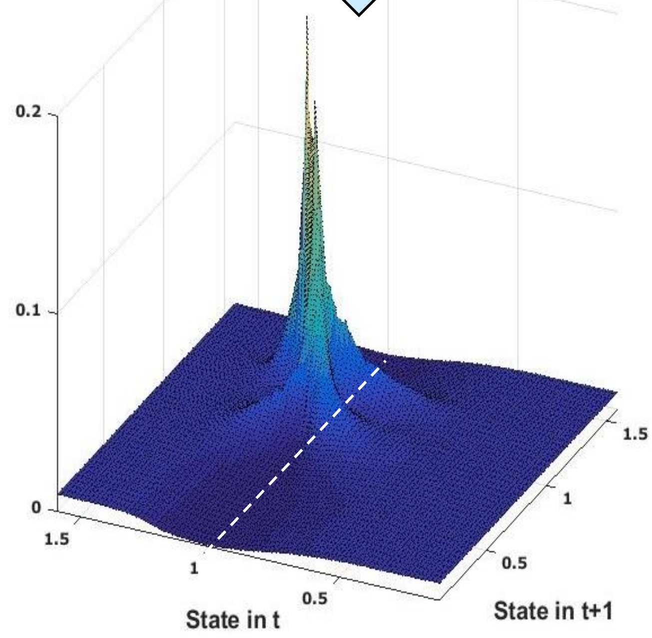
- **Three Ross recovery approaches: Piecewise adding more structure**

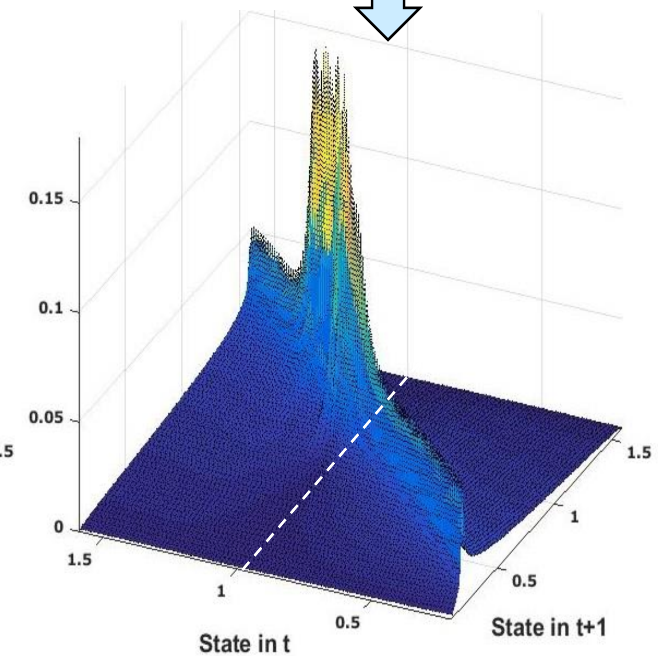
Ross Basic
$\pi_{ij} > 0$

Ross Bounded
$\pi_{ij} > 0$, Rowsum $\in$ [0.9, 1]

