# Distributive justice for behavioral welfare economics

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**Abstract**

The incompleteness of behavioral preferences can lead many or even all allocations to qualify as Pareto optimal. But the incompleteness does not undercut the precision of utilitarian policy recommendations. Utilitarian methods can be applied to groups of goods or to the multiple social welfare functions that arise when individual preferences are incomplete, and policymakers do not need to provide the preference comparisons that individuals are unable to make for themselves. The utilitarian orderings that result, although also incomplete, can generate a unique optimum. Nonseparabilities in consumption reduce the precision of policy advice but in all cases the dimension of the utilitarian optima drops substantially relative to the Pareto optima.

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# 1 Introduction

Economic models of irrational decision-making have increasingly interpreted an individual who fails to choose consistently as a set of agents acting at different frames. An individual whose value for a good depends on his endowment is seen as a set of preference relations, one for each endowment (Tversky and Kahneman (1991)). A hyperbolic discounter becomes a set of agents that apply discount rates that depend on the date they choose (Laibson (1997)). But when individuals are viewed as sets of agents, it is no longer clear how to define their welfare. The most popular approach has been to narrow the set of welfare judgments for an individual to a behavioral preference that all versions of the individual agree on (Bernheim and Rangel (2007, 2009)).[1] When the frame-based versions of an individual $i$ disagree about how to order $x$ and $y$ then $i$ is declared to not have a preference between $x$ and $y$: the behavioral preference for individual $i$ is incomplete. This incompleteness can lead Pareto optimality to become highly indecisive. In an economy of individuals with incomplete behavioral preferences, the set of Pareto optima can be vast with the same dimension as the entire set of allocations (Mandler (2014)). Every allocation in the neighborhood of an optimum will be another optimum and the characteristic lessons of policy analysis break down: if an economy's initial allocation is Pareto optimal and a small externality is introduced the allocation will remain optimal. In some cases every allocation will be Pareto optimal.

This paper argues that maximizing the sum of utilities – the second-most popular welfare criterion – can close the decisiveness gap. Standard utilitarian models cannot make this case, since they require completeness, but the construction offered here shows how to make interpersonal comparisons of utility when preferences are incomplete. Without some remedy, utilitarianism is left in a precarious position: preferences presumably display incompleteness in at least a corner of their domain.

Individuals will have utilities defined on groups of goods, which can take the consumption of goods in other groups as arguments, but do not take a decisive stand on how to aggregate these utilities. Their attitudes to aggregation can depend on their decision-making frames, leaving preferences incomplete. For example, an individual might have von Neumann-

[1] See also Rubinstein and Salant (2008) and Mandler (2004, 2005).

Morgenstern utilities for the goods delivered at particular states but use different probabilities to weight these utilities as the decision-making frame varies.

The planner meanwhile follows classical utilitarian guidelines: for each group of goods, the planner judges how the utility functions of different individuals for that group should be combined. The planner can but does not have to fill in individuals' incomplete preferences: a solution to the decisiveness problem would not be convincing if planners had to impose the preference judgments individuals themselves were unable to supply.

If the planner comes equipped with a full set of judgments, both across individuals and across groups for single individuals, then as in classical utilitarianism a single objective function rules. Less dictatorial planners will apply a family of utilitarian objective functions and judge allocation $x$ to be superior to $y$ only if they all recommend $x$ over $y$. The resulting 'utilitarian superiority' ordering will be incomplete but this does not lead to a large set of optima. I first show that when the individuals' utilities for any group depend only on the consumption of the goods in that group – 'separability' – and are strictly concave, there is a unique utilitarian optimal allocation. In practice advocates and policymakers presume separability when they debate policy questions; arguments about, say, public health expenditures are rarely conditioned on individuals' consumption of other goods.

When separability is not satisfied, there can be multiple utilitarian optima but the set of optima has measure 0 and a dimension that never rises above the number of goods minus 1. Utilitarianism thus escapes the extreme dimensional expansion of the Pareto optima in behavioral settings. The dimension of the utilitarian optima in fact compares favorably with the dimension of the Pareto optima in a complete preference model.

These results illustrate the broader principle that Pareto optimality delivers sharp advice only in artificial settings – such as a general equilibrium model where the policymaker has certain knowledge of the economy's primitives – while utilitarian methods are robust to a wide array of modeling environments. Even a little preference incompleteness is enough of a wedge for this usefulness gap to appear.

The utilitarian project is hardly trouble-free. A utilitarian planner must compare the utilities of different individuals and come to a normative judgment as to who gains the most from an extra increment of a good. The difficulty of making these comparisons is made neither harder nor easier by preference incompleteness.

Though formal models of utilitarianism assume complete preferences, the classical utilitarians from the beginning recognized the difficulty individuals face in comparing different types of satisfaction: John Stuart Mill (1863) famously acknowledged the diversity of kinds of pleasure and the early neoclassical economists recognized that individuals waver in their weighting of present versus future consumption and the costs of uncertainty. The presence of conflict within individuals did not however undermine the utilitarians' confidence that marginal utility diminishes and is interpersonally comparable. This paper models exactly this combination of positions: individuals who do not weight groups of goods consistently and planners that nevertheless can judge how goods should be distributed across individuals.

This paper's reformulation of utilitarianism clarifies the Sen (1979, 1986) charges against the 'welfarism,' the doctrine that social decisions should be a function only of individual welfare levels rather than the social and psychological content of the policy options. For a welfarist, these details matter only insofar as they feed into individual utility levels. Incomplete preferences fit with Sen's view since then there are no utility numbers that represent the whole of an individual's welfare: the raw material of welfarism is missing. The present version of utilitarianism lets deliberation about interpersonal comparisons proceed autonomously in different domains; decisions about, say, education can draw on the particulars of how schooling affects individual lives while decisions about the environment can depend on the details of that sphere. Though Sen suggests that utilitarianism must rely on welfarist foundations, the present version does not.

The final part of the paper addresses the fact that utilitarian optimal allocations will sometimes fail to be Pareto optimal. From a strict utilitarian perspective, Pareto suboptimality does not present a problem: utilitarian judgments should trump individual judgments. More helpfully, I will show that as the extent of individual preference incompleteness increases, the potential for conflict between utilitarian and Pareto optimality disappears. For sufficiently incomplete preferences, utilitarian optima are sure to be Pareto optimal.

When utility is non-separable – the utility for one group of goods is potentially affected by consumption of all of society's goods – the calculation of the dimension of the optima presents technical challenges. By interpreting each of the sums-of-utilities that the planner takes as goals as the utility function of a hypothetical agent, the utilitarian optima can be understood as the Pareto efficient allocations of an economy composed of these hypothetical agents. Smale's (1974) concept of an isolated community then provides just the right mathematical tool once it is adapted to groups of goods rather than individuals. Though some hurdles must be cleared, due to the twist that each of the hypothetical agents can experience an externality from the consumption levels of the other hypothetical agents, one message of this paper is that this old general equilibrium machinery is fruitful for a topic as far-flung as utilitarianism with behavioral preferences.

Several papers address the problem of how to make welfare decisions in the presence of behavioral preferences. Like this paper, Kahneman et al. (1997) attempt to apply a classical utilitarian rather than a Paretian approach. Fleurbaey and Schokkaert (2013) and Apesteguia and Ballester (2015) both recognize the danger of indecisiveness that accompanies the Paretian ban on policies that overrule observable individual choices. FS offers methods for distributive decision-making compatible with incomplete preferences while AB propose brokering among the conflicting preferences that individual behavior reveals. List (2004) provides a general social-choice setting that incorporates individuals with incomplete ('multidimensional') preferences. Danan et al. (2015) and Argenziano and Gilboa (2018) propose versions of utilitarianism when preferences are incomplete. In most of the above work, social welfare judgments are posited whereas in this paper they are built from comparisons of individual satisfaction, as in Edgeworth and Marshall. Moreover our main target is the size of the set of optima, which has not been the focus of preceding research.

## 2 Behavioral preferences and Pareto optima

An individual's preferences over a domain $X$ depend on how decision-making is framed. The frame could be the date the individual chooses or the individual's endowment. Let $f$ denote a frame, drawn from a set of frames $\mathcal{F}$, and let $\succsim_f$ be the preferences that rule at $f$.

**Definition 1** *The* ***behavioral preference*** $\succsim$*, a binary relation on* $X$*, is the unanimity ordering of the frame-based preferences:* $x \succsim y$ *if and only if* $x \succsim_f y$ *for all* $f \in \mathcal{F}$*.*

As usual strict preferences $\succ$ are defined by $x \succ y \Leftrightarrow (x \succsim y$ and not $y \succsim x)$. So $x \succ y$ obtains if at every frame $x$ is weakly preferred to $y$ and for at least one frame $x$ is strictly preferred to $y$: $x$ is a Pareto improvement over $y$ for the individual's frame-based selves.

The status quo, the most common frame, can induce an endowment effect or status quo bias where individuals agree to move away from their ex ante allocation only if offered a substantial reward. If an individual begins with the status quo endowment $e$ in Figure 1, a drop in good 1 consumption to the level at $x$ would require a large increase in good 2 as compensation, while given the status quo endowment $e'$ a drop in good 2 consumption to the level at $x$ would require a large increase in good 1.[2] As the Figure illustrates, the individual will then exhibit indifference curves at different frames that cross.

However the diversity of frames behind Figure 1 arises, the behavioral preference $\succsim$ labels the bundles to the northeast of both indifference curves to be $\succsim$-superior to $x$ and the bundles to the southwest of both indifference curves to be $\succsim$-inferior to $x$. The remaining bundles such as $z$ are unranked relative to $x$: the preference is incomplete. The kink in the set of $\succsim$-superior bundles that results is characteristic.

Beyond the Pareto rationale that no frame-based preference should be overruled, behavioral preferences can be given positive explanations. If the state of nature determines the benefits delivered by goods the individual is not sure how to value then the individual's decision-making frame can be viewed as a probability distribution and the Aumann (1962) and Bewley (1986) theory of multiple priors will allow for a diversity of frames. The operative prior can vary with, say, the individual's endowment or mood, and the preference judgments backed by all of an individual's priors then provide a model for $\succsim$ that can capture phenomena like status quo bias.

**A multiple priors model of behavioral preferences** *An individual at state $s$ consumes a bundle of goods $x_s = (x_s(1), ..., x_s(\ell)) \geq 0$ evaluated by the von Neumann-Morgenstern utility $u_s$. A frame $f \in \mathcal{F}$ is a probability distribution $\pi^f$ over a finite set of states $\mathcal{S}$, and the agent at frame $\pi^f$ has the preference $\succsim_f$ defined by*

$$x \succsim_f y \quad \textit{if and only if} \quad \textstyle\sum_{s \in \mathcal{S}} \pi_s^f u_s(x_s) \geq \sum_{s \in \mathcal{S}} \pi_s^f u_s(y_s).$$

[2] See Kahneman et al. (1990), Knetsch (1989), Samuelson and Zeckhauser (1988), and Thaler (1980).

