Optimal Capital Structure and Bankruptcy Choice: Dynamic Bargaining vs Liquidation*

Samuel Antill†   Steven R. Grenadier‡

June 14, 2018

Abstract

We model a firm's optimal capital structure decision in a framework in which it may later choose to enter either Chapter 11 reorganization or Chapter 7 liquidation. Creditors anticipate equityholders' ex-post reorganization incentives and price them into the ex-ante credit spreads. Using a realistic dynamic bargaining model of reorganization, we show that the off-equilibrium threat of costly renegotiation can lead to lower leverage, even with liquidation in equilibrium. If reorganization is less efficient than liquidation, the added option of reorganization can actually make equityholders worse off ex-ante, even when they liquidate on the equilibrium path.

Keywords: Capital Structure, Bankruptcy, Default, Dynamic Bargaining
JEL: C73, C78, G31, G33

1 Introduction

In 2016, the U.S. bankruptcy court system received nearly 38,000 commercial bankruptcy filings (American Bankruptcy Institute). For publicly traded firms, 80% of bankruptcies are handled under Chapter 11, while only 20% are Chapter 7 liquidations (Corbae and D'Erasmo (2017)). Given the significance of Chapter 11 filings among reasonably-sized firms, the possible contingency of a future reorganization must be priced into the debt issued by such firms. We propose a model in which equityholders can choose both their timing of default and the chapter of bankruptcy,[1] then examine how this flexibility alters capital structure decisions. In addition to providing novel mechanisms for explaining debt conservatism and the "credit spread puzzle," this model allows us to answer other important questions: how do the characteristics of the Chapter 11 process impact the capital structure of firms? Conversely, can the capital structure of firms impact their choice of bankruptcy chapter?

In this work, we develop and solve a realistic continuous-time dynamic bargaining model of Chapter 11. For tractability, we must make some simplifying assumptions, but we make every effort to ensure our model accords with the U.S. Bankruptcy Code and its implementation. We include many features of the Chapter 11 process, such as automatic stay, suspension of dividends, the exclusivity period, post-exclusivity proposals by creditors, forced conversion to Chapter 7, absolute priority rule (APR) in liquidation, and the unanimity rule (by creditor class) in approval of a reorganization plan. The reorganized firm may issue new debt and continue operating. Chapter 11 entails inefficiencies which are distinct from Chapter 7, such as professional fees and a decline in the cashflows that accumulate during reorganization. Moreover, both debtors and creditors face uncertainty over future asset values as they debate reorganization plans. In our model, creditors and equityholders are fully strategic in proposing and accepting plans, and we solve for a unique Markov perfect equilibrium outcome in closed form. This outcome turns out to be Pareto optimal, despite potential delays in agreement.

Using this equilibrium bargaining outcome, we extend the classic Leland (1994) model of endogenous default by allowing firms to choose between Chapter 7 or Chapter 11 when they default. As is standard in these models (see Strebulaev and Whited (2012) or Sundaresan (2013)), equityholders receive nothing in liquidation. It follows that at the moment of default, equityholders choose Chapter 11 if and only if the expected bargaining outcome exceeds the fixed cost they must pay to enter Chapter 11. Intuitively, in the subgame following debt issuance (ex-post), Chapter 11 is optimal for equityholders when the firm is sufficiently profitable at the moment of default. Taking into account the ex-post, strategic behavior of equityholders, the firm issues rationally priced debt at time zero (ex-ante) to exploit tax benefits.

The time zero capital structure decision depends on the relative efficiencies of Chapter 7 and Chapter 11 (traded off against the tax benefits of leverage). Specifically, depending on model parameters, there are three possible scenarios. The first two cases are rather stark and straight-

---

[1]Iverson (2017) reports that fewer than 2% of Chapter 11 filings are involuntary.

forward. In the first case, Chapter 11 is significantly more efficient than Chapter 7, so equityholders naturally find the former more attractive ex-post upon default. Thus, at time zero, debtholders demand a higher credit spread to compensate them for the rents equityholders extract in the event of a future reorganization. Notably, equityholders are willing to pay this higher spread since it is the rational expectation of their contingent Chapter 11 proceeds. The net effect is an increase in ex-ante firm value from the added option of a Chapter 11, due to the lower default costs of Chapter 11. Alternately, in the second case, Chapter 11 is extremely wasteful relative to Chapter 7. It follows that equityholders are not willing to incur the fixed cost of entering reorganization. Thus at time zero, equityholders optimally issue the same coupon as in the Leland (1994) model (which neglects Chapter 11), and ex-post liquidate at the same stopping time as well. In this case, the added option of Chapter 11 has no effect on ex-ante firm value.

The third case, in which Chapter 11 is slightly less efficient than Chapter 7, is the most interesting. In this case, if equityholders issue the optimal coupon from Leland (1994), they will ex-post find it optimal to enter Chapter 11. This is because large coupons imply the firm defaults in profitable states of the world, where the prospects of Chapter 11 for equityholders justify the fixed cost of entry. Debtholders thus demand a higher credit spread at time zero for such a coupon. Equityholders are hesitant to pay this spread since reorganization destroys more value than Chapter 7. For such parameters, equityholders have two choices. They can issue a large coupon to reap tax benefits, and accept that they will pay for the ex-post Chapter 11 inefficiencies with a higher credit spread at time zero. We call this the "optimal inefficient Chapter 11" strategy. Alternately, equityholders can issue a smaller coupon consistent with ex-post optimal Chapter 7. In this case they sacrifice the tax benefits of a larger coupon, but they enjoy a lower cost of debt due to the rational expectation of a future, more efficient liquidation. We call this the "constrained debt Chapter 7" strategy. Counterintuitively, regardless of which of these two strategies is optimal, the added option of Chapter 11 actually reduces ex-ante firm value.

Graham (2000) points out that "paradoxically, large, liquid, profitable firms with low expected distress costs use debt conservatively" and "the typical firm could double tax benefits by issuing debt until the marginal tax benefit begins to decline." Many existing dynamic models can produce low leverage (for example, Hennessy and Whited (2005); DeAngelo, DeAngelo, and Whited (2011); Strebulaev (2007) and others). However, our model generates a new mechanism for explaining low leverage, even when the firm's environment on the equilibrium path is identical to Leland (1994). If the relative inefficiencies of Chapter 11 are large compared to the tax benefits of debt, equityholders optimally use the "constrained debt Chapter 7" strategy and issue a modest coupon. For such a coupon, they will find Chapter 7 optimal ex-post, which lowers the cost of debt ex-ante. Since this entails forgoing tax benefits, equityholders issue the largest coupon consistent with future Chapter 7. In this case, for reasonable parameters, our model predicts a leverage ratio of 40%. For the same parameters, the Leland (1994) model predicts a 70% leverage ratio. To an econometrician, our model looks identical to the Leland model for such parameters: a firm issues debt then eventually liquidates. However, the off-equilibrium considerations introduced by our bargaining model lead

the firm to issue a much smaller coupon than in the standard Leland model. In this case, our model predicts lower leverage than the Leland model, even for the 65% of (public and private) firms that liquidate in Chapter 7 (Bernstein, Colonnelli, and Iverson (2017), henceforth BCI).

Endogenous default models of capital structure tend to underestimate credit spreads (Huang and Huang (2012)). Under the "optimal inefficient Chapter 11" strategy, our model suggests that high credit spreads could be due to the rational expectation of future rents extracted by equityholders in Chapter 11. For these parameter ranges, firms are unwilling to sacrifice tax benefits to get a lower cost of debt from issuing a low coupon consistent with ex-post Chapter 7. Instead, they accept the higher cost of debt and issue a large coupon for the tax shield. Since the higher default costs are internalized by equityholders when they issue debt, the overall coupon is still lower than in the Leland model. For reasonable parameter values, credit spreads can be 17 basis points higher than in the Leland model, even with an optimal leverage ratio that is 7 percentage points lower. While many other models can produce higher credit spreads than Leland (1994), ours does so simply by adding a realistic choice of bankruptcy chapter.

Finally, our model generates many other testable implications about the relationship between Chapter 11 and capital structure. Consider the following list. Creditor rights might be interpreted as the relative bargaining power of creditors in bankruptcy. Under this interpretation, stronger creditor rights lead to higher optimal leverage and firm value, consistent with empirical evidence. Firms with higher growth rates should be more likely to choose Chapter 11, so comparing Chapter 11 and Chapter 7 by the value of assets at the end of bankruptcy might overstate the efficiency of Chapter 11. Firms with more volatile cashflows or lower growth rates should have longer bankruptcies. Firms that choose Chapter 11 should have both more valuable assets and higher leverage ratios at the time of default than firms which choose Chapter 7. When the "constrained debt Chapter 7" strategy is optimal, anything that makes Chapter 11 less appealing (for example, higher legal costs) will actually improve firm value. Changes in parameter values can have surprising comparative statics when they cause the firm to shift from Chapter 11 to Chapter 7 or vice versa.

Our paper contributes to the literature on dynamic contingent claims models of capital structure. Relative to Leland (1994), we contribute by allowing equityholders to choose to file for Chapter 11 bankruptcy or Chapter 7 liquidation. Papers such as Fan and Sundaresan (2000) have extended the Leland (1994) framework to allow for costless renegotiation in private workouts.[2] Articles such as Hackbarth, Hennessy, and Leland (2007) and Hackbarth and Mauer (2012) study the optimal mixture of bank and public debt, where bank debt may be renegotiated in a private workout. These works document important links between private workouts and optimal capital structure decisions, but the details of the Chapter 11 procedure are not modeled.

François and Morellec (2004) and Broadie, Chernov, and Sundaresan (2007) are more similar to

---

[2]See also Anderson and Sundaresan (1996), Mella-Barral and Perraudin (1997). More recent papers which jointly consider renegotiation and investment include Sundaresan and Wang (2007), Shibata and Nishihara (2015). Christensen et al (2014) have a similar model with dynamic capital structure. Earlier structural models of debt pricing include Merton (1974), Brennan and Schwartz (1984), Kane et al (1984), Black and Cox (1976), and Fisher et al (1989). More recent models of dynamic capital structure with endogenous default (solely through Chapter 7) include Leland (1998), Goldstein et. al (2001), Titman and Tsyplakov (2007), Strebulaev (2007).

our work. François and Morellec (2004) extend the model of Fan and Sundaresan (2000) to study the Chapter 11 bankruptcy procedure. In their model, equityholders choose a threshold at which to enter Chapter 11. While the asset value is below this threshold, equityholders and debtholders split the cashflow according to Nash bargaining. If asset values do not rise back above the same threshold before an exogenous window of time expires, then the firm liquidates. Moraux (2002) and Galai et al (2007) use a similar formulation. Broadie, Chernov, and Sundaresan (2007) numerically extend this by keeping track of accumulated earnings and accumulated arrears during the bankruptcy. In their framework, the firm emerges from bankruptcy when accumulated earnings are sufficient to pay off the accumulated arrears, where an exogenous fraction of the debt is forgiven. They study equity and debt values when creditors pick the bankruptcy threshold compared to the same values when equityholders choose the thresholds. Both François and Morellec (2004) and Broadie, Chernov, and Sundaresan (2007) also consider a time zero capital structure decision. These papers capture the impact of bankruptcy procedure on time zero capital structure, but only allow for liquidation after the firm has already entered Chapter 11. In reality, the majority of firms go straight to Chapter 7 without ever entering Chapter 11 (BCI (2017)). By allowing equityholders to choose either Chapter 7 or Chapter 11, our model produces implications for the choice of bankruptcy procedure. This also has important implications for the time zero capital structure decision which are impossible to produce in either of the previously mentioned models. To our knowledge, the only models which allow firms to enter Chapter 7 or Chapter 11 are Bernardo, Schwartz, and Welch (2016) and Corbae and D'Erasmo (2017). Both models are extremely different from ours (for example, bankruptcies always last one period and all debt matures in one period), so our analysis complements theirs while providing novel insights.

A novel methodological contribution of our paper relative to all previous work is our bargaining model of Chapter 11. We use a new continuous-time formulation of the stochastic bargaining model from Merlo and Wilson (1995). This captures two important features of the bankruptcy process. First, all impaired classes of creditors (including equity) must unanimously agree to a reorganization plan to exit bankruptcy (see Section 3.1). The models mentioned previously assume that equity or debt unilaterally decide the timing of the exit. Second, Chapter 11 bankruptcies can take as long as 10 years, and all parties face significant uncertainty over how the value of the firm's assets will change in this time. The previously mentioned models, such as Corbae and D'Erasmo (2017), assume that the split between equity and debt is either determined exogenously or by Nash bargaining at the moment of entering Chapter 11. Our stochastic bargaining framework allows parties to change their bargaining strategies as they observe the resolution of uncertainty.[3]

---

[3]A few earlier papers have bargaining models of Chapter 11 that are more strategic than Nash bargaining. For example, Paseka (2003) considers a dynamic bargaining game in Chapter 11, but only equityholders can make take it or leave it offers, there is no accumulation of cash flows, and the firm cannot relever after Chapter 11. Eraslan (2008) structurally estimates a dynamic but deterministic bargaining model of Chapter 11 similar to Rubinstein (1982), and Annabi et al (2012) numerically solve a bargaining model of Chapter 11 with exogenously many rounds of fixed exogenous length. Earlier theoretical papers studying Chapter 11 include Franks and Torous (1989) and Longstaff (1990) who model Chapter 11 as a right to extend the maturity date of debt. Lambrecht and Perraudin (1996) study a creditor race in bankruptcy where multiple creditors might preempt one another in seizing assets (Bruche (2011) examines a similar setup). Importantly, none of these consider the decision of whether to enter Chapter 7 or Chapter

Our stochastic bargaining model produces results which cannot be replicated by a standard Nash bargaining model of Chapter 11. For example, Chapter 11 outcomes like APR violations or post-reorganization capital structure and firm value are unpredictable until the moment of reorganization. Additionally, we find that equity's expected bargaining outcome is increasing in the size of the firm, consistent with empirical evidence. This relationship is used by Garlappi, Shu, and Yan (2008) and Garlappi and Yan (2011) to explain a number of empirical facts about distress and expected equity returns, even though their models use Nash bargaining which does not endogenously produce this relationship. Our stochastic bargaining model thus provides a theoretical foundation for proxying equity bargaining power with firm size. The realistic nature of the bargaining could help improve the already strong fit of other structural contingent claims models with renegotiation (for example, Favara, Morellec, Schroth, and Valta (2017) or Morellec, Nikolov, and Schuerhoff (2018)).

The article proceeds as follows. Section 2 describes the model and reviews the Leland (1994) setup. Section 3 provides institutional details then describes and solves for the equilibrium of the Chapter 11 reorganization timing game. Section 4 derives the optimal decision to enter Chapter 11 or Chapter 7. Section 5 describes the time zero capital structure choice, provides our results on capital structure, and gives additional empirical implications. Section 6 discusses possible generalizations of the model and concludes.

2 Model setup

In this section we begin by describing the setup of our model of the leveraged firm. In particular, we focus on the timing of the decision to enter Chapter 11 reorganization or Chapter 7 liquidation, the continuous-time bargaining game that occurs during reorganization, and the endogenous emergence from reorganization. This model is solved in Sections 3 and 4, working backwards through time. As a useful benchmark, and for our own model prior to and following reorganization, in Section 2.2 we provide a brief derivation of the standard solution where Chapter 11 reorganization is not considered.
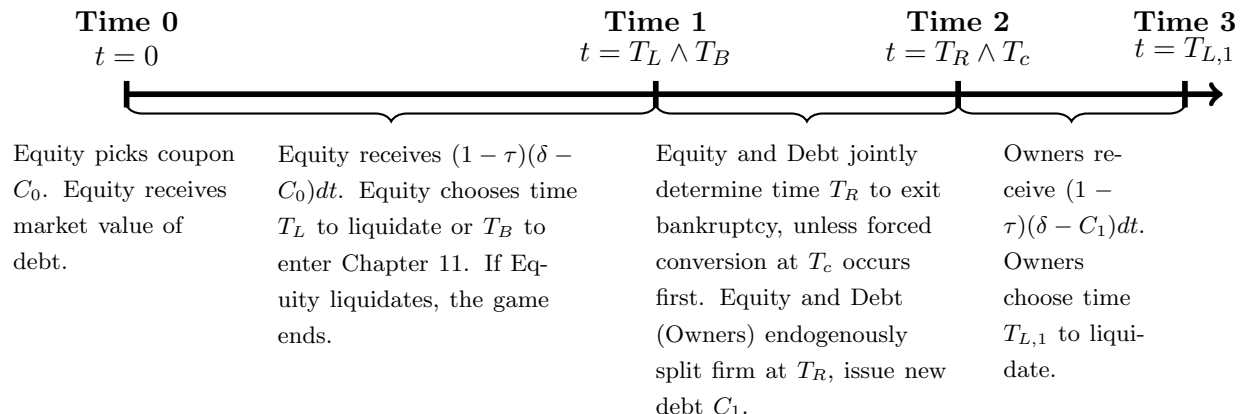
2.1 Outline and assumptions of the baseline model

We consider a continuous time infinite horizon model of a firm, whose manager maximizes shareholder value. At all times at which the firm is operating, its assets in place produce earnings before interest and taxes (EBIT) δ_t. We assume the existence of a risk neutral measure with risk free rate r under which δ_t follows a geometric Brownian motion

$$d\delta_t = \delta_t \mu dt + \delta_t \sigma dB_t, \tag{1}$$

11, and none of these papers have any capital structure decision.

Figure 1: **Timeline of the Model**

Time 0 $t=0$	**Time 1** $t = T_L \wedge T_B$	**Time 2** $t = T_R \wedge T_c$	**Time 3** $t = T_{L,1}$
Equity picks coupon C_0. Equity receives market value of debt.	Equity receives $(1-\tau)(\delta - C_0)dt$. Equity chooses time T_L to liquidate or T_B to enter Chapter 11. If Equity liquidates, the game ends.	Equity and Debt jointly determine time T_R to exit bankruptcy, unless forced conversion at T_c occurs first. Equity and Debt (Owners) endogenously split firm at T_R, issue new debt C_1.	Owners receive $(1-\tau)(\delta - C_1)dt$. Owners choose time $T_{L,1}$ to liquidate.

where $\mu < r$ and $\sigma > 0$ are constants representing the risk neutral growth rate and volatility of δ_t, respectively, and B_t is a Brownian motion under the risk neutral measure. The cashflow δ_t is subject to effective corporate tax rate τ.

The model comprises a sequence of optimizations which are separated by four distinct times, $\mathbb{T} = 0, 1, 2, 3$. Figure 1 presents a graphical timeline of the model. Time 0 represents the initial determination of the capital structure of the firm. Specifically, at Time 0, the firm can issue consol debt with total perpetual coupon C_0. Debt entails a tax shield, and we follow the literature in assuming a full-loss offset provision, so the firm subsequently pays taxes $\tau(\delta_t - C_0)dt$ per unit time. The firm chooses C_0 to maximize firm value, the determination of which will depend on expectations of future strategic decisions.

Once the firm has issued debt with coupon C_0, they progress to the period between Time 0 and Time 1. During this period, the firm is operating and equityholders receive after-tax cashflow $(1-\tau)(\delta_t - C_0)dt$ per unit time. Equityholders choose Time 1, the moment of default and the end of the period, and this can be either of two potential stopping times that the owners must contemplate. One is the standard liquidation decision that is common in the literature. At any stopping time T_L, the equityholders may choose to liquidate, at which point equityholders receive 0, and debtholders receive the liquidation value

$$\zeta \delta_{T_L} = \frac{(1-\tau)(1-\alpha)\delta_{T_L}}{r - \mu}. \tag{2}$$

The liquidation value $\zeta \delta_{T_L}$ is the expected discounted value of receiving $(1-\tau)\delta_t$ in perpetuity given the current value of δ_{T_L}, multiplied by a constant $(1-\alpha)$. As is standard in the literature, we assume a fraction $\alpha \in (0,1)$ of the firm value is lost in liquidation. If the firm chooses to liquidate, the game ends.

Our novel contribution is the additional option for the firm to enter a Chapter 11 reorganization. At any time T_B, the firm may declare bankruptcy and enter into a Chapter 11 reorganization. In

this case, they pay a fixed cost $B > 0$ and enter the next period.

If the firm enters Chapter 11, then the period between Time 1 and Time 2 represents the time spent in Chapter 11 reorganization. In this period, debtholders and equityholders play a continuous-time bargaining game to determine when to emerge from bankruptcy. During the Chapter 11 process, which Section 3.1 describes in greater detail, the automatic stay provision prevents creditors from demanding payments. At the same time, dividend suspension prevents debtors (equityholders) from paying themselves dividends. We assume that the firm continues to receive cashflows $(1 - \tau)h\delta dt$ per unit time, where $h \in [0, 1]$ is a multiplier representing the inefficiency of operations during bankruptcy, and these cashflows accumulate. At any stopping time T_R, the debtholders and equityholders may agree to a reorganization plan. At this time, the equityholders and debtholders split the firm value $V(\delta_{T_R})$ minus a fixed reorganization cost $R_0 > 0$. They also receive the accumulated earnings, for a total payment of

$$P_{T_R} = V(\delta_{T_R}) - R_0 + (1 - \tau)h \int_{T_B}^{T_R} \delta_s ds. \tag{3}$$

In Broadie, Chernov, and Sundaresan (2007), the authors provide a model of Chapter 11 that assumes equityholders receive the residual firm value after paying arrears at a time chosen by equityholders. We depart from this by modeling the reorganization as a bargaining process. Consistent with the laws of Chapter 11, equityholders and debtholders alternate filing plans for how to split the total P_t, and the process ends at the first time T_R when one party makes a proposal the other party accepts.

The period ends at Time 2 by either the debtholders and equityholders agreeing to a reorganization plan at stopping time T_R, or by a judge-mandated liquidation. With probability ιdt per unit time, the judge converts the Chapter 11 reorganization into a Chapter 7 liquidation. In the event of liquidation, debtholders receive the liquidation value $\zeta\delta$ plus the accumulated earnings net of fees, equityholders receive nothing consistent with APR, and the game ends. While this occurs exogenously, agents anticipate the possibility of liquidation and may endogenously increase the likelihood of liquidation by stalling. If equityholders and debtholders agree to a reorganization prior to liquidation, the game proceeds to the final period.

If the firm reorganizes, the period between Time 2 and Time 3 represents the operation of the reorganized firm. Just as at Time 0, the new equityholders of the reorganized firm issue new debt C_1 to maximize firm value at Time 2. For the remainder of the period, equityholders receive a payment $(1 - \tau)(\delta_t - C_1)dt$ per unit time. For simplicity, we assume that the option to reorganize no longer exists and the firm may only exit through liquidation (Section 6.1 discusses relaxing this assumption). Thus, at any stopping time $T_{L,1}$, equityholders may choose to liquidate the firm. As described previously, at the time of liquidation (Time 3) equityholders receive 0, the new debtholders receive $\zeta\delta_{T_{L,1}}$, and the game ends.

2.2 Benchmark model: The Leland model with only Chapter 7 liquidation

In the standard Leland model, a levered firm with coupon C chooses a liquidation time T_L to maximize equity value:

$$E^L(\delta) = \sup_{T_L \in F^\delta} \mathbb{E}^\delta \left[\int_0^{T_L} e^{-rt}(1-\tau)(\delta_t - C)dt \right], \tag{4}$$

where throughout the paper, $\mathbb{E}^\delta$ represents expectation with respect to the probability law of the process δ_t starting at $\delta_0 = \delta$. We require that T_L is a stopping time with respect to the filtration F^δ generated by δ_t. It is worth noting that there could be a time $t < T_L$ such that the cashflow to equity is negative. Consistent with the prior literature, we assume in this case that equityholders issue new shares and dilute their equity in order to pay the coupon to debtholders. For the optimal T_L, the value of equity will always be positive for $t < T_L$, consistent with limited liability, so such dilution is possible.

In the region where liquidation is not optimal, standard dynamic programming arguments show the value of equity $E^L(\delta)$ satisfies the following ordinary differential equation (ODE):

$$rE^L(\delta) = \mathcal{D}E^L(\delta) + (1-\tau)(\delta - C), \tag{5}$$

where, defining "smooth" to mean continuously differentiable and twice continuously differentiable almost everywhere, $\mathcal{D}$ is the differential operator from Ito's lemma for smooth functions of δ_t:

$$\mathcal{D}f(\delta) = f'(\delta)\mu\delta + f''(\delta)\frac{\sigma^2}{2}\delta^2. \tag{6}$$

As $\delta \to \infty$, the value of the option to liquidate should become worthless. This implies the value of equity $E^L(\delta)$ should approach the value of receiving the after-tax cashflows less the debt payments in perpetuity, which is $(1-\tau)\left[\delta/(r-\mu) - C/r\right]$. Imposing this condition, the relevant solution of (5) is

$$E^L(\delta) = A_1 \delta^\psi + (1-\tau)\left[\frac{\delta}{r-\mu} - \frac{C}{r}\right],$$

where A_1 is an arbitrary constant and ψ is the negative root of the characteristic polynomial

$$r - \mu z - \frac{\sigma^2}{2}z(z-1) = 0.$$

It can be verified that the optimal liquidation time T_L is a hitting time $T_L = \inf\{t : \delta_t \leq \delta_L\}$ for some barrier δ_L. The constant A_1 and the liquidation threshold δ_L are determined by value matching and smooth pasting at δ_L. Since equity receives nothing in liquidation, the value matching and smooth pasting conditions are

$$A_1 \delta_L^\psi + (1 - \tau)[\frac{\delta_L}{r - \mu} - \frac{C}{r}] = 0 \tag{7}$$

$$A_1 \psi \delta_L^{\psi-1} + \frac{(1 - \tau)}{r - \mu} = 0. \tag{8}$$

This system of equations has the usual unique solution

$$\delta_L = \frac{\psi}{\psi - 1} \frac{r - \mu}{r} C \tag{9}$$

$$A_1 = \delta_L^{-\psi}(1 - \tau)[\frac{C}{r} - \frac{\delta_L}{r - \mu}]. \tag{10}$$

Taking the liquidation threshold δ_L as given, the value of the debt $D^L(\delta)$ satisfies an ODE similar to (5) prior to liquidation:

$$r D^L(\delta) = \mathcal{D} D^L(\delta) + C, \tag{11}$$

and similar logic shows the relevant solution of this ODE is

$$D^L(\delta) = \frac{C}{r} + A_2 \delta^\psi \tag{12}$$

for an arbitrary constant A_2. As discussed in the previous section, at the time of liquidation T_L a fraction of firm value α is lost, leaving value $\zeta \delta_L$ for the debtholders. Imposing that $D^L(\delta_L) = \zeta \delta_L$ uniquely determines the constant A_2, which gives the rational expectations value of consol debt with coupon C:

$$D^L(\delta) = \frac{C}{r} + \delta^\psi \delta_L^{-\psi}[-\frac{C}{r} + \zeta \delta_L]. \tag{13}$$

The standard Leland model features a time zero capital structure decision. Specifically, at time 0 equityholders choose the coupon C for their consol debt to maximize the total firm value $E^L(\delta_0) + D^L(\delta_0)$. As in the standard tradeoff theory, the firm weighs the tax benefits of debt with the loss of firm value in liquidation. For any arbitrary δ_0, we can plug in (9) for δ_L and the resulting expression for $E^L(\delta_0) + D^L(\delta_0)$ is concave in C. Solving the first order condition gives the unique optimal C^* :

$$C^* = \delta_0 \frac{r}{r - \mu} \frac{\psi - 1}{\psi} [\frac{-\tau}{\psi(1 - \tau)\alpha + (\psi - 1)\tau}]^{\frac{-1}{\psi}}. \tag{14}$$

Note that the optimal coupon C^* is linear in the starting cashflow δ_0. Since the liquidation barrier δ_L is linear in C, it can be seen from equations (9, 10, 13, 14) that at the optimal coupon,

$$E^L(\delta_0) + D^L(\delta_0) = \theta \delta_0 \tag{15}$$

for a constant θ that is a known function of the model primitives given in Appendix A.

3 Chapter 11 as a stochastic bargaining game

Recall that our model is divided by four distinct times. Since we rule out a second reorganization, in the period between Time 2 and Time 3 the equityholders solve the standard liquidation problem described in Section 2.2. Likewise, at Time 2 they issue an optimal level of debt as described above. In this section, we describe and solve the period between Time 1 and Time 2, the Chapter 11 process.

We first discuss some features of the Chapter 11 process that are important for our model in Section 3.1. We then set up our model of bargaining in reorganization in Section 3.2. Notably, the payoff is simplified by our Leland model benchmark from above. In Section 3.3, we determine the optimal timing of reorganization, which coincides with the equilibrium timing due to a Pareto-optimality result. Then, conditional on this optimal timing, we solve for the equilibrium bargaining split in Section 3.4. Section 3.5 discusses implications of our dynamic bargaining model, and Section 3.6 compares our dynamic bargaining model to Nash bargaining.

3.1 Relevant features of Chapter 11

In order to be tractable, our model makes some simplifying assumptions regarding the Chapter 11 process. However, we make every attempt to ensure that our model is broadly consistent with some of the most salient features of the actual Bankruptcy Code. In this section we summarize some of the most important aspects of the Chapter 11 process that inform much of our modeling assumptions. A comprehensive description of Chapter 11 is far beyond the scope of this paper.

First, over 98% of Chapter 11 cases begin with a voluntary filing (Iverson (2017)). In a voluntary filing, the debtor (management on behalf of equityholders) chooses the bankruptcy chapter. In some cases, creditors can file for an involuntary bankruptcy in their chosen chapter under 11 USC § 303. However, courts only enforce a controverted filing if "the debtor is generally not paying such debtors debts as such debts become due," and in this case the debtor still "may file an answer to a petition under this section" to choose the chapter (11 USC § 303(d,h)).

Second, the automatic stay provision of Chapter 11 (11 USC § 362) prohibits all entities from "any act to obtain possession of property of the estate." In particular, debtholders stop receiving coupons and equityholders stop receiving dividends.

