Intertemporal Substitution, Precautionary Saving, and the Currency Premium

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Abstract

When the expected consumption growth depends positively on the consumption volatility, the interest rate will depend positively on the consumption volatility, due to the intertemporal substitution effect, and negatively on the square of the volatility (consumption variance), due to the precautionary saving effect. In a two-country model of such economies, the interest-rate spread is positively correlated with currency premium but negatively correlated with expected currency premium of long horizon. Thus, the Engel (2016) paradox is resolved with this model.

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1 Introduction

Our paper tries to explain a curious feature of currency markets, which consists of two puzzles. First, the currency premium is positively correlated with the interest rate differential between foreign and home countries. This is called the forward premium puzzle or uncovered interest parity puzzle (see Fama, 1984). Second, the expected future currency premium over a long horizon (the long currency premium) and the sum of expected future currency premiums over all horizons (the cumulative currency premium) are negatively correlated with the interest rate differential. This is called the excess co-movement puzzle (see e.g. Evans, 2012). Engel (2016) shows that these two puzzles cannot be accounted for simultaneously by existing models, and thus constitute a paradox.

In two-country models of currency study, the correlation between interest-rate differentials and the currency premium is given by the negative of the correlations between the interest rate and the consumption variance, because the currency premium is proportional to the differential between home and foreign consumption variance. Thus, the two currency puzzles noted above can be solved simultaneously if the interest rate is negatively correlated to the consumption variance and positively correlated to the expected consumption variance of long horizons. We build a model with these two asset-pricing features.

One feature of our model is that the expected variance of long horizons depends on the volatility only. One way to generate this feature is to assume the volatility follows an AR(1) process. In this paper, we make use of the fact that volatility is not perfectly correlated with variance, even though the variance is the square of the volatility. The existing currency models do not have this feature because the expected variance of long horizons depends on the variance itself.

Another feature is that the mean consumption growth depends positively on the consumption volatility. This implies that the interest rate depends positively on the consumption volatility because the intertemporal-substitution component is proportional to the mean consumption growth. Meanwhile, as is standard in the literature, the interest rate also negatively depends on the square of the consumption volatility (the consumption variance) because the precautionary-saving component is negatively proportional to the consumption variance. In this setting, then, the interest rate has two terms: one that is positively proportional to the consumption volatility and another that is negatively proportional to the consumption variance.

In a two-country model with the above two features, the correlation between the interest rate and the consumption variance is negative. Therefore, the correlation between interest-rate differentials and the currency premium is positive. On the other hand, the correlation between the interest rate and the expected consumption variance of long horizons is positive. Therefore, the correlation between interest-rate differentials and the expected currency premium of long horizons is negative. In this way, our model resolves the Engel (2016) paradox.

The two features of the model have been studied in asset pricing literature. On the one hand, the consumption volatility in the consumption mean is documented empirically in Bekaert and Liu (2004), among others. On the other hand, the stochastic volatility model for consumption in our paper is first proposed by Stein and Stein (1991) as a stochastic volatility model of stock return. We remark that our model leads to a tractable term structure of interest rates, as in Constantinides (1992).

The currency market is important for our understanding of asset-pricing models. The existing literature shows that it is important for the consumption volatility to be stochastic to solve the forward premium puzzle. Our results show that to account for the excess comovement puzzle, the expected consumption variance in the long horizon needs to be different from consumption variance itself, and should enter into the mean of consumption growth.

Prior studies have relied on richer features of preference, such as habit formation, recursive utility and long-run risks (see, for example, Bansal and Shaliastovich, 2013; Colacito and Croce, 2011, 2013; Lustig et al., 2015; Verdelhan, 2010). Engel (2016) shows that these models can not account simultaneously for the two puzzles described above. In our paper, the preference is the standard Constant Relative Risk Aversion (CRRA) utility, but the consumption process is assumed to have the additional features discussed, which are absent from standard models.

The paper is organised as follows. In section 2, we introduce the Engel's paradox. These are the key empirical regularities that we want to simultaneously solve. In section 3, we introduce a two-country model. The interest rates, currency premium cumulative premium and under this model are then studied. In section 4, we solve the two currency puzzle simultaneously. In section 5, we discuss further the economic intuition and potential extension of our model. Section 6, concludes.

2 Engel's Paradox

In this section, we introduce the currency puzzles and fix notations.

We denote the home country by the superscript h and the foreign country by the superscript f. We express the level of the real exchange rate S_t in home currency per unit of foreign currency, with s_t its logarithm. Let r_t^f and r_t^h denote the foreign and home real interest rates, respectively. While most papers in the literature on the forward premium puzzle study the nominal exchange rate and the nominal interest rates (see, e.g., Fama (1984), Backus et al. (2001), Brennan and Xia (2006)), the literature on the excess comovement of the level of the exchange rate is about the real exchange rate and the real interest rates (see e.g. Evans (2012)). Engel (2016) documents that the real version of the forward premium puzzle also holds, forming a crucial ingredient in a series of the real version of puzzles that seems paradoxically. In this paper, the main goal is to simultaneously solve these real version of puzzles. Thus, we only study the real exchange rate and the real interest rate and refer to them as "exchange rate" and "interest rate", respectively. We only discuss the extension to the nominal version of our model at the end of the paper.

We define the currency excess return in log form,

$$\rho_{t+1} = s_{t+1} - s_t + r_t^f - r_t^h \ , \tag{2.1}$$

and define the currency premium as the conditional expected currency excess return,

$$E_t[\rho_{t+1}] = E_t[s_{t+1}] - s_t + r_t^f - r_t^h \ . \tag{2.2}$$

The expected currency premium over horizon j ($j \in \{0, 1, 2, \cdots\}$) is given by $E_t[\rho_{t+j+1}]$. For large j, we refer to it as the "long premium". We use the term "cumulative premium" to refer to the cumulative expected currency premium

$$\sum_{j=0}^{\infty} E_t[\rho_{t+j+1}] \ .$$

Economically speaking, the expected log return (2.2) is only the first-order approximation of the risk premium as defined in the asset pricing theory. However, in this paper, we study the log currency premium (2.2) because the puzzles we want to solve, as well as most of the other empirical regularities in this literature, are documented in the form of log excess return (2.1); see, e.g., Fama (1984). In addition, many of the theory models study the log currency premium (2.2); see, e.g., Backus et al. (2001).

The uncovered interest parity (UIP) implies that the currency premium is zero. Thus, if UIP holds, both the long premium and the cumulative premium should also be zero:

$$\begin{aligned} E_t[\rho_{t+j+1}] &= 0, \qquad for\ all\ j, \\ \sum_{j=0}^{\infty} E_t[\rho_{t+j+1}] &= 0 \ . \end{aligned}$$

However, empirical tests find that the currency premium, long premium, and cumulative premium are all nonzero and stochastic. Interestingly, the covariance of the currency premium with the interest rate differential has opposite signs from the covariance of the long premium (as well as the cumulative premium) with the interest rate differential.

On the one hand, Engel (2016) documents that the covariance between the currency premium and the interest rate differential is positive,

$$cov\left[E_t[\rho_{t+1}], r_t^f - r_t^h\right] > 0 \ . \qquad \text{(forward premium puzzle)} \tag{2.3}$$

This is the forward premium puzzle for the real exchange rate. It implies that a high-interest-rate currency tend to have a higher premium and therefore, justifies the carry trade. The risk-based explanation for the forward premium puzzle explains the currency premium as compensation for risk; see, e.g., Backus et al. (2001) and Brennan and Xia (2006). A stronger version of the forward premium puzzle is the result of the Fama (1984) regression:

$$cov\left[E_t[s_{t+1} - s_t], r_t^f - r_t^h\right] > 0 \ . \qquad \text{(Fama regression)} \tag{2.4}$$

The Fama regression is another statement of the empirical regularity that the high-interest-rate currencies tend to appreciate.

