THE CUTOFF STRUCTURE OF TOP TRADING CYCLES IN SCHOOL CHOICE

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TOP TRADING CYCLES FOR SCHOOL CHOICE

- School Choice: Assigning students to schools
  - Allow students to choose schools
  - Account for siblings, neighborhood status
- Top Trading Cycles (TTC) is an attractive mechanism
  - Pareto efficient and strategy-proof for students
  - Policy lever: school priorities can guide the allocation
- But TTC is rarely used
  - Difficult to assess how changes in input (priorities and preferences) affect the TTC allocation

THE CUTOFF STRUCTURE OF TTC

- Characterizing the TTC assignment
  - TTC assignment given by n^2 admissions cutoffs
- Calculating the TTC cutoffs
  - Solve for sequential trade by looking at trade balance equations
  - TTC cutoffs are solutions to a differential equation
- Structure of the TTC assignment
  - Comparative statics
  - Welfare comparisons with other school choice mechanisms
  - Designing TTC priorities

RELATED LITERATURE

- School choice – theory and practice
  - Abdulkadiroğlu & Sönmez (2003)
  - Abdulkadiroğlu, Pathak, Roth, Sönmez (2005), Abdulkadiroğlu, Pathak, Roth (2009), Pathak & Shi (2017), Pathak & Sönmez (2013)
- Cutoff representations of school choice mechanisms
  - Abdulkadiroğlu, Angrist, Narita, Pathak (2017), Agarwal & Somaini (2017), Kapor, Neilson, Zimmerman (2016)
  - Azevedo & Leshno (2016), Ashlagi & Shi (2015)
- Characterizations of TTC mechanism
  - Shapley & Scarf (1973), attributed to David Gale
  - Abdulkadiroğlu, Che & Tercieux (2010), Morrill (2013), Abdulkadiroğlu et al.(2017), Dur & Morrill (2017)

THE TTC ALGORITHM

School Priorities

1	**2**	**3**
s_1	s_2	s_5
s_2	s_4	s_7
s_3	s_6	s_9
$\vdots$	$\vdots$	$\vdots$
s_7	s_1	s_1

Step 1:

- Schools point to their favorite student
- Students point to their favorite school
- Choose a cycle, assign included students to their favorite school.

THE TTC ALGORITHM

School Priorities

1	2	3
s_1	s_2	s_5
s_2	s_4	s_7
s_3	s_6	s_8
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s_7	s_1	s_1

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THE TTC ALGORITHM

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1	2	3
s_1	s_2	s_5 → **3**
s_2	s_4	s_7
s_3	s_6	s_8
⋮	⋮	⋮
s_7	s_1	s_1

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THE TTC ALGORITHM

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1	**2**	**3**
s_1	s_2	s_5
s_2	s_4	s_7
s_3	s_6	s_8
$\vdots$	$\vdots$	$\vdots$
s_7	s_1	s_1

Step k:

- Schools point to their favorite remaining student
- Students point to their favorite remaining school
- Choose a cycle, assign included students to their favorite school.

THE TTC ALGORITHM

School Priorities

1	2	3
~~s_1~~ $\rightarrow$ **2**	~~s_2~~ $\rightarrow$ **1**	s_5
s_2	s_4	s_7
s_3	s_6	s_8
$\vdots$	$\vdots$	$\vdots$
s_7	s_1	~~s_1~~

Step k:

- Schools point to their favorite remaining student
- Students point to their favorite remaining school
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THE TTC ALGORITHM

School Priorities

1	2	3
~~s_1~~	~~s_2~~	$\boxed{s_5}$
$\boxed{s_2}$	s_4	s_7
s_3	s_6	s_8
$\vdots$	$\vdots$	$\vdots$
s_7	$\boxed{s_1}$	~~s_1~~

Step k:

- Schools point to their favorite remaining student
- Students point to their favorite remaining school
- Choose a cycle, assign included students to their favorite school.

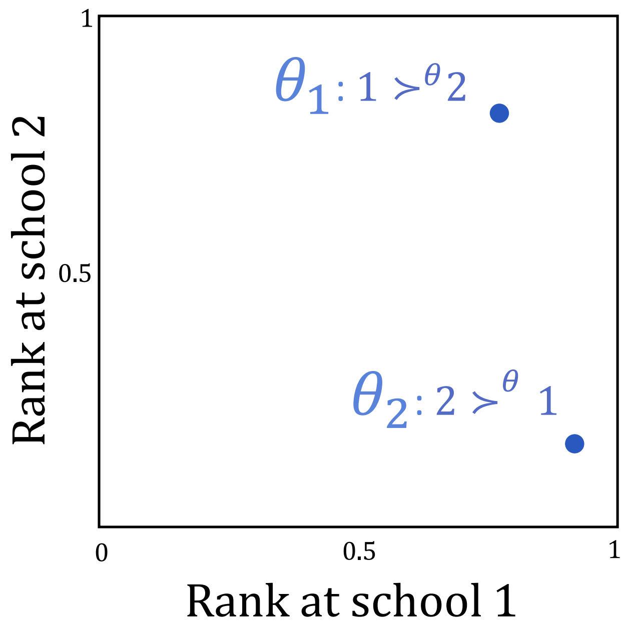
CHARACTERIZING THE TTC ASSIGNMENT

SCHOOL CHOICE MODEL

- Finite number of students $\theta = (\succ^{\theta}, r^{\theta})$
  - Student θ has preferences $\succ^{\theta}$ over schools
  - $r_c^{\theta} \in [0,1]$ is the rank of student θ at school c (percentile in c's priority list)

- Finite number of schools c
  - School c can admit q_c students
  - $\succ^{c}$ a strict ranking over students

SCHOOL CHOICE VISUALIZATION



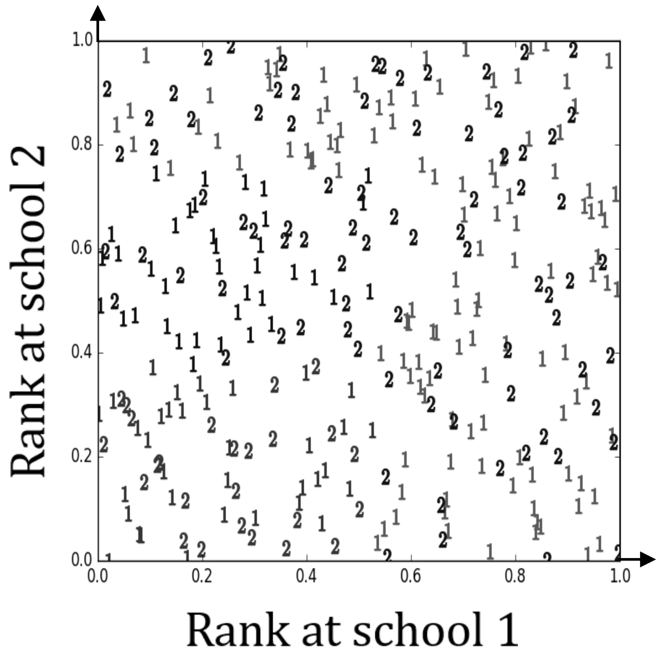
Student θ_1

- prefers 1 to 2
- highly ranked at 1
- highly ranked at 2

Student θ_2

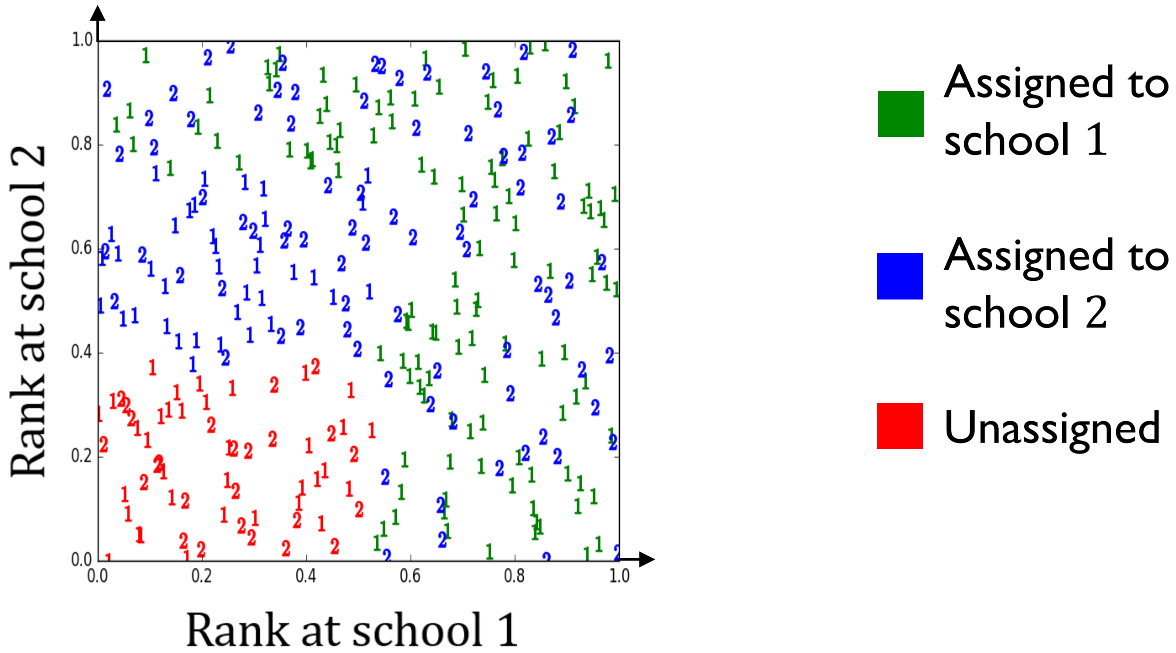
- prefers 2 to 1
- highly ranked at 1
- poorly ranked at 2