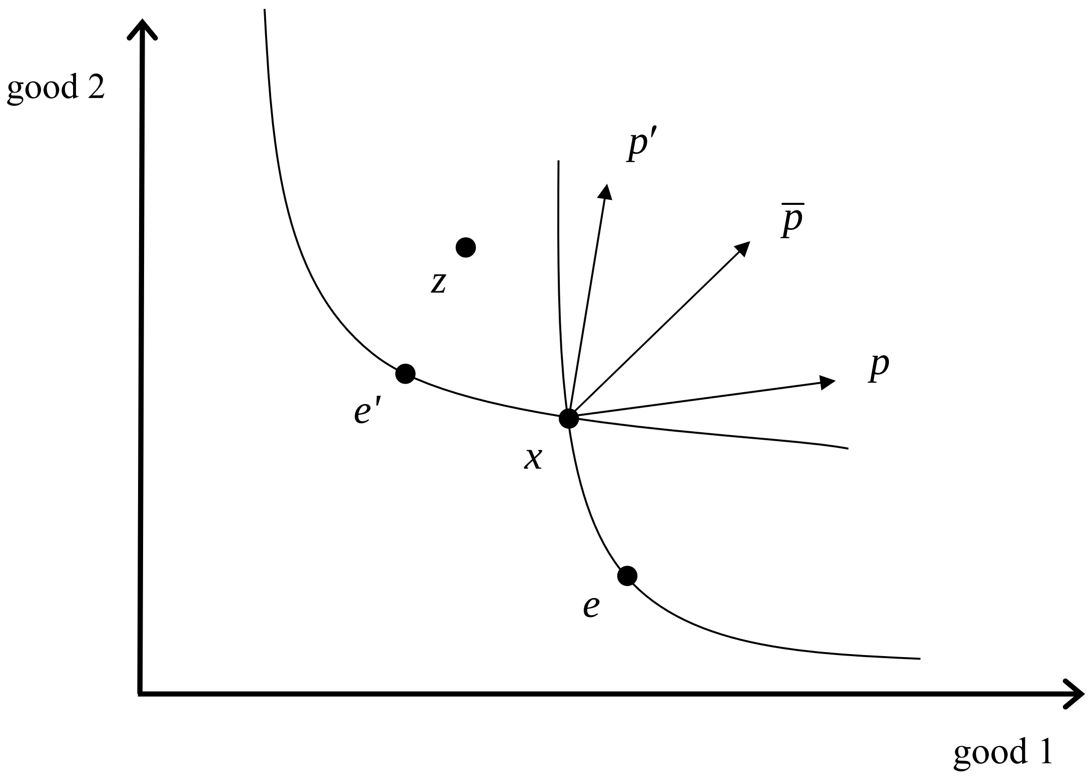


Figure 1: a behavioral preference with multiple supporting prices

*The behavioral preferences* $\succsim$ *that result are then given by Definition 1.*[3]

*For status quo bias, let the first of two goods have uncertain value and suppose the utility at state* $s$ *equals*

$$u_s(x_s(1), x_s(2)) = \gamma_s v_1(x_s(1)) + v_2(x_s(2)).$$

*The coefficient* $\gamma_s$ *is the realization of the uncertain good's benefit at state* $s$*. If when the endowment of good* 1 *is large the agent's frame* $\pi^f$ *assigns high probability to the states* $s$ *where* $\gamma_s$ *is large then the agent will display endowment-driven status quo bias. If we restrict the individual to the two-dimensional domain where consumption is state-independent, the upper contours of* $\succsim$ *will display the kinks pictured in Figure 1. The overlap of the indifference curves that hold at different frames/priors indicate that* $\succsim$ *is incomplete.*

*In Aumann (1962) or Bewley (1986), the individual entertains all priors in* $\mathcal{F}$ *simultaneously and* $x \succsim y$ *holds only if* $x$ *defeats* $y$ *at every prior. Here each prior is interpreted as a distinct decision-making frame.* ■

With a judicious definition of states, the multiple priors model can cover diverse behavioral influences on preferences, including for example mood-driven variations in the extent of diminishing marginal utility or risk aversion. These influences are embedded in an individual's cardinal vNM utilities while the frame affects only the probability weights on the utilities. But the weights that vary by frame need not be probabilities and individuals do not, even hypothetically, have to foresee the possible frames. With hyperbolic discounting, the cardinal utilities are given by the individual's preferences at any fixed date and the weights are the discount rates that agents apply to future goods.

**Hyperbolic discounting** *An individual consumes at several time periods, dates* 1 *through* 3 *for concreteness. A consumption bundle is thus a* $x = (x_1, x_2, x_3) \geq 0$ *where each* $x_i$ *consists of* $\ell$ *goods. The frame is the date* $t = 1, 2, 3$ *at which decisions are made. Under hyperbolic discounting, for each* $t$ *the preference* $\succsim_t$ *of the date-*$t$ *decision-maker can be represented by*

[3] So $x \succsim y$ if and only if

$$\sum_{s\in\mathcal{S}} \pi_s^f u_s(x_s) \geq \sum_{s\in\mathcal{S}} \pi_s^f u_s(y_s)$$

for all $f \in \mathcal{F}$, or, equivalently,

$$\sum_{s\in\mathcal{S}} \widetilde{\pi}_s u_s(x_s) \geq \sum_{s\in\mathcal{S}} \widetilde{\pi}_s u_s(y_s)$$

for all $\widetilde{\pi} = \sum_{f\in\mathcal{F}} \alpha^f \pi^f$ with $\left(\alpha^f\right)_{f\in\mathcal{F}} \geq 0$ and $\sum_{f\in\mathcal{F}} \alpha^f = 1$ (assuming $\mathcal{F}$ is finite). The individual could therefore instead have a decision-making frame for each of the convex combinations of the $\pi^f$.

*the utility function* $u_t : \mathbb{R}_+^{(4-t)\ell} \to \mathbb{R}$ *defined by*

$$u_1(x) = u(x_1) + \beta \left( \delta u(x_2) + \delta^2 u(x_3) \right),$$
$$u_2(x) = u(x_2) + \beta\delta u(x_3),$$
$$u_3(x) = u(x_3),$$

*where* $u : \mathbb{R}_+^\ell \to \mathbb{R}$ *is the agent's concave within-period utility and* $\beta$ *and* $\delta$ *lie in* $[0, 1]$.

*Since consumption prior to* $t$ *has already occurred for the date* $t$ *agent, that agent chooses only among bundles that specify the same consumption from* $1$ *to* $t-1$. *Almost every pair of bundles specifies different values of* $x_1$ *and only the date* $1$ *agent can reveal a preference between such pairs: there is no conflict among the different dated versions of the individual for almost every* $x$.

*To give the model some bite, we restrict the domain by letting one time period pass with* $\overline{x}_1$ *consumed and consider the individual's preferences over the remaining two goods yet to be consumed, i.e., over the set* $\overline{X} = \{x : x_1 = \overline{x}_1\}$. *The dates* $1$ *and* $2$ *agents have complete preferences over* $\overline{X}$ *represented by the utilities* $u_1(\overline{x}_1, \cdot, \cdot)$ *and* $u_2$ *respectively. For bundles* $x$ *and* $y$ *in* $\overline{X}$ *with* $x_2 \neq y_2$, *the preferences of the date* $3$ *agent are irrelevant (since again that agent does not choose from such pairs). Following Bernheim and Rangel (2009), the behavioral preference* $\succsim$ *consists of the preference judgments that the individual's multiple selves agree on: set* $\mathcal{F} = \{1, 2\}$ *and, on the domain of pairs* $x, y \in \overline{X}$ *such that* $x_2 \neq y_2$, *define* $\succsim$ *by Definition 1.*

*As in the multiple priors model,* $\succsim$ *will be incomplete and its upper contour sets will display kinks, as in Figure 1 when* $\ell = 1$. ■

In a society where a typical individual $i$ has one of the behavioral preference relations $\succsim^i$ we have considered, the set of Pareto optima will expand markedly. The kinks in the sets of bundles $\succsim^i$-superior to an arbitrary reference bundle imply that an interval of prices supports the reference bundle: for $x$ in Figure 1, the boundaries of this interval are $p$ and $p'$. Given a Pareto optimal allocation $(x^1, ..., x^I)$ for a society of $I$ individuals, the second welfare theorem reports that there must be a common supporting price vector: the individuals' price intervals intersect. If moreover the intersection is robust in that one of these supporting price vectors, $\overline{p}$ in Figure 1, does not lie exactly on the boundary of any individual $i$'s interval and if the boundaries of each $i$'s interval varies continuously with $x^i$ then, as $(x^1, ..., x^I)$ changes slightly, the intervals of supporting prices will continue to intersect. By the first welfare theorem, the new allocation must be Pareto optimal too. The Pareto optimal allocations with these properties therefore have the same dimension as the entire set of allocations and hence have positive measure.[4]

**Theorem 1 (Mandler 2014)** *If the Pareto optimum $(x^1, ..., x^I)$ has a common supporting price vector $\overline{p}$ such that, for each individual $i$, $\overline{p}$ lies in the interior of the set of prices that support $x^i$ and $i$'s set of supporting prices varies continuously in the allocation then any allocation sufficiently near $x$ is Pareto optimal.*

The expansion of the Pareto optima is the problem this paper addresses.

# 3 Classical utilitarianism

Classical utilitarianism, most importantly Edgeworth (1881), argued that the incremental or marginal value of resources to individuals should be equalized: a shift of a resource from agents with low marginal values for the resource to agents with high marginal values leads to an improvement. Though their discussions of measurement were casual, the classicals understood that a rescaling of units of pleasure or utility would not (and should not) change any ranking of allocations.

We use the following measurement terminology throughout the paper.[5] If $u : Y \to \mathbb{R}$ is a function and $a$ and $b$ are real numbers, let $au + b$ denote the function $h : Y \to \mathbb{R}$ defined by $h(y) = au(y) + b$ for all $y \in Y$. Call $U$ a **cardinal set of functions** if there is a $u : Y \to \mathbb{R}$ such that $U$ equals $u$ and its increasing affine transformations, that is, for all $\widehat{u} : Y \to \mathbb{R}$,

$$\widehat{u} \in U \iff \text{there exist real numbers } a > 0 \text{ and } b \text{ such that } \widehat{u} = au + b.$$

[4] To qualify as an allocation, there can be no disposal of goods. Throughout the paper, 'dimension' refers to the dimension of $C^0$-manifolds with boundary.

[5] The following account follows the tradition of modeling measurement via sets of utilities. See Sen (1970), d'Aspremont and Gevers (1977), Roberts (1980), Bossert and Weymark (1996).

Define $\widehat{\mathcal{U}}$ to be a **cardinal selection** from a Cartesian product of cardinal sets of functions $\mathcal{U} = U^1 \times ... \times U^n$ if there is a $u \in \mathcal{U}$ such that, for all $\widehat{u} \in \mathcal{U}$,

$$\widehat{u} \in \widehat{\mathcal{U}} \iff \text{there exist real numbers } a > 0, b^1, ..., b^n \text{ such that } \widehat{u}^{\,i} = au^i + b^i \text{ for } i = 1, ..., n.$$

If both $u$ and $\widehat{u}$ lie in the same cardinal selection then the 'units' of the functions in $\widehat{u}$ share a common rescaling relative to the functions in $u$: each $\widehat{u}^{\,i}$ is rescaled from $u^i$ using the same constant $a$. Viewing each $u^i$ as a utility, the ratios of utility increments (or marginal utilities) are consequently preserved across $u$ and $\widehat{u}$ drawn from the same cardinal selection:

$$\frac{\widehat{u}^{\,i}(x) - \widehat{u}^{\,i}(y)}{\widehat{u}^{\,j}(w) - \widehat{u}^{\,j}(z)} = \frac{u^i(x) - u^i(y)}{u^j(w) - u^j(z)}$$

for all $i$ and $j$ and bundles $x, y, w$, and $z$. The constancy of these ratios leads to two different types of decisiveness, both of which will be important. If $i$ and $j$ represent individuals, the common ratio arises when a utilitarian planner can weigh one individual's gain against another individual's gain. If $i$ and $j$ denote groups of consumption goods for a single individual then that individual or a planner can weigh one group's benefit relative to another group's benefit. As we will see, a planner can be decisive in the first sense even when an individual or planner fails to be decisive in the second sense.

