Information Aggregation in Competitive Markets*

Lucas Siga† Maximilian Mihm‡

December 9, 2018

Abstract

We study when equilibrium prices can aggregate information in a market with a large population of privately informed buyers and sellers. Our main result identifies a property of information—the *betweenness property*—that is both necessary and sufficient for aggregation. The characterization provides predictions about equilibrium prices in complex, multidimensional environments.

1 Introduction

When do prices aggregate information? This question is central to understanding a market economy where information about unknown fundamentals is dispersed over a large number of market participants, and prices are the primary channel by which information is aggregated and transmitted in the economy.

In this paper, we study information aggregation in a competitive market with common-value assets, and a large (non-atomic) population of privately informed buyers and sellers. Trade occurs through an auction mechanism that closely resembles the call market used to set daily opening prices on the New York Stock Exchange. After observing signals, traders submit sealed bids and an auctioneer determines the market-clearing price. With their bids, traders determine their chances of trading,

*We thank Nageeb Ali, Vladimir Asriyan, Ayelen Banegas, Paulo Barelli, Pablo Beker, Larry Blume, Aaron Bodoh-Creed, Vince Crawford, Mehmet Ekmekci, Vijay Krishna, Mark Machina, Larry Samuelson, Tom Sargent, Jeroen Swinkels, Juuso Välimäki, and Joel Watson for helpful suggestions. Special thanks to Joel Sobel for his invaluable guidance at the various stages of this project.

†Division of Social Science, New York University Abu Dhabi. Email: lucas.siga@nyu.edu.

‡Division of Social Science, New York University Abu Dhabi. Email: max.mihm@nyu.edu.

but the the large population implies that individuals have negligible impact on prices and total trading volume. Accordingly, our model formalizes the key price-taking assumption of competitive equilibrium models, but with an explicit price formation process based on strategic auction models.[1]

Our main result provides a characterization of the information environments where there exists equilibrium prices that aggregate information in this market. On one hand, our result shows that equilibrium prices can aggregate information even in complex information environments where the previous auction literature makes no predictions about the information efficiency of prices. On the other hand, our result establishes limitations of market trading mechanisms by identifying when Bayes-Nash equilibrium prices cannot implement a fully-revealing rational expectations equilibrium (REE).

To fix ideas our approach, we start by considering two simple examples.

Example 1. Consider a market for an asset X, which depends on two independent inputs A and B. For instance, the value of asset X could reflect the real returns from an investment in two different sectors, or the yields of a commodity in two different locations. The value of the asset is the sum of the two inputs. Traders are ex-ante identical, but receive specialized information (e.g, by industry or region). With equal probability, a trader receives a signal that is perfectly informative about one input but conveys no information about the other input. In a market with a large population of traders, public signals reveal the value of the asset because half of the population is perfectly informed about input A and the other half is perfectly informed about input B. The question is whether, in a market with private information, prices can aggregate the information dispersed over traders.

This market has a fully-revealing REE. But when traders condition directly on fully-revealing prices, they can ignore their private signals. It is therefore unclear where prices originate, or how they incorporate information (Hellwig, 1980; Milgrom, 1981). The auction literature addresses this problem by providing a complete description of the trading mechanism. However, this literature relies on strong information assumptions. In particular, in order to establish an equilibrium in monotone bidding strategies signals must satisfy the monotone likelihood-ratio property (MLRP), and

[1] We therefore follow Aumann (1964, p.39), who argues that "*a mathematical model appropriate to the intuitive notion of perfect competition must contain infinitely many participants*" and Milgrom (1981, p.923), who argues that "*to address seriously such questions as how do prices come to reflect information...one needs a theory of how prices are formed.*"

nothing is known about whether auction prices can aggregate information when the MLRP is not satisfied. In the market for asset X signals do not satisfy the MLRP and so the previous auction literature provides no prediction about the information conveyed by equilibrium prices.

An auction with a large population of traders provides an alternative approach to the aggregation problem. For instance, it is straightforward to show that equilibrium prices *can* aggregate information in the market for asset X. To illustrate, suppose there are four possible states $\{(1,1),(1,2),(2,1),(2,2)\}$, corresponding to the realization of the two inputs, and so there are three possible values $\{2,3,4\}$ for the asset. There are four possible signals, $\{L_A, H_A, L_B, H_B\}$, where L_c indicates that input $c \in \{A,B\}$ has low realization 1, and H_c indicates that input c has the high realization 2. Half of the traders are sellers endowed with one unit of the asset, and the other half are buyers with unit demand. Now consider the following strategy. With a low signal, a trader submits a bid of 2 with probability $\frac{2}{3}$ and 3 with probability $\frac{1}{3}$; with a high signal, the trader submits a bid of 3 with probability $\frac{1}{3}$ and 4 with probability $\frac{2}{3}$. When all traders follow this strategy, we can appeal (informally for now) to the law of large numbers to describe aggregate demand and supply. For each state, the aggregate demand $D(p)$ represents the mass of buyers who submit a bid of p or above, and the aggregate supply $S(p)$ represents the mass of sellers who submit an ask of p or below. When the value is 2, all traders receive a low signal; two-thirds then submit a bid of 2 and one-third submit a bid of 3 (Figure 1a). When the value is 3, half of the traders receive a high signal and the other half receive a low signal; one-third then submit a bid of 2, one-third submit a bid of 3, and one-third submit a bit of 4 (Figures 1b). When the value is 4, all traders receive high signals; one-third then submit a bid of 3, and two-thirds submit a bid of 4 (Figure 1c). As Figure 1 illustrates, the market-clearing price equals the value of asset X in every state. Moreover, since individual traders cannot impact prices, there are no profitable deviations and the strategy is an equilibrium. □

Example 2. Are there also markets where prices *cannot* aggregate information? Consider the market for an alternative asset Y that has value 4 when both inputs are equal and value 2 otherwise. The information signals convey about states is the same as for asset X but the payoff structure is different: inputs are substitutes for asset X and complementary for asset Y.

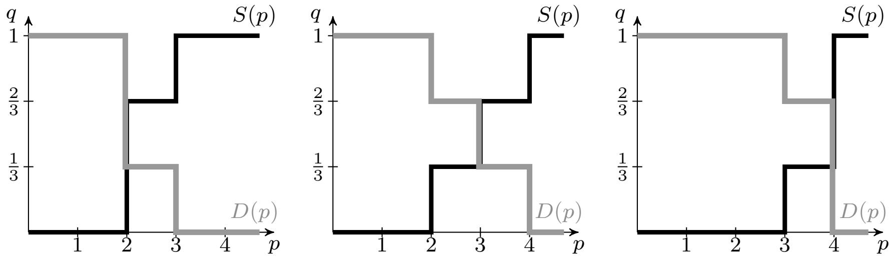


Figure 1: Aggregate demand and supply.

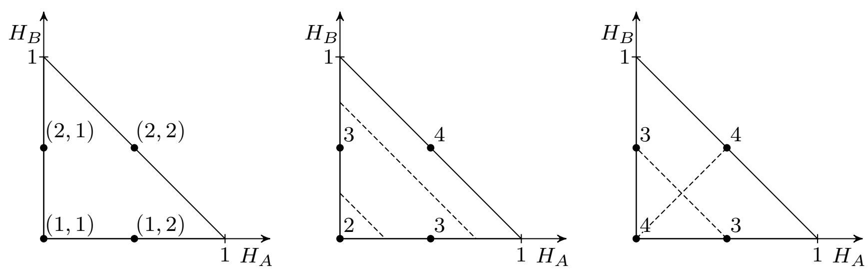
Bayes-Nash equilibrium prices *cannot* aggregate information in the market for asset Y. To illustrate, consider any strategy-profile where aggregate supply and demand cross at $p = 4$ when all traders receive a low signal, and also when all traders receive a high signal. Suppose that, on aggregate, traders submit higher bids when they receive a high signal for input A than when they receive a low signal for input A.[2] In order for the price to equal 4 in both states where the value is 4, it must be the case that (on aggregate) traders submit lower bids when they receive a high signal for input B than when they receive a low signal for input B. Now consider the state where the value of the asset is 2 and traders either receive a high signal on input A or a low signal on input B (i.e., in the state $(2, 1)$). Since aggregate bids are highest in this state, the price cannot be less than 4. As a result, there is no strategy where the market-clearing price is equal to the value in every state. There are strategies where the market-clearing price is different in every state, but these strategies present traders with arbitrage opportunities. If traders predict a price that is strictly less than the value in some state, buyers have an incentive to increase their bids locally to increase their chances of trading, and sellers have an incentive to increase their asks locally to decrease their chances of trading. Likewise, if the price is strictly greater than the value, buyers have an incentive to decrease bids and sellers have an incentive to decrease asks. Competitive forces therefore apply upward pressure on prices in states where the asset is undervalued, and downward pressure on prices in states where the asset is overvalued. As equilibrium prices cannot equal values, the only escape is that equilibrium prices do not aggregate information. □

[2] A symmetric argument applies when, on aggregate, traders submit higher bids when they receive a low signal for input A than when they receive a low signal for input A.

The example of asset X shows that, in a competitive market, the MLRP is not necessary for information aggregation. In fact, information aggregation is possible even in complex information environments where signals have no meaningful order properties. On the other hand, the example of asset Y shows that some conditions must be satisfied, otherwise a fully-revealing REE can not be implemented as an equilibrium of our market. In environments with finite states and signals, our main result shows that a property of information that we call the *betweenness property* is both necessary and sufficient for information aggregation.

The betweenness property is a condition on information primitives. In our environment, nature chooses a state which determines (i) the common-value of a unit of asset, and (ii) the conditional distribution over signals. A *betweenness order* is a ranking on the simplex of conditional distributions with the defining characteristic that level curves are linear.[3] The betweenness property is satisfied if there is a betweenness order such that higher value states generate higher ranked conditional distributions.

To illustrate, consider the conditional probability that a trader receives one of the high signals in Examples 1 and 2. In state $(1,1)$, the probability of receiving either signal H_A or H_B is 0; in state $(2,1)$, the probability for H_A is $\frac{1}{2}$, and the probability for H_B is 0; in state $(1,2)$, the probability for H_A is 0, and the probability for H_B is $\frac{1}{2}$; and in state $(2,2)$, the probability for either high signal is $\frac{1}{2}$. Figure 2a illustrates this information structure.



(a) Information structure (b) Asset X (c) Asset Y

Figure 2: The betweenness property in Examples 1 and 2.

In Figure 2b, we replace states with the values of asset X. The dashed lines indicate level curves of a betweenness order that is monotone in values. As the figure

[3]As such, betweenness orders are a generalization of expected utility where level curves are linear and parallel. See Section 3.1.

illustrates, the betweenness property is satisfied, and this is why equilibrium prices can aggregate information. In Figure 2c, we replace states with the values of asset Y. The dashed lines indicate that the convex hull of high value states intersects the convex hull of low value states. In that case, there is no betweenness order that is monotone in values, and equilibrium prices cannot aggregate information.

The intuition for our characterization result comes from three important insights about large markets. First, if prices aggregate information they must equal values; otherwise there are arbitrage opportunities (as in market Y). Second, the law of large numbers provides a powerful representation of aggregate bidding behavior (as in the market X). In particular, cumulative bid distributions for both buyers and sellers are separable in a component that depends only on strategic behavior and a separate component that depends only on information primitives. Finally, the strategic component of a bid distribution has a dual representation as a betweenness order, and vice versa. For prices to equal values, the betweenness order must be monotone in values, which is exactly what the betweenness property requires.

The betweenness property is much weaker than the MLRP. In particular, while the MLRP is a restrictive condition in environments with a large number of states, we show that the betweenness property is generic as long as the number of signals is as large as the number of states. This illustrates the power of the market in environments where signals are more numerous than states. On the other hand, in environments with more states than signals, the betweenness property is also restrictive. While a fully-revealing REE always exists in these markets, it generally cannot be implemented in a Bayes-Nash equilibrium.This highlights limitations of the market when prices must distinguish between many values with limited signals.

Our results are especially relevant in multidimensional environments where signals generally do not satisfy strong order properties such as the MLRP. By focussing on properties of the *distribution* over signals, rather than the signals themselves, our results do not restrict the dimensionality of the states or signals. As an application, we consider a class of environments where states have multiple inputs and signals are specific to inputs (as in the markets for assets X and Y). A signal then conveys information for only one dimension of the asset's value, and traders must rely on prices to aggregate the fragmented information diffused in the marketplace. We show that the MLRP is never satisfied in such environments. On the other hand, when the value is separable in inputs (as it is for asset X but not Y), the betweenness property is generic whenever there are at least as many signals as states for each input.

The paper is organized as follows. Section 2 discusses related literature. Section 3 defines the betweenness property and describes the market. Section 4 presents our main result and a detailed proof sketch. We also show how the equilibrium in a large market can be interpreted as the limit of approximate equilibria in finite markets, and how our result can be adapted to a market with divisible assets. Section 5 presents our genericity results and our multi-input example. Section 7 concludes. Formal proofs are given in an appendix.

2 Related literature

Our work primarily contributes to a literature that uses common-value auctions to study the information revealed by prices in competitive markets, and thereby provide microfoundations for REE.[4]

In a seminal contribution, Wilson (1977) shows how equilibrium prices in a single-unit auction can converge in probability to the value as the population of bidders grows. Milgrom (1979) provides the first characterization of environments that permit aggregation and Milgrom (1981) extends the analysis to general Vickrey auctions. To overcome the winner's curse—which intensifies when assets become increasingly scarce—aggregation requires that the information of the winning bidder's signal is arbitrarily precise. This imposes a strong restriction on information. Pesendorfer and Swinkels (1997) therefore consider auctions where both the number of traders n and the number of assets g increases, which is a natural assumption for a competitive market. When traders receive conditional *i.i.d.* signals that satisfy the MLRP, they show that the classic strategy-profile in Milgrom and Weber (1982)—where traders submit bids equal to the expected value conditional on being pivotal—is the unique symmetric equilibrium. Moreover, the equilibrium price converges in probability to the value if and only if $g \to \infty$ and $(n-g) \to \infty$. The double-largeness condition is necessary and sufficient for a loser's curse to offsets the winner's curse. Kremer (2002) simplifies and extends the analysis to characterize the asymptotic distribution of

[4] A parallel literature has studied information aggregation in common-value elections (Condorcet, 1785; Austen-Smith and Banks, 1996; Feddersen and Pesendorfer, 1997). The closest work in this literature to ours is Barelli, Bhattacharaya, and Siga (2018), who analyze a multi-candidate election with private information and, employing a similar geometric approach to ours, show when a voting strategy can aggregate information.

prices for various auction formats in a unified framework. To address limitations of a market with exogenous supply, Reny and Perry (2006) consider a double-sided auction. In an environment with affiliated common and private-values (which implies the MLRP), they show that when the population is sufficiently large there is a monotone equilibrium where prices are arbitrarily close to a fully-revealing REE.

This prior literature highlights two distinct questions about REE. (1) *Market power*: In a finite market, each trader has some market power. If traders internalize this market power, then they may strategically adjust bids so as not to reveal private information, thereby distorting the information conveyed by equilibrium prices. Do these distortions vanish as the market grows? (2) *Price formation*: Competitive equilibrium models do not provide an explicit description of the trading-mechanism, and therefore do not show how individual actions and information translate into prices. Is there a fully specified price formation process where traders condition only on their private signals and yet equilibrium prices are fully-revealing?

By focusing on a large market, our sufficiency result sidelines the question of market power in order to focus on the question of price formation.[5] The large population implies that competition in our market manifests in the arbitrage behavior of traders who can only impact their chances of buying and selling. This reflects the important economic idea that, in a large anonymous market, traders believe they cannot impact prices, and the competition for resources—rather than market power—drives individual and aggregate behavior. In such a market, we show that Bayes-Nash equilibrium prices can aggregate information even in complex, multidimensional environments where signals have no meaningful total order (such as the MLRP).[6]

On the other hand, our necessity result is relevant for both questions of market power and price formation. In particular, as we show in Section 4.2, the restrictions we identify on the market trading mechanism apply also to approximate (and therefore exact) equilibria in finite markets. Regardless whether or not market power distorts how individual traders reveal information in a finite market, the trading mechanism simply cannot aggregate information when the betweenness property is not satisfied.

[5] Our sufficiency result does not address the question of market power directly because we are unable to show whether the equilibria we construct in a large market can be approximated by a sequence of *exact* equilibria in finite auctions. In Section 4.2 we do show how the equilibria we construct can be interpreted as the limit of a sequence of *approximate* equilibria in finite auctions.

[6] Serrano-Padial (2012) and Bodoh-Creed (2013) also study auctions with an infinite population of traders but focus exclusively on environments where signals satisfy the MLRP.

In particular we are able to identify the information environments where a REE exists, but cannot be implemented as a Bayes-Nash equilibrium of an auction trading-mechanism. The failure of information aggregation is very strong in the sense that, when the betweenness property is not satisfied, there is no arbitrage-fee and invertible mapping from information into prices, and so the market mechanism necessarily loses information.[7]

There are also alternative approaches to provide microfoundations for REE. A literature following Kyle (1985) studies markets with strategic traders who receive private information, non-strategic noise traders who supply liquidity and prevent the market from collapsing, and a market maker who determines the price. Trading is dynamic and information revelation occurs over time. The information aggregation process is therefore quite different from the auction approach because there is feedback from prices. There are also significant differences in the trading mechanism. In Kyle models, all orders are executed; in an auction, bids are conditional orders that are executed only when the price is either above (for sellers) or below (for buyers) a threshold. To solve for an equilibrium in Kyle models, strong information assumptions are needed. The standard assumption is that random variables are jointly normal, which implies the MRLP, and that signals are *i.i.d* conditional on the value.

In an important recent contribution, Lambert, Ostrovsky, and Panov (2018) consider a single-period version of the Kyle model, maintaining joint-normality but relaxing the *i.i.d.* conditions. Their model admits a unique linear equilibrium. In this equilibrium, prices aggregate information asymptotically if and only if noise trade is positively correlated with the value. There are significant differences with our work: (i) our trading mechanism is very different, (ii) our model does not have noise traders, (iii) our large population implies that individual traders have no price impact, and (iv) our environment has finite states and signals, but we impose no distributional assumption on the the joint-probability over states and signals.

There is also a literature that studies strategic foundations for REE in markets where traders submit monotone supply and demand schedules (Kyle, 1989; Vives,

[7]In this regard, we also add to a literature on failures of information aggregation in markets. For instance, costly information acquisition (Jackson, 2003), uncertainty about the number of bidders (Harstad, Pekeč, and Tsetlin, 2008), costly bidder solicitation (Lauermann and Wolinsky, 2017), state-dependent actions (Atakan and Ekmekci, 2014), or decentralized bilateral trading (Wolinsky, 1990), have all been shown to impede information aggregation even in environments where the MLRP is satisfied. Our aggregation result is strong: the market mechanism loses necessary information for aggregation.

2011, 2014).[8] Perhaps the closest paper in this literature to ours is Palfrey (1985), who studies Cournot competition as the population of firms grows. He also considers an environment with finite states and signals, but fixes an exogenous demand for assets. He does not provide a complete characterization of the environments where information aggregates, but shows that a necessary condition (which is also almost sufficient) is that the matrix of conditional distributions has full-rank. In a market where traders do not have price impact, we show that this condition is sufficient for information aggregation because it implies a linear property, which implies the betweenness property. However, the full-rank condition is not necessary for aggregation because (i) the full-rank condition is sufficient but not necessary for the linear property, and (ii) the linear property is sufficient but not necessary for the betweenness property.

3 Model

We study a double-sided auction with a large population of traders. The common value of an asset depends on an unknown state, and traders receive private signals that are *i.i.d.* conditional on the state. In this market, we are interested in the information that equilibrium prices convey about values.

3.1 The environment

The environment has a finite set of states $\Omega=\{\omega_1, ..., \omega_M\}$ and signals $S=\{s_1, ..., s_K\}$, with a probability distribution P on $\Omega \times S$. In state ω, an asset has value $v(\omega)$ and the conditional distribution over signals is P_ω. To simplify exposition, we assume that P has full support and states with different values generate different conditional distributions over signals (i.e., $v(\omega) \neq v(\omega')$ implies $P_\omega \neq P_{\omega'}$). The key primitives are the value function $v : \Omega \to \mathbb{R}_{++}$ and information structure $\{P_\omega : \omega \in \Omega\}$.

The previous auction literature generally imposes an order on signals that is strongly correlated with values, and uses this order to obtain an equilibrium in

[8] In particular, Vives (2014) also considers a market with an infinite population of traders. To address the well-known Grossman-Stigliz critique, he shows that a fully revealing REE can be implemented as a Bayes-Nash equilibrium when traders acquire costly information about both a private and common value component of the asset. In his model, random variables are jointly normal. As a result, signals satisfy the MLRP, and it is possible to construct a linear, monotone equilibrium. In contrast, our objective is to understand the information conveyed by equilibrium prices in environments where signals do not necessarily satisfy strong order properties.

monotone bidding strategies. We depart from this approach by imposing no order on the signals. However, as we show in the introduction, some property of information is necessary for aggregation: values must be related in some way to the information structure, so that competitive forces can guide aggregate behavior and ensure that equilibrium prices aggregate information. Below, we define the required property.

We denote by $\succeq$ a continuous weak order on the set of distributions over signals $\Delta(S)$, with the asymmetric part $\succ$ and the symmetric part $\sim$.[9] The following definition recalls two prominent classes of continuous weak orders.

Definition 1. The continuous weak order $\succeq$ is (i) a *linear order* if, for all $\theta \in (0,1)$ and $\ell, \ell', \ell'' \in \Delta(S)$, $\ell \succeq \ell'$ implies $\theta\ell + (1-\theta)\ell'' \succeq \theta\ell' + (1-\theta)\ell''$; (ii) a *betweenness order* if $\ell \succ \ell'$ implies $\ell \succ \theta\ell + (1-\theta)\ell' \succ \ell'$, and $\ell \sim \ell'$ implies $\ell \sim \theta\ell + (1-\theta)\ell' \sim \ell'$.