EXAMPLE

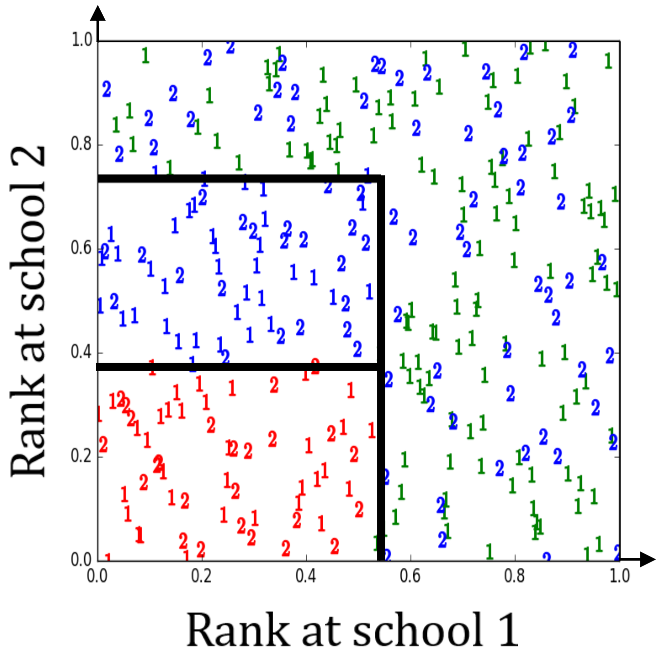


- 2/3 students prefer school 1
- Ranks are uniformly i.i.d. across schools
- $q_1 = q_2$

EXAMPLE – TTC ASSIGNMENT



EXAMPLE – TTC ASSIGNMENT



Assigned to school 1

Assigned to school 2

Unassigned

TTC ASSIGNMENT VIA CUTOFFS

Theorem.

The TTC assignment is given by cutoffs $\{p_b^c\}$ where:

- Each student θ has a budget set

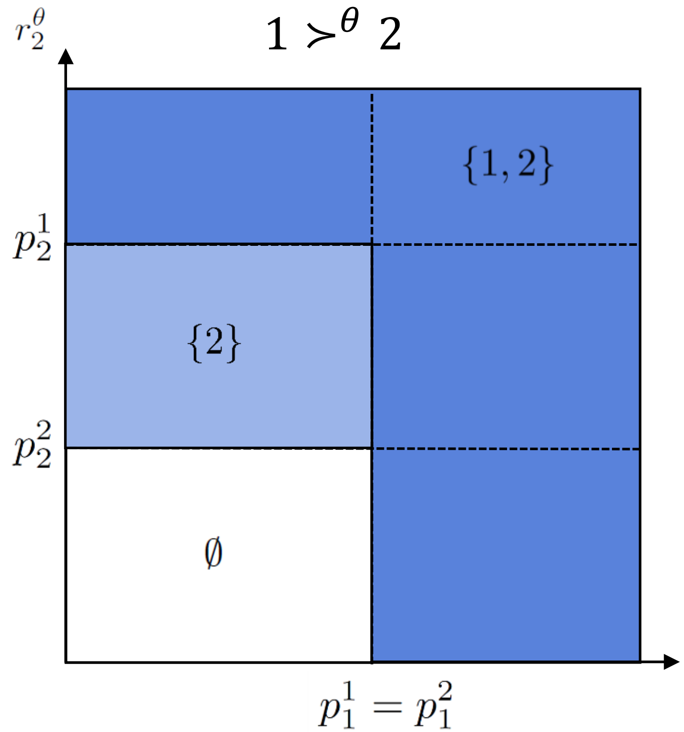
$$B(p,\theta) = \{c \mid \exists b \text{ s.t. } r_b^\theta \geq p_b^c\}$$

- Students assigned to their favorite school in their budget set

$$\mu(\theta) = \max_{\succ^\theta}(B(p,\theta))$$

Interpretation: p_b^c is the minimal priority at school b that allows trading a seat at school b for a seat at school c

EXAMPLE – ASSIGNMENT VIA CUTOFFS

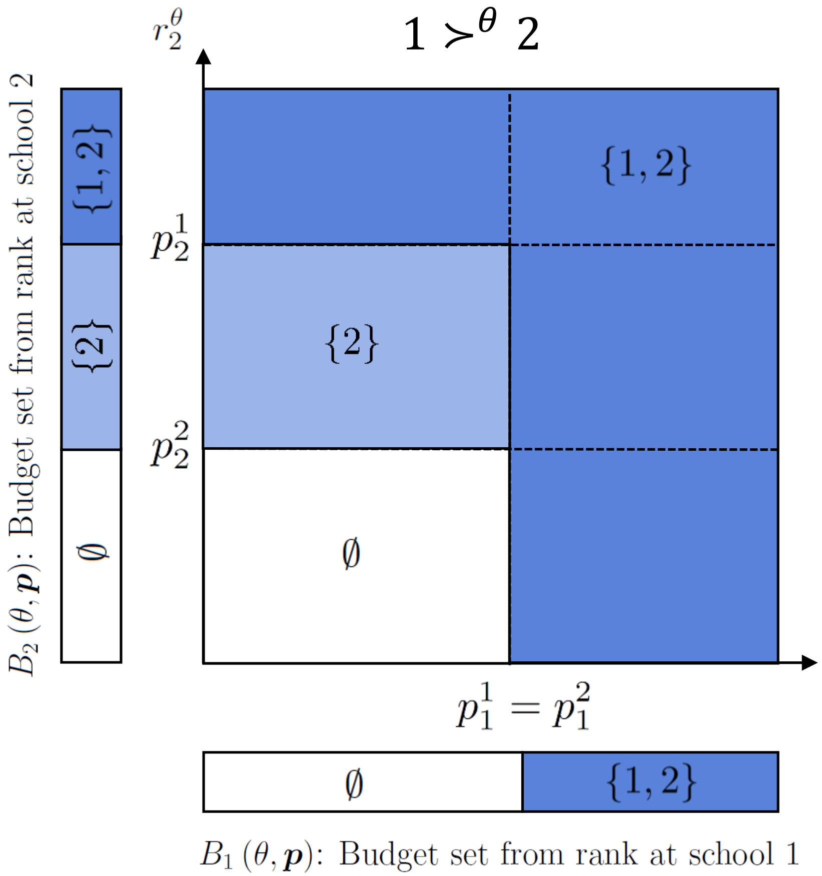


$$B(p,\theta) = \{c \mid \exists b \ \text{ s.t. } r_b^\theta \geq p_b^c\}$$

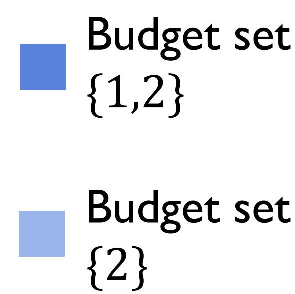
Budget set {1,2}

Budget set {2}

EXAMPLE – ASSIGNMENT VIA CUTOFFS



$$B(p, \theta) = \{c \mid \exists b \text{ s.t. } r_b^{\theta} \geq p_b^c\}$$



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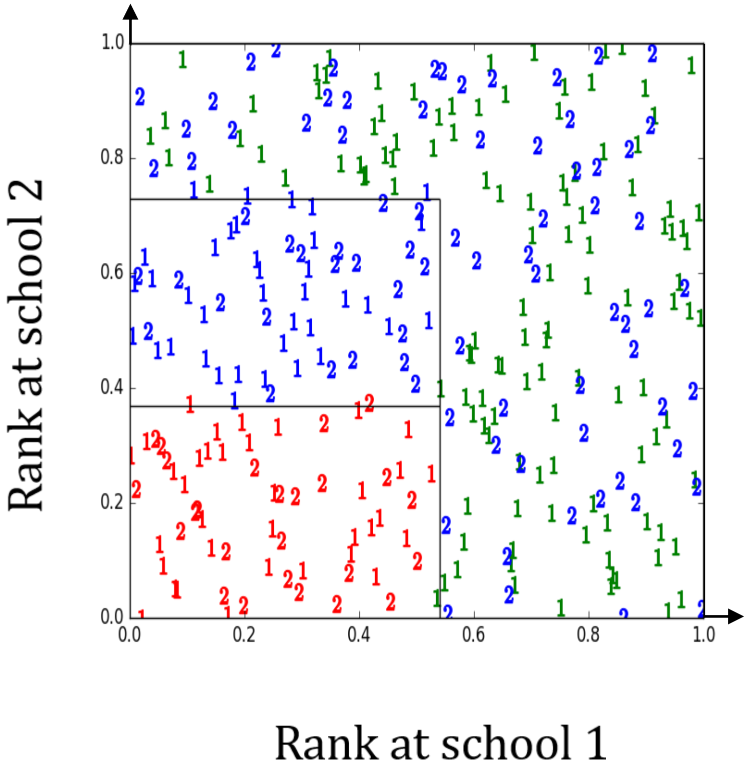


$$\mu(\theta) = \max_{\succ_\theta}(B(p,\theta))$$

- Assigned to school 1
- Assigned to school 2
- Unassigned

GENERAL STRUCTURE OF CUTOFFS

There is a renaming of the schools such that

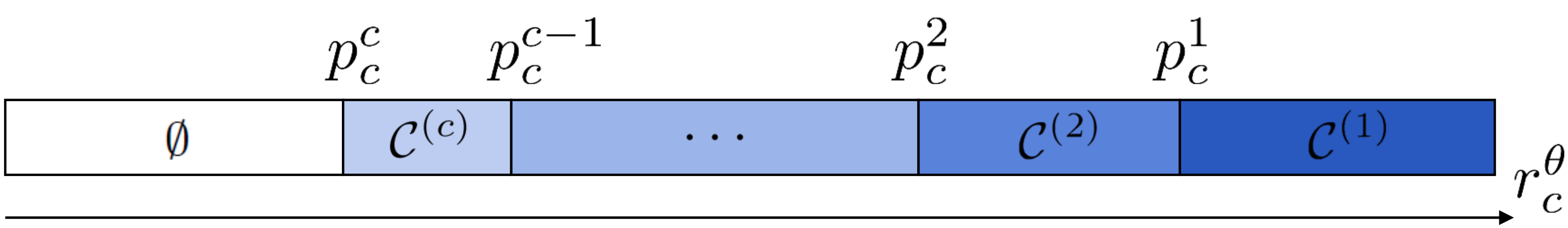
- Each student's budget set is

$$\mathcal{C}^{(\ell)} = \{\ell, \dots, n\}$$

- The cutoffs are ordered

$$p_c^1 \geq p_c^2 \geq \cdots \geq p_c^c = p_c^d$$

for all $c < d$



CALCULATING TTC CUTOFFS

CONTINUUM MODEL

- Finite number of schools $c \in C = \{1, \ldots, n\}$
  - School c can admit a mass q_c of students

