Walras's Theories of Exchange and Production Equilibrium in the 1870s and Beyond

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Abstract

During the Winter 1871-72, in preparation of a series of conferences to be soon after delivered in Geneva, Walras jotted down some extended Notes where several new concepts and constructs that would characterize the subsequent developments of general equilibrium theory can be found. Yet, such novel ideas often appear to be mixed up with traditional conceptions inherited from earlier schools of economic thought, especially from J.S. Mill's *Principles*. What is altogether missing in these Notes is the principle of optimizing behavior, which, in fact, came to Walras's mind only in the Fall 1872. Then, with Paul Piccard's help, Walras was eventually able to establish an analytical link between utility maximization and the demand functions of consumers-traders, in the context of a pure-exchange, two-commodity economy.

All the theoretical advances made by Walras over the highly productive period 1872-1877 can be viewed as attempts to revise his preexisting theory in the light of the Fall 1872 discovery. Yet, the peculiar way in which Walras came to learn how to use the new approach, together with the persistence of received ideas, often incompatible with a generalized assumption of optimizing behavior and its implications, can be shown to have conditioned and constrained Walras's endeavors. In this paper we critically assess the exchange and production equilibrium models put forward over the 1870s, explaining how their inconsistencies and shortcomings, of which Walras will try to get rid by incessantly revising his theory for the rest of his scientific life, up to 1900 and beyond, can be traced back to the mixed origins of Walras's general equilibrium approach.

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1 Introduction

In Winter 1871-72, in view of a series of ten lectures to be delivered in Geneva in the first few months of 1872, Walras jotted down an extended outline of what

can be regarded as the earliest comprehensive sketch of his general equilibrium theory. In view of their width, scope, and systematic character, such lecture notes, to be called the "Geneva Notes" in the sequel, can be viewed as the true starting point for Walras's scientific quest. From that moment onwards, for about three decades, Walras will relentlessly revise his theory, through an endless process of change affecting, to a greater or lesser degree, his treatment of all the four "big problems of economics", i.e., in Walras's own words, the problems of exchange, production, capitalization, and money.

Over such a long stretch of time, Walras's amendments of his theoretical system will find their chief outlet in the several editions of his *magnus opus*, the *Éléments d'économie politique pure* (henceforth *Éléments*): the first edition will appear in two instalments, respectively published in 1874 and 1877; the second in 1889; the third in 1896; and finally the fourth, the last in Walras's own lifetime, in 1900. A fifth edition, also called *édition définitive*, will be published posthumously in 1926. Apart from the various editions of the *Éléments*, the intervening changes in Walras's theoretical ideas will also make themselves manifest in a conspicuous number of *mémoires*, articles, and publications, coming out in scientific journals, bulletins of learned societies, readings and collections of writings, which will often anticipate the revisions to be encompassed in the subsequent editions of his main book.

Over the last century, such a tortured evolution of Walras's theoretical ideas has repeatedly attracted the attention of scholars and commentators, historians of economic thought and general equilibrium theorists. A number of alternative suggestions have been put forward to explain Walras's theoretical restlessness. Yet, almost invariably, such accounts have been focusing on special issues or specific shortcomings of individual portions of Walras's imposing theoretical construction, requiring or justifying particular corrections or independent adjustments. To the best of our knowledge, no attempt has yet been made at providing a unified explanation of the phenomenon under scrutiny.

In this paper we shall try to offer such an integrated explanation. Our fundamental idea is that, at the start of his intellectual journey, when drafting the Geneva Notes, Walras held a quite traditional, yet relatively articulated, conception of economic equilibrium, pervading all of the four main fields of inquiry listed above. In spite of the obvious differences among the four areas, such traditional conception was characterized by a common negative trait: for it nowhere envisaged a state of economic equilibrium as the outcome of the optimizing behavior of the individuals participating in the economic system or any of its subsystems.

Yet, in the Fall 1872, with the fundamental help of Paul Piccard, a professor of mechanics and one of Walras's colleagues at the Academy (later University) of Lausanne, Walras fortuitously stumbled upon the idea that the demand behavior of consumers-traders in a two-commodity economy may be explained as the result of their attempts at maximizing utility under budget and price uniformity constraints. From this discovery, a new conception of equilibrium as an array of mutually compatible optimizing plans emerged in the framework of a pure-exchange, two-commodity economy populated by an arbitrary finite num-

ber of consumers-traders. The corresponding model, quickly formalized and presented by Walras at a meeting of the Académie des sciences morales and politiques held in Paris in August 1873, was published a few months later. By 1874 the new equilibrium conception had conquered the entire field of pure exchange, extending its grip to the model of an economy where an arbitrary finite number of constrained utility maximizing consumers-traders exchange several commodities among themselves.

This model lies at the center of the first instalment of the 1^{st} edition of the *Éléments* (1874), chiefly devoted to the analysis of the problem of exchange. The analysis is essentially carried out by making use of the new equilibrium conception, resting on the combination of individual optimization and inter-individual compatibility of plans. Yet, some relevant features of the pre-existing traditional equilibrium conception persist in the new setting: in particular, the idea survives that an equilibrium state must be conceived as the stationary outcome of an equilibration process, called *tâtonnement* process, taking place in 'real' time, i.e., in the same time dimension as the one over which the economy, or any of its subsystems, is supposed to evolve, under the assumption of unchanging data.

Over the highly creative time span 1874-1877, Walras strived to extend the application of the new ideas from the exchange field to the other branches of economic activity. By 1877, the date when the second instalment of the 1^{st} edition of the *Éléments* eventually saw the light, he was able to put forward a model with production of consumer goods (henceforth, production model) and a model with capital formation (henceforth, capitalization model), where capital goods proper are produced. Yet, no properly formalized model concerning the fourth "big problem", namely, the problem of money, was added to the sundry observations on this topic already contained in the first instalment of the 1^{st} edition of the *Éléments*.

In the production model, two markets are at work, one for the services of the various types of capital and the other for products, here only represented by consumer goods. In the market for services, consumers, who are also the owners of all types of capital, sell the services rendered by the capital assets they own, while entrepreneurs buy such services, in order to combine them in the production processes from which the products come out; in the market for products, consumers buy the consumer goods that emerge from the production processes, while entrepreneurs sell the products they obtain. Here the assumption of optimizing behavior is applied to consumers-owners, regarded now as both the buyers of the consumer goods that are being produced and the sellers of the services of the capital assets they own. Yet, such assumption of optimizing behavior is not extended to entrepreneurs, who are instead supposed to act as quantity adaptors, according to a rule which cannot be traced back to any simple kind of constrained optimization.

Similar remarks apply to the capitalization model, where a third market on which newly produced capital goods proper are traded is added to the already examined markets for produced consumer goods and services. On such additional market, entrepreneurs are once again viewed as quantity adaptors, rather

than as optimizing agents, while the behavior of consumers-owners, in so far as they act as suppliers of savings and demanders of the newly produced capital goods proper, is not explained on the basis of the same optimizing principles as the ones already employed in the production model to explain their behavior when they act as purchasers of consumer goods and suppliers of capital services. In both models, moreover, the equilibrium state, assumedly reached at the end of an appropriate *tâtonnement* process, is conceived as the stationary outcome of an equilibration process taking place in 'real' time, once again under the assumption of unchanging data.

As can be seen, by 1877, when the second instalment of the 1^{st} edition of the *Éléments* appeared and the 1^{st} edition of Walras's *Théorie mathématique de la richesse sociale*, collecting his first four *mémoires* (1874, 1876a, 1876b, 1877a) was published, the structure of Walras's general equilibrium theory was almost completely specified and three of its constituent models (namely, the models of exchange, production, and capitalization) were fully formalized. Yet, only the exchange model could be said to fully encompass, with only one significant exception to be discussed later on, the characteristic features of the new equilibrium conception, centered on the assumption that the agents' behavior is optimizing. In the other two formalized models, the behavioral assumptions characterizing the new equilibrium conception were intermingled with the remnants of pre-existing ideas. Moreover, no sort of optimizing behavior was assumed to hold in the still poorly formalized parts of Walras's writings dealing with money.

As will be shown in the following, this mixture of old and new conceptions and assumptions explains the existence of a number of contradictions and true and proper mistakes, marring the theoretical framework emerging from the extraordinary feat accomplished by Walras over the period 1872-1877. The incessant revisions that Walras will bring about over more than thirty years testify to his attempts at correcting the mistakes that impaired his theory at its start, pruning the old ideas that survived his discovery of the fundamental role of optimization in economics occurring in the Fall 1872 (henceforth, for short, Walras's 1872 discovery), and progressively extending the new equilibrium conception, with its optimizing slant and behavioral consequences, to all the models and topics to which such conception had remained partly or fully extraneous at the beginning.

Although such process of revision involved all the parts of Walras's theoretical construction, in the following we shall focus attention on the exchange and production models only. The capitalization and money models (the latter not really formalized before 1899), though certainly affected by the same general process of change as that acting upon the exchange and production models, will not be considered in the critical account that follows: for they also raise such a large number of specific issues as to make it preferable to set them aside for a separate discussion. Moreover, the greatest attention will be paid to reconstructing the development of Walras's thought over the 1870s, especially focusing on the transition from the theory of exchange and production to be found in the Geneva Notes of Winter 1871-72 to the exchange and production models put

forward by Walras in both his first three *mémoires* and the two instalments of the 1^{st} edition of the *Éléments*, all appearing between 1874 and 1877: for, as will be argued, the foundations of Walras's own version of general equilibrium theory are precisely laid down during that relatively short time span. Yet, whenever a controversial issue will emerge during our rational reconstruction of Walras's thought over the 1870s, we shall provide a detailed account of the attempts made by Walras over the following decades, up to the beginning of the Twentieth century, to fill the gaps and amend the weak points in his theoretical system, as put together in the mid-1870s.

The paper is structured as follows. Section 2 summarizes the traditional equilibrium conception, largely inherited from past economists, that permeates the Geneva Notes, shaping so much of Walras's pre-analytic vision and contributing so many lasting traits to his more mature theoretical thought. Then we discuss the dramatic change of perspective brought about in the Fall 1872 by Walras's finding of a possible theoretical link between utility maximization and demand theory, a discovery that paved the way to the development of a new equilibrium conception. At first we focus on the impact of Walras's 1872 discovery on the exchange model put forward between 1872 and 1876: specifically, Section 3 examines the so-called "mathematical" solution of the model and the associated equilibrium determination issue; Section 4, instead, is devoted to the so-called "empirical" solution and the related equilibrium establishment issue. We then move to analyze the limited impact of Walras's 1872 breakthrough on the production model, as developed from 1874 to 1877: precisely, proceeding in a way similar to that already followed with regard to the exchange model, in Section 5 we examine the so-called "mathematical" solution of the model and the connected equilibrium determination issue; Section 6, instead, is devoted to the so-called "empirical" solution and the associated equilibrium establishment issue. Section 7 concludes.

2 Walras's Original Equilibrium Conception: The Geneva Notes

As already hinted at, the Geneva Notes can be viewed as the actual starting point for the development of Walras's theory of general equilibrium. They are important for what can be found therein, in terms of assumptions, definitions, theoretical constructs that will persist, either unaltered or with some changes, for most of Walras's scientific life[1]. But what is missing there is no less important for the understanding of both Walras's original vision and the subsequent evolution of his thought: for in those Notes one cannot find any formalization,

[1] A few ideas developed in great detail in the Geneva Notes had already been cursorily discussed in some of Walras's earlier surviving manuscripts, mostly drafted in the period 1869-71, which have been recently edited and published in Vol. XI of the *Œuvres Économiques Complètes d'Auguste et de Léon Walras*. To some such documents we shall come back in due time. Yet, none of them is even remotely comparable to the Geneva Notes in terms of width, depth, and scope.

not even a loose one, of the idea of individual optimizing behavior as applied to any one of the four great problems (exchange, production, capitalization, and money) into which, according to Walras, the subject-matter of economics can be partitioned.

Walras's analysis of exchange starts from a "law", called "loi de l'offre effective et de la demande effective" (1871-72, 423)[2]. In order to illustrate such "law", Walras makes use of an example concerning a specific market for financial assets, "la Bourse de Paris", where the "Rente Française 3%" is traded against money (1871-72, 420-3). This illustration, taken up again in very similar terms (with "blé" instead of "Rente Française 3%") in Walras's first published theoretical paper, a *mémoire* called "Principe d'une théorie mathématique de l'échange" (1874, 32), will then recur almost unaltered in all the editions of the *Éléments* (1988, 71-2)[3]. The only significant change, to which we shall come back in Section 4 below, will consist in the addition, in the 2^{nd} and following editions of the *Éléments* (1988, **2-5**, 71-2), of a few well-chosen words, introducing a very important qualification.

In this illustration price changes are formally represented as the images of a sign preserving function of the excess demands prevailing in the market. Such changes, brought about by the very participants in the market, namely, the buyers ("acheteurs") or the sellers ("vendeurs"), who "vont à l'enchère" or "au

[2] In this paper all references to Walras's works other than the *Éléments d'économie politique pure*, or, in short, the *Éléments*, are simply given by mentioning, between parentheses, the year of first publication or, in case of materials unpublished at the time of their drafting, as is the case here, by mentioning the year or years when such materials were presumably written, without specifying the name of the author. Page indications always refer to the reprint of Walras's work under question in the relevant volume of the *Œuvres économiques complètes d'Auguste et de Léon Walras*, quoted in full in the References at the end of the paper. In the various versions of Walras's works reprinted in the Volumes of the *Œuvres* there occasionally occur minor changes in either the titles of the writings or the wording of some sentences with respect to the original publications. In any case, all such changes are signaled in the critical edition we refer to. Occasionally we shall cite sentences or documents due to the editor(s) of one or the other of the volumes of the *Œuvres*. In such a case, our quotation will be (Walras, year of publication of the volume concerned, page(s) of the sentence or document we are referring to). As to the *Éléments*, we direct the reader to the following footnote.

[3] As already mentioned, during Walras's lifetime there appeared four editions of the *Éléments*: the 1^{st} was published in two instalments in 1874 and 1877, respectively; the 2^{nd} in 1889; the 3^{rd} in 1896; the 4^{th} in 1900. A 5^{th} edition, containing a few minor changes and some additions to the 4^{th} edition, written by Walras himself at the beginning of the 20^{th} century (probably in 1901 or 1902) in view of a planned reprint of the *Éléments*, was posthumously published in 1926, sixteen years after Walras's death. When quoting from the *Éléments*, we shall invariably refer to the comparative edition, published in 1988 as vol. VIII of the *Œuvres économiques complètes d'Auguste et de Léon Walras*. Each such reference in either the text or the footnotes of this paper will have the following structure: (1988, **n** or **n1-n2**, p or p1-p2), where **n**, **n1**, and **n2** are edition numbers, with **n** running from **1** to **5**, **n1** from **1** to **4**, and **n2** from **2** to **5**, while p, p1, p2 are page numbers, always referring to the 1988 comparative edition. When no edition number is specified, it is implied that no change occurred in the quoted passage or expression over all the five editions; on the contrary, when just one edition number is specified (resp., when two edition numbers joined by a hyphen are specified), it is implied that the quoted passage or expression only occurs in the specified edition (resp., in all the editions between the specified two, themselves included). Jaffé's English translation of the *Éléments* will be referred to as (Walras 1954).

rabais", as the case may be, tend to correct market disequilibrium, producing a situation where

> [la] demande effective est égale à l'offre effective. Chaque agent vendeur ou acheteur trouve sa contrepartie. Il y a *prix courant stationnaire* or *équilibre du marché*. (1871-72, 423; italics in the original)

As can be seen, price changes take place in 'real' time, meaning by this the time over which the market evolves and the actual trades take place. When an equilibrium state is reached, the price stops changing, becoming "stationary". An equilibrium is identified as a situation where the plans of all the participants in the market are mutually compatible and can be carried out. No hint is provided, however, as to the motivations underlying the agents' bids.

In Walras's own words, the "law of supply and demand" is said to be "la base de toute la théorie de la valeur d'échange". As can be gathered from this early, concise statement of the "law" and the associated "theory", a statement that will be reiterated over and over again in all of Walras's later writings, the "theory" itself is called to answer two distinct, yet interrelated, questions: on the one hand, there is the issue of 'equilibrium determination', subsequently identified with the problem of finding the "mathematical" solution of a system of ordinary algebraic equations describing the basic structure of either the economic system under scrutiny or any of its subsystems; on the other, there is the issue of 'equilibrium establishment', successively evolving into the analysis of the so-called tâtonnement processes, through which the market is supposed to "empirically" find out, by trial and error, the "stationary" solution, assumedly coinciding with the "mathematical" one, of a system of functional equations supposedly depicting the working over time of the competitive mechanism.

It is worth dwelling upon Walras's initial answers to both questions since, as will be seen, the embryonic solutions put forward by him in the Geneva Notes will largely condition the subsequent evolution of his thought for about three decades. As to the issue of 'equilibrium determination', what emerges from the illustration summarized above is the statement that in the market for the "Rente Française 3%" the equilibrium price is the one equating demand and supply. On the nature of the demand and supply functions analyzed by Walras in the Geneva Notes we shall say more in a while. For the moment, we just want to stress that the statement that the equation of demand and supply provides the market equilibrium condition, a statement sounding quite trivial nowadays, represents instead a veritable achievement for Walras in the Winter 1871-72: for, up to a few months before, he was still striving to show, obviously without success, that the equilibrium price in a market depends on the "ratio between demand and supply" ("rapport de la demande à l'offre")[4].

[4]Such hopeless attempts can be found in a manuscript titled "Applications des mathématiques à l'économie politique ($2^{ème}$ tentative 1869-70)" (1869-70, 353-4), where Walras tries, *inter alia*, to counteract Cournot's criticism of the time-honored assertion that "[l]e prix des choses [...] est en raison inverse de la quantité offerte, et en raison directe de la quantité demandée" (Cournot 1838, 35-6). It is interesting to note that a few years later, in his first the-

In this regard, it is worth noting that, between the late 1860s and the early 1870s, when Walras was progressively changing his mind about the price equilibrium condition, the most influential economist attacking the idea that the equilibrium price depends on the "ratio [...] between demand and supply", suggesting instead that "the proper mathematical analogy is that of an *equation*", was precisely J. S. Mill (1848, 467, italics in the original). In view of this, it seems quite likely that, in spite of the remarkable differences between Mill's and Walras's theories of value and prices, differences on which we shall come back later in this Section, Walras's switch from the "ratio" to the "equation" condition for market equilibrium at the turn of the 1870s might have been prompted by his contemporary reading of Book III, Chapter II of of J. S. Mill's *Principles of Political Economy*, where Mill discusses at length the issue under question.

If Mill's influence on Walras's change of mind concerning the 'equilibrium determination' problem is only probable, the former's major role in shaping the latter's approach to the issue of 'equilibrium establishment' is instead absolutely certain. For, about twenty years after drafting the Geneva Notes, in the midst of the raging controversy with Edgeworth about, *inter alia*, the tâtonnement construct[5], Walras himself will take care to underline, in two distinct occasions, that his own theory of tâtonnement in exchange is nothing but a reworked version of Mill's analysis of the price adjustment process due to the play of supply and demand in a competitive market. The first clear recognition of such lineage can be found in a letter to Bortkiewicz, dated February 27, 1891, where Walras writes:

> Dans la question du tâtonnement [...] je prends le mode presque universel de libre concurrence en matière d'échange, celui qu'a décrit John Stuart Mill, et qui consiste à faire la hausse en cas d'excédent de la demande sur l'offre et la baisse en cas d'excédent de l'offre sur la demande, et je démontre que ce procédé amène l'équilibre par l'égalité de l'offre et de la demande. (Jaffé, 1965, II, letter 999, 434-5)

An even more explicit statement that his own theory of tâtonnement in the exchange model directly descends from Mill's paradigmatic discussion is made by Walras in a handwritten note, discovered by Jaffé among Walras's manuscripts, which, though undated, can almost certainly be traced back to the same period as the one when the above letter to Bortkiewicz was written (beginning of 1891). In such a note, now published as an enclosure to a much posterior letter to Pareto, dated January 9, 1895, Walras, referring to his own description of the mode of operation of the tâtonnement process in the exchange model, writes:

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oretical *mémoire*, Walras will take just the opposite stance towards Cournot's criticism (1874, 30). The first document where the equation of demand and supply is recognized as providing the market equilibrium condition is a manuscript titled "Applications des mathématiques à l'économie politique (1871). 3^e tentative (la bonne, v.p. 15)" (1871, 395).

[5] See Edgeworth (1889a, 1889b, 1891) and Bortkiewicz (1890), who participates in the controversy, at Walras's instigation, siding wholeheartedly with the latter.

> Ce mode est celui qui a été défini par John Stuart Mill au livre III, ch. II, paragraphe 4 de ses *Principes* et qui peut être considéré comme le mode principal qui se pratique sur tous les marchés [...]. (Jaffé, 1965, II, letter 1200, 628-32)

In order to properly contextualize Walras's reference to a specific paragraph of Mill's *Principles*, it is convenient to recall that Book III of the *Principles*, titled "Exchange", deals with the theory of value and prices; Chapter II of that Book and Paragraph 4 of that Chapter, respectively titled "Of Demand and Supply, in Their Relation to Value" and "The Equation of Demand and Supply is the law of their value", confront the issues alluded to by their titles. To better understand the content of the paragraph referred to by Walras, it is expedient to make a short digression about the basic premises of Mill's theory of value[6].