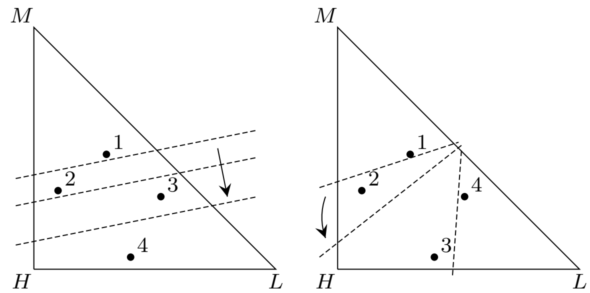
The defining characteristic of a linear order is that level curves can be represented by parallel hyperplanes. Betweenness orders are a generalization where level curves are also represented by hyperplanes but not necessarily by parallel ones (Figure 3).[10] The following monotonicity properties formalize the intuitive idea that better states generate better conditional distributions.

Definition 2. An environment satisfies the *betweenness (resp., linear) property* if there is a betweenness (resp., linear) order $\succeq$ such that $v(\omega) > v(\omega')$ implies $P_\omega \succ P_{\omega'}$.

The betweenness property is central for our information aggregation result; the linear property is useful as a reference and also plays an important role in our genericity analysis. As betweenness orders are more general, the linear property implies the betweenness property and not vice versa (Figures 3 and 4). A betweenness order is characterized by an infinite collection of level sets, which cover the simplex. Since we focus on environments with finite states and signals, it is sufficient for us to consider a finite number of these level sets. Crucial for the betweenness property is that (i) the level sets are linear, (ii) the upper contour sets are nested in the unit simplex, and (ii) the order over states is co-monotone with the order over conditional distributions.

[9] The binary relation $\succeq$ is a *continuous weak order* if it is (i) complete and transitive; (ii) $\ell \succ \ell'$ for some $\ell, \ell' \in \Delta(S)$; and (iii) $\ell \succ \ell' \succ \ell''$ implies $\theta\ell + (1-\theta)\ell'' \sim \ell'$ for some $\theta \in (0,1)$. Such orders are studied in the literature on decision-making under risk, where S is a finite set of prizes, ℓ is a lottery over prizes, and $\succeq$ is a preference relation.

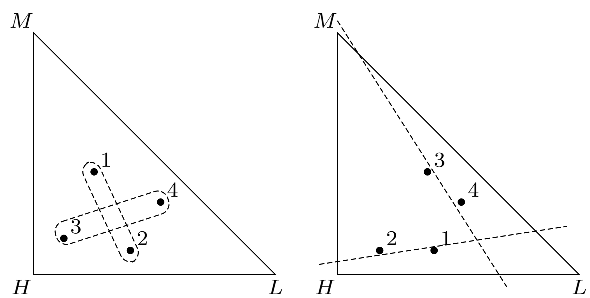
[10] von Neumann and Morgenstern (1944) show that a preference relation over lotteries has an expected utility representation if and only if it is a linear order. Linear orders are therefore central in the theory of decision-making under risk. Betweenness orders are a generalization of expected utility that can accommodate behavioral anomalies such as the Allais paradox (Chew, 1983; Dekel, 1986).



(a) Linear property (b) Betweenness property

Figure 3: Linear and betweenness properties.

A point labeled m represents the conditional distribution over signals in a state with value m. The environment in Figure 3a satisfies the linear property: there is a linear order where better states generate better conditional distributions. The environment in Figure 3b does not satisfy the linear property, but does satisfy the betweenness property.



(a) Non-separation (b) Non-nesting

Figure 4: Failure of the betweenness property.

In Figure 4a, the convex hulls of $\{P_1, P_2\}$ and $\{P_3, P_4\}$ intersect and so a hyperplane cannot separate $\{P_1, P_2\}$ from $\{P_3, P_4\}$. In Figure 4b hyperplanes can separate high from low states, but a hyperplane that separates P_1 from $\{P_2, P_3, P_4\}$ and one that separates P_4 from $\{P_1, P_2, P_3\}$ must intersect inside the simplex.

3.2 The market

There is an infinite set of traders $\mathcal{I}$ endowed with a non-atomic probability distribution.[11] The auction format provides an explicit protocol for the price formation

[11] Our formal model of the large population follows Al-Najjar (2008), where $\mathcal{I}$ is countably infinite and endowed with a finitely-additive probability measure λ on the power-set. This population model overcomes significant challenges with measurability and the law of large numbers in continuum

process, and the large population ensures that individual traders have negligible impact on prices.

Nature chooses a state ω according to the marginal distribution on Ω. Traders do not observe the state, but receive a private signal drawn independently from the conditional distribution P_ω. After receiving their signals, each trader submits a sealed bid from a compact interval $B \equiv [0, \bar{b}]$, which contains $v(\Omega)$. The traders are divided into a set of buyers with mass $\kappa \in (0,1)$ and a set of sellers with mass $1-\kappa$. Each seller owns a unit of an indivisible asset, and each buyer has unit demand. For a buyer, a bid represents the maximum price at which they are willing to trade; for a seller, it represents the minimum price at which they willing to trade.

Given a bid-profile $a : \mathcal{I} \to B$, where $a(i)$ represents the bid for trader i, the auctioneer determines a price and an allocation of assets.[12] The price $p(a)$ is the lowest bid at which the mass of sellers willing to trade exceeds the mass of buyers, and all trade occurs at this price. A buyer trades if her bid is strictly above the price and does not trade if her bid is strictly below the price, and vice versa for sellers. To clear the market, the auctioneer uniformly randomizes over bids equal to the price in order to maximize total trading volume. The payoff for a buyer is $v(\omega) - p(a)$ if she trades and 0 otherwise; for a seller it is $p(a) - v(\omega)$ if she trades and 0 otherwise.[13]

A strategy-profile $\sigma : \mathcal{I} \times S \to \mathcal{B}$ is a mapping from types to Borel probability distributions over bids, where $\sigma(i,s)$ is the (mixed) bidding strategy for trader i when they receive signal s. A strategy-profile σ and conditional distribution P_ω generate a unique probability measure P_ω^σ over bid-profiles in state ω.[14] The expected payoff for type (i,s) is $\Pi_i(\sigma|s) \equiv \sum_\omega \Pi_i(\sigma|\omega) P_s(\omega)$, where $P_s(\omega)$ is the probability of state ω conditional on signal s, $\Pi_i(\sigma|\omega) \equiv \int_{\mathcal{A}} \pi_i(a|\omega) dP_\omega^\sigma$ is the expected payoff conditional on state ω, and $\pi_i(a|\omega)$ is trader i's payoff in state ω for the bid-profile a. A strategy-profile is a Bayes-Nash equilibrium (henceforth, *equilibrium*) if each type maximizes their expected payoff given the strategy of other types.[15]

agent models (see, e.g., Judd 1985). We discuss the population model in detail in Appendix A.2.1. For intuition, there is no loss in suspending problems related to measurability and the law of large numbers, and thinking of the population as a continuum endowed with Lesbegue measure.

[12] The set of bid-profiles $\mathcal{A} = \{a : \mathcal{I} \to B\}$ is endowed with the σ-algebra $\mathbb{A}$ generated by cylinder sets of the form $\{a : a(i) = b\}$ for some $i \in \mathcal{I}$ and $b \in B$.

[13] A more detailed description of the auction format is given in Appendix A.2.2.

[14] Given the formal model of the large population in Appendix A.2.1, a unique countably-additive measure P_ω^σ on $(\mathcal{A}, \mathbb{A})$ is guaranteed by the Hahn-Kolmogorov Extension Theorem.

[15] Our result also holds if equilibrium requires *almost all* types to best-respond.

In principle, a state ω and strategy-profile σ generate a distribution over prices derived from the distribution P_ω^σ over bid-profiles. However, in our market, the Strong Law of Large Numbers (SLLN) implies that the price is almost surely constant.

Proposition 1. *For every strategy-profile σ there exists a unique price-function $p_\sigma : \Omega \to B$ such that, in state ω, the price is equal to $p_\sigma(\omega)$ almost surely.*[16]

4 Main result

We are interested in strategy-profiles where prices convey the same information about values as would be obtained from public signals. By the SLLN, the proportion of traders who receive signal s in state ω is almost surely equal to $P_\omega(s)$. Public signals therefore reveal the value almost surely, and a strategy-profile conveys the same information if there is a one-to-one mapping between values and prices.

Definition 3. Strategy-profile σ *aggregates information* if $v(\omega) \neq v(\omega')$ implies $p_\sigma(\omega) \neq p_\sigma(\omega')$.

It is always possible to construct a strategy-profile that aggregates information. However, we are interested in strategies where traders respond to incentives generated by the competition for assets. While an individual trader has negligible impact on the price and total trading volume, she can affect her allocation through her bids and thereby influence her expected payoff. In an equilibrium, traders will therefore try to exploit arbitrage opportunities based on their predictions about prices and values. Accordingly, the aggregate supply and demand for assets depends on the incentives of the traders, and equilibrium requires that these competitive forces are resolved. Our main result characterizes when *equilibrium* prices convey the same information about values as would obtain if signals were public.

Theorem 1. *There is an equilibrium strategy-profile that aggregates information if and only if the betweenness property is satisfied.*

By connecting the aggregation problem directly with the information primitives, the result distinguishes between two types of environments. When the betweenness

[16]Formally, this means that for every state ω there is a measurable subset $A \subset \mathcal{A}$ such that $P_\omega^\sigma(A) = 1$ and $p(a) = p_\sigma(\omega)$ for all $a \in A$.

property is satisfied, there are equilibrium prices that aggregate all private information in the market. This highlights the potential of the market. Even if individual traders are poorly informed about the value, competitive forces can coordinate individual behavior so that prices are perfectly informative. On the other hand, when the betweenness property is not satisfied, information aggregation necessarily fails. This highlights the limitations of the market. Even if the population as a whole is perfectly informed, the market cannot guide traders' to reveal their collective information.

Remark 1 (Existence and uniqueness)**.** The market always has a no-trade equilibrium where prices are completely uninformative. To illustrate, consider the following strategy-profile: regardless of their signals, all sellers ask for $\bar{b}$ and all buyers bid 0. In that case, the price is equal to 0 in every state. Buyers would like to trade at these prices but there is no supply, and so they cannot increase their chances of trading by submitting a higher bid. Sellers do not want to trade, and so have no incentive to ask for a lower price. We have been unable to characterize the set of equilibria in this market. Such a characterization would be desirable for at least to reasons: (i) to establish whether the betweenness property is sufficient to ensure that prices aggregate information in every equilibrium with strictly positive trade, and (ii) to get a sense of the failures of information aggregation that occur in trade equilibria when the betweenness property is not satisfied. Given the considerable difficulty of constructing equilibria with strictly positive trade when prices do not aggregate information, we leave this as an open question for further research.

Remark 2 (Risk preferences)**.** The assumption that traders are risk neutral simplifies exposition, but the result extends to a market where traders have heterogenous risk preferences. Suppose that each trader $i \in \mathcal{I}$ has a strictly-increasing utility function $u_i : \mathbb{R} \to \mathbb{R}$, where marginal utilities are uniformly bounded away from 0. Given a bid-profile $a : \mathcal{I} \to B$, the payoff for buyer x in state ω is then $\pi_x(a|\omega) = w_x(a|\omega)u_x(v(\omega) - p(a)) + (1 - w_x(a|\omega))\, u_x(0)$, where $w_x(a|\omega)$ is the probability that buyer x will trade in state ω given bid-profile a. Likewise, the payoff for seller y in state ω is $\pi_y(a|\omega) = w_y(a|\omega)u_y(p(a) - v(\omega)) + (1 - w_y(a|\omega))\, u_y(0)$. We can adjust the definition of equilibrium accordingly, and our main result applies as stated. The reason is that, in an equilibrium where the price equals the value, there is in fact no risk for individual traders, and so risk preferences are irrelevant. $\square$

Remark 3 (Asymmetric signals)**.** The sufficiency result is easily adapted to an environment where traders are not ex-ante exchangeable. For example, suppose there is a finite partition $(T_1, ..., T_J)$ of the traders, where each group T_j contains a strictly positive mass of buyers and sellers. Signals are independent conditional on the state, but the information structure is different for each group. Specifically, let each group T_j have a set of signals S_j and denote their information structure by $\{P^j_\omega : \omega \in \Omega\} \subset \Delta(S_j)$. It is straightforward to adjust our arguments to show that, if the environment for each group satisfies the betweenness property, then there is an equilibrium that aggregates information.[17] Moreover, by allowing the asset to have the same value in multiple states, our framework can accommodate environments where signals are not independent conditional on values. To illustrate, consider the market for asset X in the introduction. Conditional on a state, the signals of any two traders i and j are independent. But note that $P\left(s_i{=}H_A, s_j{=}H_B | v(\omega){=}3\right) {=}0 \neq \frac{1}{4}{=}P\left(s_i{=}H_A | v(\omega){=}3\right) P\left(s_j{=}H_B | v(\omega){=}3\right)$, and so signals are not independent conditional on the value, i.e., the dimension of uncertainty that is payoff-relevant for traders. □

4.1 Proof sketch

An important advantage of modeling the trading mechanism explicitly is that it allows us to show where prices originate, and why the betweenness property is necessary and sufficient to aggregate information. Our proof is constructive and consists of three key steps. We provide a sketch of the argument and illustrate the equilibrium construction with an example.

The first step in the argument identifies the restrictions that competition imposes in our environment. If an equilibrium strategy-profile σ aggregates information, then prices must *equal* values almost surely (i.e., $p_\sigma = v$). To see why, consider a strategy-profile σ that aggregates information and suppose there is a state ω such that $p_\sigma(\omega) < v(\omega)$. Since the price is strictly less than the value, it would be good for buyers to trade in state ω, and bad for sellers to trade. In general, there could be another state ω' where the price is strictly higher than the value, and it is bad for buyers to trade and good for sellers. However, because σ aggregates information,

[17] Given our main result, the construction is simple. For each group, $j = 1, ..., J$, one can construct a group-specific strategy-profile so that, in each state ω, supply for group j equals demand for group j exactly when the price is equal to the value $v(\omega)$. Since supply equals demand at the value for each group, a price equal to the value also ensures market-clearing for the whole population.

$p_\sigma(\omega') \neq p_\sigma(\omega)$, and so a buyer who submits a bid equal to $p_\sigma(\omega)$ can decrease their bid marginally below $p_\sigma(\omega)$, thereby guaranteeing that they trade in state ω (where trading is good) without changing the likelihood that they trade in state ω' (where trading is bad). Likewise, a seller who submits a bid equal to $p_\sigma(\omega)$ can increase their bid marginally above $p_\sigma(\omega)$, thereby guaranteeing that they do not trade in state ω (where trading is bad) without changing the likelihood that they trade in state ω' (where trading is good). As buyers and sellers respond to these opposing arbitrage opportunities, competitive forces exert upward pressure on the price in state ω, and downward pressure on the price in state ω'. These competitive pressures are only resolved when prices are equal to values in every state.

The second step in our argument uses the SLLN to characterize aggregate bidding behavior. For a strategy-profile σ let σ_B and σ_S denote, respectively, the restriction to buyers and sellers. We use the SLLN to show that the aggregate bidding behavior of sellers can be characterized by a vector of cumulative distribution functions $F^{\sigma_S} \equiv \left(F^{\sigma_S}_{s_1}, ..., F^{\sigma_S}_{s_K}\right)$, where $F^{\sigma_S}_{s_k}(b)$ represents the normalized share of sellers who submit an ask price less than b when they receive signal s_k. The total mass of sellers who submit an ask price less than b depends on the strategy-profile (chosen by traders) and the distribution over signals (chosen by nature). In particular, the mass of ask prices less than b in state ω is (almost surely) equal to $(1-\kappa)F^{\sigma_S}_\omega(b) \equiv (1-\kappa)F^{\sigma_S}(b) \cdot P_\omega$. Similarly, the mass of buyers who submit a bid strictly greater than b is described by $\kappa(1-F^{\sigma_B}_\omega(b)) \equiv \kappa(1-F^{\sigma_B}(b)) \cdot P_\omega$. Accordingly, aggregate supply and demand first cross in state ω at the lowest price where $\kappa(1-F^{\sigma_B}_\omega(p)) \leq (1-\kappa)F^{\sigma_S}_\omega(p)$; that is, $\kappa \leq \kappa F^{\sigma_B}_\omega(p) + (1-\kappa)F^{\sigma_S}_\omega(p) \equiv F^\sigma_\omega(p)$. Hence, the market-clearing price is given by the κ-quantile of a cumulative distribution functions F^σ_ω that is separable in terms of a component $F^\sigma \equiv \kappa F^{\sigma_B} + (1-\kappa)F^{\sigma_S}$, which depends only on strategic behavior, and another component P_ω, which depends only on information primitives.

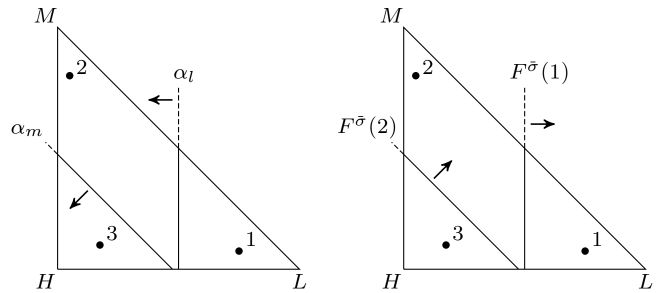
The final step in the argument establishes a duality between bidding strategies and betweenness orders: the quantiles of *any* bidding strategy can be approximated by a betweenness order, and vice versa. This step of the argument is geometric. Let $\sigma_i : S \to \mathcal{B}$ be bidding strategy for trader i, and $F^{\sigma_i} \equiv \left(F^{\sigma_i}_{s_1}, ..., F^{\sigma_i}_{s_K}\right)$ denote the trader's bidding strategy in cumulative form. Given a bid b, we can interpret the vector $F^{\sigma_i}(b)$ as the norm of a hyperplane in $\mathbb{R}^K$. By varying the bid, we obtain a collection of hyperplanes that provides a geometric characterization of the bidding strategy. Moreover, we show that (i) any quantile of the cumulative bidding strategy can be represented as the intersection of these hyperplanes with the unit simplex, and (ii) when we look at the intersection of these hyperplanes with the simplex $\Delta(S)$ they have essentially the same properties as the level curves of a betweenness order. When we apply this duality to the aggregate bidding strategy F^σ obtained in step 1, it follows that a strategy-profile induces a price-function that is monotone in values if and only if it is represented by a betweenness order that is also monotone in values.

These three steps allow us to show the following. If there is an equilibrium strategy-profile that aggregates information, equilibrium prices must equal values (by step 1); the hyperplanes that represent the aggregate bidding strategy are therefore monotone in values (by step 2); and so there is a betweenness order that is also monotone in values (by step 3). This establishes that the betweenness property is necessary for information aggregation. On the other hand, when the betweenness property is satisfied, we can use the level curves of the betweenness order to construct a symmetric strategy profile σ so that $p_\sigma = v$. Clearly, this strategy-profile aggregates information. Moreover, since individual traders have negligible market power, the expected payoff for each trader is zero for any deviation, and so all types are best-responding. As such, σ is also an equilibrium.

To illustrate the equilibrium construction, consider an environment with three states $\boldsymbol{\Omega} = \{\omega_1, \omega_2, \omega_3\}$, three signals $S = \{s_L, s_M, s_H\}$, and a value function where $v(\omega_m) = m$ for each state. In Figure 5a, the vectors α'_l and α'_m are norms of two hyperplanes, $H(\alpha'_l, c'_l)$ and $H(\alpha'_m, c'_m)$, that represent level curves of a betweenness order $\succeq$.[18] Because higher values generate better conditional distributions, the betweenness property is satisfied.

To construct the equilibrium strategy-profile, we first need to manipulate the hyperplanes $H(\alpha'_l, c'_l)$ and $H(\alpha'_m, c'_m)$ in way that does not change their intersection with the unit simplex. By the manipulations, the new hyperplanes $H(\alpha_l, c_l)$ and $H(\alpha_m, c_m)$ still represent the same betweenness order. However, the manipulation ensures that the new constants satisfy $c_l = c_m = \kappa$, and the norms satisfy $\alpha_l, \alpha_m \in [-1, 0]^3$ and $\alpha_l >> \alpha_m$. It is difficult to provide intuition for this step of the construction, and we refer the reader to the formal arguments developed in Lemmas 1 and 2 in Appendix A.1. However, to indicate how we manipulate hyperplanes without changing the

[18] We denote by $H(\alpha, c) \equiv \{z \in \mathbb{R}^K : z \cdot \alpha = c\}$ a hyperplane in $\mathbb{R}^K$, defined by the norm $\alpha \in \mathbb{R}^K$ and constant $c \in \mathbb{R}$, with strict upper and lower half-spaces $\mathring{H}_+(\alpha, c)$ and $\mathring{H}_-(\alpha, c)$, respectively.



(a) Betweenness property (b) Equilibrium strategy

Figure 5: Duality of bidding strategies and betweenness orders.

In Figure 5a, vectors α_l and α_m are norms of hyperplanes that represent level curves of a betweenness order $\succeq$. As higher values generate better conditional distributions, the betweenness property is satisfied. In Figure 5b, the vectors $F^{\bar{\sigma}}(1)$ and $F^{\bar{\sigma}}(2)$ are norms of hyperplanes that represent the aggregate bidding strategy. As higher values generate higher κ-quantiles, the strategy-profile aggregates information.

underlying weak order, it is useful to consider the simpler case of a linear order $\succeq$ represented by a collection of parallel hyperplanes $\{H(\alpha, \alpha \cdot \ell) : \ell \in \Delta(S)\}$ for $\alpha \in \mathbb{R}^K$. There is an alternative way to represent the linear order $\succeq$ on the simplex in terms of non-parallel hyperplanes in $\mathbb{R}^K$. For instance, for each distribution ℓ, define the hyperplane $H(\alpha - \alpha \cdot \ell, 0)$. Then $\ell' \in H_+(\alpha - \alpha \cdot \ell, 0)$ if and only if $\alpha \cdot \ell' - \alpha \cdot \ell \geq 0$, i.e., $\ell' \succeq \ell$. Thus the collection of hyperplanes $\{H(\alpha - \alpha \cdot \ell, 0) : \ell \in \Delta(S)\}$ also represents the same linear order, but these hyperplanes are not parallel, they have the same constants, and the norms are strictly ordered.