Third, in order to confirm a reorganization plan and exit Chapter 11, every impaired class of creditors must accept the plan by 11 USC § 1129(a) (we ignore 11 USC § 1129(b) cram downs). This gives equityholders, who constitute a class of claims, some power to hold up the reorganization process and potentially extract rents. The APR refers to the idea, in Chapter 7 or 11, that each creditor should only be compensated once all senior creditors are paid in full. However, equityholders are often able to use their bargaining power to violate this in Chapter 11. Bris, Welch, and Zhu (BWZ, 2006) find in their sample that APR is always followed in Chapter 7, while

it is violated in 12% of Chapter 11 cases. Weiss (1990) finds violations in 29 of the 37 Chapter 11 cases he studies.

Fourth, at the start of the Chapter 11 process, equityholders enjoy an "exclusivity period." Specifically, the debtor-in-possession (DIP) enjoys the exclusive right to propose reorganization plans for 120 days under 11 USC § 1121(a). Small businesses have 180 days. The debtors then have another 60 days to get the plan approved by creditors. After this window, any party in interest may file a plan. Under 11 USC § 1121(d), the court may reduce or increase this window. Since this is at the judge's discretion, both equityholders and creditors face uncertainty as to the length of the exclusivity period, although it cannot exceed 18 months (20 months for small businesses).

Fifth, it is common for bankruptcy cases which begin as Chapter 11 reorganizations to be converted to Chapter 7 liquidations. In the sample analyzed in BWZ (2006), 14% of the cases which began in Chapter 11 were converted, while as many as 40% of cases were converted in the sample of BCI (2017). While debtors may in principle choose to convert to Chapter 7, and creditors may petition for such a conversion, the ultimate decision lies with the judge. This suggests that modelling the conversion as exogenous and random is a reasonable approximation of reality. Indeed, BCI (2017) use the random assignment of judges to bankruptcy cases as exogenous variation in the probability of conversion:

> U.S. bankruptcy courts use a blind rotation system to assign cases to judges, effectively randomizing filers to judges within each court division. While there are uniform criteria by which a judge may convert a case from Chapter 11 to Chapter 7, there is significant variation in the interpretation of these criteria across judges.

Finally, Chapter 11 entails significant costs, some of which are fixed and invariant to the size of the firm or length of the bankruptcy. Further, some nontrivial amount of these costs are borne by the equityholders and may not be reimbursed from the estate. We discuss this in greater detail in Appendix E.

3.2 The dynamic reorganization game

At Time 1, the firm enters Chapter 11, and the period ends at Time 2 with a reorganization or a forced liquidation. Based on the analysis of Section 2.2, the total firm value available to be split among debtholders and equityholders if a reorganization occurs at time T_R is $\theta \delta_{T_R}$. This expression takes into account the value of the debt the new equityholders will issue.

As discussed previously, there are no payments during the Chapter 11 process. We assume that after-tax earnings accumulate into an account, and that the accumulated earnings $\int_{T_B}^{T_R} (1-\tau) h \delta_t dt$ are split among equityholders and debtholders. We allow for the possibility that the firm operates less efficiently during bankruptcy by including a multiplier $h \in [0,1]$, so $h < 1$ implies a haircut. This also can include a flow of professional fees.

We assume that with probability ιdt per unit time, exogenous to the decisions of any agent, the bankruptcy case is converted and the firm is liquidated. The parameter ι can be positive or

zero. While debtors may in principle choose to convert to Chapter 7, and creditors may petition for such a conversion, the ultimate decision lies with the judge. If a conversion occurs at time T_c, we assume APR is upheld so equityholders receive 0 and debtholders receive the liquidation value $\zeta \delta_{T_c}$ plus the accumulated earnings net of fees described below. While this occurs exogenously, agents anticipate the possibility of liquidation and may endogenously increase the likelihood of liquidation by stalling. However the firm emerges from bankruptcy, they must pay a fixed cost $R_0 > 0$ where R_0 is a parameter. This represents the costs discussed in detail in Appendix E.

In summary, if the reorganization occurs at a time $T_R < T_c$, then equityholders and debtholders split P_{T_R}, where P_t is as defined in (3):

$$P_t = \theta \delta_t - R_0 + (1 - \tau) \int_{T_B}^t h \delta_s ds.$$

The accumulated earnings complicate the problem, since now the current value δ_t is not sufficient to determine the potential payoff. To handle this, we introduce a second state variable R_t which measures the fixed cost of emerging net of the accumulated earnings:

$$R_t = R_0 - (1 - \tau) \int_{T_B}^t h \delta_s ds. \tag{16}$$

Introducing this "net exercise price" allows us to write the reorganization payoff as $P_t = \theta \delta_t - R_t$ and the liquidation payoff as $\zeta \delta_t - R_t$.

One of our primary contributions relative to the literature is modelling Chapter 11 reorganization as a bargaining process. As discussed in Section 3.1, both the debtors and creditors have opportunities to propose reorganization plans, and approval must be unanimous. Further, the bargaining process is inherently dynamic. The average Chapter 11 case lasts two and a half years (BWZ (2006)), so it is inevitable that the value of underlying assets fluctuates stochastically during this period. In light of this, we model the Chapter 11 process as a dynamic, stochastic bargaining game between debtholders and equityholders. Section 3.6 discusses the benefits of this dynamic bargaining model relative a static model like Nash bargaining.

The bargaining procedure is the continuous-time equivalent of the bargaining game in Merlo and Wilson (1995, 1998). The two players bargain over a time T_R to emerge from bankruptcy, which must be agreed upon unanimously, and a split of the firm value $\theta \delta_{T_R} - R_{T_R}$. If a forced conversion occurs, the game ends and debtholders receive the entire liquidation payoff $\zeta \delta_{T_c} - R_{T_c}$. At any moment in the game, exactly one player (equity or debt) is the proposer. The proposer may make offers to the other player in any second, and the receiving player instantaneously decides to accept or reject their proposed share of the payoff. The game ends when a proposed split is accepted. The proposer in any instant is given exogenously by a time-homogeneous Markov chain s_t taking values in two states which we label $\{e, d\}$. When $s_t = e$, equityholders get to propose splits, and when $s_t = d$, debtholders get to propose splits. For simplicity, we assume the Markov chain has constant transition intensities, so the probability of transitioning from state i to state j per unit time is $\lambda_i dt$, $i = e, d$.

The stochastic proposer bargaining protocol is standard in the literature (see Merlo and Wilson (1995, 1998); Baron and Ferejohn (1989); Yildiz (2003); Hart and Mas-Colell (1996); Rubinstein and Wolinsky (1985); Binmore and Dasgupta (1987)). The rates of transitions are a tractable representation of bargaining power. In this setting, equityholders have a strong bargaining position if λ_e, the rate of transition away from state e, is low, and if the rate of transition λ_d into state e is high. Likewise, equityholders have a weak bargaining position if s_t leaves state e quickly and transitions into state e infrequently. Virtually all bargaining models (including continuous time models like Perry and Reny (1993) and Admati and Perry (1987)) assume there is some discrete length of time during which one player cannot make offers. In our model, that length is stochastic, but for any fixed dt there exist transition probabilities such that all players have the chance to make offers within the interval $[t, t + dt]$ with arbitrarily high probability. Merlo (1997) uses a structural estimation to show the stochastic proposer model fits empirical data on government negotiations well.

The main advantage of the stochastic proposer model is that it facilitates the analysis of time-homogeneous strategies and equilibria. However, giving equityholders a window of exogenously stochastic length during which they have the exclusive right to propose splits is actually a highly realistic model of the exclusivity period. After the exclusivity period, creditors may file a competing plan, and equityholders may file additional plans. If the reader would prefer a model in which equityholders and debtholders may both make offers in any instant, letting λ_e, λ_d approach infinity accomplishes this.

Given this bargaining protocol and the model primitives, equityholders (player e) and debtholders (player d) form strategies. We will focus on equilibria in stationary strategies that only depend on the current state (δ, R, s). A stationary strategy for player i consists of

1. A region $O_i \subset \mathbb{R}^2$ of (δ, R) values for which they make an offer when they are the proposer.

2. An offer function $\omega_i : O_i \to \mathbb{R}$ such that they offer $\omega_i(\delta_t, R_t)$ to player j when $(\delta_t, R_t) \in O_i$.

3. A correspondence $A_i : \mathbb{R}^2 \to \mathbb{R}$ mapping current (δ, R) values to the set of offers that they will accept when they are the receiver.

Stationary strategies allow for a great deal of flexibility. Each player chooses a triple of infinite dimensional objects. However, restricting attention to stationary strategies does rule out some possibilities. For example, players may not condition their actions on previous offers. They also may not make decisions as explicit functions of the elapsed time since the start of the bargaining, so without loss of generality we may take the starting time as $t = 0$ rather than $t = T_B$.

The benefit of focusing on stationary strategies is that they clearly induce outcomes. If we fix a stationary strategy (O_i, A_i, ω_i) for each player, we can define

$$\mathcal{T}_i = \inf\{t : s_t = i, (\delta_t, R_t) \in O_i, \omega_i(\delta_t, R_t) \in A_j(\delta_t, R_t)\}$$

as the first time that player i is proposer and the value of (δ_t, R_t) is such that player i makes a proposal which is accepted by player j. It follows that $\mathcal{T} = \mathcal{T}_e \wedge \mathcal{T}_d$ is the time at which the game

ends (unless liquidation occurs first), according to the fixed strategies. When the game ends in reorganization, the payoff to player i depends on whose proposal is accepted. It will be convenient to define the terminal payoff for player i, given fixed strategies, as

$$J_i(\delta, R, s) = \mathbf{1}(s = i)[\theta\delta - R - \omega_i(\delta, R)] + \mathbf{1}(s = j)\omega_j(\delta, R).$$

Intuitively, $J_i(\delta, R, s)$ equals the offer which player j makes to player i if $s = j$, while if the game ends with a proposal from player i then it equals the stochastic payoff minus the offer made by player i. Finally, given these definitions of $\mathcal{T}, J_i$, we can define the outcome induced by the fixed strategies. The expected payoff to equityholders, conditional on a starting state (δ, R, s) and following the fixed stationary strategies, can be written as

$$E(\delta, R, s) = \mathbb{E}^{(\delta,R,s)}[\mathbf{1}(\mathcal{T} < T_c)e^{-r\mathcal{T}}J_e(\delta_{\mathcal{T}}, R_{\mathcal{T}}, s_{\mathcal{T}})], \tag{17}$$

while the expected payoff to creditors is

$$D(\delta, R, s) = \mathbb{E}^{(\delta,R,s)}[\mathbf{1}(\mathcal{T} < T_c)e^{-r\mathcal{T}}J_d(\delta_{\mathcal{T}}, R_{\mathcal{T}}, s_{\mathcal{T}}) + \mathbf{1}(\mathcal{T} \geq T_c)e^{-rT_c}(\zeta\delta_{T_c} - R_{T_c})]. \tag{18}$$

The expected payoffs take into account the possibility of a forced conversion, in which case equityholders receive 0 and debtholders receive $\zeta\delta_{T_c} - R_{T_c}$. Given the expected payoffs $E(\delta, R, s), D(\delta, R, s)$ induced by stationary strategies, we can define our equilibrium concept.

Definition: A Markov Perfect Equilibrium (MPE) consists of a stationary strategy (O_i, A_i, ω_i) for each player such that

1. Taking the opponents' strategies as given, for every (δ, R, s), player e's strategy maximizes $E(\delta, R, s)$ and player d's strategy maximizes $D(\delta, R, s)$.

2. Player e finds it optimal to set an acceptance policy $A_e(\delta, R) = [E(\delta, R, d), \infty)$ and player d finds it optimal to set an acceptance policy $A_d(\delta, R) = [D(\delta, R, e), \infty)$.

Our definition of a MPE is highly intuitive. Condition 1 ensures that the equilibrium strategies correspond to a Nash equilibrium in stationary strategies for any starting values. Condition 2 is our notion of subgame perfection in continuous time: players must optimally accept offers if and only if the offer exceeds their continuation value in the equilibrium.

The value functions $E(\delta, R, s), D(\delta, R, s)$ corresponding to a MPE solve a fixed point problem. Given the strategies, the expected equilibrium payoffs are $E(\delta, R, s), D(\delta, R, s)$, and given the opponent's strategy, each player finds it optimal to set an acceptance cutoff equal to their value function. Nonetheless, the fixed point problem simplifies the calculation of such equilibria, since now we only need to search for value functions, offer regions O_i, and offer functions ω_i. The following lemma simplifies analysis further:

Lemma 3.1 *In any MPE, $\omega_e(\delta, R) \leq D(\delta, R, e)$ and $\omega_d(\delta, R) \leq E(\delta, R, d)$ for all δ, R. For any MPE, there exists another MPE with identical value functions in which all equilibrium offers are accepted and the above inequalities hold with equality for all δ, R.*

The lemma is sufficiently obvious that we do not provide a proof. As a consequence of this lemma and the definition of MPE, it is without loss of generality to characterize a MPE by a collection of value functions $E(\delta, R, s), D(\delta, R, s)$ and offer regions O_i, with the interpretation that the game ends the first time $(\delta, R, s) \in O_i \times \{i\}$ for any i (unless liquidation occurs first). The outcome is player i proposing an offer equal to player j's value function, and player j accepting. Given this lemma, we can prove the bargaining outcome must be Pareto optimal:

Proposition 1 *In any MPE, $E(\delta, R, s) + D(\delta, R, s) = V(\delta, R)$, where $V(\delta, R)$ is the value function of a social planner who picks the efficient reorganization time:*

$$V(\delta, R) = \sup_{T_R \in F^{\delta, R}} \mathbb{E}^{(\delta, R)}[\mathbf{1}(T_R < T_c)e^{-rT_R}(\theta\delta_{T_R} - R_{T_R}) + \mathbf{1}(T_c \leq T_R)e^{-rT_c}(\zeta\delta_{T_c} - R_{T_c})]. \quad (19)$$

The proof is given in Appendix A, but it follows from three simple observations. First, the sum of the value functions cannot exceed V. Second, letting T_R denote the optimal reorganization time solving (19), any player can deviate to force the game to end at the maximum of T_R and the equilibrium time $\mathcal{T}$ (unless liquidation occurs first). For this to not be profitable, each player must weakly prefer to receive their terminal payoff at $\mathcal{T}$ rather than $\mathcal{T} \vee T_R$. The final observation is that in any cases where $\mathcal{T} > T_R$, it must be that waiting until $\mathcal{T}$ is just as good as waiting until T_R or else the proposer at time T_R has a profitable deviation.

3.3 The optimal timing of reorganization

In light of Proposition 1, the first step in calculating the equilibrium is to find the social planner's value function defined by the optimal stopping problem in (19). By standard dynamic programming arguments, in the region where continuation is optimal, the continuation value $V(\delta, R)$ solves a partial differential equation (PDE):

$$rV(\delta, R) = \mathcal{L}V(\delta, R) + \iota[\zeta\delta - R - V(\delta, R)] \quad (20)$$

where, letting subscripts denote partial derivatives, $\mathcal{L}$ is a differential operator defined on smooth functions of δ, R by

$$\mathcal{L}f = \delta\mu f_\delta + \frac{\sigma^2}{2}\delta^2 f_{\delta\delta} - (1 - \tau)h\delta f_R. \quad (21)$$

The first two terms are familiar from Ito's lemma, and represent the sensitivity of the value function to changes in EBIT. The third term represents the fluctuations in the continuation value due to the accumulation of earnings. The final term in (20) captures the compensation for the risk of a forced conversion to Chapter 7 liquidation.

Following Bartolini and Dixit (1991), we solve the PDE by making a change of variables. In Appendix B, we solve for the general solutions of equation (20). We impose the intuitive boundary condition that for fixed $R > 0$, the value function stays bounded by the liquidation value as $\delta \to 0$, since reorganization could never be optimal if $\delta = 0$ and $R > 0$. The unique solution of equation (20) satisfying this is $V(\delta, R) = \delta v(R/\delta)$, where the function $v : \mathbb{R} \to \mathbb{R}$ is defined by

$$v(x) = A_3 x^\gamma M(-\gamma, -2(\gamma - 1) + \frac{2\mu}{\sigma^2}, \frac{-2h(1-\tau)}{\sigma^2 x}) + \frac{\iota\zeta + \frac{h(1-\tau)\iota}{r+\iota}}{r + \iota - \mu} - \frac{\iota x}{r + \iota}. \tag{22}$$

In this definition, A_3 is an arbitrary constant, γ is the negative root of the polynomial

$$0 = [-(r + \iota - \mu) - \mu z + \frac{\sigma^2}{2} z(z - 1)],$$

and $M(a, b, z)$ is the confluent hypergeometric function

$$M(a, b, z) \equiv 1 + \frac{a}{b} z + \frac{1}{2!} \frac{a(a+1)}{b(b+1)} z^2 + \frac{1}{3!} \frac{a(a+1)(a+2)}{b(b+1)(b+2)} z^3 + \ldots$$

The confluent hypergeometric function can be thought of as a generalization of the exponential function.

In the region where the social planner finds it optimal to immediately reorganize, we have $V(\delta, R) = \theta\delta - R$. Given the form of the value function in the continuation region, we conjecture there exists a threshold $\bar{x}$ such that immediate reorganization is optimal if and only if $x \equiv R/\delta \leq \bar{x}$. In this case, $\delta v(R/\delta)$ should value match and smooth paste with $\theta\delta - R = \delta(\theta - R/\delta)$ on the curve $R/\delta = \bar{x}$. This is equivalent to the following system:

$$A_3 \bar{x}^\gamma M(-\gamma, -2(\gamma - 1) + \frac{2\mu}{\sigma^2}, \frac{-2h(1-\tau)}{\sigma^2 \bar{x}}) + \frac{\iota\zeta + \frac{h(1-\tau)\iota}{r+\iota}}{r + \iota - \mu} - \frac{\iota\bar{x}}{r + \iota} = \theta - \bar{x} \tag{23}$$

$$\frac{d}{dx}\left(A_3 \bar{x}^\gamma M(-\gamma, -2(\gamma - 1) + \frac{2\mu}{\sigma^2}, \frac{-2h(1-\tau)}{\sigma^2 \bar{x}})\right) = \frac{\iota}{r + \iota} - 1. \tag{24}$$

This system of algebraic equations is simple to solve numerically. However, we still must verify that the optimal policy is in fact a barrier policy as conjectured. We prove the following proposition in Appendix B:

Proposition 2 *Suppose $A_3, \bar{x}$ solve (23, 24), and the following two conditions are met:*

$$\bar{x} \leq -\frac{h(1-\tau) + \mu\theta + \iota(\zeta - \theta) - r\theta}{r} \tag{25}$$

$$v(x) \geq \theta - x, \tag{26}$$

where $v(x)$ is the function given in (22). Then the stopping time $T_R = \inf\{t : R_t < \bar{x}\delta_t\}$ solves (19) with associated value function

$$V(\delta, R) = \begin{cases} \delta v(\frac{R}{\delta}), & R \geq \bar{x}\delta \\ \theta\delta - R, & R \leq \bar{x}\delta. \end{cases} \tag{27}$$

The conditions of Proposition 2 are intuitive: equation (25) ensures that reorganization does not happen too early according to the barrier strategy, while equation (26) ensures it does not occur too late. The conditions are easy to check numerically for a candidate $A_3, \bar{x}$, and we have yet to find a case where they are not satisfied.

In summary, a social planner would watch the movement of the EBIT and the net exercise price and emerge from bankruptcy when the current EBIT is large or the net exercise price is low (i.e., when the accumulated earnings have offset enough of the fixed cost of emerging from bankruptcy). To be clear, the fixed cost of exiting bankruptcy makes this analogous to a real option. For some firms, this option is "in the money" at default so the reorganization is instantaneous, while for other firms, the option value of reorganizing in the future leads to efficient delay and lengthy reorganizations.

3.4 Calculating the split

From Proposition 2, the social planner chooses to emerge from bankruptcy when $(\delta, R) \in O^* \equiv \{(\delta, R) : R \leq \bar{x}\delta\}$. Proposition 1 then implies that the game cannot end when $(\delta, R) \notin O^*$ (except by forced liquidation). Intuitively, in the region where a single agent would optimally choose to wait, in equilibrium the proposer chooses to wait. Then the value function of each player in this region must equal the discounted expectation of receiving their value function a second later. If we conjecture that both value functions are smooth, then by a standard dynamic programming argument, this implies the following system of linked PDEs:

$$rE(\delta, R, e) = \mathcal{L}E(\delta, R, e) + \lambda_e[E(\delta, R, d) - E(\delta, R, e)] + \iota[0 - E(\delta, R, e)] \tag{28}$$

$$rE(\delta, R, d) = \mathcal{L}E(\delta, R, d) + \lambda_d[E(\delta, R, e) - E(\delta, R, d)] + \iota[0 - E(\delta, R, d)] \tag{29}$$

$$rD(\delta, R, e) = \mathcal{L}D(\delta, R, e) + \lambda_e[D(\delta, R, d) - D(\delta, R, e)] + \iota[\zeta\delta - R - D(\delta, R, e)] \tag{30}$$

$$rD(\delta, R, d) = \mathcal{L}D(\delta, R, d) + \lambda_d[D(\delta, R, e) - D(\delta, R, d)] + \iota[\zeta\delta - R - D(\delta, R, d)], \tag{31}$$

which must hold for all $(\delta, R) \notin O^*$. Next, recall that in the definition of a MPE, each player i

18

must find it optimal in every instant where $s_t \neq i$ to accept an offer equal to their value function. Player i's outside option should they reject would be to wait a second and receive their value function. This suggests that for the players receiving offers, their value functions should always equal the discounted expectation of receiving their value function a moment later, even in the region where offers are made. If player i's value function in state $s \neq i$ were ever strictly less than the expected discounted value of waiting a second, it is suboptimal for player i to follow their equilibrium strategy of accepting offers equal to their value function. Likewise, if player i's value function in state $s \neq i$ were ever strictly greater than the expected discounted value of waiting a second, then player i should be willing to accept an offer just below their value function. This suggests that in an MPE with smooth value functions, the value functions should satisfy (28)-(31) for all $(\delta, R) \notin O^*$, and the receiving value functions $E(\delta, R, d), D(\delta, R, e)$ should satisfy (29, 30) everywhere. The following proposition, which is proved in Appendix C, shows this constitutes a MPE.

Proposition 3 *Assume the conditions of Proposition 2 hold. Let $E(\delta, R, s), D(\delta, R, s)$ be smooth functions such that $E(\delta, R, s) + D(\delta, R, s) = V(\delta, R)$. Assume (28)-(31) are satisfied for all $(\delta, R) \notin O^*$, and (29, 30) hold everywhere. Then the following strategy for each player i constitutes a MPE, with value functions $E(\delta, R, s), D(\delta, R, s)$:*

1. *Offer player j their value function if and only if $(\delta, R) \in O^*$.*
2. *Accept an offer equal to or greater than player i's value function at any time, for any (δ, R).*

Proposition 3 allows us to calculate the MPE for our bargaining game. In Appendix C we prove Proposition 3 and calculate the unique smooth MPE value functions $E(\delta, R, s), D(\delta, R, s)$ in closed form. The solution, which requires solving a linked system of PDEs, combines the methods of Proposition 2 with Markov chain techniques appearing in Guo, Miao, Morellec (2005), among other papers.

3.5 Dynamic bargaining outcomes

In this section, we provide intuition for the outcome of our dynamic bargaining game. First, we briefly motivate our baseline parameters. Our model shares parameters $\mu, \sigma, \tau, r, \alpha$ with the standard Leland model. For these parameters, we follow the literature (see Strebulaev and Whited (2012) Table 3). For the bargaining power parameters, we choose λ_e to correspond to the exclusivity period in Chapter 11. Specifically, since equityholders begin with a 120 day window (or longer) to exclusively make offers, we choose $\lambda_e = 3$ so that in expectation the equityholders' first offer window lasts 120 days. All parameters are annualized. To start the analysis, we set $\lambda_d = \lambda_e$. For the rate of forced conversion to Chapter 7, we set a baseline value of $\iota = 0.06$, which corresponds to the 14% of Chapter 11 cases converted to liquidations in the sample of BWZ (2006), given the average length of 2.5 years for a Chapter 11 case. For ease of interpretation, we use $\delta_0 = 1$ for all our numerical analysis. Firm values may then be interpreted as years of earnings. These baseline parameters are listed in Table 1, and unless otherwise stated all analysis uses these values.

Table 1: **Baseline Parameter Values**

Common Parameters	
μ	0.02
σ	0.25
r	0.05
τ	0.2
α	0.1
ι	0.06
λ_d	3
λ_e	3
δ_0	1
(a) Constrained Debt Chapter 7	
R_0	4
h	0.9
B	0.3
(b) Optimal Inefficient Chapter 11	
R_0	3.3
h	0.95
B	0.12

This table shows the baseline parameters we use throughout the paper. When we reference Table 1(a), we are using the h, R_0, B values corresponding to panel (a), for which the firm liquidates on the equilibrium path. When we reference Table 1(b), we are using the h, R_0, B values corresponding to panel (b), for which the firm enters Chapter 11 on the equilibrium path.

The inefficiencies of Chapter 11, captured by h and R_0, are difficult to quantify, so we consider different values for these. For now we use $h = 0.95$ and $R_0 = 3.3$ which correspond to Table 1(b). Finally, it will be helpful at times to keep the EBIT in the moment of default fixed while we vary other parameters. When we do this, we set δ to the exogenous value $\delta_{def} = .3674$, which is the ratio of average firm assets for firms entering Chapter 11 to average firm assets of healthy firms in Corbae and D'Erasmo (2017). In the next section we allow equityholders to endogenously choose the EBIT at which the firm defaults, but using this exogenous value can be helpful for intuition.

Figure 2(a) shows how the firm value in Chapter 11, evaluated at δ_{def} and R_0, changes with each parameter. Specifically, we calculate $V(\delta_{def}, R_0)$, then for each parameter, we increase the parameter by 5% of its baseline value and calculate $V(\delta_{def}, R_0)$ again. Figure 2(a) plots the elasticity of firm value with respect to each parameter, calculated as the percent change in $V(\delta_{def}, R_0)$ from a 5% increase in each parameter. An increase in the cost of reorganizing (R_0) decreases firm value while an increase in the multiplier on cashflows in bankruptcy (h) increases firm value. Once the costs of entering Chapter 11 B have been paid, the firm is always better off reorganizing than liquidating, so firm value is decreasing in the rate of liquidation ι. Since the Chapter 11 outcome is Pareto efficient, the bargaining power parameters λ_e, λ_d have no impact on firm value. The other comparative statics are similar to the Leland model and thus omitted.
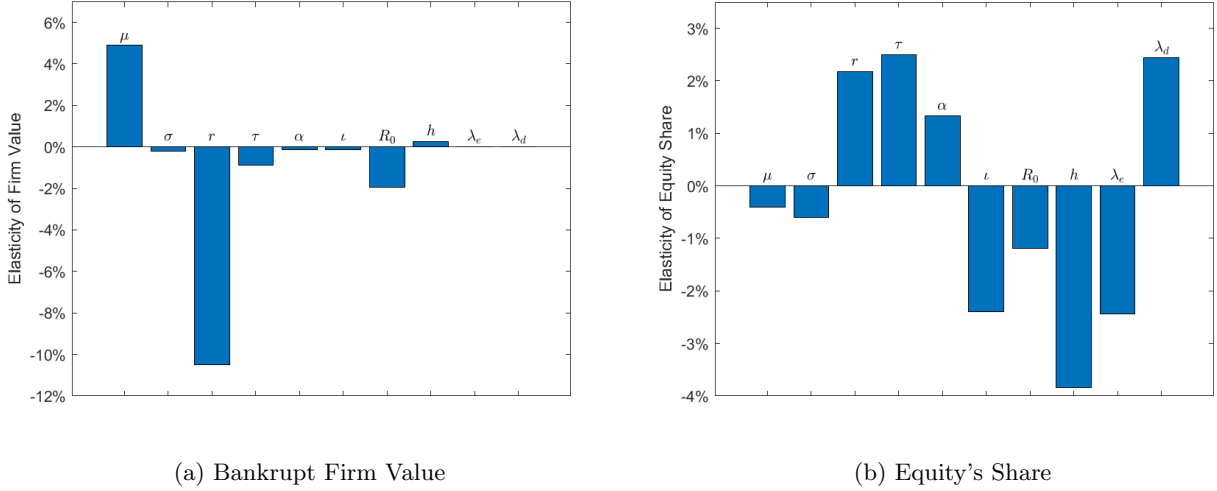
(a) Bankrupt Firm Value

(b) Equity's Share

Figure 2: **Chapter 11 Equilibrium Comparative Statics**

Using the baseline parameters of Table 1(b), we calculate the value of the firm at the moment of entering bankruptcy at $\delta = \delta_{def}$, $V(\delta_{def}, R_0)$, as in the text. We also compute the corresponding expected fraction of value that equity will receive, $E^{share}(\delta_{def}, R_0, e)$, as in the text. Then, for each parameter individually, we increase that parameter by 5% of its baseline value, and recalculate both of these quantities. Panel (a) plots the percentage change in the bankrupt firm value from increasing each parameter, one at a time, by 5%. Panel (b) plots the corresponding changes for the expected fraction of value that equity will receive.

Figure 2(b) shows how the underlying parameters of the model affect the bargaining split in Chapter 11. We calculate the expected fraction of firm value accruing to equity as

$$E^{share}(\delta_{def}, R_0, e) \equiv \frac{E(\delta_{def}, R_0, e)}{E(\delta_{def}, R_0, e) + D(\delta_{def}, R_0, e)}.$$

Just as above, we increase each parameter one at a time by 5% of its baseline value and recalculate the same quantity. Figure 2(b) plots the percentage change in $E^{share}(\delta_{def}, R_0, e)$ from a 5% increase in each parameter. The typical intuition in dynamic bargaining models is that the more patient party extracts more of the surplus (Rubinstein (1982)). Because of the exclusivity window, equityholders have the first opportunity to make offers. Intuitively then, anything which makes reorganization during the exclusivity window more attractive should improve equity's bargaining outcome. Accordingly, increasing r improves equity's expected outcome since it makes everyone more impatient. Conversely, increasing μ reduces the fraction of value that equity can extract, despite improving firm value, since it speeds up reorganization. Increasing R_0 (the cost of exiting bankruptcy) or h (the profitability of the firm during Chapter 11) makes it optimal to wait longer to exit bankruptcy, which in turn reduces equity's share of the firm value.