- Measure η specifying a distribution of a continuous mass of students
  - A student $\theta \in \Theta$ is given by $\theta = (\succ^\theta, r^\theta)$
  - Student θ has preferences $\succ^\theta$ over schools
  - $r_c^\theta \in [0,1]$ is the student's rank at school c (percentile in c priority list)

TTC ASSIGNMENT VIA CUTOFFS

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The TTC assignment is given by cutoffs $\{p_b^c\}$ where:

- Each student θ has a budget set

$$B(p,\theta) = \{c \mid \exists b \ \text{ s.t. } r_b^\theta \geq p_b^c\}$$

- Students assigned to their favorite school in their budget set

$$\mu(\theta) = \max_{\succ^\theta}(B(p,\theta))$$

Cutoffs p_b^c are the solutions to a differential equation

CALCULATING TTC CUTOFFS

Theorem.

The TTC cutoffs $\{p_b^c\}$ are given by

$$p_b^c = \gamma_b\left(t^{(c)}\right)$$

where γ satisfies the *marginal trade balance equations*

$$\sum_{a \in C} \gamma_a'(t) H_a^c\left(\gamma(t)\right) = \sum_{a \in C} \gamma_c'(t) H_c^a\left(\gamma(t)\right) \ \forall t, c.$$

$H_b^c(x)$ is the marginal density of students who have rank $\leq x$, are top ranked at school b and most prefer school c.

TRADE BALANCE EQUATIONS

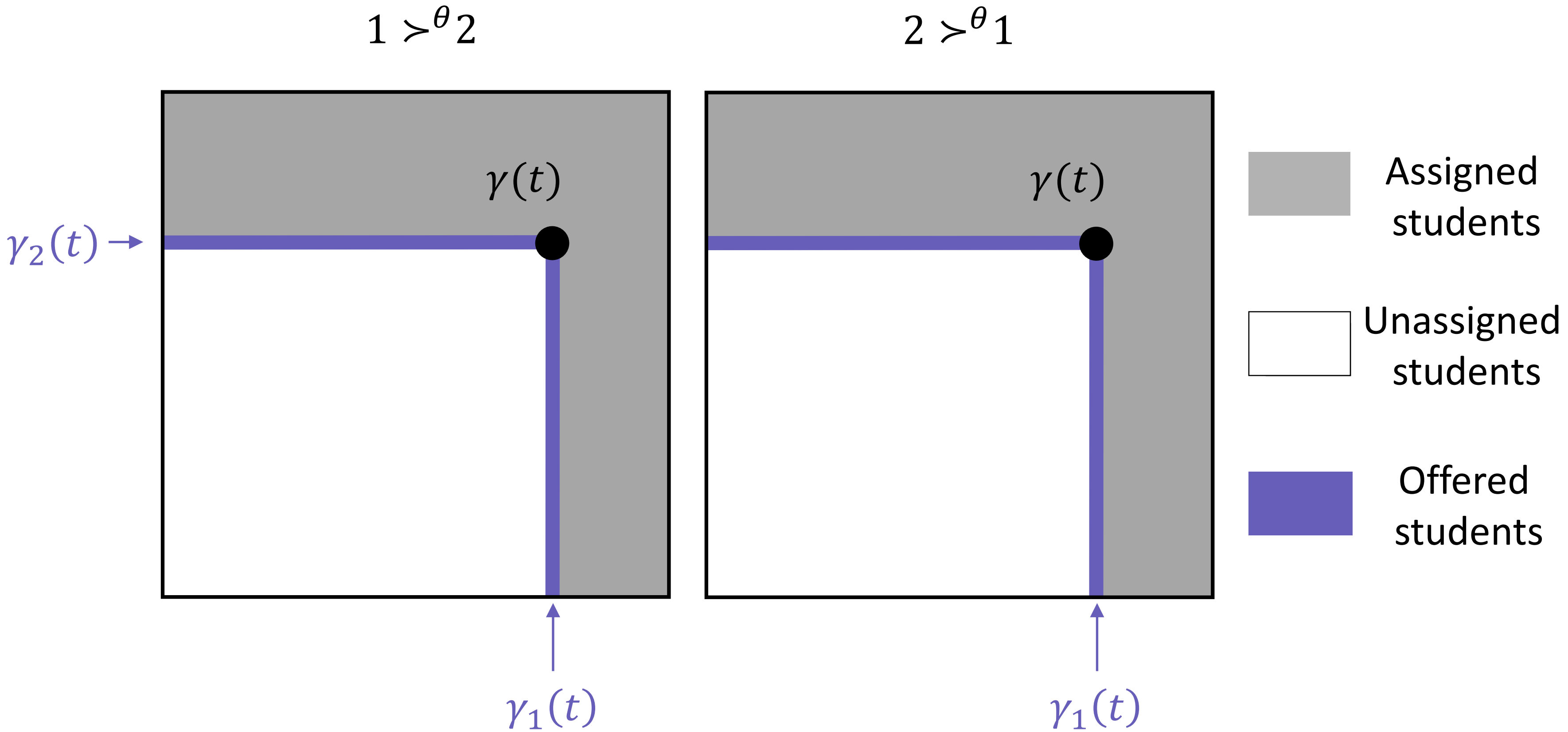
$$\#\left\{\begin{array}{c}\text{Students}\\ \text{assigned to } c\\ \text{by time } t\end{array}\right\} = \#\left\{\begin{array}{c}\text{Students}\\ \text{who traded } c\\ \text{by time } t\end{array}\right\}$$

for all times t.

- Necessary condition for aggregate trade
- Equivalent to the differential equation $\gamma'(t) = d(\gamma(t))$, where $\gamma_c(t)$ is the rank of students pointed to by school c at time t.
- γ is the TTC path