Building, with some changes, upon Ricardo's tradition (Ricardo 1821, 11-127), Mill distinguishes three categories of commodities: the first category owns those "commodities which are absolutely limited in quantity"; the second, those "which are susceptible of indefinite multiplication without increase of cost"; the third, those "which are susceptible of indefinite multiplication, but not without increase of cost" (Mill 1848, 464-5, 471, 488). When taken in its strictest sense, the first category comprises a limited number of commodities of minor relevance, such as "ancient sculptures" or "pictures by old masters"; yet, under a more liberal interpretation of its scope, the first category gains much wider boundaries, to the point that, "potentially, all land whatever is a commodity of this class". In spite of this admission, however, it is the second category that remains for Mill by far the most important of the three, since it "embrac[es] the majority of all things that are bought and sold". Finally, the third category, which is "intermediate between the two preceding", encompasses all "agricultural produce [...] and generally all the rude produce of the earth" (Mill 1848, 464).

Having introduced the above classification, it should be stressed that the paragraph of Mill's *Principles* cited by Walras discusses the "laws" of both 'equilibrium determination' and 'equilibrium establishment' concerning the prices of commodities supposedly belonging to the first class, that is, of those commodities "which are absolutely limited in quantity". Since the aggregate quantity of any such commodity is fixed, all that its potential buyers or sellers can do is to bid its price up or down, according to whether excess demand or excess supply dominates the market, and they proceed like this up until demand becomes equal to supply and market equilibrium is eventually reached. This is the reason why, in the first of the two quotations reproduced above, Walras speaks of "tâtonnement [...] *en matière d'échange*", since the kind of adjustment process

[6] A more detailed discussion of Mill's theory of value is postponed to the end of this Section. Yet, neither here nor in the sequel, we shall lay any claim to provide a complete exposition or a critical discussion of Mill's thought. On the contrary, we shall confine our reconstruction to what is strictly required for a comparison between Mill's approach and Walras's. Moreover, with a view to minimizing all possible misunderstandings, we shall resort to Mill's own words as far as possible.

originally envisaged by Mill in the cited paragraph, and later taken up by Walras in what will become the tâtonnement process in his pure-exchange model, is such that, for any given commodity, the only possibility open to the agents participating in the market for that commodity is to trade among themselves, at progressively adjusting prices, a fixed aggregate quantity of the commodity concerned.

Yet, even if in Book III, Chapter II, Paragraph 4 of his *Principles* Mill introduces the "law of supply and demand" with reference to commodities whose quantity is "absolutely limited", just a few pages ahead, in the immediately following Paragraph 5, he is ready to admit that, under regularly occurring circumstances, the very same "law", with appropriate qualifications, applies also to commodities which are "susceptible of indefinite multiplication". This is due to the following reason:

> In the case of most commodities, it requires a certain time to increse their quantity; and if the demand increases, then until a corresponding supply will be brought forward, that is, until the supply can accommodate itself to the demand, the value will so rise as to accommodate the demand to the supply. (Mill 1848, 649)

As we shall see, this mode of operation of the "law" will play an important, though somewhat disguised, role in the construction of Walras's analysis of the process of adjustment to equilibrium in theory of production. Yet, the detailed examination of this issue must be postponed to a later stage of our discussion, when Walras's theory of production, as developed in the Geneva Notes, will have been introduced. For the moment, it is convenient to focus attention on the issue of 'equilibrium determination' in the theory of exchange.

In this regard, in the Winter 1871-72 Walras eventually succeeds in bringing to conclusion his long-lasting and winding process of elaboration of a theory of exchange of two commodities for one another, a process which, having started more than a decade before, had gone through three largely unsuccessful intermediate steps or "attempts" (1860, 1869-70, and 1871), only the last of which can be credited of some positive results. The final outcome, eventually reached in the Geneva Notes (1871-72, 424-9), can be summarized as follows.

Given any two commodities, denoted (A) and (B), Walras assumes the existence of individuals endowed with specified quantities of either commodity. At any relative price $p_{ab} \equiv \frac{1}{p_{ba}}$, each agent endowed with a given quantity of (A), say i_a, demands a specified quantity of (B), $B^d_{i_a}(p_{ab})$, thereby offering a quantity $A^o_{i_a}(p_{ab}) = B^d_{i_a}(p_{ab})p_{ab}$ of (A). A similar reasoning applies to every agent i_b, endowed with a given quantity of (B).

Aggregating the individual functions over all the agents endowed with quantities of (A) and (B), respectively, one obtains the aggregate demand and supply functions for the two commodities, $A^d(p_{ba})$, $A^o(p_{ab})$, $B^d(p_{ab})$, $B^o(p_{ba})$. For any pair of commodities (A) and (B), the individual, hence the aggregate, demand functions are assumed to be exogenously given; they are supposed to be continuous and monotonically decreasing, and to take a zero value for a finite maximum

price and a finite maximum value for a zero price. Hence, the graph of any such aggregate demand function, drawn in a diagram with the appropriate relative price on the abscissa axis and the quantity demanded on the ordinate axis, is a downward sloping curve, called "courbe d'échange", cutting both axes. The individual and aggregate supply functions can be derived from the corresponding demand functions. The supply curves can similarly be drawn in appropriate diagrams.

Under such assumptions, an equilibrium pair of reciprocal relative prices, p^*_{ab} and p^*_{ba}, where $p^*_{ab} \equiv \frac{1}{p^*_{ba}}$, is such that $A^d(p^*_{ba}) = A^o(p^*_{ab})$ and $B^d(p^*_{ab}) = B^o(p^*_{ba})$, i.e., such that aggregate demand equals aggregate supply on either market. In view of the equilibration process mentioned above, Walras calls "prix courants stationnaires" the reciprocal relative prices associated with an equilibrium state (1871-72, 429).

While taken as given for the purposes of equilibrium determination, the aggregate demand curves, or "courbes d'échange", are assumed to depend on four types of more basic data. For each commodity, such data are the following: "utilité extensive", "utilité intensive", "quantité existante", "mode de répartition" (1871-72, 430-5). Even if Walras loosely discusses how these data and their variations can possibly affect the determination and variation of the equilibrium prices and quantities, he makes no attempt to formally derive the "courbes d'échange" from the data by means of any theory explaining individual or aggregate behavior. In particular, no optimizing behavior is assumed on the part of individual traders.

Walras's next goal is to construct a theory of exchange of several commodities among themselves. Assuming the existence of m commodities in the economy, with $m \geq 3$, Walras pursues his task in two steps. At first, supposing all $\frac{m(m-1)}{2}$ pairs of commodities to be directly traded with one another on corresponding pairwise particular markets, he imagines the same number of pairs of equilibrium reciprocal relative prices to be established on such special markets. Then, through an arbitrage process, whose idea is borrowed from Cournot (1838, chap. 3), Walras supposes that the $m(m-1)$ particular equilibrium relative prices will be brought to satisfy a consistency condition, regarded as the hallmark of what deserves to be named "équilibre général", which can be spelled out as follows: for every triple of commodities (A), (B), and (C), one must have $p_{cb} = p_{ca}/p_{ba}$ (1871-72, 438-41)[7].

Walras strives to show that, as long as the consistency condition is not met, any trader would gain from indirect exchanges involving at least three commodities. Any state different from the "general equilibrium" one would then be disrupted by actions taken by one trader or another. Hence, only the "general equilibrium" state can be justified as a persistent state. It should be noted that this sort of justification of the establishment and persistence of the "general equilibrium" condition, while resting on individual incentives and self-interested

[7] The idea of an arbitrage process bringing about the consistency condition among relative prices specified in the text had already been briefly discussed by Walras in both his second and his third "attempt" to apply mathematics to political economy (1869-70, 345-6, and 1871, 408-9).

behavior, neither attempts at identifying the individual traders responsible for taking the actions that will result in the emergence of the "general equilibrium" condition, nor assumes the behavior of the acting traders to be optimizing.

A sort of dynamical process is sketched, but the path followed during the process, as well as the time dimension over which the process itself is supposed to take place, are left unspecified. Only the final outcome is described, as summarized by the "general equilibrium" condition, of which one cannot know whether it is path-independent or not. Hence, for the purposes of the theory, the "general equilibrium" condition may be regarded as instantaneously satisfied. It is precisely at this juncture that the idea of an 'instantaneous' adjustment process creeps into Walras's theoretical system for the first time.

Assuming now the "general equilibrium" condition to be satisfied, the $\frac{m(m-1)}{2}$ "equilibrium" pairs of reciprocal relative prices for the m commodities in terms of one another can be equivalently expressed as a system of m prices of all commodities in terms of one of them chosen as the standard of measurement ("étalon pour la mesure des valeurs"), or the numéraire of the system, whose price is set equal to 1 (1871-72, 440-2).

Now, let us suppose that a "new" commodity enters the economy from outside. Then, for any such commodity, by exploiting the properties of the price system normalized by means of the choice of a numéraire, and taking such "equilibrium" price system as fixed, Walras constructs two functions and the corresponding curves. At first he constructs a "courbe d'achat", which is interpreted as the graph of the functional relationship between the demand for the "new" commodity and its numéraire price, under the assumption that the "new" commodity is exchanged against the numéraire value, at the given numéraire prices, of the quantities of the other commodities with which the holders of the "new" commodity are willing to make exchanges. Then, from such "courbe d'achat", he derives a "courbe de prix", which is interpreted as the graph of the functional relationship between the existing quantity of that commodity and its numéraire price (1871-72, 441-7).

The entire construction is cumbersome and, at least as regards the "courbe de prix", definitely wrong. The existing mistakes will be corrected in the first edition of the *Éléments* (1988, 239, 342), where Walras will also clarify why the "courbe de prix" should be regarded as so important in his overall theoretical system and explain which is the role effectively assigned to such construction therein: as we shall see shortly, this role will turn out to be the same as the one actually played by the "courbe de prix" in the Geneva Notes. When freed from the mistaken connotations marring it when first introduced, Walras's "courbe de prix" is nothing but a partial equilibrium demand curve for a consumer good, drawn under the *ceteris paribus* assumption that the prices of all the other commodities be taken as fixed. As Walras himself will subsequently point out (1988, 239), such "courbe" coincides with Cournot's "courbe de demande ou du debit" (Cournot 1838, chap. 4).

Let us now turn to the theory of production. A first fundamental distinction, that Walras inherits from his father August, is that between "capitaux",

or capital assets, and "revenus", or incomes: capital assets outlive their first use; incomes, instead, can be used just once. Three types of capital are distinguished: "terres ou capitaux fonciers", i.e., lands or landed capital, denoted by T or t; "hommes ou capitaux personnels", i.e., personal capital, denoted by P or p; "capitaux mobiliers ou capitaux proprement dits", i.e., capital goods proper, denoted by K or k. Lands are imperishable, non-producible and marketable; personal capital is perishable, non-producible and non-marketable; capital goods proper are perishable, producible and marketable. Each capital repeatedly renders a service, consisting in the use of the very same capital for a specified period of time. Services may be used in production, in which case they are said to be productive services, or in consumption, in which case they are said to be consumable services. Services are one of the two types of incomes, the other one being represented by consumer goods (1871-72, 465-73).

Production activity involves the functioning of two markets: the market for services and the market for products. Four categories of agents act on these two markets. The first three are the owners of the three types of capital: landowners, who own the lands; workers, who own personal capital; and capitalists, who own capital goods proper. The fourth category consists of entrepreneurs. On the market for services, entrepreneurs act as buyers of the various types of services, which are sold by the owners of the corresponding types of capital. On the market for products, entrepreneurs act as sellers of produced commodities, which may be either consumer goods or capital goods proper (but the two categories of products are not clearly distinguished at this stage); produced commodities are bought by the owners of the various types of capital, who are also consumers, by means of the incomes (rents, wages, and interest payments) they receive in exchange for the services they sell (1871-72, 465-70).

The peculiar role of the entrepreneur, misconceived by that school of thought that Walras calls the "English School" ("école anglaise"), as represented by "A. Smith, Ricardo, MacCulloch" (1871-72, 448), is repeatedly stressed by Walras in passages that, though being just rough drafts, clearly anticipate what will become a tenet of Walras's system of thought:

> L'entrepreneur achète les services producteurs (aux propriétaires foncièrs, travailleurs et capitalistes). Il les combines. Il est le propriétaire des produits. Les produits se vendent sur le marché des produits (aux consommateurs). S'il les vend plus cher : *Bénéfice*. Si moins cher : *Perte*. Le bénéfice est corrélatif à la perte - à l'état normal, ni bénéfice ni perte, ou du moins bénéfice égal à l'intérêt du capital et au salaire des facultés personnelles de l'entrepreneur. Mieux vaut dire bénéfice *nul*. [...] L'école anglaise n'a pas compris le rôle de l'entrepreneur. C'est un intermédiaire dont on peut faire abstraction. (1871-72, 469, 473, italics in the original)

In the normal state of the market for products, entrepreneurs are expected to make neither gains ("bénéfices") nor losses ("pertes")[8]. Hence, when the market

[8] It is worth noting that, in the above quoted passage, the absence of entrepreneurial gains

for a consumer good is in equilibrium, the unit cost of production ("prix de revient") must equal the selling price ("prix de vente" or "prix courant"). This equality is brought about by competition ("concurrence") among producers, as Walras clearly states in the following sentence:

> Sous l'influence de la liberté de la production et de la concurrence, le prix courant et le prix de revient tendent à concorder l'un avec l'autre. (1871-72, 449)

The operation of the competitive mechanism in production is illustrated by means of a price-quantity diagram[9] where, in line with Walras's standard graphical conventions, the price (in numeraire) of a consumer good, say, commodity (T), is measured on the abscissa axis, while the quantity of that good is measured on the vertical axis. The "courbe de prix", that is, the demand curve, relative to commodity (T) is represented in the diagram as a monotonically decreasing curve Q_tP_t cutting both axes, with intercepts Q_t and P_t with the vertical and the horizontal axis, respectively. Descending along such curve from Q_t to P_t three points are successively plotted in the diagram, namely, the points $\theta'' = (p_t'', q_t'')$, $\theta = (p_t, q_t)$, and $\theta' = (p_t', q_t')$, so that along the horizontal axis one has $0 < p_t'' < p_t < p_t' < P_t$, while along the vertical axis one has $0 < q_t' < q_t < q_t'' < Q_t$.

As no endowments of the good are supposed to be in storage, all the quantity of the good available for consumption comes from production. The entrepreneurs engaged in the production of that good are supposed to throw their entire production onto the market. Hence, the *supply curve* is represented by a horizontal straight line, whose intercept with the ordinate axis corresponds to the available output. The consumers are supposed to absorb all the quantity supplied, clearing the market. Hence, the market clearing price, i.e., the "prix courant" or "prix de vente", is that price that corresponds to the projection onto the abscissa axis of the intersection point between the "courbe de prix" and the horizontal *supply curve*; therefore, in Walras's diagram, if the quantity produced and brought to the market were q_t', the corresponding market-clearing price would be p_t'; and similarly with q_t and p_t, or with q_t'' and p_t''[10].

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or losses is said to represent the "better way" to describe the "normal state" of the market; yet, an alternative, though less satisfactory, way, where the "normal" entrepreneurial gains are identified with the interest payments on capital and the salary of the entrepreneur, is not ruled out altogether. We shall come back to Walras's still wavering stance on this issue at the end of this Section.

[9] In the Geneva Notes this diagram is reproduced twice, in two consecutive pages (1871-72, 449-50), the two versions of it slightly differing from one another in the notation employed to identify the relevant values of the variables involved (prices and quantities of the product concerned). To fix the ideas, we shall refer to the diagram on page 449.

[10] We have used the italicized expression *supply curve* to denote a horizontal straight-line segment connecting an ordinate on the vertical axis (such as q_t', q_t, or q_t'') with the corresponding point on the demand curve (namely, θ', θ, or θ''), since any such segment does in effect correspond to what would be called nowadays, in a partial equilibrium framework of Marshallian inspiration, a 'perfectly inelastic supply curve'. It should be noted, however, that Walras nowhere makes use of the expression 'supply curve' (or, in French, 'courbe d'offre') to

By making use of the same diagram, Walras then undertakes to analyze the relation "entre le *prix* et les *frais de production*" (1871-72, 450, italics in the original), with a view to explaining how the competitive mechanism in production succeeds in bringing about the convergence of the market price to the unit cost of production of the commodity concerned. The first step would be to explain how such cost is determined. Yet no such explanation is provided at this juncture. Walras seems to recognize that an explanation is needed and to suggest that it will be put forward in due time. In fact he writes: "Nous ne savons *encore* en quoi consistent les frais de production" (1871-72, 450; italics added), where the word "encore" would appear to allude to a forthcoming clarification. But such implicit promise will never be really honored in the Geneva Notes. Postponing to the end of this Section the attempt to explain why in 1871-72 Walras is still unable to fulfil his tacit commitment, we now proceed to show what he is actually able to do in this respect.

Among the three values plotted along the abscissa axis (p''_t, p_t, p'_t), the intermediate one, p_t, is taken to represent the "prix de revient", or the "frais de production", or the unit cost of production. As just recalled, no explanation of such assumption is provided. Yet, for the purposes of the analysis of the working of the competitive mechanism in production, such unit cost p_t is not only taken as exogenously given, but also assumed to be independent of the level of output: as a matter of fact, in Walras's diagram, what has all the appearance of a unit cost curve, even if Walras does not us this expression, is drawn as a vertical straight line segment, parallel to the quantity axis, joining the point p_t on the horizontal axis with the point θ on the demand curve. By referring to the diagram we have just described, Walras writes:

> Quant au rapport du prix courant avec les *frais de production*, la courbe de prix permet de l'apercevoir avec un parfait netteté.
>
> Si les frais de production, ou le prix de revient, soit [*sic*] égaux a p_t, et que le prix courant soit p'_t, il y a un bénéfice $p'_t - p_t$ pour le

denote the graph of a functional relation between the aggregate quantity of a product brought to the market and its price (this being the reason why we have italicized the expression in the text).

As a matter of fact, almost twenty years after the Geneva Notes, Walras will express his sharp disapproval of the use, then starting to spread, of the expression 'supply curve' (or 'courbe d'offre') to denote, in a partial equilibrium framework, the graph of the functional relation between the individual or aggregate quantity supplied of a given commodity and its price, assumed to coincide with its marginal cost of production. Walras levels his criticism at such use of the expression 'supply curve' (or 'courbe d'offre'), and of course also at the underlying theoretical conception, at first indirectly in a letter to Edgeworth of October 29, 1889 (Jaffé 1965, II, letter 931, 369), and then more directly, a few months later, in a note about some issues raised by Auspitz and Lieben's book *Untersuchungen über die Theorie des Preises* (1889), where in effect the authors make use of such construct. In such note, originally published in the *Revue d'économie politique* (1890) and later reprinted as Appendix II of the 3^{rd} and following editions of the *Éléments* (1988, **3-5**, 710-4), he brands as "unacceptable" the authors' use of the "courbe d'offre ou courbe de prix de revient en function de la quantité fabriquée" (1988, **3-5**, 712) for the purpose of determining the equilibrium of the market for a consumer good in a partial equilibrium framework. The reasons for such deeply-rooted hostility will be made clear in Section 5 below.

fabricant. L'industrie se porte vers la production de la merchandise et en augmente la quantité. Mais au fur et à mesure que la quantité croît de $p'_t\theta'$ à $p_t\theta$, le prix courant diminue de p'_t à p_t.

Le phénomène inverse a lieu si le prix de revient excède le prix courant. Il y a perte $p_t - p''_t$ pour le fabricant. La marchandise se consomme sans être reproduite et la quantité décroissant de $p''_t\theta''$ à $p_t\theta$, le prix courant augmente de p'_t à p_t. (1871-72, 449)

Therefore, the mechanism driving the market towards an equilibrium where selling price equals unit cost rests on a quantity adjustment rule: if the "prix de vente" is less than the "prix de revient", the entrepreneurs participating in the market concerned make losses, what induces them to reduce their output or to leave the market altogether; vice versa, if the "prix de vente" exceeds the "prix de revient", the entrepreneurs already operating in that market make gains, what induces them to increase their output; moreover, new entrepreneurs may be induced to enter that market from outside, thereby increasing the aggregate output further. The process will go on, in 'real' time and through adjustments of actual production, as long as selling price and unit cost differ. When they are equal, the quantity adjustment stops, output matches consumption, assumed to be constant over time, and the price becomes "stationnaire" (1871-72, 468). Gains and losses represent positive or negative incentives explaining entrepreneurs' behavior, which is self-interested, though not optimizing. The acting individuals are not identified: some of them may even come from outside, others may exit the market. The exact path of the adjustment process is unknown. One can only know the direction of change and the final state, which is however fixed and path-independent, so long as the unit cost does not change.

This being the state of Walras's theory of 'equilibrium establishment' in the field of production as of 1871-72, it may be interesting to check whether there exists some sort of relationship between Walras's approach and the theory of production, value, and costs, developed about two decades before by J.S. Mill, that is, by the economist who, already at the centre of Walras's criticism in the Geneva Notes, shortly after will become the most prominent target for Walras's cutting remarks in the *Éléments*[11]. In view of the deeply-rooted contrast between the two authors, *a priori* one would expect no significant relation to exist between the two in the field of production, particularly so since in this field Walras nowhere pays any special tribute to Mill's ideas, differently from what he had done, as has been seen, with respect to the field of exchange. Yet, this expectation turns out to be wholly unfounded.