On the other hand, the covariance between the long premium and the interest rate differential is negative in the data, that is, for large j,

$$cov\left[E_t[\rho_{t+j+1}], r_t^f - r_t^h\right] \ < \ 0. \qquad \text{(long premium puzzle)} \tag{2.5}$$

We term this the long premium puzzle. Finally, we consider the cumulative premium puzzle, which is even stronger than the long premium puzzle. Empirical studies find that the covariance of the cumulative premium and the interest rate differential is negative,

$$cov\Big[\sum_{j=0}^{\infty} E_t[\rho_{t+j+1}], r_t^f - r_t^h\Big] \ < \ 0 \ . \qquad \text{(excess comovement puzzle)} \tag{2.6}$$

This puzzle is closely related to the excess comovement puzzle described in the literature.

The excess comovement puzzle has important implication for the level of the exchange rate. As shown in Engel (2016), by telescoping (2.1) forward, the level of the exchange rate can be expressed as the difference between the cumulative expected interest rate differential and the cumulative currency premium, if we assume that the exchange rate, interest rate differential, and currency excess return are all stationary,

$$s_t - \lim_{j\to\infty} E_t[s_{t+j}] = E_t \sum_{j=0}^{\infty} \left[r_{t+j}^f - r_{t+j}^h - \overline{r^f - r^h}\right] - E_t \sum_{j=0}^{\infty} \left[\rho_{t+j+1} - \overline{\rho}\right] \ , \tag{2.7}$$

where $\overline{r^f - r^h}$ is the unconditional mean of the interest rate differential and $\overline{\rho}$ is the unconditional mean of the excess return.

Taken together with equation (2.7), the inequality (2.6) states that the covariance between the exchange rate level and the interest rate differential is higher than the covariance between the cumulative expected currency premium and the interest rate differential, which is the covariance under UIP. This higher covariance implies that the real exchange rate level is excessively volatile - in other words, the exchange rate levels exhibit excess comovement, implying higher overshooting than the classical Dornbusch and Mundell-Fleming models propose.

Inequalities (2.4), (2.5), and (2.6) have important implications for the currency market, as we alluded to above. However, the inequalities (2.5) and (2.6) seem to contradict inequalities (2.3) and (2.4). Engel (2016) shows that a variety of models - including recursive utility, habit formation, long-run risk, and others - cannot simultaneously accommodate the forward premium puzzle and the excess comovement puzzle. He refers to this fact as a paradox. In this paper, we refer to the pair of inequalities (2.3) and (2.5) as Engel's paradox in long premium form, and refer to the inequalities (2.3) and (2.6) as the corresponding cumulative premium form. In this paper, we resolve the paradox in both forms. We discuss the level puzzle and Fama regression at the end of this paper.

3 A Two-Country Exchange-Rate Model

Many papers on currency studies use two-country models with each country being a representative agent economy. We also follow this approach in our paper. Assume that in each country (home and foreign) there exists a representative agent with consumption C_t^i, $i \in \{h, f\}$. For parsimony, we assume that C_t^h and C_t^f are independent but have an identical distribution.

The consumption C_t^i could be interpreted as a quantity index of multiple goods as in Lucas (1982), Cole and Obstfeld (1991), Backus and Smith (2013), Colacito and Croce (2013), and others. In our paper, we specify the distributions of C_t^h and C_t^f exogenously and leave aside the specification of multiple goods dynamics that generate such consumption index. A similar approach has been used in a "long-run risks" model with the Epstein and Zin (1989) recursive utility. Bansal and Shaliastovich (2007, 2013), Colacito and Croce (2011), Lustig and Verdelhan (2007), Backus et al. (2001), and others use this approach. As will be shown later, a time-additive preference is sufficient to jointly solve the two puzzles of interest, so the recursive utility is not needed[2]. We remark that, without much change, we can extend our model to cases where C_t^h and C_t^f also have a common component. This will not change either the exchange rate or the currency premium, but may leads to a better fit for the consumption data

[2]We have also built a recursive-utility version of our model with the same main story. By assuming EIS=1, closed-form solutions are obtained. This model is available upon request.

of each country.

3.1 Representative Agent Economy

We assume that the representative agent in each country $i \in \{h, f\}$ has a constant relative risk aversion utility

$$\sum_{t=0}^{\infty} \mathrm{E}_0 \left[e^{-\beta t} \frac{C_t^{i\,1-\gamma}}{1-\gamma} \right] ,$$

where C_t^i is the consumption for representative agent i, $\beta > 0$ is the subjective discount coefficient, and $\gamma > 0$ is the risk-aversion coefficient. For parsimony, we assume that β and γ are identical across both countries.

Consumption Process

Let, for country $i \in \{h, f\}$, $c_t^i = \ln C_t^i$ be the log-aggregate consumption for the corresponding representative agent. We assume that the difference in log-aggregate consumption $c_{t+1}^i - c_t^i$ (which we will call consumption growth in this paper) satisfies the following process:

$$c_{t+1}^i - c_t^i = \mu_{ct}^i + \sigma_{ct}^i \varepsilon_{ct+1}^i , \tag{3.1}$$

where the conditional mean μ_{ct}^i of the consumption growth is given by

$$\mu_{ct}^i = \lambda \sigma_{ct}^i + (h - 1/2) \sigma_{ct}^{i2} , \tag{3.2}$$

and the conditional "volatility" σ_{ct}^i of the consumption growth is given by

$$\sigma_{ct}^i = (x_t^i + \theta) \quad ,$$

where λ, h, and θ are constants. We assume that x_t^i follows an AR(1) process:

$$x_{t+1}^i = \varphi x_t^i + \sigma \varepsilon_{xt+1}^i . \tag{3.3}$$

We also assume that the innovations ε_{ct+1}^i and ε_{xt+1}^i are i.i.d, following a $N(0,1)$ distribution across country and time. The constants φ and σ satisfy $0 < \varphi < 1$ and $\sigma > 0$.

Note that because x_t^i is normal, $(x^i + \theta)$ can be negative; thus, consumption volatility equals $|x_t^i + \theta|$. Without loss of generality, we assume $\theta > 0$. When $\theta > 0$ is large enough, $x_t^i + \theta$ is positive with probability close to 1, and we will refer to $x_t^i + \theta$ as consumption volatility. This model of stochastic volatility was first used in Stein and Stein (1991)[3].

Note that the consumption mean μ_{ct}^i depends on both conditional volatility $x_t^i + \theta$ and conditional variance $(x_t^i + \theta)^2$. As will be shown in this paper, the conditional volatility term in μ_{ct}^i, $\lambda(x_t^i + \theta)$, leads to a positive correlation between r_t and the risk premium through the intertemporal substitution component of the interest rate and is key to solving the two puzzles simultaneously. This type of consumption volatility in consumption mean model is documented empirically in the literature; see, e.g., Bekaert and Liu (2004).

The parameter h is not crucial for our results, but it provides some flexibility without sacrificing tractability. In most currency studies, the consumption mean does not depend on consumption volatility, which is the key difference from our formulation. We remark that most existing "long-run risk" models use two variance processes with different AR(1) coefficients to generate two decay modes. Our setup generates two decay modes as long as θ is positive.