TRADE BALANCE – VISUALIZATION

$\gamma_c(t)$: Rank of students pointed to by school c at time t



TRADE BALANCE – VISUALIZATION

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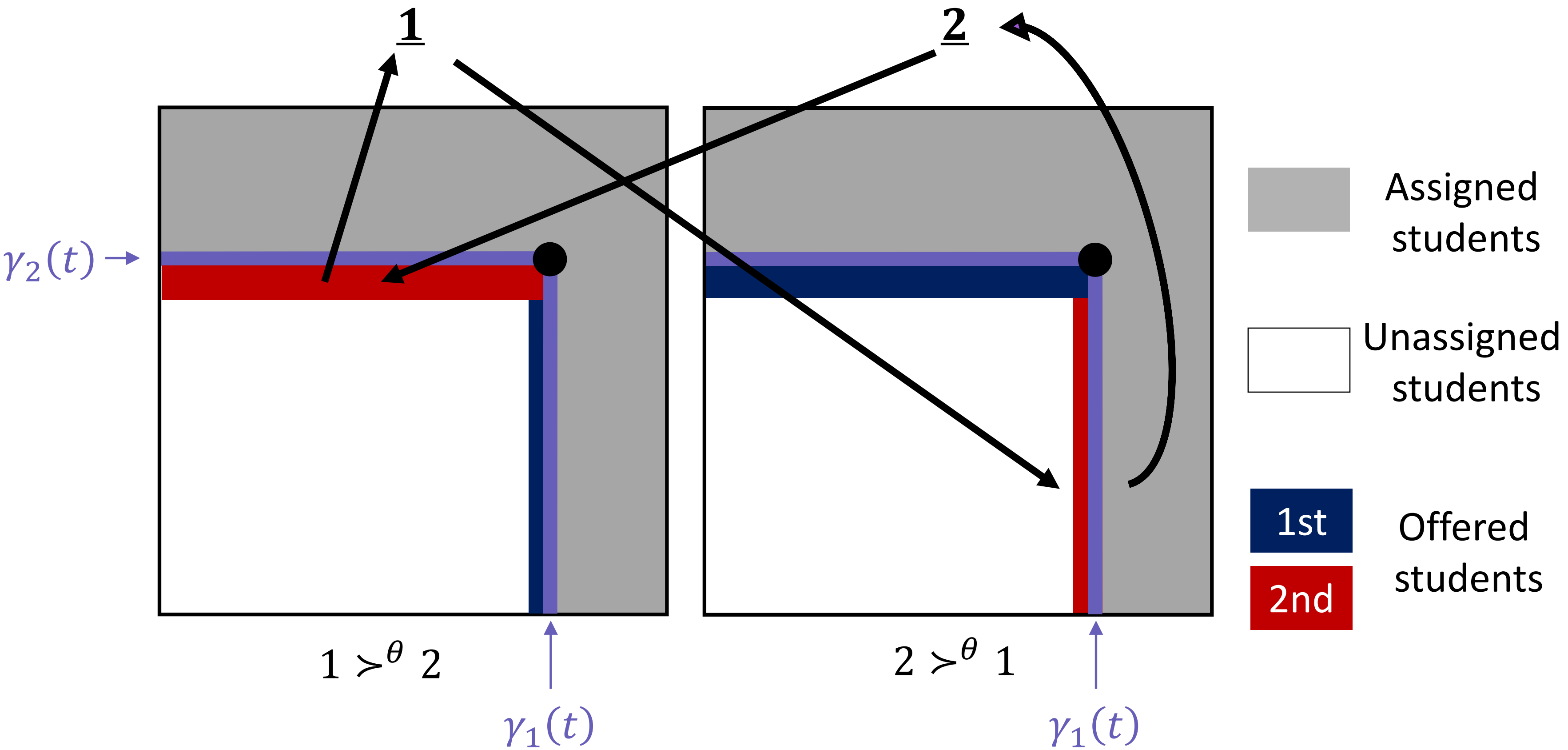
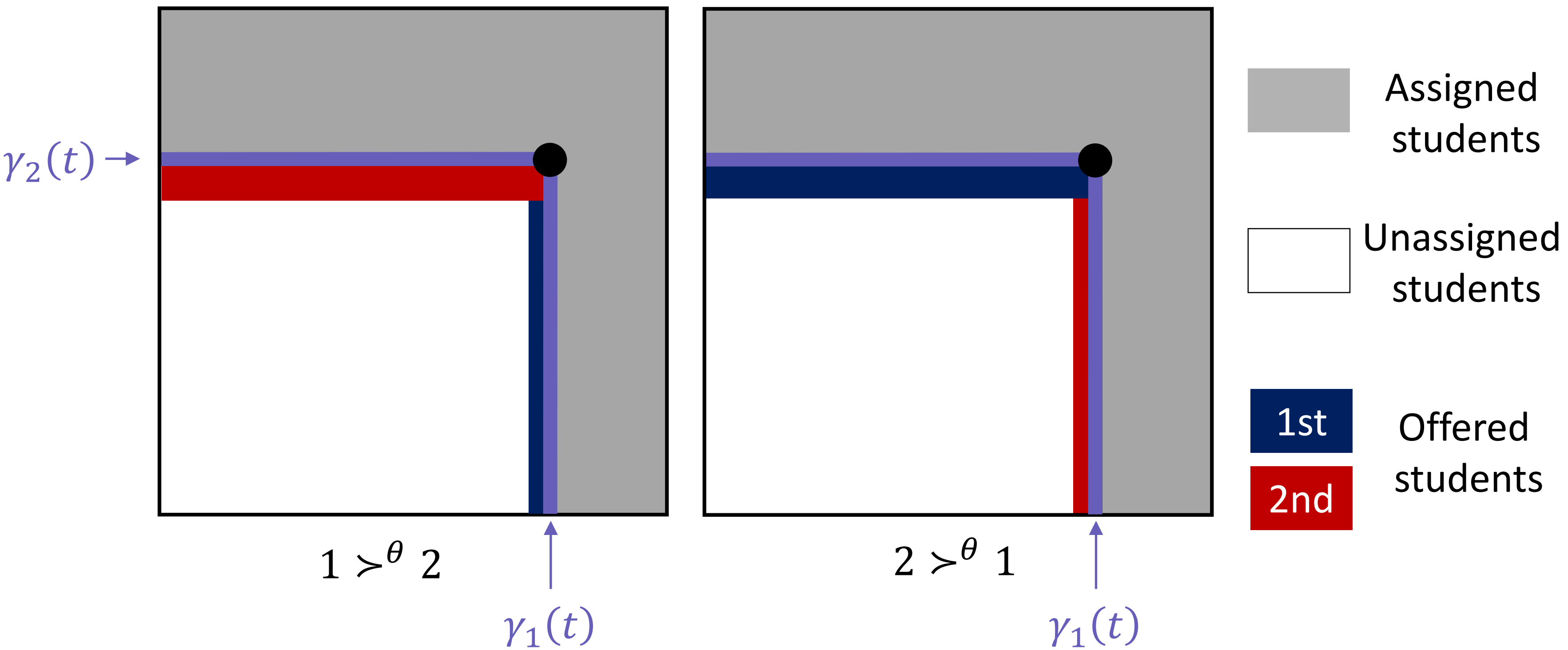


TRADE BALANCE – VISUALIZATION

$\gamma_c(t)$: Rank of students pointed to by school c at time t

$$\gamma_2'(t)(density\ of\ 1 \succ 2) = \gamma_1'(t)(density\ of\ 2 \succ 1)$$



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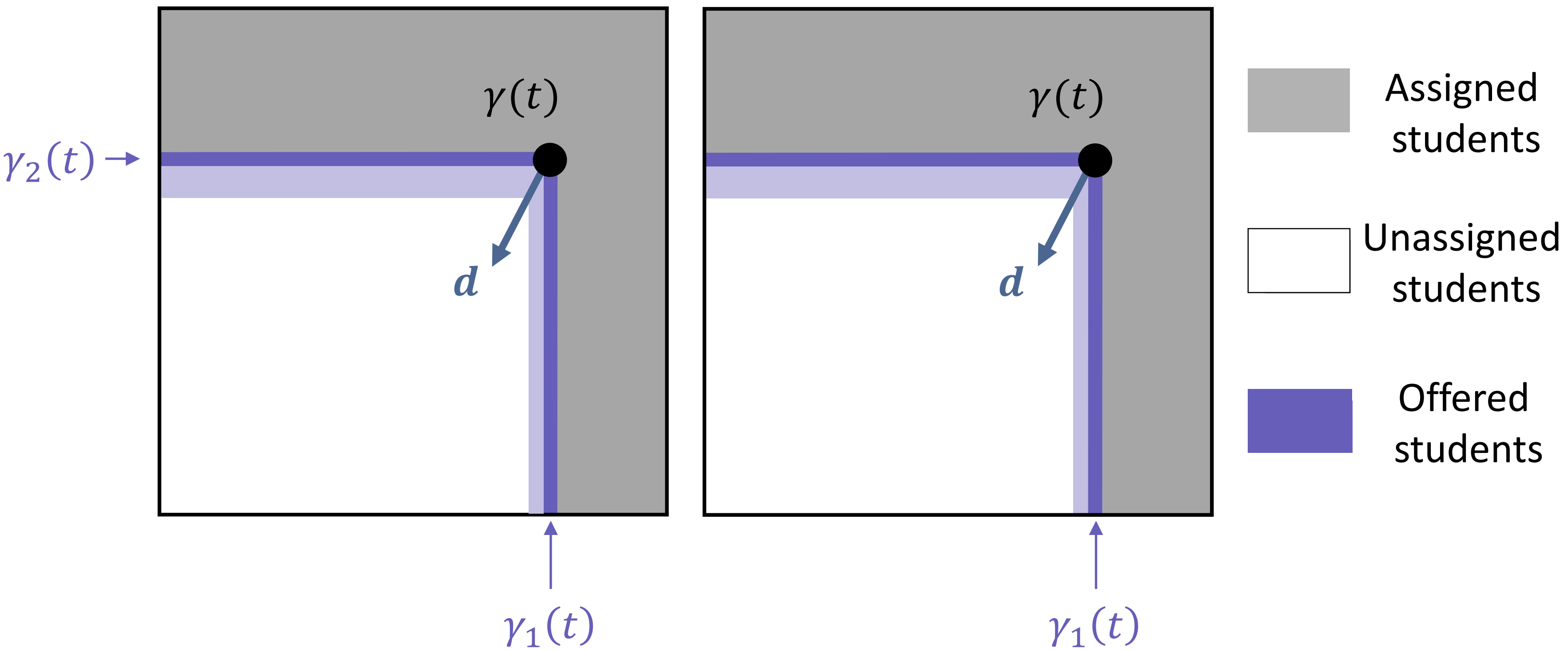
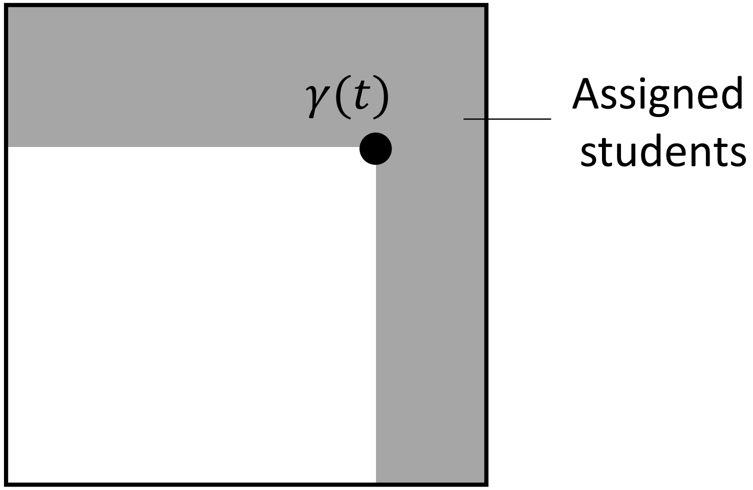


CAPACITY EQUATIONS

Stopping times $t^{(c)}$

$$t^{(c)} = min\left\{t: \#\left\{\begin{matrix}\text{Students} \\ \text{assigned to } c \\ \text{by time } t\end{matrix}\right\} \geq q_c\right\}$$

- Necessary condition for market clearing
- Equivalent to equations involving $\gamma\left(t^{(c)}\right)$



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Theorem.