In order to carry out the proposed comparative assessment, it is necessary, first of all, to summarize Mill's stance in this regard. Once again we lay no claim

[11] As regards the Geneva Notes, see (1871-72, 472-3). As regards the *Éléments*, see, in particular, the Lessons numbered 38 and 40 in the 4^{th} and 5^{th} editions (1988, 609-16 and 639-51), respectively titled "Exposition and réfutation de la théorie anglaise du prix de produits" and "Exposition and réfutation des théories anglaises du salaire et de l'intérêt", where Walras wryly criticizes Mill for his faulty logic and inconsistent theory (1988, especially 615-6 and 646-8).

to completeness. In particular, in the following we shall focus attention exclusively on the second of Mill's three classes of commodities, to which there belong those commodities "which are susceptible of indefinite multiplication without increase of cost". As Mill writes (1848, 472-6), the value of each commodity belonging to this class is governed by its unit cost of production:

> Adam Smith and Ricardo have called that value of a thing which is proportional to its cost of production, its Natural Value (or its Natural Price). They meant by this, the point about which the value oscillates, and to which it always tends to return; the centre value, towards which, as Adam Smith expresses it, the market value of a thing is constantly gravitating; and any deviation from which is but a temporary irregularity, which, the moment it exists, sets forces in motion tending to correct it. [...]
>
> The latent influence by which the values of things are made to conform in the long run to the cost of production, is the variation that would otherwise take place in the supply of the commodity. The supply would be increased if the thing continued to sell above the ratio of its cost of production, and would be diminished if it fell below that ratio. [...]
>
> [I]n all things which admit of indefinite multiplication, demand and supply only determine the perturbations of value, during a period which cannot exceed the length of time necessary for altering the supply. While thus ruling the oscillations of value, they themselves obey a superior force, which makes value gravitate towards Cost of Production [...].

But how do demand and supply concretely act to determine such "perturbations of value"? The fact is that, while for those "commodities which are absolutely limited in quantity" the supply, that is, "the quantity offered for sale", is permanently fixed, for those "commodities which are susceptible of indefinite multiplication without increase of cost", the supply is similarly fixed, but only "temporarily", "at a given time and place". For any commodity belonging to the latter category, it often occurs that the supply be "temporarily" fixed at a level different from the level of demand that would correspond to its natural value, that is, to the unit cost of production of that commodity. But, whenever this occurs, since "the quantity demanded [...] varies according to the value" of the commodity (Mill 1848, 465, 469),

> a value different from the natural value [turns out] to be necessary to make the demand equal to the supply, [so that] the market value will deviate from the natural value; but only for a time; for the permanent tendency of supply is to conform itself to the demand which is found by experience to exist for the commodity when selling at its natural value. If the supply is either more or less than this, it is so accidentally, and affords either more or less than the ordinary

> rate of profit; which, under free and active competition, cannot long continue to be the case. (Mill 1848, 475)
>
> [For], if the value of a commodity is such that it repays the cost of production not only with the customary, but with a higher rate of profit, capital rushes to share in this extra gain, and by increasing the supply of the article, reduces its value. (Mill 1848, 472)

It can now be easily seen that Walras's diagram commented upon above, together with the related symbolic discussion, are nothing but a moderately analytic attempt at formalizing Mill's theory of the quantity adjustment process supposedly driving the market price of a commodity "susceptible of indefinite multiplication without increase of cost" towards its unit cost of production: in Walras's diagram and symbolic discussion, as in Mill's literary argument, the 'supply' is the "quantity of the commodity offered for sale"; the market price is that price that equates the "temporarily" given supply with the demand for the commodity concerned, itself assumed to be a decreasing function of market price; the unit cost is taken as given and independent of output; any divergence between market price and unit cost is supposed to set in motion a quantity adjustment process which, acting on the market price, tends to remove the gap between market price and unit cost. Finally, Walras's insistence on the "stationary" character of the price eventually arrived at (1871-72, 468), identical with the unit cost, is a further proof that Walras's conception of an equilibrium price is patterned after Mill's (or, for that matter, Ricardo's or Smith's) notion of a natural price as a persistent, time-invariant centre of gravitation around which market prices oscillate, or towards which they converge.

As regards the issue of 'equilibrium establishment' in the field of production, the only noticeable difference between the two authors' approaches, as of 1871-72, seems to lie in both the nature of the agents who are supposed to bring about the quantity adjustment process and the incentives to which such agents are supposed to react: for, while in Mill the acting characters are the capitalists, who react to any difference between actual and ordinary rate of profit, in Walras, instead, the acting characters are the entrepreneurs, who react to the gains or losses that may occur with respect to a standard which, in the "normal state", is assumed to be nil. Yet, also in this respect Walras's wavering interpretation of the characteristics of a "normal state", as offered in the already quoted sentence ("à l'état normal, ni bénéfice ni perte, ou du moins bénéfice égal à l'intérêt du capital et au salaire des facultés personnelles de l'entrepreneur". Mieux vaut dire bénéfice *nul*.), shows that, still in 1871-72, he had not yet succeeded in freeing himself completely of the influence of the English School, especially of Mill[12], as regards the interpretation of profits and the role of the entrepreneur.

In the light of the above remarks, therefore, one is bound to conclude that,

[12] The central part of Walras's sentence recalled above can almost certainly be traced back to Mill's writings. To convince oneself of this lineage, it is enough to look at Mill's *Principles* (Book II: Distribution, Chap. XV. Of Profits), where the eloquent title of Section 1 is: "Profits resolvable into three parts: interest, insurance, and wages of superintendence" (Mill 1848, 400).

in the Geneva Notes, Walras's analysis of the quantity adjustment process, supposedly bringing about the convergence of the market price of a producible commodity to its cost of production, is unquestionably molded on Mill's analysis of the same process in the *Principles*. Yet, the basic agreement between Mill's *Principles* and Walras's Geneva Notes concerning the theory of 'equilibrium establishment' in the field of production does not extend to the theory of 'equilibrium determination' in the same field. To put it bluntly: while Mill does possess a theory, no matter how incomplete and even contradictory, of cost determination, hence also of equilibrium price determination, for those "commodities which are susceptible of indefinite multiplication without increase of cost", Walras, in the Geneva Notes, is still unable to put forward any theory of cost determination, hence to explain how the price of products can be determined, being forced, as we have seen, to take the unit costs of producible commodities as exogenously given.

Mill's theory of costs rests on his theory of distribution, that is, on his theory of wages, profits, and rents. But rents can be excluded from the computation of costs, since, in line with the Ricardian tradition, Mill states that "[r]ent is not an element in the cost of production of the commodity which yields it", at least as far as agricultural produce is concerned. Therefore,

> things which admit of indefinite multiplication, naturally and permanently exchange for each other according to the comparative amount of wages which must be paid for producing them, and the comparative amount of profits which must be obtained by those capitalists who pay those wages. (Mill 1848, 498)

Hence, by determining wages by means of the so-called Wage Fund Doctrine, still dominating the *Principles* (Book II, Ch. XI, 343-60) before Mill's retractation, and profits by means of a mixture of a theory of surplus product of Ricardian derivation and an abstinence theory of profits inspired by Senior and others (Book II, Ch. XV, 405-21), Mill is apparently able to explain the costs of "things which admit of indefinite multiplication", hence their natural values and prices, since "[t]he natural value is synonymous with the Cost Value" (Mill 1848, 498).

However, the route followed by Mill is barred for Walras, for, since the Geneva Notes, he totally rejects the theory of distribution of the English School. In the draft of the text of the 9^{th} conference, titled "Du mécanisme de la production, ou de l'association des services producteurs dans les entreprises. - Rôle de l'entrepreneur. - Bénéfice. - Perte. - Capital fixe et capital circulant.", referring to Ricardo's and Mill's theory of rent, and more generally to their theory of distribution, Walras writes:

> Avant d'aller plus loin, je rèfuterai les diverses théories de la rente. Dans mon système, il n'y a pas de théorie de la rente ; il y a une théorie du capital et du revenue. La rente est le revenu du capital foncier, comme le travail est le revenu du capital personnel, comme le profit est le revenu du capital mobilier [...]. (1871-72, 468)

Walras's refutation of the theory of distribution of the English School, involving a thorough criticism of the theories of rent, interest and wages put forward by the supporters of that School, will become a permanent trait of his reconstruction of the history of economics and will be reproposed, with very small changes and additions, chiefly concerning the theory of rent, in all the editions of the *Éléments* (1988, 618-51, where the changes across the various editions involve just a few lines at pages 626-7, 630, and 633-7).

At the same time, as can be gathered by the passage reproduced above, he suggests an alternative theory of distribution, resting on the pricing of the services of the various types of capital. Yet, as regards such pricing, no satisfactory explanation is provided. What can be found in the Geneva Notes is the mere statement that the prices of productive services are determined on the market for such services by the play of demand and supply, in a way similar to the one in which the prices of consumer goods are determined on the market for products (1871-72, 451, 468, 473). In this respect, what clearly appears from perusing the Geneva Notes is that, at the moment of drafting them, Walras had not yet been able to progress beyond the conception developed half a century before by J.-B. Say in his *Traité d'économie politique* (1803, Livre I, Chap. V)[13].

As will be made clear by Walras himself in his third *mémoire* on the "Équations de la production" (1876a), the "prix de revient", i.e., the unit cost of production, of a given consumer good depends on the technical conditions of production and the prices of the services entering as inputs into the production of that good. In the Geneva Notes, Walras does not discuss in any detail the technical conditions of production. This is somewhat surprising, if one considers that in 1870, in an untitled manuscript now published in Volume XI of the

[13] Upon his appointment as Professor of Political Economy at the Académie de Lausanne, Walras, as was customary, was asked to deliver an inaugural lecture, which took place on October 20, 1871. Part of the text of this lecture is now published as an enclosure to the letter of appointment sent to Walras on July 27, 1871 (Jaffé 1964, I, letter 187, 280-2). In this lecture Walras, beyond criticizing Mill's Wages Fund theory and adumbrating his later refutation of the latter's theory of distribution, also praises J.-B. Say for "sa conception des trois services producteurs", a conception that he declares to be willing to borrow, "en la rectifiant par la théorie de la valeur d'échange et par la théorie du capital et du revenu qu'a données mon père" (282). A similar praise of J.-B. Say's approach to the theory of production, mitigated by a number of qualifications, can be found also in the *Éléments*, where, referring to J.-B. Say's *Traité*, Walras writes:

> Il y avait [...] une conception assez nette et assez exacte de l'association des trois services producteurs dans l'œuvre de la production. Les dénominations adoptées étaient bonnes ; aussi les avons-nous reproduites. Mais il restait cependant à combler de lacunes importantes. [...] Say n'explique qu'imparfaitement de quels services le salaire, l'intérêt et le fermage sont le prix ; et sa théorie n'indique pas plus que celle des physiocrates comment le prix se détermine. Il fallait faire intervenir ici une bonne théorie de la valeur et du mechnisme de l'échange, une bonne théorie du capital et du revenu et du méchanisme de la production, la conception de l'entrepreneur, celle du marché des produits et des services [...]. (1988, 648-9)

It appears that the inadequacies that Walras will blame upon J.-B. Say in the mid-1870s, when writing the text of the 1^{st} edition of the *Éléments*, are the same as those of which he himself can be found guilty in 1871-72, when drafting the Geneva Notes.

Œuvres complètes (1870, 361-78)[14], Walras had explicitly introduced, without naming them, his future *coefficients de fabrication* (1870, 374-5), i.e., those fixed technical coefficients that will become, for most of Walras's scientific life, the standard way by which he will represent the technology to be used in production[15].

In the 1870 manuscript, Walras had arrived very close at writing the equation defining the unit cost of production of a produced capital good, i.e., of a capital good proper, as the sum of the expenditures to be incurred in the purchase of the quantities of the services to be employed in the production of one unit of that capital good proper. What was still lacking in the 1870 manuscript, in order to arrive at a correct definition of the unit cost of production of the product concerned, was a clear specification and determination of the prices of the productive services to be employed in the process: the latter prices, in fact, were still confused with the so-called "absolute values" of the corresponding services, allegedly determinable as the "ratios" of their vaguely defined demands ("besoins") and supplies ("provisions"). Since in 1870 Walras was still dreaming of a theory of values and prices based on the "ratios" of demands and supplies, the attempt made in his 1870 manuscript at determining the prices of productive services by means of such "ratios" couldn't but end up in a failure, exactly as his contemporary attempt ($2^{ème}$ tentative 1869-70) at determining the prices of consumer goods. But while in 1871-72, in the Geneva Notes, Walras will eventually be able to correct his previously mistaken view as far as the prices of consumer goods are concerned, he will unfortunately be unable to do the same as regards the prices of productive services. Hence, in the Geneva Notes, one can find no theory at all of the prices of productive services, and consequently no consistent theory of costs either. This failure will prove to be so dramatic as to induce Walras to put aside even that innovative way of representing technology by means of technical coefficients of production that he had already discovered in 1870, but that he choose to completely neglect in his Notes of the Winter 1871-72.

To sum up, while lacking a theory of costs, in the Geneva Notes Walras is ready to accept Mill's idea that the competitive mechanism will make market prices converge to unit costs at equilibrium. Moreover, in discussing the quantity adjustment process assumed to bring about such convergence, he tacitly accepts another idea coming from the same source: namely, that the unit cost of production of any given product is independent of its level of output. However, while in Mill the assumed scale independence of the unit cost of production is somehow justified by his underlying theory of cost determination, at least as far as that class of "commodities which are susceptible of indefinite multiplica-

[14] Even if the manuscript is untitled, its sections are however headed by the acronym "C & R", followed by a number, where "C & R" patently stands for 'Capitaux and Revenus".

[15] As is well-known, Walras's technical coefficients express the quantities of inputs to be employed per unit of output, for each of the production processes characterizing a given single-output technology, where the inputs are the productive services rendered by the various types of capitals and the outputs are either consumer goods or capital goods proper. On the technical coefficients and their assumed fixity we shall come back in much greater detail in Section 5 below.

tion without increase of cost" is concerned, in Walras no such theory, however unsatisfactory, is put forward, so that the assumption of a constant unit cost appears to be wholly unfounded and deceptive.

Moreover, by accepting Mill's idea of the convergence of market prices of producible commodities to their constant unit costs, Walras appears to come dangerously close to endorsing also Mill's tenet that the price of a product is ultimately *determined* by its cost of production. However, though still lacking a fully-fledged alternative theory of the *determination* of the prices of products, and lacking altogether a theory of the prices of productive services, Walras is by no means willing to accept the so-called "cost theory of value" of products suggested by the economists of the English School. In the Geneva Notes this is made repeatedly clear by him, with sharp, unambiguos statements, such as the following two:

> Il est faux que les frais de production déterminent le prix des produits agricoles. C'est le marché. (1871-72, 468)
>
> Mais il est clair que la circostance des frais de production n'agit sur le *prix* qu'en agissant sur la *quantité existante*. L'utilité et la quantité existante reste donc les seules circostances intrinsèques. (1871-72, 450)

The rejection of the "cost theory of value" will remain one of the recurrent themes in all of Walras's subsequent writings. Yet, to make his condemnation stronger than it is in the Geneva Notes, where it is made unavoidably weak by the lack of theoretical support, Walras is bound to put forward a complete alternative theory of price determination, what he sets out to do starting from the major breakthrough of the Fall 1872.

3 The New Equilibrium Conception and the Exchange Model: Equilibrium Determination

As can be gathered from the preceding Section, Walras arrives very early at characterizing his notion of "équilibre général", identifying several of its properties particularly in the field of exchange. Yet, when drafting the Geneva Notes, though devoting a lot of remarks to the notion of utility and its possible applications to the theory of exchange (1871-72, 3^e and 5^e Leçons, 430-5, 447-51), Walras is still far away from discovering the existence of what will soon after appear as the fundamental relation between marginal utility and price, and consequently also from succeeding in deriving the individual demand and supply functions of consumer goods from the solution of constrained utility maximization problems. As already hinted at, such discovery and derivation will only occur in the Fall 1872, thanks to the essential help of Paul Piccard[16].

[16] Paul Piccard's crucial contribution to the solution of Walras's problem of deriving individual demand functions from utility and quantity considerations was first disclosed by Jaffé, in his monumental edition of Léon Walras's correspondence (Jaffé 1965, I, 309-11), where the

When, in the Fall 1872, Paul Piccard put forward his solution of the utility maximization and demand derivation problems, he was certainly conditioned by both the kind of question he had been asked to answer and the degree of mathematical knowledge he might have expected from his correspondent. Even if we do not know the exact terms of Walras's original question, we can infer from Piccard's answer that Walras had asked him to solve a problem concerning the derivation of the demand and supply functions of only two commodities (consumer goods) from the utility functions of a specified set of competitive traders. The individual utility functions appearing among the data of the problem were surely conceived, as always in Walras, as additively separable functions. Moreover, Piccard was certainly aware that Walras's mathematical skills were limited. Hence the proposed solution was very elementary in character: it employed just the indispensable minimum amount of calculus, being chiefly couched in geometrical terms. Moreover, given the additive separability of the traders' utility functions, their marginal utility functions (Walras's "raretés") turned out to be functions of one variable only. Hence, no need to employ any calculus of functions of several variables, or to use more advanced constrained maximization methods, such as Langrange's, could possibly arise. So that, not surprisingly, no such kind of calculus or method was employed by either Piccard or, *a fortiori*, Walras.

Starting from these premises, Piccard's manuscript analyzes the choice problem of a consumer-trader participating in a two-commodity economy. The consumer-trader concerned is characterized by an endowment, consisting in a given amount of only one of the two commodities, and an additively separable, differentiable utility function, mapping quantities of the two commodities into utility numbers. The economy is supposed to be competitive, in the sense that, in making their choices, consumers-traders take prices (as a matter of fact, just the relative price of one commodity in terms of the other) as given parameters and plan to trade at uniform rates of exchange. Under these assumptions, Piccard quickly arrives at solving the consumer-trader's constrained utility maximization problem, identifying the proportionality to prices of the decision maker's marginal utilities (or, what is the same, the equality of the latter's weighted marginal utilities) and the budget equation as the two conditions allowing the analyst to derive the excess demands for the two commodities as functions of the relative price. Piccard explicitly writes down the equation of the supply function of the commodity of which the consumer-trader is endowed. Walras completes the picture, by writing down in the margin of Piccard's manuscript the equation of the demand function of the other commodity (Walras 1993, 695).

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text of Piccard's manuscript, containing his answer to Walras's request for help, is published in full. Jaffé's remarks on this issue in his 1972 paper (Jaffé 1972, 303-5) are very useful, too. The text of Piccard's manuscript is also reproduced, together with some glosses in the margins by Walras and a few comments by the Editors, in (Walras 1993, 693-5). Walras thanks Piccard for his help in a letter addressed to the latter on October 25, 1873. However, such letter, which will be published in November of the same year in the *Bulletin of the Société vaudoise des sciences naturelles* (1873, 479-81), does not appear to do full justice to Piccard, in the light of the latter's fundamental role in the construction of Walras's theoretical system.

Once the individual demand and supply functions of the consumers-traders participating in the two-commodity competitive economy under scrutiny have been obtained as the solution of individual constrained utility maximization problems, Walras can easily proceed to determine the equilibrium price and traded quantities of such an economy by exploiting the same procedure already adopted in the Geneva Notes: first, the individual functions are aggregated over all the consumers-traders; then, aggregate demand and supply are set equal to one another for either commodity, thereby providing the market-clearing equations required for determining the equilibrium relative price and, as a consequence, also the traded quantities.

Even if the steps to be taken are the same as in the Geneva Notes, a few fundamental differences emerge: instead of being taken as "empirically given" relations, only loosely connected with some set of underlying data, the individual and aggregate demand and supply functions turn out to be the definite outcome of the consumers-traders' optimizing choices, which are in turn precisely associated with well-specified characteristics of the very same individuals. Similarly, the equilibrium of the two-commodity exchange model, instead of appearing as the upshot of the balancing of elusive 'forces' of demand and supply, grasped by the investigator on the basis of the latter's empirical experience and no further probed or explained from a theoretical point of view, must now be interpreted as a state of the economy in which the individual plans of action (trade plans, in the case concerned), optimally chosen by agents (consumers-traders) pursuing their subjective interests in a competitive setting, are mutually compatible and consequently, by virtue of such compatibility, objectively feasible, too.

Walras quickly incorporates the results of this discovery into his first *mémoire*, called "Principe d'un théorie mathématique de l'échange" (1874, 26-46), and, shortly afterwards, into the first instalment of the 1^{st} edition of the *Éléments* (1988, 69-117, 127-148, 150-1[17]). Yet, some traits of the original non-optimizing approach, characteristic of Walras's Geneva Notes, persist in these writings, too. For example, in the 1874 *mémoire* on the theory of exchange between two commodities, as in all the Leçons of the *Éléments* devoted to this specific topic, one finds a permutation in the logical order in which one would expect that the arguments ought to be developed, in the light of Walras's 1872 discovery: for Walras starts from the determination of the competitive equilibrium solution on the basis of "empirically given" demand functions, exactly as he had done in the Geneva Notes; and only after that he suggests that the individual demand and supply functions can be derived from the solution of individual utility maximization problems, without however using any calculus and without providing any analytical formula (even if, as we know, he was acquainted with such formulas).