We could also extend our model by assuming x_t^i to be the square of the OU process, so that $x_t^i + \theta$ is guaranteed to be positive. In this case, the economic intuition remains unchanged. In this paper, we assume that x_t^i follows the Ornstein-Uhlenbeck (OU) process (3.3) for simplicity and tractability.

Pricing Kernel

We assume that the financial market is complete so that there exists a unique pricing kernel for all agents. For each agent $i \in \{h, f\}$, by using the composite good index of agent i as numeraire, the ith pricing kernel can be written as

$$\pi_{t+1}^i = e^{-\beta} e^{-\gamma(c_{t+1}^i - c_t^i)} = e^{-\beta} e^{-\gamma(\mu_t^{ic} + \sigma_{ct}^i \varepsilon_{ct+1}^i)} \ . \tag{3.4}$$

The pricing kernel of home country π_{t+1}^h is different from the pricing kernel of foreign country π_{t+1}^f because the numeraire is different. Note that the pricing kernel of the home country depends only on the home country risk ε_{ct+1}^h; thus only the home country risk is priced by the home country pricing kernel, with a market price of risk $x_t^h + \theta$. The foreign country risk, in contrast, is not priced by the home country pricing kernel: ε_{ct+1}^f does not appear in the pricing kernel π_{t+1}^h.

Interest Rates

[3]Stein and Stein (1991) call $x_t^i + \theta$ the stochastic volatility. Strictly speaking, it is the signed volatility, because, $x_t^i + \theta$ can become negative. The stochastic volatility is the absolute value of $x_t^i + \theta$.

The one-period real interest rate for country $i \in \{h, f\}$ in our model is

$$\begin{aligned} r_t^i &= -\ln E_t[\pi_{t+1}^i] \\ &= \beta + \gamma\lambda(x_t^i + \theta) - \gamma\left(\frac{1+\gamma}{2} - h\right)(x_t^i + \theta)^2 \ . \end{aligned} \tag{3.5}$$

The interest rate r_t^i can be decomposed into two components: the precautionary saving component, which is represented[4] by $-\frac{\gamma(\gamma+1)}{2}\sigma_{ct}^{i2}$, and the intertemporal substitution component $\gamma(\lambda\sigma_{ct}^i + h\sigma_{ct}^{i2})$.

The precautionary saving component of the interest rate in our model is

$$-\frac{\gamma(\gamma+1)}{2}\sigma_{ct}^{i2} = -\frac{\gamma(\gamma+1)}{2}(x_t^i + \theta)^2 \ . \tag{3.6}$$

which depends negatively on the conditional variance $(x_t^i + \theta)^2$. The intertemporal substitution component of the interest rate

$$\gamma(\lambda\sigma_{ct}^i + h\sigma_{ct}^{i2}) = \gamma\lambda(x_t^i + \theta) + \gamma h(x_t^i + \theta)^2 \ , \tag{3.7}$$

where γ is the inverse of the elasticity of intertemporal substitution (EIS), depends on conditional volatility $x_t^i + \theta$ in addition to conditional variance. When $\lambda > 0$, this component implies that the interest rate depends positively on conditional volatility. Similarly, when $h > 0$, this term of the interest rate depends positively on conditional variance.

As a side note, the above interest rate leads to a tractable term structure of interest rates, which was first studied by Constantinides (1992).

3.2 Currency Premium

We now solve for currency premium.

Exchange Rate

Since we assumed the market to be complete, the exchange rate is proportional to the ratio of the pricing kernels of the two countries. The log growth rate of the real exchange rate is

$$s_{t+1} - s_t = \ln \pi_{t+1}^f - \ln \pi_{t+1}^h \ ,$$

[4]Note that the precautionary saving component depends on $(\gamma + 1)$, which is proportional to the prudence (the third derivative of the utility function; see Leland (1968) and Kimball (1990)).

where for $i \in \{h, f\}$, π^i_{t+1} is the pricing kernel with the consumption bundles of country i as numeraire. This relationship was studied in Backus et al. (2001) and Brennan and Xia (2006). As mentioned earlier, if the pricing kernels have a common component, their difference is canceled.

Currency Premium

An investment in foreign currency can be compared to an investment in a s-tock that continuously pays dividends. In this analogy, the exchange rate S_t corresponds to the stock price, and the foreign exchange rate r^f_t corresponds to the continuous compound dividend yield of the stock.

In our model, the one-period excess return on the investment in foreign currency is

$$\begin{aligned} R_{t+1} &= \frac{S_{t+1} e^{r^f_t - r^h_t}}{S_t} = \frac{\pi^f_{t+1}}{\pi^h_{t+1}} e^{r^f_t - r^h_t} \\ &= \exp\left\{ \frac{1}{2}\gamma^2 \sigma^{h2}_{ct} + \gamma \sigma^h_{ct} \varepsilon^h_{ct+1} - \frac{1}{2}\gamma^2 \sigma^{f2}_{ct} - \gamma \sigma^f_{ct} \varepsilon^f_{ct+1} \right\} \quad . \end{aligned}$$

From the above equation, R_{t+1} has home country risk ε^h_{ct+1} with exposure (beta coefficient) $\gamma\sigma^h_{ct}$, and foreign country risk ε^f_{ct+1} with exposure (beta coefficient) $-\gamma\sigma^f_{ct}$. However, only home country risk is priced, because the home pricing kernel has a market price of risk $\gamma\sigma^h_{ct}$ for home country risk ε^h_{ct+1} and a market price of risk 0 for foreign country risk ε^f_{ct+1} Thus, the risk premium is

$$(\gamma\sigma^h_{ct}) \cdot (\gamma\sigma^h_{ct}) + 0 \cdot (-\gamma\sigma^f_{ct}) = (\gamma\sigma^h_{ct})^2$$

and

$$E_t[R_{t+1}] = e^{\gamma^2 \sigma^{h2}_{ct}} \quad .$$

The log currency premium can be expressed in terms of the consumption variance,

$$\begin{aligned} E_t[\rho_{t+1}] &= E_t[\ln R_{t+1}] \\ &= \gamma^2 \sigma^{h2}_{ct} - \frac{\gamma^2 \sigma^{h2}_{ct} + \gamma^2 \sigma^{f2}_{ct}}{2} = \frac{\gamma^2}{2}(\sigma^{h2}_{ct} - \sigma^{f2}_{ct}) \quad , \end{aligned} \tag{3.8}$$

where $-\frac{\gamma^2}{2}(\sigma^{h2}_{ct} + \sigma^{f2}_{ct})$ is due to Jensen's effect. The logarithmic form makes the risk premium symmetric.

We remark that the consumption variance $(x^i_t + \theta)^2$ is negatively proportional to the precautionary saving component of the interest rate (3.6), $-\frac{\gamma(\gamma+1)}{2}(x^i_t + \theta)^2$.

Term Structure of the Consumption Variance

We now study the term structure of the expected consumption variance, $E_t[\sigma_{ct+j}^{i2}]$, of j-periods in the future.

Lemma 3.1. *For country $i \in \{h, f\}$, the term structure of the expected consumption variance $j \in \{0, 1, 2, \cdots\}$ periods in the future is*

$$E_t[\sigma_{ct+j}^{i2}] = \varphi^{2j} x_t^{i2} + 2\ \theta\ \varphi^j x_t^i + \Theta_1 \quad , \tag{3.9}$$

where $\Theta_1 = \frac{1}{2}\left[\frac{1-\varphi^{2j}}{1-\varphi^2} + \theta^2\right]$ is a constant.