The TTC assignment is given by computing cutoffs $\{p_b^c\}$

$$p_b^c = \gamma_b\left(t^{(c)}\right)$$

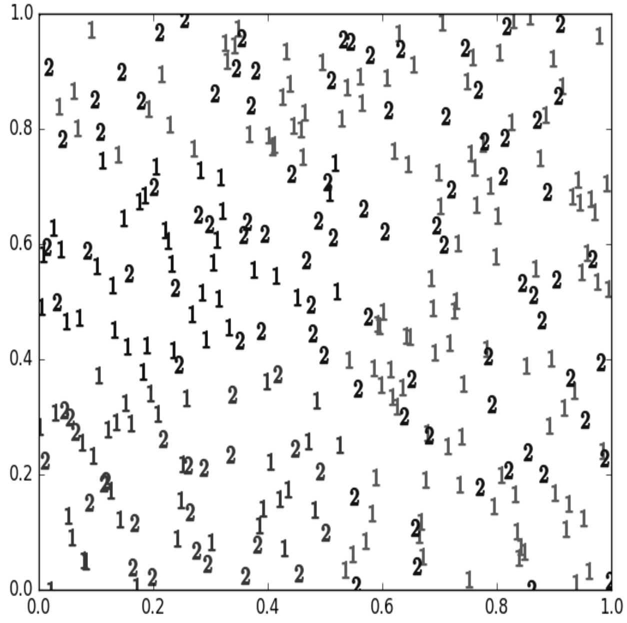
where γ satisfies the *marginal trade balance equations*, and assigning students to their favorite school in their budget set

$$B(p,\theta) = \{c \mid \exists b \ \text{ s.t. } r_b^\theta \geq p_b^c\}$$

$$\mu(\theta) = \max_{\succ^\theta}\left(B(p,\theta)\right).$$

- Closed form solutions, comparative statics
- Admissions probabilities

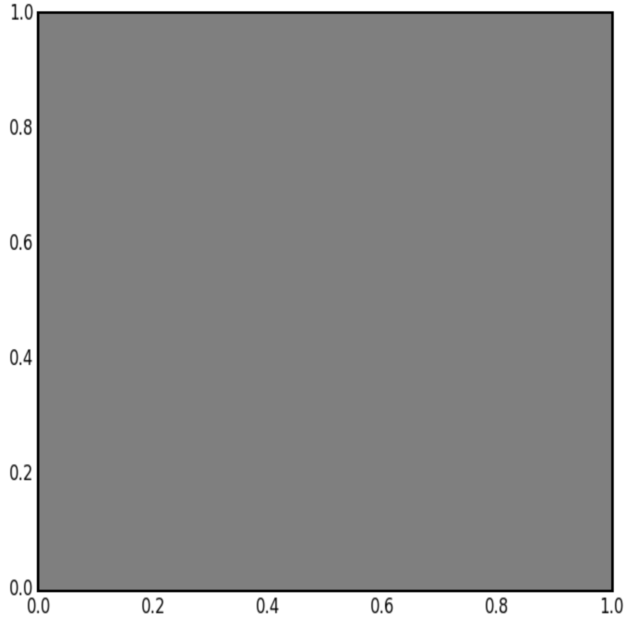
EXAMPLE: CALCULATING TTC CUTOFFS



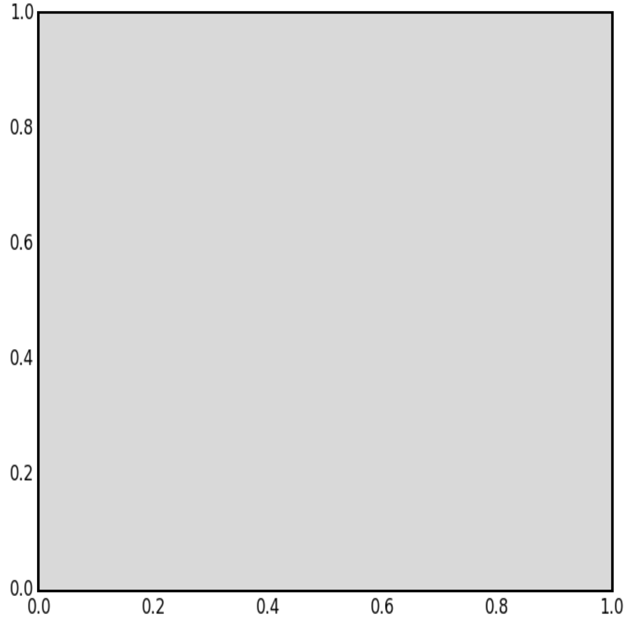
2/3 of students prefer school 1*, ranks are uniformly i.i.d. across schools,* $q_1 = q_2$

EXAMPLE: CALCULATING TTC CUTOFFS

$1 \succ^{\theta} 2$



$2 \succ^{\theta} 1$



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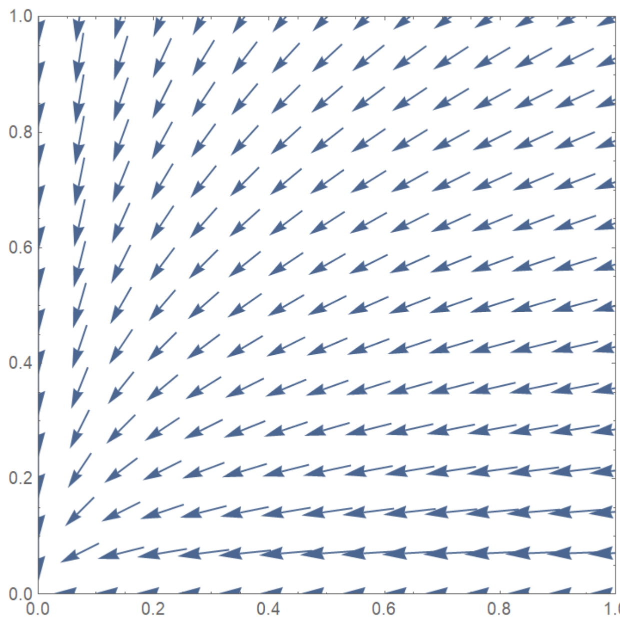
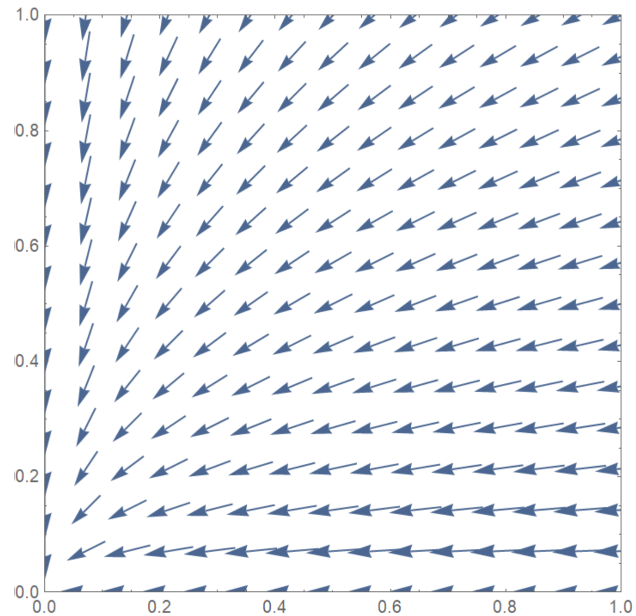
EXAMPLE: CALCULATING TTC CUTOFFS

$1 \succ^{\theta} 2$



$2 \succ^{\theta} 1$



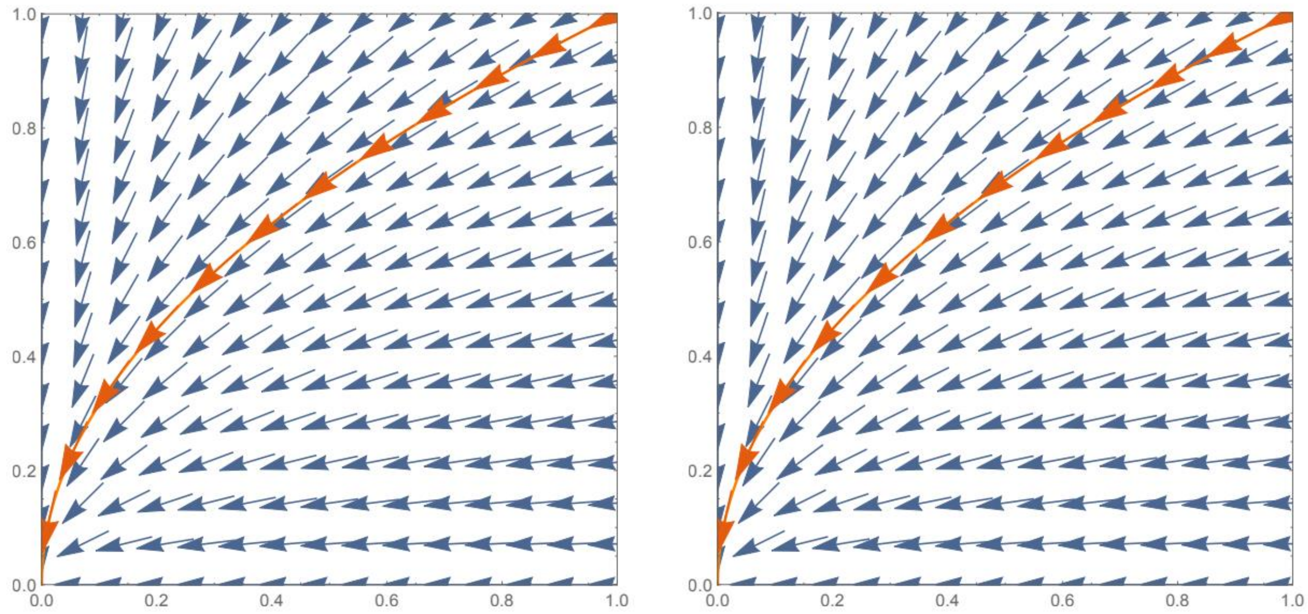
2/3 of students prefer school 1*, ranks are uniformly i.i.d. across schools,* $q_1 = q_2$