We use the new hyperplanes $H(\alpha_l, \kappa)$ and $H(\alpha_m, \kappa)$ to construct a bidding strategy $\sigma_i : S \to \mathcal{B}$ for trader i, where, for each signal, the trader randomizes over the finite set of values $\{1, 2, 3\}$. As a result, σ_i is fully described by a 2×3 matrix,

$$\begin{pmatrix} F^{\sigma_i}(1) \\ F^{\sigma_i}(2) \end{pmatrix} \equiv \begin{pmatrix} F^{\sigma_i}_{s_L}(1) & F^{\sigma_i}_{s_M}(1) & F^{\sigma_i}_{s_H}(1) \\ F^{\sigma_i}_{s_L}(2) & F^{\sigma_i}_{s_M}(2) & F^{\sigma_i}_{s_H}(2) \end{pmatrix}.$$

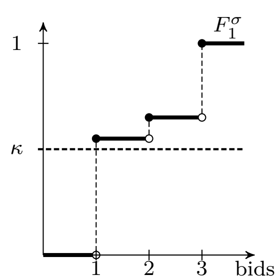
In particular, because $-\alpha_l(s), -\alpha_m(s) \in [0, 1]$ and $-\alpha_l(s) < -\alpha_m(s)$, we can choose σ_i so that $F^{\sigma_i}(1) = -\alpha_l$ and $F^{\sigma_i}(2) = -\alpha_m$. Hence, we construct the bidding strategy from the underlying betweenness order given by the betweenness property.

Finally, we can show that the symmetric strategy-profile σ, where every trader follows σ_i, ensures that, almost surely, the price is equal to the value in every state. This follows because the SLLN implies that aggregate bidding strategy F^σ derived in step 1 of the proof sketch is (almost surely) equal to the cumulative distribution function F^{σ_i} derived from the betweenness order. As a result:

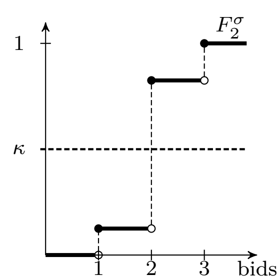
(a) As $P_1 \in \mathring{H}_+(-\alpha_l, \kappa)$, it follows that $F^\theta(1) \cdot P_1 > \kappa$. In state ω_1, the mass of bids less or equal to 1 is strictly greater than κ, and so the price can be no higher than 1. On the other hand, no trader bids strictly lower than 1, and so the price can be no lower than 1. Therefore, $p_\sigma(\omega_1) = 1$ (Figure 6a).

(b) As $P_2 \in \mathring{H}_-(-\alpha_l, \kappa)$, it follows that $F^\theta(1) \cdot P_2 < \kappa$. In state ω_2, the mass of bids less than or equal to 1 is therefore strictly less than κ, and so the price must be strictly greater than 1. On the other hand, $P_2 \in \mathring{H}_+(-\alpha_m, \kappa)$, and so $F^\theta(2) \cdot P_2 < \kappa$. As a result, the mass of bids less than or equal to 2 is (almost surely) greater than κ, and so price can be no higher than 2. Because no trader submits a bid in the interval $(1, 2)$, it follows that $p_\sigma(\omega_2) = 2$ (Figure 6b).

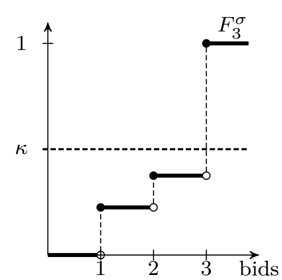
(c) As $P_3 \in \mathring{H}_-(-\alpha_m, 1-g)$, it follows that $F^\theta(2) \cdot P_3 < \kappa$. In state ω_3, the mass of bids less than or equal to 2 is strictly less than κ, and so the price must be strictly greater than 2. On the other hand, no trader submits a bid greater than 3, and so the price can be no higher than 3. Because no trader submits a bid in the interval $(2, 3)$, it follows that $p_\sigma(\omega_3) = 3$ (Figure 6c).



(a) CDF in state 1



(b) CDF in state 2



(c) CDF in state 3

Figure 6: Cumulative bid distributions.

4.2 Finite approximation

To illustrate how an equilibrium in the large market can be approximated by finite markets, consider an increasing sequence of finite populations indexed by $n = 1, ..., \infty$. Every population is divided into buyers and sellers with constant proportion of buyers $\kappa \in (0, 1)$. Nature chooses state ω and, in each population, traders draw independent signals from the conditional distribution P_ω. Given signals, traders submit bids and the auctioneer determines a market-clearing price.[19]

We denote by $\{\sigma_n\}_{n=1}^{\infty}$ a sequence of strategy-profiles, where σ_n is a strategy for the n-th population. A strategy σ_n and distribution over signals P_ω generate a distribution over bid-profiles $P_\omega^{\sigma_n}$ in state ω, and a corresponding random price denoted $p(\sigma_n, \omega)$. A sequence of strategy-profiles aggregates information asymptotically if the random prices eventually provide arbitrarily precise information about the value.

Definition 4. $\{\sigma_n\}_{n=1}^{\infty}$ *aggregates information asymptotically* if there is a price-function $p_{\sigma_\infty} : \Omega \to B$ such that (i) $v(\omega) \neq v(\omega')$ implies $p_{\sigma_\infty}(\omega) \neq p_{\sigma_\infty}(\omega')$, and (ii) in state ω, the sequence of prices $\{p(\sigma_n, \omega)\}_{n=1}^{\infty}$ converges in probability to $p_{\sigma_\infty}(\omega)$.[20]

We are again interested in strategy-profiles where traders respond to arbitrage opportunities. For a strategy-profile σ_n, let $\Pi_i(\sigma_n|s)$ denote the expected payoff of a type $(i, s) \in \mathcal{I}_n \times S$, and $\Pi_i^*(\sigma_n|s)$ denote the expected payoff if type (i, s) were to play a best-response. Then σ_n is an *ε-equilibrium* if $\Pi_i(\sigma_n|s) \geq \Pi_i^*(\sigma_n|s) - \varepsilon$ for all types. A 0-equilibrium is a standard Bayes-Nash equilibrium; ε-equilibrium allows for bounded profitable deviations. A sequence of strategy-profiles approximates equilibrium if the bound vanishes.

Definition 5. A sequence of strategy-profiles $\{\sigma_n\}_{n=1}^{\infty}$ *approximates equilibrium* if there is a sequence $\{\varepsilon_n\}_{n=1}^{\infty} \to 0$ such that, for all n, σ_n is a ε_n-equilibrium.

For a sequence of symmetric strategy profiles, the following proposition shows that the betweenness property is necessary and sufficient to aggregate information asymptotically.[21]

Proposition 2. *There is a sequence of symmetric strategy-profiles that approximates equilibrium and aggregates information asymptotically if and only if the betweenness property is satisfied.*

Proposition 2 reflects essential same economic intuitions as our aggregation result for the large market. (1) For a sufficiently large population, the law of large numbers disciplines aggregate bidding behavior, so that prices are stable. (2) When prices

[19] A detailed description of the auction format is given in Appendix A.3.1.

[20] Formally, for $\varepsilon>0$ there is n_ε so that $P_\omega^{\sigma_n}(p(\sigma_n,\omega)\in[p_{\sigma_\infty}(\omega)-\varepsilon, p_{\sigma_\infty}(\omega)+\varepsilon]) \geq 1-\varepsilon$ when $n \geq n_\varepsilon$.

[21] Our approximation result extends to finite asymmetries. Arbitrary asymmetries raise technical difficulties with our application of central limit arguments.

convey precise information, competition ensure that prices must be close to values to prevent traders from pursuing arbitrage opportunities. (3) There is a direct connection between bidding strategies and betweenness orders, which is central to understanding when aggregate bidding strategies correctly order values. It is the combination of these three insights that establishes the role of the betweenness property.

4.3 Walrasian market

While we illustrate our main result in the context of a double-sided auction with unit supply and demand, our findings apply to a broad class of market environments. In this section, we consider a Walrasian market where assets are divisible and, instead of bids, traders compete in monotone supply and demand schedules.

Suppose a buyer is endowed with a unit of wealth and can submit a non-increasing demand schedule, which represents the quantity of asset she is willing to buy at each price. A seller is endowed with a unit of asset and can submit a non-decreasing supply schedule, which represents the quantity of assets she is willing to sell at each price. Individual demand and supply schedules can be interpreted as buy and sell limit orders, and the profile of schedules can be interpreted as an order book. Given the order book, a clearing-house sets the minimum price p at which aggregate supply exceeds aggregate demand, and all trade occurs at this price. Buy limit orders strictly above the price and sell limit orders strictly below the price are executed, and the clearing house uniformly randomizes over limit orders equal to the price to clear the market. In state ω, the payoff for a buyer who purchases q units at price p is then $q(v(\omega) - p)$; the payoff for a seller is $q(p - v(\omega))$.

The strategy of a trader is a mapping from signals to monotone schedules. As in the large double-sided auction, the SLLN implies that, for every strategy-profile σ, there is a unique price function $p_\sigma : \Omega \to \mathbb{R}$ so that the price in state ω is equal to $p_\sigma(\omega)$ almost surely. In a Bayes-Nash equilibrium, traders observe their private signals, predict prices and values, and try to exploit arbitrage opportunities.

Proposition 3. *The Walrasian market has an equilibrium strategy-profile that aggregates information if and only if the betweenness property is satisfied.*

The equilibrium construction is analogous to the double-sided auction because the strategy of a trader in the Walrasian market is isomorphic to the decumulative strategy in an auction. The main difference is in the interpretation of the equilibrium arguments. In the double-sided auction, competitive forces generate incentives for buyers to shift probability mass towards higher bids when the auction price is below the value, to increase their chances of trading, and vice versa for sellers. In the market with divisible assets, buyers instead shift quantities towards higher bids when the price is below the value, to increase the quantity they trade, and vice versa for sellers. Analogous forces occur when the price is above the value.

5 Genericity

In this section, we provide a way to quantify how likely it is that an environment will satisfy the betweenness property. To simplify exposition, we assume that the value-function v is injective. The information structure for M states and K signals can be represented by a real matrix of dimension $K \times M$, where column m represents the distribution over signals conditional on state ω_m. As a result, we can quantify information structures with the Lebesgue measure on $\mathbb{R}^{(K-1)M}$.[22]

Proposition 4. *The betweenness property has full measure if and only if $K \geq M$.*

Together with our aggregation result, Proposition 4 establishes when information aggregation is a generic equilibrium property in a large market. As long as the cardinality of signals is larger than the cardinality of states, the betweenness property is generic and there is an equilibrium strategy-profile that aggregates information. On the other hand, in environments where the number of states is strictly greater than the number of signals, there is always a strictly positive measure of information structures where the betweenness property fails, and equilibrium prices cannot aggregate information. Moreover, from the proof it follows that the measure of information structures where the betweenness property is satisfied vanishes if the number of signals is held constant and the number of values increases.

To provide intuition for Proposition 4, suppose $K = M$. Let $P_\Omega = (P_{\omega_1}, ..., P_{\omega_K})$ be the $K \times K$ matrix that represents an information structure with K signals and K states. It is well-known that the set of invertible $K \times K$ matrices has full measure. For an invertible matrix, the system of equations $\alpha \cdot P_\Omega = \beta$ has a solution for $\beta \in \mathbb{R}^K$.

[22]In Appendix A.4, we show that the set of all information structures, and the subsets satisfying the betweenness property and the MLRP, respectively, are Lebesgue measurable.

Now choose β such that $\beta(m) > \beta(m')$ whenever $v(\omega_m) > v(\omega_{m'})$; then β defines an expected utility function $\alpha \in \mathbb{R}^K$ satisfying the linear property, which implies the betweenness property. A similar argument can be applied when $K > M$ by completing rectangular matrices appropriately. On the other hand, when $K < M$, there is a strictly positive measure of information structures where a high value state is in the convex hull of lower value states, which is inconsistent with the betweenness property.

By way of contrast, we can show that the MLRP—which is imposed by the prior literature on information aggregation in auctions—is generally not satisfied in environments with many states, regardless of the number of signals. The following is a formal definition of the MLRP in our environment.

Definition 6. An environment satisfies the MLRP if there is a weak order $\trianglerighteq$ on signals such that $v(\omega) > v(\omega')$ implies $\frac{P_\omega(s)}{P_\omega(s')} \geq \frac{P_{\omega'}(s)}{P_{\omega'}(s')}$ whenever $s \trianglerighteq s'$.

It is straightforward to show that the MLRP implies the betweenness property.[23] However, the MRLP imposes much stronger conditions on the environment, as the following proposition shows.

Proposition 5. *The MLRP has measure bounded above by* $\frac{2}{M!}$.

Proposition 5 implies that as the number of states grows the measure of information structures satisfying the MLRP quickly converges to 0, regardless of the number of signals. As a result, there are many environments where the MLRP fails and yet equilibrium prices can aggregate information in our market.

6 Multi-input environments

The MLRP is especially restrictive when the value of an asset depends on multiple unrelated sources of uncertainty, and there is therefore no meaningful total order on the set of signals. By replacing orders on signals with orders on distributions over signals, our approach to model information imposes no restrictions on the dimensionality of states and/or signals. In particular, since the orders over the signals is not relevant for the betweenness property, our main result shows that equilibrium prices can aggregate

[23] By the MLRP, $v(\omega) > v(\omega')$ implies that P_ω first-order stochastically dominates $P_{\omega'}$. By the well known characterization of first-order stochastic dominance, in terms of expected utility, it follows that the MLRP implies the linear property, which in turn implies the betweenness property.

information in environments where the value of the asset depends on multiple sources of uncertainty, and traders have access to specialized information. As an illustration, we consider a class of environments where a state has multiple inputs, but each signal is informative about only one input.

Definition 7. An environment (Ω, S, P, v) is a *multi-input environment* if

(1) there are C inputs such that $\Omega = \Omega_1 \times ... \times \Omega_C$ and $S = S_1 \cup ... \cup S_C$;

(2) for $\omega \in \Omega$, $s \in S$, and $c \in \{1, ..., C\}$: (i) $P(s|\omega, s_c \in S_c) = P(s|\omega_c)$, (ii) $P(s_c \in S_c|\omega) = P(s_c \in S_c)$, and (iii) $P(\omega) = \prod_{c=1}^{C} P(\omega_c)$;

(3) $v(\omega) = \psi(\phi_1(\omega_1) + ... + \phi_C(\omega_C))$ for strictly increasing $\psi : \mathbb{R}_{++} \to \mathbb{R}_{++}$, and injective functions $\phi_c : \Omega_c \to \mathbb{R}_{++}$ for $c = 1, ..., C$.

Multi-input environments are a special case of the environments we have considered thus far. By condition (1), the states are multidimensional with one dimension for each input, and every input has a set of signals. By condition (2i), a signal on input c depends only on the realization of the c-th input of a state. By condition (2ii), the likelihood that a signal is drawn for one of the inputs is independent of the state. By condition (2iii), the realization of states for each of the inputs are independent. Finally, condition (3) imposes a separability condition on the value function, which includes cases where $v(\omega) = \sum_{c=1}^{C} \phi_c(\omega_c)$ or $v(\omega) = \prod_{c=1}^{C} \phi_c(\omega_c)$ for any injective functions $\{\phi_c\}_{c=1}^{C}$. A multi-input environment therefore provides a stylized model of a market where the value of the asset depends on multiple sources of uncertainty, but each trader receives noisy information about only one source of uncertainty. We say that a multi-input environment is *non-trivial* if $|\Omega_c| > 1$ for at least two inputs; that is, the value depends on at least two distinct sources of uncertainty. In non-trivial multi-input environments, there is no natural order on signals and the information structure therefore cannot satisfy strong order properties such as the MLRP.

Proposition 6. *A non-trivial multi-input environment does not satisfy the MLRP.*

To illustrate, suppose there are two inputs, $\Omega_1 = \{0, 1\}$ and $\Omega_2 = \{0, 2\}$, and the value is given by the sum of inputs: $v(\omega) = \omega_1 + \omega_2$. There are two signals per input, $S_1 = \{L_1, H_1\}$ and $S_2 = \{L_2, H_2\}$, and the signal in each dimension c is perfectly informative about ω_c. In particular, for each input c, H_c conveys better news than L_c. As a result, the MLRP is satisfied for each input. However, a trader receives a signal for only one of the inputs, and so the MLRP is not satisfied (Figure 7). By contrast,

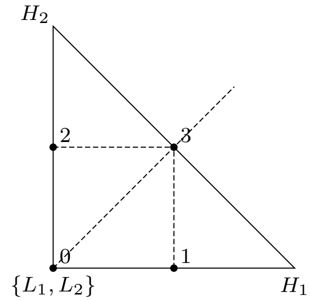


Figure 7: Two inputs, four states and four signals.

The dashed line from the origin represents the likelihood-ratio between signals H_1 and H_2 for $\omega = (1, 2)$. Since the likelihood-ratio is infinite for $\omega = (0, 2)$, and 0 for $\omega = (1, 0)$, the likelihood-ratio is not monotone in values.

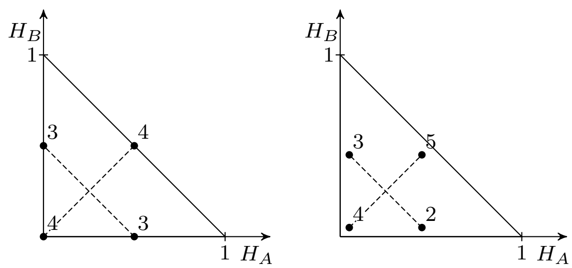
the betweenness property can be satisfied generically in a multi-input environment. To show this, we do not apply the Proposition 4 directly because it quantifies the betweenness property in relation to the set of all information structures. Instead, we consider only multi-input environments with a fixed number of inputs C, and fixed cardinalities for the states $|\Omega_c| = M_c$ and signals $|S_c| = K_c$ per input. We therefore measure the betweenness property in relation to a smaller set of information structures.

The key feature of this multi-input environment is that each signal conveys information about only one input, and information is therefore highly fragmented. In general, this cannot be good for the prospects of information aggregation, and yet a simple condition ensures that the betweenness property is generic.

Proposition 7. *The betweenness property is generic if and only if $K_c \geq M_c$ for all c.*

Together with our main result, Proposition 7 establishes when information aggregation is a generic equilibrium property in multi-input environments. With C inputs there are $\prod_{c=1}^{C} M_c$ states, but only $\sum_{c=1}^{C} K_c$ signals. As a result, there are generally far more values than signals. While the information that individual signals can convey about values is very limited, there is also additional structure on the value function (the separability condition). As a result, less information is needed to reveal the value, and Proposition 7 shows that the second effect dominates. Multi-input environments therefore provide a stark illustration of our main result: while the MLRP is never satisfied in these environments, there are natural conditions under which equilibrium prices in a large market aggregate information almost surely.

To illustrate why the separability condition is required, we can adapt Example 2 from the introduction. Asset Y attains a high value when the two inputs match ($v(l_A, l_B) = v(h_A, h_B) = 4$), and a low value when they mis-match ($v(l_A, h_B) = v(h_A, l_B) = 3$). The inputs are independent, and with equal probability a trader receives a signal that is informative about one of the inputs. The example therefore satisfies properties (1) and (2) of Definition 7 (Figure 8a).



(a) Asset Y (b) Asset Z

Figure 8: Robust failure of information aggregation without the separability condition.

Although there are as many signals as states, Proposition 4 does not apply to asset Y because of the restriction that each signal is informative about only one input. While there are as many signals as states for each input, Proposition 7 also does not apply because the value function is not separable. Indeed, as we argue in Example 2, equilibrium prices cannot aggregate information in the market for asset Y. There are, however, some features of asset Y that appear fragile. In particular, while signals are informative about the state, individual signals provide no information about the value. For instance, $P(v(\omega) = 4|s_i = H_A) = P(v(\omega) = 2) = \frac{1}{2}$. If the individual signals provide no information about values, it may be unsurprising that equilibrium prices cannot aggregate information. However, this special feature of asset Y is not the reason that information aggregation fails. To illustrate, we perturb the example.

Example 3. Figure 8b illustrates the information structure for an alternative asset Z. Properties (1) and (2) of Definition 7 hold, but signals are not perfectly correlated with inputs. Moreover, by adjusting the value function, individual signals are now informative about values. For instance, conditional on H_A, it is more likely that Z has value 5 than value 4. However, the value function is not separable because the value of Z is still strictly higher in states where the inputs match than in states where they mismatch. By Theorem 1 it is straightforward to see that equilibrium prices cannot aggregate information: since the convex hull of lower value states (l_A, h_B) and (h_A, l_B) intersects the convex hull of higher value states (l_A, l_B) and (h_A, h_B), the environment does not satisfy the betweenness property. Moreover, there is clearly an open ball around the conditional distribution for each state such that—as long as properties (1) and (2) of Definition 7 are maintained—the convex hull of lower value states will still intersect the convex hull of higher value states, and equilibrium prices therefore cannot aggregate information. When the value function is not separable—property (3) in Definition 7—it is therefore possible to have robust failures of information aggregation even when there are as many signals as states for each input. □

7 Conclusion

In this paper, we address a fundamental question of market exchange: when do prices aggregate information? By studying a double-sided auction with an infinite population of traders, our approach to this question combines insights from both strategic auction and competitive equilibrium models.

Our main result identifies a simple condition on information primitives that is both necessary and sufficient for equilibrium prices to aggregate all private information dispersed over market participants. Intuitively, some conditions on information primitives are necessary for the market to coordinate aggregate behavior. However, information aggregation does not require a strong order property on signals directly, but instead requires an order property on distributions over signals: for some betweenness order, higher value states must generate higher ranked conditional distributions. We call this the betweenness property.