The order in which the two problems (namely, the equilibrium determination and the demand functions derivation problem) are tackled is then reverted in

[17] With only a few exceptions, in the specified page intervals of the comparative edition of the *Éléments* the text of the 1^{st} edition remains unchanged in all the following editions. Yet, on few occasions, short passages or sentences or individual words belonging to editions different from the 1^{st} can be found scattered in the main text.

the Leçons of the *Éléments* dealing with the theory of the exchange of several commodities among themeselves (1988, **1**, **2-3**, **2-5**, and **4-5**, 152-260[18]), to which we shall turn in a moment, as well as in the second of Walras's *mémoires*, titled "Équations de l'échange", published two years later (1876a, 54). Yet, the expository choice made by Walras in his first theoretical publication confirms that, at least as regards the exchange model with two commodities, the equilibrium conception he originally had in mind was independent of any optimizing presupposition, which in fact entered the picture only at a later stage, and that such old conception went on affecting Walras's theoretical reasoning for a while, even in the very field, that of exchange, in which the new conception had made its grand entrance.

Such a dependence on pre-existing ideas and entrenched lines of reasoning can also be easily detected in the next step taken by Walras, which consists in generalizing the exchange model with two commodities to a model with an arbitrary finite number of commodities greater than or equal to three. This extension is at first developed in 1874, in the second part of Section II of the first instalment of the *Éléments*, then comprising eighteen Leçons from the 11^e to the 28^e[19], and subsequently taken up again, with some changes, in his second *mémoire* (1876a, 53-72).

Here Walras follows a three-step procedure which is very similar to the one he had already adopted in the Geneva Notes. Namely, in the first step, for each pair of commodities, a price-quantity partial equilibrium is determined on the particular market where the two commodities are traded for one another. In the second step, an arbitrage process brings about a "general equilibrium" state where all relative prices are made to satisfy the "general equilibrium" consistency condition on price triples that has been recalled in the previous Section[20]. Finally, the demands and supplies of the various commodities, expressed as functions of a consistent price system, are equalized, thereby providing a system of market-clearing equations, one for each commodity, which enable the theorist to determine the equilibrium prices and traded quantities.

At the first step of the procedure, the only significant difference between the

[18] In the specified page interval of the comparative edition of the *Éléments*, pages or passages reproducing the text of the 1^{st} edition alternate with pages or passages reproducing the text of all the following editions. Such textual differences, however, have nothing to do with the issue discussed in the text above.

[19] In the 4^{th} and 5^{th} editions of the *Éléments* the part of the text dealing with the exchange model with several commodities will be separated from Section II, providing the contents of a new Section III: see (1988, **1**, **2-3**, **2-5**, and **4-5**, 152-260).

[20] In the 1^{st} edition of the *Éléments*, at this stage of the argument, Walras does not exploit his "general equilibrium" condition to normalize the price system by means of the choice of a numéraire, postponing such choice to the Section on money (1988, **1**, 437-568, where the passages drawn from the 1^{st} edition alternate with those belonging to the subsequent ones). Hence, in the 1^{st} edition, the demands and supplies of the m commodities are expressed as functions of the $m(m-1)$ consistent relative prices, instead of being expressed in terms of the m numéraire prices, as will be done by Walras in his second *mémoire* (1876a, 53-72), as well as in the 2^{nd} and all the subsequent editions of the *Éléments* (1988, **1** and **2-5**, 153-260, where the passages drawn from the 1^{st} edition alternate with those belonging to the following ones). In the sequel, for the sake of simplicity, we shall confine attention to the solution adopted by Walras in his second *mémoire* (1876a).

new approach embraced by Walras after his 1872 discovery and the one adopted in the Geneva Notes is that now the demand and supply functions for pairs of commodities are no longer taken as "empirically given", but derived from the solution of utility maximization problems. It is of course a very important change, which however does not call into question the basic idea that, in order to tackle the equilibrium determination issue, one must necessarily start from the analysis of particular markets for commodity pairs. This drawback, of which Walras will never be able to free himself completely, is due, at least in part, to his limited mathematical skills, preventing him from attacking a multivariate optimization problem directly. From this point of view, Piccard's simple approach, involving just two variables at a time, offers Walras an easy escape, relieving him from the need to explore more complicated solutions.

At the second step of the procedure, Walras proceeds exactly as before: also here, there is a hint at the functioning of an arbitrage process, supposedly driving the price system to eventually satisfy the "general equilibrium" condition. Yet, once again, the concrete mode of operation and the time path of the process are left unspecified. Hence, in the end, the "arbitrage" operations are supposed to carry their effects through before anything else has had time to occur; so that, at least from the second *mémoire* onwards (1876a, 60-9), a consistent price system, satisfying the "general equilibrium" condition, is supposed to be *instantaneously* reached, for all theoretical purposes.

Finally, as regards the third step, the market-clearing condition, already present in the Geneva Notes in a market-by-market framework, is generalized to all markets, so that the simultaneous character of the system of market-clearing equations is made explicit. According to Walras, by solving such an equation system, the theorist can in principle determine the equilibrium prices, one for each commodity, expressed in terms of a commodity chosen as the numéraire, whose price is set identically equal to 1. Plugging these equilibrium prices into the individual demand functions, one can determine the equilibrium quantities of all commodities demanded and supplied by all the consumers-traders participating in the exchange economy under question. Hence, at the end, the equilibrium solution consists of a system of prices and an array of trade plans, one for each consumer-trader, that are optimally chosen at the equilibrium prices by the competitive participants in the economy and that, being mutually compatible, can also be carried out.

4 The New Equilibrium Conception and the Exchange Model: Equilibrium Establishment

As we have seen at the end of the previous Section, solving the equilibrium determination problem essentially consists, for Walras, of finding the roots of as many ordinary algebraic equations as there are unknowns to be determined. Any solution obtained in this way is labelled by Walras as "la solution mathématique", or "théorique", or even "scientifique" of the problem at hand, that

is, in the present case, of the exchange problem with several commodities. Yet, finding such a solution is not enough for Walras, for at least as important, if not more important, is to supplement such a "mathematical" solution with another sort of solution, which is called by him "la solution pratique" or "empirique". This idea is clearly expressed in Walras's second *mémoire*, by means of a sentence appearing at the end of the section where the "mathematical" solution of the exchange problem with several commodities has supposedly been achieved:

> Voilà comment, les équations de demande étant données, les prix en résultent mathématiquement. Reste seulement à montrer, *et c'est là le point essentiel*, que ce même problème dont nous venons de fournir la solution théorique est aussi celui qui se résout pratiquement sur le marché par le mécanisme de la libre concurrence. (1876a, 64; italics added)

It should be noted that the above sentence, with the appropriate adjustments required to fit it to the specific problem at hand, will recur over and over again in all of Walras's later writings, with reference to any one of the four "big problems" identified by him. But this is not all: for it is not only necessary to supplement the "mathematical" or "theoretical" solution with the "empirical" or "practical" one, but it is also fundamental that the two solutions should coincide with one another. This requirement is clearly expressed by the following passage, appearing in the same *mémoire* as the preceding one, at the end of Walras's discussion of the way in which the market concretely discovers the "empirical" solution, namely, by trial and error or, to use the expression made famous by Walras in the *Éléments* (1988, **1**, 188; **2-5**, 189), "par tâtonnement":

> Il résulte de la démonstration précèdent que, pour plusieurs marchandises comme pour deux, les éléments nécessaires et suffisants de l'établissement des prix courants ou d'équilibre sont l'utilité des marchandises pour les échangeurs et la quantité de ces marchandises possédée par les porteurs. Ces éléments constitutifs étant déterminés, les prix des marchandises [...] en l'une quelconque d'entre elles se déterminent empiriquement sur le marché, par le mécanisme de la libre concurrence, exactement comme ils se déterminent mathématiquement suivant les trois conditions 1° de la satisfaction maximum des besoins, 2° de l'égalité de l'offre et de la demande effective, 3° de l'équilibre général du marché. (1876a, 69)

In order to establish the identity of the two solutions, the "mathematical" and the "empirical", Walras complements the formally specified system of ordinary algebraic equations, whose roots provide the "mathematical" solution of the model, with a less formally specified system of functional equations (in essence, difference equations), allegedly describing the "empirical" working of the "mechanism of competition", whose evolution over time supposedly portrays the equilibration process driving the economic system under question towards

the very same “mathematical" solution as the one obtained by solving the system of ordinary algebraic equations describing the structure of the model.

The operation of the "mechanism of competition" is still largely based on that "loi de l'offre et de la demande" on which Walras had already founded his reasoning in the Geneva Notes. Such persistence of old ideas in the new theoretical context is by no means surprising, as it reflects Walras's deeply rooted and long-lasting attitude towards the analysis of the adjustment processes in the field of exchange which, as he himself will confirm in the letters and documents quoted in the last Section, largely consists in a reformulation of the traditional approach to the equilibration issue, as developed, in particular, by J.S. Mill.

The link with preceding ideas is also confirmed by a circumstance already mentioned in the previous Section: in the first instalment of the first edition of the *Éléments* (1988, 71-2), the true and proper analysis of the tâtonnement process in exchange is preceded by an illustration referring to the trading of a financial asset at the Paris Stock Exchange, almost identically reproducing a passage of the Geneva Notes (1871-72, 423). Yet, in the *Éléments*, differently from the Geneva Notes, such illustration has a purely introductory purpose: its aim is just to provide a relatively realistic representation of a market equilibration process. The theoretical analysis of the tâtonnement process in the exchange model with several commodities is developed in both a few pages of the first instalment of the 1^{st} edition of the *Éléments* (1988, **1**, 188, 190, 192, 194, 196) and a paragraph of Walras's second *mémoire* (1876a, 64-9)[21].

In accordance with the basic presupposition of the "law of supply and demand", the equilibration process is supposed to be driven by changes in prices, which are the state variables of the adjustment process, reacting to the occurrence of excess demands in the various markets: namely, the numéraire price of any commodity is supposed to increase or decrease according to whether the excess demand for that commodity is positive or negative; when the excess demand for a given commodity is nil, that is, when the demand for that commodity equals its supply, the price does not change. Hence, a state of equilibrium is established in this model when, the demands for all commodities being equal to the corresponding supplies, all prices remain unchanged, i.e., all prices are stationary ("prix stationnaires") (1988, **1**, 194, 196; 1876a, 69).

The process is supposed to take place as follows. At the start, prices are cried at random ("criés au hazard"), under the assumption that they satisfy the "general equilibrium" condition. At the quoted prices, the consumers-traders make their optimizing choices, which will generally result in positive or negative excess demands on the various markets. Excess demand functions are assumed to be monotonically decreasing and to take a zero value for a finite positive price. During the equilibration process, markets are visited sequentially, according to a preassigned order. Hence, when a visited market exhibits a positive (resp., negative) excess demand, the corresponding price increases (resp., decreases),

[21] For the reasons already explained in footnote 20 above, we shall follow the exposition of the second *mémoire*, where prices are normalized by means of a numéraire.

the other prices remaining unchanged, up until the market-clearing value is reached. At this point, the consumers-traders carry out their planned transactions on that market. The process repeats itself sequentially. When all the markets have been visited, and all prices have adjusted to their market-clearing values, a new price system emerges which, according to Walras (1988, **1**, 392, 394; 1876a, 68), for reasons that he strives to explain, is "certainly" closer to the equilibrium price system corresponding to the "mathematical" solution. So that, by iterating the procedure as many times as it is necessary, the "empirical" solution will converge to the "mathematical" one.

This sort of equilibration mechanism is not free from inconsistencies, which have been repeatedly pointed out in the literature since the early 1880s[22]. At present, however, we are less concerned with the analytical inconsistencies of the model, to which we shall come back later in this Section, than with the rationale of the equilibration process under discussion and the role it is called to play within Walras's theory of exchange. As a matter of fact, from a purely logical and methodological point of view, a number of serious difficulties arise, for the newly adopted optimizing assumptions concerning the traders' behavior can hardly be reconciled with a conception of the equilibration process dating back to a pre-existing non-optimizing approach.

The first question that Walras is bound to answer in the new setting is the following: Given that, as in the Geneva Notes, the state variables of the equilibration process are commodity prices, and given that, as before, the adjustment towards equilibrium is brought about by price changes that obey the "law of supply and demand", who is now in control of the state variables, implementing their changes? In the Geneva Notes, as we have seen, price changes were directly effected by the traders acting on either side of the market, i.e., by the buyers or the sellers. This idea also persists in Walras's writings on the theory of exchange following his 1872 discovery, but is now confined to the introductory illustration we have just recalled. Outside this illustration, however, that assumption can no longer hold. For now the traders are supposed to behave as competitive utility maximizers, what they were not in the Geneva Notes. But this means that, in making their choices, the traders are now compelled to take prices as given parameters, what in turn implies that they cannot directly control prices or change them[23].

Walras appears to be aware of this problem, at least to a certain degree. In fact, in his second *mémoire*, in discussing the issue of price adjustment as a

[22] The first two authors to raise doubts or objections about the convergence of the process and the determinateness of the equilibrium, granting that it is eventually reached, were Bertrand (1883), in a review-article concerning the 1883 edition of Walras's collection of essays titled *Théorie mathématique de la richesse sociale*, where his second *mémoire* on the theory of exchange of several commodities had been reprinted, and Wicksteed, in a private letter to Walras (letter 619 of December 1, 1884, published in (Jaffé 1965, II, 16-18)), critically commenting the passage of the 1^{st} edition of the *Éléments* referred to at the end of the last paragraph in the text. Such criticisms, as well as Walras's reactions to them, will be discussed later in this Section.

[23] The first to point out this implication of the competitive assumption in the context of Walras's equilibration process was Goodwin (1951). See also Arrow (1959).

reaction to the occurrence of positive or negative excess demand in the market for a given commodity, instead of assigning the responsibility for changing the corresponding price to any specific market participant, he uses the impersonal expressions "on fait la hausse du prix de la marchandise en numéraire" and "on fait la baisse" (1876a, 66[24]). This means to assign the responsibility for adjusting prices to an impersonal, anonymous entity or mechanism: namely, "the market" or "the mechanism of competition". There is, of course, nothing objectionable in such an assumption, provided that one is ready to accept that not all that occurs in the economy is directly traceable to the optimizing behavior of the individuals participating in it. Yet, as we are well aware nowadays with the benefit of hindsight, accepting that some such super-individual mechanism is at work in the model of the economy under scrutiny is a step that will prove very hard to take for general equilibrium theorists and many other economics practitioners.

In any case, for the discussion's sake, let us admit that price changes are brought about by "the market". Then, a striking number of problems arise, some of which Walras strives to solve at his best, often, but not always, with some success. The major problems that crop up at this point, all connected with each other, may be spelled out by means of the following questions: Is it possible to specify a quantitative rule of price change? What do the consumers-traders do when they act at out-of-equilibrium prices? What is the path followed by the economy during the equilibration process? Does such path actually converge to a stationary solution? Does the out-of-equilibrium behavior of the consumers-traders change the data of the theory? And, finally, if there occurs any such change in the data, can it possibly jeopardize the eventual convergence of the "empirical" solution to the "mathematical" one? We shall now try to answer these questions in turn, taking for granted, as apparently Walras himself did in his writings of the period under scrutiny (1871-1877), that not only does the equilibration process evolve in 'real' time, but it is also characterized by actual, observable, out-of-equilibrium behavior of all market participants. This latter assumption will only be questioned at the end of our discussion.

Walras tackles the first question by means of a clever device, already hinted at above, which consists in imagining an equilibration mechanism working market by market, where the markets for all commodities are visited sequentially in a preassigned order. Proceeding in this way, under his assumptions about the continuity and monotonicity of the excess demand functions, he is able, at least in principle, to specify the exact amount of change that each price must undergo at each round of the equilibration process: for, under Walras's rule and assumptions, given all the other prices, each price will change up to the point

[24] An identical expression appears also in the *Éléments*, but only since the 2^{nd} edition (1988, **2-5**, 189). This expression does not appear in the first instalment of the 1^{st} edition only due to 'technical' reasons: for in 1874, when the first instalment was published, Walras had decided not to employ the numéraire normalization of prices at this stage of development of his argument, as recalled in footnote 20 above. Therefore he could not envisage a tâtonnement process in exchange based on changes in numéraire prices, what will instead become possible two years later, in his 1876 *mémoire* on the "Équations de l'échange".

at which the corresponding market clears. When the entire list of markets has been visited, then the issue of the re-normalization of prices will of course arise, something of which Walras seems to be partly aware. But this does not appear to raise any insurmountable problem[25].

By means of the above apparatus, Walras is also able to provide tentative answers to the next two questions listed above, namely, those respectively concerning the out-of-equilibrium behavior of the consumers-traders and the out-of-equilibrium path followed by the economy. It should be stressed that these questions are certainly not easy to answer. For, assuming, as Walras undoubtedly did in his exchange models following the 1872 discovery, that the consumers-traders behave as competitive utility maximizers, then the theory at hand is apparently unable to explain what the consumers-traders actually do when they are called to act at out-of-equilibrium prices. For, at such prices, the consumers-traders can indeed choose their utility-maximizing *plans of action*, but, barring exceptional assumptions or circumstances to which we shall come back in a moment, they are generally unable to carry out their plans, since such plans are typically incompatible with each other and the external environment, so that they cannot be implemented as observable *actions*.

This is a very serious predicament for any theory pretending to explain out-of-equilibrium actions on the basis of out-of-equilibrium optimizing plans: for, by construction, out-of-equilibrium plans cannot be carried out as they are; hence, any such attempt at explaining the agents' out-of-equilibrium behavior would require to supplement the theory of optimizing behavior with a wholly different sort of theory, capable of converting non-executable optimizing plans into actual, non-optimal actions. Walras seems to perceive this difficulty. As a matter of fact, his market-by-market equilibration procedure may be viewed as an ingenious manner of finding a reasonable, though only approximate, way out of the said dilemma. For, let us suppose that, at the quoted prices, excess demands are not nil on all markets. According to Walras's assumptions, markets are visited sequentially and market-by-market price adjustments are supposed to take place. Now Walras supposes that, at any visited market, the corresponding price changes in such a way as to clear the market itself. But then, even if the situation prevailing in the overall economy is one of general disequilibrium, that specific market, provided that it can be assumed to be isolated from the others, as it is typical in a partial equilibrium framework, can be regarded as if it were in equilibrium. Therefore, under the standard assumptions characteristic of partial equilibrium analysis, the consumers-traders can be supposed to carry out their partial equilibrium optimizing plans, as far as that market only is concerned.

This obviously provides only an approximate, local solution to the problem of explaining actual behavior in a general disequilibrium situation, a solution which ought not to be generalized outside the strict boundaries of partial equilibrium analysis. Yet, Walras deludes himself into thinking that, by stringing together the partial equilibrium solutions obtained by means of his market-by-market procedure, he can not only trace the overall path followed by the economy

[25] For a modern treatment, see, e.g., Uzawa (1960).

during the tâtonnement process, but also avail himself of such construction to discuss the issue of the convergence of the process to a stationary solution. Such mixture of partial and general equilibrium analysis, however, is illegitimate.

The true and proper mistakes that, as already mentioned above, are made by Walras in his endeavors to prove the convergence of the tâtonnement process, are essentially linked to the unwarranted attempts he makes at generalizing a few results that, perhaps acceptable as useful approximations in a partial equilibrium setting, become misleading or utterly wrong in a general equilibrium one. As the subsequent evolution of general equilibrium analysis will demonstrate, the issue of the convergence of an equilibration process to any stationary solution whatsoever cannot be satisfactorily confronted unless Walras's mixing of partial and general equilibrium analysis is dropped. Yet, giving up such blend also means shelving Walras's unfortunate efforts to trace the path of the equilibration process under the assumption that observable actions do actually take place during the process itself.

Such an outcome, however, was inconceivable for Walras in the mid-1870s, when he was busy trying to complete and polish up his theory of equilibrium establishment in the exchange model. At that time, he was still striving to show that his market-by-market sequential adjustment process would "certainly" converge to a definite "empirical" stationary solution, coinciding with the "mathematical" solution of the exchange equilibrium model. No formal proof of such convergence, however, will ever be provided. Yet, from the very start of his endeavors, Walras perceived that one necessary, though not sufficient, condition for the coincidence of the two solutions is that the data of the problem must not change during the equilibration process. This data invariance assumption is made explicit in his second *mémoire* with the following words :

> C'est le droit du théoricien de supposer les éléments des prix invariables durant le temps qu'il emploi à formuler la loi d'établissement des prix d'équilibre. Mais c'est son devoir, une fois cette opération terminée, de se souvenir que les éléments des prix sont essentiellement variables et de formuler en conséquence la loi de variation des prix d'équilibre. (1876a, 70)[26]

This passage is particularly interesting since it reveals the sort of dilemma that Walras was forced to face. Let us focus on the first part of the quotation first. As recalled above, in the exchange model the data of the problem, i.e., "les éléments nécessaires et suffisants de l'établissement des prix courants ou d'équilibre", are "l'utilité des marchandises pour les échangeurs et la quantité de ces marchandises possédée par les porteurs". According to Walras, it is the "right of the theorist" to suppose that such "éléments constitutifs" do not change during the equilibration process. In this regard, it is important to distinguish those data whose eventual changes are exogenous, that is, in the case under discussion, independent of the working of the equilibration process, from those

[26] An identical passage can be found also in the *Éléments*, from the 2^{nd} edition onwards (1988, **2-5**, 146).

that instead may endogenously depend on what occurs during the process; of course, in principle, there might also exist data whose changes may depend on both exogenous and endogenous causes. In this regard, it should be clear that the theorist has the "right" to assume the invariance only of those data that can be legitimately supposed to exclusively depend on exogenous factors. On the contrary, no such "right" can be claimed concerning data that may change for endogenous reasons, too.