Any improvement in the outside option of creditors (waiting for Chapter 7) helps the creditors in the bargaining game. An increase in ι thus leads to a worse outcome for equity since, in the event of forced conversion to Chapter 7, creditors receive everything. Conversely, an increase in

α or τ helps equityholders, since the tax benefits of debt and the liquidation costs α are lost in Chapter 7. It is well known that the ex-ante firm value in Leland (1994) decreases in σ while the liquidation value does not depend upon σ, so an increase in volatility makes Chapter 7 relatively more attractive and harms equity's outcome. Finally, higher λ_e means equity's offer window is shorter, while lower λ_d means the creditors' offer window is longer, so both of these reduce equity's expected share of the firm value.

3.6   Benefits of the dynamic bargaining framework

The remainder of this section provides three arguments for the benefits of our dynamic bargaining model relative to Nash bargaining. First, in many existing models that rely on Nash bargaining, all outcomes of the reorganization are known with certainty. For example, in Fan and Sundaresan (2000), the bargaining occurs when the asset value hits a lower threshold and is instantaneous. It follows that the firm value upon exiting bargaining, and the respective fractions of the firm value which go to equity and creditors, is known with certainty even before the bargaining begins. Later models featuring bargaining, like that in François and Morellec (2004), allow for the possibility that the firm is liquidated prior to reorganizing. However, conditional on emerging, the value of the firm and the split are both known with certainty. In contrast, the empirical evidence suggests nothing about bankruptcy is predictable. Gilson, Hotchkiss, and Ruback (2000) find that the post-reorganization market value of bankrupt firms is difficult to forecast. Using management's cash flow projections and a methodology employed by practitioners, they find that value estimates are unbiased, but with a very large variance. The estimated values range from 20% to over 250% of the realized market value.

The outcomes of Chapter 11 bargaining are similarly difficult to predict. Eberhart and Sweeney (1992) look at whether post-bankruptcy-announcement bond prices are unbiased estimates of the final settlement prices. They fail to reject the null hypothesis that the post-announcement bond prices are unbiased estimates. However, in their Table 2, expected bond returns can only explain 42%-76% of realized bond returns over the bankruptcy. Wong et al (2007) try to predict cases in which shareholders receive a nonzero payout at the end of Chapter 11 with little success - they obtain a psuedo R^2 of less than .18 with their Cox's proportional hazards model (Table 6). BWZ (2006) examine the determinants of creditor recovery rates (Table XV). The R^2 in their regressions ranges between .21 and .46. Their regressions which seek to predict APR violations have R^2 values between .18 and .6. All their regressions include pre-bankruptcy variables like assets and leverage, which simpler Nash bargaining models predict should be sufficient to exactly determine these quantities. Our model predicts that these Chapter 11 outcomes cannot be perfectly forecast, consistent with the empirical evidence. Likewise, BWZ (2006) show regressions which seek to predict time in bankruptcy have R^2 values between .07 and .26. In their sample, the days in bankruptcy vary from 56 to 2,215 days, while Fan and Sundaresan (2000) model an instantaneous bargaining process and the models of François and Morellec (2004) and Broadie, Chernov, and Sundaresan (2007) assume an exogenous upper limit on the length of Chapter 11. Our model

allows for any length Chapter 11 to occur with positive probability, and this length is uncertain. Additionally, all earlier models of Chapter 11 bargaining we are aware of assume that the capital structure upon emerging from bankruptcy is known with certainty. Gilson (1997) finds adjusted R^2 values between .14 and .24 when trying to predict post-Chapter 11 leverage ratios (Table II) with a variety of explanatory variables (including pre-Chapter 11 leverage ratios). In our model of Chapter 11, the accumulation of cash flows introduces path dependence such that, in equilibrium, the EBIT upon emerging from Chapter 11 is not known with certainty until the firm exits Chapter 11. Since this EBIT determines the post-reorganization capital structure, this means the post-reorganization capital structure cannot be perfectly predicted, consistent with empirical evidence.

Figure 3: $E^{share}(\delta, R_0, e)$
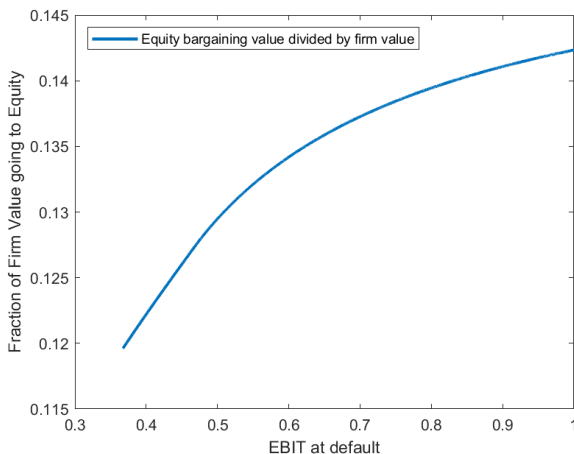
This figure shows $E^{share}(\delta, R_0, e)$, the ratio of equity's Chapter 11 value function to the total firm value in bankruptcy if they default at a value δ and choose Chapter 11, as a function of δ. The parameters correspond to Table 1(b).

Second, our dynamic bargaining model produces an endogenous link between firm size and equity's bargaining outcome. In a standard Nash bargaining model like Fan and Sundaresan (2000), equityholders rationally expect to receive a constant fraction (equal to their bargaining power parameter) of the firm value, no matter when they begin the bargaining. However, the empirical literature suggests that equityholders enjoy larger APR violations when the bankrupt firm is larger. In Table 7 of Franks and Torous (1994), the authors find that a 1% increase in the size (liabilities) of a firm at the time of default is associated with an approximate 1 percentage point increase in the absolute priority deviation going to equityholders. Table 6 of Betker (1995) shows that equity receives much larger APR violations when the firm is closer to solvency. Eberhart, Moore, and Roenfeldt (1990) find a positive correlation between APR violations received by equity and the market capitalization of the firm at the announcement of bankruptcy. Consistent with this empirical fact, our model predicts that when the firm is larger (closer to the reorganization threshold) at the time of default, equityholders receive a better outcome in bargaining. Figure 3 plots the expected

fraction of firm value accruing to equity $E^{share}(\delta, R_0, e)$ as a function of δ. It can be clearly seen that equity's expected share of firm value is increasing in the EBIT. The intuition for why dynamic bargaining produces this endogenous link between solvency and bargaining outcome is the same one which explains Figure 2. If the firm's condition improves to the point that exiting Chapter 11 is efficient before the exclusivity period expires, then creditors are less willing to wait for their turn in the bargaining game. Anything which makes reorganization more likely during the exclusivity period (like a higher starting EBIT) will thus improve equity's outcome.

It is interesting in itself that our dynamic bargaining matches this empirical link between size and shareholder bargaining outcome. However, by providing a strategic foundation for this result, our model also lends theoretical support to other papers which rely on this fact. Garlappi, Shu, and Yan (2008) and Garlappi and Yan (2011) use a modified version of Fan and Sundaresan (2000) to produce testable implications on equity returns, default probability, and shareholder bargaining power. In their empirical evidence, they show that if size is a proxy for expected shareholder recovery, then debt renegotiation can explain the concentration of momentum profits among low credit quality firms, as well as the lower expected returns and stronger book-to-market effects exhibited by distressed firms. While the model they use has a constant bargaining outcome for shareholders, our dynamic bargaining model endogenously produces exactly the link between size and shareholder recovery which they need to explain all of these phenomena. A recent literature has shown that the possibility for debt renegotiation can help explain empirical patterns in leverage (Morellec, Nikolov, and Schuerhoff (2018)), investment (Favara, Morellec, Schroth, and Valta (2017)), and equity returns (Hackbarth, Haselman, and Schoenherr (2015); Favara, Schroth, and Valta (2012)). Since our dynamic bargaining adds this additional realism to the negotiation process, it is conceivable that variations of our dynamic bargaining model could match the data even better.

Third, stochastic dynamic bargaining is inherently a more realistic description of Chapter 11. The Nash bargaining model itself is an axiomatic, not strategic, model of bargaining which does not describe noncooperative agents. It is well known that the Nash solution coincides with the outcome of the strategic Rubinstein bargaining model. However, when the object of the bargaining evolves stochastically, this is an unsatisfying answer. The Nash outcome is the one that would occur if equity and debt could, off the equilibrium path, exchange infinitely many offers while the rest of reality remains frozen in time. Our dynamic bargaining model allows participants to watch uncertainty resolve while considering proposals, as in reality.

4 Analysis of the decision to reorganize or liquidate and capital structure

In this section, we work backwards in time and solve for the firm's optimal strategy prior to defaulting. We first consider the firm's optimal stopping problem for when to default, and whether to file for Chapter 11 or Chapter 7. When the firm defaults, equityholders prefer to enter Chapter 11 if and only if their prospects in the bargaining equilibrium of the previous section justify the

fixed cost they must pay to enter Chapter 11. Section 4.1 studies the problem of when to default when only Chapter 11 is available. Section 4.2 solves the full problem in which equityholders may choose either chapter, and Section 4.3 describes the time zero capital structure decision.

4.1 The levered firm with the option to reorganize

In this section we consider the decision of the equityholders between Time 0 and Time 1 to enter Chapter 11. Ultimately, equityholders will have the option to reorganize or liquidate, but the first step to solve this problem is to ignore the option to liquidate. We assume the bargaining game starts with equityholders making proposals in the exclusivity period (i.e., in state $s_0 = e$). Thus if equityholders choose to enter Chapter 11, they receive $E(\delta, R_0, e) - B$, where $E(\delta, R, s)$ is the unique smooth MPE value function for equityholders and B is a fixed cost of entering bankruptcy.[4] The function $E(\delta, R, e)$ is calculated in closed form in Appendix C, for simplicity of notation we define $\mathcal{E}(\delta) \equiv E(\delta, R_0, e)$.

Prior to bankruptcy, equityholders receive cashflow $(1 - \tau)(\delta - C_0)dt$ per unit time, where C_0 is the optimally chosen coupon at time 0. In this section, we assume the firm only has the option to enter Chapter 11. In this case, equityholders choose a stopping time T_B at which point the firm enters Chapter 11 to solve

$$E^B(\delta) = \sup_{T_B \in F^\delta} \mathbb{E}^\delta \Big[\int_0^{T_B} e^{-rt}(1 - \tau)(\delta_t - C_0)dt + e^{-rT_B}(\mathcal{E}(\delta_{T_B}) - B) \Big]. \tag{32}$$

To solve for the optimal time to enter Chapter 11, we conjecture a lower barrier δ_B such that equityholders declare bankruptcy the first time $\delta_t \leq \delta_B$. Following the logic of Section 2.2, the value of equity prior to entering bankruptcy must be

$$E^B(\delta) = A_4 \delta^\psi + (1 - \tau)\Big[\frac{\delta}{r - \mu} - \frac{C_0}{r} \Big],$$

where A_4 is an arbitrary constant and ψ is again the negative root of

$$0 = -r + \mu z + \frac{\sigma^2}{2} z(z - 1).$$

---

[4]In Chapter 11, both debtors and creditors hire professionals. Professional fees incurred during bankruptcy are typically reimbursed from the firm's assets through §330(a) awards. Weiss (1990) estimates that such fees average 3.1% of firm value, but LoPucki and Doherty (2011) give many reasons why this is an underestimate. In extreme cases like the bankruptcy of Allied Holdings, fees can reach 22% of firm assets (LoPucki and Doherty (2011) Appendix A).

Firms also hire professionals prior to entering bankruptcy, and these prepetition fees are not reimbursed. It is thus reasonable to think of these prepetition fees, which average 43% of total fees, as being incurred by equityholders (LoPucki and Doherty (2011)).

Finally, there is empirical evidence that a substantial component of these fees are fixed costs, which do not vary with the size of the firm or length of bankruptcy (Warner (1977); Guffey and Moore (1991); LoPucki and Doherty (2004); LoPucki and Doherty (2011); BWZ (2006)). In the sample of BWZ (2006), firms with less than $100,000 in pre-bankruptcy assets incur expenses that average 31.5% of assets, while for firms with more than $10 million in assets, fees average 1.3% of assets. See Appendix E for more details and evidence.

The constant A_4 is determined by value matching and smooth pasting on the bargaining value at the point of bankruptcy. Using the closed form for $\mathcal{E}(\delta)$, we solve the nonlinear system

$$A_4 \delta_B^{\psi} + (1-\tau)[\frac{\delta_B}{r-\mu} - \frac{C_0}{r}] = \mathcal{E}(\delta_B) - B \tag{33}$$

$$A_4 \psi \delta_B^{\psi-1} + (1-\tau)[\frac{1}{r-\mu}] = \mathcal{E}'(\delta_B). \tag{34}$$

Proposition 4 provides conditions analogous to those in Proposition 2 under which the barrier strategy is optimal:

Proposition 4 *Assume the conditions of Proposition 2 hold. Suppose A_4, δ_B solve (33, 34), and the following two conditions are met:*

1. *On the set $[0, \delta_B]$, the function $\mathcal{E}(\delta)$ satisfies*

$$- r(\mathcal{E}(\delta) - B) + \mu \delta \mathcal{E}'(\delta) + \frac{\sigma^2 \delta^2}{2} \mathcal{E}''(\delta) \leq -(1-\tau)(\delta - C_0). \tag{35}$$

2. *On the set $[\delta_B, \infty)$, the function $\mathcal{E}(\delta)$ satisfies*

$$A_4 \delta^{\psi} + (1-\tau)[\frac{\delta}{r-\mu} - \frac{C_0}{r}] \geq \mathcal{E}(\delta) - B. \tag{36}$$

Then the stopping time $T_B = \inf\{t : \delta_t < \delta_B\}$ solves (32) with associated value function

$$E^B(\delta) = \begin{cases} A_4 \delta^{\psi} + (1-\tau)[\frac{\delta}{r-\mu} - \frac{C_0}{r}], & \delta \geq \delta_B \\ \mathcal{E}(\delta) - B, & \delta \leq \delta_B. \end{cases} \tag{37}$$

The proof appears in Appendix D. Finally, once we have solved for δ_B, the calculation for the value of debt is straightforward. Debt has value

$$D^B(\delta) = A_5 \delta^{\psi} + \frac{C_0}{r}, \tag{38}$$

and A_5 is calculated by value matching at δ_B:

$$A_5 \delta_B^{\psi} + \frac{C_0}{r} = D(\delta_B, R_0, e). \tag{39}$$

Once we plug in the closed form solutions for $D(\delta, R_0, e), \mathcal{E}(\delta), \mathcal{E}'(\delta)$, equations (33-39) represent a system of algebraic equations which are easily solved numerically. Likewise, the second derivative in (35) is available in closed form, allowing us to numerically check the conditions for the verification.

Figure 4: $\mathcal{E}(\delta) - B$
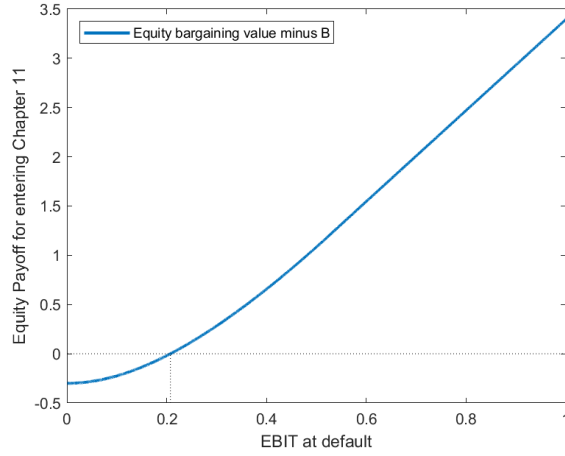
This figure shows $\mathcal{E}(\delta) - B$, the payoff to equity if they default at a value δ and choose Chapter 11, as a function of δ. The parameters correspond to Table 1(a).

4.2 The levered firm with the option to reorganize or liquidate

In this section, we consider the decision of the equityholders prior to Time 1 to enter Chapter 11 or enter Chapter 7. In the previous subsection we solved for the equity value E^B when equityholders may only choose Chapter 11, and showed the corresponding optimal stopping time T_B is a first hitting time with threshold δ_B. In Section 2.2, we derived the equity value E^L when equityholders could only liquidate, with corresponding optimal liquidation time T_L and associated threshold δ_L. In this section, we study the decision of how to optimally choose a time of liquidation T_L and time of bankruptcy T_B to maximize

$$E_0(\delta) = \sup_{T_L, T_B \in F^\delta} \mathbb{E}^\delta \Big[\int_0^{T_B \wedge T_L} e^{-rt}(1-\tau)(\delta_t - C_0)dt + \mathbf{1}(T_B < T_L)e^{-rT_B}[\mathcal{E}(\delta_{T_B}) - B]\Big]. \tag{40}$$

This decision is equivalent to picking a time of default $T_D = T_L \wedge T_B$ and whether to enter Chapter 7 or Chapter 11 at that time. Using our results from Section 3, the latter decision is trivial: either the bargaining value net of fixed costs $\mathcal{E}(\delta_{T_D}) - B$ is larger than zero, so Chapter 11 is optimal, or it is less than 0, so liquidation is optimal. Define

$$g(\delta) \equiv \max(\mathcal{E}(\delta) - B, 0). \tag{41}$$

Then the decision of when to enter Chapter 7 or Chapter 11 is equivalent to

$$E_0(\delta) = \sup_{T_D \in F^\delta} \mathbb{E}^\delta \Big[\int_0^{T_D} e^{-rt}(1-\tau)(\delta_t - C_0)dt + e^{-rT_D}g(\delta_{T_B})\Big]. \tag{42}$$

27

Since g is continuous and nonnegative, standard results (Øksendal (2003) Chapter 10) show that $E_0(\delta)$ exists, with associated exercise region $S \equiv \{\delta : E_0(\delta) = g(\delta)\}$.

In reality, firms default in bad states of the world. However, if creditors have no rights in Chapter 11, then equityholders might use Chapter 11 in good states of the world as an opportunity to default on their existing debt, issuing more debt afterward to take advantage of the tax shield. Since Chapter 11 is an opportunity to reduce, not increase a firm's debt, so we rule out this unrealistic case with the following assumption:

Assumption 1. *The bargaining power of debtholders is high enough that*

$$\lim_{\delta \to \infty} \mathcal{E}(\delta) - \frac{(1 - \tau)\delta}{r - \mu} = -\infty.$$

This intuitive assumption says that as firms become infinitely profitable, the unlevered firm value exceeds the value to equity of defaulting and entering Chapter 11. We give a specific condition on underlying parameters that is sufficient for this in Appendix D. When this assumption holds, we can obtain a clean characterization for the equityholders' optimal policy in (40).

Proposition 5 *Suppose the conditions of Propositions 2 and 4 are met, and in addition Assumption 1 holds. For any fixed C, let $S(C) \equiv \{\delta : E_0(\delta) = g(\delta)\}$ denote the set of δ values where the firm defaults immediately, and let δ_L, δ_B be the optimal liquidation and reorganization thresholds from Sections 2.2 and 4.1. Then $\bar{\delta}(C) \equiv \sup S(C)$ is finite. Further, $\bar{\delta}(C)$ equals the liquidation trigger δ_L if and only if $\mathcal{E}(\bar{\delta}(C)) \leq B$ and it equals the bankruptcy threshold δ_B if and only if $\mathcal{E}(\bar{\delta}(C)) \geq B$.*

This proposition says that for any fixed C, at a large enough δ the firm knows with certainty which of Chapter 11 or Chapter 7 they will eventually enter, and it will occur at a lower threshold. We next show that which of these occurs will depend on C:

Proposition 6 *Suppose the conditions of Propositions 2 and 4 are met, and in addition Assumption 1 holds. The default threshold $\bar{\delta}(C)$ is a weakly increasing and continuous function of C, and $\lim_{C \to \infty} \bar{\delta}(C) = \infty$. There exists $\bar{C}$ such that $\mathcal{E}(\bar{\delta}(\bar{C})) = B$ and $C > \bar{C}$ implies $\mathcal{E}(\bar{\delta}(C)) > B$.*

Proposition 6 delivers the central intuition of the choice between Chapter 7 and Chapter 11 in our model. When the firm has a larger coupon, they default at higher δ values. We see from Figure 4 that equity's value in Chapter 11 $\mathcal{E}(\delta)$ is strictly increasing in δ, so equity's prospects in Chapter 11 are more likely to justify the fixed cost B of entering Chapter 11 when δ is high. Proposition 6 shows the existence of a $\bar{C}$ such that when the firm has issued more debt than $\bar{C}$, they will default at a sufficiently profitable δ that Chapter 11 is preferable to Chapter 7 at that δ. Given the strict monotonicity of $\mathcal{E}$ and $\bar{\delta}(C)$ we observe numerically, equityholders will strictly prefer liquidation for $C < \bar{C}$, by the same logic.

4.3 Analysis of the capital structure with Chapter 11 reorganization

In this section, we consider the decision of equityholders at Time 0 of how much debt to issue. Let

$$F^j(\delta_0, C_0) \equiv E^j(\delta_0, C_0) + D^j(\delta_0, C_0), \ j = L, B$$

denote firm value for a given coupon C_0, assuming either a future Chapter 7 ($j = L$) or Chapter 11 ($j = B$) bankruptcy. Since equityholders receive the proceeds of the initial debt issue, at time 0 equity chooses the optimal coupon C_0 to maximize the sum of the values of equity and debt, subject to the constraint that equity will subsequently decide between Chapter 7 and Chapter 11 to maximize equity value. Under Assumption 1, as long as δ_0 is large relative to C_0, equity will know immediately after they issue debt whether they will eventually enter Chapter 7 or Chapter 11 (Proposition 5), so the time zero value of equity equals the maximum of $E^B(\delta_0, C_0)$ and $E^L(\delta_0, C_0)$. Under rational expectations, the time zero value of the firm for a given coupon C_0 is then

$$F_0(\delta_0, C_0) \equiv \begin{cases} F^B(\delta_0, C_0), & E^B(\delta_0, C_0) > E^L(\delta_0, C_0) \\ F^L(\delta_0, C_0), & E^B(\delta_0, C_0) < E^L(\delta_0, C_0) \\ F^{B\vee L}(\delta_0, C_0), & E^B(\delta_0, C_0) = E^L(\delta_0, C_0), \end{cases} \tag{43}$$

where $F^{B\vee L}(\delta, C) \equiv \max(F^L(\delta, C), F^B(\delta, C))$. In words, the value of the firm for a given coupon is either the value of the firm conditional on eventual liquidation, or the value of the firm conditional on eventual Chapter 11. Which of these cases occurs is determined by which is better ex-post for equityholders. As is standard in dynamic models of capital structure, equityholders lack commitment power. For a given coupon C_0, equityholders might be able to get a better price on debt (and higher overall time 0 value) if they could commit to a future Chapter 7. However, if that C_0 implies equity will prefer Chapter 11, the debt will be priced at time 0 under the rational expectation of a future Chapter 11. From Proposition 6, and the observation that in all numerical examples $\mathcal{E}(\delta)$ is strictly increasing, we can obtain a cleaner characterization of how debt will be priced at a given coupon: there will always exist $\bar{C}$ such that for large δ_0,

$$F_0(\delta_0, C_0) = \begin{cases} F^B(\delta_0, C_0), & C_0 > \bar{C} \\ F^L(\delta_0, C_0), & C_0 < \bar{C} \\ F^{B\vee L}(\delta_0, C_0), & C_0 = \bar{C}. \end{cases}$$

It follows that at time 0, the firm has two options. They may choose any coupon larger than $\bar{C}$ and receive the firm value $F^B(\delta_0, \cdot)$ under the rational expectation of a future Chapter 11 reorganization. Alternately, they may choose a coupon weakly less than $\bar{C}$ and receive the firm value $F^L(\delta_0, \cdot)$ under the rational expectation of a future Chapter 7 liquidation.

Using the solutions derived in the last two sections (and Section 2.2), we calculate $F_0(\delta_0, \cdot)$ according to equation (43). We then numerically maximize $F_0(\delta_0, \cdot)$ on a grid of possible coupons to find the optimal coupon C^*. In the cases we consider in the following section, C^* is equal to one

of the following three coupons:

$$\bar{C} = \max\{C_0 : E^B(\delta_0, C_0) \leq E^L(\delta_0, C_0)\} \tag{44}$$

$$C_L = argmax_{C_0 \in [0,\infty)} F^L(\delta_0, C_0) \tag{45}$$

$$C_B = argmax_{C_0 \in [0,\infty)} F^B(\delta_0, C_0). \tag{46}$$

In these equations, $\bar{C}$ is the threshold coupon from Proposition 6. C_L and C_B are the optimal coupons equity would choose if they were constrained to only use Chapter 7 or only use Chapter 11, respectively.

5   Capital structure and empirical predictions

5.1   Capital structure decisions and the relative efficiency of Chapter 11

In this section, we analyze the optimal capital structure of the firm with the option to reorganize or liquidate. As is standard in capital structure models, the equityholders internalize the inefficiency of their ex-post optimal bankruptcy procedure when they issue debt. Put differently, the price equityholders can charge for their debt will exactly reflect the inefficiency of their future preferred bankruptcy procedure. When one form of bankruptcy (Chapter 11 or Chapter 7) is so inefficient relative to the other that equityholders would never find it optimal ex-post, equityholders can credibly ignore that option. In these cases, equityholders are unconstrained by their lack of commitment when choosing the coupon to maximize the tax benefits given their preferred future bankruptcy form. Debtholders will correctly infer the future strategy of equityholders when they price the debt.

However, when Chapter 11 is slightly less efficient than Chapter 7, our model predicts a more nuanced capital structure decision. In this region of the parameter space, debtholders prefer Chapter 7 liquidation, since Chapter 11 reorganization only allows them to capture a fraction of a slightly smaller pie. Equityholders would like to issue a large coupon to take advantage of tax benefits, and commit to future liquidation to obtain a low cost of debt. However, for these parameters, the result of Proposition 6 implies that these two goals conflict with each other: large coupons imply equityholders will ex-post find Chapter 11 optimal. Debtholders recognize this and pay less for debt with such a coupon at time 0. Since equityholders cannot formally commit to Chapter 7, they have two choices. They can issue a large coupon to maximize tax benefits, and accept that debtholders will charge extra for the future Chapter 11 inefficiencies. Alternately, equityholders can issue the largest coupon $\bar{C}$ consistent with Chapter 7 being optimal for equity ex-post. This allows equityholders to get a better price for the debt they issue, but they forgo tax benefits since for these parameters $\bar{C}$ is smaller than the coupon they would otherwise issue.

To an econometrician, in this latter case our model looks identical to the Leland model: a firm issues debt then eventually liquidates. However, the off-equilibrium considerations introduced by our bargaining model lead the firm to issue a much smaller coupon than in the standard Leland

model. In this case, our model predicts lower leverage than the Leland model, even for the 65% of firms that liquidate in Chapter 7 (BCI (2017)).

To illustrate the capital structure decision in more detail, we now present examples of each case. The parameters h, R_0, B capture the inefficiencies of Chapter 11. To succinctly describe regions of the (h, R_0, B) parameter space, we introduce two measures of efficiency. The first measure is RelEff, which is the ratio of total firm value upon entering Chapter 11 to the total firm value at the moment of liquidation:

$$\text{RelEff} \equiv \frac{V(\delta_{def}, R_0) - B}{\zeta \delta_{def}}. \tag{47}$$

Recall the numerator is the total firm value in Chapter 11 reorganization, which incorporates a partial loss of earnings during Chapter 11 and a fixed cost of exiting Chapter 11, minus the fixed cost B of entering Chapter 11. The denominator is the liquidation value debtholders receive by selling the assets for their perpetuity value minus proportional liquidation costs. Since Chapter 11 entails fixed costs in our model while Chapter 7 does not, the value of this ratio is sensitive to the δ at which it is evaluated. Intuitively, spending several years in court over a firm worth one dollar would be extraordinarily wasteful relative to liquidating such a firm, regardless of the overall efficiency of each procedure. This is why both are evaluated at the exogenous value δ_{def}.
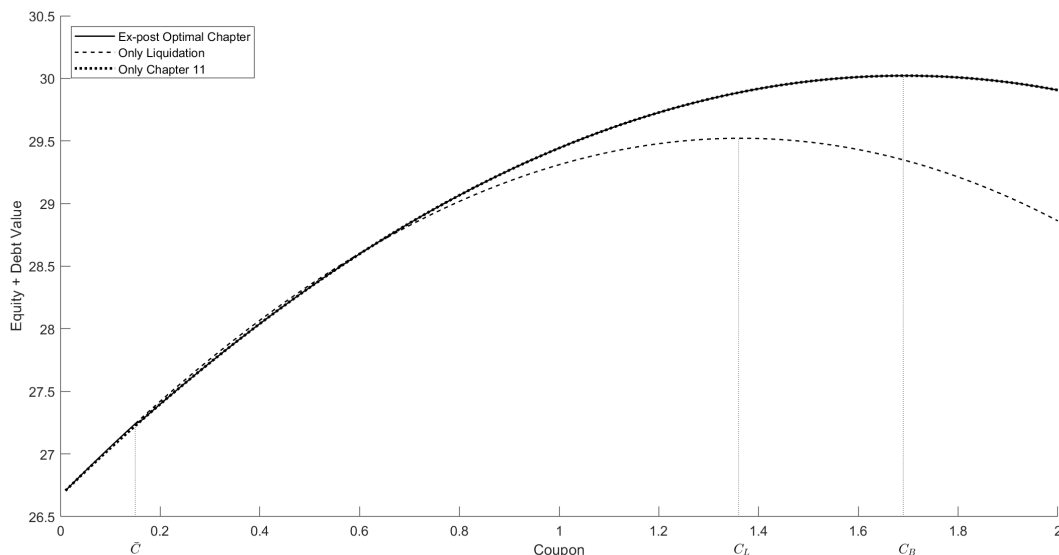
RelEff has no direct significance in the solution of our model, but is helpful for concisely summarizing the inefficiencies of Chapter 11 and Chapter 7 without delving into the optimal strategy of equityholders. The extent to which these inefficiencies impact firm value is of course endogenous. It will be helpful to define a second measure which measures how much firm value is changed by the added option of Chapter 11:

$$\text{Choicevalue} \equiv \frac{F_0(\delta_0, C^*)}{F_L(\delta_0, C_L)}. \tag{48}$$

The numerator is the time 0 value of the firm with the option to liquidate or enter Chapter 11, evaluated at the optimal coupon. The denominator is the time 0 firm value in the Leland model with only Chapter 7, evaluated at the corresponding optimal coupon. This measure provides clearer intuition on how the optimal strategy changes with the Chapter 11 parameters. We will use Choicevalue to partition the space of (h, R_0, B) values into cases corresponding to distinct optimal strategies, then reference RelEff to describe the exogenous inefficiencies which induce each case.