Throughout the paper, individuals will be indexed by the finite set $\mathcal{I} = \{1, ..., I\}$ and goods by the finite set $\mathcal{L} = \{1, ..., L\}$, a consumption for individual $i$ is a $x^i = (x^i(1), ..., x^i(L))$, and an **allocation** is a profile $x = (x^1, ..., x^I) \in \mathbb{R}^{IL}_+$ of consumptions for each individual. A superscript attached to an object will now indicate the individual associated with that object, e.g., $\succsim^i$ for individual $i$'s behavioral preference.

In the usual complete preferences understanding of classical utilitarianism, an individual $i \in \mathcal{I}$ is described by a cardinal set of utilities $U^i$, where each $u^i \in U^i$ is a utility function on the non-negative bundles of $L$ goods. A classical utilitarian planner decides on a cardinal selection $\mathcal{Y}$ from $U^1 \times ... \times U^I$ and judges allocation $x$ to be weakly superior to $y$ if, for any (and therefore all) $(u^1, ..., u^n) \in \mathcal{Y}$, $\sum_{i\in\mathcal{I}} u^i(x^i) \geq \sum_{i\in\mathcal{I}} u^i(y^i)$. This 'weak superiority relation' is complete: for any pair $x, y \in \mathbb{R}^{IL}_+$ either $x$ is weakly superior to $y$ or $y$ is weakly superior to $x$. If we specify a feasible set of allocations $\{x \in \mathbb{R}^{IL}_+ : \sum_{i\in\mathcal{I}} x^i(k) \leq e(k)$ for

$k \in \mathcal{L}\}$, where $e(k)$ is society's endowment of good $k$, and if, for each individual $i$, some and therefore every $u^i$ in $U^i$ is strictly concave and continuous there will be exactly one utilitarian optimum.

When individual preferences are even a 'little' incomplete, the starting point of this standard model of utilitarianism, the $U^i$, will be missing. One task ahead is to fill this void. As it happens, the classical utilitarians did not suppose that individuals had a global utility assessment of all possible alternatives and so the overhaul applies to their theories too.

# 4 Utilitarianism with behavioral preferences

To extend utilitarianism to behavioral or other forms of incomplete preferences, consider first individuals for whom various disjoint groups of goods deliver satisfaction separably. For the early neoclassical economists, each good formed such a group while in expected utility theory or in most intertemporal models, the goods delivered at a particular state or date form separable groups.

Suppose each individual can specify a cardinal set of utility functions for each group. For example, in expected utility theory these functions consist of the increasing affine transformations of the vNM utility that holds at a particular state $s$ ($u_s$ in the multiple priors model). As in a complete preferences model, the planner takes these sets as data but now for every individual there is one cardinal set for each group of goods. If an individual $i$ weights the utilities of groups differently at the various frames then the behavioral preference $\succsim^i$ that results will be incomplete: $i$ cannot pin down a cardinal selection from the utilities of the different groups. But since individuals possess cardinal judgments about the utilities that govern any single group $g$, a utilitarian planner can compare the utilities of different individuals for group $g$ goods, that is, make a cardinal selection from the individuals' sets of utilities for the goods in $g$. If anything, it should be easier to compare the utilities of individuals for a group of goods than for all goods.

The planner can in addition but does not have to make comparisons of an individual's utilities for groups. With or without these further comparisons, there will be a unique utilitarian optimum. For dictatorial planners ready to make utility comparisons across all individuals and groups, this result reproduces Edgeworth's conclusions and his reasoning. For agnostic planners, the result shows that policy-making does not require planners or individuals to make the difficult decisions that compare the values of different groups.

## 4.1 Individual utilities and preferences

A group of goods will have utilities that planners can interpersonally compare and that may satisfy a separability assumption. Formally, a **group** $g$ is a subset of the set of goods $\mathcal{L}$ and the entire set of groups $\mathcal{G}$ is a partition of $\mathcal{L}$ with $G$ cells. Each individual $i$ is endowed with a cardinal set of utility functions $V_g^i$ for each group $g \in \mathcal{G}$, where each $v_g^i \in V_g^i$ maps $\mathbb{R}_+^L$ to $\mathbb{R}$ and indicates $i$'s utility for goods in $g$. Expected utility theory furnishes a canonical example: if we associate a state $s$ with the group $g$ of contingent goods delivered at $s$ then the functions in $V_g^i$ are the von Neumann Morgenstern cardinal utilities for these goods multiplied by a probability of state $s$. Let $x_g^i \in \mathbb{R}_+^{|g|}$ denote $i$'s consumption of the goods in $g$ and $x^i(k) \in \mathbb{R}_+$ denote $i$'s consumption of good $k$.

The utilities of individual $i$ for the groups are aggregated into a behavioral preference by a (not necessarily cardinal) selection $\mathcal{V}^i \subset V_1^i \times ... \times V_G^i$ with typical element $v^i = (v_1^i, ..., v_G^i)$. Any $v^i \in \mathcal{V}^i$ preference defines a utility function $\sum_{g\in\mathcal{G}} v_g^i$ on $\mathbb{R}_+^L$ that weights $i$'s utilities of groups and represents one of $i$'s complete frame-based preferences $\succsim_f^i$.[6]

Each $\sum_{g\in\mathcal{G}} v_g^i$ formed from a $v^i \in \mathcal{V}^i$ or the $\succsim_f^i$ it represents is an equally legitimate way for $i$ to evaluate bundles and $i$ therefore prefers $x^i$ to $y^i$ only when all of these objective functions concur. Define $\mathcal{V}^i$ to **generate** the behavioral preference $\succsim^i$ on $\mathbb{R}_+^L$ when

$$x^i \succsim^i y^i \Longleftrightarrow \sum_{g\in\mathcal{G}} v_g^i(x^i) \geq \sum_{g\in\mathcal{G}} v_g^i(y^i) \text{ for all } v^i \in \mathcal{V}^i.$$

If $\mathcal{V}^i$ is a cardinal selection from $V_1^i \times ... \times V_G^i$, then $\left\{\sum_{g\in\mathcal{G}} v_g^i : v^i \in \mathcal{V}^i\right\}$ forms a cardinal set of functions and the $\succsim^i$ that $\mathcal{V}^i$ generates will be complete. But when $\mathcal{V}^i$ consists of a

[6] I have assumed that the $\succsim_f^i$ are represented by sums of $v_g^i$'s to fit with the expected utility and intertemporal preference models and with the Jevons-Marshall tradition. As long as utilities do not have to satisfy the separability condition I introduce below, the assumption brings only a modest loss of generality in terms of the $\succsim^i$'s that can be generated: for example any $\succsim^i$ with a utility representation, whether additive or not, can be generated by some $\mathcal{V}^i$. See Appendix A.

larger set of functions (in the extreme, all of $V_1^i \times ... \times V_G^i$), individual $i$ backs conflicting ways of weighting the group utilities. The frame-based preferences then conflict and the $\succsim^i$ that $\mathcal{V}^i$ generates will be incomplete. For example, if $\succsim^i$ is a behavioral preference for the multiple priors model then, as $v^i \in \mathcal{V}^i$ varies, the probabilities implicit in the expected utility function $\sum_{g \in \mathcal{G}} v_g^i$ will vary as well. In all cases, $\succsim^i$ will be transitive.

Though it plays no formal role, it is natural to associate each decision-making frame $f$ with a cardinal selection $\mathcal{V}_f^i \subset V_1^i \times ... \times V_G^i$ interpreted as a representation of the complete preferences $\succsim_f^i$ that hold for the individual at $f$: $x^i \succsim_f^i y^i \Leftrightarrow \sum_{g \in \mathcal{G}} v_g^i(x^i) \geq \sum_{g \in \mathcal{G}} v_g^i(y^i)$ for all $v^i \in \mathcal{V}_f^i$. Each $\mathcal{V}^i$ then equals a union of cardinal selections from $V_1^i \times ... \times V_G^i$ and $x^i \succsim^i y^i$ obtains if and only if the utilities formed by draws from the $\mathcal{V}_f^i$ all judge $x^i$ to be superior to $y^i$.[7]

Utilities satisfy **separability** relative to the set of groups $\mathcal{G}$ when each $v_g^i \in V_g^i$ can vary only with respect to $i$'s consumption of the goods in $g$: for any $g \in \mathcal{G}$ and $x_g^i \in \mathbb{R}_+^{|g|}$, $v_g^i \in V_g^i$ must be constant on $\{y^i \in \mathbb{R}_+^L : y_g^i = x_g^i\}$.[8] In the absence of separability, the consumption levels of the goods in non-$g$ groups can be complements or substitutes for the $g$ goods.

To illustrate how the $V_g^i$ sets of cardinal utilities, the frame-based preferences $\succsim_f^i$, and the $\succsim^i$ interrelate and to see how the $V_g^i$ can be inferred from behavior, we return to the multiple-priors and hyperbolic discounting models. The utilities in both satisfy separability.

**Multiple priors redux** *An agent with von Neumann-Morgenstern utility $u_s^i(x_s)$ at state $s \in \mathcal{S}$ chooses at various frames $f$ where each $f \in \mathcal{F}$ is a probability $\pi^f$ on $\mathcal{S}$.*

*Each group $g \in \mathcal{G}$ consists of the goods delivered at some state $s \in \mathcal{S}$ and so $\mathcal{G}$ can be identified with $\mathcal{S}$. To adjust the domain of the utilities to equal the entire set of goods, define $v_s^i : \mathbb{R}_+^L \to \mathbb{R}$ by $v_s^i(x) = u_s^i(x_s)$ for each $s$. A $V_s^i$ then equals the cardinal set of functions that contains $v_s^i$. Separability is satisfied. Since a decision-making frame $f$ is a probability*

[7] A $\mathcal{V}^i$ that does not equal a union of cardinal selections $\bigcup_f \mathcal{V}_f^i$ would weaken the frame interpretation but otherwise have no effect. Such a $\mathcal{V}^i$ deletes some $v^i$'s from some $\mathcal{V}_f^i$'s without changing the $\succsim^i$ that $\bigcup_f \mathcal{V}_f^i$ represents.

[8] Although I call this assumption 'separability' for brevity, it is a limit on the operative domain of each $v_g^i$. The link to standard usage is that the assumption implies that the $\sum_{g \in \mathcal{G}} v_g^i$ functions, which collectively represent $\succsim^i$, will satisfy 'additive separability' in Gorman (1959) or 'groupwise separability' in Jorgenson and Lau (1975). Since $\succsim^i$ might not have a utility representation, however, none of the preference definitions of separability apply to $\succsim^i$. I reserve 'additive' for functions equal to a sum of group utilities that do represent a preference.