Ross Unimodal
$\pi_{ij} > 0$, Rowsum $\in$ [0.9, 1], Unimodal Rows

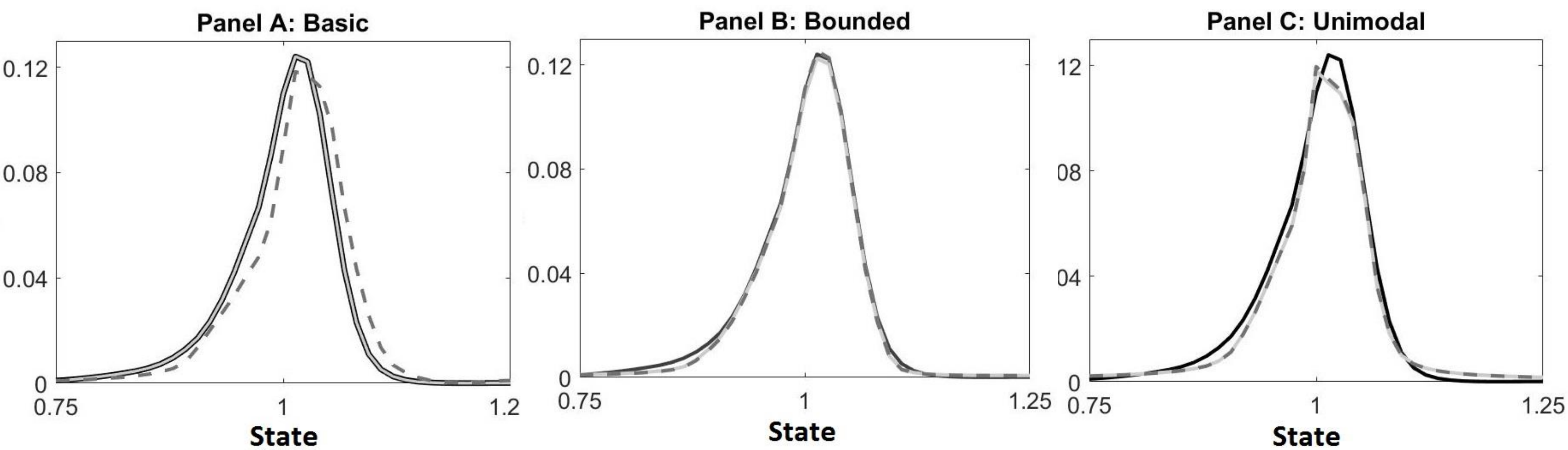






(2) Recovered Probabilities (February 17, 2010)

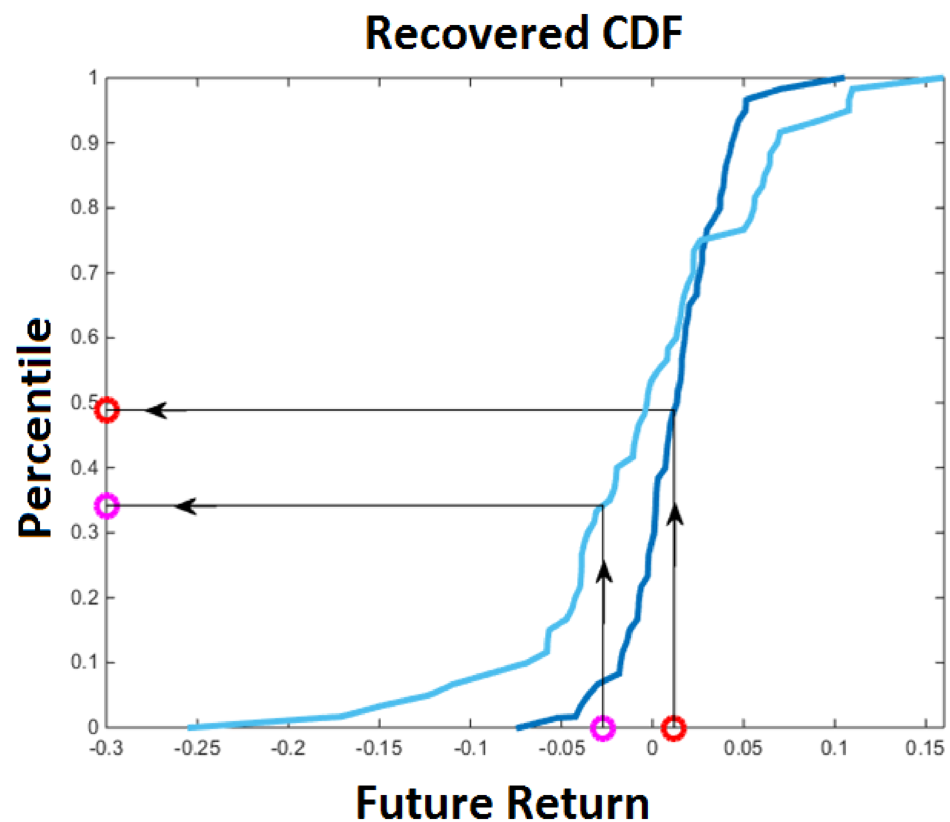
- **Black line: One-month spot state prices**
- **Light gray line: Transition state prices with current initial state**
  - Can deviate from spot state prices due to optimization constraints
- **Gray dashed line: Recovered physical probabilities**