As far as the utility functions are concerned, Walras tacitly, yet consistently (since the Geneva Notes), assumed that their possible changes do not depend on economic activities; hence, the consumers-traders' utility functions can be put into the category of purely exogenous data, whose time-invariance can be legitimately assumed by the theorist (from Walras's point of view, of course). As to the quantities of the commodities owned by the traders, the question is instead more complex. First of all, it should be noticed that in this model the commodities to be traded are exclusively consumer goods, that is, in Walras's terminology, "revenus",or objects that do not survive their first use. Hence, to assume that such data do not change over the equilibration process implies to assume, at the very least, that those commodities can indeed be traded, but cannot be consumed up until the equilibrium is reached, for otherwise they would disappear altogether[27].

Yet, assuming the invariance of the aggregate quantities of commodities available in the economy, since no consumption but only trading can take place during the equilibration process, is not enough to guarantee that the data of the model remain unchanged. For such data comprise not only the aggregate quantities of commodities existing in the economy, but also the individual quantities owned by the individual traders; and, while the aggregate quantities may be protected from change by assuming that no consumption takes place during the trading process, the individual quantities cannot be so protected: for the very trading process is an endogenous factor of change in the distribution of the endowments among the consumers-traders, provided, of course, that actual trades are allowed to occur over the equilibration process, as Walras indeed was ready to assume in his writings of the mid-1870s.

Almost certainly, in those years Walras was still unable to perceive that the market-by-market tâtonnement process in exchange he was so busy designing,

[27] In all probability, this is the kind of data invariance assumption that Walras really had in mind concerning the fate of consumer goods during the equilibration process in his original exchange model: that such goods can be traded, but not consumed. This interpretation is confirmed by what Walras will write in this regard in the 2^{nd} edition of the *Éléments*, where, in comparing the equilibration process concerning production with that concerning exchange, he underlines a difference between the two that, in his opinion, is particularly relevant:

> [L]e tâtonnement en matière de production rencontre une complication qui n'existait pas en matière d'échange. Dans l'échange, il n'y a pas de modification des marchandises. Un prix étant crié, et la demande et l'offre effective correspondent à ce prix n'étant pas égales, on crie un autre prix auquel correspondent une autre demande et une autre offre effectives. Dans la production, il y a transformation de services producteurs en produits. (1988, **2-3**, 308)

and that was apparently so well conceived as to allow him to solve both the problem of determining the out-of-equilibrium behavior of the traders and that of predicting the out-of-equilibrium path of the economy, had such disruptive implications on his own crucial data invariance assumption. His very likely unawareness of this unexpected and disturbing effect is also revealed by the following blunder marring the tâtonnement construct in his original exchange model: the demand and supply functions appearing therein, which are made to explicitly depend on prices only, remain the same over the trading process, whereas they should change with any change in the distribution of the endowments among the traders, since such individual endowments, not differently from prices, are among the arguments of the functions.

This mistake is a further indirect proof that, after adopting his new optimizing perspective in the Fall 1872, Walras still found it difficult to get rid of a number of pre-existing ideas and attitudes that were no longer consistent with his new approach. In fact, also in the Geneva Notes the individual, hence the aggregate, demand and supply functions had been made to depend on a set of data, among which also the quantities of the various commodities available in the economy and their distribution among the traders. Yet, since in the Geneva Notes no clear analytical link was established between such data and the demand and supply functions, which were simply taken as "empirically" given in that manuscript, it was then easy to overlook the causal effects on the data, hence on the individual and aggregate demand and supply functions, brought about by trading among consumers. This neglect, however, would become wholly unjustified in the new analytical framework, where individual optimizing behavior, hence individual data, ought to occupy the centre of the stage. Yet, in spite of the change in outlook, the traditional forgetfulness tended to persist even there almost by inertia.

As a matter of fact, it will take Walras a long time to realize that the tâtonnement construct in his original exchange model was flawed. Moreover, such awareness, perhaps less than full, will have to wait for the external impulse provided by the critical remarks on related issues made by the French mathematician Bertrand in his scathing review (Bertrand 1883) of the second enlarged edition of Walras's *Théorie mathématique de la richesse sociale* (1883). Walras's reaction to Bertrand's criticism will be discussed at the end of this Section. For the time being, there still remains one important point to analyze concerning the nature of the equilibrium allegedly reached at the end of the equilibration process.

As already mentioned, in the mid-1870s Walras, though unable to provide any formal proof of the convergence of the trading process to a "stationary" solution, was still convinced that such convergence would "certainly" occur[28]: in

[28] As recalled in footnote 22 above, Walras's confidence in the "certain" convergence to a stationary solution of the trading process will be shaken by the critical remarks leveled at the tâtonnement construct in the exchange model by Wicksteed in his letter of December 1, 1884, to which Walras will reply two weeks later, after due reflection, by recognizing the validity of Wicksteed's observations (Jaffé 1965, II, letter 627, 24-6). As a likely consequence of this exchange of letters, since the 2^{nd} edition of the *Éléments* (1889) Walras will replace

fact, not differently from the Geneva Notes, also in his writings of the mid-1870s he qualified the equilibrium prices eventually arrived as "prix stationnaires" (1876a, 69; 1988, **1**, 196). Yet, a sort of puzzle arises at this juncture.

For one might think to justify the "stationary" character of the equilibrium solution on grounds of 'realism': namely, in order that the equilibration process should display a 'realistic' appearance, one might be led to suppose that it evolves over 'real' time under unchanging conditions, whence the stationarity of the equilibrium solution. Yet, this would make it necessary to assume that the data of the economy be invariant over the same 'real' time over which the economy evolves. But not only is this assumption illegitimate, at least in the case under discussion, but it is also, from Walras's own viewpoint, highly 'unrealistic': for, according to Walras, real-world data change continuously, so that assuming their time-invariance clashes with a basic feature of all real economies. Such belief in the continual change in the data clearly emerges from the second part of the passage quoted above, where Walras bluntly states that it is the theorist's "duty" to keep in mind that the data underlying the price determination problem are "essentially variable" (1876a, 70)[29]. So the paradoxical result is that, in order to exhibit a 'realistic' adjustment process, Walras ends up by making a data-invariance assumption and adopting a "stationary" equilibrium concept that, from his own viewpoint, are utterly 'unrealistic'.

Here again one can find blatant traces of the persistence of old ideas and pre-existing conceptions within the new theoretical approach stemming from Walras's 1872 discovery. For an equilibration process in 'real' time leading to a stationary or quasi-stationary equilibrium is indeed consistent with the classical conception of natural prices as centres of gravitation, around which market prices oscillate or towards which they converge. This conception, characteristic of the English School, underlies the equilibration process envisaged by J.S.Mill, from which, as we have seen, Walras's idea of the tâtonnement process in exchange originates. Yet, such a stationary or quasi-stationary interpretation of the equilibrium solution of an adjustment process in 'real' time is fully at variance with the new equilibrium conception discovered by Walras in the Fall 1872 and soon after applied to his exchange model. For, according to the new conception, an equilibrium state should be viewed as an array of optimally chosen plans, one for each individual belonging to the economy, which are compatible

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the word "certain" with the word "probable" in the passages discussing the convergence of the tâtonnement process non only in exchange, but also in production (1988, **2-5**,194-5, 326, 328).

[29] The same belief is confirmed by a famous passage, appearing at the end of the Lesson called "Du marché permanent" in all the editions of the *Éléments* (1988, 579-80), where Walras writes:

> Tel est le marché permanent, tendant toujours à l'équilibre sans y arriver jamais par la raison qu'il ne s'y achemine que par tâtonnements et qu'avant même que ces tâtonnements soient achevés, ils sont à recommencer sur nouveaux frais, toutes les données du problème [...] ayant changé.

It might be interesting to note that the idea of a "marché permanent", where the data of the problem change "de jour en jour, d'heure en heure, du minute en minute", is already present in the Geneva Notes (1871-72, 470).

with each other and with the external data. But plans are mentalistic concepts, which should be naturally referred to the instant at which the individuals take their decisions. Similarly, since the data of the economy vary from instant to instant, they should be referred to a given instant of 'real' time. Hence, according to the new equilibrium conception, an equilibrium state ought to be referred to a specific instant of 'real' time, precisely, to the instant at which the plans are made on the basis of the existing data. To superimpose upon such an 'instantaneous' equilibrium conception a stationary interpretation, allegedly more in line with the needs of an equilibration process in 'real' time, is the source of unending conflicts, which explain so much of the tormented evolution of Walras's thought on this issue.

As a matter of fact, the very same logical inconsistency referred to above can ultimately be traced back to the conflict between the assumed stationarity of the data and the instantaneous character of the equilibrium conception descending from the 1872 discovery. It is certainly not accidental that Walras, when in the mid-1880s makes his first public attempt to rebut Bertrand's criticism, should drop the assumption of invariance of the data in 'real' time, turning the equilibration process in the exchange model into a purely virtual process in 'logical' time, where no observable actions are allowed to take place out of equilibrium, and that, at the same time, he should explicitly recognize that the equilibrium eventually arrived at is 'instantaneous' in character. All this is explained in a long footnote, somewhat incongrously appearing in a paper on the German economist Hermann-Henri Gossen (1885, 312, fn. 1), of which the central statement runs as follows:

> M. Bertrand m'object que le problème de l'échange n'est pas déterminé [...]. Je réponde à cela que, sur le marché théorique, en cas d'excédent de la demande sur l'offre ou de l'offre sur la demande, [...] l'échange demeure suspendu jusq'à ce que la hausse ou la baisse ait amené l'égalité de l'offre et de la demande [...]. Le prix courant théorique est essentiellement un prix unique résultant, à un moment donné, d'un échange général.[30]

As can be seen, only in the mid-1880s will Walras be ready to admit that, as far as the exchange model is concerned, an instantantaneous equilibrium interpretation is logically more consistent with his newly embraced perspective than the stationary interpretation inherited from the past. The persistence of the old ideas is so strong as to hinder Walras's natural propensity to believe

[30] Similar remarks can be found in the already cited manuscript, probably drafted by Walras at the beginning of 1891, now published as an enclosure to the letter from Walras to Pareto dated January 9, 1895 (Jaffé 1965, II, letter 1200, 627-329). Moreover, since the 2^{nd} edition of the *Éléments*, with a view to controverting Bertrand's criticism, Walras decided to slightly amend the already quoted passage, dating back to the Geneva Notes, that illustrates the working of the law of supply and demand by means of the example based on the trading of the "Rente Française 3%" at the "Bourse de Paris": the amendment consists in the addition of a few (literally, thirteen) words, implying that the exchange is suspended whenever there exists excess demand or supply in the market, being permitted to take place only at equilibrium (1988, **2-5**, 71-2).

that, in the competitive markets of the real world, equilibrium prices are so quickly established as to appear almost instantaneously arrived at. This idea is clearly expresed in the following passage of the 1^{st} edition of the *Éléments*, dating back to 1874:

> On voit clairement à présent ce qu'est le mécanisme de la concurrence sur le marché ; c'est la solution pratique, et pour hausse et baisse des prix, du problème de l'échange dont nous avons fourni la solution théorique et mathématique. On doit comprendre d'ailleurs que notre intention n'est aucunement de substituer une solution à l'autre. La solution pratique est d'une rapidité et d'une sûreté que ne laissent rien à désirer. On peut voir, sur des grands marchés fonctionnant même sans courtiers ni crieurs, le prix courant d'équilibre se déterminer en quelques minutes, et des quantités considérables de marchandise s'échanger à ce prix en deux ou trois quart d'heure. (1988, 93)

5 The New Equilibrium Conception and the Production Model: Equilibrium Determination

The new equilibrium conception makes its way into Walras's theory of production, too, even if in this field dropping the old ideas and embracing the new ones proves even more difficult than in the field of exchange, where the new approach had seen its birth. The new theory of production makes its first public appearance in 1876, in Walras's third *mémoire*, titled "Équations de la production" (1876b, 73-99), and shortly afterwards is taken up again, with some changes, in the second instalment of the 1^{st} edition of the *Éléments* (1988, **1**, 312, 314, 316, 319-22, 324-30[31]), appearing in 1877.

In line with the parts on production of the Geneva Notes, also in the writings of the period 1876-1877 two markets are distinguished: a market for products and a market for services. Yet, while in the Geneva Notes the market for services was mentioned, but hardly analyzed, in the production model of the years 1876 and 1877 it is thoroughly discussed, representing the chief analytical innovation in the field. Before entering into the analytical details of the working of the two markets, it is convenient to survey the main conceptual features of the overall model.

In the production model the only kinds of outputs taken into consideration are consumer goods. Also in the exchange model discussed in the previos two Sections consumer goods were the only kind of commodities that consumers-traders were allowed to exchange among themselves. But here, differently from the exchange model, such consumer goods are the outcome of production processes run by entrepreneurs. The inputs to the production processes are the services of the various types of capital, distinguished, as in the Geneva

[31] In the specified pages or page intervals, short passages or sentences or individual words belonging to editions different from the 1^{st} are occasionally scattered in the main text.

Notes, in landed capital ("terres"), personal capital ("facultés personnels") and capital goods proper ("capitaux proprement dits"). In the production model the owners of the various types of capital are not allowed to sell their capital assets as such, not even those types of capital, i.e., landed capital and capital goods proper, that in principle are saleable in a private-ownership, market economy, as the one envisaged by Walras in his writings of the period 1876-1877; the owners can only sell the services rendered by the capital assets they own, or, what is the same, they can lend their capital assets for specified periods of time. The owners of the various types of capital can decide whether to sell their respective services to entrepreneurs, who will employ them in the production processes as productive services, or to consume them directly or sell them to other consumers-owners as consumable services.

The value of the services sold to entrepreneurs or other consumers represents the incomes of the owners of the capital assets rendering such services, who can only use them to buy consumer goods. Consumers' behavior in the production model is explained in the same way as in the exchange model: in either case consumers are constrained utility maximizers, the only differences being that here they can purchase not only consumer goods, but also consumable services, and that their budget constraints depend on the value of the services rendered by the capital assets they own, rather than on the value of the commodity endowments in their hands. Therefore, the behavior of consumers-owners in the production model is assumed to be as optimizing as the behavior of consumers-traders in the exchange model: consumers-owners are competitive utility maximizers, who, taking the prices of both services and consumer goods as given parameters, make their choices as to the quantities of consumer goods to be demanded in the product market and of services to be supplied in the market for services in such a way as to maximize their utility functions under their budget constraints. Hence, the assumptions concerning consumers-owners allow the theorist to explain the working of both the demand side of the product market and the supply side of the market for services. To complete the picture there remains to take care of both the supply side of the product market and the demand side of the market for services. The agents operating on these sides of the two markets are the entrepreneurs, whose behavior must therefore be specified.

In this regard, it should be stressed at once that the assumption of optimizing behavior, characterizing consumers-owners in the production model, does not extend to the other category of agents participating in the economy, namely, the entrepreneurs. As a matter of fact, in the production model of the period 1876-1877 the behavioral assumptions concerning the entrepreneurs are quite similar to the assumptions governing their behavior in the Geneva Notes. Yet, since the issue of entrepreneurs' behavior in Walras's theory of production is highly controversial[32], it is convenient to postpone our examination of this topic, analyzing first the definition of equilibrium proposed by Walras for the

[32] The first leading economist to openly criticize Walras's assumptions concerning entrepreneurs' behavior and the associated definition of equilibrium in production is Edgeworth, in his already cited essays of 1889 and 1891 (Edgeworth 1889a, 1889b, and 1891).

production model:

> L'état d'équilibre de la production, contenant implicitement l'état d'équilibre de l'échange, est à présent facile à définir. C'est celui, d'abord, où l'offre et la demande effectives des services producteurs sont égales, et où il y a prix courant stationnaire, sur le marché de ces services. C'est celui, ensuite, où l'offre et la demande effectives des produits sont égales, et où il y a prix courant stationnaire, sur le marché des produits. C'est celui, enfin, où le prix de vente des produits est égal à leur prix de revient en services producteurs. Les deux premières conditions se rapportent proprement à l'équilibre de l'échange ; la troisième est spécialement relative à l'équilibre de la production.
>
> Cet état d'équilibre de la production est, comme l'état d'équilibre de l'échange, un état idéal et non réel. [...] Mais c'est l'état normal en ce sens que c'est celui vers lequel les choses tendent d'elles-mêmes sous le régime de la libre concurrence appliqué à la production comme à l'échange. (1988, 283[33])

This definition of equilibrium is particularly interesting because it mixes properties which are characteristic of what Walras calls "equilibrium determination", to which there corresponds, as we have seen, the so-called "solution mathématique", with properties which are characteristic of what Walras calls "equilibrium establishment", to which there corresponds the so-called "solution empirique": properties of the first type are those represented by the equality conditions between, on the one hand, demand and supply in both the market for services and the market for products, and, on the other, between selling price ("prix de vente") and unit cost ("prix de revient") of products; properties of the second type are those related with the stationarity of prices ("prix courant stationnaire") in both the market for services and the market for products. A third property of the second type, the stationarity of the quantities produced of the various types of consumer goods, is not explicitly mentioned here, but, as will be seen, can be associated with the third property of the first type, namely, with the equality of selling price and unit cost for each consumer good.

This twofold characterization of the equilibrium concept employed by Walras in his production model reveals the intimate connection existing for him between "equilibrium determination" and "equilibrium establishment". A time-honored and well-known expression, the "loi de l'offre et de la demande", is employed by Walras to concisely refer to the first two equilibrium conditions, which "se rapportent proprement à l'équilibre de l'échange". Similarly, a less common, though not entirely unheard of, expression, the "loi des frais de production" or "du prix de revient" (1876b, 73), is employed to refer to the third equilibrium condition, which is "spécialement relative à l'équilibre de la production". An even more synthetic expression, the "loi de l'offre et de la demande et du prix

[33] A very similar, though not identical, passage can be found also in Walras's third *mémoire* (1876b, 78)

de revient", is employed to jointly refer to all the three equilibrium conditions (1988, 330).

By exploiting the above definition of equilibrium, we can now go back to examine the behavioral assumptions concerning the entrepreneurs in the production model. As in the Geneva Notes, entrepreneurs are supposed to buy productive services on the market for services and employ such services in technologically feasible production processes, obtaining products that they sell to consumers in the market for products. In performing their typical activities, given the prices of both services and products, the entrepreneurs incur costs for the purchase of services and get revenues from the sale of products. They react to differences between revenues and costs by changing their output levels, in the way described by Walras in the following passage, which immediately follows the one quoted above:

> [S]i, dans certains entreprises, le prix de vente des produits est supérieur à leur prix de revient en services producteurs, d'où résulte un *bénéfice*, les entrepreneurs affluent ou développent leur production, ce qui augmente la quantité des produits, en fait baisser le prix et réduit l'écart, et si, dans certaines entreprises, le prix de revient en services producteurs est supérieur à leur prix de vente, d'où résulte une *perte*, les entrepreneurs se détournent ou restreignent leur production, ce qui diminue la quantité des produits, en fait hausser le prix et réduit encore l'écart. (1988, 283-4; 1876b, 78)

Hence, by acting on the output levels of the consumer goods they produce, the enterpreneurs are instrumental in making the prices of such consumer goods converge to their respective unit costs, thereby realizing the third equilibrium condition. Therefore,

> à l'état d'équilibre de la production, les entrepreneurs ne font ni bénéfice ni perte. (1876b, 79)

But then, since the entrepreneurs change the quantities of the goods they produce in reaction to incentives that are represented by gains or losses, when such gains and losses are nil, the entrepreneurs do not change their output. Hence, as anticipated above, the stationarity of the quantities produced of consumer goods becomes another characteristic condition of equilibrium in production, associated to the third equilibrium condition of the first type (equality between price and unit cost).

As can be seen, exactly as supposed in the Geneva Notes, also in the production model of the years 1876 and 1877 entrepreneurs are assumed to behave as quantity adaptors, reacting to incentives represented by the gains they expect to obtain or the losses they expect to suffer. Hence, also in the theory of production developed by Walras *after* his discovery of utility maximization in the Fall 1872, entrepreneurs' behavior continues to differ from that of consumers (either consumers-traders or consumers-owners, as the case may be) in two fundamental respects: first, unlike consumers, entrepreneurs are not optimizing agents;

secondly, again unlike consumers, who, being competitive, have no control over the time changes of price variables, which play the role of state variables in one type of adjustment process envisaged by Walras in both the exchange and the production model, entrepreneurs do have some control over the time changes of output variables, which play the role of state variables in an adjustment process specific to the production model, as conceived by Walras. (Yet, as we shall see in the next Section, entrepreneurs' control over output variables is far from complete.)

Having stressed the similarities between the behavioral assumptions concerning entrepreneurs underlying the Geneva Notes and those underlying the new production model of the period 1876-1877, we must now underline that the theory of production contained in the latter model is by far more developed than the one of the Geneva Notes: for, as already recalled, no assumptions about technology, no theory of pricing of productive services, and therefore no theory of costs can be found in the Geneva Notes, while all these topics are carefully tackled in the new production model.

Let us start from technology. In this regard, the basic assumption made by Walras in 1876 and 1877, an assumption that will be kept basically unaltered for most of his scientific life, is that in the economy there exists a single-output, fixed-coefficient technology, on which all production processes are based: every produced commodity is the only output of a specified production process, employing fixed quantities of inputs, i.e., of productive services, per unit of output. As already remarked, such fixed coefficients of production, called "coefficients de fabrication" in the *Éléments* (1988, 304), are not to be found in the Geneva Notes, even if they had already been introduced by Walras in his 1870 manuscript on "Capitaux & Revenus"[34].