*$\pi^f$, the individual's frame-based preference $\succsim_f^i$ is represented by the cardinal selection from $V_1^i \times ... \times V_S^i$ given by*

$$\mathcal{V}_{\pi^f}^i = \left\{ \left( a\pi_1^f v_1^i + b_1, ..., a\pi_S^f v_S^i + b_S \right) : a \in \mathbb{R}_{++} \text{ and } b_1, ..., b_S \in \mathbb{R} \right\}.$$

*The multiple-priors behavioral preference $\succsim^i$ is then generated by the selection $\mathcal{V}^i = \bigcup_{\pi^f \in \mathcal{F}} \mathcal{V}_{\pi^f}^i$.*

*The $V_s^i$ can be inferred from individual i's preferences over lotteries with objective probabilities. Alternatively, each $v_s^i$ and hence each $V_s^i$ can be deduced from the vNM representation of the $\succsim_f^i$ that holds at one of the frames $\pi^f$.* ■

**Hyperbolic discounting redux** *At dates 1 and 2, which are the frames, an individual i has the preferences $\succsim_1^i$ and $\succsim_2^i$ on the domain $\overline{X}$ (defined in section 2) that are represented by the utilities $u_1^i(x) = u^i(x_2) + \delta u^i(x_3)$ and $u_2^i(x) = u^i(x_2) + \beta\delta u^i(x_3)$. The two groups are the goods that appear at dates 2 and 3 respectively.*

*For the cardinal selections, define $v_t^i : \overline{X}^i \to \mathbb{R}$ by $v_t^i(x^i) = u^i(x_t^i)$ for $t = 1, 2$. Letting t indicate the date t group, $V_t^i$ equals the cardinal set of functions that contains $v_t^i$. Separability is again satisfied. The frame-based preferences $\succsim_1^i$ and $\succsim_2^i$ are represented by the cardinal selections*

$$\begin{aligned} \mathcal{V}_1^i &= \left\{ \left( av_2^i + b_2, a\left(\delta v_3^i\right) + b_3 \right) : \ a \in \mathbb{R}_{++} \text{ and } b_2, b_3 \in \mathbb{R} \right\} \text{ and} \\ \mathcal{V}_2^i &= \left\{ \left( av_2^i + b_2, a\left(\beta\delta v_3^i\right) + b_3 \right) : \ a \in \mathbb{R}_{++} \text{ and } b_2, b_3 \in \mathbb{R} \right\}, \end{aligned}$$

*and $\mathcal{V}^i = \mathcal{V}_1^i \cup \mathcal{V}_2^i$ generates the hyperbolic behavioral preference $\succsim^i$.*

*Both $V_1^i$ and $V_2^i$ can be inferred from the additive representations of either $\succsim_1^i$ or $\succsim_2^i$.* ■

Finally, in the less formal early neoclassical model each group consists of a single good and utilities again satisfy separability: for each good $k$, the utilities in $V_k^i$ vary only with respect to $x_k^i = x^i(k)$. The difficulty recognized by J.S. Mill that individuals may not know how to weight the utilities of goods can lead individuals to adopt different weights on the $v_k^i$ in various circumstances and thus to different cardinal selections. Each selection in effect corresponds to a frame-based preference.

## 4.2 Social welfare

Our utilitarian planner takes as data a set of cardinal utilities $V_g^i$ for each individual $i$ and group $g$. Just as a classical utilitarian selects interpersonally comparable utilities for individuals, the planner selects interpersonally comparable utilities for individuals for each group $g$, a cardinal selection from $V_g^1 \times ... \times V_g^I$, denoted $\mathcal{W}_g$. A typical element of $\mathcal{W}_g$ is a $v_g = \left(v_g^1, ..., v_g^I\right)$, a group $g$ utility for every individual $i$. Each $\mathcal{W}_g$ defines a **group** $g$ **utilitarian ordering** that deems allocation $x$ weakly superior to $y$ if, for any and hence all $v_g \in \mathcal{W}_g$, $\sum_{i \in \mathcal{I}} v_g^i(x^i) \geq \sum_{i \in \mathcal{I}} v_g^i(y^i)$. Each $\sum_{i \in \mathcal{I}} v_g^i$ coincides substantively and mathematically with a classical utilitarian objective function.

The planner may also compare an individual's utilities across groups. All of the planner's judgments taken together are given by a selection $\mathcal{W} \subset \prod_{(g,i) \in \mathcal{G} \times \mathcal{I}} V_g^i$ that places individual and group comparisons on the same mathematical footing: a vector of functions in $\mathcal{W}$, denoted $v = (v_g^i)_{(g,i) \in \mathcal{G} \times \mathcal{I}}$, identifies a utility $v_g^i$ for each group-individual pair $(g, i)$. The selection $\mathcal{W}$ must be compatible with the $\mathcal{W}_g$: we require for each $g$ that the projection of $\mathcal{W}$ onto $V_g^1 \times ... \times V_g^I$ equals $\mathcal{W}_g$. Comparisons of individual utilities across groups thus do not dilute the planner's within-group comparisons of individual utilities, in line with the classical utilitarian position that individuals' vacillations about the worth of goods do not undermine interpersonal comparisons.

The model gives the planner wide latitude to embrace or abstain from comparisons of an individual's utilities for groups. At the abstention end of the spectrum, a planner refrains from all across-group comparisons when, for each $i$, the projection of $\mathcal{W}$ onto $V_1^i \times ... \times V_G^i$ equals $V_1^i \times ... \times V_G^i$ itself, and we then say the planner or $\mathcal{W}$ is **group agnostic**. At the opposite pole, a **dictatorial** $\mathcal{W}$ imposes a single relative weighting of each individual's utilities for groups by requiring $\mathcal{W}$ to be a cardinal selection.

While we have assumed implicitly that a planner begins with the individual welfare comparisons – the $\mathcal{W}_g$ – the alternative where a planner constructs $\mathcal{W}$ in one integrated step works equally well. The sequential view more closely resembles Edgeworth (1881) where the case for redistributing resources to high marginal utility individuals applies to any resource or 'means' that can generate pleasure. There is no fiction of an aggregate consumption good in Edgeworth nor any claim that a single all-encompassing utility function covers all sources of pleasure at all dates. The utilitarian optima moreover are determined solely by the $\mathcal{W}_g$ when separability holds rather than by all of $\mathcal{W}$.

Each $v \in \mathcal{W}$ defines a welfare function $\sum_{i\in\mathcal{I}} \sum_{g\in\mathcal{G}} v_g^i$, the sum across individuals $i$ of one of the utilities that $\mathcal{W}$ assigns to $i$. For the early neoclassical economists, adding individual utilities to form social welfare functions was an assumption; for choice under uncertainty, the Harsanyi (1955) aggregation theorem and specifically Hammond (1981) provide rationales.[9]

Each of the $\sum_{i\in\mathcal{I}} \sum_{g\in\mathcal{G}} v_g^i$ objective functions represents a complete ordering and thus can compare any pair of allocations.[10] Under separability each $\sum_{i\in\mathcal{I}} \sum_{g\in\mathcal{G}} v_g^i$ gives classical utilitarian advice: welfare increases when the goods in some group $g'$ are transferred from an agent $i$ with low marginal utility for these goods (according to $v_{g'}^i$) to an agent $j$ with high marginal utility (according to $v_{g'}^j$). But the different welfare functions drawn from $\mathcal{W}$ can rank allocations differently. For example if $x$ is an improvement over $y$ with respect to the group $g$ utilitarian ordering but a worsening with respect to the $g'$ ordering and $\mathcal{W}$ is group agnostic then the welfare functions drawn from $\mathcal{W}$ that assign large weight to the group $g$ utilities will approve a move from $y$ to $x$ while other welfare functions drawn from $\mathcal{W}$ will reject the move. Due to this diversity, we require unanimous consent before declaring an allocation to be an improvement. Letting $e = (e_1, ..., e_G) \geq 0$ be the economy's endowment of goods, define the feasible allocations $F = \{x \in \mathbb{R}_+^{IL} : \sum_{i\in\mathcal{I}} x^i \leq e\}$.

**Definition 2** *Allocation $x$ is **utilitarian superior** to $y$ if $\sum_{i\in\mathcal{I}} \sum_{g\in\mathcal{G}} v_g^i(x^i) \geq \sum_{i\in\mathcal{I}} \sum_{g\in\mathcal{G}} v_g^i(y^i)$ for all $v \in \mathcal{W}$ and strict inequality holds for at least one $v \in \mathcal{W}$, and is a **utilitarian optimum** if $x$ is feasible and there is no feasible $y$ that is utilitarian superior to $x$.*

Though the unanimity in Definition 2 leads to a cautious ranking, utilitarian optimality discriminates with precision. We put aside optima that deliver the same utility levels by

[9]The probabilities in the utility representations in Hammond coincide across individuals. Since individual utilities in the present paper are state-dependent (as in Hammond), we may let probabilities vary by individual, as we must when a change in the vector of utilities drawn from $\mathcal{W}$ indicates only a change in the probability frame of a single individual.

[10]Each $\sum_{i\in\mathcal{I}} \sum_{g\in\mathcal{G}} v_g^i$ arises at one frame and it is natural, as in individual decision-making, to associate each frame $f$ with a cardinal selection $\mathcal{W}_f \subset \prod_{(g,i)\in\mathcal{G}\times\mathcal{I}} V_g^i$. Any two $v$'s drawn from the same $\mathcal{W}_f$ will then generate, up to affine transformation, the same welfare function and $\mathcal{W}$ will equal the union $\bigcup_{f\in\mathcal{F}} \mathcal{W}_f$.

assuming that the $v_g^i \in V_g^i$ are strictly concave on the goods in $g$. Let $v_g^i : \mathbb{R}_+^L \to \mathbb{R}$ be **strictly concave on** $g$ if it is strictly concave when restricted to the goods in $g$.[11]

**Theorem 2** *If separability is satisfied and, for each group $g$ and individual $i$, $v_g^i \in V_g^i$ is strictly concave on $g$ and continuous then there is a unique utilitarian optimum.*

**Proof.** Fix some $g \in \mathcal{G}$, $y_{-g} = (y_{g'}^i)_{i \in \mathcal{I}, g' \in \mathcal{G} \setminus \{g\}}$, and $v \in \mathcal{W}$. For any $i$,

$$\sum\nolimits_{g' \in \mathcal{G}} v_{g'}^i(x_g^i, y_{-g}^i) = v_g^i(x_g^i, y_{-g}^i) + \sum\nolimits_{g' \neq g} v_{g'}^i(x_g^i, y_{-g}^i)$$

and, due to separability, $\sum_{g' \neq g} v_{g'}^i(x_g^i, y_{-g}^i)$ equals the same constant for all $x_g^i \geq 0$. Thus $x'$ solves $\max_{x \geq 0} \sum_{i \in \mathcal{I}} \sum_{g' \in \mathcal{G}} v_{g'}^i(x_g^i, y_{-g}^i)$ s.t. $\sum_{i \in \mathcal{I}} x^i \leq e$ if and only if, for each $g$, $x_g'$ solves $\max_{x_g \geq 0} \sum_{i \in \mathcal{I}} v_g^i(x_g^i, y_{-g}^i)$ s.t. $\sum_{i \in \mathcal{I}} x_g^i \leq e_g$.

Due to continuity, there is a $\overline{x}_g = \left(\overline{x}_g^1, ..., \overline{x}_g^I\right)$ that solves the latter problem and, due to strict concavity on $g$, this solution $\overline{x}_g$ is unique. Due to separability the solution does not depend on the choice of $y_{-g}$ and, since the projection of $\mathcal{W}$ onto $V_g^1 \times ... \times V_g^I$ equals a cardinal selection, the solution does not depend on the choice of $v \in \mathcal{W}$. Hence $(\overline{x}_1, ..., \overline{x}_G)$ is the unique optimum. ■

Theorem 2 places no restrictions on $\mathcal{W}$ beyond each $\mathcal{W}_g$ being a cardinal selection: the planner can fall anywhere between the group agnostic and dictatorial extremes. With group agnosticism, planners make no comparisons of groups and utilitarian superiority reduces to the unanimity ordering of the group-by-group utilitarian orderings. Define allocation $x$ to be **group-unanimously superior** to $y$ when $\sum_{i \in \mathcal{I}} v_g^i(x^i) \geq \sum_{i \in \mathcal{I}} v_g^i(y^i)$ for all $g \in \mathcal{G}$ and $v_g \in \mathcal{W}_g$ with at least one strict inequality and a **group-unanimity optimum** if $x$ is feasible and no feasible $y$ is group-unanimously superior to $x$.