(3) Test if recovered Probabilities predict Future Returns: Berkowitz (2001)

- **H0: Future realized returns are drawn from the recovered distribution**

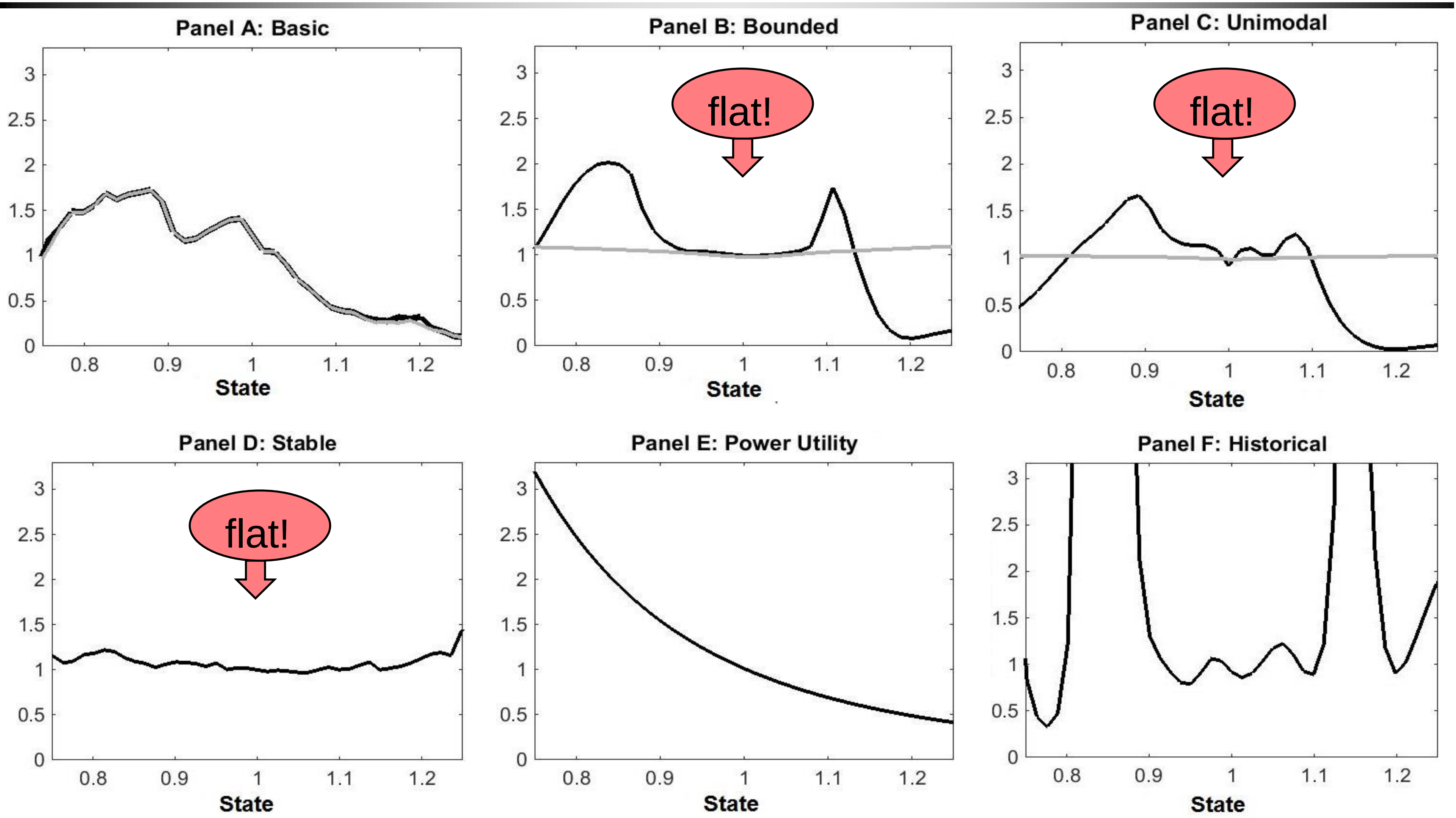
- **For the first date, plug in the next month out-of-sample return into the recovered physical cdf and find the percentile (in between 0 and 1)**
- **Shift one month and repeat the above sequence until the end of the sample (223 months in total)**
- **Test all 223 percentiles for uniformity (Berkowitz test)**