The assumption of fixed production coefficients is stated by Walras in his third *mémoire* in the following terms:

> Nous supposons [...] les coefficients [de fabrication] déterminés *a priori*. En réalité ils ne le sont pas [...]. Les quantités respectives de chacun des services producteurs qui entrent [...] dans une unité de chacun des produits sont déterminées seulement après la détermination des prix des services producteurs, par la condition que le prix de revient des produits soit minimum. Il serait facile d'exprimer cette condition par une système d'équations ; mais, comme ce système serait en quelque sorte indépendant des autres que nous considérons, nous en faisons abstraction, pour plus de simplicité, en supposant

[34] A possible reason for the absence of the "coefficients de fabrication" in the Geneva Notes has been suggested in Section 2 above. Yet, in such Notes one can find some parameters, called "coefficients de circulation" (1871-72, 457), expressing the fraction of the quantity of each commodity that is traded during a specified period of time. From such coefficients one can derive, according to Walras, the transactions demand for money. Hence, the "coefficients de circulation" of the Geneva Notes are strictly related to both the technical coefficients of the 1870 manuscript and the "coefficients de fabrication" of the production model of the 1876-1877 period.

que les coefficients ci-dessus figurent parmi les données du problème. (1876b, 83[35])

In Walras's third *mémoire*, this is all that can be found on the subject of the variability of production coefficients and cost minimization. In spite of Walras's optimistic statement that "it would be easy" to solve the problem of the determination of cost-minimizing coefficients of production, nothing is added in this respect in his 1876 *mémoire*. One year later, when the second instalment of the 1^{st} edition of the *Éléments* is eventually published, Walras will insert one single page on this topic into one of the last Leçons of that edition, Leçon 51 "De l'augmentation dans la quantité des produits", belonging to the then penultimate Section V of the *Éléments*, titled "Conditions et consequences du progrès économique" (1988, 585-6).

In this added page Walras focuses on the production process of one single commodity, say commodity (B), denoting the production coefficients for that commodity by b_t, b_p, b_k, ..., where the subscripts t, p, and k respectively stand for generic types of services of landed capital ("terres"), personal capital ("personnes"), and capital goods proper ("capitaux proprement dits"). Then Walras supposes that such production coefficients, instead of being taken as given and unchanging, may vary as a result of the substitution of one service for another, under the technological constraint represented the following equation[36]:

$$\varphi(b_t, b_p, b_k, ...) = 0.$$

He then suggests that, given the prices of the services employed in the production of commodity (B), the cost-minimizing values of the production coefficients characterizing the production process of that commodity can be obtained by minimizing its unit cost of production, under the specified technological constraint. Yet, in spite of the help requested from, and immediately obtained by, Hermann Amstein, a professor of mathematics at the École d'ingénieurs of Lausanne, Walras does not succeed in solving the cost minimization problem, let alone in determining the cost-minimizing production coefficients he is apparently looking for[37]. Hence, the issue is left unanswered in the 1^{st} edition of the *Éléments*, where it is nowhere touched upon again.

[35] A similar, though not identical, passage appears also in (1988, **1**, 305-6).

[36] Only much later, in the 4^{th} edition of the *Éléments*, published in 1900, this equation will be called "*équation de fabrication*" (1988, **4**, 586).

[37] At the end of 1876, when revising his theory of production in view of the publication of the second instalment of the 1^{st} edition of the *Éléments*, Walras writes to Hermann Amstein asking for his help in solving the cost minimization problem mentioned above. Amstein's answer, contained in a letter to Walras dated January 6, 1877, provides two formally equivalent solutions to Walras's problem, one of which based on Lagrange's method for solving constrained optimization problems. Walras, however, proves unable to exploit either one of Amstein's suggested solutions.

Amstein's letter is published in (Jaffé 1965, I, letter 364, 516-20). In the same place, one can also find the transcript of the draft of the question that Walras had put to Amstein, as well as the transcripts of an undated note, written by Walras at either the end of 1894 or the beginning of 1895, and of a few penciled notations added by Walras to Amstein's autograph letter, certainly handwritten immediately before or soon after 1900. All these documents are also reproduced in (Walras 1993, 660-665), together with some remarks by the Editors. Jaffé's

As a matter of fact, the whole question will never be taken up again up until the mid-1890s, when, taking advantage of his improved mathematical skills, under the stimulus provided by Pareto's[38] and especially Barone's[39] concomitant reflections on the theory of production and costs, and finally also under the spur of the mounting controversy with Wicksteed (1894) over the theory of marginal productivity, Walras eventually goes back to the issue of the variability of production coefficients that he had put aside for about two decades, yet attacking it from a perspective significantly different from the one hinted at twenty years before[40].

It might be interesting to investigate the reasons why Walras did not take advantage of Amstein's early 1877 suggestions in revising his theory of production in preparation of the second instalment of the 1^{st} edition of the *Éléments* to appear later in the same year. In the same vein, one might ask the following questions: Why did Walras decide to relegate the few lines on that topic added to the second instalment of the 1^{st} edition to a marginal Leçon placed towards the end of the book? Why did it take him almost two decades to go back to an issue that, according to his own bold statement in the third *mémoire*, it would have been so "easy" to solve?

According to Jaffé (1964, 206-7, and 1965, I, 519, note 5), Walras failed to exploit Amstein's suggestions because his poor mathematical competence at

notes to Amstein's letter are very useful. One should also consult (Walras 1954, Translator's Notes to Lesson 36, 549-54; Collation of editions, lesson 36, 604-6) and (Jaffé 1964).

[38] In a series of papers published over the years 1894 and 1895, focusing first on the theory of foreign exchanges (Pareto 1894a), then on what will come to be known as 'Pareto optimality' (Pareto 1894b), and finally on international trade theory (Pareto 1895), Pareto tackles and solves Walras's problem of cost minimization and the variability of production coefficients, providing the formal conditions for the determination of the cost-minimizing coefficients of production. Pareto will take up again this issue in Volume II of his *Cours d'économie politique* (Pareto 1897, 83-4, note (714)).

[39] Barone exerts an important influence in the mid-1890s on the development of Walras's ideas on cost minimization, variability of production coefficients, and marginal productivity theory, both through his lively exchange of letters with Walras over the period 1894-5 and through his draft, in 1895, of a paper critically reviewing, from Walras's own perspective, Wicksteed's booklet, *An Essay on the Coordination of the Laws of Distribution*, published in the preceding year (Wicksteed 1894). The 1894-5 relevant correspondence between Barone and Walras can be found in (Jaffé 1965, II, letters 1191, 1215-1224, 1228, 1237; 619-21, 643-662, 665-6, 671-4). Barone's article, originally written in view of its publication on *The Economic Journal*, never appeared in that journal and subsequently went lost. Yet, a French translation by Walras of the Italian version of the article has survived among Walras's papers and is now published in (Jaffé 1965, II, letter 1215, editorial note (4), 644-8). Barone's studies on the theory of production and distribution appeared in 1896 in the Italian journal *Il Giornale degli Economisti* (Barone 1896). The whole sequence of events is beautifully reconstructed by (Jaffé 1964).

[40] Walras's new reflections of the mid-1890s on the topic of cost minimization and variable production coefficients find a first expression as a postscript to a short essay titled "Note sur la réfutation de la théorie anglaise du fermage du M. Wicksteed", published in a collection of essays edited by the University of Lausanne in 1896, and soon after reprinted as Appendix III to the 3^{rd} edition of the *Éléments* (1988, **3**, 715-22). In the 4^{th} and 5^{th} editions Appendix III of the 3^{rd} will be suppressed. However, in those two editions, the content of the postscript of the suppressed Appendix will be reproduced, with some changes, not always the same in the two editions, in the pages immediately following the one discussed in the text (1988, **4-5**, 586-7; **4**, 588, 590; **5**, 589, 591).

the time prevented him from fully understanding their meaning. This sort of argument certainly contributes to the explanation of the peculiar phenomena listed above. Yet other considerations are probably no less significant.

First of all it should be stressed that the issue of the variability of production coefficients did not come to Walras's mind as an issue related to entrepreneurs' optimizing choices. On the contrary, it occurred to him as the unavoidable, objective result of the progressive changes in the proportions in which the various types of capital turn out to be available in the economy, due to the accumulation process, which increases the total amount of capital goods proper as well as their weight with respect to the other types of capital, especially lands; to the process of demographic growth, which similarly increases the total amount and weight of personal capital; and, finally, also to technological and economic progress, viewed as an exogenous factor, which changes the proportions in which the services of the various types of capital enter into the production processes. This is the reason why, as already recalled, the question of the variability of production coefficients was at first tackled by Walras in a marginal Leçon, dealing with the issue of "The increase in the quantity of products" and belonging to a Section of the *Éléments* devoted to the analysis of "The conditions and consequences of economic progress", where it will remain forever. In the last analysis, therefore, cost minimization played at first only a secondary role in the explanation of variable production coefficients; in the explanation of this phenomenon, in fact, a paramount role was assigned by Walras to factors excluding any optimizing behavior.

But there is also another likely reason why Walras hesitated to insist on cost minimization as the main factor behind the variability of production coefficients. As we have recalled above, Walras's fundamental assumption about entrepreneurs' behavior, in both the Geneva Notes and his writings of the years 1876 and 1877, when he was developing his production model, was that they act as quantity adaptors, increasing or decreasing the scale of the production processes they control according to the occurrence of gains or losses in their activities; this means that they are not assumed to behave as profit maximizers or, more generally, as optimizing agents. Now, to suppose that they choose their production techniques in such a way as to minimize their costs means to assign them another behavioral rule, different from the first and fundamental one singled out by Walras from the very beginning. The two rules are not necessarily incompatible, as Walras himself will come to vaguely perceive at the very end of his scientific life[41]. But coordinating them into a coherent whole would have required analytical skills and a broad conceptual vision that he certainly did not possess in the years 1876 and 1877, nor would ever come to possess in the

[41] A possible way to reconcile the two rules is apparently suggested by Walras in what are probably the last few lines written by him in 1902, in view of a planned future edition of the *Éléments*, an edition that will only materialize in 1926, well after Walras's death. Such few lines are contained in a footnote (1988, **5**, 591, fn. 1) aimed at rebutting the criticism, advanced by Pareto in a slim pamphlet published one year before (Pareto 1901), of Walras's own version of marginal productivity theory, as put forward in the 4^{th} edition of the *Éléments* (1900).

following quarter of a century.

Moreover, Walras did not want to run the risk that the new rule might disrupt the theoretical arrangement he had so laboriously succeeded in constructing under the assumption of fixed production coefficients. This is the reason why in 1876, when he did not yet have the faintest idea of how to solve the problem of the determination of variable production coefficients through cost minimization, he rushed anyhow to play down the role and significance of such as yet inexistent solution. Such attitude can be easily detected by looking at the following two statements contained in the quotation on the variability vs. fixity of production coefficients reproduced above (1876b, 86): the first statement is that the quantities of the services entering into one unit of a given product (i.e., the production coefficients) "*sont déterminés seulement après la détermination des prix des services*"; the second is that the system of equations allowing the theorist to determine the cost-minimizing production coefficients can be set aside, for the sake of simplicity, since such system would be "*en quelque sorte indépendant des autres que nous considérons*".

Now, the italicized expressions (reproduced in French, to avoid all possible misunderstanding) are both wrong, because they patently collide with the interdependence among all relations and variables which is the distinctive trait of Walrasian general equilibrium theory. After a (long) while, Walras himself will realize the mistaken character of such expressions: the second will in fact be dropped since the 2^{nd} edition of the *Éléments* (1889); for the first expression, on the contrary, it will be necessary to wait for the 4^{th} edition (1900), where the words "*sont déterminés seulement après la détermination des prix des services*" are replaced by the words "*sont déterminés en même temps que les prix des services*"[42]. But their original insertion reveals Walras's desire to protect his theoretical system from the disarray that might have been provoked by the introduction of an assumption, that of cost minimization or, more generally, of an optimizing behavior on the part of entrepreneurs, that, as of 1876 (or 1877 or even later), was still perceived by Walras as largely extraneous to the logic of his theory of production.

Keeping therefore to the single-output, fixed-coefficient assumption about technology, the only one on which Walras's theory of production is effectively based not only in the 1^{st}, but also in all the subsequent editions of the *Éléments*, if one looks at the theoretical core of the book, we give now a full account of such theory, remaining absolutely faithful to the substance of Walras's formulation, but changing the notation in such a way as to bring it more in line with contemporary standards.

Hence, let m be the number of products (consumer goods) in the economy, indexed by $l = 1, ..., m$, and n the number of services, indexed by $j = 1, ..., n$. Let $p = (p_1, ..., p_l, ..., p_m)$ and $c = (c_1, ..., c_l, ..., c_m)$ be the row vectors of the prices (in numéraire) and unit costs (in numéraire) of the m products, respectively. Let $q_s = (q_{1s}, ..., q_{ls}, ..., q_{ms})$ and $q_d = (q_{1d}, ..., q_{ld}, ..., q_{md})$ be the column vectors

[42] This correction was due to a very precise suggestion advanced by Barone, in a letter to Walras dated 13 novembre 1895 (Jaffé 1965, II, letter 1220, 653-5).

of the quantities to be produced and of the quantities demanded of the m products, respectively. Then let $w = (w_1, ..., w_j, ..., w_n)$ be the row vector of the prices (in numéraire) of the n services. Let $z_d = (z_{1d}, ..., z_{jd}, ..., z_{nd})$ and $z_s = (z_{1s}, ..., z_{js}, ..., z_{ns})$ be the column vectors of the quantities demanded and supplied of the n services, respectively. Finally, let the single-output, fixed-coefficient technology be represented by the n by m matrix A:

$$A_{n\times m} = [a_{jl}] = [a_1 ... a_l ... a_m] = \begin{bmatrix} a_{11} & ... & a_{1l} & ... & a_{1m} \\ \vdots & & \vdots & & \vdots \\ a_{j1} & ... & a_{jl} & ... & a_{jm} \\ \vdots & & \vdots & & \vdots \\ a_{n1} & ... & a_{nl} & ... & a_{nm} \end{bmatrix},$$

where the generic element a_{jl} is the production coefficient of input j in the production of output l, representing the quantity of service j entering in the production of one unit of product l; moreover the generic column vector a_l is the production technique of output l, representing the quantities of the n services entering as inputs in the production of one unit of product l.

Now, given the quantities of all the services in the hands of the owners of the corresponding capital assets, consumers-owners, taking the prices of consumer goods, p, and the prices of services, w, as given parameters, maximize their utility functions under their respective budget constraints, determining their individual demand functions of consumer goods and supply functions of services. Aggregating over all consumers-owners, the aggregate demands for consumer goods and supplies of services are obtained as functions of the prices of consumer goods and services, that is, $q_d(p, w)$ and $z_s(p, w)$. At the same time, given the row vector of prices of services, w, and the column vector of quantities of consumer goods to be produced, q_s, the row vector of unit costs of the products and the column vector of demands for services are obtained by pre-multiplying and post-multiplying the matrix A by w and q_s, respectively, that is:

$$c = \begin{pmatrix} c_1, & ..., & c_l, & ..., & c_m \end{pmatrix} = wA$$

and

$$z_d = \begin{bmatrix} z_{1d} \\ \vdots \\ z_{jd} \\ \vdots \\ z_{nd} \end{bmatrix} = Aq_s.$$

Then an equilibrium in the production model consists of a vector of prices of services, w^*, and a vector of prices of consumer goods, p^*, such that the demand for services is equal to the supply of services, i.e., $z_d^* = z_s(p^*, w^*)$, the demand for consumer goods is equal to the supply of consumer goods, i.e., $q_d(p^*, w^*) = q_s^*$, and the prices of products are equal to their unit costs, i.e.,

$p^* = c^* = w^*A$. When these systems of equations are satisfied, all the three conditions for an equilibrium in production, as identified by Walras in the definition recalled above, are met. Hence, the "equilibrium determination" problem in the production model can be solved by simply finding the solution of the above systems of algebraic equations, which would be nothing other than Walras's "solution mathématique" (1876b, 82-3; 1988, 304-5).

As can be seen, also here, as in the Geneva Notes, equilibrium in production requires the equality of product prices and unit costs, namely, $p^* = c^*$. Hence, also in the production model of the mid-1870s, as in the Geneva Notes of the Winter 1871-72, the equality between the prices of produced consumer goods and their unit costs of production may appear, from Walras's own point of view, to be dangerously reminiscent of that equilibrium condition which had led the economists of the English School, J.S. Mill in the first place, to endorse the so-called "cost theory of value", according to which it is the unit cost of production that *determines* the price of a product. As we have seen, already in the Geneva Notes Walras had declared his stern opposition to such "cost theory of value". But at that time his critical weapons could be hardly effective, since he could not rely on any consistent representation of technology, any fully-fledged theory of prices of services, or finally any theory of costs.

At the time of elaborating the production model of the years 1876 and 1877, however, Walras's situation was much more favorable: for, at that time, given the then available adequate representation of technology, the equilibrium unit costs could be traced back to the equilibrium prices of services, that is, $c^* = w^*A$, while the prices of services could in turn be determined by means of the condition that, at equilibrium, the demands for services be equal to their supplies, that is, $z_d^*(p^*, w^*) = z_s(p^*, w^*)$. Hence, in the last analysis, prices of consumer goods, prices of services, and unit costs of production could all be shown to be co-determined at equilibrium. In the mid-1870s, therefore, Walras could have counteracted the "cost theory of value" by simply pointing out that, in the framework of general equilibrium theory, all endogenous variables must be regarded as being simultaneously determined at equilibrium, so that costs cannot be said to determine prices any more than prices can be said to determine costs.

Walras, however, did not follow this route, which would have probably required a fuller methodological awareness than that on which he could count at that time. He tried however to use a somewhat similar argument, by maintaining that the reasoning of the English School could be justified only as an "approximation" or, what one would perhaps say nowadays, only from a partial equilibrium prspective. Walras's argument appears just at the end of the Section titled "Théorie de la production" in the 1^{st} edition of the *Éléments* (1988, **1**, 342, editorial footnote *l*). In the preceding pages Walras had reproduced the same partial equilibrium diagram as that employed in the Geneva Notes to explain the convergence of the price of a produced commodity to its unit cost, assumed given and independent of the output level. As recalled towards the end of Section 2 above, where that diagram was described and commented upon, such convergence was supposed to be brought about by changes in the quantity

produced of the commodity concerned. Referring to that representation, Walras concluded his partial equilibrium exercise as follows:

> On peut remarquer qu'au point de vue que nous avons adopté, mais à ce point de vue seulement, il est exact que *le prix des produits est déterminé par le montant des frais de production.* Ainsi, l'école anglaise, en posant son principe, a pris une vérité d'approximation pour une vérité absolue. (Walras's italics)

This statement is indeed correct, under the assumption that the technical coefficients are fixed and that the prices of services are (provisionally) taken as given, what can be justified from a partial analysis perspective. Yet, since the 2^{nd} edition of the *Éléments*, Walras decided to remove the entire sentence, probably because it appeared to him to make an eccessive concession to the English School's viewpoint. So that, in the end, barring both the general equilibrium critical argument and the partial equilibrium tentative justification, the only solution which remained open to Walras was to try, uncorrectly, to reverse the causality direction between prices of products and prices of productive services. Precisely, instead of assuming that the prices of productive services determine the prices of products, as the economists of the English School would have pretended, Walras ended up by suggesting that just the opposite causality nexus holds true, namely, that the prices of products are the factors determining the prices of productive services (1988, 264). But this assertion, though understandable as a polemical retort, is unfounded from the perspective of general equilibrium analysis.

6 The New Equilibrium Conception and the Production Model: Equilibrium Establishment

As with the exchange model, also in the case of the production model finding the "mathematical solution" is not enough for Walras: also here, in fact, the "mathematical solution" must be shown to coincide with the "empirical" one, directly attained by the market "par tâtonnement":

> Reste seulement à montrer, en ce qui concerne l'équilibre de la production comme en ce qui concernait celui de l'échange, que ce même problème dont nous avons donné la solution théorique est aussi celui qui se résout pratiquement sur le marché par le mécanisme de la libre concurrence. (1876b, 84; 1988, 307)

Walras's description of the equilibration process in the production model is highly tormented: the version provided in both the third *mémoire* (1876b) and the 1^{st} edition of the *Éléments* is significantly altered in the 2^{nd} edition and again modified in the 4^{th}. As will be explained in the following, all these changes have to do with analytic and conceptual difficulties marring the representation of the adjustment process in this model.

Walras imagines the equilibration process to evolve through a sequence of alternating steps (or "phases"), involving at first the market for products, while leaving the market for services unaffected, and then, at the following step, involving the market for services, while leaving the market for products unchanged. In summary, given the prices of services, which are provisionally kept constant, the adjustment process in the market for products brings about the equality of prices and unit costs, on the one hand, and of quantities demanded and supplied of all products, on the other. Once these two (partial) equilibrium conditions have been attained, the third condition, requiring the equality between quantities demanded and supplied of all services, is generally not met. Then, according to Walras, a new step of the adjustment process gets started in the market for services, going on through changes in the prices of services until the third (partial) equilibrium condition is met. Should this disrupt the (partial) equilibrium initially reached in the first step in the market for products, the process will start again there, and so on. The process will only end when all the three equilibrium conditions are simultaneously satisfied.