Case 1: Choiceval > 1. Suppose that Chapter 7 liquidation is less efficient than Chapter 11 reorganization. Since the tax benefits of debt are large empirically, equityholders like to issue large coupons. Such coupons imply equityholders will default in profitable states of the world (Proposition 6), and these are the states of the world where equity's prospects in Chapter 11 justify the fixed costs of entering Chapter 11 (Proposition 5). Debtholders might dislike sharing the firm with equityholders in Chapter 11, but since Chapter 11 is more efficient, the overall pie is bigger. Equityholders are thus willing to pay a higher cost of debt associated with Chapter 11 being ex-post optimal, since they are compensated by the rents they eventually extract in Chapter

31

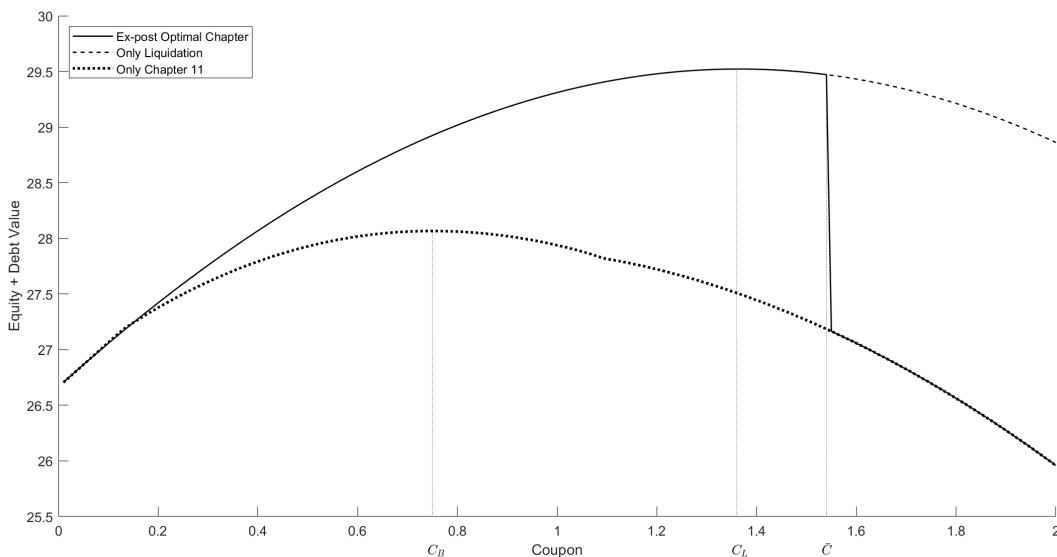Figure 5: **Capital Structure Choice: RelEff=115%**

This figure shows ex-ante firm value (Equity + Debt) as a function of the coupon C_0. The dashed curve plots $F^L(\delta_0, C_0)$, the firm value under the assumption of future liquidation. The vertical dotted line descending from the peak of the dashed curve marks C_L, the optimal coupon under future liquidation, on the x-axis. The dotted curve plots $F^B(\delta_0, C_0)$, the firm value under the assumption of future Chapter 11. The vertical dotted line descending from the peak of the dotted curve marks C_B, the optimal coupon under future Chapter 11 reorganization, on the x-axis. The solid curve plots the actual firm value $F_0(\delta_0, C_0)$ as a function of the coupon C_0. Finally, the vertical dotted line ending in $\bar{C}$ on the x-axis marks the largest coupon for which equity finds liquidation optimal ex-post. The parameters corresponding to this figure are the baseline parameters from Table 1, with $R_0 = 0.6, h = 1, B = 0.1$.

11, and they issue a large coupon to take full advantage of the tax shield.

Figure 5 plots firm value as a function of C_0, the time 0 perpetual coupon on the consol debt. We use the parameters of Table 1, except we use different (h, R_0, B) values such that Chapter 11 is 15% more efficient than Chapter 7 by our metric RelEff. The dashed curve plots $F^L(\delta_0, C_0)$, the firm value under the assumption of future liquidation, as a function of the coupon C_0. As usual, the tradeoff between the tax shield of debt and the efficiency loss in liquidation leads to an inverted U shape for firm value as a function of the coupon. The vertical dotted line descending from the peak of the dashed curve marks C_L, the optimal coupon under future liquidation, on the x-axis. The dotted curve plots $F^B(\delta_0, C_0)$, the firm value under the assumption of future Chapter 11. Again there is a tradeoff between the tax shield of debt and the inefficiency of Chapter 11. The vertical dotted line descending from the peak of the dotted curve marks C_B, the optimal coupon under future Chapter 11 reorganization, on the x-axis. Since we have assumed here that the bankruptcy costs in Chapter 11 are less extreme than those in Chapter 7, the optimal coupon C_B is larger than C_L as expected.

32

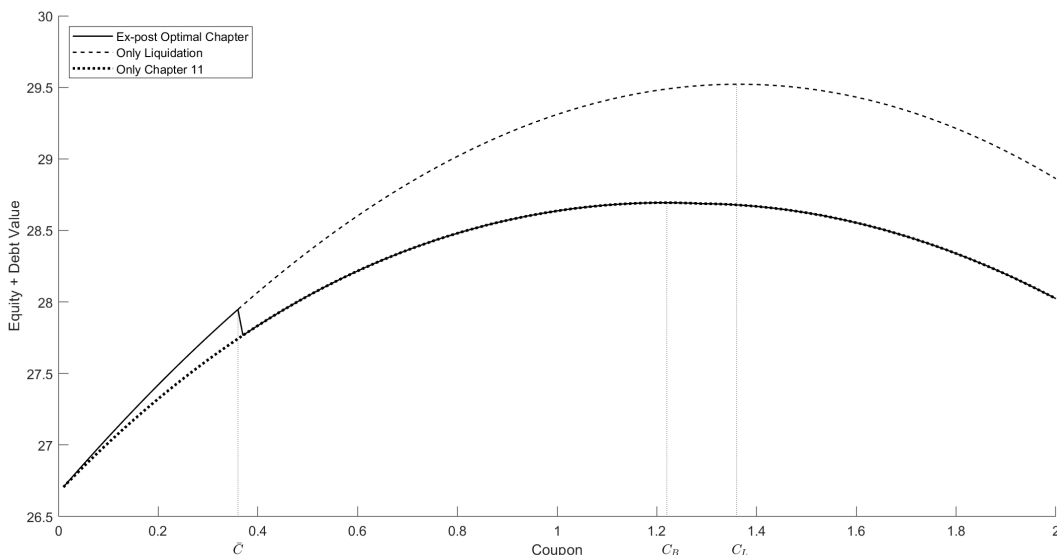Figure 6: **Capital Structure Choice: RelEff=50%**

This figure shows ex-ante firm value (Equity + Debt) as a function of the coupon C_0. The dashed curve plots $F^L(\delta_0, C_0)$, the firm value under the assumption of future liquidation. The vertical dotted line descending from the peak of the dashed curve marks C_L, the optimal coupon under future liquidation, on the x-axis. The dotted curve plots $F^B(\delta_0, C_0)$, the firm value under the assumption of future Chapter 11. The vertical dotted line descending from the peak of the dotted curve marks C_B, the optimal coupon under future Chapter 11 reorganization, on the x-axis. The solid curve plots the actual firm value $F_0(\delta_0, C_0)$ as a function of the coupon C_0. Finally, the vertical dotted line ending in $\bar{C}$ on the x-axis marks the largest coupon for which equity finds liquidation optimal ex-post. The parameters corresponding to this figure are the baseline parameters from Table 1, with $R_0 = 4.4, h = 0.4, B = 2$.

The solid curve plots the actual firm value $F_0(\delta_0, C_0)$ as a function of the coupon C_0. Since equityholders lack commitment power, F_0 is either equal to F^B or F^L, depending upon whether equityholders will subsequently find it optimal to enter Chapter 11 or liquidate. The first vertical dotted line marks $\bar{C}$, the threshold coupon for Chapter 11 vs Chapter 7, on the x-axis. For $C \leq \bar{C}$, the solid curve F_0 follows the liquidation value F^L since the firm will subsequently find it optimal to liquidate. For $C > \bar{C}$, the solid curve follows F^B, since equityholders will subsequently enter Chapter 11. In particular, if equityholders want to sell debt for the value under liquidation D^L, the largest coupon they may issue is $\bar{C}$. Since Chapter 11 is good for firm value in this instance, equityholders find it optimal to issue C_B and credibly signal a future Chapter 11 reorganization, since this is the maximal point on the solid curve. Equityholders are thus unconstrained by their lack of commitment when they decide to maximize the firm value under future Chapter 11.

Case 2: Choiceval = 1. Another possible case is that Chapter 7 liquidation is much more efficient than Chapter 11 reorganization. In this case, if equity had commitment power, they would

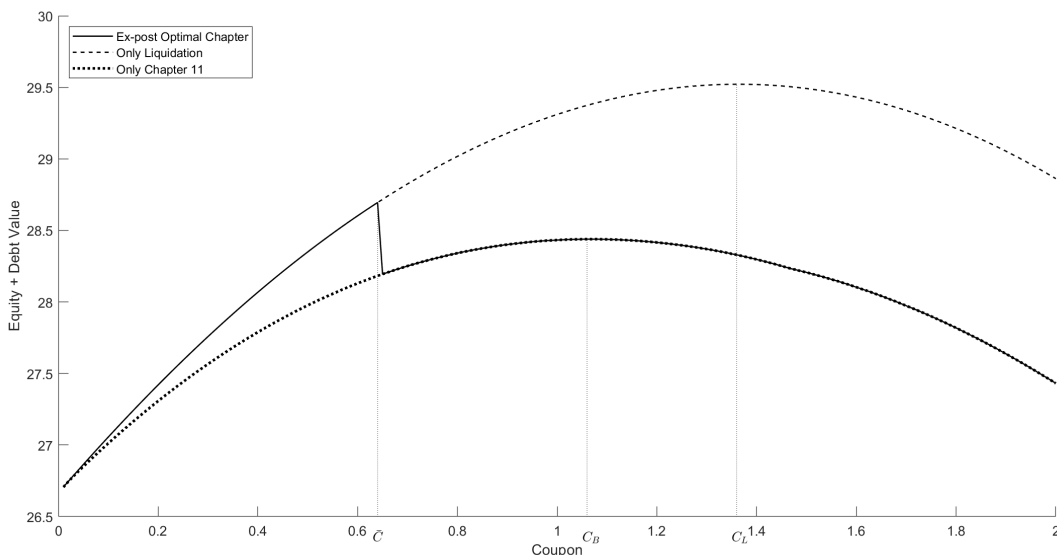Figure 7: **Capital Structure Choice: RelEff=85%**

This figure shows ex-ante firm value (Equity + Debt) as a function of the coupon C_0. The dashed curve plots $F^L(\delta_0, C_0)$, the firm value under the assumption of future liquidation. The vertical dotted line descending from the peak of the dashed curve marks C_L, the optimal coupon under future liquidation, on the x-axis. The dotted curve plots $F^B(\delta_0, C_0)$, the firm value under the assumption of future Chapter 11. The vertical dotted line descending from the peak of the dotted curve marks C_B, the optimal coupon under future Chapter 11 reorganization, on the x-axis. The solid curve plots the actual firm value $F_0(\delta_0, C_0)$ as a function of the coupon C_0. Finally, the vertical dotted line ending in $\bar{C}$ on the x-axis marks the largest coupon for which equity finds liquidation optimal ex-post. The parameters corresponding to this figure are the baseline parameters from Table 1(b).

get the greatest time 0 value by committing to a future Chapter 7, and issuing debt with the coupon C_L that maximizes the firm value conditional on future Chapter 7. However, if Chapter 11 reorganization is very inefficient, there will be no commitment problem. Specifically, with an inefficient Chapter 11 process, equityholders would have to default in a very profitable state of the world in order for their bargaining prospects to justify the fixed costs of Chapter 11. They will only default in such a state of the world if they have issued debt with a coupon $\bar{C}$ much larger than C_L. So even without formal commitment power, equity can issue their favorite coupon C_L and credibly promise a future Chapter 7 liquidation. This is depicted graphically in Figure 6, with (h, R_0, B) values that correspond to a Chapter 11 process that is 50% less efficient than Chapter 7.

The interpretation of Figure 6 is exactly the same as Figure 5. The point on the x-axis where the solid curve F^0 drops down from the dashed curve F^L to the dotted curve F^B corresponds to the threshold coupon $\bar{C}$. In this figure, we see that the solid curve F_0 is maximized at the point where F^L is maximized, with corresponding coupon C_L. Since $C_L < \bar{C}$, equityholders are able to issue C_L, credibly promise a future liquidation, and receive F^L. In this sense, equityholders are

Figure 8: **Capital Structure Choice: RelEff=75%**

This figure shows ex-ante firm value (Equity + Debt) as a function of the coupon C_0. The dashed curve plots $F^L(\delta_0, C_0)$, the firm value under the assumption of future liquidation. The vertical dotted line descending from the peak of the dashed curve marks C_L, the optimal coupon under future liquidation, on the x-axis. The dotted curve plots $F^B(\delta_0, C_0)$, the firm value under the assumption of future Chapter 11. The vertical dotted line descending from the peak of the dotted curve marks C_B, the optimal coupon under future Chapter 11 reorganization, on the x-axis. The solid curve plots the actual firm value $F_0(\delta_0, C_0)$ as a function of the coupon C_0. Finally, the vertical dotted line ending in $\bar{C}$ on the x-axis marks the largest coupon for which equity finds liquidation optimal ex-post. The parameters corresponding to this figure are the baseline parameters from Table 1(a).

unconstrained in their decision to maximize the firm value under future liquidation.

Case 3: Choiceval < 1. Perhaps the most interesting case is when the Chapter 11 procedure is less efficient than Chapter 7, but efficient enough that equityholders still find it attractive ex-post for reasonable levels of debt. The inefficiency of Chapter 11, combined with equity's lack of commitment power, will actually reduce firm value in this region, relative to the value if Chapter 11 were not an option. To see this intuitively, suppose that Chapter 7 is slightly more efficient than Chapter 11. In such a situation, equityholders might be able to get a much lower cost of debt by committing to a future Chapter 7 liquidation. If equityholders had commitment power, in this case they would promise a future Chapter 7 liquidation and issue the coupon C_L that optimally trades off tax benefits with the liquidation costs. However, such a coupon C_L would imply that equityholders default in a profitable state of the world, where the reasonably efficient Chapter 11 is appealing to equityholders. Since equityholders lack commitment power, debtholders recognize that the coupon C_L will correspond to a future Chapter 11, and charge a higher cost of debt.

This leaves the equityholders with two choices. If the higher cost of debt associated with

35

Chapter 7 is small in magnitude relative to the tax benefits of debt, equityholders will optimally issue the coupon C_B which maximizes tax benefits relative to Chapter 11 inefficiencies. In this case, equityholders are optimally choosing an inefficient Chapter 11 process and a high cost of debt, because it allows them to capture tax benefits. An example of this case is shown in Figure 7, which plots firm value as a function of the coupon C_0, for the parameters in Table 1(b) which induce RelEff= 0.85.

Figure 7 has the same interpretation as Figures 5 and 6. The highest firm value, corresponding to the coupon C_L, is on the dashed curve depicting firm value under future liquidation. However, Chapter 11 is sufficiently attractive that $\bar{C} < C_L$, so equity's lack of commitment power prevents them from obtaining this firm value. The highest attainable value on the solid curve corresponds to C_B, the optimal coupon given a future Chapter 11. Thus, even though Chapter 11 is less efficient than Chapter 7, the tax benefits of a large coupon outweigh the increased cost of debt so equity optimally chooses Chapter 11. We call this the "optimal inefficient Chapter 11" strategy.

The other choice that equity can make in Case 3 is to issue $\bar{C}$. If the higher cost of debt associated with Chapter 11 is large relative to the tax benefits of a larger coupon, equityholders will sacrifice some tax benefits to issue a coupon that credibly commits them to a future Chapter 7. Specifically, equityholders will optimally issue $\bar{C}$, the largest coupon such that they will subsequently find Chapter 7 optimal.

Figure 8 plots firm value as a function of the initial coupon C_0, with a parameter set (Table 1(a)) corresponding to RelEff = 75%. Once again, the highest point on the graph occurs on the dashed curve, at $F^L(\delta_0, C_L)$, which corresponds to the firm value if equity could issue C_L and commit to Chapter 7. However, this value is unattainable for equityholders. If they issue debt with a coupon larger than $\bar{C}$, which is the point on the x-axis where the solid curve drops from the dashed curve to the dotted curve, then the value they receive is F^B. As a result, the best equityholders can do is to issue $\bar{C}$ and receive $F^L(\delta_0, \bar{C})$. We refer to this as the "constrained debt Chapter 7" strategy.

We reiterate that the equityholders optimally issue a lower coupon than in the Leland (1994) model, even though they subsequently face the exact same liquidation costs. This is because we have found a novel agency cost of debt: it encourages equityholders to destroy firm value in Chapter 11 bankruptcy. The coupon which optimally trades off tax benefits with liquidation costs and this novel agency cost is lower than the one predicted by the Leland model.

5.2 Results on capital structure

In Graham (2000), he finds that "paradoxically, large, liquid, profitable firms with low expected distress costs use debt conservatively" and "the typical firm could double tax benefits by issuing debt until the marginal tax benefit begins to decline." Contingent claims models like that in Leland (1994) have historically had a difficult time matching the 20-25% quasi-market leverage ratios typical of Compustat firms (Strebulaev and Whited (2012)) without assuming extreme liquidation costs or adding much more complicated assumptions.

We solve our model for the parameters of Table 1(a), which correspond to the "constrained debt

Chapter 7" case of our model, with a RelEff value of 75%. In this case, equityholders optimally issue $\bar{C}$, which is 0.64 for these parameters, in order to credibly commit to a future liquidation. The optimal coupon with just liquidation (C_L) is 1.36, roughly twice as large as $\bar{C}$. We follow the literature in evaluating leverage as $D_0(\delta_0, C^*)/F_0(\delta_0, C^*)$, the value of debt at issuance divided by the sum of equity and debt values at issuance, all evaluated at the optimal coupon. For these exact parameters, the Leland model (see Section 2.2) predicts a leverage ratio of 70%, while in our equilibrium the leverage ratio is just 40%. To be clear, many models predict lower leverage ratios than the Leland model. However, simply by adding a realistic choice between Chapter 11 and Chapter 7, our model can lead to a leverage ratio 30 percentage points lower than Leland (1994), even though the equilibrium behavior is indistinguishable.
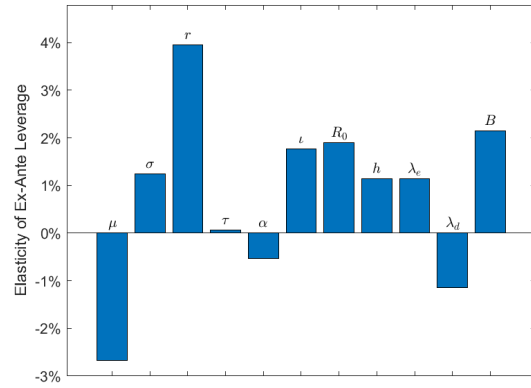
We calculate the unlevered firm value as $U = (1-\tau)\delta_0/(r-\mu)$, the perpetuity value of the cashflows. The ratio of the levered firm value to the unlevered firm value, calculated as $F_0(\delta_0, C^*)/U$, captures the value of debt in our model. Without the Chapter 11 option, the tax benefits of debt (net the liquidation inefficiencies) would add 11% to the unlevered firm value. However, in order to credibly commit to Chapter 7, in our model firms can only add 8% to their unlevered firm value. For these parameters, our model thus suggests the option to reorganize costs firms 3% of their unlevered firm value.

Chapter 11 efficiency and capital structure: Figure 9 shows comparative statics for the capital structure implied by our model. Starting with the parameter values in Table 1(a), we compute the optimal coupon, leverage, and ratio of levered firm value to unlevered firm value as above. Then, one parameter at a time, we increase the parameter by 5% of its value and recalculate these quantities. The figure plots elasticities (the percent change resulting from a 5% change in each parameter) for each of these quantities. In this "constrained debt Chapter 7" case, anything which makes Chapter 11 less appealing will increase $\bar{C}$. Intuitively, when Chapter 11 gets worse for equityholders, it is easier for them to promise not to file for Chapter 11, which lets them issue a higher coupon while receiving the lower cost of debt corresponding to Chapter 7. When we increase R_0 by 5%, panels (a) and (b) of Figure 9 show that the optimal coupon and leverage increase. Since the marginal benefit of debt is positive, this increases the levered firm value as well (panel (c)). Counterintuitively, a less efficient Chapter 11 process is actually increasing firm value. This general effect can be observed in many parameters which affect the relative attractiveness of Chapter 11 for equityholders. Increasing the rate of conversion ι to Chapter 7 or increasing the cost to equity B of entering Chapter 11 both lead to higher leverage and firm value. Increasing h, which improves the efficiency of Chapter 11, actually increases leverage and firm value, but this is because it makes equityholders worse off in the bargaining (Figure 2) which loosens their constraint.

Of course, all of these results depend upon the firm finding the "constrained debt Chapter 7" strategy optimal. In Figure 10, we present the same comparative statics exercise for the parameters of Table 1(b), for which the "optimal inefficient Chapter 11" strategy is optimal (RelEff=85%). Here the firm is optimally choosing Chapter 11, so for small declines in the efficiency of Chapter 11 (for example, higher R_0, B or lower h) the firm optimally reduces debt. This is the standard tradeoff
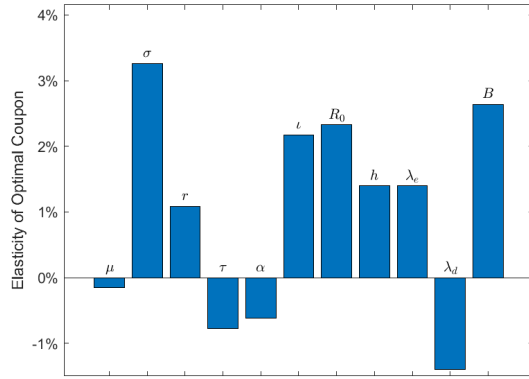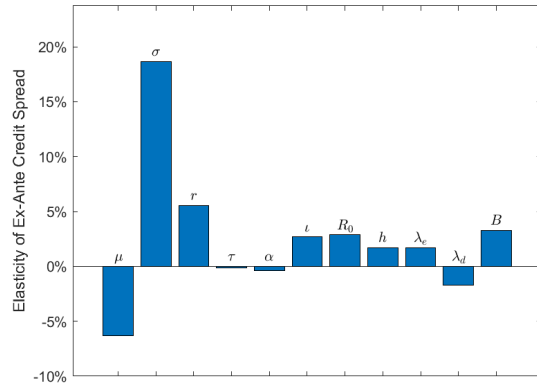
(a) Optimal Coupon

(b) Ex-Ante Leverage
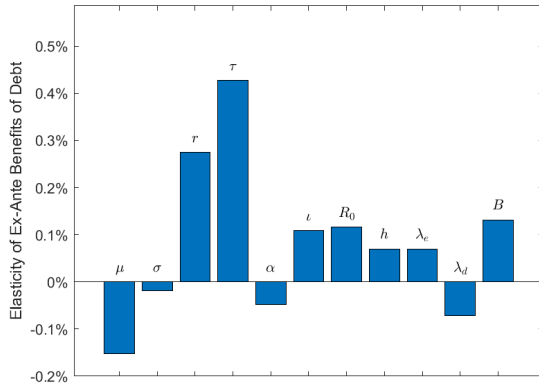
(c) $\frac{\text{Levered Firm Value}}{\text{Unlevered Firm Value}}$

(d) Ex-Ante Credit Spread

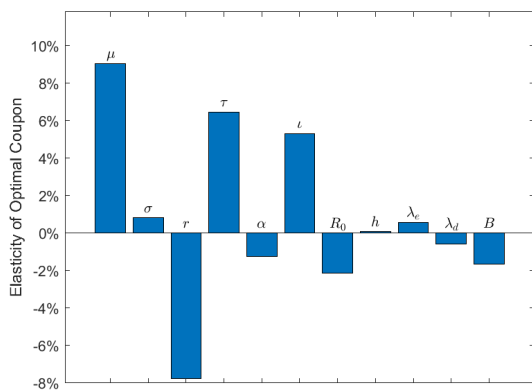Figure 9: **Capital Structure Comparative Statics, "Constrained Debt Chapter 7" Case**
Using the baseline parameters of Table 1(a), we calculate the optimal coupon and the corresponding ex-ante leverage, ex-ante credit spread, and the ratio of the firm value to the unlevered firm value U as in the text. Then, for each parameter individually, we increase that parameter by 5% of its baseline value, and recalculate each of the four quantities. Panel (a) plots the percentage change in the optimal coupon from increasing each parameter, one at a time, by 5%. Panels (b), (c), and (d) plot the corresponding changes for optimal ex-ante leverage, the ratio of the time 0 levered firm value to the perpetuity value of the cashflows, and the ex-ante credit spread, respectively.

theory logic, and this decline in efficiency is accompanied by a decline in firm value. However, an increase in ι, the rate of conversion to Chapter 7, can still increase leverage. While such an increase makes Chapter 11 slightly less efficient, it also makes Chapter 11 much better for debtholders since it endogenously increases their outside option and thus their bargaining position. As a result, equityholders take advantage of the lower cost of debt by issuing more debt, increasing firm value.

Creditor rights and capital structure: There is a growing empirical literature examining the real effects of creditor rights. Li, Whited, and Wu (2016) study the enactment of antirecharacterization laws in seven states in the late 1990s and early 2000s. These laws protected the rights of creditors who used special purpose vehicles to conduct secured borrowing, and several papers argue these represent an exogenous increase in creditor rights. Li et al (2016) find this led to an increase in leverage. Mann (2015) studies the same laws and also finds an increase in long term debt over assets.

In our model, creditor rights might reasonably be interpreted as the relative bargaining power of debtholders. Laws like the antirecharacterization laws certainly reduce the relative bargaining power of equityholders since they may no longer hold up creditors with the threat of recharacterizing assets held in special purpose vehicles. As noted in Section 3, the timing in the dynamic bargaining game is not affected by the relative bargaining power of creditors and debtors. However, the bargaining power of creditors affects the fraction of firm value that can be captured by equityholders in Chapter 11, which factors into the capital structure decision of equityholders. Recall that higher λ_e values correspond to shorter offer windows for equityholders and stronger creditor rights. In the "constrained debt Chapter 7" strategy considered in Figure 9, an increase in creditor rights (increasing λ_e by 5%) increases leverage by approximately 1%. Similarly, increasing λ_d lowers leverage. This is the same mechanism discussed above: when creditor rights improve, Chapter 11 becomes less attractive to equityholders, so they may issue a larger coupon $\bar{C}$ while still credibly committing to a Chapter 7. In Figure 10 where Chapter 11 is optimal, better creditor rights still lead to an increase in leverage, because it makes Chapter 11 more appealing to debtholders and lowers the cost of debt. Thus our model generally predicts that stronger creditor rights lead to higher leverage, consistent with the empirical evidence. The only exception to this is when an increase in creditor rights pushes equityholders from Chapter 11 to Chapter 7. In unreported results we have found that a decline in λ_d can make equityholders prefer the lower cost of debt associated with Chapter 7 to the rents they can extract in Chapter 11. They then drastically reduce their coupon from C_B to $\bar{C}$ and choose liquidation instead of Chapter 11.

In most cases, when creditor rights improve, the resulting increase in leverage leads to an improvement in firm value. Under the "constrained debt Chapter 7" strategy, this is because the marginal benefits of debt for the firm are positive at the optimum and equity's constraint becomes looser with stronger creditor rights. Thus the increase in debt has a net positive effect on firm value. Under the "optimal inefficient Chapter 11" strategy, when creditor rights improve, the expected costs of default endogenously decline. This is because creditors get a better Chapter 11 outcome so equity waits longer to default for any given coupon. As a result, our model predicts that stronger
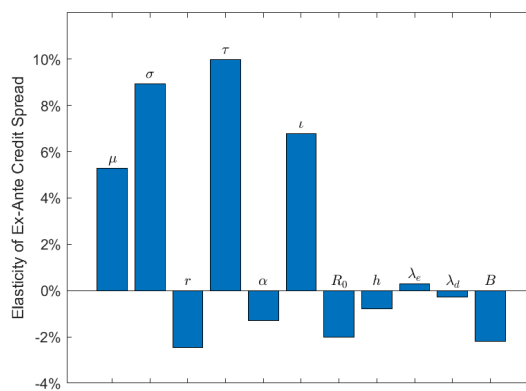
(a) Optimal Coupon
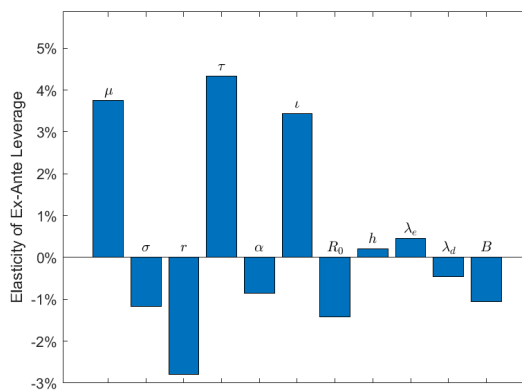
(b) Ex-Ante Leverage
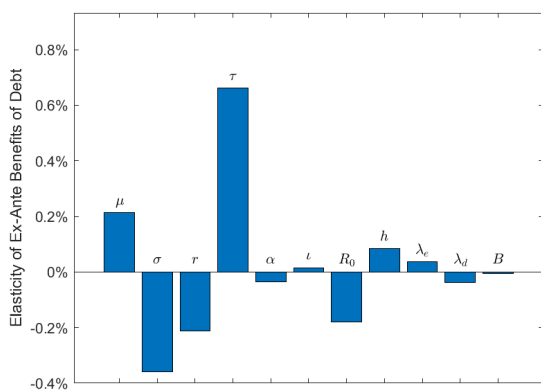
(c) $\frac{\text{Levered Firm Value}}{\text{Unlevered Firm Value}}$

(d) Ex-Ante Credit Spread

Figure 10: **Capital Structure Comparative Statics, "Optimal Inefficient Chapter 11" Case**

Using the baseline parameters of Table 1(b), we calculate the optimal coupon and the corresponding ex-ante leverage, ex-ante credit spread, and the ratio of the firm value to the unlevered firm value U as in the text. Then, for each parameter individually, we increase that parameter by 5% of its baseline value, and recalculate each of the four quantities. Panel (a) plots the percentage change in the optimal coupon from increasing each parameter, one at a time, by 5%. Panels (b), (c), and (d) plot the corresponding changes for optimal ex-ante leverage, the ratio of the time 0 levered firm value to the perpetuity value of the cashflows, and the ex-ante credit spread, respectively.

creditor rights should improve firm value. There is empirical evidence for this comparative static. Ponticelli and Alencar (2016) find an increase in value after an increase in the enforceability of creditor rights, and Ersahin (2017) finds greater productivity after the antirecharacterization laws discussed previously. However, these empirical results have different mechanisms than the tax benefits of debt which drives the result in our model.