**Proposition 1** *If $\mathcal{W}$ is group agnostic then the group-unanimity and utilitarian superiority orderings coincide. For any $\mathcal{W}$, any utilitarian optimum is a group-unanimity optimum.*[12]

Proposition 1 clarifies just how conservative the utilitarian superiority relation can be: when $\mathcal{W}$ is group agnostic, a change in allocations must be recommended by every group

[11] That is, for all $x^i \in \mathbb{R}_+^L$, $v_g^i$ is strictly concave on $\left\{y^i \in \mathbb{R}_+^L : y^i(k) = x^i(k) \text{ for all } k \notin g\right\}$.
[12] Omitted proofs are in Appendix B.

utilitarian ordering. In a model of uncertainty, for example, a change must deliver an improvement for the group $g$ utilitarian ordering even when the group $g$ goods appear at a highly unlikely state. While cautious orderings ordinarily make it easier to declare allocations optimal and therefore invite indecisiveness, Theorem 2 shows that under separability there is a unique utilitarian optimum. In fact under the assumptions of Theorem 2 any across-group comparisons made by $\mathcal{W}$ end up being irrelevant: the proof of Theorem 2 (or the second sentence of Proposition 1) shows that the utilitarian optimum is the unique group-unanimity optimum.

For dictatorial planners, Theorem 2 and its proof restate the uniqueness reasoning of a standard model of utilitarianism: the planner judges every trade-off across both goods and individuals via one utilitarian ordering, as a traditional utilitarian would, and the unique optimum is found by maximizing one function subject to the resource constraints. If in particular there is just one group, planners are necessarily dictatorial and the $\succsim^i$ are complete. But when there is more than one group, planners have good reason not to go to the dictatorial extreme: they may be stymied by the same decisions that individuals cannot resolve consistently for themselves. A planner's judgment that the marginal utility of a group of goods is higher for the poor than for the rich need not make it any easier to decide which types of goods and pleasures deserve priority, decisions that the classical utilitarians themselves considered challenging. Even when planners feel they can make such decisions, they may want to respect individuals' equivocations about how groups of goods should be weighed. A middle ground therefore has some appeal, for example, a $\mathcal{W}$ such that for each individual $i$ the projection of $\mathcal{W}$ onto $V_1^i \times ... \times V_G^i$ equals $\mathcal{V}^i$, the representation of $i$'s behavioral preferences.

**Hyperbolic discounting continued**. A society of $I$ individuals with the hyperbolic preferences of section 2 will have a large set of Pareto optima. Suppose for concreteness that there is one good per time period, $\ell = 1$, which implies that each good is a group. If the feasible set $\overline{F}$ is $\{x \in \overline{X}^1 \times ... \times \overline{X}^I : \sum_{i \in \mathcal{I}} x_t^i \leq e_t \text{ for } t = 2, 3\}$, the dimension of the Pareto optima will equal $2(I-1)$, which is also the dimension of the frontier of $\overline{F}$ (the non-disposal points where the inequalities that define $\overline{F}$ hold with equality). As explained in section 2, each indifference curve for each $\succsim^i$ has a continuum of supporting prices, at a Pareto optimum those continua for the $I$ individuals will intersect, and the intersection will typically persist following any small adjustment in the allocation. By the first welfare theorem, the new allocation must then also be Pareto optimal.

A utilitarian planner specifies for each $t = 2, 3$ a cardinal selection $\mathcal{W}_t$ from $V_t^1 \times ... \times V_t^I$, possibly supplemented by further intrapersonal comparisons of utility across goods. For $t = 2, 3$, there will be a unique distribution of the date-$t$ good that is utilitarian optimal, found by selecting some $v_t \in \mathcal{W}_t$ and maximizing $\sum_{i \in \mathcal{I}} v_t^i(x_t^i)$ subject to $\sum_{i \in \mathcal{I}} x_t^i \leq e_t$ and $\left(x_t^1, ..., x_t^I\right) \geq 0$. Specifying the cardinal selection may present a normative challenge, but the difficulties should be no more formidable than with complete preferences. ■

## 4.3 Social welfare: nonseparable utilities

So far utilities have satisfied separability: each $v_g^i$ has been a function only of goods in group $g$. Without this assumption there need not be a unique utilitarian optimum. For example, suppose each good is a singleton group and that the utility of good 1 depends on the consumption of good 2 and vice versa while the utilities for other goods are functions only of their own consumption levels. When $\mathcal{W}$ is group agnostic, the utilitarian optima will coincide with the Pareto efficient allocations of a society of hypothetical agents with the group $g$ utilitarian orderings, one for each $g \in \mathcal{G}$ (Proposition 1). If the groups 1 and 2 utilitarian orderings display sufficient diversity regarding how goods 1 and 2 are optimally distributed across individuals there will be a one-dimensional set of utilitarian optima for the economy that consists of just goods 1 and 2: given a pair $\sum_{i \in \mathcal{I}} v_1^i$ and $\sum_{i \in \mathcal{I}} v_2^i$ that define these two orderings, the utility possibility frontier will be one-dimensional just as the utility possibility frontier of a standard two-agent economic model is one-dimensional. As there remains a unique optimum for the distribution of goods $3, ..., L$, the utilitarian optima for the economy of all $L$ goods will also be one-dimensional.

Our goal will be to show that even in worst cases the size of the set of utilitarian optima compares favorably with the size of the set of Pareto optima (as defined by the $\succsim^i$ not by the hypothetical agents above) both in terms of dimension and measure. Since the utilitarian optima always form a subset of the group unanimity optima (Proposition 1), we can find a bound on the dimension or measure of the utilitarian optima by considering the group-unanimity optima. The above example lies far from the worst case: it displays the minimum extent of nonseparability in consumption and accordingly the dimension of the group-unanimity optima expands modestly, from 0 to 1. The greatest expansion occurs when, for each group $g$, the $\sum_{i \in \mathcal{I}} v_g^i$ that define the group $g$ utilitarian ordering are nontrivial functions of all $L$ goods in the model and each good is its own group: the dimension of the group-unanimity optima can then rise to $G - 1 = L - 1$. This case is similar but not identical to a standard general equilibrium model where the dimension of the Pareto efficient allocations would normally equal the number of individuals $I$ minus 1. In our setting, the role of an individual is played by a group $g$ with the associated 'utility' $\sum_{i \in \mathcal{I}} v_g^i$. The planner thus has $G$ objective functions which suggests that the dimension of the group-unanimity optima will be $G - 1$. But while standard utilities are functions of different variables (the agents' private consumptions) each $\sum_{i \in \mathcal{I}} v_g^i$ is potentially a function of all $LI$ of the model's consumption variables. Externalities in effect appear since the goods that affect the 'utility' $\sum_{i \in \mathcal{I}} v_g^i$ can also affect the 'utility' $\sum_{i \in \mathcal{I}} v_{g'}^i$, where $g' \neq g$. These externalities imply that the set of optima need not have a well-defined dimension (it need not be a manifold) and consequently Theorem 3 below will provide only an upper bound on the dimension of the optima.[13] 'Upper' is the bound of interest since we are interested in worst-case scenarios.

Dimension is neither the sole nor indisputable way to gauge the size of a set. Our result on dimension, Theorem 3, will imply that the utilitarian optima also form a measure-0 subset of society's allocations. The Pareto optima in contrast can have positive measure as we saw in section 2. There are other yardsticks, however, for example the diameter of the set of optima, that Theorem 3 will not address.

To find a bound for the dimension of the group-unanimity optima, which in turn will bound the dimension of the utilitarian optima, I adapt the concept of an 'isolated community' (Smale (1974)) from the general equilibrium theory of Pareto efficiency. A classical isolated community is a subset of individuals that consumes only goods that individuals outside the

[13] For a simple case of the complications introduced by these externalities, notice that it is possible to have $\sum_{i \in \mathcal{I}} v_g^i = \sum_{i \in \mathcal{I}} v_{g'}^i$ for distinct groups $g$ and $g'$. There are then effectively fewer than $G$ objective functions in the model and the dimension of the group-unanimity optima accordingly falls.

community do not consume, for example when community members and nonmembers have utilities that are increasing on disjoint sets of goods. An 'isolated basket' will be a set of groups of goods where for each group $g$ in the basket the $v_g^i$ can be nontrivial functions only of goods in the groups the basket contains. Define $h : \mathbb{R}_+^L \to \mathbb{R}$ to be **variable on group** $g$ if there is a pair $x, y \in \mathbb{R}_+^L$ such that $x_{g'} = y_{g'}$ for each $g' \neq g$ and $h(x) \neq h(y)$.

**Definition** *The nonempty pairwise-disjoint subsets of groups $\mathcal{B}_1, ..., \mathcal{B}_n \subset \mathcal{G}$ form **isolated baskets** if, for each $\mathcal{B}_j$, (i) any $v_g^i \in V_g^i$, where $g \in \mathcal{B}_j$ and $i \in \mathcal{I}$, is variable only on a nonempty set of goods in the groups in $\mathcal{B}_j$ and (ii) no nontrivial partition of $\mathcal{B}_j$ has cells that satisfy (i).*

A model $\left(V_g^i\right)_{i \in \mathcal{I}, g \in \mathcal{G}}$ has a unique family of isolated baskets, which may consist of just one basket.[14] Each $\mathcal{B}_j$ in a family defines a corresponding set of objective functions, the $\sum_{i \in \mathcal{I}} v_g^i$ such that $g \in \mathcal{B}_j$ and $v_g \in \mathcal{W}_g$, that are variable only on groups drawn from the same basket. Separability is the prominent example. Each $\sum_{i \in \mathcal{I}} v_g^i$ then varies only as a function of $x_g^1, ..., x_g^I$ while the remaining $\sum_{i \in \mathcal{I}} v_{g'}^i$, $g' \neq g$, are constant in these variables: each group $g$ by itself forms an isolated basket. As the extent of nonseparabilities in consumption increase from the floor given by separability, the sizes of the isolated baskets increase in tandem.

In general equilibrium theory, nontrivial isolated communities cause the dimension of the Pareto efficient allocations to fall below the number of individuals minus 1. The individuals in each isolated community form a free-standing economic model and the dimension of the Pareto efficient allocations for this community equals the number of individuals in the community minus 1. The dimension of the optima for the unified model then equals the sum of the community-specific dimensions, which must be less than the total number of individuals minus 1. For example, with two communities of $I_1$ and $I_2$ individuals, where $I_1 + I_2 = I$, the dimension of the optima will equal $I_1 - 1 + I_2 - 1 < I - 1$. In our model, where groups play the role of individuals, this accounting continues to hold: the dimension of the group-unanimity optima will fall below $G - 1$ when some of the isolated baskets contain more than one group. Theorem 2 is a case in point.

[14] The family need not form a partition of $\mathcal{G}$: if, for some $g \in \mathcal{G}$ and every $i \in \mathcal{I}$, $v_g^i \in V_g^i$ is a constant function then $g$ is not an element of any basket.

It will be easier to work directly with welfare vectors for the group utilitarian orderings rather than the allocations that give rise to those vectors, which will also allow us to drop the strict concavity assumption. Given $\mathcal{W}$, fix an arbitrary $v_g = \left(v_g^1, ..., v_g^I\right) \in \mathcal{W}_g$ for each $g \in \mathcal{G}$. The set of **feasible welfare vectors** is

$$W = \{w \in \mathbb{R}^G : \text{ there exists } x \in F \text{ such that } \sum_{i \in \mathcal{I}} v_g^i(x^i) = w_g \text{ for } g \in \mathcal{G}\}$$

and the set of **optimal welfare vectors** is

$$W_O = \{w \in W : \text{ there does not exist } x \in F \text{ such that } \sum_{i \in \mathcal{I}} v_g^i(x^i) \geq w_g \text{ for } g \in \mathcal{G}$$
$$\text{and with strict equality for some } g \in \mathcal{G}\}.$$

Any change in the choice of $v_g \in \mathcal{W}_g$ will only rescale the vectors in $W$ and $W_O$ and/or change each coordinate by a constant, leaving the dimension of the sets unaffected.