(4) Recovered Probabilities do NOT predict Future Returns

Approach	p-Value: Berkowitz test	Comment
Ross Basic	0.018	Positivity of Π
Ross Bounded	0.005	Positivity of Π, Rowsums in [0.9, 1]
Ross Unimodal	0.001	Positivity of Π, Rowsums in [0.9, 1] , Rows in Π are unimodal
Ross Stable	0.010	Do not use transition matrix Π but work with spot state prices
Power Utility, $\gamma=4$	0.697	Use a power utility function with $\gamma=4$ to obtain the pricing kernel
Historical	0.294	Use the 5 year historical monthly return distribution

(5) Why the Ross Recovery Theorem Fails: Pricing Kernels



(5) Why the Ross Recovery Theorem Fails: Decomposition of Pricing Kernels

- **Borovicka et al. (2016) decompose transition state prices:**

$$\pi_{ij} = m_{ij} \times p_{ij} = m_{ij} \times \frac{p_{ij}}{p_{ij}^{true}} \times p_{ij}^{true} = m_{ij} \times m_{ij}^{perm} \times p_{ij}^{true}$$

- **Ross sets the permanent pricing kernel component to one. What is left is the <u>transitory component</u> of the pricing kernel**

- **We follow Bakshi et al. (2016) to extract the transitory component using 30-year Treasury bond futures (Datastream)**

- **Ross recovery should also deliver the transitory component**

- **We compare both approaches to each other in a regression analysis, but find no similarity**

(5) Why the Ross Recovery Theorem Fails: Time-Homogeneity / Fit to Option Prices

Approach	One-Month Implied Volatility Error: MRMSE	One-Year Implied Volatility Error: MRMSE	Comment
Ross Basic	0.009	0.050	Positivity of Π
Ross Bounded	0.141	0.062	Positivity of Π, Rowsums in [0.9, 1]
Ross Unimodal	0.160	0.065	Positivity of Π, Rowsums in [0.9, 1] , Rows in Π are unimodal
Ross Stable	(0.008)	(0.004)	Do not use transition matrix Π but work with spot state prices
Power Utility, $\gamma=4$	(0.008)	(0.004)	Use a power utility function with $\gamma=4$ to obtain the pricing kernel

(5) Why the Ross Recovery Theorem Fails: A Simulated Ross Recovery Economy

- **Do small data errors in the option prices might cause the recovery theorem to fail?**

- **To check, we simulate economies, where a particular recovery approach holds true: Draw future realized returns from the <u>recovered</u> physical distribution**

- **We get a 95% non-rejection rate for testing at the 5% level that future returns are compatible with the recovery approach**

- **Next we perturb option prices with several different perturbation levels and apply Ross recover. We then test with our simulated economy**

- **Non-rejection rates for Ross Basic decrease even for small imposed errors (<u>Instability</u> - affected by small errors in the data). Other approaches are much more stable**

References

- **Audrino, F., Huitema, R., and Ludwig, M. (2015). An Empirical Analysis of the Ross Recovery Theorem. Working Paper.**
- **Bakshi, G., Chabi-Yo, F., and Gao, X. (2016). A Recovery That We Can Trust? Deducing and Testing the Restrictions of the Recovery Theorem. Working Paper.**
- **Berkowitz, J. (2001). Testing Density Forecasts, With Applications to Risk Management. Journal of Business and Economic Statistics, 19(4):465-474.**
- **Borovicka, J., Hansen, L. P., and Scheinkman, J. A. (2016). Misspecified Recovery. Journal of Finance, 71(6):2493-2544.**
- **Jensen, C. S., Lando, D., and Pedersen, Lasse, H. (2017). Generalized Recovery. Working Paper.**
- **Ross, S. (2015). The Recovery Theorem. Journal of Finance, 70(2):615-648**.