- Marginal trade balance equations given valid gradient:

$$\gamma'(t) = d\big(\gamma(t)\big)$$

EXAMPLE: CALCULATING TTC CUTOFFS

$1 \succ^{\theta} 2$ $\qquad\qquad$ $2 \succ^{\theta} 1$

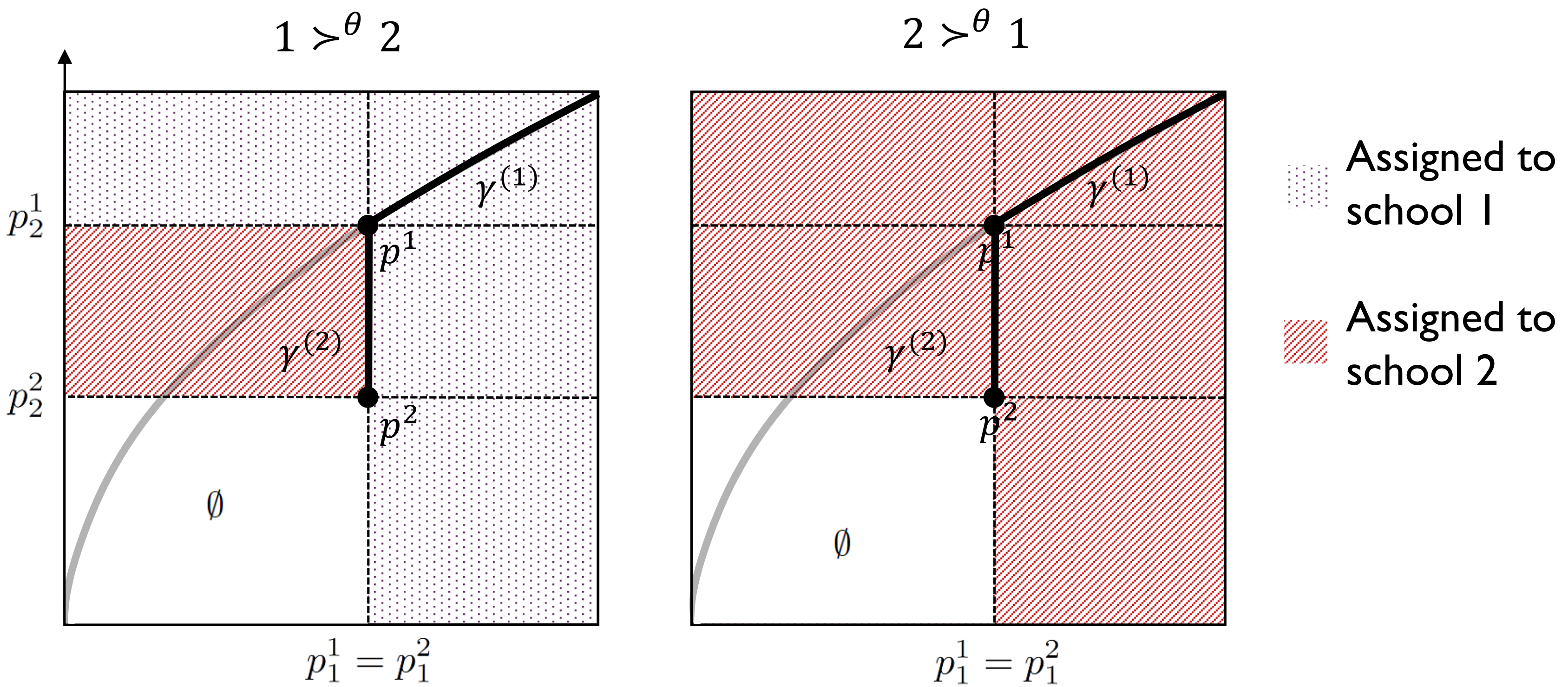


2/3 of students prefer school 1, *ranks are uniformly i.i.d. across schools,* $q_1 = q_2$

- TTC path γ with initial condition $\gamma(0) = \mathbf{1}$ and satisfying

$$\sum_{a \in C} \gamma_a'(t) H_a^c(\gamma(t)) = \sum_{a \in C} \gamma_c'(t) H_c^a(\gamma(t))$$

EXAMPLE: CALCULATING TTC CUTOFFS



2/3 of students prefer school 1, ranks are uniformly i.i.d. across schools, $q_1 = q_2$

- TTC path γ indicates the run of TTC
- Cutoffs p are the points at which schools reach capacity

EXAMPLE: CALCULATING TTC CUTOFFS

- Valid gradient

$$d(x) = -\begin{bmatrix} \frac{x_1}{x_1 + 2x_2} & \frac{2x_2}{x_1 + 2x_2} \end{bmatrix}$$

($d(\cdot)$ balances marginal densities)

- TTC path

$$\gamma(t) = \left(t^{1/3}, t^{2/3}\right)$$

($\gamma'(t) = d\left(\gamma(t)\right)$)

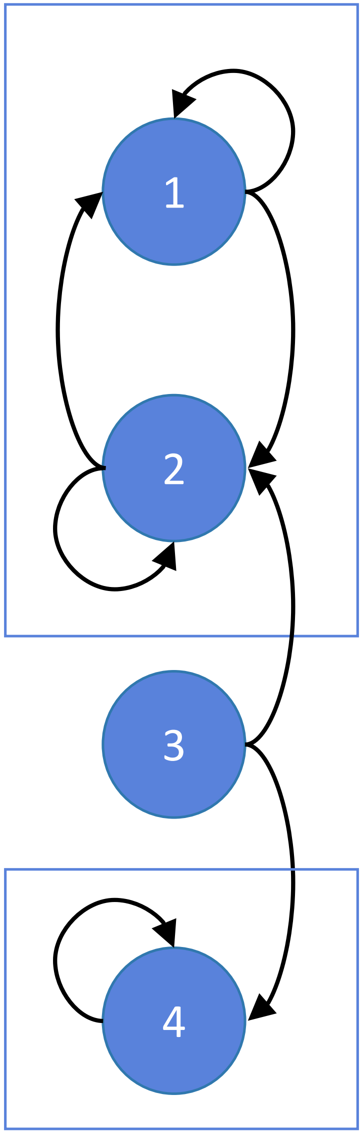
- TTC cutoffs

$$p^1 = \left((1 - 3q_1)^{1/3}, \left((1 - 3q_1)^{2/3}\right)\right)$$

($p_b^c = \gamma_b\left(t^{(c)}\right)$)

TRADE BALANCE IS SUFFICIENT

- Trade balance of gradient is mathematically equivalent to stationarity of a Markov chain
  - schools ⇔ states
  - transition probability p_{bc} ⇔ mass of students b points to, who want c
  - trade balance ⇔ stationarity
- Unique solution within each communicating class
- Different solutions yield the same allocation
  - Multiplicity only because of disjoint trade cycles
  - Different paths clear the same cycles at different rates



CONTINUUM TTC GENERALIZES DISCRETE TTC

- Trade Balance Uniquely Determines the Allocation
  - Differential equation and TTC path may not be unique, but all give the same allocation

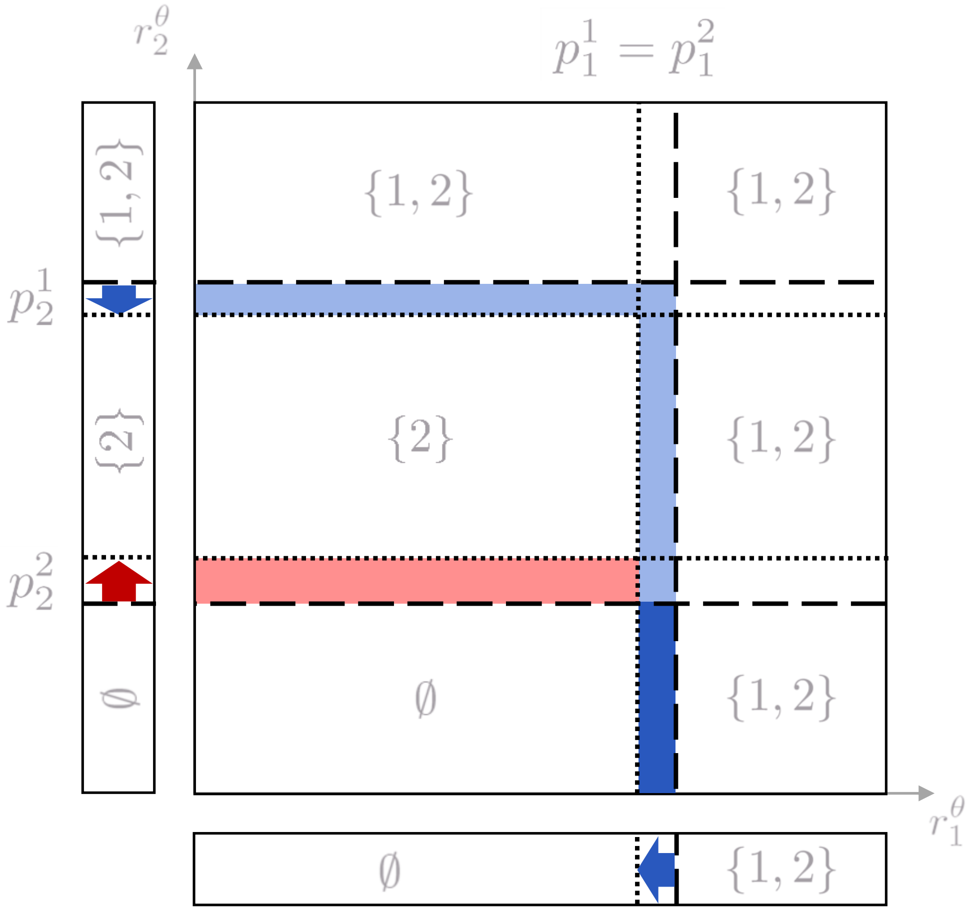
- Consistent with Discrete TTC
  - Can naturally embed discrete TTC in the continuum model
  - The continuum embedding gives the same allocation as TTC in the discrete model

- Convergence
  - If two distributions of students have full support and total variation distance ε, then the TTC allocations differ on a set of students of measure $O(\varepsilon|C|^2)$.