If we were to assume that the $v_g^i$ are strictly concave (on the variables $x^i(k)$ that affect $v_g^i$) then, for each $w \in W_O$, there would be only one feasible allocation that achieves $w$, that is, one $x \in F$ such that $\sum_{i \in \mathcal{I}} v_g^i(x^i) = w_g$ for $g \in \mathcal{G}$. The dimension of the optimal welfare vectors would then coincide with the dimension of the group-unanimity optimal allocations which in turn must be at least as great as the dimension of the utilitarian optimal allocations. Think of the contract curve of an Edgeworth box and its utility possibility frontier: under strict concavity both are one-dimensional.[15]

**Theorem 3** *Suppose, for any group $g$ and individual $i$, that each $v_g^i \in V_g^i$ is continuous and there are $n$ isolated baskets with $d_1, ..., d_n$ groups. Then the set of optimal welfare vectors is contained in a set of dimension $\sum_{j=1}^n (d_j - 1)$, a number that cannot exceed $G - 1$.*

The dimensional expansion of the set of optimal welfare vectors thus depends on the sizes of the isolated baskets, which in turn depend on the extent of the nonseparabilities in consumption and on the sizes of the groups. The worst case, where the dimension of the

[15] The set $W_O$, however, need not be a manifold which stops us from pinning down the dimension of the set of optimal allocations.

optimal welfare vectors reaches $L-1$, occurs when the entire set of goods forms the sole isolated basket and each good is a singleton group. Even here the utilitarian optima have not undergone the $L(I-1)$ explosion of dimensionality that occurs for the Pareto optima when preferences are incomplete (see section 2). If, as one presumes in market settings, the number of individuals is larger than the number of goods, $I > L$, then with incomplete preferences the dimension of the utilitarian optimal welfare vectors will be less than the dimension of the utility possibility frontier that obtains when preferences are complete. Moreover with incomplete preferences the Lebesgue measure of the Pareto optima will typically be positive while the Lebesgue measure of the utilitarian optima will typically be 0.

Theorem 2 is not a corollary of Theorem 3, first due to our transition to welfare vectors and second since Theorem 3 states only that when each $d_j = 1$ the set of optima is a discrete set of points (a 0-dimensional set) rather than a single point.

# 5 Utilitarian versus Pareto optimality

Utilitarian optima can fail to be Pareto optimal when individual preferences are given by the $\succsim^i$. A utilitarian planner weights an individual $i$'s utility for group $g$ according to the planner's judgment about the satisfaction $i$ derives from $g$ and that weighting might not be compatible with the weightings implicit in $\succsim^i$. For example, suppose there are two individuals $a$ and $b$, two goods each of which forms a group, and that utilities are separable and differentiable. Let $\mathcal{W}_1$ place equal weight on $v_1^a$ and $v_1^b$ and let $\mathcal{W}_2$ place equal weight on $v_2^a$ and $v_2^b$ which, with strictly concave utilities that coincide across individuals, would lead to a utilitarian optimum where $x^a = x^b$. Depending on the $\succsim^i$, Pareto improvements may be possible. If, at some frame $f$, the $v^a \in \mathcal{V}_f^a$ assigns weights 2 and 1 respectively to $v_1^a$ and $v_2^a$ while $v^b \in \mathcal{V}_f^b$ assigns weights 1 and 2 to $v_1^b$ and $v_2^b$ then the marginal rates of substitution of the agents at $f$ cannot align at the utilitarian optimum.[16] Some transfer between the individuals will therefore deliver a Pareto improvement for $\succsim_f^a$ and $\succsim_f^b$. Whether each individual $i$ is $\succsim^i$-better off with this transfer will depend on the diversity they display across frames. But if at every frame each individual places nearly the same relative weights on the

[16] If $Dv_1^a(x_1^a) = Dv_1^b(x_1^b)$ and $Dv_2^a(x_2^a) = Dv_2^b(x_2^b)$ then $\frac{2Dv_1^a(x_1^a)}{Dv_2^a(x_2^a)} \neq \frac{Dv_1^b(x_1^b)}{2Dv_2^b(x_2^b)}$.

utilities for goods then both individuals' behavioral preferences will back the transfer.[17] As this example suggests, both the compatibility and incompatibility of utilitarian and Pareto optimality are robust possibilities.

For the orthodox utilitarian, a failure of utilitarian optimality to achieve Pareto optimality is not a problem: the planner's judgments take precedence over the agents. The hyperbolic discounting example in section 4 is a case in point. In the pre-WWII heyday of utilitarianism, it was common to hold that impatience and discounting amounted to failures of rationality – an offense 'against the rules of economic reason' in Schumpeter's words (1912, p. 35). A utilitarian planner therefore might well ignore individuals' time preferences altogether and, for *every* date $t$, use the same cardinal selection $\mathcal{W}_t$ from $V_t^1 \times ... \times V_t^I$.

Whether or not utilitarian-Pareto conflicts are problematic, planners might want to avoid them. A liberal planner could deliberately choose weights on an individual's utility for groups to match the individual's weights. Or the planner could just choose $\mathcal{W}$ and a utilitarian optimum $x$ such that $x$ is Pareto optimal. Since this choice would impose an additional restriction beyond utilitarian optimality, it will not expand the set of optima.

More productively, I show that the potential for a clash between utilitarian and Pareto optimality diminishes as preference incompleteness increases. First I define what it means for a liberal planner to match the weights of an incomplete preference.

**Definition 3** *Given $\mathcal{V}^i$ for $i \in \mathcal{I}$, **preference compatibility** is satisfied for $\mathcal{W}$ if there exists $\overline{v} \in \mathcal{W}$ such that $\overline{v}^i \in \mathcal{V}^i$ for $i \in \mathcal{I}$.*

Suppose that we now adjust the preferences, letting them get progressively more incomplete by expanding the selections $\mathcal{V}^i$ that define the $\succsim^i$ while keeping the sets $V_g^i$ and the selection $\mathcal{W}$ fixed. If each selection $\mathcal{V}^i$ is large enough – there is enough incompleteness – then preference compatibility must be satisfied. In particular, when $\mathcal{V}^i = V_1^i \times ... \times V_L^i$ for each $i$, any $\mathcal{W}$ is preference compatible: with sufficient incompleteness, every planner becomes a liberal. As we will see, if incompleteness is in this sense sufficiently substantial and separability holds then any utilitarian optimum will be Pareto optimal.

[17] This argument bears some similarity to the difficulties of implementing Pareto efficient outcomes with separable preferences identified by Le Breton and Sen (1999).

It's worth pausing to consider the other extreme: if each $\mathcal{V}^i$ is a cardinal selection from $V_1^i \times ... \times V_L^i$, which implies that the $\succsim^i$ generated by $\mathcal{V}^i$ is complete, then preference compatibility effectively eliminates a planner's ability to specify the $\mathcal{W}_g$ independently. Given $V_g^i$ for $i \in \mathcal{I}$ and $g \in \mathcal{G}$, define group $g'$ to be **nontrivial** if each $V_{g'}^i$ contains nonconstant functions.

**Proposition 2** *If each $\mathcal{V}^i$ is a cardinal selection then, given $\mathcal{W}_{g'}$ for some nontrivial group $g'$, there is only one $\mathcal{W}_g$ for each $g \neq g'$ such that $\mathcal{W}$ satisfies preference compatibility.*

As one would expect, the combination of preference compatibility and complete preferences will stifle a planner's latitude to impose interpersonal comparisons.

The **Pareto optimality** of an allocation $x$ has the standard definition: $x \in F$ and there does not exist $y \in F$ such that $y^i \succsim^i x^i$ for all $i \in \mathcal{I}$ and $y^j \succ^j x^j$ for some $j \in \mathcal{I}$, where each $\succsim^i$ is generated by $\mathcal{V}^i$.

Whether preference compatibility holds due to preferences being substantially incomplete or the careful choice of a liberal planner, the assumption is not quite enough to guarantee that utilitarian optima are Pareto optimal. The following example, driven by a failure of separability, illustrates.

**Example** *Suppose there are two individuals $a$ and $b$ and two goods, each of which is a group, and define*

$$\begin{aligned}
\overline{v}_1^a(x_1^a, x_2^a) &= \ln x_1^a + 2\ln x_2^a, \;\; \overline{v}_2^a(x_1^a, x_2^a) = 2\ln x_1^a + \ln x_2^a, \\
\overline{v}_1^b(x_1^b, x_2^b) &= 2\ln x_1^b + \ln x_2^b, \;\; \overline{v}_2^b(x_1^b, x_2^b) = \ln x_1^b + 2\ln x_2^b.
\end{aligned}$$

*Let these four functions be the profile given by preference compatibility. For each $i$, one of the $v_1^i + v_2^i$ formed by the $v^i \in \mathcal{V}^i$ is then $\ln x_1^i + \ln x_2^i$. So, if $\mathcal{V}^i$ is a cardinal selection the $\succsim^i$ it generates is the complete preference represented by $\ln x_1^i + \ln x_2^i$. If $e_1 = e_2$ (the Edgeworth box is square) it is then easy to confirm that the Pareto optima satisfy $x_1^i = x_2^i$ for $i = a, b$ (the $45°$ line). On the other hand, the planner's good $1$ objective function is $\ln x_1^a + 2\ln x_2^a + 2\ln x_1^b + \ln x_2^b$ which is maximized subject to the resource constraints at the allocation $x^a = (\frac{1}{3}e_1, \frac{2}{3}e_2)$, $x^b = (\frac{2}{3}e_1, \frac{1}{3}e_2)$. Since this allocation is the unique global optimum* *for the good 1 objective function, it must be a utilitarian optimum when $\mathcal{W}$ is group-agnostic.* ■

The problem in the Example is that each individual $i$'s utility for one good is affected by $i$'s consumption of the other good and though this 'side effect' is cancelled in the construction of $\succsim^i$ by $i$'s utility for the other good, the cancellation does not enter into the planner's maximization of the good 1 objective function. Separability blocks this path for trouble.