Appendix: The Eigenvalue Problem

- **Physical transition probabilities p_{ij} of moving from one specific state to any other state have to sum up to 1:**

$$p_{11} + p_{12} = 1 \quad \longrightarrow \quad \frac{1}{\delta} \times \frac{1}{u'_1} \times 0.9 \times u'_1 + \frac{1}{\delta} \times \frac{1}{u'_2} \times 0.05 \times u'_1 = 1$$

$$p_{21} + p_{22} = 1 \quad \longrightarrow \quad \frac{1}{\delta} \times \frac{1}{u'_1} \times 0.95 \times u'_2 + \frac{1}{\delta} \times \frac{1}{u'_2} \times 0.02 \times u'_2 = 1$$

- **This results in an Eigenvalue problem:**

$$\begin{pmatrix} \pi_{11} & \pi_{12} \\ \pi_{21} & \pi_{22} \end{pmatrix} \times \begin{pmatrix} \frac{1}{u'_1} \\ \frac{1}{u'_2} \end{pmatrix} = \delta \times \begin{pmatrix} \frac{1}{u'_1} \\ \frac{1}{u'_2} \end{pmatrix} \quad \longrightarrow \quad \begin{pmatrix} 0.9 & 0.05 \\ 0.02 & 0.95 \end{pmatrix} \times \begin{pmatrix} \frac{1}{u'_1} \\ \frac{1}{u'_2} \end{pmatrix} = \delta \times \begin{pmatrix} \frac{1}{u'_1} \\ \frac{1}{u'_2} \end{pmatrix}$$

Appendix: Recover the Pricing Kernel directly, without using Transition Matrix Π (I)

- **We start with the Eigenvalue problem, where state 1 is the current state:**

$$\begin{pmatrix} \pi_{11} & \pi_{12} \\ \pi_{21} & \pi_{22} \end{pmatrix} \times \begin{pmatrix} \dfrac{1}{u'_1} \\ \dfrac{1}{u'_2} \end{pmatrix} = \begin{pmatrix} \dfrac{1}{u'_1} \\ \dfrac{1}{u'_2} \end{pmatrix} \times \delta$$

- **We then can multiply Π from the left hand side and get:**

$$\begin{pmatrix} \pi_{11} & \pi_{12} \\ \pi_{21} & \pi_{22} \end{pmatrix} \times \begin{pmatrix} \pi_{11} & \pi_{12} \\ \pi_{21} & \pi_{22} \end{pmatrix} \times \begin{pmatrix} \dfrac{1}{u'_1} \\ \dfrac{1}{u'_2} \end{pmatrix} = \begin{pmatrix} \pi_{11} & \pi_{12} \\ \pi_{21} & \pi_{22} \end{pmatrix} \times \begin{pmatrix} \dfrac{1}{u'_1} \\ \dfrac{1}{u'_2} \end{pmatrix} \times \delta = \begin{pmatrix} \dfrac{1}{u'_1} \\ \dfrac{1}{u'_2} \end{pmatrix} \times \delta^2$$

- **Multiplying Π with Π gives us a transition matrix for two transitions (i.e. for two months)**

Appendix: Recover the Pricing Kernel directly, without using Transition Matrix Π (II)

- **Now, we only account for the first row in both systems of equations. If we are in the current state, then the one month transition state prices equal the one month (spot) state prices**

- **Combining the equations gives us a non-linear system of equations with *n* equations and *n* unknowns (*n=2* in this example):**

$$\begin{pmatrix} \pi_{11} & \pi_{12} \\ \pi_{11}^{(2)} & \pi_{12}^{(2)} \end{pmatrix} \times \begin{pmatrix} \frac{1}{u'_1} \\ \frac{1}{u'_2} \end{pmatrix} = \begin{pmatrix} \delta \times \frac{1}{u'_1} \\ \delta^2 \times \frac{1}{u'_1} \end{pmatrix} \quad \Leftrightarrow \quad \begin{pmatrix} \pi_{11} & \pi_{12} \\ \pi_{11}^{(2)} & \pi_{12}^{(2)} \end{pmatrix} \times \begin{pmatrix} 1 \\ \frac{u'_1}{u'_2} \end{pmatrix} = \begin{pmatrix} \delta \\ \delta^2 \end{pmatrix}$$

- **Estimates the pricing kernel directly (minimize squared errors). Pricing Kernel has to be scaled due to this approximation error**