While no individual trader observes the conditional distribution, the betweenness property is sufficient for competitive market forces to guide individual and aggregate behavior so that prices are perfectly revealing. On the other hand, when the betweenness property is not satisfied, we show that information aggregation necessarily fails. This highlights the limitations of the market, especially in environments with many states and relatively few signals. In such environments, even if collectively the population is perfectly informed, the market cannot coordinate behavior so that equilibrium prices reveal their collective information.

A Appendix

The appendix is organized as follows. Section A.1 provides important preliminary results about hyperplanes. Section A.2 provides a formal description of our large market. Section A.3 proves the main result, as well as the finite approximation. Section A.4 establishes our genericity results, including the multi-input environment.

A.1 Preliminaries

For vector $\alpha \in \mathbb{R}^K$, let $\alpha(i)$ denote the i-th component of α; $\mathbf{0} \equiv (0, ..., 0)$ is the origin; $\mathbf{e} \equiv (1, ..., 1)$ is the vector of 1's; and $\mathbf{e}_i$ is the unit vector with $\mathbf{e}_i(j) = \mathbb{1}[j = i]$, where $\mathbb{1}[.]$ is the indicator function.

For vector $\alpha \in \mathbb{R}^K/\{\mathbf{0}\}$ and scalar $c \in \mathbb{R}$, $H(\alpha, c) \equiv \{\ell : \alpha \cdot \ell = c\}$ is the hyperplane in $\mathbb{R}^K$ defined by norm α and constant c; $H_+(\alpha, c)$ is the corresponding upper half-space; $\mathring{H}_+(\alpha, c)$ is the strict upper half-space; $H_-(\alpha, c)$ is the lower half-space; and $\mathring{H}_-(\alpha, c)$ is the strict lower half-space. When $c = 0$, we omit c from the notation (e.g., $H(\alpha) \equiv H(\alpha, 0)$). For a set $A \subset \mathbb{R}^K$, $co(A)$ denotes the convex hull of A, and $A^\Delta \equiv A \cap \Delta^K$ is the intersection of A with the unit-simplex in $\mathbb{R}^K$, denoted $\Delta^K \equiv \{z \in \mathbb{R}^K_+ : \mathbf{e} \cdot z = 1\}$.

A.1.1 Intersection of hyperplanes and the unit-simplex

We first provide two general results regarding the intersection of hyperplanes in $\mathbb{R}^K$ and the unit-simplex Δ^K. Lemma 2, in particular, is central to our main result.

Lemma 1. *For vector $\alpha \in \mathbb{R}^K/\{\mathbf{0}\}$, and scalars $c \in \mathbb{R}$ and $\hat{c} \neq 0$, we have the following: (i) $H^\Delta(\hat{c}\alpha, \hat{c}c) = H^\Delta(\alpha, c)$, and (ii) $H^\Delta(\hat{c}\mathbf{e} + \alpha, \hat{c} + c) = H^\Delta(\alpha, c)$.*

Proof. The proof of part (i) is trivial. Part (ii) follows from the following chain of equalities: $H^\Delta(\hat{c}\mathbf{e} + \alpha, \hat{c} + c) = \{\ell : (\hat{c}\mathbf{e} + \alpha) \cdot \ell = \hat{c} + c, \ell \cdot \mathbf{e} = 1, \ell \geq 0\} = \{\ell : \hat{c}\mathbf{e} \cdot \ell + \alpha \cdot \ell = \hat{c} + c, \ell \cdot \mathbf{e} = 1, \ell \geq 0\} = \{\ell : \hat{c} + \alpha \cdot \ell = \hat{c} + c, \ell \cdot \mathbf{e} = 1, z \geq 0\} = \{\ell : \alpha \cdot \ell = c, \ell \cdot \mathbf{e} = 1, \ell \geq 0\} = H^\Delta(\alpha, c)$. □

Lemma 2. *Let $\alpha, \alpha' \in \mathbb{R}^K/\{\mathbf{0}\}$ be such that $H^\Delta_+(\alpha') \neq \Delta^K$. There exists $\lambda > 0$ such that $\lambda\alpha' \geq \alpha$ if and only if $H^\Delta_+(\alpha) \subset H^\Delta_+(\alpha')$.*

Proof. We first show that $H^\Delta_+(\alpha) \subset H^\Delta_+(\alpha')$ guarantees existence of $\lambda > 0$ such that $\lambda\alpha' \geq \alpha$. We argue the contrapositive: suppose there is no $\lambda > 0$ such that

$\lambda\alpha' \geq \alpha$. Then we want to show that there is some $\ell \in \Delta^K$ with $\ell \cdot \alpha \geq 0 > \ell \cdot \alpha'$. By assumption, $\alpha' \notin Z \equiv \{\tilde{z} \in \mathbb{R}^K : \lambda\tilde{z} \geq \alpha, \text{ for some } \lambda > 0\}$. Since Z is closed and convex, by the Separating Hyperplane Theorem, there is some $z \in \mathbb{R}^K/\{\mathbf{0}\}$ such that $z \cdot \alpha' < 0 \leq z \cdot \tilde{z}$ for all $\tilde{z} \in Z$. Furthermore, $z \geq 0$. If not, then $z \cdot \mathbf{e}_i < 0$ for some i, and we can argue to the following contradiction: if $\tilde{z} \in Z$, then $z' = \tilde{z} + t\mathbf{e}_i \in Z$ for $t > 0$; but $z \cdot (\tilde{z} + t\mathbf{e}_i)$ can be made arbitrarily small by increasing t, thereby contradicting that $z \cdot z' \geq 0$. Since $z > 0$, we can normalize z so that $z \cdot \mathbf{e} = 1$, i.e., $z \in \Delta^K$. As $\alpha \in Z$, $z \cdot \alpha \geq 0$ (because $\tilde{z} \cdot \alpha \geq 0$ for all $\tilde{z} \in Z$), and so $z \in H_+(\alpha)$. But $z \cdot \alpha' < 0$, and so $z \notin H_+(\alpha')$. Hence, $H_+^\Delta(\alpha)$ is not a subset of $H_+^\Delta(\alpha')$.

For the converse, suppose $\lambda\alpha' \geq \alpha$ for some $\lambda > 0$. It suffices to show that $\lambda\alpha' \cdot z \geq 0$ whenever $z \in H_+^\Delta(\alpha)$ (since this implies that $\alpha' \cdot z \geq 0$). To see this, note that $\lambda\alpha' \cdot z = \alpha \cdot z + (\lambda\alpha' - \alpha) \cdot z$. The first term is non-negative because $z \in H_+(\alpha)$. The second term is non-negative because $(\lambda\alpha' - \alpha) \geq 0$ by assumption, and $z \geq 0$. As a result, $z \in H_+^\Delta(\alpha)$ implies $z \in H_+(\alpha')$. $\square$

A.1.2 Nesting and the betweenness property

A *level curve* of an order $\succeq$ on Δ^K is an equivalence class of $\sim$. For a betweenness order, a level curve can be represented by a hyperplane: it is the intersection of a hyperplane in $\mathbb{R}^K$ with the unit-simplex Δ^K. We now provide a characterization of the betweenness property in terms of a finite collection of (strictly) nested hyperplanes.

Definition 8. A collection of hyperplanes $\{H(\alpha_r) : r = 1, ..., R\}$ is *nested* if, for all $r = 2, ..., R$, either (i) $H_+^\Delta(\alpha_r) \subset H_+^\Delta(\alpha_{r-1})$, or (ii) $H_+^\Delta(\alpha_r) \supset H_+^\Delta(\alpha_{r-1})$. The collection of hyperplanes is *strictly nested* if, for all $r = 2, ..., R$, either (i) $H_+^\Delta(\alpha_r) \subset \mathring{H}_+^\Delta(\alpha_{r-1})$, or (ii) $\mathring{H}_+^\Delta(\alpha_r) \supset H_+^\Delta(\alpha_{r-1})$.

Let $\boldsymbol{\Omega}_1, ..., \boldsymbol{\Omega}_{R+1}$ be a partition of $\boldsymbol{\Omega}$ such that, for $\omega \in \boldsymbol{\Omega}_r$ and $\omega' \in \boldsymbol{\Omega}_{r'}$, we have $v(\omega) > v(\omega')$ if and only if $r > r'$. For each $r = 1, ..., R+1$, let $P_{[r]} \equiv \{P_\omega : \omega \in \boldsymbol{\Omega}_r\}$.

Definition 9. An environment satisfies the *nested hyperplane separation property (NHSP)* if there is a nested collection of hyperplanes $\{H(\alpha_r) : r = 1, ..., R\}$ such that $v(\omega) > v(\omega')$ implies $P_\omega \in \mathring{H}_-(\alpha_r)$ and $P_{\omega'} \in H_+(\alpha_r)$ for some $r = 1, ..., R$. It satisfies the *strict NHSP* if there is a strictly nested collection of hyperplanes $\{H(\alpha_r) : r = 1, ..., R\}$ such that $v(\omega) > v(\omega')$ implies $P_\omega \in \mathring{H}_-(\alpha_r)$ and $P_{\omega'} \in \mathring{H}_+(\alpha_r)$ for some $r = 1, ..., R$.

Clearly, the strict NHSP implies the NHSP. The next lemma shows the converse.

Lemma 3. *If the NHSP is satisfied, then the strict NHSP is satisfied.*

Proof. Suppose P satisfies the NHSP and let $\{H(\alpha_r) : r = 1, ..., R\}$ be the corresponding collection of nested hyperplanes. By the NHSP, $co\left(P_{[R+1]}\right) \cap H^{\Delta}_{+}(\alpha_R) = \emptyset$. Therefore, by the Separating Hyperplane Theorem, there exists $\hat{\alpha}_R \in \mathbb{R}^K/\{\mathbf{0}\}$ such that $P_{[R+1]} \subset \mathring{H}^{\Delta}_{-}(\hat{\alpha}_R)$, and $H^{\Delta}_{+}(\alpha_R) \subset \mathring{H}^{\Delta}_{+}(\hat{\alpha}_R)$.

By the NHSP, $co\left(P_{[R]} \cup H^{\Delta}_{-}(\hat{\alpha}_R)\right) \cap H^{\Delta}_{+}(\alpha_{R-1}) = \emptyset$. Therefore, by the Separating Hyperplane Theorem, there exists $\hat{\alpha}_{R-1} \in \mathbb{R}^K/\{\mathbf{0}\}$ such that $co\left(P_{[R]} \cup H^{\Delta}_{-}(\hat{\alpha}_R)\right) \subset \mathring{H}^{\Delta}_{-}(\hat{\alpha}_{R-1})$, and $H^{\Delta}_{+}(\alpha_{R-1}) \subset \mathring{H}^{\Delta}_{+}(\hat{\alpha}_{R-1})$. Continuing this procedure generates a strictly nested collection of hyperplanes $\{H(\hat{\alpha}_r) : r = 1, ..., R\}$ such that the strict NHSP is satisfied. □

Lemma 4. *The betweenness property is satisfied iff the (strict) NHSP is satisfied.*

Proof. (1) We first show that the betweenness property implies the strict NHSP. By the betweenness property, there exists a betweenness order $\succeq$ such that $v(\omega) > v(\omega')$ implies $P_\omega \succ P_{\omega'}$. That means there exists a level curve of $\succeq$, described by $H^{\Delta}(\hat{\alpha}_1, c_1)$ for some $\hat{\alpha}_1 \in \mathbb{R}^K/\{\mathbf{0}\}$ and $c_1 \in \mathbb{R}$, such that $P_{[1]} \subset \mathring{H}^{\Delta}_{+}(\hat{\alpha}_1, c_1)$ and $\bigcup_{r=2}^{R+1} P_{[r]} \subset \mathring{H}^{\Delta}_{-}(\hat{\alpha}_1, c_1)$. Since the separation is strict, c_1 can be chosen so that $\alpha_1 \equiv \hat{\alpha}_1 - c_1\mathbf{e} \neq \mathbf{0}$. By Lemma 1, $H^{\Delta}(\alpha_1) = H^{\Delta}(\hat{\alpha}_1, c_1)$, and so $P_{[1]} \subset \mathring{H}^{\Delta}_{+}(\alpha_1)$ and $\bigcup_{r=2}^{R+1} P_{[r]} \subset \mathring{H}^{\Delta}_{-}(\alpha_1)$. Likewise, there exists a level curve of $\succeq$, $H^{\Delta}(\hat{\alpha}_2, c_2)$, such that $P_{[1]} \cup P_{[2]} \subset \mathring{H}^{\Delta}_{+}(\hat{\alpha}_2, c_2)$ and $\bigcup_{r=3}^{R+1} P_{[r]} \subset \mathring{H}^{\Delta}_{-}(\hat{\alpha}_2, c_2)$. Again, since the separation is strict, c_2 can be chosen so that $\alpha_2 \equiv \hat{\alpha}_2 - c_2\mathbf{e} \neq \mathbf{0}$. By Lemma 1, $H^{\Delta}(\alpha_2) = H^{\Delta}(\hat{\alpha}_2, c_2)$, and so $P_{[1]} \cup P_{[2]} \subset \mathring{H}^{\Delta}_{+}(\alpha_2)$ and $\bigcup_{r=3}^{R+1} P_{[r]} \subset \mathring{H}^{\Delta}_{-}(\alpha_2)$. Repeating for $r = 3, ..., R$ yields a collection of hyperplanes $\{H(\alpha_r) : r = 1, ..., R\}$ as required by the strict NHSP; the hyperplanes are strictly nested because they represent distinct level curves of $\succeq$.

(2) We now show that the strict NHSP implies the betweenness property. From the strict NHSP, let $\{H(\alpha_r) : r = 1, ..., R\}$ be the collection of strictly nested hyperplanes. Then $\{H(-\alpha_r) : r = 1, ..., R\}$ is also a strictly nested collection of hyperplanes. Moreover, $v(\omega) > v(\omega')$ implies $P_\omega \in \mathring{H}_{+}(-\alpha_r)$ and $P_{\omega'} \in \mathring{H}_{-}(-\alpha_r)$ for some $r = 1, ..., R$. We augment this strictly nested collection of hyperplanes with hyperplanes $H(-\alpha_0)$ and $H(-\alpha_{R+1})$ such that the simplex is contained in the strict upper half-space of $H(-\alpha_0)$ and in the strict lower half-space of $H(\alpha_{R+1})$.

For every distribution $\ell \in \Delta(S)$, there exists a unique $r_\ell \in \{0, ..., R+1\}$ such that $\ell \in H_+(-\alpha_{r_\ell}) \cap \mathring{H}_-(-\alpha_{r_\ell+1})$. Hence, there exists a unique $\theta_\ell \in [0,1]$ such that $\ell \in H(\theta_\ell \alpha_{r_\ell} + (1-\theta_\ell)\alpha_{r_\ell+1})$. We now define the binary relation $\succeq$ on $\Delta(S)$ by $\ell \succeq \ell'$ iff $r_\ell + \theta_\ell \geq r_{\ell'} + \theta_{\ell'}$. It is straightforward to verify that $\succeq$ is a betweenness order, and the betweenness property is satisfied with respect to this order. $\square$

A.2 The large market

In this section, we provide a formal description of the large auction, and establish implications of the strong law of large numbers. We then prove results in Section 3.

For a cumulative distribution function $G : B \to [0,1]$, let $\vec{G}(b) \equiv \lim_{b' \uparrow b} G(b')$. When $G(b)$ is the probability of a bid less than or equal to b, $\vec{G}(b)$ is the probability of a bid strictly less than b. The cumulative distribution function G is non-decreasing and right-continuous, and $\vec{G}$ is non-decreasing and left-continuous.

A.2.1 Proper large population

Following Aumann (1964), competitive market models often consider a continuum of agents endowed with a non-atomic probability measure (e.g., Lebesgue measure on $[0,1]$). There are, however, some well-known limitations of the continuum-agent framework (see, e.g., Judd 1985; Al-Najjar 2008). First, there is a measurability problem when agents and/or nature randomize independently, which poses a challenge in strategic settings (where agents randomize) and environments with incomplete information (where nature randomizes). Second, standard laws of large numbers do not extend to a continuum of random variables, which poses a challenge when describing aggregate outcomes such as prices.

As we are interested in prices for a strategic setting with incomplete information, we use the alternative population model in Al-Najjar (2008). In this model, a *large population* consists of a tuple $(\mathcal{I}, \mathbb{I}, \lambda)$, where $\mathcal{I} \subset [0,1]$ is a *countable* set of agents, $\mathbb{I}$ is the *power-set*, and λ is a *finitely*-additive probability measure with $\lambda(i) = 0$ for all $i \in \mathcal{I}$. As in a continuum-agent framework, $\lambda(i) = 0$ means agent i has negligible impact on aggregate outcomes. However, because $\mathbb{I}$ is the power-set, there are no measurability restrictions. Moreover, when the large population is a suitable limit of finite populations, a SLLN applies and provides a simple characterization of aggregate behavior. We present the formal definition below and refer to Al-Najjar (2008) for a detailed discussion (including proof of existence).[24]

Consider a sequence $\{\mathcal{I}_n\}_{n=1}^{\infty}$ of finite subsets of $[0,1]$, where each $\mathcal{I}_n$ can be interpreted as a finite set of traders. The sequence $\{\mathcal{I}_n\}_{n=1}^{\infty}$ is *proper* if $\mathcal{I}_n \subsetneq \mathcal{I}_{n+1}$ for all n, and $\lim_{n\to\infty} \frac{|\mathcal{I}_n|}{|\mathcal{I}_{n+1}|} = 0$ (i.e., the population grows, and at an increasing rate). The following definition describes a large population that can be viewed as the limit of a proper sequence of finite populations.

Definition 10. The large population $(\mathcal{I}, \mathbb{I}, \lambda)$ is *proper* if there is a proper sequence of finite populations $\{\mathcal{I}_n\}_{n=1}^{\infty}$ such that $\mathcal{I} = \bigcup_{n=1}^{\infty} \mathcal{I}_n$ and, for any finite collection $\{I'_r \in \mathcal{I} : r = 1, ..., R\}$, there exists a subsequence $\{\mathcal{I}_h\}_{h=1}^{\infty}$ of $\{\mathcal{I}_n\}_{n=1}^{\infty}$ such that $\lambda(I'_r) = \lim_{h\to\infty} \frac{|I'_r \cap \mathcal{I}_h|}{|\mathcal{I}_h|}$ for all r.

A.2.2 The auction format

Let $(\mathcal{I}, \mathbb{I}, \lambda)$ be a proper large population. The set of traders $\mathcal{I}$ is divided into a set of buyers $\mathcal{X}$ with mass $\kappa \in (0,1)$ and a set of sellers $\mathcal{Y}$ with mass $1-\kappa$. Each seller $y \in \mathcal{Y}$ owns a unit supply of the asset, and a bid represents the minimum price at which y willing to sell. Each buyer $x \in \mathcal{X}$ has unit demand for the asset, and a bid represents the maximum price at which x is willing to buy.

For bid-profile $a : \mathcal{I} \to B$, the (normalized) cumulative bid distributions of the buyers and sellers are denoted as follows:[25]

$$X_a(b) \equiv \frac{1}{\kappa} \int_{i\in\mathcal{X}} \mathbb{1}[a(i) \le b]\lambda(di) \quad \text{and} \quad Y_a(b) \equiv \frac{1}{1-\kappa} \int_{i\in\mathcal{Y}} \mathbb{1}[a(i) \le b]\lambda(di).$$

Given cumulative distribution functions X_a and Y_a, $\vec{X}_a(b)$ and $\vec{Y}_a(b)$ are, respectively, the normalized mass of buyers and sellers submitting a bid strictly less than b, and $F_a(b) \equiv \kappa X_a(b) + (1-\kappa)Y_a(b)$ is a cumulative distribution function obtained by weighting the cumulative distribution of buyers and sellers with their respective

[24] Al Najjar (2008) provides a detailed analysis and discussion of the connection between asymptotic equilibria in finite games, equilibria in a large population game, and equilibria in a continuum-agent game. There is an error in the result relating asymptotic equilibria in finite games with the equilibria in a large population game (see Tolvanen and Soultanis, 2012). This error is inconsequential for our analysis because our approximation result, provided in the supplementary appendix, is based on entirely different arguments.

[25] For a bounded function $f : \mathbb{R} \to \mathbb{R}$, the integral with respect to a finitely-additive measure λ is defined as for countably additive measures, by constructing integrals for simple functions and then taking a limit of a sequence of simple functions $\{f_n\}_{n=1}^{\infty}$ converging to f (see, e.g., Al-Najjar 2008, Section 2.3.1).

population shares. For $\gamma \in [0,1]$, let $\mathcal{Q}_a(\gamma) \equiv \inf\{b \in B : \gamma \leq F_a(b)\}$ denote the γ-quantile of the weighted cumulative distribution.

Given a bid-profile a, the auctioneer determines a price and an allocation of assets. The price $p(a)$ is the lowest bid that ensures the mass of sellers willing to sell at the price exceeds the mass of buyers who are willing to buy:

$$p(a) \equiv \inf\{b \in B : \kappa\,(1 - X_a(b)) \leq (1-\kappa)Y_a(b)\} = \inf\{b \in B : \kappa \leq F_a(b)\} \equiv \mathcal{Q}_a(\kappa).$$

For cumulative distribution functions X_a and Y_a, right-continuity implies that the infimum is attained, and (i) $\kappa(1 - X_a(p(a))) \leq (1-\kappa)Y_a(p(a))$ and (ii) $\kappa(1 - \vec{X}_a(p(a))) \geq (1-\kappa)\vec{Y}_a(p(a))$. We impose a condition later that ensures right-continuity.