We call $E_t[\sigma_{ct+j}^{i2}]$ the long consumption variance for large j, and $\sum_{j=0}^{\infty} E_t[\sigma_{ct+j}^{i2} - \Theta_1]$ the cumulative consumption variance.

Note that when $\theta \neq 0$, ignoring the constant term Θ_1, which is irrelevant to the correlation, $E_t[\sigma_{ct+j}^{i2}]$ has two terms x_t^i and x_t^{i2} that decay as functions of j, φ^j and φ^{2j}, respectively.

The term x_t^i in the expected future variance is one of the key features of our model. In the literature, consumption variance is modeled as a CIR process or a square root process, which implies that expected future variance depends only on itself. This is one reason that previous models fail to jointly solve the forward premium puzzle and the excess comovement puzzle in Engel (2016).

4 Resolution of Engel's Paradox

In this section, we solve the forward premium puzzle and simultaneously the long premium or cumulative premium puzzle.

In our one-state-variable setup, either the interest rate, the expected currency premium, or both need to be non-monotonic functions of the state variable in order to solve the two currency puzzles simultaneously, as shown in the following proposition.

Proposition 4.1. *If the interest rate r and the expected consumption variance are both monotonic functions of x, then the covariance between the expected currency premium and the interest rate differential has the same sign for all horizons j.*

In our paper, x_t has a mean of 0, so most of its probability mass is concentrated near 0. When $\theta > 0$ is large, the probability mass for $x_t + \theta < 0$ is small, so both $x_t + \theta$ and $(x_t + \theta)^2$ are effectively increasing in x_t. The expected consumption variance, given by equation (3.9), is a linear function of $x_t+\theta$ and $(x_t+\theta)^2$ with positive coefficients, and thus is increasing in x_t. However, the interest rate $r(x_t)$ given by (3.5) depends both on $x_t+\theta$ and $-(x_t+\theta)^2$, which are generated by the intertemporal substitution effect and the precautionary saving effect respectively, and are thus a non-monotonic function of x_t. This non-monotonic dependence of the interest rate on x_t is key to resolving Engel's paradox.

Note that the covariance between the expected currency premium and the interest rate differential is related to the covariance between the consumption variance and the interest rate,

$$\begin{aligned}\mathrm{cov}\left(E_t[\rho_{t+j+1}],\ r_t^f - r_t^h\right) &= \frac{\gamma^2}{2}\mathrm{cov}\left(E_t[\sigma_{ct+j}^{h2}] - E_t[\sigma_{ct+j}^{f2}],\ r_t^f - r_t^h\right) \\ &= -\gamma^2 \mathrm{cov}\left(E_t[\sigma_{ct+j}^{h2}],\ r_t^h\right)\ .\end{aligned}$$

The first equality follows from the independent consumption assumption between countries, while the second equality follows from the independence and the identical distribution assumptions.

From the equations for the interest rate (3.5) and the consumption variance (3.9), we can prove the following result.

Lemma 4.2. *The term structure of the covariance between the expected currency premium and the interest rate differential over horizon $j \in \{0,1,2,\cdots\}$ is*

$$\begin{aligned}&\mathrm{cov}\left(E_t[\rho_{t+j+1}],\ r_t^f - r_t^h\right) \\ &= \gamma^3 \left(\frac{1+\gamma}{2} - h\right)\ \varphi^{2j}\ \frac{2\sigma^4}{(1-\varphi^2)^2} + 2\ \gamma^3 \left[2\theta^2 \left(\frac{1+\gamma}{2} - h\right) - \lambda\theta\right]\ \varphi^j\ \frac{\sigma^2}{1-\varphi^2}\ .\end{aligned} \tag{4.1}$$

The proof of this lemma is given in the Appendix. This lemma will be used repeatedly below.

For a better understanding of equation (4.1), we first consider some special cases.

Corollary 4.3. *If $\theta = 0$ and $\lambda \neq 0$, the covariance between the expected future currency premium and the interest rate differential is positively proportional to the currency premium:*

$$\begin{aligned}\mathrm{cov}\left(E_t[\rho_{t+j+1}],\ r_t^f - r_t^h\right) &= \gamma^3\left(\frac{1+\gamma}{2} - h\right)\ \varphi^{2j}\frac{2\sigma^4}{(1-\varphi^2)^2} \\ &= \varphi^{2j}\mathrm{cov}\left(\rho_{t+1},\ r_t^f - r_t^h\right)\end{aligned}$$

for all $j \in \{0,1,2,\cdots\}$.

When $\theta = 0$, the expected consumption variance, $E_t[\sigma^{i2}_{ct+j}] = x^{i2}_t \varphi^{2j}$, is proportional to the consumption variance itself. This is also true in widely used affine models of stochastic variance. For example, affine models of stochastic variance are used in Engel (2016). In this case, the above covariance has the same sign for all j, and thus the two currency puzzles cannot be solved simultaneously.

Corollary 4.4. *If $\lambda = 0$ and $\theta \neq 0$, the covariance of the expected future currency premium with the interest rate differential is positively proportional to $\left(\frac{1+\gamma}{2} - h\right)$:*

$$\begin{aligned} &\operatorname{cov}\left(E_t[\rho_{t+j+1}],\ r^f_t - r^h_t\right) \\ &= \gamma^3 \left[\varphi^{2j}\ \frac{2\sigma^4}{(1-\varphi^2)^2} + 4\theta^2\ \varphi^j \frac{\sigma^2}{1-\varphi^2}\right]\left(\frac{1+\gamma}{2} - h\right), \end{aligned}$$

for all $j \in \{0, 1, 2, \cdots\}$.

In this case, even though the expected currency premium has two decay modes, the covariance still has the same sign for all j. This is because the interest rate depends negatively on consumption variance and does not depend on consumption volatility.

Engel (2016) considers an extensive list of models, such as recursive utility, habit formation, delayed overshooting, and long-run risks. In these models, the consumption growth processes are affine. Two decay modes for the expected currency premium can be achieved by assuming multiple factors in the volatility processes. However, in these models, the consumption mean does not depend on consumption volatility; thus, there is no intertemporal substitution effect and no positive dependence of interest rate on consumption volatility. Due to Corollary 4.4, the covariance of the expected currency premium with the interest rate differential remains positive or negative for all horizons, depending on the sign of $\left(\frac{1+\gamma}{2} - h\right)$.

4.1 Forward Premium Puzzle

The forward premium puzzle (2.3) states that the currency premium is positively correlated with the interest rate differential. Noting that the covariance between the currency premium $E_t[\rho_{t+1}]$ and the interest rate differential is given by (4.1) with $j = 0$, we have the following property.