**Theorem 4** *If separability is satisfied, $\mathcal{W}$ is preference compatible, and any $v_g^i \in V_g^i$ is strictly concave on $g$ for all $g$ and $i$, then any utilitarian optimum is Pareto optimal.*

**Proof.** Suppose $x$ Pareto dominates $y$: $x^i \succsim^i y^i$ for all $i \in \mathcal{I}$ and $x^j \succ^j y^j$ for some $j \in \mathcal{I}$, where each $\succsim^i$ is generated by $\mathcal{V}^i$. Then for the $(\overline{v}_1^i, ..., \overline{v}_G^i) \in \mathcal{V}^i$, $i \in \mathcal{I}$, given by preference compatibility, $\sum_{g\in\mathcal{G}} \overline{v}_g^i(x^i) \geq \sum_{g\in\mathcal{G}} \overline{v}_g^i(y^i)$ for all $i \in \mathcal{I}$. Due to strict concavity, the $z$ defined by $z^i = \frac{1}{2}x^i + \frac{1}{2}y^i$ for each $i \in \mathcal{I}$ satisfies $\sum_{g\in\mathcal{G}} \overline{v}_g^i(z^i) \geq \sum_{g\in\mathcal{G}} \overline{v}_g^i(y^i)$ for all $i$ and, since $x^j \neq y^j$, $\sum_{g\in\mathcal{G}} \overline{v}_g^j(z^j) > \sum_{g\in\mathcal{G}} \overline{v}_g^j(y^j)$. Hence $\sum_{i\in\mathcal{I}} \sum_{g\in\mathcal{G}} \overline{v}_g^i(z^i) > \sum_{i\in\mathcal{I}} \sum_{g\in\mathcal{G}} \overline{v}_g^i(y^i)$. But if $x$ is feasible then $y$ cannot be a utilitarian optimum: if it were then $y$ and hence $z$ would be feasible, by Proposition 1 separability would therefore imply $\sum_{i\in\mathcal{I}} \overline{v}_g^i(y^i) \geq \sum_{i\in\mathcal{I}} \overline{v}_g^i(z^i)$ for each $g \in \mathcal{G}$, and hence $\sum_{g\in\mathcal{G}} \sum_{i\in\mathcal{I}} \overline{v}_g^i(y^i) \geq \sum_{g\in\mathcal{G}} \sum_{i\in\mathcal{I}} \overline{v}_g^i(z^i)$, a contradiction. Any utilitarian optimum is therefore Pareto optimal. ■

In the absence of separability, utilitarian-Pareto disagreements will still typically disappear as preference incompleteness increases. Call $y$ a **maximum for** $\widehat{v}$ if $y$ is feasible and

$$\sum_{i\in\mathcal{I}} \sum_{g\in\mathcal{G}} \widehat{v}_g^{\,i}(y^i) \geq \sum_{i\in\mathcal{I}} \sum_{g\in\mathcal{G}} \widehat{v}_g^{\,i}(x^i)$$

for any feasible $x$. Proposition 3 in Appendix A shows that it will normally be the case that a utilitarian optimum is a maximum for some $\widehat{v} \in \mathcal{W}$. Comparably to utilitarian-Pareto clashes under separability, it may well be that some $\widehat{v}^{\,i}$ will fail to be in individual $i$'s selection $\mathcal{V}^i$. But if we again let preferences become more incomplete by expanding the selection $\mathcal{V}^i$ then eventually $\mathcal{V}^i$ will contain $\widehat{v}^{\,i}$ for each $i$. The following Theorem implies that there will then be no utilitarian-Pareto disagreements. Define $v_g^i : \mathbb{R}_+^L \to \mathbb{R}$ to be **strictly coordinately concave** if it is strictly concave on the variables that affect $v_g^i$, that is, there is a nonempty set of coordinates $\mathcal{K} \subset \mathcal{L}$ such that, for all $x^i \in \mathbb{R}^L_+$, $v^i_g$ is strictly concave on $\{y^i \in \mathbb{R}^L_+ : y^i(k) = x^i(k) \text{ for } k \notin \mathcal{K}\}$ and constant on $\{y^i \in \mathbb{R}^L_+ : y^i(k) = x^i(k) \text{ for } k \in \mathcal{K}\}$.

**Theorem 5** *If $y$ is a maximum for $\widehat{v}$, where $\widehat{v}^i \in \mathcal{V}^i$ for each $i$, and $v^i_g \in V^i_g$ is strictly coordinately concave for each $g$ and $i$, then $y$ is Pareto optimal.*

**Proof.** Since $y$ is a maximum for $\widehat{v}$, $\sum_{i\in\mathcal{I}}\sum_{g\in\mathcal{G}} \widehat{v}^i_g(y^i) \geq \sum_{i\in\mathcal{I}}\sum_{g\in\mathcal{G}} \widehat{v}^i_g(z^i)$ for any feasible $z$. But, as in the proof of Theorem 4, if the feasible $x$ Pareto dominates $y$ then, for the feasible $z$ defined by $z^i = \frac{1}{2}x^i + \frac{1}{2}y^i$ for each $i$, $\sum_{i\in\mathcal{I}}\sum_{g\in\mathcal{G}} \widehat{v}^i_g(z^i) > \sum_{i\in\mathcal{I}}\sum_{g\in\mathcal{G}} \widehat{v}^i_g(y^i)$. ■

# 6 Conclusion

The métier of utilitarianism is the division of a resource among competing individual claims. It is accordingly the precision of decisions about the optimal distribution of goods that is compatible with the incompleteness of individual preferences. The theory provided here offers no help for nondistributive decisions that individuals themselves do not know how to make. If each individual cannot decide how much current output should be invested in the future – how much consumption should be sacrificed to lessen carbon pollution? – then a utilitarian planner will likely also be unable to judge those trade-offs. But the incompleteness of preferences that inevitably accompanies challenging decisions, such as carbon pollution, need not interfere with distributive decision-making: utilitarian advisors can recommend how to divide up any given stock of goods.

# Appendix A: Further results

### Generatable preferences

Any $\succsim^i$ that has a utility representation $u$ can be generated by some $\mathcal{V}^i$ regardless of the partition of groups $\mathcal{G}$: set $V^i_g = \{au + b : a \in \mathbb{R}_{++}, b \in \mathbb{R}\}$ for each $g \in \mathcal{G}$ and let $\mathcal{V}^i$ be any selection from $V^i_1 \times ... \times V^i_G$. At the other end of spectrum, as long as $L > 1$ we can admit the extreme incomplete preference relation $\succsim^i$ that orders no pair of distinct bundles: letting $u$ satisfy $u(x^i) = u(y^i)$ if and only if $x^i = y^i$ (see Mandler (2015)), set $V^i_{\{1\}} = \{au + b : a \in \mathbb{R}_{++}, b \in \mathbb{R}\}$, $V^i_{\{2,...,L\}} = \{-au + b : a \in \mathbb{R}_{++}, b \in \mathbb{R}\}$ and $\mathcal{V}^i = V^i_1 \times ... \times V^i_G$. Less dramatically, we can admit the $\succsim^i$ with $x^i \succsim^i y^i$ if and only if $x^i \geq y^i$ by setting

$\mathcal{G} = \{\{1\}, ..., \{L\}\}$, $V^i_{\{k\}} = \{ah_k + b : a \in \mathbb{R}_{++}, b \in \mathbb{R}\}$ for each good $k$, where $h_k : \mathbb{R}^L_+ \to \mathbb{R}$ is defined by $h_k(x^i) = x^i_k$, and $\mathcal{V}^i = V^i_{\{1\}} \times ... \times V^i_{\{L\}}$.

**Utilitarian optima maximize a social welfare function**

We show that the assumption that a utilitarian optimum is a maximum for some $\left(\widehat{v}^{\,i}_{g}\right)_{g\in\mathcal{G},i\in\mathcal{I}}$ is mild. Call the utilitarian optimum $x$ **interior** if $\sum_{i\in\mathcal{I}} v^i_g(x^i) > \sum_{i\in\mathcal{I}} v^i_g(0)$ for each $g \in \mathcal{G}$, where $(v^1_g, ..., v^I_g) \in \mathcal{W}_g$ for each $g$. Define the function $v^i_g$ to be **strongly increasing** if $x^i \geq y^i$ and $x^i \neq y^i$ imply $v^i_g(x^i) > v^i_g(y^i)$.

**Proposition 3** *If each $v^i_g \in V^i_g$ is concave and strongly increasing for all $i$ and $g$ and if $x$ is an interior utilitarian optimum then there exists $\left(\widehat{v}^{\,i}_{g}\right)_{g\in\mathcal{G},i\in\mathcal{I}}$, where each $\widehat{v}_g \in \mathcal{W}_g$, such that $x$ is a maximum for $\left(\widehat{v}^{\,i}_{g}\right)_{g\in\mathcal{G},i\in\mathcal{I}}$.*

**Proof.** Letting $x$ be a utilitarian optimum, Proposition 1 implies $x$ is a group-unanimity optimum. Hence, for each $k$, $x$ is a solution to $\max_y \sum_{i\in\mathcal{I}} v^i_g(y^i_g)$ s.t. $\sum_{i\in\mathcal{I}} v^i_{g'}(y^i_{g'}) \geq \sum_{i\in\mathcal{I}} v^i_{g'}(x^i_{g'})$ for each $g' \in \mathcal{G}\backslash\{g\}$, $\sum_{i\in\mathcal{I}} y^i_g \leq e_g$ for each $g \in \mathcal{G}$, and $y \geq 0$, where $(v^1_g, ..., v^I_g) \in \mathcal{W}_g$. Due to the strong increasingness and interiority assumptions there is a $y'$ such that each inequality constraint is slack. Hence the Slater (1950) constraint qualification is satisfied (Boyd and Vandenberghe (2004, 5.2.3 and 5.3.2)) and hence there exist $\lambda^g_{g'} \geq 0$ for $g' \in \mathcal{G}\backslash\{g\}$ such that

$$\sum_{i\in\mathcal{I}} v^i_g(x^i_g) + \sum_{g'\in\mathcal{G}\backslash\{g\}} \lambda^g_{g'} \sum_{i\in\mathcal{I}} v^i_{g'}(x^i_{g'}) \geq \sum_{i\in\mathcal{I}} v^i_g(y^i_g) + \sum_{g'\in\mathcal{G}\backslash\{g\}} \lambda^g_{g'} \sum_{i\in\mathcal{I}} v^i_{g'}(y^i_{g'})$$

for all $y$. Summing over $g$,

$$\sum_{g\in\mathcal{G}} \sum_{i\in\mathcal{I}} v^i_g(x^i_g) + \sum_{g\in\mathcal{G}} \sum_{g'\in\mathcal{G}\backslash\{g\}} \lambda^g_{g'} \sum_{i\in\mathcal{I}} v^i_{g'}(x^i_{g'}) \geq \sum_{g\in\mathcal{G}} \sum_{i\in\mathcal{I}} v^i_g(y^i_g) + \sum_{g\in\mathcal{G}} \sum_{g'\in\mathcal{G}\backslash\{g\}} \lambda^g_{g'} \sum_{i\in\mathcal{I}} v^i_{g'}(y^i_{g'})$$

for all $y$. For each $g \in \mathcal{G}$ and $i \in \mathcal{I}$, set $\widehat{v}^{\,i}_{g} = v^i_g + \sum_{g'\in\mathcal{G}\backslash\{g\}} \lambda^{g'}_g v^i_g = \left(1 + \sum_{g'\in\mathcal{G}\backslash\{g\}} \lambda^{g'}_g\right) v^i_g$. Since $1 + \sum_{g'\in\mathcal{G}\backslash\{g\}} \lambda^{g'}_g$ does not depend on $i$, $\left(\widehat{v}^{\,1}_{g}, ..., \widehat{v}^{\,I}_{g}\right) \in \mathcal{W}_g$. ■

# Appendix B: Remaining proofs

**Proof of Proposition 1.** Fix some $\mathcal{W}$ which need not be group agnostic, let $x$ be group-unanimously superior to $y$, and set some $(v_g)_{g\in\mathcal{G}} \in \mathcal{W}$. Then $\sum_{i\in\mathcal{I}} v^i_g(x^i) \geq \sum_{i\in\mathcal{I}} v^i_g(y^i)$ for all $g$ and with strict inequality for some $g$. Hence $\sum_{i\in\mathcal{I}} \sum_{g\in\mathcal{G}} v^i_g(x^i) > \sum_{i\in\mathcal{I}} \sum_{g\in\mathcal{G}} v^i_g(y^i)$: $x$ is utilitarian superior to $y$. Any utilitarian optimum $x$ is therefore a group-unanimity optimum. If not there would be a feasible $y$ that is group-unanimously superior to $x$ and hence utilitarian superior to $x$.