All trade occurs at the price $p(a)$. A buyer trades if her bid is strictly above the price, and does not trade if her bid is strictly below the price. A seller trades if her bid is strictly below the price, and does not trades if her bid is strictly above the price. To clear the market, the auctioneer uniformly randomizes over bids equal to the price. This allocation-rule defines the likelihood $w(i,a)$ that i trades. In particular, for bid-profile a with cumulative distribution functions X_a (for buyers) and Y_a (for sellers), and a price $p(a) = p$, the likelihood that buyer $x \in \mathcal{X}$ trades is

$$w(x,a) = \begin{cases} 0 & \text{if } a(x) < p \text{ or } Y_a(p) = 0 \\ \frac{(1-\kappa)Y_a(p) - \kappa(1-X_a(p))}{\kappa(X_a(p) - \vec{X}_a(p))} & \text{if } a(x) = p,\ \kappa(1-\vec{X}_a(p)) > (1-\kappa)Y_a(p) > 0\,, \\ 1 & \text{otherwise} \end{cases}$$

the likelihood that seller $y \in \mathcal{Y}$ trades is

$$w(y,a) = \begin{cases} 0 & \text{if } a(y) > p \text{ or } (1-\vec{X}_a(p)) = 0 \\ \frac{\kappa(1-\vec{X}_a(p)) - (1-\kappa)\vec{Y}_a(p)}{(1-\kappa)(Y_a(p) - \vec{Y}_a(p))} & \text{if } a(y) = p,\ (1-\kappa)Y_a(p) > \kappa(1-\vec{X}_a(p)) > 0\,, \\ 1 & \text{otherwise} \end{cases}$$

where all randomizations are independent. The price-rule $p : \mathcal{A} \to B$ and allocation-rule $w : \mathcal{I} \times \mathcal{A} \to [0,1]$ ensure that, almost surely, the market clears.[26] The volume $t_a \equiv \min\{\kappa(1 - \vec{X}_a(p(a))), (1-\kappa)Y(p(a))\}$ is the mass of assets that are traded at price $p(a)$.

In state ω, the payoff for buyer x is $\pi_x(a|\omega) \equiv w(x,a)(v(\omega) - p(a))$ and the payoff for seller y is $\pi_y(a|\omega) \equiv w(y,a)(p(a) - v(\omega))$.

A.2.3 Bidding strategies

For strategy-profile $\sigma : \mathcal{I} \times S \to (B)$, let $G^{\sigma(i,s)}$ denote the cumulative distribution function for the strategy of type (i,s), and denote by $G^{\sigma_i} \equiv \left(G^{\sigma(i,s_1)}, ..., G^{\sigma(i,s_K)}\right)$ the vector-valued function that describes trader i's cumulative distribution for each signal. Then G^{σ_i} is a complete description of trader i's strategy. In state ω, trader i's cumulative distribution over bids depends on their strategy G^{σ_i} and the distribution over signals P_ω, and is defined by $G^{\sigma_i}_\omega(b) \equiv G^{\sigma_i}(b) \cdot P_\omega$.

The *mean* cumulative distribution over bids for types $(i,s) \in \mathcal{X} \times S$ (i.e., buyers who receive signal s) is defined by $X^\sigma_s(b) \equiv \frac{1}{\kappa}\int_{\mathcal{X}} G^{\sigma(i,s)}(b)\lambda(di)$. Denote by $X^\sigma \equiv \left(X^\sigma_{s_1}, ..., X^\sigma_{s_K}\right)$ be the corresponding vector-valued function that gives the mean cumulative distribution of buyers for each signal. In state ω, the mean cumulative distribution of buyers is defined by $X^\sigma_\omega(b) \equiv \frac{1}{\kappa}\int_{\mathcal{X}} G^{\sigma_i}_\omega(b)\lambda(di)$. By Bayes rule, $X^\sigma_\omega = X^\sigma \cdot P_\omega$ because

$$\begin{aligned}
X^\sigma_\omega(b) \equiv & \frac{1}{\kappa}\int_{\mathcal{X}} G^{\sigma_i}_{\omega_i}(b)\,\lambda(di) \equiv \frac{1}{\kappa}\int_{\mathcal{X}} \left[G^{\sigma_i}(b)\cdot P_\omega\right]\lambda(di) \\
= & \frac{1}{\kappa}\int_{\mathcal{X}} \left[\sum_{k=1}^{K} G^{\sigma(i,s_k)}(b) P_\omega(s_k)\right]\lambda(di) = \sum_{k=1}^{K} P_\omega(s_k)\left[\frac{1}{\kappa}\int_{\mathcal{X}} G^{\sigma(i,s_k)}(b)\lambda(di)\right] \\
\equiv & \sum_{k=1}^{K} P_\omega(s_k) X^\sigma_{s_k}(b) = X^\sigma(b)\cdot P_\omega.
\end{aligned}$$

[26]To see that the market clears, consider the cases where $\kappa(1 - \vec{X}_a(p(a))) > (1-\kappa)Y_a(p(a)) > 0$:

$$\begin{aligned}
\int_{i\in\mathcal{X}} w(i,a)\lambda(di) = & \int \Big(0\mathbb{1}[a(i)<p(a)] + \frac{(1-\kappa)Y_a(p) - \kappa(1-X_a(p))}{\kappa(X_a(p) - \vec{X}_a(p))}\mathbb{1}[a(i)=p(a)] + 1\mathbb{1}[a(i)>p(a)]\Big)d\lambda \\
& = \kappa\big(X_a(p(a)) - \vec{X}_a(p))\big)\frac{(1-\kappa)Y_a(p) - \kappa(1-X_a(p))}{\kappa(X_a(p) - \vec{X}_a(p))} + \kappa(1 - X_a(p(a)) \\
& = (1-\kappa)Y_a(p(a)) = \int_{i\in\mathcal{Y}} 1\mathbb{1}[a(i) \le p(a))]\lambda(di) = \int_{i\in\mathcal{Y}} w(i,a)\lambda(di).
\end{aligned}$$

Analogously, Y_s^σ, Y^σ, and Y_ω^σ denote, respectively, the mean cumulative distribution for sellers with signal s, the vector-valued mean cumulative distribution for sellers for each signal, and the mean cumulative distribution for sellers in state ω.

The *weighted* mean cumulative distribution for traders who receive the signal s is defined by $F_s^\sigma(b) \equiv \kappa X_s^\sigma(b) + (1-\kappa) Y_s^\sigma(b)$, with the corresponding vector-valued function $F^\sigma \equiv \left(F_{s_1}^\sigma, ..., F_{s_K}^\sigma\right)$. The weighted mean cumulative distribution over bids in state ω is defined by $F_\omega^\sigma \equiv \kappa X_\omega^\sigma + (1-\kappa) Y_\omega^\sigma$, and so $F_\omega^\sigma = F^\sigma \cdot P_\omega$.[27] For $\gamma \in (0,1)$, we denote by $\mathcal{Q}_\omega^\sigma(\gamma)$ the γ-quantile of the cumulative distribution F_ω^σ, defined by $\mathcal{Q}_\omega^\sigma(\gamma) \equiv \inf\{b \in B : \gamma \leq F_\omega^\sigma(b)\}$. We focus on strategy-profiles where F_ω^σ is a cumulative distribution function and $\vec{F}_\omega^\sigma(b)$ is the mean likelihood of submitting a bid strictly less than b. This ensures that the infimum in $\mathcal{Q}_\omega^\sigma(\gamma)$ is attained and, by the following lemma, the realized distributions over bids by buyers and sellers are almost surely cumulative distribution functions.

Lemma 5. *Consider a strategy profile σ and a countable collection of bids $\{b_j\}_{j=1}^\infty$. For every state ω, there exists a measurable subset of bid-profiles $A \in \mathcal{A}$ such that $P_\omega^\sigma(A) = 1$ and, for all $a \in A$ and $j \geq 1$, $X_a(b_j) = X_\omega^\sigma(b_j)$, $\vec{X}_a(b_j) = \vec{X}_\omega^\sigma(b_j)$, $Y_a(b_j) = Y_\omega^\sigma(b_j)$, $\vec{Y}_a(b_j) = \vec{Y}_\omega^\sigma(b_j)$, and $F_a(b_j) = F_\omega^\sigma(b_j)$.*

Proof. The key step in the proof of Lemma 5 follows directly the argument in the proof of Theorem 1 in Al-Najjar (2008). As the proof requires additional concepts that are not used elsewhere in our arguments, we provide formal details in a supplementary appendix. □

A.2.4 Proof of Proposition 1

Proof. Fix a strategy-profile σ, state ω and $\gamma \in (0,1)$. The result follows by establishing that there is a measurable subset of bid-profiles $A_\omega \subset \mathcal{A}$ such that $P_\omega^\sigma(A_\omega) = 1$ and, for all $a \in A_\omega$, $\mathcal{Q}_a(\gamma) = \mathcal{Q}_\omega^\sigma(\gamma)$. We then define $p_\sigma(\omega') = \mathcal{Q}_\omega^\sigma(\kappa)$ for state ω'.

For $\varepsilon > 0$, let $A_\varepsilon^+ = \{a \in \mathcal{A} : \mathcal{Q}_a(\gamma) > \mathcal{Q}_\omega^\sigma(\gamma) + \varepsilon\}$, and let $b_\varepsilon^+ = \mathcal{Q}_\omega^\sigma(\gamma) + \frac{\varepsilon}{2}$. For every $a \in A_\varepsilon^+$, $\mathcal{Q}_a(\gamma) > b_\varepsilon^+$, and therefore $F_a(b_\varepsilon^+) < \gamma$. On the other hand, $\mathcal{Q}_\omega^\sigma(\gamma) < b_\varepsilon^+$, and therefore $F_\omega^\sigma(b_\varepsilon^+) \geq \gamma$. Hence, $F_a(b_\varepsilon^+) \neq F_\omega^\sigma(b_\varepsilon^+)$. By Lemma 5, there is a set $\tilde{A}_\varepsilon^+$ such that $P_\omega^\sigma\left(\tilde{A}_\varepsilon^+\right) = 1$ and $F_a(b_\varepsilon^+) = F_\omega^\sigma(b_\varepsilon^+)$. Hence, $\tilde{A}_\varepsilon^+ \cap A_\varepsilon^+ = \emptyset$, and so $\mathcal{Q}_a(\gamma) \leq \mathcal{Q}_\omega^\sigma(\gamma) + \varepsilon$ for all $a \in \tilde{A}_\varepsilon^+$.

[27] This follows from $F_\omega^\sigma(b) \equiv \kappa X_\omega^\sigma(b) + (1-\kappa) Y_\omega^\sigma(b) = \kappa \left(X^\sigma(b) \cdot P_\omega\right) + (1-\kappa)\left(Y^\sigma(b) \cdot P_\omega\right) = \left(\kappa X^\sigma(b) + (1-\kappa) Y^\sigma(b)\right) \cdot P_\omega \equiv F^\sigma(b) \cdot P_\omega$.

Now let $A_\varepsilon^- = \{a \in \mathcal{A} : \mathcal{Q}_a(\gamma) < \mathcal{Q}_\omega^\sigma(\gamma) - \varepsilon\}$, and let $b_\varepsilon^- = \mathcal{Q}_\omega^\sigma(\gamma) - \frac{\varepsilon}{2}$. For every $a \in A_\varepsilon^-$, $\mathcal{Q}_a(\gamma) < b_\varepsilon^-$, and therefore $F_a(b_\varepsilon^-) \geq \gamma$. On the other hand, $\mathcal{Q}_\omega^\sigma(\gamma) > b_\varepsilon^-$, and therefore $F_\omega^\sigma(b_\varepsilon^-) < \gamma$. Again, by Lemma 5, there is a set $\tilde{A}_\varepsilon^-$ such that $P_\omega^\sigma\left(\tilde{A}_\varepsilon^-\right) = 1$, $\tilde{A}_\varepsilon^- \cap A_\varepsilon^- = \emptyset$, and so $\mathcal{Q}_a(\gamma) \geq \mathcal{Q}_\omega^\sigma(\gamma) + \varepsilon$ for all $a \in \tilde{A}_\varepsilon^-$.

Let $\tilde{A}_\varepsilon = \tilde{A}_\varepsilon^+ \cap \tilde{A}_\varepsilon^-$. Then $\tilde{A}_\varepsilon$ is the intersection of two measure 1 sets and so $P_\omega^\sigma\left(\tilde{A}_\varepsilon\right) = 1$. Moreover, $\mathcal{Q}_a(\gamma) \in [\mathcal{Q}_\omega^\sigma(\gamma) - \varepsilon, \mathcal{Q}_\omega^\sigma(\gamma) + \varepsilon]$ for all $a \in \tilde{A}_\varepsilon$.

Now fix a sequence $\{\varepsilon_j\}_{j=1}^\infty$ such that $\varepsilon_j \downarrow 0$. By the preceding argument, there exists a sequence $\left\{\tilde{A}_{\varepsilon_j}\right\}_{j=1}^\infty$ measurable such that, for every $j \geq 1$, $P_\omega^\sigma\left(\tilde{A}_{\varepsilon_j}\right) = 1$, and $\mathcal{Q}_a(1-g) \in [\mathcal{Q}_\omega^\sigma(\gamma) - \varepsilon_j, \mathcal{Q}_\omega^\sigma(\gamma) + \varepsilon_j]$ for all $a \in \tilde{A}_{\varepsilon_j}$. Let $A_\omega = \bigcap_{j=1}^\infty \tilde{A}_{e_j}$. Then, A_ω is the intersection of a countable collection of measure 1 sets, and so $P_\omega^\sigma(A_\omega) = 1$ (because P_ω^σ is countably additive). Moreover, because $\bigcap_{j=1}^\infty [\mathcal{Q}_\omega^\sigma(\gamma) - \varepsilon_j, \mathcal{Q}_\omega^\sigma(\gamma) + \varepsilon_j]$ $= \{\mathcal{Q}_\omega^\sigma(\gamma)\}$, we have $\mathcal{Q}_a(\gamma) = \mathcal{Q}_\omega^\sigma(\gamma)$ for all $a \in A_\omega$. □

A.2.5 Expected payoffs and trading volume

As a corollary of Lemma 5 and Proposition 1, we can characterize expected payoffs and the volume of trade for any strategy-profile. Consider a strategy-profile $\sigma : \mathcal{I} \times S \to B$, signal $s \in S$, state $\omega \in \Omega$, and price $p \in B$. For a buyer $x \in \mathcal{X}$, define

$$W(x,\sigma|s,\omega,p) \equiv \begin{cases} 0 & \text{if } Y_\omega^\sigma(p)=0 \\ 1-\vec{G}^{\sigma(x,s)}(p) & \text{if } (1-\kappa)Y_\omega^\sigma(p) \geq \kappa(1-\vec{X}_\omega^\sigma(p)), Y_\omega^\sigma(p)>0 \\ 1-G^{\sigma(x,s)}(p) + (G^{\sigma(x,s)}(p) - \vec{G}^{\sigma(x,s)}(p)) \frac{(1-\kappa)Y_\omega^\sigma(p) - \kappa(1-X_\omega^\sigma(p))}{\kappa(X_\omega^\sigma(p) - \vec{X}_\omega^\sigma(p))} & \\ & \text{if } \kappa(1-\vec{X}_\omega^\sigma(p)) > (1-\kappa)Y_\omega^\sigma(p) > 0 \end{cases}$$

and for a seller $y \in \mathcal{Y}$, define

$$W(y,\sigma|s,\omega,p) \equiv \begin{cases} 0 & \text{if } \vec{X}_\omega^\sigma(p)=1 \\ G^{\sigma(x,s)}(p) & \text{if } \kappa(1-\vec{X}_\omega^\sigma(p)) \geq (1-\kappa)Y_\omega^\sigma(p), \vec{X}_\omega^\sigma(p)<1 \\ \vec{G}^{\sigma(x,s)}(p) + (G^{\sigma(x,s)}(p) - \vec{G}^{\sigma(x,s)}(p)) \frac{\kappa(1-\vec{X}_\omega^\sigma(p)) - (1-\kappa)\vec{Y}_\omega^\sigma(p)}{\kappa(Y_\omega^\sigma(p) - \vec{Y}_\omega^\sigma(p))} & \\ & \text{if } (1-\kappa)Y_\omega^\sigma(p) > \kappa(1-\vec{X}_\omega^\sigma(p)) > 0 \end{cases}$$

and let $t_\sigma(\omega) \equiv \min\{\kappa(1 - \vec{X}_\omega^\sigma(p_\sigma(\omega))), (1-\kappa)Y_\omega^\sigma(p_\sigma(\omega))\}$.

Corollary 1. *For a strategy-profile σ, (i) the expected payoff of type (i,s) is*

$$\Pi_i(\sigma|s) = \begin{cases} \sum_{\omega\in\Omega} W\left(i,\sigma|s,\omega,\mathcal{Q}_\omega^\sigma(\kappa)\right)\left(v(x) - \mathcal{Q}_\omega^\sigma(\kappa)\right)P_s(\omega) & \text{if } i \in \mathcal{X} \\ \sum_{\omega\in\Omega} W\left(i,\sigma|s,\omega,\mathcal{Q}_\omega^\sigma(\kappa)\right)\left(\mathcal{Q}_\omega^\sigma(\kappa) - v(\omega)\right)P_s(\omega) & \text{if } i \in \mathcal{Y} \end{cases},$$

and (ii) there is $A_\omega \in \mathbb{A}$ such that $P^\sigma_\omega(A_\omega) = 1$ and, for all $a \in A_\omega$, $t_a = t_\sigma(\omega)$.

Proof. By Lemma 5 and Proposition 1, for every state ω, there is a subset of bid-profiles $A_\omega \in \mathcal{A}$ such that $P^\sigma_\omega(A_\omega) = 1$ and, for all $a \in A_\omega$, $p(a) = \mathcal{Q}^\sigma_\omega(\kappa)$, $X_a(p(a)) = X^\sigma_\omega(\mathcal{Q}^\sigma_\omega(\kappa))$, $\vec{X}_a(p(a)) = \vec{X}^\sigma_\omega(\mathcal{Q}^\sigma_\omega(\kappa))$, $Y_a(p(a) = Y^\sigma_\omega(\mathcal{Q}^\sigma_\omega(\kappa))$, $\vec{Y}_a(p(a)) = \vec{Y}^\sigma_\omega(\mathcal{Q}^\sigma_\omega(\kappa))$, and we also have that $F_a(p(a)) = F^\sigma_\omega(\mathcal{Q}^\sigma_\omega(\kappa))$. Hence, $t_a = t^\sigma(\omega)$ for all $a \in A_\omega$, which establishes part (ii). Moreover, for a buyer $x \in \mathcal{X}$,

$$\begin{aligned}
\Pi_x(\sigma|s) &\equiv \sum_\omega \Pi_x(\sigma|\omega)P_s(\omega) \equiv \sum_\omega \left(\int_{\mathcal{A}} w(i,a)\Big(v(\omega) - p(a)\Big)P^\sigma_\omega(da)\right) P_s(\omega) \\
&= \sum_\omega \left(\int_{A_\omega} w(x,a)\Big(v(\omega) - p(a)\Big)P^\sigma_\omega(da)\right) P_s(\omega) \\
&= \sum_\omega \Big(W(x,\sigma|s,\omega,\mathcal{Q}^\sigma_\omega(\kappa))(v(\omega) - \mathcal{Q}^\sigma_\omega(\kappa))\Big)P_s(\omega);
\end{aligned}$$

where the first and second equalities are by definition; the third equality follows because $P^\sigma_\omega(A_\omega) = 1$ for every ω; and the last equality follows because, for every state ω and every $a \in A_\omega$, we can replace empirical moments with their theoretical counterparts. The analogous argument applies for a seller. $\square$

As a corollary of Lemma 5 and Proposition 1, we can also show that, for any strategy-profile, if there is no trade in any state, then there is no trade in every state and prices are uninformative.

Corollary 2. *If $t_\sigma(\omega) = 0$, then $t_\sigma(\omega') = 0$ and $p_\sigma(\omega) = p_\sigma(\omega')$.*

Proof. Let $p \equiv p_\sigma(\omega)$, $p' \equiv p_\sigma(\omega')$, $t \equiv t_\sigma(\omega) = 0$, and $t' \equiv t_\sigma(\omega')$. We first show that $p = p'$ by deriving a contradiction when either $p' < p$ or $p' > p$.

(1) Suppose $p' < p$. If $Y^\sigma_\omega(p) = 0$, our full support assumption implies $Y^\sigma(p) = \mathbf{0}$, and so $Y^\sigma(p') = \mathbf{0}$, which implies $Y^\sigma_{\omega'}(p') = 0$. Since $(1-\kappa)Y^\sigma_{\omega'}(p') \geq \kappa(1 - X^\sigma_{\omega'}(p'))$, it follows that $X^\sigma(p') = \mathbf{e}$, and so $(1 - X^\sigma_\omega(p')) = 0$. Hence, $\kappa(1 - X^\sigma_\omega(p')) \leq (1-\kappa)Y^\sigma_\omega(p')$ and so $\mathcal{Q}^\sigma_\omega(\kappa) \leq p'$, which contradicts $p = p_\sigma(\omega)$. If $Y^\sigma_\omega(p) > 0$, then $t = 0$ implies $1 - \vec{X}^\sigma_\omega(p) = 0$.

Since $(1-\kappa)\vec{Y}^\sigma_\omega(p) \leq \kappa(1 - \vec{X}^\sigma_\omega(p))$, it follows that $\vec{Y}^\sigma_\omega(p) = 0$, and therefore $Y^\sigma_\omega(p') = 0$. But since $p' = \mathcal{Q}^\sigma_{\omega'}(p')$, this implies $1 - X^\sigma_{\omega'}(p') = 0$, which implies $X^\sigma(p) = \mathbf{e}$, and so $\kappa(1 - X^\sigma_\omega(p')) = 0$. Hence, $\kappa(1 - X^\sigma_\omega(p')) \leq (1-\kappa)Y^\sigma_\omega(p')$, and so $\mathcal{Q}^\sigma_\omega(\kappa) \leq p'$, which contradicts $p = p_\sigma(\omega) > p'$.