Proposition 4.5. *The forward premium puzzle is solved if*

$$\lambda\theta < \left(2\theta^2 + \frac{\sigma^2}{1-\varphi^2}\right)\left(\frac{1+\gamma}{2} - h\right) \quad . \tag{4.2}$$

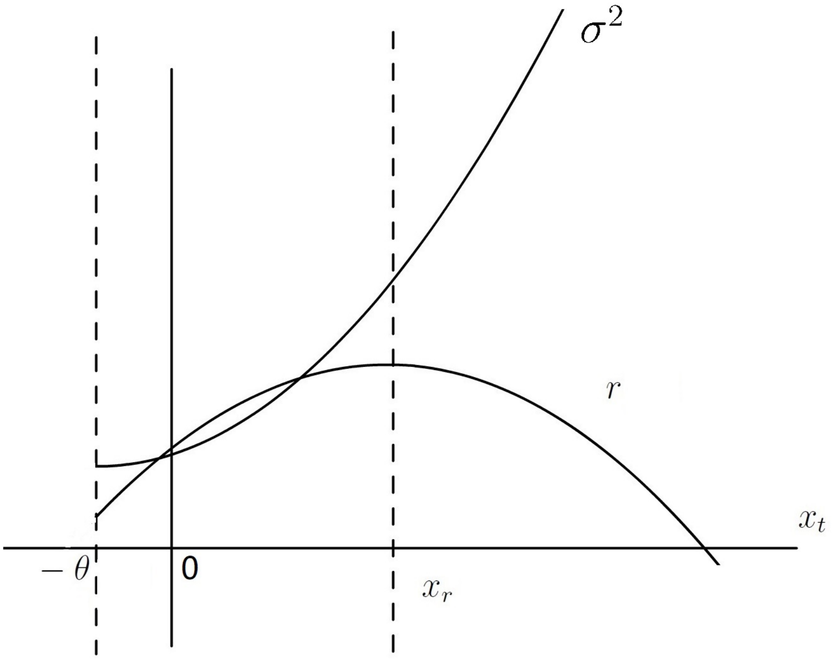

Figure 1: $r(x_t)$ and $\sigma^2(x_t)$ are both non-monotone in x_t. For $x_t \in (-\theta, x_r)$, $x_r = -\theta + \frac{\lambda}{\gamma+1-2h}$, both $r(x_t)$ and $\sigma^2(x_t)$ increase with x_t and thus increase with each other. For $x_t > x_r$ or $x_t < -\theta$, $r(x_t)$ decrease with $\sigma^2(x_t)$. Thus the unconditional covariance between the two is negative if λ is small enough. Note that the region $x_t < -\theta$ has negligible probability mass if $\theta \gg 0$.

To understand this condition intuitively, note that the interest rate r_t can be written as a non-monotone function of x_t,

$$r_t = -\gamma \left(\frac{\gamma+1}{2} - h \right) (x_t - x_r)^2,$$

where $x_r = -\theta + \frac{\lambda}{\gamma+1-2h}$. The consumption variance $\sigma_{ct}^2 = (x_t + \theta)^2$ is also a non-monotone function of x_t. These two functions of x_t are plotted in Figure 1.

Given our assumptions that $\theta > 0$ and $\lambda > 0$, and if we further assume that $h < \frac{\gamma+1}{2}$, then we have $x_r > -\theta$. Note that for $x_t \in (-\theta, x_r)$, r_t and σ_{ct}^2 are positively correlated and the intertemporal substitution effect dominates. In contrast, outside this interval, r_t and σ_{ct}^2 are negatively correlated and the precautionary saving effect dominates. The unconditional covariance between r_t and σ_{ct}^2 is the average of the covariances for all x_t. When λ approaches 0, the interval $(-\theta, x_r)$ becomes the zero set, and the covariance between r_t and σ_t^2 is negative. In contrast, when λ increases, the interval also increases and the covariance becomes positive. Thus, for the unconditional covariance to be negative, λ cannot be too large, as specified by equation (4.2).

4.2 Long Premium Puzzle

We now study the long premium puzzle and provide conditions for resolving the long premium form of Engel's paradox. The long premium puzzle states that for large j, the expected future premium must be negatively correlated with the interest rate differential.

Proposition 4.6. *The condition for solving the long premium puzzle is*

$$\lambda\theta \;>\; 2\theta^2\left(\frac{1+\gamma}{2}-h\right). \tag{4.3}$$

Putting conditions (4.2) and (4.3) together, we have the following proposition, which specifies the condition for the resolution of the long premium form of Engel's paradox.

Proposition 4.7. *The simultaneous resolution of the forward premium puzzle and the long premium puzzle requires the following conditions:*

$$2\theta^2\left(\frac{1+\gamma}{2}-h\right) < \lambda\theta < \left(2\theta^2+\frac{\sigma^2}{1-\varphi^2}\right)\left(\frac{1+\gamma}{2}-h\right)\;. \tag{4.4}$$

The first and second inequalities are the conditions for solving the long and forward premium puzzles, respectively. When both inequalities are satisfied, the long formulation of Engel's paradox is resolved.

Taken together, conditions (4.2) and (4.3) imply the following corollary.

Corollary 4.8. *To simultaneously resolve the forward premium puzzle and the long premium puzzle, for terms in the interest rate that depend on $(x_t+\theta)^2$, the precautionary saving component must dominate the intertemporal substitution component*

$$h < \frac{1+\gamma}{2}\;. \tag{4.5}$$

In many existing models, consumption growth follows affine processes and the consumption mean depends on consumption variance only. Thus, both the intertemporal substitution component and the precautionary saving component depend on consumption variance only. In this setup, the condition for the precautionary saving effect to dominate the intertemporal substitution effect is equation (4.5). This condition is assumed in Engel (2016). In our model, the consumption mean depends on both the consumption volatility and the consumption variance, so the condition in (4.5) is insufficient. In stead, condition (4.4) is needed.

Note that the consumption variance $(x_t + \theta)^2$ appears in r_t as

$$-\gamma\left(\frac{1+\gamma}{2} - h\right)(x_t + \theta)^2 \quad .$$

Here, $\gamma h(x_t + \theta)^2$ is from the intertemporal substitution component, while $-\frac{\gamma(1+\gamma)}{2}$ is from the precautionary saving component. Equation (4.5) states that, for terms that depend on $(x_t+\theta)^2$, the precautionary saving component dominates the intertemporal substitution component.

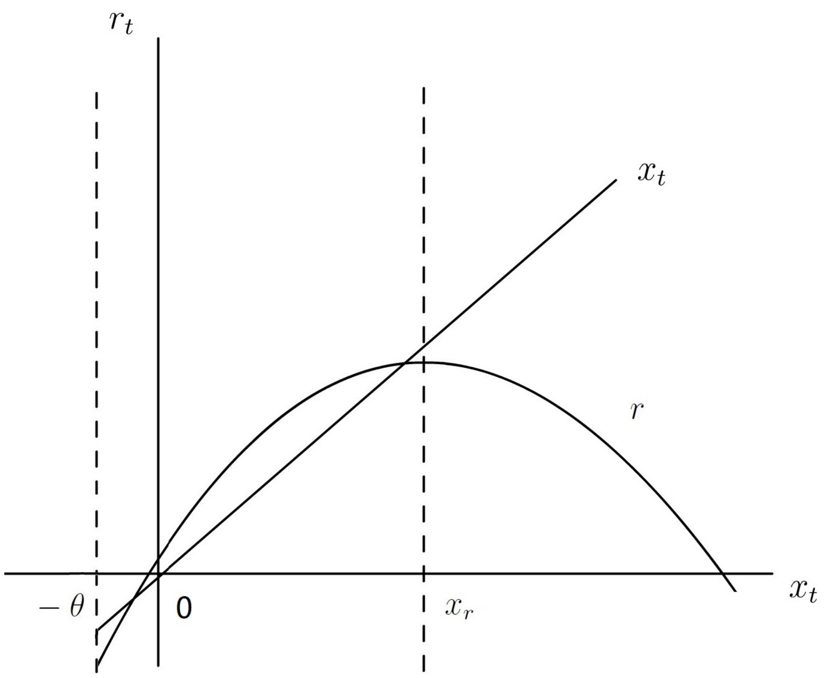

Figure 2: The long consumption variance is proportional to x_t. $r(x_t)$ is non-monotone in x_t. It increases with x_t for $x_t \le x_r$ and decrease for $x_t > x_r$, $x_r = -\theta + \frac{\lambda}{\gamma+1-2h}$. The overall correlation between $r(x_t)$ and x_t is positive if $x_r > 0$.