The peculiar two-step structure of the equilibration process in the production model, which will remain unchanged for a quarter of a century in spite of the many changes affecting other aspects of the process, deserves some explanation. Starting from the first step, it is immediate to notice that the kind of adjustment process that is supposed to take place therein is basically the same as the quantity adjustment process already envisaged in the Geneva Notes, where, as will be recalled, such process was supposed to drive the price of each product towards its unit cost of production, assumed to be constant. But in the Geneva Notes, as we have seen, unit costs were taken as fixed partly because of the persistent influence of the theory of production and costs advocated by the economists of the English School and partly for want of an adequate theory of the prices of services, alternative to that put forward by the same economists. Now, in 1876, at the moment of writing his third *mémoire*, Walras's overall situation would appear to be quite different, for in the meanwhile he had succeeded in developing a suitable account of technology, combining it with a fully-fledged theory of the prices of services. Hence, one might expect that in 1876 Walras should eventually be able to devise a comprehensive equilibration process, where prices of services, prices of products, quantities demanded of services, and quantities supplied of products would simultaneously adjust in a unique all-embracing process. For, after all, as Walras himself will recognize many years later, in a *mémoire* subsequently reprinted as Appendix I in the 3^{rd} edition of the *Éléments* (1988, **3-5**, 693-710), such simultaneous adjustment involving all prices and quantities at one and the same time is what actually occurs in the markets of the real world.

Yet, in the mid-1870s, when writing his third *mémoire* and soon after the Sections on production of the second instalment of the 1^{st} edition of the *Éléments*, Walras was not really in the condition to conceive of such a unified adjustment process, for that would have demanded a radical change in his behavioral assumptions concerning the entrepreneurs, a change that, as will be explained in the sequel, Walras was both unable and unwilling to make then,

and that he will never be able or willing to make for the rest of his scientific life. Being unable to put forward a unified theory of the tâtonnement process in the production model, in 1876 and 1877 he concocted the above-mentioned two-step procedure, allegedly justified, in spite of its acknowledged 'unrealism', "pour les besoins de la démonstration" (1988, **3-5**, 704), using the first step to deal with the quantity and price adjustment process in the market for products, under the assumption, already made in the Geneva Notes, of given unit costs, while reserving the second step to separately deal with the price adjustment process in the market for services, an issue which could not have possibly arisen in the Geneva Notes, due to the lack of any proper theory of the prices of services therein.

Let us now examine in greater detail the two alternating steps envisaged by Walras, starting from the first. At the beginning of the process, a vector of prices of services, $\bar{w}$, and a vector of quantities of products to be supplied by the entrepreneurs, $\bar{q}_s$, are cried out at random. From these initial data a vector of unit costs of products, $c' = \bar{w}A$, and a vector of demands for services, $z'_d = A\bar{q}_s$, can be obtained (1876b, 85-6; 1988, 314, 316). Moreover, let us assume that the consumers-owners be competitive utility maximizers; that they have no endowments of consumer goods, but only of services (more precisely, of the various types of capital, rendering proportional quantities of services); that their demand functions for consumer goods, $q_d(\cdot, \bar{w})$, be invertible with respect to the suppressed argument, and finally that the entrepreneurs throw on the market the entire quantities of products that they produce. Then, under the stated assumptions, from the equations expressing the equality between supplies and demands of the various products, $\bar{q}_s = q_d(\cdot, \bar{w})$, one can draw the market-clearing prices of products as functions of their quantities, given the prices of services, i.e., $p' = q_d^{-1}(\bar{q}_s, \bar{w})$. Proceeding in this way, Walras is simply generalizing to a multi-commodity, general equilibrium framework the approach he had already exploited in his Geneva Notes with reference to a single-commodity, partial-equilibrium context, where the functional relationship between price and quantity of a consumer good was summarized, as seen in Section 2 above, by the so-called "courbe de prix".

At this point one has to consider an apparently bewildering assumption made by Walras: for he assumes the existence of a "foreign market" ("marché étranger") where, at the quoted prices of services, $\bar{w}$, the entrepreneurs can find productive services "en quantités indéfinies" (1876b, 86; 1988, **1**, 312, 314). This assumption is so alien to the logic of general equilibrium theory that Walras himself will find it more sensible to drop it since the 2^{nd} edition of the *Éléments* (1988, **2-3**, 312; **4-5**, 313). Nevertheless, the reasons why Walras was initially led to make such a questionable assumption are quite clear. Given the assumptions on the behavior of consumers-owners, when the prices of services are $\bar{w}$ and the prices of consumer goods are p', the utility-maximizing quantities of productive services to be supplied are $z_s(p', \bar{w})$. At the same time, given the fixed-coefficient technology, A, and the assumptions on the behavior of entrepreneurs, when the quantities of products to be supplied by them are $\bar{q}_s$, the quantities of productive services they need in order to produce the prescribed quantities of

products are $z'_d = A\bar{q}_s$. Obviously, since both $\bar{w}$ and $\bar{q}_s$ are randomly cried out, there is no reason why $z_s(p', \bar{w})$ should be equal to z'_d. Yet, if, given the fixed-coefficient technology, the vector z'_d of quantities of services were not to be made available, the vector $\bar{q}_s$ of quantities of products could not be produced. But then the tâtonnement process would stop even before starting. The need for an accommodating entity providing the required inputs, such as the "foreign market" of the 1^{st} edition, becomes inescapable.

The problem stressed in the previous paragraph arises from a number of conflicting assumptions and prescriptions. In the first place, the original version of the tâtonnement process in production stipulates that, at each round of the tâtonnement, the entrepreneurs actually transform specified quantities of inputs into specified quantities of outputs: in other words, actual physical production is supposed to occur. In the second place, the two broad categories of agents confronting themselves on the markets for services and products are assumed to obey entirely different behavioral rules: while entrepreneurs must mechanically comply with the purely objective requirements of technology, consumers-owners act instead as competitive utility maximizers, choosing their optimal consumption plans, including their supplies of productive services, subject to their competitive budget constraints. When the two kinds of behavior do not match, what is always the case during the tâtonnement process before a general equilibrium is reached, physical production can only go on if either one of the following two alternatives is pursued: either consumers-owners are forced to do what they do not want to do, but this would conflict with one of the basic tenets, 'voluntary behavior', of the new optimizing approach discovered by Walras in the Fall 1872; or, alternatively, some entity extraneous to the economy described in the model, such as the accommodating "foreign market" of both the third *mémoire* and the 1^{st} edition of the *Éléments*, is asked to replace the reluctant consumers-owners in their supposed duties[43].

Neither alternative is of course satisfactory. In particular, the "foreign market" solution adopted in the production model of the years 1876 and 1877, while providing an immediate relief to the issue of letting physical production at least start, leaves a myriad of major problems open. In particular, if 'domestic' entrepreneurs buy the quantities of services required for production from the accommodating "foreign" agents at the specified prices of services, $\bar{w}$, what will happen of the numéraire counterpart of such purchases? With what resources will 'domestic' consumers-owners be able to purchase the produced quantities of consumer goods? It is clear that, if not physical production, all the other

[43] Hence, when in the 2^{nd} and 3^{rd} editions of the *Éléments* the "foreign market" assumption is dropped, but actual physical production is still supposed to take place out of equilibrium, Walras will be forced to accept that the 'voluntary behavior' postulate be violated: for, under the stated circumstances, in order that the entrepreneurs be able to carry out the required productive activities before an equilibrium is reached, consumers-owners will generally be compelled to supply quantities of productive services different from the utility-maximizing ones. As will be seen at the end of this Section, this serious flaw of the production model of the 2^{nd} and 3^{rd} editions will be remedied only in the 4^{th} edition of the *Éléments*, with the adoption of the so-called "hypothèse des *bons*" and the consequent 'virtualization' of the entire production process out of equilibrium.

components of an ongoing economic concern would be jeopardized. Yet, setting these problems aside, and provisionally assuming that the consumers-owners can somehow find the resources necessary to make their desired purchases of consumer goods, let us follow Walras's reasoning further.

As we have seen above, under the stated assumptions, the market-clearing prices of consumer goods turn out to be $p' = q_d^{-1}(\bar{q}_s, \bar{w})$. Yet, in general, p' will differ from $c' = \bar{w}A$, the vector of the unit costs of production of consumer goods. For every commodity l, with $l = 1, ..., m$, such discrepancy would give rise to gains or losses in its production according to whether p'_l is greater or less than c'_l. The entrepreneurs already involved in the production of commodity l will increase or decrease their production of commodity l according to whether they make gains or losses in producing it; similarly, the prospect of gains may induce entrepreneurs who are not yet participating in the market for commodity l to enter it from outside, while the occurrence of losses may lead some entrepreneurs already in that market to leave it. This adjustment process will continue up to when the price of each commodity is brought into equality with its unit cost, which is independent of the scale of production and constant, as long as the production coefficients are fixed and the prices of services are (temporarily) taken as given.

The quantities of the various consumer goods to be produced to make such equality effective can be obtained by solving the equations $q'_s = q_d(c', \bar{w})$, stating the equality between the quantities produced and demanded of all commodities, when the prices of all such commodities are set equal to their respective unit costs. Once again, proceeding in this way, Walras is simply generalizing to a multi-commodity, general equilibrium context the approach he had already exploited in his Geneva Notes with reference to a single-commodity, partial-equilibrium context, where, as recalled at the end of Section 2 above, the quantity produced of any given consumer good was so adjusted as to bring into equality the price of that good, to be read on its "courbe de prix", and the unit cost of the same good, assumed to be fixed and independent of output.

In his third *mémoire*, as well as in the second instalment of the 1^{st} edition of the *Éléments*, Walras, though assuming that the equality of prices and unit costs of production of consumer goods is brought about "par tâtonnement", confines himself to simply stating the final result, which we have summarized above by means of the equation $q'_s = q_d(c', \bar{w})$, what would actually mean that the adjustment is instantaneously effected (1876b, 87-8; 1988, 316, 319). In the 2^{nd} and following editions, however, Walras will add a couple of pages with the purpose of showing how a market-by-market quantity adjustment process, similar to the market-by-market price adjustment process employed in the exchange model to explain the progressive emergence of the equality between demand and supply in all markets, would eventually bring about the equality between prices and unit costs for all products (1988, **2-5**, 317-8). In any case, whether instantaneously or "par tâtonnement", through a market-by-market procedure, the equilibration process in the product market ends with the realization of two equilibrium conditions out of the three that characterize the production model, namely, the equality between the quantities demanded and supplied and

between the prices and the unit costs of all products. While the first equality implies that the prices of all products stop changing, the second implies that entrepreneurs make neither gains nor losses, so that the quantities produced of all consumer goods stop changing as well.

As can be seen, both the first step of the equilibration process contemplated by Walras in his production model and the distinctive features of the state of "équilibre partiel" (1876b, 89; 1988, **1**, 322) eventually reached at the end of such a first step display strong similarities with the adjustment process and the characteristic traits of the equilibrium state envisaged by J.S. Mill in his theory of production strictly speaking, that is, in that part of his theoretical system that deals with quantities and prices of produced commodities and production costs, as distinguished from those other parts that deal with the prices of services and the distribution of income among the participants in the production process[44].

Yet, the situation eventually resulting on the product market at the end of the first step of the procedure can only correspond to a state of "équilibre partiel", since, for a truly general equilibrium, also the third equilibrium condition, requiring that demands and supplies of services be equal, must be met. It is precisely at this juncture that the differences existing between Walras's theoretical approach in the mid-1870s and the approach advocated by the English School, especially by J.S. Mill, make themselves patent: for Walras's theory of the determination and establishment of the prices of services, hence also his theory of income distribution, eventually put forward in the mid-1870s as by-products of his 1872 discovery, significantly differ from those of the English School.

The cumbersome design of the tâtonnement process in the production model is the device by means of which Walras tries to reconcile the two conflicting viewpoints co-existing in his production model: for, by envisaging a procedure based on the alternation of two sharply distinguished steps, Walras deludes himself that a quantity adjustment process in the market for products, drawing its inspiration from J.S. Mill's approach and taking place exclusively in the first step of the procedure, may be reconciled with a price adjustment process in the market for services, more in line with the new equilibrium conception, which is instead exclusively confined to the second step. The attempted reconciliation is of course deceptive. Yet, this is the way-out devised by Walras in the mid-1870s, to which he will stick for the rest of his scientific life.

In the second step of the procedure, therefore, Walras assumes an equilibration process to get started in the market for services, where the prices of services, which had been taken as (temporarily) fixed up to that point, are now allowed to change: each such price will increase or decrease according to whether the excess demand on the corresponding market is positive or negative. This equilibration process, being very similar to the one already illustrated with reference to the price adjustment process in the market for products, will not be discussed in detail here. We just confine ourselves to pointing out that when

[44] Such similarity is stressed also by Schumpeter, an author who cannot certainly be suspected of any negative bias against Walras, when he writes that "on an infinitely higher level of rigor, Walras really reformulated the theories of production of A. Smith, J. B. Say, and J. S. Mill" (1954, 1010, footnote 30).

the third equilibrium condition is eventually reached, the prices of services will stop changing. However, even at this point a full general equilibrium state is not generally achieved, for the equilibration process concerning the prices of services might have unbalanced the market for products. In such a case, the process must start anew, with the markets for products and services alternating with one another during the twofold tâtonnement process.

When the triple equilibrium condition is eventually satisfied, the general equilibrium of the market is established, implying the stationarity of the prices ("prix stationnaire") of all the services and all the products (1876b, 95; 1988, 329-30). Yet, since also the third equilibrium condition is met, entrepreneurs make neither gains nor losses; hence, they have no incentive to change their levels of production. Therefore, as hinted at above, a third stationarity condition is met at a state of general equilibrium of production, concerning the quantities produced of all commodities. All this, as already remarked, is summarized by Walras under the heading of the double "Loi de l'offre et de la demande et du prix de revient" (1988, 330).

Also with reference to the production model, Walras repeatedly professes his faith in the ability of the market or competition mechanism to bring about "par tâtonnement" an "empirical" solution which is identical with the "mathematical" solution obtained by solving the equation systems summarized in the previous Section. He is so confident in this outcome as to proclaim in his writings of 1876 and 1877 that its realization is "certain" (1876b, 94; 1988, **1**, 326, 328). Since the 2^{nd} edition of the *Éléments*, as already pointed out when examining the exchange model (see footnote 28 above), Walras will weaken his claim, replacing the word "certain" with the word "probable" in the cited passages. Yet, in the case of the production tâtonnement, the situation is even more complicated than that arising in the tâtonnement in exchange: for, in the production case, the inconsistencies impairing Walras's equilibration analysis are such as to jeopardize not only the convergence of the process, but even its feasibility. As we have seen, the main difficulties arise from two separate sources: the coexistence of two different behavioral rules, respectively concerning consumers-owners and entrepreneurs, and the assumption of a 'real' time adjustment process, with observable actions taking place out of equilibrium. Let us consider the two issues in turn.

As regards the first issue, it should be noted that Walras occasionally tries to establish a parallelism between the motivations underlying the behavior of entrepreneurs and those explaining the behavior of consumers-owners in the markets for services and products. Such attempts are witnessed, e.g., by the following passage:

> [N]ous trouvons ici, dans le désir d'éviter des pertes et de faire des bénéfices, la raison déterminante de demande des services producteurs et d'offre des produits par les entrepreneurs, comme nous avons déjà, dans le désir d'obtenir la satisfaction maxima des besoins, la raison déterminante d'offre des services producteurs et de demande des produits par les propriétaires fonciers, travailleurs et

capitalistes. (1988, 284)

Yet, such parallelism is hardly justified. As far as the consumers-owners are concerned, their individual characteristics are fully specified; their individual choices of utility-maximizing consumption and trade plans are perfectly determined, at both equilibrium and disequilibrium prices; moreover, at equilibrium, their individual actions are precisely determined, too. On the contrary, as far as the entrepreneurs are concerned, their individual characteristics are never specified: we are only informed about the overall production technology, as represented by the matrix of technical coefficients, which is the same for all the entrepreneurs participating in the economy; no individual choices of production plans are ever specified, either at equilibrium or out of equilibrium: as a matter of fact, we are told that the aggregate production of any commodity whose price is greater or less than its unit cost will be increased or decreased, but we do not know which individual entrepreneurs will bring about such increases or decreases; finally, we know that, at both partial and full equilibrium, aggregate outputs will be determined in such a way as to equalize prices and unit costs, as implied by the equations $q'_s = q_d(c', \bar{w})$, but once again we cannot know which individual entrepreneurs will be instrumental in making such outcome happen. This helps explain why, as already suggested, entrepreneurs, though assumed to act on output levels, cannot be said to fully control the state variables of the quantity adjustment processes at work in Walras's production model.

There is just one somewhat cryptic passage where Walras, in briefly considering the otherwise wholly neglected issue of fixed costs, cursorily hints at the distribution of aggregate output among firms:

> [N]ous négligeons une autre circonstance, celle de la distinction entre les frais fixes et les frais variables dans le entreprises. Mais, puisque nous supposons les entrepreneurs ne faisant ni bénéfices ni pertes, nous pouvons bien les supposer aussi fabriquant des quantités égales de produits, auquel cas tous les frais de toute nature peuvent être considérés comme proportionnels. (1988, 306)

Yet, as can be seen, such passage adds nothing to our knowledge, simply confirming that entrepreneurs do not possess any individuality whatsoever in Walras's system of thought. In a sense, the entrepreneurs of the production model may appear to be similar to those "arbitrageurs" whose self-interested, but not optimizing, behavior is instrumental, as recalled in Sections 2 and 3 above, in making the price system satisfy the "general equilibrium" consistency condition. As a matter of fact, the individual characteristics of the "arbitrageurs", as well as their individual out-of-equilibrium behavior, are left unspecified, in both the Geneva Notes and the post-1872 exchange models. Only the final result of their efforts, a result that may well be supposed to be instantaneously reached, is disclosed. But all these traits, *mutatis mutandis*, are shared in common by the entrepreneurs of the production model, of whom the "arbitrageurs" of the Cournot-Walras tradition may now appear to be the proximate ancestors.

To sum up on this issue, Walras's entrepreneurs, unlike the consumers-traders of the exchange model and the consumers-owners of the production model, are left basically untouched by the 1872 optimizing twist: in fact, also in the production model put forward in 1876 and 1877, as in the Geneva Notes, the entrepreneur remains "un intermédiaire dont on peut faire abstraction" (1871-72, 473). In dealing with entrepreneurs after his finding of the Fall 1872, Walras appears to disregard the individualistic foundations characterizing his newly discovered equilibrium conception: somewhat paradoxically, and in spite of Walras's repeated criticism of the economists of the English School for their inability to comprehend the true role of the entrepreneur in the organization of production and the functioning of a market economy, Walrasian entrepreneurs appear to be more similar in their behavior to the capitalists of the English School than to the consumers populating Walras's own post-1872 models[45]. In view of its twofold derivation and allegiance, the production model of the years 1876 and 1877 ends up by lingering halfway between the old approach, rooted in the English classical tradition, which chiefly affects the representation of production activities, and the new approach, based on utility maximization, which instead dominates the scene as far as consumption and exchange are concerned[46].

The persistence of preexisting ideas in the production model of the mid-1870s is also evident in the characterization of the equilibration process: in

[45] Such lack of individualistic foundations and optimizing presuppositions in Walras's analysis of production activities, and especially in his conception of the role of the entrepreneur, is probably at the heart of Edgeworth's recurring criticism of Walras's ideas in this field (Edgeworth 1889a, 268; 1889b, 280-1; 1891, 370-5; 1904, 24-32; 1910, 378-81). On this issue, one should also consult Newman (1990, 525-30).

[46] It will be only at the very end of his scientific journey, when working on the final draft of the 4^{th} edition of the *Éléments* (1900), that Walras will eventually realize that the "theory of marginal productivity", discovered by him just a few years before (between 1894 and 1895) with Barone's essential help, could play a role in the theory of production and producers' choices as fundamental as that played by the "theory of final utility" in the theory of consumption and consumers' choices. The symmetrical role of the two theories is justified by Walras in the following terms:

> la *théorie de la productivité marginale*, théorie capitale en économie politique pure, [...] fournit [...] le ressort de la demande des services et de l'offre des produits par les entrepreneurs, tout comme la *théorie de l'utilité finale* fournit le ressort de la demande des produits et de l'offre des services par les propriétaires fonciers, travailleurs et capitalists [...]. (1988, **4-5**, 588-9)

On the basis of this understanding, Walras could have tried to rewrite afresh his theory of production and producers' choices, that is, his theory of supply of products and demand for productive services. This, however, would have required to revolutionize the entire analytical apparatus of the production model, as developed in the years 1876 and 1877 and preserved up to the end of the Nineteenth century with only some limited changes (barring the interpretation of the tâtonnement process, on which we shall shortly come back in the text). But in 1900 it was really too late for Walras to embark upon such a forbidding task. So that in the end he decided to leave the field, by adducing, somewhat hypocritically, the following justification:

> j'ai préféré ne pas introduire [la théorie de la productivité marginale] dans ma théorie générale de l'équilibre économique, déjà suffisamment compliquée, de peur que celle-ci ne devint trop difficile à saisir dans son ensemble. (1988, **4-5**, 588-9)

fact, also in the case of production, as in that of exchange, Walras is interested in designing a process that, though evolving in 'real' time, yet is such as to preserve the time invariance of the data. As we have seen, in the exchange model put forward in 1874 Walras had taken care of the issue of data invariance during the equilibration process by assuming both the utility functions of the consumers-traders and the aggregate quantities of consumer goods available in the economy to be exogenously given and unchanging over time; by the latter assumption Walras probably meant to imply that out of equilibrium the available quantities of consumer goods could be traded, but not consumed. We have also seen, however, that Walras's likely supposition that it is enough to block the physical activity of consumption to ensure that the data do not change over the equilibration process proves to be wrong, for any out-of-equilibrium trading of commodities necessarily alters their distribution among the traders, thereby affecting the data even if it is assumed that consumption is forbidden. This, as we have seen, will prompt Walras to rule out, since 1885, also the possibility of out-of-equilibrium trading in the exchange model.