(2) Suppose $p' > p$. If $1 - \vec{X}^\sigma_\omega(p) = 0$, our full support assumption implies $X^\sigma(p) = \mathbf{e}$, and so $X^\sigma_{\omega'}(p) = 1$. Hence, $(1-\kappa)Y^\sigma_{\omega'}(p) \geq \kappa(1 - X^\sigma_{\omega'}(p))$, and so $\mathcal{Q}^\sigma_{\omega'}(\kappa) \leq p$, which contradicts $p' = p_\sigma(\omega')$. If $1 - \vec{X}^\sigma_\omega(p) > 0$, then $t = 0$ implies $Y^\sigma_\omega(p) = 0$. Since $(1-\kappa)Y^\sigma_\omega(p) \geq \kappa(1 - X^\sigma_\omega(p))$, it follows that $X^\sigma_\omega(p) = 1$, and so $X^\sigma(p) = \mathbf{e}$. As a result, $1 - X^\sigma_{\omega'}(p) = 0$, and so $\kappa(1 - X^\sigma_{\omega'}(p)) \leq (1-\kappa)Y^\sigma_{\omega'}(p)$. Hence, $\mathcal{Q}^\sigma_{\omega'}(\kappa) \leq p$, which contradicts $p' = p_\sigma(\omega') > p$.

Finally, since $t = 0$, either (i) $Y^\sigma_\omega(p) = 0$ or (ii) $1 - \vec{X}^\sigma_\omega(p) = 0$. In case (i), $Y^\sigma(p) = \mathbf{0}$, and so $Y^\sigma_{\omega'}(p) = 0$. Since $p = p'$, it follows that $(1-\kappa)Y^\sigma_{\omega'}(p') = 0$, and so $t' = 0$. In case (ii), $\vec{X}^\sigma = \mathbf{e}$, and so $\vec{X}^\sigma_{\omega'}(p) = 1$. Since $p = p'$, $\kappa(1 - \vec{X}^\sigma_{\omega'}(p')) = 0$, and so $t' = 0$. $\square$

A.3 Proof of Theorem 1

Proof. The proof consists of three steps. The first step shows that a strategy-profile σ is an equilibrium that aggregates information if and only if $p_\sigma = v$. The second step shows that, if the betweenness property is satisfied, there is a symmetric strategy-profile σ where $p_\sigma = v$. The last step shows that, if there is a strategy-profile σ where $p_\sigma = v$, then the betweenness property is satisfied.

Step 1: Let σ be a strategy-profile such that $p_\sigma = v$. Clearly, σ aggregates information. Now fix some trader $i \in \mathcal{I}$ and bid $b \in B$, and let σ^i_b be the strategy-profile where $\sigma^i(j,.) = \sigma(j,.)$ for all $j \in \mathcal{I}/\{i\}$ and $\sigma^i_b(i,s) = \delta_b$. Then $p_{\sigma^i_b} = v$ because $\lambda(i) = 0$. Moreover, $\Pi_i(\sigma|s) = \Pi_i(\sigma^i_b|s) = 0$ for all s. As a result, type (i,s) is playing a best response and, as the same holds for all types, σ is an equilibrium.

To establish the converse, we proceed by contradiction. Suppose σ is an equilibrium strategy that aggregates information and there is some state ω with $v \equiv v(\omega) \neq p_\sigma(\omega)$ $\equiv p$. Let $\boldsymbol{\Omega}' = \{\omega' : v(\omega') = v\}$ and $\varepsilon = \frac{1}{2}\min\{|v-p|, \min\{|p - p_\sigma(\omega'')| : \omega'' \notin \boldsymbol{\Omega}'\}\}$, i.e., ε is half the distance between the price induced in state ω and the price induced in any state with a value not equal v. Since σ aggregates information and $v \neq p$, it follows that $\varepsilon > 0$. We first show that it is not possible for $p < v$, by considering three collectively exhaustive cases.

(1) Suppose $\kappa(1 - \vec{X}^\sigma_\omega(p)) > (1-\kappa)Y^\sigma_\omega(p)$. Fix a buyer i and define an alternative

strategy-profile σ^i by the following cumulative distribution function for each type:

$$G^{\sigma^i(j,s)}(b) = \begin{cases} \vec{G}^{\sigma(j,s)}(p) & \text{if } j = i, b \in [p, p+\varepsilon) \\ G^{\sigma(j,s)}(b) & \text{otherwise} \end{cases}.$$

By Corollaries 1 and 2, for signal s,

$$\begin{aligned} &\Pi_i(\sigma^i|s) - \Pi_i(\sigma|s) \\ &\geq \left(G^{\sigma(i,s)}(p) - \vec{G}^{\sigma(i,s)}(p)\right)\left(1 - \frac{(1-\kappa)Y^\sigma_\omega(p) - \kappa(1 - X^\sigma_\omega(p))}{\kappa(X^\sigma_\omega(p) - \vec{X}^\sigma_\omega(p))}\right)(v-p)P_s(\omega). \end{aligned}$$

This follows because (i) the probability that (i, s) submits a bid strictly greater than p is $1 - \vec{G}^{\sigma(i,s)}(p)$ in σ^i, and $1 - G^{\sigma(i,s)}(p)$ in σ; and (ii) the probability that (i, s) submits a bid equal to p is 0 in σ^i, and $G^{\sigma(i,s)}(p) - \vec{G}^{\sigma(i,s)}(p)$ in σ. Since σ aggregates information, by Corollary 2, $Y^\sigma_\omega(p) = Y^{\sigma^i}_\omega(p) > 0$. Hence, by Corollary 1,

$$\begin{aligned} &\Pi_i(\sigma^i|s) - \Pi_i(\sigma|s) \\ &= \sum_{\omega' \in \Omega'} \left(G^{\sigma(i,s)}(p) - \vec{G}^{\sigma(i,s)}(p)\right)\left(1 - \frac{(1-\kappa)Y^\sigma_{\omega'}(p) - \kappa(1 - X^\sigma_{\omega'}(p))}{\kappa(X^\sigma_{\omega'}(p) - \vec{X}^\sigma_{\omega'}(p))}\right)(v-p)P_s(\omega') \\ &\geq \left(G^{\sigma(i,s)}(p) - \vec{G}^{\sigma(i,s)}(p)\right)\left(1 - \frac{(1-\kappa)Y^\sigma_\omega(p) - \kappa(1 - X^\sigma_\omega(p))}{\kappa(X^\sigma_\omega(p) - \vec{X}^\sigma_\omega(p))}\right)(v-p)P_s(\omega). \end{aligned}$$

(A similar argument applies in other parts of the proof, and we therefore omit details for brevity). Since this payoff-difference is non-negative and (by full support of P) there is a strictly positive mass of buyers receiving signal s, and so it must be the case that

$$\begin{aligned} &\int_{i \in \mathcal{X}} \left(G^{\sigma(i,s)}(p) - \vec{G}^{\sigma(i,s)}(p)\right)\left(1 - \frac{(1-\kappa)Y^\sigma_\omega(p) - \kappa(1 - X^\sigma_\omega(p))}{\kappa(X^\sigma_\omega(p) - \vec{X}^\sigma_\omega(p))}\right)\lambda(di) \\ &= \left(\kappa X^\sigma_s(p) - \vec{X}^\sigma_s(p)\right)\left(1 - \frac{(1-\kappa)Y^\sigma_\omega(p) - \kappa(1 - X^\sigma_\omega(p))}{\kappa(X^\sigma_\omega(p) - \vec{X}^\sigma_\omega(p))}\right) = 0, \end{aligned}$$

otherwise a strictly positive mass of buyer types has a strictly profitable deviation (and σ is not an equilibrium). Since the above equality holds for all s, it also holds in expectation over s in state ω. Hence, $\left(\kappa X^\sigma_\omega(p) - \vec{X}^\sigma_\omega(p)\right)\left(1 - \frac{(1-\kappa)Y^\sigma_\omega(p) - \kappa(1 - X^\sigma_\omega(p))}{\kappa(X^\sigma_\omega(p) - \vec{X}^\sigma_\omega(p))}\right) = 0$, which contradicts the supposition that $\kappa(1 - \vec{X}^\sigma_\omega(p)) > (1-\kappa)Y^\sigma_\omega(p) \geq \kappa(1 - X^\sigma_\omega(p))$.

(2) Suppose $\kappa(1-\vec{X}^\sigma_\omega(p)) < (1-\kappa)Y^\sigma_\omega(p)$. Fix a seller i and define the alternative strategy-profile σ^i as in case (1). By Corollaries 1 and 2, for signal s, $\Pi_i(\sigma^i|s)-\Pi_i(\sigma|s) \geq \Big(G^{\sigma(i,s)}(p)-\vec{G}^{\sigma(i,s)}(p)\Big)\left(\frac{\kappa(1-\vec{X}^\sigma_\omega(p))-(1-\kappa)\vec{Y}^\sigma_\omega(p)}{\kappa(Y^\sigma_\omega(p)-\vec{Y}^\sigma_\omega(p))}\right)(v-p)P_s(\omega)$. Since this payoff-difference is non-negative and there is a strictly positive mass of sellers receiving signal s, it must be the case

$$\begin{aligned}&\int_{i\in\mathcal{Y}}\Big(G^{\sigma(i,s)}(p)-\vec{G}^{\sigma(i,s)}(p)\Big)\left(\frac{\kappa(1-\vec{X}^\sigma_\omega(p))-(1-\kappa)\vec{Y}^\sigma_\omega(p)}{(1-\kappa)(Y^\sigma_\omega(p)-\vec{Y}^\sigma_\omega(p))}\right)\lambda(di)\\&=(1-\kappa)\Big(Y^\sigma_s(p)-\vec{Y}^\sigma_s(p)\Big)\left(\frac{\kappa(1-\vec{X}^\sigma_\omega(p))-(1-\kappa)\vec{Y}^\sigma_\omega(p)}{(1-\kappa)(Y^\sigma_\omega(p)-\vec{Y}^\sigma_\omega(p))}\right)=0\end{aligned}$$

otherwise there is a strictly positive mass of seller types with a strictly profitable deviation (and σ is not an equilibrium). Since the above equality holds for all s, it also holds in expectation over s in state ω. Hence, $(1-\kappa)\Big(Y^\sigma_\omega(p)-\vec{Y}^\sigma_\omega(p)\Big)\left(\frac{\kappa(1-\vec{X}^\sigma_\omega(p))-(1-\kappa)\vec{Y}^\sigma_\omega(p)}{(1-\kappa)(Y^\sigma_\omega(p)-\vec{Y}^\sigma_\omega(p))}\right)$ $=\kappa(1-\vec{X}^\sigma_\omega(p))-(1-\kappa)\vec{Y}^\sigma_\omega(p)=0$. By Corollary 2, $\vec{Y}^\sigma_\omega(p)>0$ and so $p>0$. Now let $\varepsilon'=\frac{1}{2}\min\{\varepsilon,p\}>0$, and define another strategy-profile $\hat{\sigma}^i$ by the cumulative distribution functions

$$G^{\hat{\sigma}^i(j,s)}(b)=\begin{cases}\vec{G}^{\sigma(j,s)}(p-\varepsilon') & \text{if } j=i, b\in[p-\varepsilon',p+\varepsilon')\\ G^{\sigma(j,s)}(b) & \text{otherwise}\end{cases}.$$

By Corollary 1, $\Pi_i(\hat{\sigma}^i|s)-\Pi_i(\sigma|s)\geq\Big(G^{\sigma(i,s)}(p-\varepsilon')-\vec{G}^{\sigma(i,s)}(p)\Big)(p-v)P_s(\omega)$ for signal s. Since this payoff-difference is non-negative and there is a strictly positive mass of sellers receiving signal s, it must be the case that $\int_{i\in\mathcal{Y}}\Big(G^{\sigma(i,s)}(p-\varepsilon')-\vec{G}^{\sigma(i,s)}(p)\Big)\lambda(di)$ $=(1-\kappa)\Big(Y^\sigma_s(p-\varepsilon')-\vec{Y}^\sigma_s(p)\Big)=0$, otherwise there is a strictly positive mass of seller types with a strictly profitable deviation (and σ is not an equilibrium). Since the above equality holds for all s, it also holds in expectation over s in state ω. Hence, $Y^\sigma_\omega(p-\varepsilon')=\vec{Y}^\sigma_\omega(p)$.

Now suppose trader i is a buyer rather than a seller. Then, for s, $\Pi_i(\sigma^i|s)-\Pi_i(\sigma|s)$ $\geq\Big(G^{\sigma(i,s)}(p)-\vec{G}^{\sigma(i,s)}(p-\varepsilon')\Big)(v-p)P_s(\omega)$.

Since this payoff-difference is non-negative and there is a strictly positive mass of buyers receiving signal s, we have $\int_{i\in\mathcal{X}}\Big(G^{\sigma(i,s)}(p)-\vec{G}^{\sigma(i,s)}(p-\varepsilon')\Big)\lambda(di)$ $=\kappa\Big(X^\sigma_s(p)-\vec{X}^\sigma_s(p-\varepsilon')\Big)=0$, otherwise there is a strictly positive mass of buyer types with a strictly profitable deviation (and σ is not an equilibrium). Since the above equality holds for all s, it also holds in expectation over s in state ω. Hence, $X^\sigma_\omega(p) = \vec{X}^\sigma_\omega(p-\varepsilon')$. As a result, we have

$$\begin{aligned}(1-\kappa)Y(p-\varepsilon') &= (1-\kappa)\vec{Y}^\sigma_\omega(p) = \kappa(1-\vec{X}^\sigma_\omega(p)) \\ &\geq \kappa(1-X^\sigma_\omega(p)) = \kappa(1-\vec{X}^\sigma(p-\varepsilon')) \geq \kappa(1-X^\sigma(p-\varepsilon')).\end{aligned}$$

But then $\mathcal{Q}^\sigma_\omega(\kappa) \leq p-\varepsilon'$, and since $\varepsilon' > 0$, this contradicts $p = p_\sigma(\omega)$.

(3) Suppose $\kappa(1-\vec{X}^\sigma_\omega(p)) = (1-\kappa)Y^\sigma_\omega(p)$. Since $(1-\vec{X}^\sigma_\omega(0)) = 1$ it follows that $p > 0$; otherwise we would have $Y^\sigma_\omega(p) = 1$ which, by full support of P, implies $Y^\sigma_s(0) = 1$ for all s, and therefore $Y^\sigma_{\omega''}(0) = 1$ for all ω'' so that $p_\sigma = \mathbf{0}$ (contradicting that σ aggregates information). Moreover, $\kappa(1-\vec{X}^\sigma_\omega(p)) = (1-\kappa)Y^\sigma_\omega(p)$ implies that $\kappa(1-\vec{X}^\sigma_\omega(p)) = \kappa(1-X^\sigma_\omega(p))(1-\kappa) = (1-\kappa)Y^\sigma_\omega(p) = (1-\kappa)\vec{Y}^\sigma_\omega(p)$. Define ε' and $\hat{\sigma}^i$ as in case (2). If i is a seller, then by Corollaries 1 and 2, for signal s, $\Pi_i(\hat{\sigma}^i|s) - \Pi_i(\sigma|s) \geq \Big(G^{\sigma(i,s)}(p-\varepsilon') - \vec{G}^{\sigma(i,s)}(p)\Big)(p-v)P_s(\omega)$. Following the same argument as in case (2), this implies $Y^\sigma_\omega(p-\varepsilon') = \vec{Y}^\sigma_\omega(p)$. If i is buyer, then by Corollaries 1 and 2, for signal s, $\Pi_i(\sigma^i|s) - \Pi_i(\sigma|s) \geq \Big(G^{\sigma(i,s)}(p) - \vec{G}^{\sigma(i,s)}(p-\varepsilon')\Big)(v-p)P_s(\omega)$. Following the same argument as in case (2), this implies $X^\sigma_\omega(p) = \vec{X}^\sigma_\omega(p-\varepsilon')$. As a result,

$$\begin{aligned}(1-\kappa)Y(p-\varepsilon') &= (1-\kappa)\vec{Y}^\sigma_\omega(p) = \kappa(1-\vec{X}^\sigma_\omega(p)) \\ &= \kappa(1-X^\sigma_\omega(p)) = \kappa(1-\vec{X}^\sigma(p-\varepsilon')) = \kappa(1-X^\sigma(p-\varepsilon')).\end{aligned}$$

But then $\mathcal{Q}^\sigma_\omega(\kappa) \leq p-\varepsilon'$, and since $\varepsilon' > 0$ this contradicts $p = p_\sigma(\omega)$. By cases (1)-(3) it follows that there no states in which the price induced by σ is strictly less than the value.

Now suppose that $p > v$. We again consider cases. For buyer i, define strategy-profile $\tilde{\sigma}^i$ by

$$G^{\tilde{\sigma}^i(j,s}(b) = \begin{cases} 1 & \text{if } j = i \\ G^{\sigma(j,s)}(b) & \text{otherwise} \end{cases}.$$

(1') Suppose $(1-\kappa)Y^\sigma_\omega(p) \geq \kappa(1-\vec{X}^\sigma_\omega(p))$. Since there are no states where price is strictly lower than value, Corollary 1 implies, for signal s, $\Pi_i(\tilde{\sigma}^i|s) - \Pi_i(\sigma|s) \geq -\Big(1-\vec{G}^{\sigma(i,s)}(p)\Big)(v-p)P_s(\omega)$. Since this payoff-difference is non-negative and there is a strictly positive mass of buyers receiving signal s, it must be the case that

$\int_{i\in\mathcal{X}} \Big(1 - G^{\sigma(i,s)}(p)\Big)\lambda(di) = \kappa\Big(1 - \vec{X}^{\sigma}_{s}(p)\Big) = 0$, otherwise there is a strictly positive mass of buyer types with a strictly profitable deviation (and σ is not an equilibrium). Since the above equality holds for all s, it also holds in expectation over s in state ω. Hence, $1 - \vec{X}^{\sigma}_{\omega}(p) = 0$. Hence, there is no trade in state ω, contradicting Corollary 2.

(2') Suppose $(1-\kappa)Y^{\sigma}_{\omega}(p) < \kappa(1 - \vec{X}^{\sigma}_{\omega}(p))$. Since there are no states where the price is strictly lower than the value, it follows by Corollary 1 that, for signal s, $\Pi_i(\tilde{\sigma}^i|s) - \Pi_i(\sigma|s)$ is greater than

$$\left(-\Big(1-\vec{G}^{\sigma(i,s)}(p)\Big)-\Big(G^{\sigma(i,s)}(p)-\vec{G}^{\sigma(i,s)}(p)\Big)\frac{(1-\kappa)Y^{\sigma}_{\omega}(p)-\kappa(1-X^{\sigma}_{\omega}(p))}{\kappa(X^{\sigma}_{\omega}(p)-\vec{X}^{\sigma}_{\omega}(p))}\right)(v-p)P_s(\omega)$$

Since this payoff-difference is non-negative and there is a strictly positive mass of buyers receiving signal s, it must be the case that

$$\int\left(\Big(1-\vec{G}^{\sigma(i,s)}(p)\Big)+\Big(G^{\sigma(i,s)}(p)-\vec{G}^{\sigma(i,s)}(p)\Big)\frac{(1-\kappa)Y^{\sigma}_{\omega}(p)-\kappa(1-X^{\sigma}_{\omega}(p))}{\kappa(X^{\sigma}_{\omega}(p)-\vec{X}^{\sigma}_{\omega}(p))}\right)\lambda(di)=0,$$

otherwise there is a strictly positive mass of buyer types with a strictly profitable deviation (and σ is not an equilibrium). Since the above equality holds for all s, it also holds in expectation over s in state ω. Hence, $\kappa(X^{\sigma}_{\omega}(p) - \vec{X}^{\sigma}_{\omega}(p)) + (1-\kappa)Y^{\sigma}_{\omega}(p) = 0$. But the first term is strictly positive by supposition, and the second term is strictly positive by Corollary 2, yielding the desired contradiction.

For the remaining steps we use the characterization of the betweenness property in terms of the (strict) NHSP (Definition 9 and Lemma 4). As in the definition of the NHSP, let $\Omega_1, ..., \Omega_{R+1}$ be a partition of Ω such that, for $\omega \in \Omega_r$ and $\omega' \in \Omega_{r'}$, we have $v(\omega) > v(\omega')$ if and only if $r > r'$. Then, for each $r = 1, ..., R+1$, there exists a value v_r such that $v_r = v(\omega)$ for all $\omega \in \Omega_r$. Let $v_0 \equiv 0$ and note that $v_0 < v_1 < .... < v_{R+1}$.

Step 2: We first show that when P satisfies the strict NHSP, we can construct a symmetric equilibrium strategy-profile that aggregates information. By the strict NHSP, there exist non-zero vectors $\hat{\alpha}_1, ..., \hat{\alpha}_R \in \mathbb{R}^K$ such that that (i) the collection of hyperplanes $\{H(\hat{\alpha}_r) : 1, ..., R\}$ is nested, and (ii) for $r = 1, ..., R$, $v(\omega) \le v_r$ implies $P_\omega \in \mathring{H}_+(\alpha_r)$ and $v(\omega') > v_r$ implies $P_{\omega'} \in \mathring{H}_-(\alpha_r)$. Because the hyperplanes are nested, Lemma 2 implies that, without loss of generality, we can assume that $\hat{\alpha}_r < \hat{\alpha}_{r+1}$ for $r = 1, ..., R-1$. Moreover, since $\kappa \in (0,1)$, we can choose $\varepsilon > 0$ such that $\mathbf{0} < \varepsilon\hat{\alpha}_r + (\kappa) < \mathbf{e}$ for all $r = 1, ..., R$. Now define a new collection of vectors $\alpha_0, ..., \alpha_{R+1} \in \mathbb{R}^K$ as follows: $\alpha_0 \equiv \mathbf{0}$, $\alpha_{R+1} \equiv \mathbf{e}$, and, for $r = 1, ..., R$, $\alpha_r \equiv \varepsilon\hat{\alpha}_r + (\kappa)$. By Lemma 1, $H^{\Delta}(\alpha_r, 1-g) = H^{\Delta}(\hat{\alpha}_r)$ for all $r = 1, ..., R$.