The expected consumption variance $\mathrm{E}_t[\sigma^2_{ct+j}]$ for large j is dominated by x_t (see equation (3.9),which is plotted in Figure 2 together with the interest rate r_t). Note that r_t and $\mathrm{E}_t[\sigma^2_{ct+j}]$ are positively correlated for $x_t < x_r$ and the intertemporal substitution effect dominates in this interval; in contrast, r_t and σ^2_t are negatively correlated and the precautionary saving effect dominates if $x_t > x_r$. The unconditional covariance between r_t and σ^2_t is the average of the covariances for all $x_t \in (-\infty, \infty)$. Thus, the unconditional covariance is positive if and only if $x_r > 0$, which is equivalent to equation (4.3).

4.3 Cumulative Premium Puzzle

In this subsection, we study the cumulative premium puzzle and provide conditions for resolving the cumulative premium formulation of Engel's paradox. Using equation (3.9), one can readily show that the cumulative consumption variance is

$$\sum_{j=0}^{\infty} E_t[\sigma_{ct+j}^{i2} - \Theta_1] = \gamma^2 \; \frac{1}{1-\varphi^2} x_t^{i2} + 2\;\gamma^2\theta\;\frac{1}{1-\varphi} x_t^i \quad . \tag{4.6}$$

Using the above equation, we can prove the following property.

Lemma 4.9. *The covariance between the cumulative expected future currency premium and the interest rate differential is*

$$\mathrm{cov}\left(\sum_{j=0}^{\infty} E_t[\rho_{t+j+1}], r_t^f - r_t^h\right) = \gamma^2 \mathrm{cov}\left(\sum_{j=0}^{\infty} (E_t[\sigma_{ct+j}^{h2}] - E_t[\sigma_{ct+j}^{f2}]), r_t^f - r_t^h\right)$$
$$= \gamma^3 \left(\frac{1+\gamma}{2} - h\right) \; \frac{1}{1-\varphi^2} \frac{2\sigma^4}{(1-\varphi^2)^2} + 2\gamma^3 \left[2\theta^2\left(\frac{1+\gamma}{2} - h\right) - \lambda\theta\right] \; \frac{1}{1-\varphi} \frac{\sigma^2}{1-\varphi^2} \quad .$$

The proof of this lemma is given in the Appendix. The cumulative premium puzzle given in equation (2.6) can be solved accordingly.

Proposition 4.10. *The condition for solving the cumulative premium puzzle is*

$$\frac{1}{2}\left(\frac{1+\gamma}{2} - h\right) \; \frac{1}{1-\varphi^2} \frac{2\sigma^2}{1-\varphi^2} \;\; < \;\; \left[\lambda\theta - 2\theta^2\left(\frac{1+\gamma}{2} - h\right)\right] \; \frac{1}{1-\varphi} \quad . \tag{4.7}$$

Together with condition (4.2), we have the following proposition, which specifies the condition for the resolution of the strong form of Engel's paradox.

Proposition 4.11. *The simultaneous resolution of the forward premium puzzle and the cumulative premium puzzle requires the following conditions:*

$$\left[2\theta^2 + \frac{\sigma^2}{(1+\varphi)(1-\varphi^2)}\right]\left[\frac{1+\gamma}{2} - h\right] < \lambda\theta < \left[2\theta^2 + \frac{\sigma^2}{1-\varphi^2}\right]\left[\frac{1+\gamma}{2} - h\right]. \tag{4.8}$$

The first and second inequalities are the conditions for solving the cumulative puzzle and the forward premium puzzle, respectively. When both inequalities are satisfied, the cumulative formulation of Engel's paradox is resolved.

Condition (4.8) implies the following corollary.

Corollary 4.12. *To simultaneously resolve the forward premium puzzle and the cumulative premium puzzle, for terms that depend on $(x_t + \theta)^2$, the precautionary saving component must dominate the intertemporal substitution component:*

$$h < \frac{1+\gamma}{2} \; . \tag{4.9}$$

This condition states that the precautionary effect should dominate the intertemporal substitution effect in terms of consumption variance. This is required for solving the forward premium puzzle; see Engel (2016).

Note that (4.8) is stronger than (4.4), which is expected. By using a telescoping sum, it is possible to show that the cumulative premium is related to the level of exchange rate S_t. The negative correlation between the cumulative expected currency premium and the interest rate differential leads to a level of volatility exceeding that predicted by the UIP or Dornbusch models.

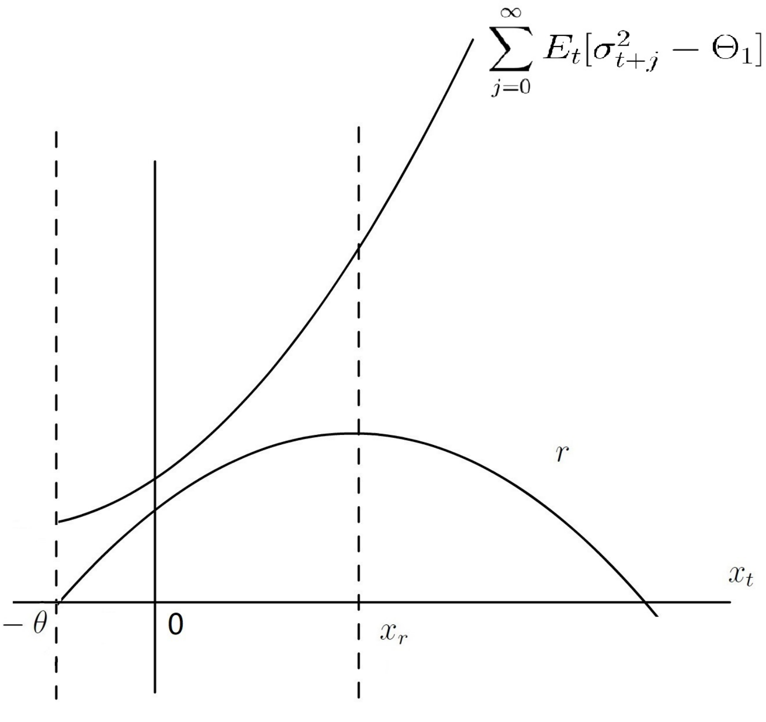

Figure 3: $r(x_t)$ and $\sum_{j=0}^{\infty} E_t[\sigma^2_{ct+j}](x_t)$ are both non-monotone in x_t. Both $r(x_t)$ and $\sum_{j=0}^{\infty} E_t[\sigma^2_{ct+j}](x_t)$ increase with x_t, and thus, increase with each other if $-\theta(1+\varphi) < x_t < x_r$, $x_r = -\theta + \frac{\lambda}{\gamma+1-2h}$. For $x_t > x_r$ or $x_t < -\theta(1+\varphi)$, $r(x_t)$ decreases with $\sum_{j=0}^{\infty} E_t[\sigma^2_{ct+j}](x_t)$. Thus, the unconditional covariance between the two is positive if λ is large enough.