Other model primitives: We briefly summarize the other predictions of our model for capital structure. All our results vary depending on the cases described above, but we focus on the results which do not appear in the standard Leland model. For example, it is possible for an increase in μ to lead to lower optimal leverage (Figure 9). This is because higher expected tax benefits make Chapter 11 more attractive. This forces equityholders to issue a lower coupon with lower leverage to credibly commit to Chapter 7, which is in some cases optimal.

Unlike in the Leland model, higher volatility can lead to higher leverage. With only liquidation, higher volatility lowers optimal leverage and the overall levered firm value. The last effect means that the reorganized firm value is lower when volatility is higher. This tends to lower the relative efficiency of Chapter 11, so for the "constrained debt Chapter 7" strategy this can lead to higher leverage in our model.

In any tradeoff model, higher taxes imply greater tax benefits of debt and thus more debt. However, Chapter 11 includes an embedded option to relever upon reorganizing, making it more appealing to equityholders when taxes are high. Figure 9 presents an example where, when taxes increase by 5%, the commitment effect outweighs the time 0 increase in tax benefits and the firm optimally lowers its coupon to commit to Chapter 7. Finally, even when liquidation inefficiencies are tiny ($\alpha = 0.005$), the commitment problem in our model can lead to leverage as low as 44%, compared to 78% in the Leland model.

Credit spreads: Our model produces credit spreads by the formula

$$\text{CS} = \frac{C^*}{D_0(\delta_0, C^*)} - r, \tag{49}$$

where $D_0(\delta_0, C^*)$ is the value of debt at the optimal coupon at issuance. Panel (d) of Figures 9 and 10 suggest that changes in Chapter 11 costs and the bargaining parameters have similar effects on credit spreads as they do on leverage and the optimal coupon. This is intuitive since higher coupons always lead to earlier default and thus riskier debt.

What is perhaps most interesting in our model is not the comparative statics of credit spreads but the levels. In cases where equityholders find Chapter 11 to be ex-post optimal, debtholders demand a higher cost of debt to compensate them for the rents that equityholders will extract in Chapter 11. The credit spread puzzle suggests that models like Leland (1994) tend to underestimate credit spreads on risky debt. By adding the option of Chapter 11, we can produce credit spreads higher than those in the Leland model. In the "optimal inefficient Chapter 11" strategy, the higher default costs lead equityholders to issue less debt than if they only were able to liquidate. This is

because they internalize the default costs when they issue debt at time 0. However, the debtholders still demand compensation for the future Chapter 11 reorganization. As a result, for the parameters in Table 1(b), the model can simultaneously generate a credit spread 17 basis points higher than the Leland model while producing an optimal leverage ratio that is 7 percentage points lower.

5.3 The decision to enter Chapter 7 or Chapter 11

There is recent interest in empirical research about the causal effect of bankruptcy procedure on future firm asset performance. The main challenge in such work is overcoming the selection bias, that firms choosing Chapter 11 are inherently different from those choosing Chapter 7. Any statements our model might generate about the causal effect of Chapter 11 vs Chapter 7 would be dependent on the parameter values we assume. However, our model generates much more general predictions about what types of firms choose Chapter 11 or Chapter 7.

Profitability, asset value, and choice of bankruptcy procedure: In our model, when equityholders default, they choose Chapter 11 if and only if their value function in the subsequent bargaining justifies their fixed cost of entering Chapter 11 (Proposition 5). Since the value function is increasing in the current EBIT (δ), this implies that firms which are more profitable at default will choose Chapter 11. It is standard in the Leland model to define the firm asset value as the unlevered firm value U, which is linear in δ, so this also implies firms with more valuable assets at default should choose Chapter 11. These predictions of our model are supported by BWZ (2006), BCI (2017), and Corbae and D'Erasmo (2017). Specifically, in Table I of BWZ (2006), they find that the average asset value of firms entering Chapter 11 is nearly four times as large as the average asset value of firms entering Chapter 7. Their Table II shows in a Probit model that conditional on being a reasonable size, firms with more valuable assets are more likely to choose Chapter 11. Table 1 of Corbae and D'Erasmo (2017) similarly shows that firms entering Chapter 11 are roughly four times as large as those entering Chapter 7, and Table 1 of BCI (2017) shows firms in Chapter 11 have four times as many plants as firms entering Chapter 7. While none of these tables show statistics on unnormalized EBIT, firms entering Chapter 11 have a higher EBITDA normalized by assets than firms entering Chapter 7 (Table 1 of Corbae and D'Erasmo (2017)). Also, multiplying the median firm's EBITDA over assets by the median firm's assets in the same table suggests a higher unnormalized EBITDA for firms entering Chapter 11.

Debt and Chapter 11: In Proposition 6, we show that firms with higher coupons tend to choose Chapter 11 (specifically, those with a coupon above some threshold $\bar{C}$). This implies the prediction that defaulting firms should be more likely to choose Chapter 11 when they have a lot of debt. Table I of BWZ (2006) shows that firms entering Chapter 11 have a higher Debt/Assets ratio, and combining this with the higher denominator for firms in Chapter 11 mentioned previously, firms entering Chapter 11 have more debt. This is also a significant predictor of Chapter 11 in their Probit regressions. Table 1 of Corbae and D'Erasmo (2017) confirms these findings, consistent with our model. In summary, Propositions 5 and 6 characterize the decision between Chapter 7 and

(a) μ

(b) σ
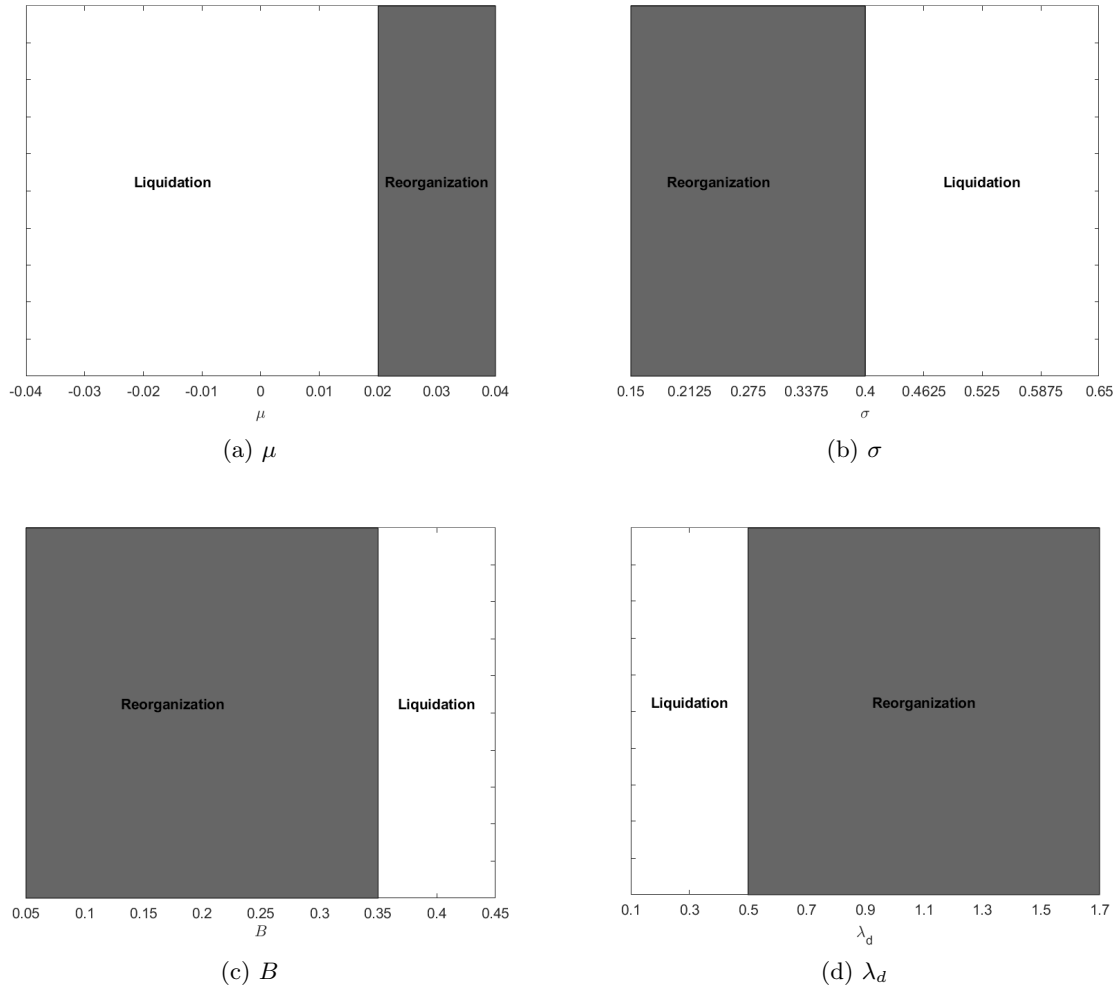
(c) B

(d) λ_d

Figure 11: **Chapter Choice**

We start with the baseline parameters of Table 1(b), for which the firm optimally chooses Chapter 11. In each subpanel, we change the pictured parameter to each of the nine values depicted on the x-axis. For each value, we evaluate the bankruptcy chapter chosen on the equilibrium path. The shaded area covers parameter values on the x-axis for which the firm chooses Chapter 11, while the white region covers parameter values for which the firm chooses Chapter 7. In all cases, shading is "right continuous" in that the firm chooses the chapter corresponding to the region immediately to the right of the x-axis value.

Chapter 11 in our model, and both of the mechanisms enjoy empirical support.

Other comparative statics: Figure 11 plots how equity's decision to choose Chapter 11 vs Chapter 7 varies with four parameters. We start with the baseline parameters of Table 1(b). Then, one parameter at a time for each of $\mu, B, \sigma, \lambda_d$, we change the parameter to nine different values and record the corresponding equilibrium chapter choice. Higher growth firms value the tax shield of debt more, which makes them want to issue a large coupon. Higher μ also increases the relative efficiency of Chapter 11 since it allows the firm to relever. This means that ceteris paribus, firms with higher μ tend to prefer Chapter 11 (panel (a)), since larger coupons encourage Chapter 11 ex-post. To our knowledge, this is a novel empirical prediction. It also suggests that estimates of the benefits of Chapter 11 might be overstated, since the firms which chose to enter Chapter 11 might have had higher growth rates on average.

Higher volatility has the opposite effect (panel (b)) as it decreases the relative efficiency of Chapter 11. Intuitively, when it is more costly for equityholders to enter Chapter 11 (B increases), equityholders are more likely to choose Chapter 7 (panel (c)). Stronger creditor rights (lower λ_d) make Chapter 11 less appealing to equityholders, and this tends to encourage Chapter 7 (panel (d)). Similarly, in unreported results we find a higher rate of conversion to Chapter 7 tends to discourage equityholders from paying for Chapter 11.

5.4 Length of Chapter 11

As we discussed in Section 3.6, our model of Chapter 11 produces the realistic result that Chapter 11 cases of any length can occur with positive probability on the equilibrium path. Our model also produces a comparative static for the length of bankruptcy which is consistent with empirical evidence. Bandopadhyaya (1994) finds that firms with a greater interest burden have a significantly higher instantaneous probability of exiting Chapter 11. In our model, firms with a larger interest burden endogenously default in more profitable states, which leads them to reach their upper reorganization threshold faster.

Our model also produces novel predictions for the length of Chapter 11 cases. The length of Chapter 11 is stochastic, since it depends upon the path of the EBIT δ_t during bankruptcy. Rather than simulate the average length, we use the metric $R_0/(\bar{x}\delta_B)$. Recall that δ_B is the endogenous threshold at which equity defaults and enters Chapter 11, while the firm emerges from Chapter 11 the first time t that $\delta_t \geq R_t/\bar{x}$, where $\bar{x}$ is endogenous. When $h = 0$, we have $R_t = R_0$ for all t, so $R_0/(\bar{x}\delta_B)$ reflects the factor by which cashflows must improve to exit Chapter 11.

Figure 12 plots elasticities of this metric $R_0/(\bar{x}\delta_B)$ with respect to a 5% change in various parameters from their baseline values (Table 1(b)). Based on Figure 12, we find that lower growth firms, higher volatility firms, and firms with greater shareholder bargaining power all have longer bankruptcy procedures. We are unaware of any empirical papers which test these findings.

Figure 12: **Comparative Statics for Equilibrium Length of Chapter 11**
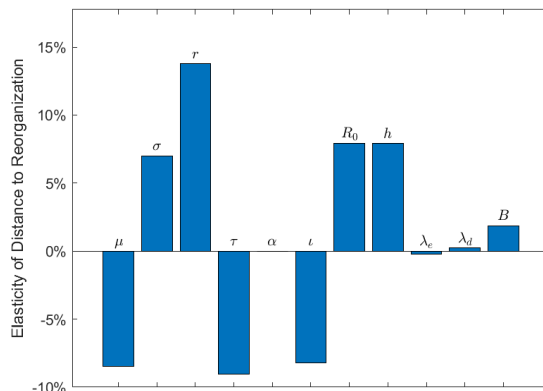
Using the baseline parameters of Table 1(b), we calculate the optimal reorganization threshold $\bar{x}$ and the optimal threshold δ_B for entering Chapter 11, and use them to calculate the ratio $R_0/(\bar{x}\delta_B)$. Then, for each parameter individually, we increase that parameter by 5% of its baseline value, and recalculate $R_0/(\bar{x}\delta_B)$. This figure plots the percentage change in $R_0/(\bar{x}\delta_B)$ from increasing each parameter, one at a time, by 5%.

6 Extensions and conclusion

In this section, we informally consider how two assumptions in our model might impact our results. First, since in reality some firms enter Chapter 11 multiple times, in Section 6.1 we discuss what might happen if we allowed firms to enter Chapter 11 more than once. Second, since empirically firms adjust their leverage while our model allows for just one capital structure decision, Section 6.2 discusses how leverage adjustments might impact our results. Section 6.3 concludes.

6.1 Multiple Chapter 11 opportunities

For tractability, we assume that firms can only enter Chapter 11 once, so after reorganizing equityholders must liquidate if they subsequently default. If we were to extend the model to allow for two Chapter 11 opportunities, we believe this would not significantly change the results. Suppose we allow two Chapter 11 opportunities, and consider a history in which the firm chose to default, entered Chapter 11, and has just emerged. When this reorganized firm is choosing its new capital structure, it looks exactly like a firm at time 0 in our model, and makes the exact capital structure decision we described in the previous section. If Choiceval > 1, then this relevered firm value is more attractive than the corresponding reorganized firm value with only Chapter 7 available in the future. Then, considering a history where the firm has not yet defaulted, Chapter 11 looks more appealing, since the reorganized firm value is even higher. Since Choiceval > 1, the firm was likely already going to choose Chapter 11, and now with two Chapter 11 opportunities this is even more appealing, so the firm chooses Chapter 11 at the first default. Thus, in this case the choice

of Chapter 7 vs Chapter 11 is unchanged, and the firm likely issues slightly more debt at time 0 since the first Chapter 11 is now less costly. It is unlikely that this increase in debt is significant, since the reduction in the cost of Chapter 11 only occurs in an unlikely state of the world (where the firm defaults and reemerges before a forced conversion) that is heavily discounted.

If Choiceval $= 1$, then when the firm emerges from the first Chapter 11, they ignore the option to enter a second Chapter 11 and issue a coupon consistent with future Chapter 7. But then in a history where the firm has not yet defaulted the first time, the first Chapter 11 looks exactly as attractive as it would if there were no second Chapter 11 opportunity. In this case, the firm still chooses Chapter 7 at the first default, and the time 0 coupon is unchanged.

If Choiceval < 1, it is possible the second Chapter 11 might change behavior. In particular, the firm value upon emerging from the first Chapter 11 will now be lower than in the current model, which makes the first Chapter 11 less appealing. This could slightly increase debt in the "constrained debt Chapter 7" strategy, since it is easier for equity to promise ex-post not to enter Chapter 11. However, it could also shift the strategy from "optimal inefficient Chapter 11" to the "constrained debt Chapter 7" strategy, since now the inefficiency of the first Chapter 11 could outweigh the tax benefits of the larger coupon $C_B > \bar{C}$.

In summary, adding a second Chapter 11 might lead to slightly higher debt in some cases, while it could also drastically reduce debt if it causes a firm to shift from the "optimal inefficient Chapter 11" strategy to the "constrained debt Chapter 7" strategy. There is no change in the intuition if we were to add a third, fourth, or general nth Chapter 11 opportunity.

6.2 Leverage adjustments and dynamic capital structure

In our model, we assume the firm can only issue debt once (as in Leland (1994)). There is empirical evidence suggesting that some firms adjust their leverage ratios infrequently (Fama and French (2002); Leary and Roberts (2005); Korteweg, Schwert, and Strebulaev (2014)). Further, debt covenants are often written with tight interest coverage ratios, such that further issuance is restricted even at the inception of the loan (Chava and Roberts (2006)). However, in reality many firms adjust their book leverage over time: the within-firm standard deviation of book leverage in Compustat is approximately 12% (see Korteweg, Schwert, and Strebulaev (2014); DeAngelo, DeAngelo and Whited (2011); DeAngelo and Roll (2015); DeAngelo, Goncalves and Stulz (forthcoming)). Our assumption that firms issue debt once is an abstraction from reality we make for tractability, which has the added benefit that it makes our results simpler to interpret.

One potential means of relaxing this assumption would be to let firms issue callable debt with an indenture restricting further issuance prior to calling existing debt. This is the assumption made in the dynamic capital structure literature (Leland (1998), Goldstein, Leland and Ju (2001), Strebulaev (2007)). The nonlinearity of equity's Chapter 11 bargaining value function would violate the scaling property needed to solve such models. However, we imagine this would not change the implications of our model too drastically. In general, the tax benefits of any particular debt issue are lower in these models since the firm may always refinance with a larger coupon if profitability

improves. Thus at time 0 firms would generally issue less debt, and in our Case 3 it is more likely that firms would find the "constrained debt Chapter 7" strategy optimal than the "optimal inefficient Chapter 11" strategy since large coupons are less valuable.

Importantly, the ability to refinance and issue more debt in such a setting would not change the ability of equityholders to credibly commit to Chapter 7. Should equityholders issue a coupon $\bar{C}$ consistent with future Chapter 7, the creditors purchasing this debt would recognize that if equityholders want to issue more debt, the firm would first need to call the outstanding debt at par, defending the existing creditors from devaluation by the shift to an ex-post optimal Chapter 11. This informal logic suggests our "constrained debt Chapter 7" strategy would be robust to allowing multiple debt issuances.

6.3   Conclusion

This paper studies the choice of bankruptcy chapter and its relationship to capital structure decisions. We provide a model of an equity value-maximizing firm that decides how much debt to issue, then subsequently chooses when and under which bankruptcy chapter to default. We model Chapter 11 reorganization with a novel continuous-time stochastic bargaining model in the style of Merlo and Wilson (1995). Specifically, equityholders and debtholders observe the firm's assets and accumulated cashflows evolve stochastically, and they must unanimously agree when to emerge from Chapter 11 and how to split the firm. The reorganized firm can then issue new debt and continue operating.

There may often be a conflict between the desires of equityholders and creditors in terms of their relative treatment in the two chapters of bankruptcy. Equityholders with larger debt obligations endogenously default in more profitable states, in which they prefer the prospect of reorganization. Creditors might enjoy higher recovery rates in Chapter 7 liquidation due to APR. Thus, when the firm issues debt, creditors take these incentives into account and demand higher credit spreads for large coupons that imply a subsequent Chapter 11. In some cases, the model predicts equityholders will optimally issue a coupon that implies a future inefficient Chapter 11, leading to lower leverage and higher credit spreads than the Leland (1994) model. In other cases, equityholders optimally issue a small coupon, such that they will find Chapter 7 optimal ex-post, to obtain a lower credit spread. For a reasonable parameterization of this case, our model predicts an optimal leverage ratio of 40% while the Leland (1994) model predicts 70%, even though the firm liquidates on the equilibrium path in both models. Stated another way, while in our model the observed bankruptcy behavior may be identical to that of the Leland model in which only liquidation may be undertaken, the off-equilibrium threat of reorganization delivers a much lower optimal leverage ratio. The added option of Chapter 11 actually reduces ex-ante firm value in these cases, since equityholders cannot commit to a future Chapter 7 liquidation.

Several extensions of our model may prove illuminating. The model could be generalized to incorporate asymmetric information between equityholders and debtholders, given the rich tradition in the corporate finance literature exploiting the implications of hidden information. In addition,

the model could be extended to a multiple-firm industry equilibrium (similar to Lambrecht (2001); Grenadier (2002); Miao (2005)). The latter extension could produce interesting interactions wherein firms might push rivals toward Chapter 7 rather than Chapter 11 in order to reduce competition. Finally, the model's framework might prove useful for empirical work aimed at estimating the relative inefficiencies of Chapter 7 and Chapter 11.

A Solving Chapter 11 efficiently

First, we provide an expression for the constant θ. In the notation of Section 2, let

$$p^1 = \frac{r}{r - \mu} \frac{\psi - 1}{\psi} [\frac{-\tau}{\psi(1 - \tau)\alpha + (\psi - 1)\tau}]^{\frac{-1}{\psi}}$$
$$p^2 = \frac{\psi}{\psi - 1} \frac{r - \mu}{r},$$

so $C^* = p^1 \delta$ and $\delta_L = p^2 C^* = p^1 p^2 \delta$. Summing the values of equity and debt,

$$\begin{aligned}
E^L(\delta) + D^L(\delta) &= \delta^\psi \delta_L^{-\psi}(1 - \tau)[\frac{C^*}{r} - \frac{\delta_L}{r - \mu}] \\
&+ \frac{(1 - \tau)}{r - \mu}\delta - \frac{(1 - \tau)C^*}{r} + \frac{C^*}{r} \\
&+ \delta^\psi \delta_L^{-\psi}[-\frac{C^*}{r} + (1 - \alpha)(1 - \tau)\frac{\delta_L}{r - \mu}] \\
&= \frac{1 - \tau}{r - \mu}\delta + \frac{\tau C^*}{r} - \delta^\psi \delta_L^{-\psi}[\tau\frac{C^*}{r} + \alpha(1 - \tau)\frac{\delta_L}{r - \mu}].
\end{aligned}$$

Evaluating at δ_0 and plugging in the above formulas, this is

$$\begin{aligned}
&= \frac{1 - \tau}{r - \mu}\delta_0 + \frac{\tau p^1 \delta_0}{r} - \delta_0^\psi (p^1 p^2 \delta_0)^{-\psi}[\tau\frac{p^1 \delta_0}{r} + \alpha(1 - \tau)\frac{p^1 p^2 \delta_0}{r - \mu}] \\
&= [\frac{1 - \tau}{r - \mu} + \frac{\tau p^1}{r} - (p^1 p^2)^{-\psi}[\tau\frac{p^1}{r} + \alpha(1 - \tau)\frac{p^1 p^2}{r - \mu}]]\delta_0 \\
&= \theta \delta_0
\end{aligned}$$

Next, in many of the proofs in this appendix, we will need to apply dominated convergence. This next lemma allows us to do so under the assumption of $r > \mu$, whenever the function in question can be bounded by an affine function of δ, R.

Lemma A.1 *For any fixed constants $\delta_0, R_0, B_1, B_2 > 0$,*

$$\mathbb{E}^{\delta_0, R_0}[\sup_t e^{-rt}(B_1 \delta_t - B_2 R_t)] < \infty.$$

48

Proof of Lemma: Since $r > \mu$, for an arithmetic Brownian motion $Z_t = (-r + \mu - \frac{\sigma^2}{2})t + \sigma B_t$, the supremum of Z_t over all t has an exponential distribution with parameter $\hat{\lambda} = \frac{2|-r+\mu-\frac{\sigma^2}{2}|}{\sigma^2} > 1$ (see, for example, Graversen and Peskir (1998)). It follows that

$$\mathbb{E}^{\delta_0}[\sup_t e^{-rt}\delta_t] = \delta_0 \mathbb{E}[\sup_t e^{Z_t}] = \delta_0 \frac{\hat{\lambda}}{\hat{\lambda}-1} < \infty,$$

and, decomposing $r = r_1 + r_2$ with $r_1 > \mu$, $r_2 > 0$,

$$
\begin{aligned}
\mathbb{E}^{\delta_0}[\sup_t e^{-rt} \int_0^t \delta_s ds] &\leq \mathbb{E}^{\delta_0}[\sup_t \int_0^t e^{-rs}\delta_s ds] \\
&\leq \mathbb{E}^{\delta_0}[\int_0^\infty e^{-rs}\delta_s ds] \\
&= \mathbb{E}^{\delta_0}[\int_0^\infty e^{-r_2 s} e^{-r_1 s}\delta_s ds] \\
&\leq \mathbb{E}^{\delta_0}[\int_0^\infty e^{-r_2 s}(\sup_s e^{-r_1 s}\delta_s)ds] \\
&= \mathbb{E}^{\delta_0}[(\sup_s e^{-r_1 s}\delta_s) \int_0^\infty e^{-r_2 s}ds] \\
&= \frac{1}{r_2}\mathbb{E}^{\delta_0}[(\sup_s e^{-r_1 s}\delta_s)],
\end{aligned}
$$

which is similarly finite. Then putting everything together, we have that

$$
\begin{aligned}
&\mathbb{E}^{\delta_0, R_0}[\sup_t e^{-rt}(B_1 \delta_t - B_2 R_t)] \\
&= \mathbb{E}^{\delta_0}[\sup_t e^{-rt}(B_1 \delta_t + B_2 h(1-\tau) \int_0^t \delta_s ds - B_2 R_0)] \\
&\leq \mathbb{E}^{\delta_0}[|\sup_t e^{-rt}(B_1 \delta_t + B_2 h(1-\tau) \int_0^t \delta_s ds)|] \\
&\leq B_1 \mathbb{E}^{\delta_0}[\sup_t e^{-rt}\delta_t] + B_2 h(1-\tau)\mathbb{E}^{\delta_0}[\sup_t e^{-rt} \int_0^t \delta_s ds] \\
&< \infty,
\end{aligned}
$$

completing the proof. As an immediate corollary, for any fixed δ, R,

$$
\begin{aligned}
V(\delta, R) = \sup_{T_R \in F^{\delta, R}} \mathbb{E}^{(\delta, R)}[&\mathbf{1}(T_R < T_c)e^{-rT_R}(\theta \delta_{T_R} - R_{T_R}) \\
&+ \mathbf{1}(T_c < T_R)e^{-rT_c}(\zeta \delta_{T_c} - R_{T_c})] < \infty
\end{aligned}
\tag{50}
$$

since, letting $T = T_R \wedge T_c$, there exist B_1, B_2 such that the expression in the expectation is less than

$$e^{-rT}(B_1\delta_T - B_2 R_T + B_2 R_0)$$

with probability 1.

Proof of Proposition 1: We introduce some simplifying notation. Let $O^* \equiv \{(\delta, R) : V(\delta, R) = \theta\delta - R\}$ be the set of values where the social planner's value function $V(\delta, R)$ equals the payoff $\theta\delta - R$. Fix a MPE with value functions E, D and equilibrium stopping time $\mathcal{T}$. It will be convenient to define the set $\mathcal{E} \equiv \cup_i O_i \times \{i\}$ so the game ends when $(\delta, R, s) \in \mathcal{E}$. From this point on, $\mathcal{T}$ is always defined as the first hitting time of $\mathcal{E}$. Let $V^e(\delta, R, s) \equiv E(\delta, R, s)$ and $V^d(\delta, R, s) \equiv D(\delta, R, s)$, and let $y \equiv \theta\delta - R$ and $z \equiv \zeta\delta - R$.