We use the vectors $\alpha_0, ..., \alpha_{R+1}$ to construct a symmetric strategy-profile. Because the strategy-profile is symmetric, it is sufficient to describe the strategy $\sigma_i \equiv \sigma(i, .)$ of one trader $i \in \mathcal{I}$ (buyer and sellers). The strategy is simple: for each signal s, $\sigma_i(s)$ has support $\{v_1, ..., v_R\}$. We use the notation $\sigma_i(v_r|s_k)$ to denote the probability that trader i submits bid v_r when she receives signal s_k. In particular, for each $r = 1, ..., R+1$ and $k = 1, ..., K$, let $\sigma_i(v_r|s_k) \equiv \alpha_r(k) - \alpha_{r-1}(k)$. Fixing a $k = 1, ..., K$, $\alpha_r > \alpha_{r-1}$ implies that $\sigma_i(v_r|s_k) > 0$ for all r, and $\sum_{r=1}^{R} \sigma(v_r|s_k) = \alpha_{R+1}(k)$ by construction. As a result, $\sigma_i(.|s_k) \in (B)$. We denote the corresponding strategy-profile by σ. Note that, for all r and k, $G^{\sigma(i,s_k)}(v_r) = \alpha_r(k)$ for every i, and so $F^{\sigma}_{s_k}(v_r) = \alpha_r(k)$. This implies that $F^{\sigma}(v_r) = \alpha_r$. Since σ is a symmetric strategy-profile, F^{σ} is a cumulative distribution function.

We now show that $v(\omega) = p_{\sigma}(\omega) \equiv \mathcal{Q}^{\sigma}_{\omega}(\kappa)$ for all ω. First consider $\omega \in \mathbf{\Omega}_1$. Then $v(\omega) = v_1$ and so $P_{\omega} \in \mathring{H}_+(\alpha_1, \kappa)$. This means that $\kappa < \alpha_1 \cdot P_{\omega} = F^{\sigma}(v_1) \cdot P_{\omega} = F^{\sigma}_{\omega}(v_1)$. Since $F^{\sigma}_{\omega}(b) = 0$ for all $b < v_1$, it follows that $\mathcal{Q}^{\sigma}_{\omega}(\kappa) = v_1$.

Next consider $\omega \in \mathbf{\Omega}_{R+1}$. Then $v(\omega) = v_{R+1}$ and so $P_{\omega} \in \mathring{H}_-(\alpha_R, \kappa)$. This means that $\kappa > \alpha_R \cdot P_{\omega} = F^{\sigma}(v_R) \cdot P_{\omega} = F^{\sigma}_{\omega}(v_R)$. Since $F^{\sigma}_{\omega}(b) = F^{\sigma}_{\omega}(v_R)$ for all $b \in [v_R, v_{R+1})$ and $F^{\sigma}_{\omega}(v_{R+1}) = 1$, it follows that $\mathcal{Q}^{\sigma}_{\omega}(\kappa) = v_{R+1}$.

Finally, consider $\omega \in \mathbf{\Omega}_r$ for $r = 2, ..., R$. Then $v(\omega) = v_r$ and $P_{\omega} \in \mathring{H}_+(\alpha_r, \kappa)$ $\cap \mathring{H}_-(\alpha_{r-1}, \kappa)$. This means that $F^{\sigma}_{\omega}(v_r) = F^{\sigma}(v_r) \cdot P_{\omega} = \alpha_r \cdot P_{\omega} > \kappa > \alpha_{r-1} \cdot P_{\omega}$ $= F^{\sigma}(v_{r-1}) \cdot P_{\omega} = F^{\sigma}_{\omega}(v_{r-1})$. Since $F^{\sigma}_{\omega}(b) = F^{\sigma}_{\omega}(v_{r-1})$ for all $b \in [v_{r-1}, v_r)$, it follows that $\mathcal{Q}^{\sigma}_{\omega}(\kappa) = v_r$. Hence, $p_{\sigma} = v$ and so, by Step 1, σ is an equilibrium strategy profile that aggregates information.

Step 3: Suppose the strategy-profile σ is an equilibrium that aggregates information. By Step 1, it follows that $v(\omega) = p_{\sigma}(\omega) = \mathcal{Q}^{\sigma}_{\omega}(\kappa)$ for all ω. For $r = 1, ..., R$, define $\alpha_r \equiv F^{\sigma}(v_r) - \kappa$. To show that the NHSP is satisfied, we show that the collection of hyperplanes $\{H(\alpha_r) : r = 1, ..., \Omega\}$ is nested, and $v(\omega) = v_r < v(\omega')$ implies $P_{\omega} \in H_+(\alpha_r)$ and $P_{\omega'} \in \mathring{H}_-(\alpha_r)$.

By definition of the quantile, $p_{\sigma} = v$ implies that, for every state ω, $\kappa \leq F^{\sigma}(b) \cdot P_{\omega}$ for $b \geq v(\omega)$ and $\kappa > F^{\sigma}(b) \cdot P_{\omega}$ for $b < v(\omega)$. Since F^{σ}_s is monotone non-decreasing, $v_r > v_{r'}$ implies $F^{\sigma}_s(v_r) \geq F^{\sigma}_s(v_{r'})$ for all s. Since, for $\omega \in \mathbf{\Omega}_r$, $F^{\sigma}(v_r) \cdot P_{\omega} > F^{\sigma}(v_{r'}) \cdot P_{\omega}$, it follows that $F^{\sigma}(v_r) > F^{\sigma}(v_{r'})$. Hence, $\alpha_1 < .... < \alpha_R$, and so by Lemma 2, the collection of hyperplanes $\{H(\alpha_r) : r = 1, ..., \Omega\}$ is nested. Now suppose $v(\omega) = v_r < v(\omega')$ for some $r = 1, ..., R$. Since $F^{\sigma}_{\omega}(v_r) \geq \kappa$, it follows that $\alpha_r \cdot P_\omega \geq 0$, and so $P_\omega \in H_+(\alpha_r)$. On the other hand, since $F^{\sigma}_{\omega'}(v_r) < \kappa$, it follows that $\alpha_r \cdot P_{\omega'} < 0$, and so $P_{\omega'} \in \mathring{H}_-(\alpha_r)$. This establishes the desired separation property. Therefore, P satisfies the NHSP and so, by Lemmas 3 and 4, the betweenness property is satisfied. $\square$

A.3.1 Finite auctions

Consider a finite population $\mathcal{I}_n = \mathcal{X}_n \cup \mathcal{Y}_n$, where $\mathcal{X}_n$ are buyers, $\mathcal{Y}_n$ are sellers, and the proportion of buyers is $\kappa \in (0, 1)$. We denote a bid-profile by $a_n : \mathcal{I}_n \to B$. Let $\tilde{X}_{a_n}(b) \equiv \sum_{x \in \mathcal{X}_n} \mathbb{1}[a(x) \leq b]$ denote the number of buyers submitting a bid less than or equal to b; $\tilde{Y}_{a_n}(b) \equiv \sum_{y \in \mathcal{Y}_n} \mathbb{1}[a(y) \leq b]$ denote the number of sellers submitting a bid less than equal to b; and $\tilde{F}_{a_n}(b) = \tilde{X}_{a_n}(b) + \tilde{Y}_{a_n}(b)$ denote the total number of bids less than or equal to b. The price $p(a_n) \equiv \inf\{b \in B : |\mathcal{X}_n| - \tilde{X}_{a_n}(b) \leq \tilde{Y}_{a_n}\}$ is the lowest bid such that number of sellers willing to trade (weakly) exceeds the number of buyers submitting a bid strictly greater than the price. Every buyer with a bid strictly greater than the price, and every seller with a bid strictly less than the price, trades at the price $p(a_n)$. The auctioneer randomizes uniformly over bids equal to the price to maximize volume under the constraint that the market must clear. This defines an allocation-rule $w(i, a_n)$ analogous to the large market. Let $X_{a_n}(b) \equiv \frac{\tilde{X}_{a_n}(b)}{\kappa|\mathcal{I}_n|}$ denote the normalized cumulative distribution of bids by the buyers; $Y_{a_n}(b) \equiv \frac{\tilde{Y}_{a_n}(b)}{(1-\kappa)|\mathcal{I}_n|}$ denote the normalized cumulative distribution of the sellers; and denote the cumulative distribution of the population by $F_{a_n}(b) \equiv \kappa X_{a_n}(b) + (1 - \kappa) Y_{a_n}(b) = \frac{\tilde{F}_{a_n}(b)}{|\mathcal{I}_n|}$. Then $p(a_n) = \inf\{b \in B : \kappa \leq F_{a_n}(b)\}$, and the infimum is attained.

Before providing the proof of theorem 2, we establish some properties of the limiting distribution of prices. A symmetric sequence of strategy-profiles $\{\sigma_n\}_{n=1}^{\infty}$ can be described by a function $\theta : S \to \mathcal{B}$ such that, for every n, $\theta(s) = \sigma_n(j, s)$ for all $j \in \mathcal{I}_n$. Let X^θ_s denote the cumulative distribution over bids for a buyer with signal s, Y^θ_s the cumulative distribution for a seller with signal s, and $F^\theta_s = \kappa X^\theta_s + (1 - \kappa) Y^\theta_s$. Then X^θ, Y^θ, and F^θ are the corresponding vectors functions. For a state ω, $X^\theta_\omega \equiv X^\theta \cdot P_\omega$, $Y^\theta_\omega \equiv Y^\theta \cdot P_\omega$, and $F^\theta_\omega \equiv F^\theta \cdot P_\omega$. Given a symmetric sequence of strategy-profiles $\{\sigma_n\}_{n=1}^{\infty}$, described by θ, a trader $i \in \mathcal{I}_1$ and $\theta_i : S \to \mathcal{B}$, we denote by $\{\sigma^{\theta_i}_n\}_{n=1}^{\infty}$ the sequence of strategy profiles where, for every n, $\sigma^{\theta_i}_n(j, s) = \theta(s)$ for $j \neq i$ and

$\sigma_n^{\theta_i}(i,s) = \theta_i(s)$. Hence, $\sigma_n^{\theta_i}$ is the strategy-profile where other traders continue to follow the symmetric strategy θ and i deviates to the strategy θ_i.

The following lemma characterizes the limiting distribution of prices for a symmetric sequence of strategy-profiles using the de Moivre-Laplace central limit theorem (see, e.g., Shiryaev (1984) pp. 62-63). It also shows that the limiting distribution does not depend on the strategy-profile followed by an individual trader; that is, price-impact vanishes.

Lemma 6. *Let $\{\sigma_n\}_{n=1}^{\infty}$ be a symmetric sequence of strategy-profiles described by $\theta : S \to \mathcal{B}$. Fix a trader $i \in \mathcal{I}_1$, a deviation θ_i for trader i, a state $\omega \in \Omega$, and a bid $b \in B$. (i) If $F_\omega^\theta(b) \leq \kappa$, then $\lim_{n\to\infty} P_\omega^{\sigma_n^{\theta_i}}(p(\sigma_n,\omega) > b) \in \left\{\frac{1}{2}, 1\right\}$, and equal to 1 if and only if $F_\omega^\theta(b) < \kappa$. (ii) If $\vec{F}_\omega^{\sigma_i}(b) \geq \kappa$, then $\lim_{n\to\infty} P_\omega^{\sigma_n^{\theta_i}}(p(\sigma_n,\omega) \leq b) \in \left\{\frac{1}{2}, 1\right\}$, and equal to 1 if and only if $\vec{F}_\omega^{\sigma_n}(b) > \kappa$.*

Proof. We show the argument for part (i), the argument for part (ii) is symmetric. For part (i), if $F_\omega^\sigma(b) = 0$, then for n sufficiently large such that $\frac{1}{|\mathcal{I}_n|} < \kappa$, we have $P_\omega^{\sigma_n^{\theta_i}}\left(\kappa \leq \tilde{F}_{a_n}(b)\right) = 0$ even when $\theta_i(s) = \delta_0$ for all s; hence $P_\omega^{\sigma_n}(p(\sigma_n,\omega) > b) = 1$. We can therefore focus on the case where $0 < F_\omega^\theta(b) \leq \kappa$. Moreover, it is without loss of generality to assume that $\theta_i(s) = \delta_{\tilde{b}}$ for some $\tilde{b} \in B$: if convergence is established for all bids $\tilde{b}$, it holds for any distribution over bids.

Let f be a generic realization of the random variable $\tilde{F}_{a_n}(b)$. For a bid-profile a_n, the price is strictly greater than b if and only if $\tilde{F}_{a_n}(b) \leq \kappa|\mathcal{I}_n| - 1$. Therefore,

$$P_\omega^{\sigma_n}(p(\sigma_n,\omega) > b) = \begin{cases} \sum_{f=0}^{\kappa|\mathcal{I}_n|-2} \binom{|\mathcal{I}_n|-1}{f} F_\omega^\theta(b)^f (1 - F_\omega^\theta(b))^{|\mathcal{I}_n|-1-f} & \text{if } \tilde{b} \leq b \\ \sum_{f=0}^{\kappa|\mathcal{I}_n|-1} \binom{|\mathcal{I}_n|-1}{f} F_\omega^\theta(b)^f (1 - F_\omega^\theta(b))^{|\mathcal{I}_n|-1-f} & \text{if } \tilde{b} > b \end{cases}.$$

By the de Moivre-Laplace central limit theorem, for n sufficiently large,

$$\sum_{f=0}^{\kappa|\mathcal{I}_n|-2} \binom{|\mathcal{I}_n|-1}{f} F_\omega^\theta(b)^f (1 - F_\omega^\theta(b))^{|\mathcal{I}_n|-1-f} \approx \Theta\left(\frac{\kappa|\mathcal{I}_n| - 2 - (|\mathcal{I}_n|-1)F_\omega^\theta(b)}{\sqrt{(|\mathcal{I}_n|-1)F_\omega^\theta(b)(1-F_\omega^\theta(b))}}\right),$$

where Θ is the cumulative distribution function of the standard normal distribution. Likewise,

$$\sum_{f=0}^{\kappa|\mathcal{I}_n|-1} \binom{|\mathcal{I}_n|-1}{f} F_\omega^\theta(b)^f (1 - F_\omega^\theta(b))^{|\mathcal{I}_n|-1-f} \approx \Theta\left(\frac{\kappa|\mathcal{I}_n| - 1 - (|\mathcal{I}_n|-1)F_\omega^\theta(b)}{\sqrt{(|\mathcal{I}_n|-1)F_\omega^\theta(b)(1-F_\omega^\theta(b))}}\right).$$

As a result, $\lim_{n\to\infty} P_\omega^{\sigma_n}\left(p(\sigma_n,\omega)>b\right)=\frac{1}{2}$ if $F_\omega^\theta(b)=\kappa$, and $\lim_{n\to\infty} P_\omega^{\sigma_n}\left(p(\sigma_n,\omega)>b\right) = 1$ if $F_\omega^\theta(b) < \kappa$. □

As a corollary of Lemma 6, we characterize when the prices converge to values.

Corollary 3. *Let $\{\sigma_n\}_{n=1}^\infty$ be a symmetric sequence of strategy-profiles described by $\theta : S \to \mathcal{B}$. Fix a trader $i \in \mathcal{I}_1$, a deviation θ_i for trader i, and a state $\omega \in \Omega$. The sequence of prices $\{p(\sigma_n,\omega)\}_{n=1}^\infty$ converges in probability to $v(\omega)$ if and only if $F_\omega^\theta(v(\omega)-\delta) < \kappa < F_\omega^\theta(v(\omega)+\delta)$ for every $\delta > 0$.*

Proof. (1) Suppose that $F_\omega^\theta(v(\omega)-\delta) < \kappa < F_\omega^\theta(v(\omega)+\delta)$ for every $\delta > 0$. Fix some $\varepsilon > 0$: we want to show that $\lim_{n\to\infty} P_\omega^{\sigma_n^{\theta_i}}\left(p(\sigma_n^{\theta_i},\omega) \in [v(\omega)-\varepsilon, v(\omega)+\varepsilon]\right) = 1$. Because F_ω^θ is monotone non-decreasing, it has a countable number of points of discontinuity. We can therefore choose $\varepsilon' \in (0,\varepsilon]$ such that F_ω^θ is continuous at $v(\omega)+\varepsilon'$ and $v(\omega)-\varepsilon'$. As $F_\omega^\theta(v(\omega)-\varepsilon') < \kappa$, it follows by Lemma 6 that $\lim_{n\to\infty} P_\omega^{\sigma_n^{\theta_i}}\left(p(\sigma_n^{\theta_i},\omega) > v(\omega)-\varepsilon'\right) = 1$. Moreover, because $\kappa > F_\omega^\theta(v(\omega)+\varepsilon') = \vec{F}_\omega^\theta(v(\omega)+\varepsilon')$, it also follows by Lemma 6 that $\lim_{n\to\infty} P_\omega^{\sigma_n^{\theta_i}}\left(p(\sigma_n^{\theta_i},\omega) \leq v(\omega)+\varepsilon'\right) = 1$. Hence, it follows that $\lim_{n\to\infty} P_\omega^{\sigma_n^{\theta_i}}\left(p(\sigma_n^{\theta_i},\omega) \in [v(\omega)-\varepsilon, v(\omega)+\varepsilon]\right) = 1$.

(2) For the converse, suppose that $F_\omega^\theta(v(\omega)+\delta) \leq \kappa$ for some $\delta > 0$. Then by Lemma 6, we have $\lim_{n\to\infty} P_\omega^{\sigma_i^n}\left(p_n^{\sigma_i}(\omega) > v(\omega)+\delta\right) \geq \frac{1}{2}$, and so the price does not converge in probability to $v(\omega)$. On the other hand, if $F_\omega^\theta(v(\omega)-\delta) \geq \kappa$ for some $\delta > 0$, then by Lemma 6, $\lim_{n\to\infty} P_\omega^{\sigma_i^n}\left(p_n^{\sigma_i}(\omega) \leq v(\omega)-\delta\right) \geq \frac{1}{2}$, and so the price does not converge to $v(\omega)$. □

A.3.2 Proof of Proposition 2

Proof. The first step of the proof is to show that a symmetric sequence of strategy-profiles aggregates information asymptotically and approximates equilibrium if and only if the price converges in probability to the value in every state. The sufficiency part follows immediately from Lemma 6, because prices converge for any deviation by trader i; thus, expected payoffs converge to zero for any deviation. The necessity part of the argument is similar to the argument for the large market. The argument for the finite approximation is actually more straightforward as we only need to show that a single trader has a profitable deviation bounded away from zero when prices do not converge in probability to values. We therefore omit this part of the proof. The following two steps then complete the argument.

(1) Suppose P satisfies the betweenness property. Consider the symmetric strategy-profile $\sigma : \mathcal{I} \times S \to \Delta(\mathbf{B})$ constructed in the proof of Theorem 1. This strategy-profile can be described by a function $\theta : S \to \mathcal{B}$ such that $\sigma(i, s) = \theta(s)$ for all $i \in \mathcal{I}$, and $F^{\theta} = F^{\sigma}$. The strategy θ therefore has the property that, for every state ω, $F^{\theta}_{\omega}(v(\omega) - \delta) < \kappa < F^{\theta}_{\omega}(v(\omega) + \delta)$ for every $\delta > 0$. Let $\{\sigma_n\}_{n=1}^{\infty}$ be the symmetric sequence of strategy-profiles described by θ. By Corollary 3, the sequence of prices converges in probability to the value in every state. Hence, $\{\sigma_n\}_{n=1}^{\infty}$ aggregates information asymptotically and therefore approximates equilibrium.

(2) For the converse, suppose that the symmetric sequence of strategy-profiles $\{\sigma_n\}_{n=1}^{\infty}$, described by $\theta : S \to \mathcal{B}$, aggregates information asymptotically and approximates equilibrium. Since prices must converge in probability to the value in every state, by Corollary 3, for every state ω, $F^{\theta}_{\omega}(v(\omega) - \delta) < \kappa < F^{\theta}_{\omega}(v(\omega) + \delta)$ for every $\delta > 0$. Since F^{θ}_{ω} is right-continuous, it follows that $\kappa \leq F^{\theta}_{\omega}(v(\omega))$. Moreover, if $v(\omega') < v(\omega)$, then $F^{\theta}_{\omega}(v(\omega')) < \kappa \leq F^{\theta}_{\omega}(v(\omega))$. Let $\mathbf{\Omega}_1, ..., \mathbf{\Omega}_{R+1}$ be the partition of $\mathbf{\Omega}$ from the proof of Theorem 1. For every $r = 1, ..., R$, let $\alpha_r \equiv F^{\theta}(v_r) - \kappa$. Then the same argument as in the proof of Theorem 1 shows that the collection of hyperplanes $\{H(\alpha_r) : r = 1, ..., R\}$ has the desired nesting and separation properties for the NHSP. Hence, by Lemma 4, P satisfies the betweenness property. $\square$

A.3.3 Proof of Proposition 3

Proof. Consider the Walrasian market described in Section 4.3. With some abuse of notation, we denote a strategy-profile in this market by σ. We assume that assets and wealth are measured in the same units, each buyer has a unit of wealth and each seller owns a unit of the asset. When trader x is a buyer, $\sigma(x, s) : B \to [0, 1]$ is a non-increasing demand schedule, where $\sigma(x, s)[p]$ denotes the quantity of the asset that the buyer demands at price p. It is without loss of generality to restrict $\sigma(x, s)[0] = 1$ and $\sigma(x, s)[\bar{b}] = 0$ for all s because, in equilibrium, buyers always demand one unit at a price of 0 and demand 0 units at a price above the greatest possible value. When trader y is a buyer, $\sigma(y, s) : B \to [0, 1]$ is a non-decreasing supply schedule, where $\sigma(y, s)[p]$ denotes the quantity of the asset that the sellers supplies at price p. It is without loss of generality to restrict $\sigma(y, s)[0] = 0$ and $\sigma(y, s)[\bar{b}] = 1$ for all s.