The interest rate $r_t = -\gamma(\frac{\gamma+1}{2} - h)(x_t - x_r)^2$, as above. The cumulative consumption variance $\sum_{j=0}^{\infty} \mathrm{E}_t[\sigma_{ct+j}^2]$ is given by equation (4.6) as a function of x_t. These two functions of x_t are plotted in Figure 3. Note that r_t and $\sum_{j=0}^{\infty} \mathrm{E}_t[\sigma_{ct+j}^2]$ have a positive covariance for $x_t \in (-(1+\varphi)\theta, x_r)$ and the intertemporal substitution effect dominates[5]. In contrast, r_t and σ_{ct}^2 have a negative covariance outside this interval and the precautionary saving effect dominates. The unconditional covariance between r_t and σ_{ct}^2 is the average of the covariances for all $x_t \in (-\infty, \infty)$.

5 Discussion

It is worthwhile to discuss the economic intuition. Let us begin with two economic mechanisms behind our solution to the currency puzzles.

First, note that the interest rate is given by

$$r_t = \beta + \gamma\mu + \gamma\lambda(x_t + \theta) - \gamma\left(\frac{1+\gamma}{2} - h\right)(x_t + \theta)^2 \ ,$$

which depends positively on the consumption volatility $x_t + \theta$ through the intertemporal substitution effect and depends negatively on the consumption variance $(x_t + \theta)^2$ through the precautionary saving effect. When $x_t + \theta$ is large, $(x_t + \theta)^2$ dominates $x_t + \theta$, so the precautionary effect dominates and the interest rate decreases with consumption variance. When $x_t + \theta$ is small, $x_t + \theta$ dominates $(x_t + \theta)^2$, so the intertemporal substitution effect dominates and the interest rate increases with consumption volatility. The competition mechanism of these two effects makes the interest rate a nonmonotonic function of $x_t + \theta$ and is the key to solving the two currency puzzles simultaneously.

Second, the currency premium of the simple return, $\frac{S_{t+1}}{S_t} e^{r_t^f}$, is

$$E_t\left(\frac{S_{t+1}}{S_t} e^{r_t^f - r_t^h}\right) = e^{\gamma^2 \sigma_{ct}^{h2}} \quad ,$$

which depends on home consumption variance $\gamma^2 \sigma_{ct}^{h2}$ but not on foreign consumption variance. Thus, only home risk is priced. The currency premium for the log return, which is the focus of this paper, is

$$E_t[\rho_{t+1}] = E_t\left(\ln\left[\frac{S_{t+1}}{S_t} - r_t^h + r_t^f\right]\right) \ .$$

It has two components,

$$E_t[\rho_{t+1}] = \gamma^2 \sigma_{ct}^{h2} - \frac{\gamma^2}{2}(\sigma_{ct}^{h2} + \sigma_{ct}^{f2}) \ ,$$

[5] Given our assumption that $\theta > 0$ and $\lambda > 0$, and further assuming $h < \frac{\gamma+1}{2}$, one can show that $x_r > -\theta$.

where the first term is compensation for risk, and the second term, which is due to Jensen's effect, is not. The currency premium can be written as the differential of home and foreign consumption variance:

$$E_t[\rho_{t+1}] = \frac{\gamma^2}{2}(\sigma_{ct}^{h2} - \sigma_{ct}^{f2}) \quad .$$

To study the forward premium puzzle, note that the covariance of the currency premium and the interest rate differential can be written as

$$\begin{aligned} \mathrm{cov}\left(E_t[\rho_{t+1}],\ r_t^f - r_t^h\right) &= \frac{\gamma^2}{2}\mathrm{cov}\left(\sigma_{ct}^{h2} - \sigma_{ct}^{f2},\ r_t^f - r_t^h\right) \\ &= -\frac{\gamma^2}{2}\left(\mathrm{cov}\left(\sigma_{ct}^{h2},\ r_t^h\right) + \mathrm{cov}\left(\sigma_{ct}^{f2},\ r_t^f\right)\right). \end{aligned}$$

The log return makes the above covariance symmetric between the home and foreign countries. We remark that the covariance $\mathrm{cov}\left(E_t[\rho_{t+1}],\ r_t^f - r_t^h\right)$ is not due to the relative sizes of the home and foreign interest rates (i.e., the sign of their differential). Rather, it depends on $\mathrm{cov}(\sigma_{ct}^{h2}, r_t^h)$ and $\mathrm{cov}(\sigma_{ct}^{f2}, r_t^f)$. The sign of these two covariances depends on whether the intertemporal substitution effect or the precautionary saving effect dominates. This is determined by λ. When λ is small, the precautionary saving effect dominates on average. In this case, higher consumption variance leads to higher precautionary saving and thus to lower interest. This mechanism creates the positive correlation between the currency premium and the interest rate differential and is the risk-based explanation of the forward premium puzzle in the literature.

On the other hand, for the long (cumulative) premium puzzle, the covariance of the long (cumulative) premium and the interest rate differential can be written as

$$\begin{aligned} \mathrm{cov}\left(E_t[\rho_{t+j+1}],\ r_t^f - r_t^h\right) &= \frac{\gamma^2}{2}\mathrm{cov}\left(E_t[\sigma_{ct+j}^{h2}] - E_t[\sigma_{ct+j}^{f2}],\ r_t^f - r_t^h\right) \\ &= -\frac{\gamma^2}{2}\left(\mathrm{cov}\left(E_t[\sigma_{ct+j}^{h2}],\ r_t^h\right) + \mathrm{cov}\left(E_t[\sigma_{ct+j}^{f2}],\ r_t^f\right)\right). \end{aligned}$$

When λ is large, the intertemporal substitution effect dominates on average. Higher consumption volatility implies higher consumption growth and thus higher interest. This mechanism creates the negative correlation between the long premium and the interest rate differential.

In this paper, we has shown that there is a range for λ such that the above two mechanisms hold simultaneously, thus resolving Engel's paradox.

5.1 Stationary Exchange Rate

Engel (2016) documents empirically that the real exchange rate is stationary and shows that if the exchange rate is stationary, the cumulative expected currency premium is linked to the excess comovement of the level of exchange rate. However, in our model (as well as in the main model in Engel), the exchange rate is not stationary. Below, we extend our model so that the exchange rate is stationary and the main economic intuition remains unchanged.

To make s_t stationary, we assume that the consumption growth for each country $i \in \{h, f\}$ is similar to that in the benchmark model (3.1) but with an additional term $-kc_t^i$,

$$c_{t+1}^i - c_t^i = -kc_t^i + \mu_t^i + \lambda(x_t^i + \theta) + (h - 1/2)(x_t^i + \theta)^2 + (x_t^i + \theta)\varepsilon_{ct+1}^i \ , \tag{5.1}$$

where $0 < k < 1$ is a constant. The term $-kc_t^i$ makes the process c_t^i stationary with mean reversion coefficient $\phi = 1 - k$.

In this case, the exchange rate satisfies

$$s_{t+1} - s_t = \ln \pi_{t+1}^f - \ln \pi_{t+1}^h = \gamma[(c_{t+1}^h - c_t^h) - (c_{t+1}^f - c_t^f)] \ ,$$

which implies that, up to an additive constant,

$$s_t = \gamma(c_t^h - c_t^f) \ .$$

Thus, s_t is mean reverting with mean reversion coefficient ϕ because c_t^h and c_t^f are stationary with mean reversion coefficient ϕ.

Note that both the consumption variance and the precautionary saving component of the interest rate are the same as those in our benchmark model (3.1). The intertemporal substitution component only differs from our benchmark model by the additional term $-kc_t^i$. If k is small[6], its impact on the covariance term structure and the conditions in Proposition 4.11 is negligible.