APPLICATIONS

COMPARATIVE STATICS

Effect of marginal increase in desirability of school 2



COMPARATIVE STATICS - WELFARE

n schools, MNL utility model (McFadden 1973)**:**

- Student preferences given by MNL utility model:

$$\mathrm{u_s}(c) = \delta_c + \varepsilon_{sc}$$

quality — idiosyncratic match value

  - δ_c is invested quality, $\varepsilon_{\theta c}$ is mean 0 random EV iid
  - Random priority, independent for each school

- Constraints on total quality
- What are the welfare maximizing quality levels $\sum_c \delta_c \leq N$?

COMPARATIVE STATICS - WELFARE

Effects of increasing school quality on student welfare:

(under MNL model, for $n = 2$ and $\delta_1/q_1 > \delta_2/q_2$)

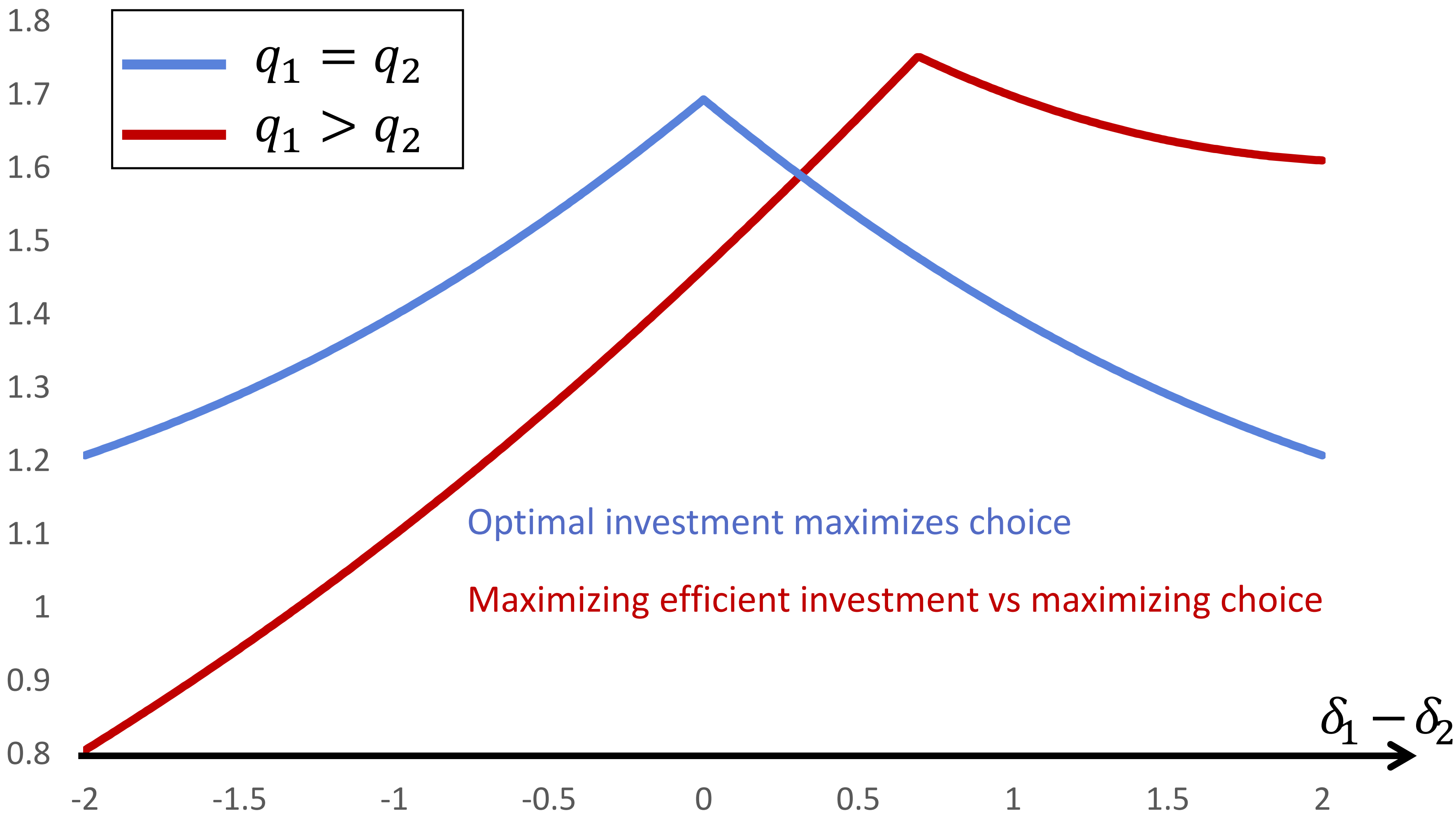
$$\frac{dSW}{d\delta_1} = \underbrace{q_1}_{\text{Direct effect}} \underbrace{- q_1 e^{\delta_2 - \delta_1} \ln\left(1 + e^{\delta_1 - \delta_2}\right)}_{\text{Indirect effect from changes in budget sets}}$$

Direct effect

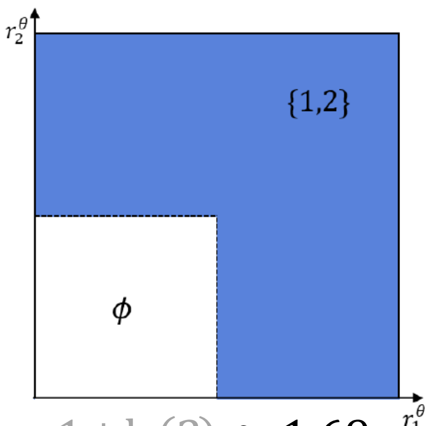
Indirect effect from changes in budget sets

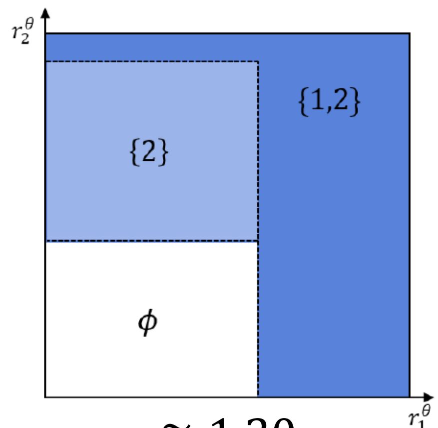
- Directly improves welfare of those who stay at the school
- Indirectly affects welfare through changing the allocation

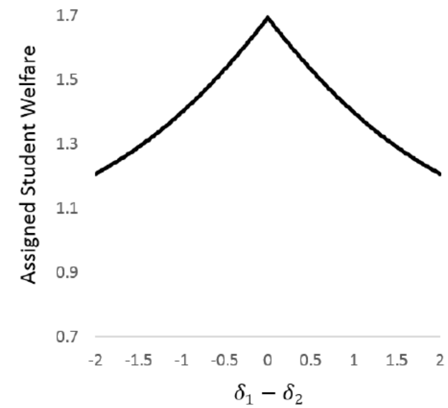
TTC WELFARE GIVEN $n = 2, \delta_1 + \delta_2 = 2$



COMPARING TTC & DA, $q_1 = q_2 = \frac{3}{8}$

<table>
<tr><th></th><th>$\delta_1 = \delta_2 = 1, OPT$</th><th>$\delta_1 = 2, \delta_2 = 0$</th><th>$\delta_1 - \delta_2$</th></tr>
<tr><td>TTC</td>
<td>

$1 + \ln(2) \cong 1.69$</td>
<td>

$\cong 1.20$</td>
<td>
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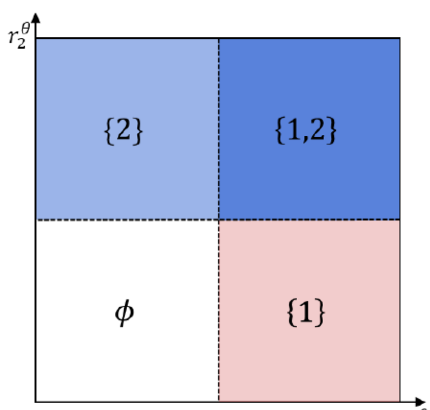
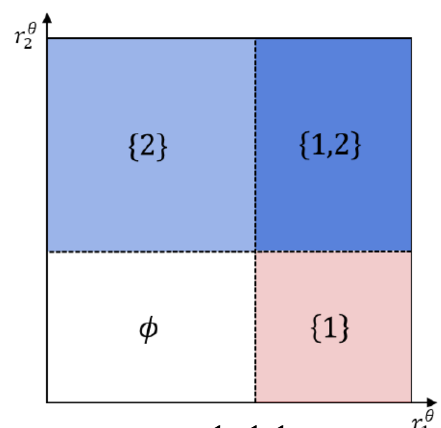
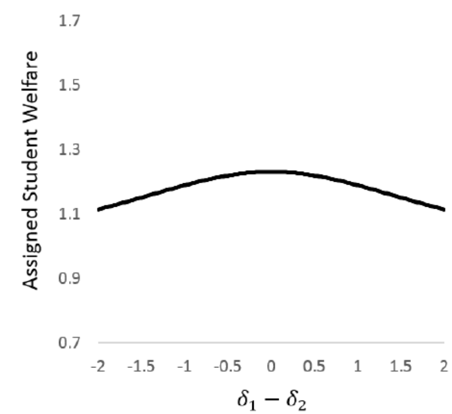
<tr><td>DA</td>
<td>