Again abusing notation, let $X^{\sigma}_s(b) \equiv \frac{1}{\kappa}\int_{\mathcal{X}} \sigma(i, b)\lambda(di)$ and $X^{\sigma} \equiv \left(X^{\sigma}_{s_1}, ..., X^{\sigma}_{s_K}\right)$ be the corresponding vector-valued function that gives the normalized aggregate demand of buyers for each signal. As in the double-sided auction, in state ω, the aggregate demand is given by $X^\sigma_\omega = X^\sigma \cdot P_\omega$. Analogously, Y^σ_s, Y^σ, and Y^σ_ω denote, respectively, the normalized aggregate supply for sellers with signal s, the vector-valued aggregate supply for sellers for each signal, and the aggregate supply in state ω. When we impose the mild technical assumption that X^σ_ω and Y^σ_ω are right-continuous, they have all the properties of the mean cumulative bidding distributions in the double-sided auction, and so the arguments from Theorem 1 can be adapted essentially verbatim to establish that the betweenness property is necessary and sufficient for equilibrium prices to aggregate information. $\square$

A.4 Genericity analysis

Let $\mathcal{P}(K, M)$ denote the set of all the information structures $\{P_\omega : \omega \in \Omega\}$ (or $\{P_\omega\}$ for short) such that P_ω has full support for every ω. An information structure $\{P_\omega\} \in \mathcal{P}(K, M)$ can be identified with a real $K \times M$ matrix, where column m represents the distribution over signals conditional on state ω_m. Let $\mathcal{P}^{BP}(K, M) \subset \mathcal{P}(K, M)$ be the subset of information structures that satisfies the betweenness property and $\mathcal{P}^{MLRP}(K, M) \subset \mathcal{P}(K, M)$ be the set of information structures that satisfy the MLRP. Let μ_{KM} denote the Lebesgue measure on $\mathbb{R}^{(K-1)M}$. When K and M are clear from the context, we omit the subscript.

The sets $\mathcal{P}$ and $\mathcal{P}^{BP}$ are open in $\mathbb{R}^{(K-1)M}$ and therefore Lebesgue measurable. The boundary of $\mathcal{P}^{MLRP}$ (the set difference between the closure of $\mathcal{P}^{MLRP}$ and $\mathcal{P}^{MLRP}$) is a Lebesgue null-set, and so $\mathcal{P}^{MLRP}$ is measurable. Using Fubini Theorem, it is straightforward to show that $\mu(\mathcal{P}) = 1$. This describes how we quantify the betweenness property and MLRP relative to all information structures.

For the multi-input environments, let $\mathcal{P}\left(\{K_c, M_c\}_{c=1}^C\right) \subset \mathcal{P}\left(\sum_{c=1}^C K_c, \prod_{c=1}^C M_c\right)$ denote the set of multi-input information structures with C inputs, satisfying conditions (1) and (2) in Definition 7. By condition (2), inputs are independent. A multi-input information structure $\{P_\omega\} \in \mathcal{P}\left(\{K_c, M_c\}_{c=1}^C\right)$ is therefore fully described by a $\gamma = (\gamma_1, ..., \gamma_C) \in \Delta^C$ and a collection $\left\{\{P_{\omega_c}\}_{c=1}^C\right\}$, where, for every input $c = 1, ..., C$, $\gamma_c = P(s \in S_c)$ is the probability of receiving a signal on input c and $P_{\omega_c} \in \Delta(S_c)$ is the conditional distribution on signals for input c conditional on the c-th dimension of the state. To measure sets of multi-input environments, we fix the number of inputs C, the number of states and signals per input $\{K_c, M_c\}_{c=1}^C$, and the likelihood of receiving a signal for each input γ, and denote by $\mathcal{P}\left(\{K_c, M_c\}_{c=1}^C, \gamma\right)$ the corresponding subset of multi-input information structures. For a subset $\mathcal{P}' \subset \mathcal{P}\left(\{K_c, M_c\}_{c=1}^C, \gamma\right)$, we take every information structure $\{P_\omega\} \in \mathcal{P}'$ and associate its vector $(\{P_{\omega_1} : \omega_1 \in \Omega_1\}, ..., \{P_{\omega_C} : \omega_c \in \Omega_c\})$. In this way, $\mathcal{P}'$ defines a vector of subsets of information structures for each input $(\mathcal{P}'_1, ..., \mathcal{P}'_C)$. We then measure $\mathcal{P}'$ by $\mu^\gamma_{\{K_c M_c\}_{c=1}^C}(\mathcal{P}') = \sum_{c=1}^C \gamma_c \, \mu_{K_c M_c}(\mathcal{P}'_c)$. Note that

$$\mu^\gamma_{\{K_c M_c\}_{c=1}^C}(\mathcal{P}')\left(\mathcal{P}\left(\{K_c, M_c\}_{c=1}^C, \gamma\right)\right) = \sum_{c=1}^C \gamma_c \, \mu_{K_c M_c}\left(\mathcal{P}(K_c M_c)\right) = \sum_{c=1}^C \gamma_c = 1,$$

i.e., the set of all multi-input environments with C inputs, a fixed number of states and signals per input $\{K_c, M_c\}_{c=1}^C$, and a fixed marginal γ over the signals for inputs, has measure 1. Let $\mathcal{P}^{BP}\left(\{K_c, M_c\}_{c=1}^C, \gamma\right) \subset \mathcal{P}\left(\{K_c, M_c\}_{c=1}^C, \gamma\right)$ denote the subset of multi-input information structures, with fixed γ, which satisfy the betweenness property. As this set is open, it is measurable with respect to $\mu^\gamma_{\{K_c M_c\}_{c=1}^C}$.

A.4.1 Proof of Proposition 4

Proof. (1) We first show that $K \geq M$ implies that $\mu_{KM}(\mathcal{P}^{BP}(K, M)) = 1$. It is well-known that $\mu_{KM}\Big(\{(z_1, ..., z_M) : z_1, ..., z_M \in \mathbb{R}^K \text{ are linearly independent}\}\Big) = 1$. Now fix $P_{\omega_1}, ..., P_{\omega_M} \in \Delta(S)$ linearly independent vectors, and choose a vector $\beta \in \mathbb{R}^K$, with $\beta_i < \beta_j$ if $\omega_i < \omega_j$. Define $z_i(j) \equiv P_{\omega_i}(j)$ for all $j \leq M$, and $z_i(j)$ be arbitrary for $j = M+1, ..., K$ such that $Z = (z_1, ..., z_K)$ is invertible. Define $\alpha_\beta = \beta Z^{-1}$. By construction, $\beta_m = \alpha_\beta \cdot P_{\omega_m}$, and thus, $U(P_\omega) = \alpha_b \cdot P_\omega$ is consistent with the EU property (defined by α_b), and therefore the betweenness property as well. Then, $\mu_{KM}(\mathcal{P}^{BP}(K, M)) = 1$.

(2) We now show that $M > K$ implies $\mu_{KM}(\mathcal{P}^{BP}(K, M)) < 1$. To do so, we show that, when $M > K$, there is a strictly positive mass of information structures where the conditional distribution for a higher value is in the convex hull of the conditional distributions for lower values. This convex containment is inconsistent with the betweenness property. Hence, when $M > K$, there is a strictly positive mass of information structures that do not satisfy the betweenness property. It is understood in the following that $\mu = \mu_{KM}$. Fix $\theta \leq \frac{1}{K}$. For $j = 1, ..., K$, let $A_j = \{z \in \Delta^K : z(j) \geq 1 - \theta\}$, and $A_{K+1} = \{z \in \Delta^K : z(j) \geq \theta \text{ for all } i\}$. If $z_j \in A_j$ for $j = 1, ..., K$, then

$A_{K+1} \in \text{co}(\bigcup_{j=1}^{K} z_j)$. When $\theta = 1$, $\mu(A_{K+1}) = 0 < \mu(A_j)$. When $\theta = 0$, $\mu(A_j) = 0 < \mu(A_{K+1})$. Then, there exists $\bar{\theta}$ such that $\mu(A_j) = \mu(A_{K+1})$. Draw an arbitrary information structure $\{P_\omega : \omega \in \Omega\}$ and let E be the event that, for each $j = 1, ..., K+1$, there exists ω such that $P_\omega \in A_j$. A direct application of the multinomial formula implies that

$$P(E) = \sum_{y_1=1}^{M-K} \sum_{y_2=1}^{M-(K-1)-y_1} \ldots \sum_{y_{K+1}=1}^{M-\sum_{k=1}^{K} y_k} \binom{M}{y_1 \ldots y_{K+1}} f\left(\sum_{k=1}^{K+1} y_k, \bar{\theta}\right),$$

where $\binom{M}{y_1 \ldots y_{K+1}} \equiv \frac{M!}{y_1! \ldots y_{K+1}!(M - \sum_{k=1}^{K+1} y_k)!}$ is the multinomial coefficient and $f(\bar{y}, \bar{\theta}) \equiv \bar{\theta}^{\bar{y}}(1-(K+1)\bar{\theta})^{M-\bar{y}}$. Clearly, $P(E) > 0$. There is a finite number of ways to assign P_ω's to A_j's, and all of them with the same (positive) measure. In particular, this implies that there exists an event E' with $P(E') > 0$, where for $v(\omega_1) < \ldots < v(\omega_{K+1})$, we have $P_{\omega_i} \in A_i$ for all $i < K$, $P_{\omega_K} \in A_{K+1}$, $P_{\omega_{K+1}} \in A_K$, and $P(E') > 0$. In the event E', $P_{\omega_K} \in \text{co}(\bigcup_{i \neq K} P_{\omega_i})$, so the betweenness property is not satisfied. □

A.4.2 Proof of Proposition 5

Proof. For any (M, K) with $K \geq 2$,

$$\mu_{KM}\left(\left\{\{P_\omega\} : \frac{P_\omega(s)}{P_\omega(s')} = \frac{P_{\omega'}(s)}{P_{\omega'}(s)} \text{ for some } P_w, P_{\omega'}, s, s'\right\}\right) = 0,$$

because the equality restriction defines a lower dimensional set. Now fix $s, s' \in S$ and define an equivalence $\sim$ as follows: $P \sim P' \implies \frac{P_\omega(s)}{P_\omega(s')} > \frac{P_{\omega'}(s)}{P_{\omega'}(s')} \iff \frac{P'_\omega(s)}{P'_\omega(s')} > \frac{P'_{\omega'}(s)}{P'_{\omega'}(s')}$ for all ω, ω'. An equivalence class for a distribution P on $\Omega \times S$ is denoted by $[P] = \{P' : P \sim P'\}$. There are M distinct states in Ω and therefore, there are $M!$ distinct equivalence classes, one for each possible strict ordering on the likelihood-ratios $\frac{P_\omega(s)}{P_\omega(s')}$. Then, $\mu_{K \times M}([P]) = \frac{1}{M!}$ for all P. There are only two equivalence classes that are consistent with the MLRP, namely: $[P] = \{P' : \frac{P'_\omega(s)}{P'_\omega(s')} > \frac{P'_{\omega'}(s)}{P'_{\omega'}(s')} \forall v(\omega) > v(\omega')\}$, and $[\hat{P}] = \{P' : \frac{P'_\omega(s)}{P'_\omega(s')} < \frac{P'_{\omega'}(s)}{P'_{\omega'}(s')} \forall v(\omega) > v(\omega')\}$. Then, the measure of information structures that satisfies MLRP, $\mu_{KM}\left(\mathcal{P}^{MLRP}(K, M)\right) \leq \mu_{KM}([P] \cup [\hat{P}]) = \frac{2}{M!}$. □

A.4.3 Proof of Proposition 6

Proof. Suppose (Ω, S, P, v) is a multi-input environment, and let c and d be non-trivial inputs. Then, there exists $\omega, \omega', \omega''$, and $\hat{\omega}_c, \tilde{\omega}_c$, $\hat{\omega}_d, \tilde{\omega}_d$ such that $v(\omega) > v(\omega')$, and $v(\omega) > v(\omega'')$, where $\omega_c = \omega''_c = \hat{\omega}_c$, $\omega_d = \omega'_d = \hat{\omega}_d$, $\omega'_c = \tilde{\omega}_c$, $\omega''_d = \tilde{\omega}_d$, and $\omega_i = \omega'_i = \omega''_i$ for all $i \neq c, d$. We need to consider three cases.

(1) Suppose there are $s_c \in S_c$, $s_d \in S_d$ such that $P_\omega(s_c) > P_{\omega'}(s_c)$ and $P_\omega(s_d) > P_{\omega''}(s_d)$. By condition (2i), $P_\omega(s_d) = P_{\omega'}(s_d)$, and $P_\omega(s_c) = P_{\omega''}(s_c)$. Then, $\frac{P_{\omega''}(s_c)}{P_{\omega''}(s_d)} > \frac{P_\omega(s_c)}{P_\omega(s_d)} > \frac{P_{\omega'}(s_c)}{P_{\omega'}(s_d)}$. Since $v(\omega) > v(\omega')$, and $v(\omega) > v(\omega'')$, the MLRP fails.

(2) Suppose there is no $s_c \in S_c$ such such that $P_\omega(s_c) > P_{\omega'}(s_c)$. Then it must be the case that $P_\omega(s_c) = P_{\omega'}(s_c)$ for all $s_c \in S_c$ By condition (2i) in Definition 7, $\frac{P_\omega(s)}{P_\omega(s')} = \frac{P_{\omega'}(s)}{P_{\omega'}(s')}$, for all $s, s' \in S$. Since $v(\omega) > v(\omega')$ the MLRP fails.

(3) Finally, the case where there is no $s_d \in S_d$ such that $P_\omega(s_d) > P_{\omega''}(s_d)$ is analogous to case (2), establishing the result. □

A.4.4 Proof of Proposition 7

Proof. Consider the set of multi-input environments $\mathcal{P}\left(\{K_c, M_c\}_{c=1}^C, \gamma\right)$ with C inputs, $\{K_c, M_c\}_{c=1}^C$ signals and states per input, and a fixed γ such that $P(s \in S_c) = \gamma_c$ for every input c and every joint distribution on $\Omega \times S$. Fix a value function $v : \Omega \to \mathbb{R}_{++}$ satisfying condition (3) in Definition 7 for some strictly increasing $\psi : \mathbb{R}_{++} \to \mathbb{R}_{++}$ and collection of injective functions $\{\phi_c : \Omega \to \mathbb{R}_{++}\}_{c=1}^C$.

By analogous reasoning as in the proof of Theorem 4, for $c = 1, .., C$, $M_c \leq K_c$ if and only if $\mu_{K_c M_c}\left(\left\{\left(P_{\omega_c^1}, ..., P_{\omega_c^{M_c}}\right) \text{linearly independent}\right\}\right) = 1$. For $c = 1, ..., C$, fix $P_{\omega_c^1}, ..., P_{\omega_c^{M_c}} \in \Delta(S_c)$ linearly independent vectors. Define (i) $z_c^i(j) \equiv P_{\omega_c^i}(s_j)$ and, if $M_c < K_c$, let $z_c^i(j)$ be arbitrary for $j = M_c + 1, ..., K_c$ such that $Z = (z_c^1, ..., z_c^{K_c})$ is invertible; (ii) $b_c = (b_c^1, ..., b_c^{K_C}) \in \mathbb{R}^{K_C}$ with $b_c^i = \phi_c(\omega_c^i)/\gamma_c$; (iii) $\alpha_c = b_c Z^{-1}$; and (iv) $\alpha = (\alpha'_1, ..., \alpha'_C)'$, where $'$ is the transpose operator. Then, for all $\omega = (\omega_1, ..., \omega_C) \in \Omega$, $\alpha \cdot P_\omega = \sum_{c=1}^C \alpha_c P_{\omega_c} \gamma_c = \sum_{c=1}^C \phi_c(\omega_c) = \psi(v(\omega))$. Since $\psi(.)$ is strictly increasing in $v(.)$, $U(P_\omega) = \alpha \cdot P_\omega$ is consistent with an EU order (defined by α). As a result, the EU property is satisfied, which implies the betweenness property.

For the converse, suppose there is a characteristic c' such that $M_{c'} > K_{c'}$. By relabelling characteristics as needed, it is without loss of generality to assume that c' is the first characteristic. Let $\mathcal{P}_1 \equiv \{P_{\omega_1} : \omega_1 \in \Omega_1\}$ be the set of all information structures restricted to the first characteristic. For each of the remaining characteristics $c = 2, ..., C$, fix some arbitrary information structure P_{ω_c}. By part (2) of the proof of Theorem 4, there is a subset $\mathcal{P}_1' \subset \mathcal{P}_1$ with $\mu^{\gamma}_{\{K_1 M_1\}}(\mathcal{P}_1') > 0$ such the conditional distribution for a higher value of the first characteristic is in the convex hull of the conditional distributions for lower values of the first characteristic. Fix some state for each of the remaining characteristic $\bar{\omega}_{-1} = (\bar{\omega}_2, ..., \bar{\omega}_C)$, and let $\bar{\mathcal{P}} \equiv \left\{(P_{\omega_1}, P_{\bar{\omega}_2}, ..., P_{\bar{\omega}_C}) : P_{\omega_1} \in \mathcal{P}_{[1]}'\right\}$. By the separability condition of the value function (property (3) of Definition 7), for each information structure $P_\omega \in \bar{\mathcal{P}}$ the conditional distribution for a higher value of the asset is in the convex hull of the conditional distributions for lower values of the asset. This means that the betweenness property is not satisfied. Moreover, by independence of the characteristics (property (2) of Definition 7), $\mu^{\gamma}_{\{K_c M_c\}_{c=1}^{C}}(\bar{\mathcal{P}}) = \gamma_{c'} \mu^{\gamma}_{\{K_1 M_1\}}(\mathcal{P}_1') > 0$. □

References

Al-Najjar, N. I. (2008): "Large Games and the Law of Large Numbers," *Games and Economic Behavior*, 64, 1 – 34.

Atakan, A. E. and M. Ekmekci (2014): "Auctions, Actions, and the Failure of Information Aggregation," *American Economic Review*, 104, 2014–48.

Aumann, R. J. (1964): "Markets with a Continuum of Traders," *Econometrica*, 32, pp. 39–50.

Austen-Smith, D. and J. S. Banks (1996): "Information Aggregation, Rationality, and the Condorcet Jury Theorem," *American Political Science Review*, 90, 34–45.

Barelli, P., S. Bhattacharaya, and L. Siga (2018): "Full Information Equivalence in Large Elections," Working Paper.

Bodoh-Creed, A. (2013): "Efficiency and Information Aggregation in Large Uniform-price Auctions," *Journal of Economic Theory*, 148, 2436 – 2466.

Chew, S. H. (1983): "A Generalization of the Quasilinear Mean with Applications to the Measurement of Income Inequality and Decision Theory Resolving the Allais Paradox," *Econometrica*, 1065–1092.

Condorcet, M. d. (1785): "Essai sur l'Application de l'Analyse a la Probabilité des Decisions Rendues a la Pluralité des Voix." Trans. McLean Iain and Hewitt Fiona, 1994. Paris.

Dekel, E. (1986): "An Axiomatic Characterization of Preferences under Uncertainty: Weakening the Independence Axiom," *Journal of Economic Theory*, 40, 304 – 318.

Feddersen, T. J. and W. Pesendorfer (1997): "Voting Behavior and Information Aggregation in Elections with Private Information," *Econometrica*, 65, 1029–1058.

Harstad, R. M., A. S. Pekeč, and I. Tsetlin (2008): "Information Aggregation in Auctions with an Unknown Number of Bidders," *Games and Economic Behavior*, 62, 476–508.

Hellwig, M. F. (1980): "On the Aggregation of Information in Competitive Markets," *Journal of Economic Theory*, 22, 477 – 498.

Jackson, M. (2003): "Efficiency and Information Aggregation in Auctions with Costly Information," *Review of Economic Design*, 8, 121–141.

Judd, K. L. (1985): "The Law of Large Numbers with a Continuum of IID Random Variables," *Journal of Economic Theory*, 35, 19–25.

Kremer, I. (2002): "Information Aggregation in Common Value Auctions," *Econometrica*, 70, 1675–1682.

Kyle, A. S. (1985): "Continuous Auctions and Insider Trading," *Econometrica*, 53, 1315–1335.

——— (1989): "Informed Speculation with Imperfect Competition," *The Review of Economic Studies*, 56, 317–355.

Lambert, N. S., M. Ostrovsky, and M. Panov (2018): "Strategic Trading in Informationally Complex Environments," *Econometrica (forthcoming)*.

Lauermann, S. and A. Wolinsky (2017): "Bidder Solicitation, Adverse Selection, and the Failure of Competition," *American Economic Review*, 107, 1399–1429.

Milgrom, P. R. (1979): "A Convergence Theorem for Competitive Bidding with Differential Information," *Econometrica*, 47, 679–88.

——— (1981): “Rational Expectations, Information Acquisition, and Competitive Bidding,” *Econometrica*, 49, 921–43.

Milgrom, P. R. and R. J. Weber (1982): “A Theory of Auctions and Competitive Bidding,” *Econometrica*, 50, 1089–1122.

Palfrey, T. R. (1985): “Uncertainty Resolution, Private Information Aggregation and the Cournot Competitive Limit,” *The Review of Economic Studies*, 52, 69–83.

Pesendorfer, W. and J. M. Swinkels (1997): “The Loser’s Curse and Information Aggregation in Common Value Auctions,” *Econometrica*, 65, 1247–1282.

Reny, P. J. and M. Perry (2006): “Toward a Strategic Foundation for Rational Expectations Equilibrium,” *Econometrica*, 74, 1231–1269.

Serrano-Padial, R. (2012): “Naive Traders and Mispricing in Prediction Markets,” *Journal of Economic Theory*, 147, 1882 – 1912.

Shiryaev, A. N. (1984): *Probability*, Springer, second edition ed.

Tolvanen, J. and E. Soultanis (2012): “A correction to “Large games and the law of large numbers” [Games Econom. Behav. 64 (2008) 1–34],” *Games and Economic Behavior*, 76, 340 – 343.

Vives, X. (2011): “Strategic Supply Function Competition With Private Information,” *Econometrica*, 79, 1919–1966.

——— (2014): “On the Possibility of Informationally Efficient Markets,” *Journal of the European Economic Association*, 12, 1200–1239.

von Neumann, J. and O. Morgenstern (1944): *Theory of Games and Economic Behavior*, Princeton University Press.

Wilson, R. (1977): “A Bidding Model of Perfect Competition,” *Review of Economic Studies*, 44, 511–18.

Wolinsky, A. (1990): “Information Revelation in a Market with Pairwise Meetings,” *Econometrica*, 58, 1–23.