Now, the proof proceeds in three steps. First, from the definition of $J_i(\delta, R, s)$, we see that $\sum_i J_i(\delta, R, s) = y$. Given this, we have

$$
\begin{aligned}
&V^e(\delta, R, s) + V^d(\delta, R, s) \\
&= \mathbb{E}^{(\delta,R,s)}[\mathbf{1}(\mathcal{T} < T_c)e^{-r\mathcal{T}}y_{\mathcal{T}} + \mathbf{1}(T_c < \mathcal{T})e^{-rT_c}z_{T_c}] \\
&\leq V(\delta, R)
\end{aligned}
\tag{51}
$$

by the definition of $V(\delta, R)$. Second, we claim that if $(\delta, R) \in O^*$, then $V^e(\delta, R, s) + V^d(\delta, R, s) = V(\delta, R)$. From the first observation, the leftside cannot be strictly greater. If it were strictly less, then letting $s' \neq s$,

$$y - V^{s'}(\delta, R, s) = V(\delta, R) - V^{s'}(\delta, R, s) > V^s(\delta, R, s)$$

where the first equality is the definition of O^*. It follows that player s would have a strictly profitable deviation to offer the other player their value function. This implies that if $(\delta, R) \in O^*$, then

$$
\begin{aligned}
&\mathbb{E}^{(\delta,R,s)}[\mathbf{1}(\mathcal{T} < T_c)e^{-r\mathcal{T}}y_{\mathcal{T}} + \mathbf{1}(T_c < \mathcal{T})e^{-rT_c}z_{T_c}] \\
&= V^e(\delta, R, s) + V^d(\delta, R, s) \\
&= V(\delta, R) \\
&= \mathbb{E}^{(\delta,R)}[\mathbf{1}(\tau_* < T_c)e^{-r\tau_*}y_{\tau_*} + \mathbf{1}(T_c < \tau_*)e^{-rT_c}z_{T_c}]
\end{aligned}
\tag{52}
$$

where $\tau_* = \inf\{t : V(\delta_t, R_t) = y_t\}$ solves the optimal stopping problem. Third, any player can demand the game end at the maximum $\mathcal{T} \vee \tau_*$ with payoffs J_i. Specifically, any player i can deviate to making offers when $s_t = i$ and $(\delta, R) \in O_i \cap O^*$ and accepting offers from player j when $(\delta, R) \in O_j \cap O^*$. For this to be a MPE, this cannot be a profitable deviation for each player. Summing across i, we have

$$\sum_i V^i(\delta, R, s) \tag{53}$$

$$\geq \mathbb{E}^{(\delta,R,s)}[\mathbf{1}(\mathcal{T} \vee \tau_* < T_c)e^{-r\mathcal{T}\vee\tau_*}y_{\mathcal{T}\vee\tau_*} + \mathbf{1}(T_c < \mathcal{T} \vee \tau_*)e^{-rT_c}z_{T_c}]$$

$$= \mathbb{E}^{(\delta,R,s)}[\mathbf{1}(\mathcal{T} > \tau_*)\{\mathbf{1}(\mathcal{T} < T_c)e^{-r\mathcal{T}}y_{\mathcal{T}} + \mathbf{1}(T_c < \mathcal{T})e^{-rT_c}z_{T_c}\}]$$

$$+ \mathbb{E}^{(\delta,R,s)}[\mathbf{1}(\mathcal{T} < \tau_*)\{\mathbf{1}(\tau_* < T_c)e^{-r\tau_*}y_{\tau_*} + \mathbf{1}(T_c < \tau_*)e^{-rT_c}z_{T_c}\}].$$

Now, fix (δ_0, R_0, s_0). Let F_t be the filtration generated by (δ, R, s), which are jointly Markov. We have $F_0 \subset F_{\tau_*}$ where $\tau_*, \delta_{\tau_*}, R_{\tau_*}, s_{\tau_*}, \mathbf{1}(\mathcal{T} > \tau_*)$ are all F_{τ_*} measurable. Then

$$\mathbb{E}^{(\delta_0,R_0,s_0)}[\mathbf{1}(\mathcal{T} > \tau_*)\{\mathbf{1}(\mathcal{T} < T_c)e^{-r\mathcal{T}}y_{\mathcal{T}} + \mathbf{1}(T_c < \mathcal{T})e^{-rT_c}z_{T_c}\}]$$

$$= \mathbb{E}[\mathbf{1}(\mathcal{T} > \tau_*)\{\mathbf{1}(\mathcal{T} < T_c)e^{-r\mathcal{T}}y_{\mathcal{T}} + \mathbf{1}(T_c < \mathcal{T})e^{-rT_c}z_{T_c}\}|F_0]$$

$$= \mathbb{E}[\mathbf{1}(\mathcal{T} > \tau_*)\mathbb{E}\{\mathbf{1}(\mathcal{T} < T_c)e^{-r\mathcal{T}}y_{\mathcal{T}} + \mathbf{1}(T_c < \mathcal{T})e^{-rT_c}z_{T_c}|F_{\tau_*}\}|F_0].$$

Applying the Markov property, this equals

$$= \mathbb{E}[\mathbf{1}(\mathcal{T} > \tau_*)\mathbb{E}^{(\delta_{\tau_*},R_{\tau_*},s_{\tau_*})}\{\mathbf{1}(\mathcal{T} < T_c)e^{-r\mathcal{T}}y_{\mathcal{T}} + \mathbf{1}(T_c < \mathcal{T})e^{-rT_c}z_{T_c}\}|F_0],$$

and since $(\delta_{\tau_*}, R_{\tau_*}) \in O^*$ by definition, applying (52), this is

$$= \mathbb{E}[\mathbf{1}(\mathcal{T} > \tau_*)\mathbb{E}^{(\delta_{\tau_*},R_{\tau_*})}\{\mathbf{1}(\tau_* < T_c)e^{-r\tau_*}y_{\tau_*} + \mathbf{1}(T_c < \tau_*)e^{-rT_c}z_{T_c}\}|F_0]$$

$$= \mathbb{E}^{(\delta_0,R_0,s_0)}[\mathbf{1}(\mathcal{T} > \tau_*)\{\mathbf{1}(\tau_* < T_c)e^{-r\tau_*}y_{\tau_*} + \mathbf{1}(T_c < \tau_*)e^{-rT_c}z_{T_c}\}].$$

Plugging this in to (53), we have that

$$\sum_i V^i(\delta, R, s)$$

$$\geq \mathbb{E}^{(\delta,R,s)}[\mathbf{1}(\mathcal{T} > \tau_*)\{\mathbf{1}(\mathcal{T} < T_c)e^{-r\mathcal{T}}y_{\mathcal{T}} + \mathbf{1}(T_c < \mathcal{T})e^{-rT_c}z_{T_c}\}]$$

$$+ \mathbb{E}^{(\delta,R,s)}[\mathbf{1}(\mathcal{T} < \tau_*)\{\mathbf{1}(\tau_* < T_c)e^{-r\tau_*}y_{\tau_*} + \mathbf{1}(T_c < \tau_*)e^{-rT_c}z_{T_c}\}]$$

$$= \mathbb{E}^{(\delta,R,s)}[\mathbf{1}(\tau_* < T_c)e^{-r\tau_*}y_{\tau_*} + \mathbf{1}(T_c < \tau_*)e^{-rT_c}z_{T_c}]$$

$$= V(\delta, R),$$

completing the proof.

B   Efficient Chapter 11

Recall that, suppressing arguments, the HJB is

$$rV = -h(1-\tau)\delta V_R + \delta\mu V_\delta + \frac{\sigma^2}{2}\delta^2 V_{\delta\delta} + \iota[\zeta\delta - R - V],$$

where $\zeta = (1-\alpha)(1-\tau)/(r-\mu)$ such that $\zeta\delta - R$ is the liquidation value of the firm, and ιdt is the probability of liquidation per unit time.

We will solve this PDE by using a change of variables. Define $v \equiv V/\delta$ and $x \equiv R/\delta$. Note this means we expect exercise at low values of x. Straightforward calculus shows $v' = V_R$, $-xv' = V_\delta - v$, $v''x^2 = \delta V_{\delta\delta}$. Then dividing by δ and substituting, we get

$$(r + \iota - \mu)v = -(\mu x + h(1-\tau))v' + \frac{\sigma^2}{2}x^2 v'' + \iota(\zeta - x).$$

B.1 General solution of the homogeneous equation

To start, consider the homogeneous equation

$$(r + \iota - \mu)v = -(\mu x + h(1-\tau))v' + \frac{\sigma^2}{2}x^2 v''.$$

Conjecture a solution $v = x^\beta w(x)$ for some function w and constant β. This implies derivatives

$$v' = \beta x^{\beta-1}w + x^\beta w'$$
$$v'' = \beta(\beta-1)x^{\beta-2}w + 2\beta x^{\beta-1}w' + x^\beta w''$$

Plugging this conjecture in, we get

$$(r + \iota - \mu)v = -(\mu x + h(1-\tau))[\beta x^{\beta-1}w + x^\beta w']$$
$$+ \frac{\sigma^2}{2}x^2[\beta(\beta-1)x^{\beta-2}w + 2\beta x^{\beta-1}w' + x^\beta w''].$$

First, gather v terms:

$$0 = v[-(r + \iota - \mu) - \mu\beta + \frac{\sigma^2}{2}\beta(\beta-1)].$$

We define β such that this equals 0. That is, β is a positive or negative root of

$$0 = [-(r + \iota - \mu) - \mu\beta + \frac{\sigma^2}{2}\beta(\beta-1)].$$

Let κ be the positive root and γ be the negative root, and for now let β be a placeholder for either root. Plugging in this β, we're left with

$$0 = -h(1-\tau)\beta x^{\beta-1}w - (\mu x + h(1-\tau))x^\beta w' + \frac{\sigma^2}{2}x^2[2\beta x^{\beta-1}w' + x^\beta w''].$$

Multiply through by $x^{-\beta+2}$:

$$0 = -h(1-\tau)\beta xw - (\mu x + h(1-\tau))x^2 w' + \frac{\sigma^2}{2}[2\beta x^3 w' + x^4 w'']$$

Finally, conduct a second change of variables to $z \equiv [-2h(1-\tau)]/[\sigma^2 x]$ and $f(z) = w(x)$. Then $w' = f'[2h(1-\tau)]/[\sigma^2 x^2]$ and

$$w'' = -2f'\frac{2h(1-\tau)}{\sigma^2 x^3} + f''\frac{4(h(1-\tau))^2}{\sigma^4 x^4},$$

where combining implies

$$w'' = -2\frac{w'}{x} + f''\frac{4(h(1-\tau))^2}{\sigma^4 x^4}$$

$$x^4 w'' = -2x^3 w' + f''\frac{4(h(1-\tau))^2}{\sigma^4}$$

Plugging in,

$$0 = -h(1-\tau)\beta xf - (\mu x + h(1-\tau))\frac{2h(1-\tau)}{\sigma^2}f'$$
$$+ \frac{\sigma^2}{2}[f''\frac{4(h(1-\tau))^2}{\sigma^4} + 2(\beta-1)xf'\frac{2h(1-\tau)}{\sigma^2}].$$

Multiplying by $-1/[h(1-\tau)x]$ and rearranging,

$$0 = \beta f + ((-2(\beta-1) + \frac{2\mu}{\sigma^2}) - z)f' + zf'',$$

which is Kummer's ODE, with general solutions

$$f(z) = M(-\beta, -2(\beta-1) + \frac{2\mu}{\sigma^2}, z)$$

$$f(z) = U(-\beta, -2(\beta-1) + \frac{2\mu}{\sigma^2}, z).$$

Thus for either root $\beta = \kappa, \gamma$, and either solution f to Kummer's ODE, we get a solution

$$A_3 x^\beta f(\frac{-2h(1-\tau)}{\sigma^2 x})$$

for some constant A_3.

B.2   Applying boundary conditions

Since $x = R/\delta$, and R can go negative, x starts out large and positive, then declines over time. We conjecture that the option is exercised before x becomes negative (which we verify shortly). Then we should be looking for a solution on a positive domain of x. Also, as $x = R/\delta \to \infty$, the cost of exercise is large and the payoff is small, so the value should not explode. We now impose this condition.

As $x \to \infty$, $M(a, b, -2h(1 - \tau)/[\sigma^2 x])$ converges to $M(a, b, 0) = 1$. Thus $x^\beta M$ works as a solution for the negative root γ, but will not work for the positive root κ since then $x^\beta \to \infty$.

As $x \to \infty$, applying the positive root $\beta = \kappa$ we get $U(-\beta, -2(\beta - 1) + 2\mu/\sigma^2, 0)$ is a finite constant. But then $x^\beta U$ goes to infinity. Applying the negative root $\beta = \gamma$, $U(-\beta, -2(\beta - 1) + 2\mu/\sigma^2, z)$ explodes faster than x^β goes to 0, violating the boundary condition.

In conclusion, the homogeneous solution must take the form

$$v(x) = A_3 x^\gamma M\left(-\gamma, -2(\gamma - 1) + \frac{2\mu}{\sigma^2}, \frac{-2h(1 - \tau)}{\sigma^2 x}\right)$$

for some constant A_3.

B.3   Finishing the value function

Finally, we must add back in the risk of liquidation to the single agent optimization. Recall after the change of variables, the HJB may be written

$$(r + \iota - \mu)v = -(\mu x + h(1 - \tau))v' + \frac{\sigma^2}{2}x^2 v'' + \iota(\zeta - x).$$

As discussed above, the only solution to the homogeneous equation satisfying the necessary boundary conditions is

$$v(x) = A_3 x^\gamma M\left(-\gamma, -2(\gamma - 1) + \frac{2\mu}{\sigma^2}, \frac{-2h(1 - \tau)}{\sigma^2 x}\right),$$

where γ is the negative root of

$$0 = [-(r + \iota - \mu) - \mu\beta + \frac{\sigma^2}{2}\beta(\beta - 1)].$$

The relevant particular solution including the last nonhomogeneous term is

$$\frac{\iota\zeta + \frac{h(1 - \tau)\iota}{r + \iota}}{r + \iota - \mu} - \frac{\iota x}{r + \iota},$$

leading to a solution

$$v(x) = A_3 x^\gamma M\left(-\gamma, -2(\gamma - 1) + \frac{2\mu}{\sigma^2}, \frac{-2h}{\sigma^2 x}\right) + \frac{\iota\zeta + \frac{h(1 - \tau)\iota}{r + \iota}}{r + \iota - \mu} - \frac{\iota x}{r + \iota}$$

for some constant A_3. At exercise, the firm receives $\theta\delta - R$ so this should smooth paste on $\theta - x$. Conjecturing exercise occurs at a lower barrier $\bar{x}$, the smooth pasting and value matching conditions are

$$v(\bar{x}) = \theta - \bar{x}$$
$$v'(\bar{x}) = -1.$$

Note that

$$\frac{d}{dz}M(a, b, z) = \frac{a}{b}M(a+1, b+1, z),$$

so

$$v'(x) = A_3\gamma x^{\gamma-1}M(-\gamma, -2(\gamma-1) + \frac{2\mu}{\sigma^2}, \frac{-2h(1-\tau)}{\sigma^2 x})$$
$$- \frac{\iota}{r+\iota} + A_3 x^\gamma \frac{-\gamma}{-2(\gamma-1) + \frac{2\mu}{\sigma^2}} \frac{2h(1-\tau)}{\sigma^2 x^2}$$
$$\times M(-\gamma+1, -2(\gamma-1) + \frac{2\mu}{\sigma^2} + 1, \frac{-2h(1-\tau)}{\sigma^2 x}).$$

Solving for this v, we have $V(\delta, R) = \delta v(R/\delta)$ for $\delta \le R/\bar{x}$ and $V(\delta, R) = \theta\delta - R$ for $\delta \ge R/\bar{x}$. We now prove that this V is the value function for the social planner's problem, and the optimal policy is reorganize when $R/\delta \le \bar{x}$, or when $\delta \ge R/\bar{x}$.

B.4 Proof of Proposition 2

Define an operator $\mathcal{A}$ that maps smooth functions V of δ, R to

$$-h(1-\tau)\delta V_R + \delta\mu V_\delta + \frac{\sigma^2}{2}\delta^2 V_{\delta\delta} + \iota[\zeta\delta - R - V].$$

By construction, $V(\delta, R) = \delta v(R/\delta)$ is smooth since it smooth pastes at $\delta = R/\bar{x}$. Also by construction, $\mathcal{A}V = rV$ for $\delta \le R/\bar{x}$. For $\delta \ge R/\bar{x}$ we have $V = \theta\delta - R$, so in this region

$$\mathcal{A}V = h(1-\tau)\delta + \delta\mu\theta + \iota(\zeta - \theta)\delta$$
$$= [h(1-\tau) + \mu\theta + \iota(\zeta - \theta)]\delta,$$

and thus in this region, $-rV + \mathcal{A}V \le 0$ if and only if

$$-r(\theta\delta - R) + [h(1-\tau) + \mu\theta + \iota(\zeta - \theta)]\delta \le 0$$
$$\Longleftrightarrow \frac{h(1-\tau) + \mu\theta + \iota(\zeta - \theta) - r\theta}{r}\delta \le -R$$
$$\Longleftrightarrow -\frac{h(1-\tau) + \mu\theta + \iota(\zeta - \theta) - r\theta}{r} \ge x,$$

which is guaranteed by the first condition of Proposition 2,

$$-\frac{h(1-\tau) + \mu\theta + \iota(\zeta - \theta) - r\theta}{r} \ge \bar{x}.$$

By construction of V, we have $V(\delta, R) = \theta\delta - R$ when $\delta \ge R/\bar{x}$, and by condition 2 of Proposition 2,

55

$$V(\delta, R) = \delta v(\frac{R}{\delta}) \geq \delta(\theta - x) = \delta\theta - R,$$

so putting this together, under the conditions of Proposition 2, our candidate value function is smooth and satisfies the variational inequality

$$\max(\theta\delta - R - V, -rV + \mathcal{A}V) = 0.$$

Next, we show that there is a constant C such that $x \geq \bar{x} \Rightarrow v(x) \leq C$. To see this, we can use the fact that $\gamma < 0$ to write $M(-\gamma, -2(\gamma - 1) + 2\mu/\sigma^2, -2h(1 - \tau)/[\sigma^2 x]) = M(a, b, z)$ in its integral representation

$$M(a, b, z) = \frac{\Gamma(b)}{\Gamma(a)\Gamma(b - a)} \int_0^1 e^{zu} u^{a-1}(1 - u)^{b-a-1} du.$$

For $u \in [0, 1]$ we have $e^{z_1 u} \leq e^{z_2 u}$ for $z_1 \leq z_2$ and all terms are positive, so $M(a, b, z)$ is positive and monotonically increasing in z. Then $M(-\gamma, -2(\gamma - 1) + 2\mu/\sigma^2, -2h(1 - \tau)/[\sigma^2 x])$ is monotonically increasing in x for $x \geq \bar{x}$. Since the expression converges to $M(a, b, 0) = 1$ as $x \to \infty$, it follows that $M \in (0, 1)$ for $x \geq \bar{x}$. Then since $\gamma < 0$, we have $x^\gamma M \leq x^\gamma \leq \bar{x}^\gamma$ for $x \geq \bar{x}$. It follows that

$$v(x) = A_3 x^\gamma M(-\gamma, -2(\gamma - 1) + \frac{2\mu}{\sigma^2}, \frac{-2h}{\sigma^2 x}) + \frac{\iota\zeta + \frac{h(1-\tau)\iota}{r+\iota}}{r + \iota - \mu} - \frac{\iota x}{r + \iota}$$

$$\leq A_3 \bar{x}^\gamma + \frac{\iota\zeta + \frac{h(1-\tau)\iota}{r+\iota}}{r + \iota - \mu} - \frac{\iota\bar{x}}{r + \iota} = C \tag{54}$$

whenever $x \geq \bar{x}$. Thus $V(\delta, R)$ is bounded above by $C\delta$ whenever $R \geq \delta\bar{x}$, while $V(\delta, R) = \theta\delta - R$ when $R \leq \delta\bar{x}$. Combining this, it is clear we can bound $V(\delta, R)$ from above by the affine function $B\delta - R + R_0$ as long as $B > \max(\theta, C)$ and $R \leq R_0$, where the latter inequality holds almost everywhere by definition of R_t.

We are now ready to finish the verification. Fix δ_0, R_0 and define $Y_t \equiv \mathbf{1}(t < T_c)e^{-rt}V(\delta_t, R_t) + \mathbf{1}(t \geq T_c)e^{-rt}(\zeta\delta_t - R_t)$. By Ito's lemma for semimartingales, for $t < T_c$,

$$Y_t = V(\delta_0, R_0) + \int_0^t e^{-rs}[-rV(\delta_s, R_s) + \mathcal{A}V(\delta_s, R_s)]ds + M_t \tag{55}$$

for a local martingale M_t with $M_0 = 0$. Since V satisfies the variational inequality, for $t < T_c$,

$$Y_t = e^{-rt}V(\delta_t, R_t) \leq V(\delta_0, R_0) + M_t,$$

where V is bounded below so M_t is a supermartingale. Take an arbitrary stopping time T and let T_n be a sequence of stopping times increasing to $T \wedge T_c$.[5] Applying optional sampling[6] for the

---

[5] For example, $T_n = \max(0, T \wedge T_c - \frac{1}{n})$

[6] By an application of Fatou's lemma, optional sampling for bounded below supermartingales holds for arbitrary

bounded below supermartingale M_t at T_n:

$$\mathbb{E}^{\delta_0, R_0}[Y_{T_n}] \leq V(\delta_0, R_0).$$

Since $V(\delta, R) \geq \theta\delta - R$, it follows that

$$\mathbb{E}^{\delta_0, R_0}[e^{-rT_n}\mathbf{1}(T_n < T_c)(\theta\delta_{T_n} - R_{T_n}) + e^{-rT_c}\mathbf{1}(T_n \geq T_c)(\zeta\delta_{T_c} - R_{T_c})] \leq V(\delta_0, R_0).$$

Taking n to infinity and using the bound $V(\delta, R) \leq B\delta - R + R_0$ along with the lemma of Appendix A to apply dominated convergence,

$$\mathbb{E}^{\delta_0, R_0}[e^{-rT}\mathbf{1}(T < T_c)(\theta\delta_T - R_T) + e^{-rT_c}\mathbf{1}(T \geq T_c)(\zeta\delta_{T_c} - R_{T_c})] \leq V(\delta_0, R_0).$$

Now, define $T_R = \inf\{t : R_t/\delta_t \leq \bar{x}\}$ and fix R_0, δ_0 such that $T_R > 0$.[7] Then by definition of V, $-rV + \mathcal{A}V = 0$ for $t < T_R$, so applying Ito's lemma as before gives

$$Y_t = V(R_0, \delta_0) + M_t.$$

let Q_n be a sequence of stopping times increasing to $T_R \wedge T_c$, let τ_n be the localizing sequence of stopping times for the local martingale M_t, and let $T_n = Q_n \wedge \tau_n \wedge n$. Applying optional sampling,

$$\mathbb{E}^{\delta_0, R_0}[Y_{T_n}] = V(\delta_0, R_0).$$

Taking n to infinity and using the bound $V(\delta, R) \leq B\delta - R + R_0$ along with the lemma of appendix A to apply dominated convergence,

$$\begin{aligned}
\mathbb{E}^{\delta_0, R_0}[Y_{T_R \wedge T_c}] &= \mathbb{E}^{\delta_0, R_0}[\mathbf{1}(T_R < T_c)e^{-rT_R}V(\delta_{T_R}, R_{T_R}) + \mathbf{1}(T_R \geq T_c)e^{-rT_C}(\zeta\delta_{T_C} - R_{T_C})] \\
&= \mathbb{E}^{\delta_0, R_0}[\mathbf{1}(T_R < T_c)e^{-rT_R}(\theta\delta_{T_R} - R_{T_R}) + \mathbf{1}(T_R \geq T_c)e^{-rT_C}(\zeta\delta_{T_C} - R_{T_C})] \\
&= V(\delta_0, R_0),
\end{aligned}$$

where the penultimate equality follows from the definition of T_R and V, completing the verification.

C Proof of Proposition 3, calculating equilibrium

Proof of Proposition 3: Let $V^e(\delta, R, s) \equiv E(\delta, R, s)$ and $V^d(\delta, R, s) \equiv D(\delta, R, s)$. The proof proceeds in two steps. First, we show it is without loss of generality to assume the offer strategy

---

stopping times.
[7]If $T_R = 0$, the following conclusion is immediate.

$\omega_i(\delta, R) = V^j(\delta, R, i)$ is optimal. If there were an alternate strategy $(\hat{\omega}, \hat{A}, \hat{O})$ that performed strictly better than the proposed strategy and $\hat{\omega}_i(\delta, R) > V^j(\delta, R, i)$ for some δ, R, i, j, then another strategy $(\tilde{\omega}, \hat{A}, \hat{O})$ does even better by setting $\tilde{\omega}_i(\delta, R) = V^j(\delta, R, i)$ in those cases. In words, offering more than necessary to make the opponent accept is wasteful, since there is complete information and offers cannot change future behavior according to stationary strategies. Likewise, if there were an alternate strategy $(\hat{\omega}, \hat{A}, \hat{O})$ that performed strictly better than the proposed strategy and $\hat{\omega}_i(\delta, R) < V^j(\delta, R, i)$ for some δ, R, i, j, then another strategy $(\tilde{\omega}, \hat{A}, \tilde{O})$ does just as well where $\tilde{\omega}_i(\delta, R) = V^j(\delta, R, i)$ and those cases are removed from the offer region. In words, if player i makes an offer that they know will be rejected, they do just as well by not making the offer. Therefore, when we consider profitable deviations, it is sufficient to consider deviations of $\hat{A}, \hat{O}$ where the alternate offer function is still $\omega_i(\delta, R) = V^j(\delta, R, i)$.

Second, we show that the equilibrium time $\mathcal{T}$ solves

$$\sup_{T_i \in F^{(\delta, R, s)}} \mathbb{E}^{(\delta, R, s)}[\mathbf{1}(T_i < T_c)e^{-rT_i}J_i(\delta_{T_i}, R_{T_i}, s_{T_i}) + \mathbf{1}(i = d)\mathbf{1}(T_c \leq T_i)e^{-rT_c}(\zeta\delta_{T_c} - R_{T_c})] \quad (56)$$

with associated value function $V^i(\delta, R, s)$. Since each player tries to optimize this quantity subject to constraints imposed by the opponent's strategy, and the equilibrium time $\mathcal{T}$ solves the unconstrained problem, this implies each player acts optimally in the MPE.

To show this, define $N_t = \mathbf{1}(t \geq T_c)$ and for notational convenience define an operator $\mathcal{H}_s$ mapping appropriately differentiable functions $f(\delta, R, s, N)$ to[8]

$$- h(1 - \tau)\delta f_R(\delta, R, s, N) + \mu\delta f_\delta(\delta, R, s, N) + \frac{\sigma^2\delta^2}{2}f_{\delta\delta}(\delta, R, s, N) + \lambda_s[f(\delta, R, s', N) - f(\delta, R, s, N)]$$
$$+ \iota[f(\delta, R, s, 1) - f(\delta, R, s, 0)].$$

Fix $N_0 = 0$. Defining $U^i(\delta, R, s, 0) = V^i(\delta, R, s)$ and $U^i(\delta, R, s, 1) = \mathbf{1}(i = d)(\zeta\delta - R)$, by construction U^i solves

$$-rU^i + \mathcal{H}_sU^i = 0$$

except possibly when $(\delta, R, s) \in O^* \times \{i\}$. By assumption, we have $U^i + U^j = V(\delta, R)$.[9] Also by construction, when $(\delta, R, s) \in O_i^* \times \{i\}$ we have $-rU^j + \mathcal{H}_sU^j = 0$ and $-rV(\delta, R) + \mathcal{H}_sV(\delta, R) \leq 0$ by Proposition 2. By the linearity of $\mathcal{H}_s$, it follows that

$$(-r + \mathcal{H}_s)U^i = (-r + \mathcal{H}_s)[V(\delta, R) - U^j] \leq 0.$$

For $t < T_c$, applying Ito's lemma for semimartingales (see, for example, Duffie (2010)) to U^i

---

[8]The fact that N_t does not transition from 1 to 0 is irrelevant.
[9]In a slight abuse of notation, we sometimes view V as a trivial function of s that equals $\zeta\delta - R$ when $N = 1$.

gives

$$
e^{-rt}U^i(\delta_t, R_t, s_t, N_t) = U^i(\delta_0, R_0, s_0, N_0)
$$
$$
+ \int_0^t e^{-ru}[(-r + \mathcal{H}_{s_u})U^i(\delta_u, R_u, s_u, N_u)]du + M_t
$$

for a local martingale M_t with $M_t = 0$. Applying an identical argument to that used in the proof of Proposition 2 (Appendix B.4) gives that for an arbitrary stopping time T,

$$
\mathbb{E}^{\delta_0, R_0, s_0, 0}[U^i(\delta_{T \wedge T_c}, R_{T \wedge T_c}, s_{T \wedge T_c}, N_{T \wedge T_c})]
$$
$$
= \mathbb{E}^{(\delta_0, R_0, s_0)}[\mathbf{1}(T < T_c)e^{-rT}V^i(\delta_T, R_T, s_T) + \mathbf{1}(i = d)\mathbf{1}(T_c \leq T)e^{-rT_c}(\zeta\delta_{T_c} - R_{T_c})]
$$
$$
\leq U^i(\delta_0, R_0, s_0, 0)
$$
$$
= V^i(\delta_0, R_0, s_0). \tag{57}
$$

For the equilibrium time $\mathcal{T}$, we have $t < \mathcal{T}$ implies[10] $(\delta_t, R_t, s_t) \notin O_i^* \times \{i\}$ which implies $(-r + \mathcal{H}_s)U^i = 0$, so an argument identical to that used in the proof of Proposition 2 (Appendix B.4) gives that

$$
\mathbb{E}^{\delta_0, R_0, s_0, 0}[U^i(\delta_{\mathcal{T} \wedge T_c}, R_{\mathcal{T} \wedge T_c}, s_{\mathcal{T} \wedge T_c}, N_{\mathcal{T} \wedge T_c})]
$$
$$
= \mathbb{E}^{(\delta_0, R_0, s_0)}[\mathbf{1}(\mathcal{T} < T_c)e^{-r\mathcal{T}}V^i(\delta_{\mathcal{T}}, R_{\mathcal{T}}, s_{\mathcal{T}}) + \mathbf{1}(i = d)\mathbf{1}(T_c \leq \mathcal{T})e^{-rT_c}(\zeta\delta_{T_c} - R_{T_c})]
$$
$$
= U^i(\delta_0, R_0, s_0, 0)
$$
$$
= V^i(\delta_0, R_0, s_0). \tag{58}
$$

Note that $V(\delta, R) \geq \theta\delta - R$, combined with the assumption that $\sum_{i=e,d} V^i(\delta, R, s) = V(\delta, R)$, implies that

$$
J_i(\delta, R, s) = \mathbf{1}(s = i)[\theta\delta - R - V^j(\delta, R, i)] + \mathbf{1}(s = j)V^i(\delta, R, j)
$$
$$
\leq \mathbf{1}(s = i)[V(\delta, R) - V^j(\delta, R, i)] + \mathbf{1}(s = j)V^i(\delta, R, j)
$$
$$
= V^i(\delta, R, s)
$$

so $V^i(\delta, R, s) \geq J_i(\delta, R, s)$, and by definition $V^i(\delta_{\mathcal{T}}, R_{\mathcal{T}}, s_{\mathcal{T}}) = J_i(\delta_{\mathcal{T}}, R_{\mathcal{T}}, s_{\mathcal{T}})$. Plugging this into (57, 58) completes the proof.