$1 + (1/3)\ln(2) \cong 1.23$</td>
<td>

$\cong 1.11$</td>
<td>
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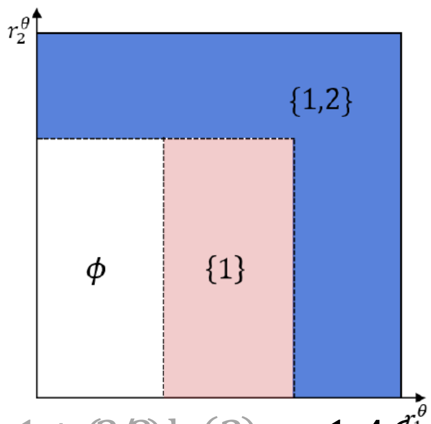
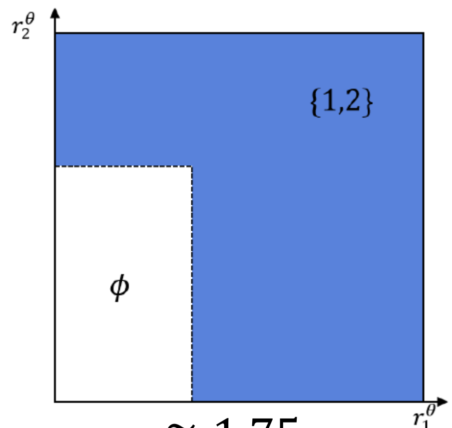
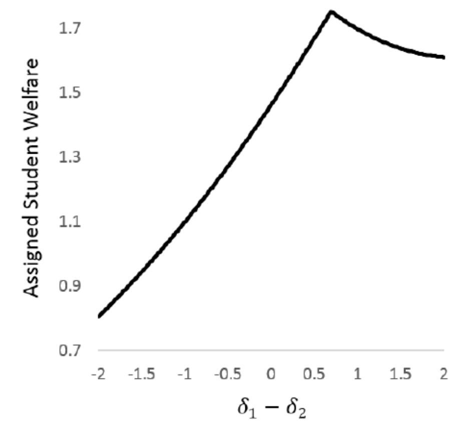
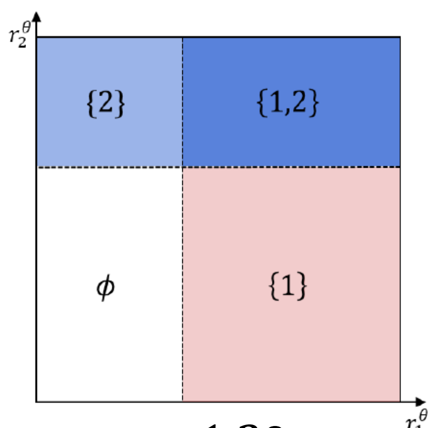
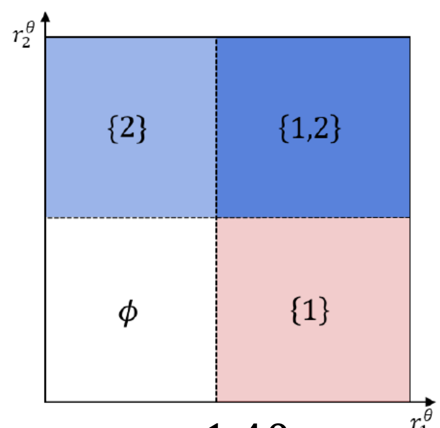
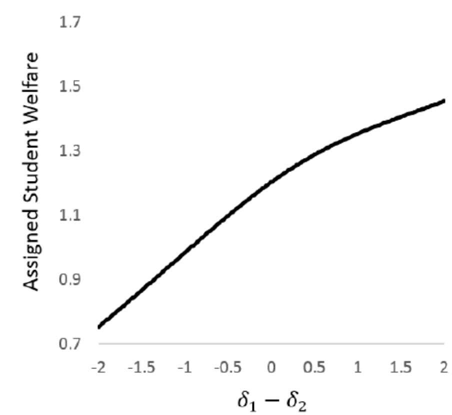
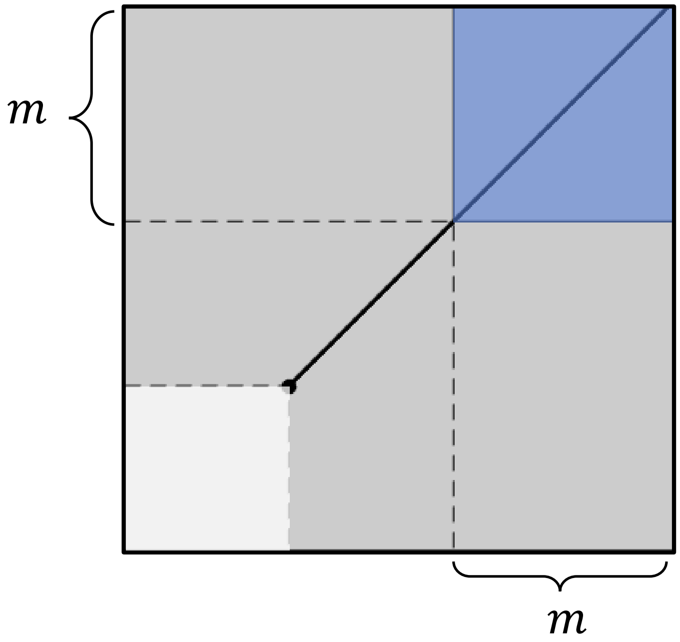
COMPARING TTC & DA, $q_1 = \frac{1}{2}$, $q_2 = \frac{1}{4}$

	$\delta_1 = \delta_2 = 1$	OPT	$\delta_1 - \delta_2$
TTC	 $1+(2/3)\ln(2) \cong 1.46$	 $\cong 1.75$	
DA	 $\cong 1.20$	 $\cong 1.40$	

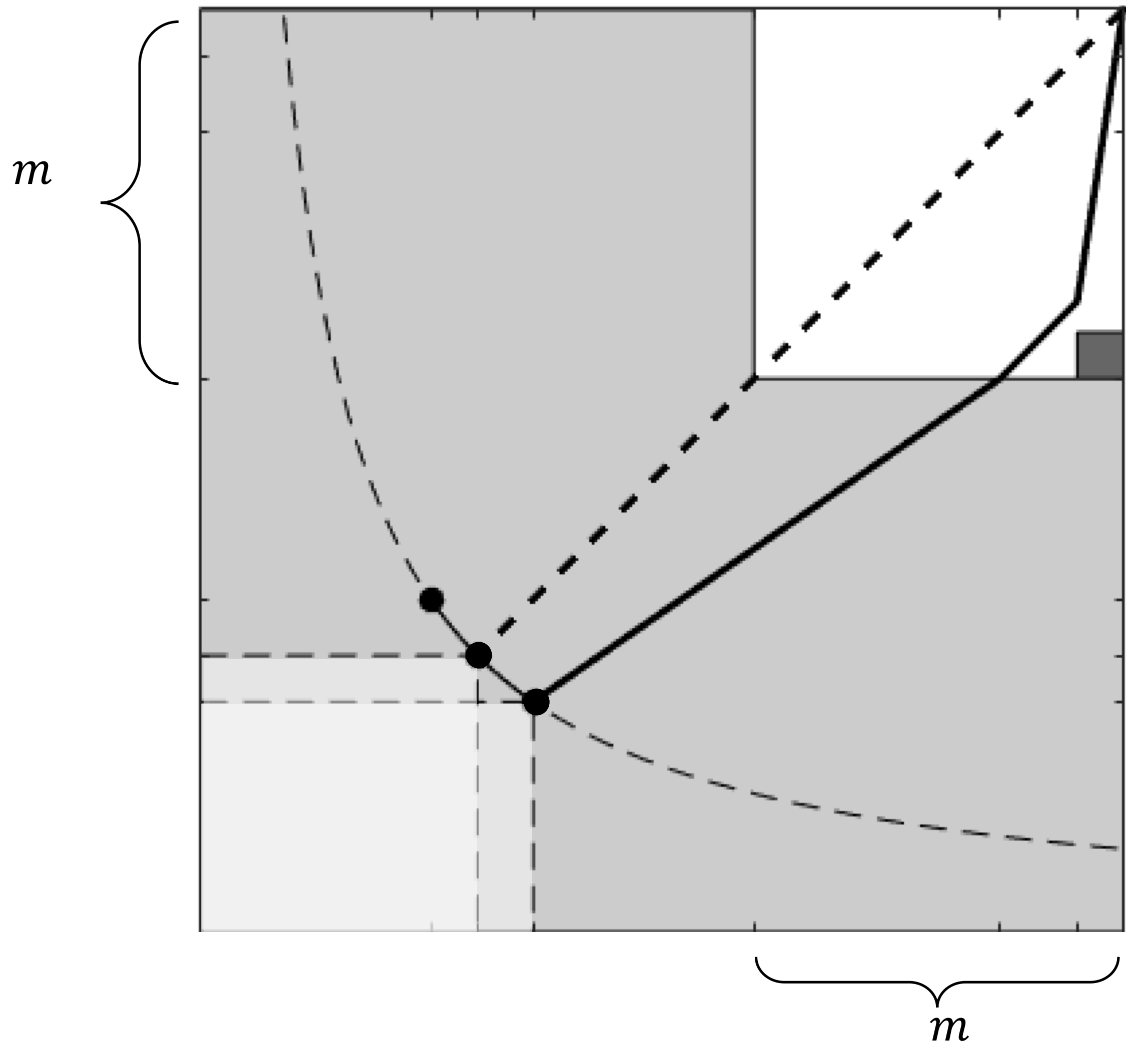
DESIGNING TTC PRIORITIES

- Symmetric economy with two schools
  - Equal capacities
  - Student equally likely to prefer either
  - priorities are uniformly random iid
- Consider changing the ranking of students with $r_c^{\theta} \geq m$ for both $c = 1,2$



TTC PRIORITIES ARE "BOSSY"

- The change affects the allocation of other students
- Changed students have the same assignment



CONCLUSIONS

- Cutoff description of TTC
  - n^2 admissions cutoffs
- Tractable framework for analyzing TTC
  - Trade balance equations
  - TTC cutoffs are a solution to a differential equation
  - Can give closed form expressions
- Structure of the TTC assignment
  - Equalizing school popularity leads to more efficient sorting on horizontal preferences
  - Welfare comparisons
  - TTC priorities are "bossy"

Thank you!