Conversely, let $x$ be utilitarian superior to $y$ and now assume in addition that $\mathcal{W}$ is group agnostic. Fix $v \in \mathcal{W}$. Suppose for some $g'$ that $\sum_{i\in\mathcal{I}} v^i_{g'}(x^i) < \sum_{i\in\mathcal{I}} v^i_{g'}(y^i)$. Then, letting $\widehat{v} \in \mathcal{W}$ equal $v$ except that $\widehat{v}^i_{g'} = \lambda v^i_{g'}$ for each $i$, we would have $\sum_{i\in\mathcal{I}} \sum_{g\in\mathcal{G}} \widehat{v}^i_g(x^i) < \sum_{i\in\mathcal{I}} \sum_{g\in\mathcal{G}} \widehat{v}^i_g(y^i)$ for $\lambda$ sufficiently large. Hence $\sum_{i\in\mathcal{I}} v^i_g(x^i) \geq \sum_{i\in\mathcal{I}} v^i_g(y^i)$ for each $g$. We also cannot have $\sum_{i\in\mathcal{I}} v_g^i(x^i) = \sum_{i\in\mathcal{I}} v_g^i(y^i)$ for all $g$ since then, by the assumption that each $\mathcal{W}_g$ is a cardinal selection, $\sum_{i\in\mathcal{I}}\sum_{g\in\mathcal{G}} \widetilde{v}_g^i(x^i) = \sum_{i\in\mathcal{I}}\sum_{g\in\mathcal{G}} \widetilde{v}_g^i(y^i)$ for all $(\widetilde{v}_g^i)_{(g,i)\in\mathcal{G}\times\mathcal{I}} \in \mathcal{W}$. Hence $x$ is group-unanimously superior to $y$. ■

**Proof of Theorem 3.** Label groups so that $\mathcal{B} = \{1, ..., d_j\}$ is an isolated basket. Given $\left(v_g^1, ..., v_g^I\right) \in \mathcal{W}_g$, let the functions $\left(\sum_{i\in\mathcal{I}} v_g^i(\cdot)\right)_{g=1}^{d_j}$ be defined on the goods in groups $1, ..., d_j$ (rather than on all $L$ goods). Due to continuity, we may choose $\left(\sum_{i\in\mathcal{I}} v_g^i(\cdot)\right)_{g=1}^{d_j}$ so that $v_g^i(x^i) > 0$ for all $i$, $g$, and $x^i$ such that $0 \leq x_g^i \leq e_g$ for $g \in \mathcal{B}$. It is sufficient to show that the utilitarian optimal welfare vectors for each isolated basket taken as an entire model are contained by a $C^0$-manifold of dimension $d_j - 1$: the utilitarian-optimal welfare vectors in a model formed by $n$ such isolated baskets will then be contained by a set of dimension $\sum_{j=1}^{n}(d_j - 1)$. Accordingly, let $x^i$ now denote $\left(x_g^i\right)_{g\in\mathcal{B}}$, and $F$ now denote $\{x \in \mathbb{R}_+^{Id_j} : \sum_{i\in\mathcal{I}} x_g^i \leq e_g$ for $g \in \mathcal{B}\}$. For positive integers $l$, let $\Delta_l$ denote the $l$-dimensional unit simplex $\left\{z \in \mathbb{R}_+^{l+1} : \sum_{k=1}^{l+1} z_k = 1\right\}$. Define the set of feasible weakly optimal welfare vectors

$$W_{\text{weak}} = \left\{w \in W : \text{ there does not exist } x \in F \text{ such that } \sum\nolimits_{i\in\mathcal{I}} v_g^i(x^i) > w_g \text{ for } g = 1, ..., d_j\right\}$$

and the convex hull of $W_{\text{weak}}$, $\operatorname{co} W_{\text{weak}}$. Due to continuity of the $v_g^i$ and the compactness of $F$, $W_{\text{weak}}$ and hence co $W_{\text{weak}}$ are compact. Define $\Delta_{\text{weak}} \subset \Delta_{d_j-1}$ by $\delta \in \Delta_{\text{weak}}$ if and only if there exist $a \geq 0$ and $w \in \operatorname{co} W_{\text{weak}}$ such that $a\delta = w$. Define $h : \Delta_{\text{weak}} \to \operatorname{co} W_{\text{weak}}$ by $h(\delta) = \max\{a\delta : a\delta \in \operatorname{co} W_{\text{weak}}\}$. The compactness of co $W_{\text{weak}}$ ensures that the max is well-defined. Since, for $\delta, \delta' \in \Delta_{d_j-1}$ with $\delta \neq \delta'$, $\left\{a\delta \in \mathbb{R}_+^{d_j}\backslash\{0\} : a \in \mathbb{R}\right\} \cap \left\{a\delta' \in \mathbb{R}_+^{d_j}\backslash\{0\} : a \in \mathbb{R}\right\} = \varnothing$, $h$ is one-to-one.

To see that $h$ is continuous and hence, since $\Delta_{\text{weak}}$ is compact, that $h$ is a homeomorphism between $\Delta_{\text{weak}}$ and $h(\Delta_{\text{weak}})$, suppose $\overline{\delta} \in \Delta_{\text{weak}}$ and let $\langle\delta_n\rangle$ be a sequence in $\Delta_{\text{weak}}$ such that $\delta_n \to \overline{\delta}$. For each $\delta_n$ there exist, by Carathéodory, $w^{\delta_n} \in \prod_{l=1}^{d_j+1} W_{\text{weak}}$ and $\lambda^{\delta_n} \in \Delta_{d_j+1}$ such that $h(\delta_n) = \sum_{l=1}^{d_j+1} w^{\delta_n}(l)\lambda^{\delta_n}(l)$. Let $x_{\delta_n} \in \prod_{l=1}^{d_j+1} F$ satisfy $\sum_{i\in\mathcal{I}} v_g^i(x_{\delta_n}^i(l)) = w_g^{\delta_n}(l)$ for $g \in \mathcal{B}$ and $l = 1, ..., d_j+1$. Since $W_{\text{weak}} \times \Delta_{d_j+1} \times \prod_{l=1}^{d_j+1} F$ is compact, there is a subsequence of $\left\langle w^{\delta_n}, \lambda^{\delta_n}, x_{\delta_n}\right\rangle$ that converges to $\left(\overline{w}, \overline{\lambda}, \overline{x}\right) \in W_{\text{weak}} \times \Delta_{d_j+1} \times \prod_{l=1}^{d_j+1} F$. Restricting ourselves to the subsequence, the continuity of the $v_g^i$ implies $v_g^i(x_{\delta_n}^i(l)) \to v_g^i(\overline{x}^i(l))$ for each $i$ and $l$ and therefore $\lim h(\delta_n) \in \operatorname{co} W_{\text{weak}}$. To exclude the possibility that $\lim h(\delta_n) < h(\overline{\delta})$, set $\delta_n \to \overline{\delta}$ so that there is a $\overline{\overline{\delta}}$ and a $\alpha_n \in [0, 1]$ for each $n$ such that $\delta_n = \alpha_n\overline{\delta} + (1-\alpha_n)\overline{\overline{\delta}}$ for each $n$. So then $\alpha_n \to 1$. For any $n$, the convexity of co $W_{\text{weak}}$ means there is a $a\delta_n \in$ co $W_{\text{weak}}$ with $a \geq 0$ such that $a\delta_n = \alpha_n h(\overline{\delta}) + (1 - \alpha_n)h(\overline{\overline{\delta}})$. Hence $h(\delta_n) \geq \alpha_n h(\overline{\delta}) + (1 - \alpha_n)h(\overline{\overline{\delta}})$ and therefore $\lim h(\delta_n) \geq h(\overline{\delta})$.

The continuity of the $v_g^i$ and compactness of $F$ imply that $\Delta_{\text{weak}}$ is closed and the convexity of $\operatorname{co} W_{\text{weak}}$ imply that $\Delta_{\text{weak}}$ is convex: given that $\Delta_{d_j-1}$ has dimension $d_j - 1$, $\Delta_{\text{weak}}$ is a $C^0$-manifold with boundary of dimension no greater than $d_j - 1$. Given that $h$ is a homeomorphism between $\Delta_{\text{weak}}$ and $h(\Delta_{\text{weak}})$, $h(\Delta_{\text{weak}})$ is also a $C^0$-manifold with boundary of dimension no greater than $d_j - 1$. Since $W_O \subset W_{\text{weak}}$, $W_O$ is a subset of a $(d_j - 1)$-manifold with boundary. ■

**Proof of Proposition 2.** Suppose $\mathcal{V}^i$ for $i \in \mathcal{I}$ are cardinal selections such that $g'$ is nontrivial and let $\mathcal{W}$ and $\mathcal{W}'$ be preference-compatible cardinal welfare selections such that $\mathcal{W}_{g'} = \mathcal{W}'_{g'}$. Let $v_g = \left(v_g^1, ..., v_g^I\right) \in \mathcal{W}_g$ and $v'_g = \left(v_g^{1\prime}, ..., v_g^{I\prime}\right) \in \mathcal{W}'_g$ for $g \in \mathcal{G}$ denote the corresponding profiles given by preference compatibility. Since $v_{g'}$ and $v'_{g'}$ are both elements of $\mathcal{W}_{g'}$, there exist $a \in \mathbb{R}_{++}$ and $b \in \mathbb{R}^I$ such that $v_{g'} = av'_{g'} + b$. Defining $\widehat{v}^{\,i} = a\left(v_1^{i\prime}, ..., v_L^{i\prime}\right) + (b^i, ..., b^i)$, we have $\widehat{v}^{\,i} \in \mathcal{V}^i$ for $i \in \mathcal{I}$ (since $(v_1^{i\prime}, ..., v_L^{i\prime}) \in \mathcal{V}^i$ by preference compatibility) and $\widehat{v}_g \equiv (\widehat{v}^{\,1}_g, ..., \widehat{v}^{\,I}_g) \in \mathcal{W}'_g$ for $g \in \mathcal{G}$.

Fix some $i \in \mathcal{I}$. Since $v^i \in \mathcal{V}^i$ and $\widehat{v}^{\,i} \in \mathcal{V}^i$ there exist $\alpha^i \in \mathbb{R}_{++}$ and $\beta^i \in \mathbb{R}^L$ such that $\widehat{v}^{\,i} = \alpha^i v^i + \beta^i$. Let $x, y \in \mathbb{R}_+^L$ satisfy $v_{g'}^i(x) \neq v_{g'}^i(y)$. Since $\widehat{v}^{\,i}_{g'} = v^i_{g'}$, we have $v_{g'}^i(x) = \alpha^i v_{g'}^i(x) + \beta_{g'}^i$ and $v_{g'}^i(y) = \alpha^i v_{g'}^i(y) + \beta_{g'}^i$ which implies $(1-\alpha^i)(v_{g'}^i(x) - v_{g'}^i(y)) = 0$ and hence $\alpha^i = 1$. As this argument applies to each $i \in \mathcal{I}$, $\widehat{v}_g = v_g + (\beta_g^1, ..., \beta_g^I)$ for each $g \in \mathcal{G}$. Given that $v_g \in \mathcal{W}_g$ for each $g \in \mathcal{G}$, we have $\widehat{v}_g \in \mathcal{W}_g$ for each $g \in \mathcal{G}$. Since for any $g \in \mathcal{G}$ and $v''_g \in \mathcal{W}'_g$, $\widehat{v}_g \in \mathcal{W}'_g$ implies there exist $a'' \in \mathbb{R}_{++}$ and $b'' \in \mathbb{R}^I$ such that $\widehat{v}_g = a'' v''_g + b''$, we conclude that $v''_g \in \mathcal{W}_g$. ■

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