C.1   Equilibrium value functions

Following Appendix B, general solutions of the homogeneous equation

---

[10]The conclusion is trivial when $\mathcal{T} = 0$.

$$(r + \iota - \mu)v = -(\mu x + h(1 - \tau))v' + \frac{\sigma^2}{2}x^2 v''$$

take the form

$$Gx^\beta f\left(\frac{-2h(1 - \tau)}{\sigma^2 x}\right)$$

for a constant G, where f is either of the solutions to Kummer's ODE:

$$f(z) = M\left(-\beta, -2(\beta - 1) + \frac{2\mu}{\sigma^2}, z\right)$$

$$f(z) = U\left(-\beta, -2(\beta - 1) + \frac{2\mu}{\sigma^2}, z\right)$$

and β is either root of

$$0 = \left[-(r + \iota - \mu) - \mu\beta + \frac{\sigma^2}{2}\beta(\beta - 1)\right].$$

First, consider general solutions in the range $x \leq 0$. As $x \to -\infty$, each player's value function should be bounded above by $\theta - x$, such that $V^i = \delta v$ is bounded above by the value of immediate exercise with all proceeds going to one player. It turns out that none of the general solutions satisfy this with $G \neq 0$,[11] so the general solution in this region is 0.

Next, consider general solutions in the range $x \in [0, \bar{x}]$. As $x \to 0$ from above, $z = -2h(1 - \tau)/[\sigma^2 x] \to -\infty$, and except for some devious corner cases $M(a, b, z)$ is asymptotically proportional to $(-z)^{-a}$. Thus for either the positive or negative root β, the product

$$Gx^\beta M\left(-\beta, -2(\beta - 1) + \frac{2\mu}{\sigma^2}, z\right)$$

is finite at $x = 0, z = -\infty$. The Tricomi U function, evaluated at a negative z, is complex valued and cannot be multiplied by a constant Z to have all real values, so we rule this function out.[12] Thus the general solution in this region is

$$G_1 x^\gamma M\left(-\gamma, -2(\gamma - 1) + \frac{2\mu}{\sigma^2}, z\right) + G_2 x^\kappa M\left(-\kappa, -2(\kappa - 1) + \frac{2\mu}{\sigma^2}, z\right)$$

where κ (γ) is the positive (negative) root β of the above quadratic.

We are now ready to solve the system of equations (28)-(31) characterizing the value functions for $(\delta, R) \notin O^*$. Recall these are

---

[11]We state this without proof, but by showing the conditions of Proposition 3 are met the final solution must be the value function.

[12]Again, all that matters in the end is that the conditions of Proposition 3 are met.

$$rE(\delta, R, e) = \mathcal{L}E(\delta, R, e) + \lambda_e[E(\delta, R, d) - E(\delta, R, e)] + \iota[0 - E(\delta, R, e)]$$

$$rE(\delta, R, d) = \mathcal{L}E(\delta, R, d) + \lambda_d[E(\delta, R, e) - E(\delta, R, d)] + \iota[0 - E(\delta, R, d)]$$

$$rD(\delta, R, e) = \mathcal{L}D(\delta, R, e) + \lambda_e[D(\delta, R, d) - D(\delta, R, e)] + \iota[\zeta\delta - R - D(\delta, R, e)]$$

$$rD(\delta, R, d) = \mathcal{L}D(\delta, R, d) + \lambda_d[D(\delta, R, e) - D(\delta, R, d)] + \iota[\zeta\delta - R - D(\delta, R, d)]$$

where

$$\mathcal{L}f = \delta\mu f_\delta + \frac{\sigma^2}{2}\delta^2 f_{\delta\delta} - (1 - \tau)h\delta f_R. \tag{59}$$

Start with equity values: letting $\hat{r} \equiv r + \iota$ and rearranging (28,29), we can use the linearity of the operator $\mathcal{L}$ to write

$$\begin{bmatrix} \hat{r} + \lambda_e & -\lambda_e \\ -\lambda_d & \hat{r} + \lambda_d \end{bmatrix} \begin{bmatrix} E(\delta, R, e) \\ E(\delta, R, d) \end{bmatrix} = \mathcal{L} \begin{bmatrix} E(\delta, R, e) \\ E(\delta, R, d) \end{bmatrix}.$$

The matrix $\begin{bmatrix} \hat{r} + \lambda_e & -\lambda_e \\ -\lambda_d & \hat{r} + \lambda_d \end{bmatrix}$ has eigendecomposition

$$\begin{bmatrix} \hat{r} + \lambda_e & -\lambda_e \\ -\lambda_d & \hat{r} + \lambda_d \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & -\frac{\lambda_d}{\lambda_e} \end{bmatrix} \begin{bmatrix} \hat{r} & 0 \\ 0 & \hat{r} + \lambda_e + \lambda_d \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & -\frac{\lambda_d}{\lambda_e} \end{bmatrix}^{-1}.$$

Define

$$\begin{bmatrix} \hat{E}(\delta, R, e) \\ \hat{E}(\delta, R, d) \end{bmatrix} \equiv \begin{bmatrix} 1 & 1 \\ 1 & -\frac{\lambda_d}{\lambda_e} \end{bmatrix}^{-1} \begin{bmatrix} E(\delta, R, e) \\ E(\delta, R, d) \end{bmatrix}$$

Then $\hat{E}$ follows the delinked system of HJBs

$$\begin{bmatrix} \hat{r} & 0 \\ 0 & \hat{r} + \lambda_e + \lambda_d \end{bmatrix} \begin{bmatrix} \hat{E}(\delta, R, e) \\ \hat{E}(\delta, R, d) \end{bmatrix} = \mathcal{L} \begin{bmatrix} \hat{E}(\delta, R, e) \\ \hat{E}(\delta, R, d) \end{bmatrix}.$$

Define

$$\xi(x, \gamma) \equiv x^\gamma M(-\gamma, -2(\gamma - 1) + \frac{2\mu}{\sigma^2}, \frac{-2h(1 - \tau)}{\sigma^2 x}).$$

As before, let γ be the negative root of

$$0 = [-(\hat{r} - \mu) - \mu\beta + \frac{\sigma^2}{2}\beta(\beta - 1)],$$

and let ν be the negative root of

$$0 = [-(\hat{r} + \lambda_e + \lambda_d - \mu) - \mu\beta + \frac{\sigma^2}{2}\beta(\beta - 1)].$$

Then as shown in Appendix B.3, in the region where $x \geq \bar{x}$, these equations have solutions $\hat{E}(\delta, R, e) = K_1 \delta \xi(\frac{R}{\delta}, \gamma)$ and $\hat{E}(\delta, R, d) = K_2 \delta \xi(\frac{R}{\delta}, \nu)$ for some constants K_1, K_2. Multiplying by $\begin{bmatrix} 1 & 1 \\ 1 & -\frac{\lambda_d}{\lambda_e} \end{bmatrix}$ delivers

$$\begin{bmatrix} E(\delta, R, e) \\ E(\delta, R, d) \end{bmatrix} = \begin{bmatrix} K_1 \delta \xi(\frac{R}{\delta}, \gamma) + K_2 \delta \xi(\frac{R}{\delta}, \nu) \\ K_1 \delta \xi(\frac{R}{\delta}, \gamma) - \frac{\lambda_d}{\lambda_e} K_2 \delta \xi(\frac{R}{\delta}, \nu) \end{bmatrix}.$$

Given these value functions, we can define $D(\delta, R, s) \equiv V(\delta, R) - E(\delta, R, s)$, and by linearity of the operator $\mathcal{L}$,

$$\begin{aligned}
(r - \mathcal{L})D(\delta, R, s) &= (r - \mathcal{L})(V(\delta, R) - E(\delta, R, s)) \\
&= (r - \mathcal{L})V(\delta, R) - (r - \mathcal{L})E(\delta, R, s) \\
&= \iota[\zeta\delta - R - V(\delta, R)] - [\lambda_s[E(\delta, R, s') - E(\delta, R, s)] - \iota E(\delta, R, s)] \\
&= \iota[\zeta\delta - R - (V(\delta, R) - E(\delta, R, s))] + \lambda_s[V(\delta, R) - E(\delta, R, s') - (V(\delta, R) - E(\delta, R, s))] \\
&= \iota[\zeta\delta - R - D(\delta, R, s)] + \lambda_s[D(\delta, R, s') - D(\delta, R, s)].
\end{aligned}$$

So $D(\delta, R, s)$ satisfies (30, 31) as desired. Thus we have determined the value functions for $(\delta, R) \notin O^*$ up to two constants K_1, K_2. Now, in the region where $(\delta, R) \in O^*$, we will solve for the value functions while receiving offers, $E(\delta, R, d)$ and $D(\delta, R, d)$. Recall these must satisfy the HJBs

$$rE(\delta, R, d) = \mathcal{L}E(\delta, R, d) + \lambda_d[E(\delta, R, e) - E(\delta, R, d)] + \iota[0 - E(\delta, R, d)]$$
$$rD(\delta, R, e) = \mathcal{L}D(\delta, R, e) + \lambda_e[D(\delta, R, d) - D(\delta, R, e)] + \iota[\zeta\delta - R - D(\delta, R, e)],$$

and since offers are made in equilibrium in this region,

$$E(\delta, R, e) = \theta\delta - R - D(\delta, R, e)$$
$$D(\delta, R, d) = \theta\delta - R - E(\delta, R, d).$$

Plugging this in and rearranging, this is

$$\begin{bmatrix} \hat{r} + \lambda_e & \lambda_e \\ \lambda_d & \hat{r} + \lambda_d \end{bmatrix} \begin{bmatrix} D(\delta, R, e) \\ E(\delta, R, d) \end{bmatrix} = \mathcal{L} \begin{bmatrix} D(\delta, R, e) \\ E(\delta, R, d) \end{bmatrix} + \begin{bmatrix} \lambda_e \\ \lambda_d \end{bmatrix} (\theta\delta - R) + \begin{bmatrix} \iota \\ 0 \end{bmatrix} (\zeta\delta - R).$$

The matrix $\begin{bmatrix} \hat{r} + \lambda_e & \lambda_e \\ \lambda_d & \hat{r} + \lambda_d \end{bmatrix}$ has eigendecomposition

$$\begin{bmatrix} \hat{r} + \lambda_e & \lambda_e \\ \lambda_d & \hat{r} + \lambda_d \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ -1 & \frac{\lambda_d}{\lambda_e} \end{bmatrix} \begin{bmatrix} \hat{r} & 0 \\ 0 & \hat{r} + \lambda_e + \lambda_d \end{bmatrix} \begin{bmatrix} 1 & 1 \\ -1 & \frac{\lambda_d}{\lambda_e} \end{bmatrix}^{-1}.$$

Define

$$\begin{bmatrix} \hat{D}(\delta, R, e) \\ \hat{E}(\delta, R, d) \end{bmatrix} \equiv \begin{bmatrix} 1 & 1 \\ -1 & \frac{\lambda_d}{\lambda_e} \end{bmatrix}^{-1} \begin{bmatrix} D(\delta, R, e) \\ E(\delta, R, d) \end{bmatrix}$$

Then $\hat{E}, \hat{D}$ follow the delinked system of HJBs

$$\begin{bmatrix} \hat{r} & 0 \\ 0 & \hat{r} + \lambda_e + \lambda_d \end{bmatrix} \begin{bmatrix} \hat{D}(\delta, R, e) \\ \hat{E}(\delta, R, d) \end{bmatrix} = \mathcal{L} \begin{bmatrix} \hat{D}(\delta, R, e) \\ \hat{E}(\delta, R, d) \end{bmatrix}$$
$$+ \begin{bmatrix} 1 & 1 \\ -1 & \frac{\lambda_d}{\lambda_e} \end{bmatrix}^{-1} \begin{bmatrix} \lambda_e \\ \lambda_d \end{bmatrix} (\theta\delta - R) + \begin{bmatrix} 1 & 1 \\ -1 & \frac{\lambda_d}{\lambda_e} \end{bmatrix}^{-1} \begin{bmatrix} \iota \\ 0 \end{bmatrix} (\zeta\delta - R).$$

Note

$$\begin{bmatrix} 1 & 1 \\ -1 & \frac{\lambda_d}{\lambda_e} \end{bmatrix}^{-1} = \frac{\lambda_e}{\lambda_e + \lambda_d} \begin{bmatrix} \frac{\lambda_d}{\lambda_e} & -1 \\ 1 & 1 \end{bmatrix},$$

so this is

$$\begin{bmatrix} \hat{r} & 0 \\ 0 & \hat{r} + \lambda_e + \lambda_d \end{bmatrix} \begin{bmatrix} \hat{D}(\delta, R, e) \\ \hat{E}(\delta, R, d) \end{bmatrix} = \mathcal{L} \begin{bmatrix} \hat{D}(\delta, R, e) \\ \hat{E}(\delta, R, d) \end{bmatrix} + \begin{bmatrix} \frac{\iota\lambda_d(\zeta\delta - R)}{\lambda_e + \lambda_d} \\ \lambda_e(\theta\delta - R) + \frac{\iota\lambda_e(\zeta\delta - R)}{\lambda_e + \lambda_d} \end{bmatrix}.$$

Let κ be the positive root of

$$0 = [-(\hat{r} - \mu) - \mu\beta + \frac{\sigma^2}{2}\beta(\beta - 1)]$$

and let ϕ be the positive root of

$$0 = [-(\hat{r} + \lambda_e + \lambda_d - \mu) - \mu\beta + \frac{\sigma^2}{2}\beta(\beta - 1)].$$

Then by the previous discussion, in the region where $\delta\bar{x} \geq R$ and $R \geq 0$ (so $x \in [0, \bar{x}]$), the homogeneous ODEs associated with this system (i.e., ignoring δ, R terms) have general solutions

$$\hat{D}(\delta, R, e) = K_3\delta\xi(\frac{R}{\delta}, \gamma) + K_4\delta\xi(\frac{R}{\delta}, \kappa)$$

$$\hat{E}(\delta, R, d) = K_5\delta\xi(\frac{R}{\delta}, \nu) + K_6\delta\xi(\frac{R}{\delta}, \phi).$$

Given constants q, c, d, one can show the particular solution to

$$(q - \mu)v = -(\mu x + h(1 - \tau))v' + \frac{\sigma^2}{2}x^2 v'' + cx + d$$

takes the form

$$v = \frac{c}{q}x + \frac{(-\frac{ch(1-\tau)}{q} + d)}{q - \mu}$$

$$V = \delta v = \frac{c}{q}R + \frac{(-\frac{ch(1-\tau)}{q} + d)}{q - \mu}\delta.$$

After carrying out the matrix multiplication, the relevant parameters for $\hat{E}(\delta, R, d)$ are

$$c = -\lambda_e - \frac{\lambda_e}{\lambda_e + \lambda_d}\iota$$

$$d = \lambda_e\theta + \frac{\lambda_e}{\lambda_e + \lambda_d}\iota\zeta$$

$$q = \hat{r} + \lambda_d + \lambda_e,$$

while for $\hat{D}(\delta, R, e)$ they are

$$c = \frac{-\lambda_d}{\lambda_e + \lambda_d}\iota$$

$$d = \frac{\lambda_d}{\lambda_e + \lambda_d}\iota\zeta$$

$$q = \hat{r}.$$

Plugging this in, the relevant particular solutions are

$$\hat{D} = \frac{\frac{-\lambda_d}{\lambda_e + \lambda_d}\iota}{\hat{r}}R + \frac{(\frac{\frac{\lambda_d}{\lambda_e + \lambda_d}\iota h(1-\tau)}{\hat{r}} + \frac{\lambda_d}{\lambda_e + \lambda_d}\iota\zeta)}{\hat{r} - \mu}\delta$$

$$\hat{E} = \frac{-\lambda_e - \frac{\lambda_e}{\lambda_e + \lambda_d}\iota}{\hat{r} + \lambda_d + \lambda_e}R + \frac{(\frac{[\lambda_e + \frac{\lambda_e}{\lambda_e + \lambda_d}\iota]h(1-\tau)}{\hat{r} + \lambda_d + \lambda_e} + [\lambda_e\theta + \frac{\lambda_e}{\lambda_e + \lambda_d}\iota\zeta])}{\hat{r} + \lambda_d + \lambda_e - \mu}\delta$$

or, adding back the general solutions,

$$\begin{bmatrix} \hat{D}(\delta, R, e) \\ \hat{E}(\delta, R, d) \end{bmatrix} = \begin{bmatrix} K_3\delta\xi(\frac{R}{\delta}, \gamma) + K_4\delta\xi(\frac{R}{\delta}, \kappa) \\ K_5\delta\xi(\frac{R}{\delta}, \nu) + K_6\delta\xi(\frac{R}{\delta}, \phi) \end{bmatrix}$$

$$+ \begin{bmatrix} \frac{\frac{-\lambda_d}{\lambda_e + \lambda_d}\iota}{\hat{r}}R + \frac{(\frac{\frac{\lambda_d}{\lambda_e + \lambda_d}\iota h(1-\tau)}{\hat{r}} + \frac{\lambda_d}{\lambda_e + \lambda_d}\iota\zeta)}{\hat{r} - \mu}\delta \\ \frac{-\lambda_e - \frac{\lambda_e}{\lambda_e + \lambda_d}\iota}{\hat{r} + \lambda_d + \lambda_e}R + \frac{(\frac{[\lambda_e + \frac{\lambda_e}{\lambda_e + \lambda_d}\iota]h(1-\tau)}{\hat{r} + \lambda_d + \lambda_e} + [\lambda_e\theta + \frac{\lambda_e}{\lambda_e + \lambda_d}\iota\zeta])}{\hat{r} + \lambda_d + \lambda_e - \mu}\delta \end{bmatrix}.$$

Multiplying by $\begin{bmatrix} 1 & 1 \\ -1 & \frac{\lambda_d}{\lambda_e} \end{bmatrix}$ gives

$$\begin{bmatrix} D(\delta, R, e) \\ E(\delta, R, d) \end{bmatrix} = \begin{bmatrix} K_3 \delta \xi(\frac{R}{\delta}, \gamma) + K_4 \delta \xi(\frac{R}{\delta}, \kappa) + K_5 \delta \xi(\frac{R}{\delta}, \nu) + K_6 \delta \xi(\frac{R}{\delta}, \phi) \\ -K_3 \delta \xi(\frac{R}{\delta}, \gamma) - K_4 \delta \xi(\frac{R}{\delta}, \kappa) + \frac{\lambda_d}{\lambda_e}[K_5 \delta \xi(\frac{R}{\delta}, \nu) + K_6 \delta \xi(\frac{R}{\delta}, \phi)] \end{bmatrix} + \begin{bmatrix} c_1 \delta + c_2 R \\ c_3 \delta + c_4 R \end{bmatrix},$$

where

$$\begin{bmatrix} c_2 \\ c_4 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ -1 & \frac{\lambda_d}{\lambda_e} \end{bmatrix} \begin{bmatrix} \frac{\frac{-\lambda_d}{\lambda_e + \lambda_d} \iota}{\hat{r}} \\ \frac{-\lambda_e - \frac{\lambda_e}{\lambda_e + \lambda_d} \iota}{\hat{r} + \lambda_d + \lambda_e} \end{bmatrix}$$

$$\begin{bmatrix} c_1 \\ c_3 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ -1 & \frac{\lambda_d}{\lambda_e} \end{bmatrix} \begin{bmatrix} \frac{(\frac{\frac{\lambda_d}{\lambda_e + \lambda_d} \iota h(1-\tau)}{\hat{r}} + \frac{\lambda_d}{\lambda_e + \lambda_d} \iota \zeta)}{\hat{r} - \mu} \\ \frac{(\frac{[\lambda_e + \frac{\lambda_e}{\lambda_e + \lambda_d} \iota] h(1-\tau)}{\hat{r} + \lambda_d + \lambda_e} + [\lambda_e \theta + \frac{\lambda_e}{\lambda_e + \lambda_d} \iota \zeta])}{\hat{r} + \lambda_d + \lambda_e - \mu} \end{bmatrix}.$$

Finally, because R can become negative, we need a different solution for the off-equilibrium region where $\delta \leq R/\bar{x}$ and $R < 0$. Luckily, the only general solution satisfying the boundary conditions is 0, so in this region

$$\begin{bmatrix} D(\delta, R, e) \\ E(\delta, R, d) \end{bmatrix} = \begin{bmatrix} c_1 \delta + c_2 R \\ c_3 \delta + c_4 R \end{bmatrix}$$

for the same constants $c_1 - c_4$. We thus have 6 unknowns, and we require $E(\delta, R, e)$ and $E(\delta, R, d)$ to value match (VM) and smooth paste (SP) at $R/\delta = \bar{x}$ (already calculated) and at $R/\delta = 0$. Recall in the exercise region, $E(\delta, R, e) = \theta \delta - R - D(\delta, R, e)$, where $\theta \delta - R$ is obviously smooth, so imposing VM and SP for $D(\delta, R, e)$ at $R/\delta = 0$ is sufficient and necessary for VM and SP of $E(\delta, R, e)$ at $R/\delta = 0$. These conditions are easiest to impose by switching back to $x = R/\delta$:[13]

$$0 = \begin{bmatrix} K_3 \xi(0, \gamma) + K_4 \xi(0, \kappa) + K_5 \xi(0, \nu) + K_6 \xi(0, \phi) \\ -K_3 \xi(0, \gamma) - K_4 \xi(0, \kappa) + \frac{\lambda_d}{\lambda_e}[K_5 \xi(0, \nu) + K_6 \xi(0, \phi)] \end{bmatrix}$$

$$0 = \begin{bmatrix} K_3 \xi'(0, \gamma) + K_4 \xi'(0, \kappa) + K_5 \xi'(0, \nu) + K_6 \xi'(0, \phi) \\ -K_3 \xi'(0, \gamma) - K_4 \xi'(0, \kappa) + \frac{\lambda_d}{\lambda_e}[K_5 \xi'(0, \nu) + K_6 \xi'(0, \phi)]. \end{bmatrix}$$

We verify two of these are redundant,[14] and thus these are actually equivalent to two equations:

---

[13]Note we arrive at these equalities by first subtracting $c_1 \delta + c_2 R$ or $c_3 \delta + c_4 R$ from both sides.

[14]Specifically, since ξ converges as $x \to 0$ ($z \to -\infty$), its derivative must converge to zero. This can be shown directly with the asymptotic properties of the Confluent Hypergeometric function.

$$\begin{bmatrix} K_3\xi(0,\gamma) + K_4\xi(0,\kappa) + K_5\xi(0,\nu) + K_6\xi(0,\phi) \\ -K_3\xi(0,\gamma) - K_4\xi(0,\kappa) + \frac{\lambda_d}{\lambda_e}[K_5\xi(0,\nu) + K_6\xi(0,\phi)] \end{bmatrix}$$
$$= \begin{bmatrix} K_3\xi'(0,\gamma) + K_4\xi'(0,\kappa) + K_5\xi'(0,\nu) + K_6\xi'(0,\phi) \\ -K_3\xi'(0,\gamma) - K_4\xi'(0,\kappa) + \frac{\lambda_d}{\lambda_e}[K_5\xi'(0,\nu) + K_6\xi'(0,\phi)] \end{bmatrix}$$

In addition to these two equations, we require the four equations corresponding to VM and SP at $\bar{x}$:

$$\begin{bmatrix} \theta - \bar{x} - [K_3\xi(\bar{x},\gamma) + K_4\xi(\bar{x},\kappa) + K_5\xi(\bar{x},\nu) + K_6\xi(\bar{x},\phi)] \\ -K_3\xi(\bar{x},\gamma) - K_4\xi(\bar{x},\kappa) + \frac{\lambda_d}{\lambda_e}[K_5\xi(\bar{x},\nu) + K_6\xi(\bar{x},\phi)] \end{bmatrix}$$
$$= -\begin{bmatrix} -c_1 - c_2\bar{x} \\ c_3 + c_4\bar{x} \end{bmatrix} + \begin{bmatrix} K_1\xi(\bar{x},\gamma) + K_2\xi(\bar{x},\nu) \\ K_1\xi(\bar{x},\gamma) - \frac{\lambda_d}{\lambda_e}K_2\xi(\bar{x},\nu) \end{bmatrix}$$
$$\begin{bmatrix} -1 - [K_3\xi'(\bar{x},\gamma) + K_4\xi'(\bar{x},\kappa) + K_5\xi'(\bar{x},\nu) + K_6\xi'(\bar{x},\phi)] \\ -K_3\xi'(\bar{x},\gamma) - K_4\xi'(\bar{x},\kappa) + \frac{\lambda_d}{\lambda_e}[K_5\xi'(\bar{x},\nu) + K_6\xi'(\bar{x},\phi)] \end{bmatrix}$$
$$= -\begin{bmatrix} -c_2 \\ c_4 \end{bmatrix} + \begin{bmatrix} K_1\xi'(\bar{x},\gamma) + K_2\xi'(\bar{x},\nu) \\ K_1\xi'(\bar{x},\gamma) - \frac{\lambda_d}{\lambda_e}K_2\xi'(\bar{x},\nu) \end{bmatrix}.$$

This is a linear system which is easily solved for $K_1 - K_6$, once one notes that

$$\xi'(x,\gamma) = \gamma x^{\gamma-1}M(-\gamma, -2(\gamma-1) + \frac{2\mu}{\sigma^2}, \frac{-2h(1-\tau)}{\sigma^2 x})$$
$$+ x^\gamma \frac{-\gamma}{-2(\gamma-1) + \frac{2\mu}{\sigma^2}} \frac{2h(1-\tau)}{\sigma^2 x^2}$$
$$\times M(-\gamma+1, -2(\gamma-1) + \frac{2\mu}{\sigma^2} + 1, \frac{-2h(1-\tau)}{\sigma^2 x}).$$

D   Period 1 decision to liquidate or enter Chapter 11

First, we provide a proof of Proposition 4, which gives the solution to the problem of optimally entering Chapter 11.

D.1   Proving Proposition 4

The conditions of Proposition 4 imply the following variational inequality holds:

$$\max(\mathcal{E}(\delta) - B - E^B(\delta), -rE^B(\delta) + \mathcal{D}E^B(\delta) + (1-\tau)(\delta - C_0)) = 0,$$

where

$$\mathcal{D}f(\delta) = f'(\delta)\mu\delta + f''(\delta)\frac{\sigma^2}{2}\delta^2. \tag{60}$$

Since the candidate E^B is smooth, applying Ito's lemma to $e^{-rt}E^B(\delta_t)$ delivers

$$e^{-rt}E^B(\delta_t) = E^B(\delta_0) + \int_0^t e^{-rs}[-rE^B(\delta_s) + \mathcal{D}E^B(\delta_s)]ds + M_t$$

for a local martingale M_t with $M_0 = 0$. By the variational inequality,

$$E^B(\delta_0) + \int_0^t e^{-rs}[-rE^B(\delta_s) + \mathcal{D}E^B(\delta_s)]ds + M_t$$

$$\leq E^B(\delta_0) + \int_0^t e^{-rs}[-(1-\tau)(\delta_s - C_0)]ds + M_t,$$

implying

$$e^{-rt}E^B(\delta_t) + \int_0^t e^{-rs}[(1-\tau)(\delta_s - C_0)]ds \leq E^B(\delta_0) + M_t.$$

Let τ_n be the sequence of localizing stopping times for M_t, let T be an arbitrary stopping time, and let $Q_n = T \wedge \tau_n \wedge n$. Then we can apply optional sampling to write

$$\mathbb{E}^{\delta_0}[e^{-rQ_n}E^B(\delta_{Q_n}) + \int_0^{Q_n} e^{-rs}[(1-\tau)(\delta_s - C_0)]ds]$$

$$\leq E^B(\delta_0) + \mathbb{E}^{\delta_0}[M_{Q_n}]$$

$$= E^B(\delta_0) + M_0$$

$$= E^B(\delta_0).$$

Since $\delta^\psi \to 0$ as $\delta \to \infty$, there exist constants k^0, k^1 such that $E^B(\delta) \leq k^0 + k^1\delta$. Also, it is clear that

$$|\int_0^t e^{-rs}[(1-\tau)(\delta_s - C_0)]ds| \leq (1-\tau)\int_0^\infty e^{-rs}\delta_s ds + \frac{C_0}{r},$$

where the rightside is integrable. Thus we can apply the lemma of Appendix A to use dominated convergence, and clearly $Q_n \to T$ as $n \to \infty$, so

$$\mathbb{E}^{\delta_0}[e^{-rT}E^B(\delta_T) + \int_0^T e^{-rs}[(1-\tau)(\delta_s - C_0)]ds]$$

$$\leq E^B(\delta_0).$$

By the variational inequality, this implies

$$\mathbb{E}^{\delta_0}[e^{-rT}(\mathcal{E}(\delta_T) - B) + \int_0^T e^{-rs}[(1-\tau)(\delta_s - C_0)]ds]$$

$$\leq E^B(\delta_0).$$

Applying the same argument for the optimal T_B, and using the fact that $\delta \geq \delta_B$ implies $-rE^B(\delta) + \mathcal{D}E^B(\delta) + (1-\tau)(\delta - C_0) = 0$ by the construction of E^B, we get

$$\mathbb{E}^{\delta_0}[e^{-rT_B}(\mathcal{E}(\delta_{T_B}) - B) + \int_0^{T_B} e^{-rs}[(1-\tau)(\delta_s - C_0)]ds] = E^B(\delta_0),$$

completing the proof.

D.2 Liquidation vs Chapter 11 Renegotiation

First, recall that asymptotically $\mathcal{E}(\delta) = E(\delta, R_0, e)$ approaches $(\theta - c_1)\delta - (1 - c_2)R_0$. Thus a sufficient condition for Assumption 1 is that $(\theta - c_1) < (1-\tau)/(r-\mu)$.