The cumulative premium is linked to the level of exchange rate by a telescoping sum, as shown in Engel (2016). The cumulative premium puzzle implies that the level of exchange rate is more volatile than is implied by UIP. As a result, on average, currency overshooting is higher than predicted by the Dornbusch model, which assumes a zero risk premium.

5.2 Fama Regression

The Fama regression implies that a high interest currency tends to appreciate. Because inequality (2.4) implies inequality (2.3), the result of the Fama regression (2.4) is stronger than the forward premium puzzle.

[6]Engel (2016) documents that the real exchange rate is mean reverting with slow speed (e.g., $k = -0.02$ for Canada and $k = -0.04$ for the UK).

Proposition 5.1. *The condition for generating the results of Fama regression is*

$$0 < \left[h - \frac{1}{2}\right]\left[\frac{1+\gamma}{2} - h\right]\frac{2\sigma^2}{1-\varphi^2} - \left[2\theta\left(\frac{1}{2} - h\right) - \lambda\right]\left[2\theta\left(\frac{1+\gamma}{2} - h\right) - \lambda\right]. \quad (5.2)$$

Furthermore, if $h < \frac{1+\gamma}{2}$ *and* $2\theta\left(\frac{1+\gamma}{2} - h\right) < \lambda$*, the inequality (5.2) becomes*

$$h \; > \; \frac{1}{2} + \frac{\left[2\theta\left(\frac{1}{2} - h\right) - \lambda\right]\left[2\theta\left(\frac{1+\gamma}{2} - h\right) - \lambda\right]}{\left(\frac{1+\gamma}{2} - h\right)\frac{2\sigma^2}{1-\varphi^2}} \; . \quad (5.3)$$

As shown in Proposition 4.7 and Corollary 4.8, the assumptions $h < \frac{1+\gamma}{2}$ and $2\theta\left(\frac{1+\gamma}{2} - h\right) < \lambda$ are needed to resolve Engel's paradox in long premium formulation.

6 Conclusion

While prior studies have emphasized the importance of the precautionary saving effect in understanding the currency premium over short horizons (e.g., the forward premium puzzle), we find that the intertemporal substitution effect plays a key role in understanding the currency premium over long horizons (e.g., the excess comovement puzzle).

We have built a model in which the interest rate depends positively on the consumption volatility and negatively on the consumption variance. The positive dependence on the consumption volatility is generated through intertemporal substitution, since we assume that the consumption mean depends positively on the consumption volatility. The negative dependence on the consumption variance is generated by precautionary saving, as proposed in the literature. Furthermore, in our model, the expected consumption variance depends on the consumption volatility as well as the consumption variance. Under this setup, the interest rate differential is positively correlated with the currency premium but negatively correlated with the long currency premium (and cumulative currency premium). Thus, our paper resolves the paradox raised by Engel (2016).

Our results suggest that the consumption mean should depend positively on consumption volatility, which has not been widely used in the literature. Furthermore, expected consumption variance should depend on current consumption volatility in addition to current consumption variance. One alternative is a multiple-factor model of consumption variance processes with different mean reversion coefficients.

Appendix: Proof of Proposition and Lemmas

Proof of Lemma 4.2

From equations (3.8), and (3.9), the expected log currency premium over horizon j is

$$\begin{aligned} E_t[\rho_{t+j+1}] &= \frac{\gamma^2}{2} E_t\left[\sigma_{ct+j}^{h2} - \sigma_{ct+j}^{f2}\right] \\ &= \frac{1}{2}\gamma^2\varphi^{2j}(x_t^{h2} - x_t^{f2}) + \gamma^2\theta\varphi^j(x_t^h - x_t^f)\ . \end{aligned}$$

The interest rate is given in equation (3.5). The interest rate differential between the foreign and home countries is

$$r_t^f - r_t^h = \gamma\left(\frac{1+\gamma}{2} - h\right)\ \left[x_t^{h2} - x_t^{f2}\right] + \gamma\left[2\theta\left(\frac{1+\gamma}{2} - h\right) - \lambda\right]\ \left[x_t^h - x_t^f\right]\ . \tag{6.1}$$

Because x_t^h and x_t^f are independent, the covariance between the expected future currency premium and the interest rate differential with different time horizons $j \in \{0, 1, 2, \cdots\}$ is

$$\begin{aligned} &\text{cov}\left(E_t[\rho_{t+j+1}],\ r_t^f - r_t^h\right) \\ &= \frac{1}{2}\gamma^3\left(\frac{1+\gamma}{2} - h\right)\ \varphi^{2j} Var\left[x_t^{h2} - x_t^{f2}\right] + \gamma^3\left[2\theta^2\left(\frac{1+\gamma}{2} - h\right) - \lambda\theta\right]\varphi^j\ Var\left[x_t^h - x_t^f\right] \\ &= \gamma^3\left(\frac{1+\gamma}{2} - h\right)\ \varphi^{2j}\ \frac{2\sigma^4}{(1-\varphi^2)^2} + 2\ \gamma^3\left[2\theta^2\left(\frac{1+\gamma}{2} - h\right) - \lambda\theta\right]\ \varphi^j\ \frac{\sigma^2}{1-\varphi^2}\ . \end{aligned}$$

Proof of Proposition 4.5

The covariance (4.1) when j = 0 is

$$\begin{aligned} &\text{cov}\left(E_t[\rho_{t+j+1}],\ r_t^f - r_t^h\right) \\ &= \gamma^3\left(\frac{1+\gamma}{2} - h\right)\ \frac{2\sigma^4}{(1-\varphi^2)^2} + 2\ \gamma^3\left[2\theta^2\left(\frac{1+\gamma}{2} - h\right) - \lambda\theta\right]\ \frac{\sigma^2}{1-\varphi^2}\ . \end{aligned}$$

Because $\gamma > 0$ and $0 < \varphi < 1$, the fact that the above covariance is positive yields condition (4.2).

Proof of Lemma 4.9

From equations (3.8), and (4.6), the cumulative expected future currency premium is

$$E_t\left[\sum_{j=0}^{\infty}\rho_{t+j+1}\right]=\frac{1}{2}\gamma^2\frac{1}{1-\varphi^2}(x_t^{h2}-x_t^{f2})+\gamma^2\theta\frac{1}{1-\varphi}(x_t^h-x_t^f)\ .$$

The interest rate differential between the foreign and home countries is given in (6.1).

Because x_t^h and x_t^f are independent, the covariance between the cumulative expected future currency premium and the interest rate differential is

$$\begin{aligned}
&\text{cov}\left(\sum_{j=0}^{\infty}E_t[\rho_{t+j+1}],r_t^f-r_t^h\right)\\
&=\frac{1}{2}\gamma^3\left(\frac{1+\gamma}{2}-h\right)\frac{1}{1-\varphi^2}Var\left[x_t^{h2}-x_t^{f2}\right]+\gamma^3\left[2\theta^2\left(\frac{1+\gamma}{2}-h\right)-\lambda\theta\right]\frac{1}{1-\varphi}Var\left[x_t^h-x_t^f\right]\\
&=\gamma^3\left(\frac{1+\gamma}{2}-h\right)\frac{1}{1-\varphi^2}\frac{2\sigma^4}{(1-\varphi^2)^2}+2\ \gamma^3\left[2\theta^2\left(\frac{1+\gamma}{2}-h\right)-\lambda\theta\right]\frac{1}{1-\varphi}\frac{\sigma^2}{1-\varphi^2}\quad .
\end{aligned}$$

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