Now, in production the situation is even more complex than in exchange, for here, on top of trading and consumption, also production activities must be taken care of. Moreover, one should consider that one of the central features of the equilibration process in the production model, as envisaged by Walras in the mid-1870s, consisted precisely of the assumption that the physical transformation of inputs into outputs is an observable phenomenon taking place in 'real' time. In view of this, it was simply inconceivable for Walras, in the mid-1870s, to suppose that the problem of the time invariance of data in the production model could simply be tackled by preventing the entrepreneurs from carrying out any production activity out of equilibrium: what appeared to Walras to be acceptable for consumption, was instead totally unacceptable for production, account being taken of the different characteristics of the two types of economic activities[47].

[47] The belief that production, unlike consumption, is a physical activity that would be meaningless to stop during the equilibration process is a conviction that Walras will reiterate over the years, at least up until the 3^{rd} edition of the *Éléments*. In particular, the difference between the exchange and the production model, as far as the equilibration process is concerned, is stressed by him in an identical additional passage of the 2^{nd} (1889) and 3^{rd} (1895) edition of the *Éléments*, supplementing his discussion of the tâtonnement process in production, as previously developed in the second instalment of the 1^{st} edition (1877):

> Il s'agit d'arriver à l'équilibre de la production de la même façon que nous sommes arrivés à l'équilibre de l'échange, c'est à dire en supposant les données du problème invariables pendant tout le temps que dureront nos tâtonnements, sauf à supposer ensuite ces données variables en vue d'étudier les effets de leur variations. Mais le tâtonnement en matière de production rencontre une complication qui n'existait pas en matière d'échange. Dans l'échange, il n'y à pas de modification des marchandise. Un prix étant crié, et la demande et l'offre effectives correspondant à ce prix n'étant pas égales, on crie un autre prix auquel correspondent une autre demande et une autre offre effectives. Dans la production, il y a transformation des services producteurs en produits. Certains prix des services étant criés, et certains quantités de produits étant fabriquées, si ces prix et ces quantités ne sont pas prix et quantités d'équilibre, il faudra non seulement crier d'autres prix, mais fabriquer d'autres quantités de produits. (1988,

Supposing therefore that services be actually transformed into products during the tâtonnement process, how can Walras manage to keep the data unaffected? As already recalled, in the 1^{st} edition of the *Éléments* (1988, **1**, 312, 314) it is assumed that, at each round of the tâtonnement, when actual production is supposed to take place, 'domestic' entrepreneurs will be able to find on an accommodating "foreign market" some unspecified agents, who are ready to sell them the quantities of services they need[48]; such services will then be transformed into consumer goods, which in turn will be sold to consumers-owners and consumed during the same round. The process will then repeat itself over a sequence of rounds, up until an equilibrium is eventually reached.

Given this time structure of the tâtonnement process, in the first three editions of the *Éléments* Walras's likely conjecture is that, in spite of all the actual trading, production, and consumption activities that are supposed to take place in each round of the process, the assumption of data invariance over 'real' time can anyhow hold true for the following reasons. In the first place, all the endogenous quantity variables involved in the process, that is, productive services and produced consumer goods, are pure flows, or, in Walras's terminology, "revenus", which do not survive their first use; therefore, since all such variables do not outlast the round of tâtonnement in which they are respectively employed or obtained, they cannot represent any possible source of endogenous change in the data. In the second place, Walras is evidently convinced that all the potential sources of exogenous change in the data can be similarly neutralized. First, he assumes that the utility functions characterizing consumers-owners and the technology available to entrepreneurs be exogenously given and unchanging over the equilibration process. Secondly, he tacitly (in the 1^{st} edition of the *Éléments*) or explicitly (in the 2^{nd} and 3^{rd} editions) assumes that the available quantities of services to be employed in production be unchanging so long as the tâtonnement process goes on (1988, **1**, 312; **2**, 308).

Yet, since the available quantities of services are proportional to the quantities of the corresponding types of capital, assuming that the quantities of services remain unchanged is tantamount to assuming that the quantities of the corresponding types of capital do not change over 'real' time. Now, in the production model no endogenous change in the amounts of the various types of capital is contemplated, since the analysis of all possible changes in the quantities of capital goods proper, the only type of capital which is susceptible of endogenous change due to investment decisions, is ruled out, being deliberately postponed to the capitalization model. However, even if endogenous changes are not allowed for in the production model, exogenous changes in the available amounts of personal capital and capital goods proper are instead to be expected over 'real' time, since, in Walras's opinion, such types of capital are intrinsically subject to either demographic changes or deterioration and wear and tear. Hence,

2-3, 308)

[48] In the 2^{nd} and 3^{rd} editions of the *Éléments*, as already said, Walras will abandon the "foreign market" assumption, imagining instead that the required quantities of services be provided by the 'domestic' consumers-owners themselves, even against their will (1988, 308).

assuming the endowments of personal capital and capital goods proper to be unchanging over time is tantamount to assuming that all types of capital can be assimilated to landed capital, the only type of capital that, according to Walras, is imperishable and unalterable.

To sum up, therefore, it is only by means of quite artificial and self-contradictory assumptions that Walras is able to construct an equilibration process in the production model that satisfies the data-invariance assumption and potentially converges to a stationary equilibrium in 'real' time. Yet, it should be clear that the pseudo-realism of the adjustment process so constructed is more than offset by the utter unrealism of the assumptions that Walras is forced to make, which are even more unpalatable here than in the exchange model.

This is all the more disturbing since Walras does not really need his equilibration process to converge to a stationary or persistent state. As a matter of fact, it was only natural to assume such a convergence in the world of the economists of English School, where market prices were supposed to gravitate around natural prices, in turn assumed to tend towards their unit costs of production. Hence, the idea that the equilibration process at work in Walras's production model should converge to a stationary outcome is almost certainly another remnant of the past, attesting to the influence exerted by Mill's theory of production, costs and value on Walras's original production model, as put forward in the mid-1870s. Yet, such a stationary conception of equilibrium is at variance with the new equilibrium conception, 'instantaneous' in character, which, after Walras's discovery of utility maximization in the Fall 1872 , makes its way with great difficulty in the production model, too[49].

It will take Walras more than twenty years to get rid of the ambiguities and true and proper mistakes marring his production model since its inception in the mid-1870s, eventually accepting to discard his long-cherished idea that out-of-equilibrium production activities be allowed to take place in 'real' time and, at the same time, jettisoning the most disturbing remnants of the classical approach to the theory of production, together with the associated stationary interpretation of the equilibrium concept. Walras's final change of mind on this issue will be sanctioned by the introduction of his well-known "hypothèse des *bons*", at first in the *mémoire* "Équations de la circulation" (1899, 581-2) and,

[49] The ambivalence of the equilibrium notion employed in Walras's production model of the mid-1870s, as well as the twofold nature of the equilibration process supposedly at work therein, are instanced by a passage appearing the 2^{nd} and 3^{rd} editions of the *Éléments*, where Walras discusses what entrepreneurs and consumers-owners will do when the tâtonnement eventually ends:

> [...] ils pourrons ou s'acquitter et en rester là, ou plutôt continuer indéfiniment la production dont la marche sera dès lors réglée pour autant qu'aucune variation ne surviendra dans les données, c'est-à-dire dans les quantitées possédées des services et dans les utilités des services et des produits. (1988, **2-3**, 308)

As can be seen, the equilibrium eventually reached can indifferently correspond, according to Walras's own words, to one single occurrence, as would be the case with the 'instantaneous' interpretation of the equilibrium concept, or to the repetition of an indefinite number of identical occurrences, as would be the case with the stationary interpretation.

just one year later, in the 4^{th} edition of the *Éléments* (1988, **4-5**, 447, 449).

Such "hypothèse" consists of supposing that out of equilibrium entrepreneurs and consumers-owners do not actually trade physical quantities of services and products, but only "bons", that is, promises to exchange specified quantities of services and products, with the agreement that all such promissory notes are null and void unless a general equilibrium actually obtains. Hence, out of equilibrium no physical transformation of services into products is allowed to take place and no observable production activities can actually be carried out by entrepreneurs and consumers-owners. Under the rule of the "hypothèse des bons", therefore, not only consumption and trading activities, already excluded since 1874 and 1885, respectively, but also production activities can no longer occur out of equilibrium. Since no observable actions are allowed to take place before an equilibrium is reached, the entire equilibration process becomes a purely 'virtual' process, evolving in a sort of 'logical' time, disconnected from the 'real' time over which the economy evolves, and taking just one instant of 'real' time to carry all its effects through. Therefore the equilibrium eventually reached, based on the data ruling at a given instant of 'real' time and necessarily unchanging during the 'virtual' equilibration process taking place in 'logical' time, becomes 'instantaneous' in nature[50].

7 Conclusions

At the beginning of the 1870s Walras had reached a remarkable set of theoretical results, especially concerning exchange and production equilibrium. During the Winter 1871-72, he systematically collected most of these results in a series of lecture notes, drafted in preparation of ten conferences to be soon after delivered in Geneva. As argued in Section 2, such notes, called Geneva Notes in the paper, represent the starting point for the subsequent developments of Walras's general equilibrium theory.

A number of novel ideas can be found therein. As regards the theory of exchange, in the Geneva Notes Walras was able to put forward a fully-fledged model of market equilibrium in a two-commodity economy populated by an

[50] With the introduction of the "hypothèse des *bons*", the 'instantaneous' character of equilibrium, already acknowledged by Walras since the mid-1880s with exclusive reference to the exchange model (see (1885, 312, fn. 1) and footnote 30 above), is now extended to a "complete equilibrium", encompassing all four of the "big problems" of economics (exchange, production, capitalization, and money). As a matter of fact, Walras introduced the "hypothèse des *bons*" in 1899, precisely when he was drafting his *mémoire* "Équations de la circulation" (1899), by means of which he meant to solve the problem of money, the last of the four "big problems" to be formalized. Referring to such equations, in a letter to H. Laurent dated March 24, 1899, Walras so described his final achievement:

> J'aurai ainsi complètement résolu le problème qui consiste, en partant de certaines *utilités* et de certaines *quantités possédées* de toutes les espèces de la richesse pour et par un certain nombre d'échangeurs, à établir rationnellement un équilibre complet de la société économique à un moment donné. (Jaffé 1965, III, letter 1396, 66)

arbitrary finite number of consumers-traders. Then, by taking the individual demand functions as "empirically" given, he obtained the equilibrium solution of the model by equating aggregate demand and aggregate supply in the market for either commodity. Moreover, starting from this two-commodity model, and resorting to an "arbitrage" process borrowed from Cournot's *Recherches*, he was also able to develop a multi-commodity model where the notion of "general equilibrium" made its first appearance. In the field of production, building upon both the distinction between "capitaux" and "revenus" inherited from his father August, and the distinction among the various types of capital and the corresponding services bequeathed by J.-B. Say, Walras sketched out a preliminary analysis of the working of both the market for products and, to a more limited extent, the market for services. Moreover, he examined the role played by entrepreneurs in running the production process and bringing about equilibrium in production.

Yet, side by side with such new constructs and conceptions, several preexisting ideas, due to a number of former economists, found their place in the Geneva Notes. In the field of exchange, Walras made a systematic use by of the time-honored notion of an "empirically" given demand function for a consumer good, assumed to be monotonically decreasing in its own price. Such notion had been formally analyzed by Cournot in his *Recherches*, from which Walras chiefly drew his inspiration, as he himself will acknowledge in his later writings. But the same notion had also been employed, though more informally, by the English classical economists and their followers, especially by J.S. Mill in his *Principles of Political Economy*. And it is precisely from Mill's *Principles* that Walras, as he himself will recognize many years later, borrowed the idea, so central in the Geneva Notes, of an adjustment process which is assumed to work in accordance with the so-called "Law of Supply and Demand", that is, through price changes induced by the excess demands prevailing in the market for a consumer good, thereby driving the market itself to a stationary price equilibrium.

In the field of production, in analyzing the market for a produced consumer good, Walras assumed in the Geneva Notes that the market equilibrium condition should be identified with the equality between the selling price of that consumer good and its unit cost of production, in turn assumed to be fixed and independent of the output level. But such condition was essentially the same as that characterizing most of J.S. Mill's theory of production, value, and costs, as expounded in the latter's *Principles*. Similarly, in analyzing the process of adjustment towards a stationary price and quantity equilibrium in the market for a consumer good, Walras supposed such process to be implemented by the entrepreneurs, assumed to react to the gains or losses ensuing from their production activities by increasing or diminishing the quantities of output produced. But this process was patently patterned after the analogous equilibration process supposedly at work in the market for a produced consumer good, as conceived by J.S. Mill in his *Principles*, where such process is driven by the capitalists' reactions to the profits, greater or smaller than normal, they expect to earn in the production processes in which they are involved.

Yet, in spite of all the innovative ideas and the received views that can be de-

tected in the Geneva Notes, not a hint of any assumption of optimizing behavior can be found therein. In fact, even if utilities and commodity endowments are listed among the characteristics of consumers, the latter are never supposed to behave as utility maximizers: neither when they act as purchasers of consumer goods, since the demands for such goods are taken as "empirically" given in the Geneva Notes; nor when they act as suppliers of services, since the supplies of such services are nowhere analyzed or explained in such Notes. Similarly, entrepreneurs are never supposed to behave as optimizing agents, that is, as either profit maximizers or cost minimizers; on the contrary, they are invariably regarded as quantity adaptors, according to a rule which once again appears to be inspired by the economists of the English School, first of all by J.S. Mill, and which can hardly be justified on optimizing grounds.

As recalled in Section 3, however, in the Fall 1872 a radical change of perspective was brought about by Walras's discovery that the principle of optimization can play a revolutionary role in economics. Such a finding, originally occurring in the field of utility maximization, consumer choice, and demand theory, produced its first fundamental effects on Walras's general equilibrium theory over the period 1872-1877, significantly transforming the exchange equilibrium model and also affecting, though less comprehensively, the production equilibrium model. The older equilibrium conceptions, inherited from past economists and schools of thought, were replaced, to a greater or lesser degree, by a new equilibrium conception, where an equilibrium state must be conceived as an array of optimally chosen and mutually compatible plans of actions. Yet, the ubiquitous persistence of preexisting ideas and constructs, often conflicting with the new equilibrium conception, hindered the development of a fully coherent new theoretical system, marring the models put forward in the mid-1870s with shortcomings and inconsistencies that Walras would strive to amend for the next quarter of a century.

The post-1872 exchange equilibrium models have been thoroughly examined in Sections 3 and 4 of the paper. In the model put forward in the mid-1870s, consumers-traders were viewed as utility maximizers, so that their demand and supply functions of consumer goods were obtained as solutions of constrained utility maximization problems. Yet, this novelty was surrounded by the remnants of past theories. In the first place, Walras did not give up his old two-step procedure, now completely irrelevant and misleading, according to which the equilibrium determination issue should at first be tackled by analyzing particular equilibria for market pairs, to be only subsequently generalized to the analysis of a multimarket economy. In the second place, and more significantly, in the model of the mid-1870s, as in the Geneva Notes, the equilibration process was still conceived as a process in 'real' time, based on actual transactions and also supposed, in J.S. Mill's wake, to drive the economy towards a stationary price equilibrium. Yet, the characteristics of such equilibration or tâtonnement process turn out to be incompatible with the distinctive features of the new equilibrium conception. Such inconsistency, not initially perceived by Walras, would only be acknowledged by him after Bertrand's 1883 criticism, what would eventually persuade Walras to publicly recognize, in 1885, that the only equi-

libration process consistent with the new equilibrium conception is a 'virtual' process, leading the economy to an 'instantaneous' equilibrium.

The post-1872 production equilibrium model has been examined in detail in Sections 5 and 6. In the model developed in the years 1876 and 1877, consumers, identified with the owners of the various types of capital, were viewed as utility maximizers, exactly as the consumers-traders of the post-1872 exchange model. Hence, their demands for consumer goods and supplies of services were obtained as solutions of constrained utility maximization problems. Yet, the optimizing assumption was not extended to the other category of agents participating in the economy, namely, entrepreneurs, who, as in the Geneva Notes, were still supposed to behave as quantity adaptors, in the tradition of J.S. Mill and the other economists of the English School.

Such asymmetrical assumptions concerning the respective behavior of consumers and entrepreneurs impinged upon the epistemological and analytical consistency of the post-1872 production model. Vaguely perceiving that some difficulties could possibly arise from such assumed asymmetry of behavioral rules, Walras quite soon made some efforts to provide the entrepreneurs with a supplementary role, more in line with the optimizing orientation of the his new approach: in fact, as early as in 1876, he started to suggest that entrepreneurs, on top of being regarded as quantity adaptors, should also be viewed as cost minimizers. Yet, for the following two decades, Walras proved wholly unable to pursue this idea any further. When, in the mid-1890s, in the wake of Pareto's and Barone's advances in the theory of production, he eventually went back to his early suggestion, he succeded in putting forward an embryonic version of marginal productivity theory, but was unable to integrate it into the main body of his theoretical system. And when, in 1900, Walras finally understood that the theory of marginal productivity could play a role in the field of production and supply similar to, and as important as, the role played by the theory of marginal utility in the field of consumption and demand, it was definitely too late for him to undertake such a demanding task as that of reconstructing from scratch the entire supply side of his general equilibrium theory.

Hence, in spite of Walras's several hints and many attempts, the different rules governing the behavior of the two types of agents populating the post-1872 production model persisted basically unaltered from 1876 up to the end of Walras's scientific life. Such twofold inspiration and allegiance undermined the foundations of the model, producing a number of negative theoretical consequences. One of the weakest features of the production model is represented by the tâtonnement process supposedly at work in the poduction economy. Here the difficulties due to the coexistence of different behavioral rules are magnified by the assumption, inherited from J.S. Mill and the economists of the English School, that the equilibration process should be regarded as an observable process, involving physical activities taking place in 'real' time. Over the years Walras tried to remedy the true and proper mistakes arising in this context by making artificial or self-contradictory assumptions, such as the assumed existence of an accommodating "foreign market", as imagined in his writings of 1876 and 1877, or as the assumed violation of the 'voluntary behavior' princi-

ple, as hypothesized in the 1889 and 1895 editions of the *Éléments*, or finally as the assumed identification of all types of capital with landed capital, as tacitly posited in all of his writings from 1876 to 1895. Yet, all such remedies proved insufficient to eradicate the underlying difficulties.

A radical solution to the persistent inconsistencies defacing the post-1872 production model over its entire life was found by Walras only at the end of his long journey, with the adoption, in 1899 and 1900, of the so-called "hypothèse des *bons*", by means of which he was eventually able to 'virtualize' the entire tâtonnement process, ruling out all out-of-equilibrium observable activities, including not only consumption and trading, but also production. By means of that "hypothèse", Walras was also able to get rid, once and for all, of that notion of stationary equilibrium that he had inherited from the J.S. Mill and other former economists, and simultaneously to generalize to his overall general equilibrium theory that notion of 'instantaneous' equilibrium that, since 1885, he had already employed with exclusive reference to the exchange model, eventually recognizing that such 'instantaneous' equilibrium notion is the only one that proves to be consistent with the assumption of optimizing behavior.

To sum up, in this paper we have shown that Walras's discovery, in the Fall 1872, of the paramount role to be potentially played by the assumption of optimizing behavior in the theory of economic equilibrium brought about a remarkable change in his theoretical outlook. At the same time, we have also shown that the limited adoption of the novel optimizing conception in the exchange and, especially, in the production model put forward by Walras in the period 1872-1877, when combined with the persistence in his system of thought of a number of his preexisting ideas, often hardly compatible with the new approach, produced a number of inconsistencies and left several unsettled questions to be taken care of. This, in the last analysis, explains Walras's incessant efforts to correct and perfect his theory over all his subsequent scientific life. As we have shown, over the years Walras was able to fill some gaps and to fix several mistakes surviving in his theory, a few of them at the very last moment. Yet, something important, from Walras's own viewpoint, remained beyond reach. An obvious example is the theory of marginal productivity, whose potential role was perceived by Walras, but left unexploited. In a letter written to Knut Wicksell on November 2, 1900 (Jaffé 1965, II, letter 1465, 129), Walras proved to be aware of this specific inadequacy, for, referring precisely to marginal productivity theory, he wrote:

> [...] n'ayant pas fait une étude approfondie de la question, je préfère me borner à l'indiquer, en la maintenant en dehors de ma théorie.

At the same time, he also proved to be not only fully aware of the origins, nature, and history of his own theory, but also quite confident about its perennial value:

> [Ma théorie] est celle du *Grenznutzen* poursuivie dans les derniers détails de l'équilibre économique. J'ai mis 40 ans à l'élaborer et je

la livre avec confiance à l'examen de la génération d'économistes à laquelle va incomber la tâche de constituer la science.

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