In period 1, the firm can decide to liquidate and receive 0 or enter Chapter 11 and receive $\mathcal{E}(\delta) - B = E(\delta, R_0, e) - B$. Specifically, they solve

$$E_0(\delta) = \sup_{T_B, T_L \in F^\delta} \mathbb{E}^\delta[\int_0^{T_B \wedge T_L} e^{-rt}(1-\tau)(\delta_t - C_0)dt + \mathbf{1}(T_B < T_L)e^{-rT_B}[\mathcal{E}(\delta_{T_B}) - B]]. \tag{61}$$

Note that choosing stopping times T_B, T_L is equivalent to choosing $T = T_B \wedge T_L$ and whether to liquidate or enter Chapter 11 at time T. The latter decision is of course trivial since the firm will always choose the larger of $E(\delta_T, R_0, e) - B$ and 0. Thus (61) can be rewritten equivalently as

$$E_0(\delta) = \sup_{T \in F^\delta} \mathbb{E}^\delta[\int_0^T e^{-rt}(1-\tau)(\delta_t - C_0)dt + e^{-rT}g(\delta_T)], \tag{62}$$

where $g(\delta) \equiv \max(0, \mathcal{E}(\delta) - B)$. Further, we can define the Ito process $G_t \equiv \int_0^t e^{-rs}(1-\tau)(\delta_s - C_0)ds$ and

$$\hat{g}(G, \delta, t) \equiv G + C_0/r + e^{-rt}g(\delta) \geq 0$$

to write

$$\hat{E}_0(\delta, G, t) = \sup_{T \in F^{\delta,G,t}} \mathbb{E}^{\delta,G,t}[\hat{g}(G_T, \delta_T, T)], \tag{63}$$

which exists by Øksendal (2003) Theorem 10.1.9. It is clear from inspection that $E_0(\delta) = \hat{E}_0(\delta, 0, 0) - C_0/r$. We can thus define, for any fixed C_0, the exercise region $S(C_0) = \{\delta : E_0(\delta) = g(\delta)\}$.

Proof of Proposition 5: For any fixed C, let $\delta_B(C), \delta_L(C)$ denote the corresponding optimal thresholds from Proposition 4 and Section 2.2, respectively. By Assumption 1, there exists $\bar{\delta}(C)$ such that

$$\frac{1-\tau}{r-\mu}\delta' - \frac{(1-\tau)C}{r} > g(\delta')$$

for all $\delta' > \bar{\delta}(C)$. Then it cannot be that $E_0(\delta') = g(\delta')$ for $\delta' > \bar{\delta}(C)$, or else deviating to $T = \infty$ would produce a reward greater than the value function, a contradiction. Thus $\bar{\delta}(C) = \sup\{\delta : E_0(\delta) = g(\delta)\}$ is finite. Suppose that $\mathcal{E}(\bar{\delta}(C)) > B$. Then, again by Øksendal (2003) Theorem 10.1.9, if we define $\overline{T} \equiv \inf\{t : \delta_t \leq \bar{\delta}(C)\}$, for $\delta > \bar{\delta}(C)$ we have

$$E_0(\delta) = \mathbb{E}^\delta\Big[\int_0^{\overline{T}} e^{-rt}(1-\tau)(\delta_t - C_0)dt + e^{-r\overline{T}}g(\delta_{\overline{T}})\Big]$$

$$= \mathbb{E}^\delta\Big[\int_0^{\overline{T}} e^{-rt}(1-\tau)(\delta_t - C_0)dt + e^{-r\overline{T}}(\mathcal{E}(\bar{\delta}(C)) - B)\Big].$$

Since $\delta_B(C)$ maximizes this by Proposition 4, it must be that $\bar{\delta}(C) = \delta_B(C)$.[15] Finally, suppose that $\mathcal{E}(\bar{\delta}(C)) \leq B$. Then for $\delta > \bar{\delta}(C)$ we have

$$E_0(\delta) = \mathbb{E}^\delta\Big[\int_0^{\overline{T}} e^{-rt}(1-\tau)(\delta_t - C_0)dt + e^{-r\overline{T}}g(\delta_{\overline{T}})\Big]$$

$$= \mathbb{E}^\delta\Big[\int_0^{\overline{T}} e^{-rt}(1-\tau)(\delta_t - C_0)dt\Big],$$

and again since $\delta_L(C)$ maximizes this, it must be that $\bar{\delta}(C) = \delta_L(C)$.

Given the existence of $\bar{\delta}(C)$, we may apply a standard formula for the first hitting time of a geometric Brownian motion to write the value function explicitly for $\delta > \bar{\delta}(C)$:

$$E_0(\delta) = (1-\tau)\Big[\frac{\delta}{r-\mu} - \frac{C}{r}\Big] + \Big(\frac{\delta}{\bar{\delta}(C)}\Big)^\psi\Big[g(\bar{\delta}(C)) - (1-\tau)\Big[\frac{\bar{\delta}(C)}{r-\mu} - \frac{C}{r}\Big]\Big].$$

It will be helpful to define

$$I(\delta, x, C) \equiv (1-\tau)\Big[\frac{\delta}{r-\mu} - \frac{C}{r}\Big] + \Big(\frac{\delta}{x}\Big)^\psi\Big[g(x) - (1-\tau)\Big[\frac{x}{r-\mu} - \frac{C}{r}\Big]\Big]$$

and note that, for fixed C, $E_0(\delta) = \sup_{x \leq \delta} I(\delta, x, C)$.

Finally, before we prove Proposition 6, the following lemma is useful.

---

[15]Specifically, we have $E^B \leq E_0$, so if δ_B were different from $\bar{\delta}(C)$ then E^B could be improved to E_0 by deviating to $\overline{T}$.

Lemma D.1 $\mathcal{E}(\delta)$ *diverges to infinity as* $\delta \to \infty$.

Proof of Lemma: Using the notation of Appendix C, we first prove $c_2 > -1$. Multiply the expression for c_2 by $\hat{r}(\hat{r} + \lambda_e + \lambda_d)$:

$$- \hat{r}\frac{\lambda_d}{\lambda_d + \lambda_e}\iota - \lambda_d\iota - \lambda_e\hat{r} - \hat{r}\frac{\lambda_e}{\lambda_d + \lambda_e}\iota$$
$$= -\hat{r}\iota - \lambda_d\iota - \lambda_e\hat{r}$$
$$> -\hat{r}\iota - \hat{r}r - \lambda_d\iota - \lambda_d r - \lambda_e\hat{r}$$
$$= -\hat{r}(\hat{r} + \lambda_d + \lambda_e).$$

Since $c_2 > -1$, it must be that $c_1 < \theta$. If not, then asymptotically as $\delta \to \infty$, the debt value function approaches $c_1\delta + c_2 R > \theta\delta - R$, the reorganized firm value. This is a contradiction, since it implies equity accepts a negative offer. Therefore, in the limit as $\delta \to \infty$, we have $\mathcal{E}(\delta) = E(\delta, R_0, e) = (\theta - c_1)\delta - (1 - c_2)R_0$ starts to increase in δ and thus also goes to infinity.

Proof of Proposition 6: First, we note that if $\hat{C} > C$, then $S(C) \subset S(\hat{C})$.[16] As a result, $\bar{\delta}(C)$ must be weakly increasing in C.

Next, we show that $\bar{\delta}$ diverges to infinity. Suppose by contradiction this weren't the case: there exists K such that $\bar{\delta}(C) \leq K$ for all large enough C. For any $\epsilon > 0$, we can pick $\delta_0 > K$ arbitrarily high so that $(\delta_0/K)^\psi < \epsilon$. For arbitrary C, the value function as above must be

$$E_0(\delta_0) = (1 - \tau)[\frac{\delta_0}{r - \mu} - \frac{C}{r}] + (\frac{\delta_0}{\bar{\delta}(C)})^\psi[g(\bar{\delta}(C)) - (1 - \tau)[\frac{\bar{\delta}(C)}{r - \mu} - \frac{C}{r}]].$$

By continuity, $g(\delta) - (1 - \tau)\delta/(r - \mu)$ attains a maximum H on the compact set $[0, K]$, so

$$E_0(\delta_0) \leq (1 - \tau)[\frac{\delta_0}{r - \mu} - \frac{C}{r}] + (\frac{\delta_0}{\bar{\delta}(C)})^\psi[H + \frac{C(1 - \tau)}{r}]$$
$$\leq (1 - \tau)[\frac{\delta_0}{r - \mu} - \frac{C}{r}] + \epsilon[\max(H, 0) + \frac{C(1 - \tau)}{r}].$$

Now, send C to infinity, keeping δ_0 constant and taking $\epsilon < 1$. This clearly becomes negative, a contradiction. Then $\bar{\delta}(C)$ increases to infinity as $C \to \infty$, and by the previous lemma $\mathcal{E}(\delta)$ increases to infinity as $\delta \to \infty$, so there must exist some $\bar{C}$ such that $\mathcal{E}(\bar{\delta}(C)) > B$ whenever $C > \bar{C}$. Since $\mathcal{E}(0) = 0$, if $\bar{\delta}(C)$ is continuous in C then we can take $\bar{C}$ to satisfy $\mathcal{E}(\bar{\delta}(\bar{C})) = B$.

The remainder of the proof shows $\bar{\delta}(C)$ is continuous in C. By the analysis of Appendix C, as $\delta \to \infty$ we have that $\mathcal{E}(\delta)$ converges to an affine function $a\delta + b$ with $0 < a$. By Assumption 1,

---

[16]The value function corresponding to $\hat{C}$ is clearly weakly smaller than that corresponding to C. If there were a point $y \in S(C)$ such that $y \notin S(\hat{C})$, then by definition of $S(C), S(\hat{C})$ the value function corresponding to $\hat{C}$ would be strictly larger at y (since the payoff is independent of $C, \hat{C}$) which is a contradiction.

$a < (1 - \tau)/(r - \mu)$. Then there exists a constant d such that $\mathcal{E}(\delta) < a\delta + d$, so

$$\frac{(1-\tau)C}{r} - \frac{(1-\tau)\delta}{r-\mu} + \mathcal{E}(\delta) - B$$
$$< \frac{(1-\tau)C}{r} - \delta\left(\frac{(1-\tau)}{r-\mu} - a\right) + d.$$

Then defining

$$\phi(C) \equiv \left(\frac{(1-\tau)}{r-\mu} - a\right)^{-1} d + \left(\frac{(1-\tau)}{r-\mu} - a\right)^{-1} \frac{(1-\tau)C}{r},$$

we have for any C that $\delta \geq \phi(C)$ implies

$$\frac{(1-\tau)C}{r} - \frac{(1-\tau)\delta}{r-\mu} + \mathcal{E}(\delta) - B < 0.$$

In this case, for $\delta > \phi(C)$, it must be that the value function $E_0(\delta)$ is defined by a lower threshold $\bar{\delta}(C)$ with $\bar{\delta}(C) < \delta$, since exercise for $\delta \geq \phi(C)$ is strictly suboptimal. We are now ready to show, for any $\tilde{C}$, that $\bar{\delta}(C)$ is continuous on $[0, \tilde{C}]$. To see this, fix $\delta > \phi(\tilde{C})$. Pick some arbitrary $C \in [0, \tilde{C}]$ and let $E_0^C(\delta)$ be the associated value function. As above,

$$E_0^C(\delta) = \sup_{x \leq \delta} I(\delta, x, C),$$

and since we just showed the value function has a lower threshold $x \leq \phi(C)$, it must be that this equals

$$= \sup_{x \leq \phi(C)} I(\delta, x, C).$$

This is a parameterized constrained optimization where the objective is continuous, and the correspondence $C \to [0, \phi(C)]$ is clearly continuous and compact valued. Define

$$I^*(\delta, x, C) \equiv \sup_{x \leq \phi(C)} I(\delta, x, C).$$

Applying Berge's Theorem, the correspondence

$$C \rightrightarrows Z(C) \equiv \{x : I^*(\delta, x, C) = I(\delta, x, C)\}$$

mapping C to the set of optimal x corresponding to C is upper hemicontinuous. Since any $x \in Z(C)$ must be in $S(C)$ (the region where $E_0^C(x) = g(x)$), we have $\bar{\delta}(C) \geq \sup Z(C)$, but if $\bar{\delta}(C) > \sup Z(C)$ then the firm is not acting optimally by definition of $Z(C)$, so $\bar{\delta}(C) = \sup Z(C)$. A standard argument shows the supremum of the image of an upper hemicontinuous correspondence is continuous, completing the proof.[17]

---

[17]For any ϵ, C, by upper hemicontinuity there exists ϵ_1 such that for all y satisfying $|y - C| < \epsilon_1$ and any $z \in Z(C), u \in Z(y)$, we have $|z - u| < \epsilon/3$. Pick $z \in Z(C)$ with $|\sup(Z(C)) - z| < \epsilon/3$. Then, for any y satisfying

E Fixed costs of Chapter 11

This appendix provides more details on the costs of Chapter 11. Both debtors and creditors hire professionals (i.e., accountants, lawyers, investment bankers, financial advisors) who charge nontrivial fees. Fees incurred during bankruptcy are typically reimbursed from the estate (the firm's assets):

"The large bulk of bankruptcy professional fees and expenses are awarded under Bankruptcy Code Section 330(a). Section 330(a) awards are to professionals employed by the DIP... or employed by an official committee... the DIP pays the awards from the estate (LoPucki and Doherty (2011))."

Creditors have further opportunities for fee reimbursement through 11 USC § 503(b) and § 506(b). Weiss (1990) estimates that such fees average 3.1% of firm value, but LoPucki and Doherty (2011) give many reasons why this is an underestimate. In extreme cases like the bankruptcy of Allied Holdings, fees can reach 22% of firm assets (LoPucki and Doherty (2011) Appendix A).

While these fees are typically awarded and thus subtracted from the total estate to be split between creditors, prepetition fees are an important exception. Firms hire professionals prior to filing for Chapter 11 (prepetition), and the court "does not award prepetition fees" (LoPucki and Doherty (2011)). Indeed, while firms are supposed to report prepetition fees and expenses in connection with a future Chapter 11 under 11 USC § 329(a), they frequently fail to report. Within the dataset used for LoPucki and Doherty (2011) (which is generously provided on LoPucki's website), prepetition fees averaged 43% of the subsequent total 11 USC § 330(a) awards in cases where the firm reported both. These fees must be paid by the firm (i.e., equityholders) since they are not awardable.

Fixed costs: Larger firms and firms with longer bankruptcies certainly pay more in professional fees. However, there is substantial empirical evidence suggesting these fees have a fixed cost component. White (1989) assumes entering Chapter 11 entails a fixed cost, citing court fees, lawyers' fees, and the lost time of management. White (2016) surveys the literature on small business bankruptcy and states "that the costs of Chapter 11 reorganization are high and that they have a fixed component that prevents small corporations from using the procedure."

Early bankruptcy studies focused on particular industries, and many found evidence of fixed costs. For example, Warner (1977) examines railroad bankruptcies and finds "this evidence suggests that there are substantial fixed costs associated with the railroad bankruptcy process, and hence economies of scale with respect to bankruptcy costs." He also finds that the length of bankruptcy cannot explain these costs. Guffey and Moore (1991) find that direct bankruptcy costs exhibit substantial economies of scale in a sample of trucking firms.

Later studies examine bankruptcies across many industries and find similar results. For example, Table 1 of LoPucki and Doherty (2004) presents results of a regression of log fees on log assets, log number of days in bankruptcy, the number of professional firms, and a constant. They find significant coefficients of 0.414 and 0.535 on assets and length of bankruptcy, respectively. The

---

$|y - C| < \epsilon_1$, choose $u \in Z(y)$ with $|\sup(Z(y)) - u| < \epsilon/3$ so by triangle inequality $|\sup(Z(y)) - \sup(Z(C))| < \epsilon$.

constant term, however, is large and positive with a t-statistic nearly three times as large as either assets or bankruptcy length. This suggests that larger and longer bankruptcies entail higher fees, but there is a significant fixed cost faced by all firms. As an extreme example, Farm Fresh Inc., with less than $200 million in assets upon declaring bankruptcy, spent three million in fees on a 44 day bankruptcy (LoPucki and Doherty (2011) Appendix A).

BWZ (2006) examine direct and indirect costs of bankruptcy in a comprehensive sample of small and large corporate bankruptcies in Arizona and New York from 1995 to 2001. In section V.B, they show that the ratio of chapter 11 expenses to pre-bankruptcy assets is drastically higher for small firms than for large firms. For firms with less than $100,000 in pre-bankruptcy assets, expenses average 31.5% of assets, while for firms with $100,000 to $1 million in assets, fees average 10.2% of assets. For firms with more than $10 million in assets, fees average 1.3% of assets. This stark result is consistent with a large fixed cost component of legal fees. Again, the economies of scale which bankruptcy fees exhibit are not driven by shorter bankruptcies for larger firms. In section IV.A they find the weakly positive relationship between firm size (measured in asset value) and bankruptcy duration is not statistically significant.

Put together, these five studies (Warner (1977); Guffey and Moore (1991); LoPucki and Doherty (2004); LoPucki and Doherty (2011); BWZ (2006)) all document economies of scale in bankruptcy fees that strongly suggest a fixed cost component to bankruptcy fees.

References

[1] Anat R Admati and Motty Perry. Strategic delay in bargaining. *The Review of Economic Studies*, 54(3):345–364, 1987.

[2] Ronald W Anderson and Suresh Sundaresan. Design and valuation of debt contracts. *The Review of Financial Studies*, 9(1):37–68, 1996.

[3] Amira Annabi, Michèle Breton, and Pascal François. Resolution of financial distress under chapter 11. *Journal of Economic Dynamics and Control*, 36(12):1867–1887, 2012.

[4] Arindam Bandopadhyaya. An estimation of the hazard rate of firms under chapter 11 protection. *The Review of Economics and Statistics*, pages 346–350, 1994.

[5] David P Baron and John A Ferejohn. Bargaining in legislatures. *American political science review*, 83(4):1181–1206, 1989.

[6] Leonardo Bartolini and Avinash Dixit. Market valuation of illiquid debt and implications for conflicts among creditors. *Staff Papers*, 38(4):828–849, 1991.

[7] Antonio E Bernardo, Alan Schwartz, and Ivo Welch. Contracting externalities and mandatory menus in the us corporate bankruptcy code. *The Journal of Law, Economics, and Organization*, 32(2):395–432, 2015.

[8] Shai Bernstein, Emanuele Colonnelli, and Ben Iverson. Asset allocation in bankruptcy. Technical report, National Bureau of Economic Research, 2017.

[9] Ken Binmore and Partha Dasgupta. *Economics of bargaining.* B. Blackwell, 1987.

[10] Fischer Black and John C Cox. Valuing corporate securities: Some effects of bond indenture provisions. *The Journal of Finance*, 31(2):351–367, 1976.

[11] Michael J Brennan and Eduardo S Schwartz. Optimal financial policy and firm valuation. *The journal of finance*, 39(3):593–607, 1984.

[12] Arturo Bris, Ivo Welch, and Ning Zhu. The costs of bankruptcy: Chapter 7 liquidation versus chapter 11 reorganization. *The Journal of Finance*, 61(3):1253–1303, 2006.

[13] Mark Broadie, Mikhail Chernov, and Suresh Sundaresan. Optimal debt and equity values in the presence of chapter 7 and chapter 11. *The journal of Finance*, 62(3):1341–1377, 2007.

[14] Max Bruche. Creditor coordination, liquidation timing, and debt valuation. *Journal of Financial and Quantitative Analysis*, 46(5):1407–1436, 2011.

[15] Sudheer Chava and Michael R Roberts. Is financial contracting costly? an empirical analysis of debt covenants and corporate investment. *RODNEY L WHITE CENTER FOR FINANCIAL RESEARCH-WORKING PAPERS-*, 19, 2006.

[16] Peter Ove Christensen, Christian Riis Flor, David Lando, and Kristian R Miltersen. Dynamic capital structure with callable debt and debt renegotiations. *Journal of Corporate Finance*, 29:644–661, 2014.

[17] Dean Corbae and Pablo D'Erasmo. Reorganization or liquidation: Bankruptcy choice and firm dynamics. Technical report, National Bureau of Economic Research, 2017.

[18] H. DeAngelo, A.S. Goncalves, and R.M Stulz. Corporate deleveraging and financial flexibility. *Review of Financial Studies*, forthcoming.

[19] Harry DeAngelo, Linda DeAngelo, and Toni M Whited. Capital structure dynamics and transitory debt. *Journal of Financial Economics*, 99(2):235–261, 2011.

[20] Harry DeAngelo and Richard Roll. How stable are corporate capital structures? *The Journal of Finance*, 70(1):373–418, 2015.

[21] Darrell Duffie. *Dynamic asset pricing theory.* Princeton University Press, 2010.

[22] Allan C Eberhart and Richard J Sweeney. Does the bond market predict bankruptcy settlements? *The Journal of Finance*, 47(3):943–980, 1992.

[23] Hülya KK Eraslan. Corporate bankruptcy reorganizations: estimates from a bargaining model. *International Economic Review*, 49(2):659–681, 2008.

[24] Nuri Ersahin. Creditor rights, technology adoption, and productivity: Plant-level evidence. 2017.

[25] Eugene F Fama and Kenneth R French. Testing trade-off and pecking order predictions about dividends and debt. *The review of financial studies*, 15(1):1–33, 2002.

[26] Hua Fan and Suresh M Sundaresan. Debt valuation, renegotiation, and optimal dividend policy. *The Review of Financial Studies*, 13(4):1057–1099, 2000.

[27] Giovanni Favara, Erwan Morellec, Enrique Schroth, and Philip Valta. Debt enforcement, investment, and risk taking across countries. *Journal of Financial Economics*, 123(1):22–41, 2017.

[28] Giovanni Favara, Enrique Schroth, and Philip Valta. Strategic default and equity risk across countries. *The Journal of Finance*, 67(6):2051–2095, 2012.

[29] Edwin O Fischer, Robert Heinkel, and Josef Zechner. Dynamic recapitalization policies and the role of call premia and issue discounts. *Journal of Financial and Quantitative Analysis*, 24(4):427–446, 1989.

[30] Pascal François and Erwan Morellec. Capital structure and asset prices: Some effects of bankruptcy procedures. *The Journal of Business*, 77(2):387–411, 2004.

[31] Julian R Franks and Walter N Torous. An empirical investigation of us firms in reorganization. *The Journal of Finance*, 44(3):747–769, 1989.

[32] Dan Galai, Alon Raviv, and Zvi Wiener. Liquidation triggers and the valuation of equity and debt. *Journal of Banking & Finance*, 31(12):3604–3620, 2007.

[33] Lorenzo Garlappi, Tao Shu, and Hong Yan. Default risk, shareholder advantage, and stock returns. *The Review of Financial Studies*, 21(6):2743–2778, 2008.

[34] Lorenzo Garlappi and Hong Yan. Financial distress and the cross-section of equity returns. *The Journal of Finance*, 66(3):789–822, 2011.

[35] Stuart C Gilson. Transactions costs and capital structure choice: Evidence from financially distressed firms. *The Journal of Finance*, 52(1):161–196, 1997.

[36] Stuart C Gilson, Edith S Hotchkiss, and Richard S Ruback. Valuation of bankrupt firms. *The Review of Financial Studies*, 13(1):43–74, 2000.

[37] Robert Goldstein, Nengjiu Ju, and Hayne Leland. An ebit-based model of dynamic capital structure. *The Journal of Business*, 74(4):483–512, 2001.

[38] John R Graham. How big are the tax benefits of debt? *The Journal of Finance*, 55(5):1901–1941, 2000.

[39] Svend Erik Graversen and Goran Peskir. Optimal stopping and maximal inequalities for geometric brownian motion. *Journal of applied probability*, 35(4):856–872, 1998.

[40] Steven R Grenadier. Option exercise games: An application to the equilibrium investment strategies of firms. *The Review of Financial Studies*, 15(3):691–721, 2002.

[41] Daryl M Guffey and William T Moore. Direct bankruptcy costs: Evidence from the trucking industry. *Financial Review*, 26(2):223–235, 1991.

[42] Xin Guo, Jianjun Miao, and Erwan Morellec. Irreversible investment with regime shifts. *Journal of Economic Theory*, 122(1):37–59, 2005.

[43] Dirk Hackbarth, Rainer Haselmann, and David Schoenherr. Financial distress, stock returns, and the 1978 bankruptcy reform act. *The Review of Financial Studies*, 28(6):1810–1847, 2015.

[44] Dirk Hackbarth, Christopher A Hennessy, and Hayne E Leland. Can the trade-off theory explain debt structure? *The Review of Financial Studies*, 20(5):1389–1428, 2007.

[45] Dirk Hackbarth and David C Mauer. Optimal priority structure, capital structure, and investment. *The Review of Financial Studies*, 25(3):747–796, 2011.

[46] Sergiu Hart and Andreu Mas-Colell. Bargaining and value. *Econometrica: Journal of the Econometric Society*, pages 357–380, 1996.

[47] Christopher A Hennessy and Toni M Whited. Debt dynamics. *The Journal of Finance*, 60(3):1129–1165, 2005.

[48] Jing-Zhi Huang and Ming Huang. How much of the corporate-treasury yield spread is due to credit risk? *The Review of Asset Pricing Studies*, 2(2):153–202, 2012.

[49] Benjamin Iverson. Get in line: Chapter 11 restructuring in crowded bankruptcy courts. *Management Science*, 2017.

[50] Alex Kane, Alan J Marcus, and Robert L McDonald. How big is the tax advantage to debt? *The Journal of Finance*, 39(3):841–853, 1984.

[51] Arthur Korteweg, Michael Schwert, and I Strebulaev. How often do firms really refinance? evidence from corporate filings. Technical report, Working paper, USC and Stanford University, 2014.

[52] Bart Lambrecht and William Perraudin. Creditor races and contingent claims. *European Economic Review*, 40(3):897–907, 1996.

[53] Bart M Lambrecht. The impact of debt financing on entry and exit in a duopoly. *The Review of Financial Studies*, 14(3):765–804, 2001.

[54] Mark T Leary and Michael R Roberts. Do firms rebalance their capital structures? *The journal of finance*, 60(6):2575–2619, 2005.

[55] Hayne E Leland. Corporate debt value, bond covenants, and optimal capital structure. *The journal of finance*, 49(4):1213–1252, 1994.

[56] Hayne E Leland. Agency costs, risk management, and capital structure. *The Journal of Finance*, 53(4):1213–1243, 1998.

[57] Shaojin Li, Toni M Whited, and Yufeng Wu. Collateral, taxes, and leverage. *The Review of Financial Studies*, 29(6):1453–1500, 2016.

[58] Francis A Longstaff. Pricing options with extendible maturities: Analysis and applications. *The Journal of Finance*, 45(3):935–957, 1990.

[59] Lynn M LoPucki and Joseph W Doherty. The determinants of professional fees in large bankruptcy reorganization cases. *Journal of Empirical Legal Studies*, 1(1):111–141, 2004.

[60] Lynn M LoPucki and Joseph W Doherty. *Professional fees in corporate bankruptcies: Data, analysis, and evaluation*. Oxford University Press, 2011.

[61] William Mann. Creditor rights and innovation: Evidence from patent collateral. *Available at SSRN*, 2015.

[62] Pierre Mella-Barral and William Perraudin. Strategic debt service. *The Journal of Finance*, 52(2):531–556, 1997.

[63] Antonio Merlo. Bargaining over governments in a stochastic environment. *Journal of Political Economy*, 105(1):101–131, 1997.

[64] Antonio Merlo and Charles Wilson. A stochastic model of sequential bargaining with complete information. *Econometrica: Journal of the Econometric Society*, pages 371–399, 1995.

[65] Antonio Merlo and Charles Wilson. Efficient delays in a stochastic model of bargaining. *Economic Theory*, 11(1):39–55, 1998.

[66] Robert C Merton. On the pricing of corporate debt: The risk structure of interest rates. *The Journal of finance*, 29(2):449–470, 1974.

[67] Jianjun Miao. Optimal capital structure and industry dynamics. *The Journal of finance*, 60(6):2621–2659, 2005.

[68] Franck Moraux. Valuing corporate liabilities when the default threshold is not an absorbing barrier. 2002.

[69] Erwan Morellec, Boris Nikolov, and Norman Schürhoff. Agency conflicts around the world. 2018.

[70] Bernt Øksendal. Stochastic differential equations. In *Stochastic differential equations*. Springer, 2003.

[71] Alexander I Paseka. Debt valuation with endogenous default and chapter 11 reorganization. 2003.

[72] Motty Perry and Philip J Reny. A non-cooperative bargaining model with strategically timed offers. *Journal of Economic Theory*, 59(1):50–77, 1993.

[73] Jacopo Ponticelli and Leonardo S Alencar. Court enforcement, bank loans, and firm investment: evidence from a bankruptcy reform in brazil. *The Quarterly Journal of Economics*, 131(3):1365–1413, 2016.

[74] Ariel Rubinstein. Perfect equilibrium in a bargaining model. *Econometrica: Journal of the Econometric Society*, pages 97–109, 1982.

[75] Ariel Rubinstein and Asher Wolinsky. Equilibrium in a market with sequential bargaining. *Econometrica: Journal of the Econometric Society*, pages 1133–1150, 1985.

[76] Takashi Shibata and Michi Nishihara. Investment-based financing constraints and debt renegotiation. *Journal of Banking & Finance*, 51:79–92, 2015.

[77] Ilya A Strebulaev. Do tests of capital structure theory mean what they say? *The Journal of Finance*, 62(4):1747–1787, 2007.

[78] Ilya A Strebulaev, Toni M Whited, et al. Dynamic models and structural estimation in corporate finance. *Foundations and Trends® in Finance*, 6(1–2):1–163, 2012.

[79] Suresh Sundaresan. A review of mertons model of the firms capital structure with its wide applications. *Annu. Rev. Financ. Econ.*, 5(1):21–41, 2013.

[80] Suresh Sundaresan and Neng Wang. Investment under uncertainty with strategic debt service. *The American economic review*, 97(2):256–261, 2007.

[81] Sheridan Titman and Sergey Tsyplakov. A dynamic model of optimal capital structure. *Review of Finance*, 11(3):401–451, 2007.

[82] Jerold B Warner. Bankruptcy costs: Some evidence. *The journal of Finance*, 32(2):337–347, 1977.

[83] Lawrence A Weiss. Bankruptcy resolution: Direct costs and violation of priority of claims. *Journal of Financial Economics*, 27(2):285–314, 1990.

[84] Michelle J White. The corporate bankruptcy decision. *Journal of Economic Perspectives*, 3(2):129–151, 1989.

[85] Michelle J White. Small business bankruptcy. *Annual Review of Financial Economics*, 8:317–336, 2016.

[86] Brad Wong, Graham Partington, Maxwell Stevenson, and Violet Torbey. Surviving chapter 11 bankruptcies: Duration and payoff? *Abacus*, 43(3):363–387, 2007.

[87] Muhamet Yildiz. Bargaining without a common prioran immediate agreement theorem. *Econometrica*, 71(3):793–811, 2003.