History-Based Choice between Consumption Streams

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Abstract

The empirical and experimental research reveals that an agent may manifest preferences which differ from the classical economics postulates. A few of such manifestations are utility from anticipation, preferences for improvement, preferences for happy endings and memorable consumptions. This paper studies those phenomena by static choices within a dynamic context. This research provides an axiomatic framework and a model which rationalises such decisions; furthermore, it shows that there is an additive utility function which represents the preferences with those specifications.

Keywords: Consumption streams, History-based representation, Diagonal independence, Constant equivalence, Additivity.

1 Introduction

Sometimes we make a decision which concerns not only to the current period but also to the upcoming periods. In other words, we choose between the sequences of alternatives, where each option in the given sequence is affordable for the consumption in the predefined further period. This kind of decisions, which correspondents to the static choice in the dynamic context, has two main challenges. The first is the uncertainty concerning later preferences, and the second, which is the primary interest of this study, is the intertemporal trade-off.

The traditional economic models of intertemporal choice, like, exponential discounting (Koopmans 1960 [4], Samuelson 1937 [12]), hyperbolic discounting (Ainslie 1992 [1], Laibson 1997 [6]), present-biased preferences (O'Donoghue and Rabin [10]) etc. assume that given two similar rewards, people always prefer reward which arrives sooner rather than later. Even though this assumption

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is quite realistic and logical, there is an experimental and empirical evidence which shows that people, sometimes, prefer a reward which arrives later rather than sooner. A few of main causes for such preferences are: utility from anticipation (Loewenstein 1987 [7]), preferences for improvement (Loewenstein and Prelec [9],[8]), preferences for happy endings (Ross and Simonson 1991 [11]) and memorable consumptions (Gilboa, Postlewaite and Samuelson [3]).

According to the aforementioned literature, the decision maker (DM) could demonstrate patience or impatience. In other terms, the DM can prefer consumption sequences with the increasing, decreasing, constant, or volatile gratification tendencies. Hence, there is no unique pattern, according to which the DM will prefer any defined order of the alternatives in the sequence. These unstable preferences induce difficulties to modelling the DM's preferences over sequences, and rationalize those kind of preferences. This study aims to develop a behavioral foundation to fill the gap between the economic theory and such empirical and experimental evidence.

The profound investigation of the mentioned behavioral manifestation sheds light on one of the the main reasons of those kind preferences. That is, the dependence between the choice alternatives over the periods. The meaning is that the DM's choice at some given period depends on her choices before or after that period, i.e., we don't have independence between the choices along periods.

To formalise all above-mentioned, assume that the DM have a preference relation over the set of sequences. Each sequence is an array of consumtion goods, where the consumption of each option takes place in a predefined order. For example, assume that the DM has to decide how to spend/consume her budget during the following n periodes. Suppose the amount of budget is B($\in R_+$), so she has to opt a tuple from $\{(c_1, c_2, ..., c_n) \in R_+^7 : c_1 + c_2 + ... + c_n \leq B\}$ set, where c_1 is her consumption at the first period, c_2 at the period, and so on.

The representation result given (1) functional form justifies aforementioned behavioral phenomena.

$$U(c_1, c_2, ..., c_n) = u_1(c_1) + u_2(c_2, c_1) + ... + u_n(c_n, c_1, c_2, ..., c_{n-1}) \qquad (1)$$

In this model the utility of sequence $(c_1, c_2, ..., c_n)$ is defined by the collection of the utility functions, $\{u_1, u_2, ..., u_n\}$, where each utility, u_i, is the present value from the consumption at the corresponding period, i. Moreover, in this model the DM's utility depends not only on the consumption of that period, but also the consumptions before that period. The intuition of this representation is that today's consumption could change the DM's tomorrow's preferences. That is, the DM's consumption in each period is a reference point for the further periods' preferences, so, for a given period, any previous consumption of the DM, has a potential effect on her current consumption.

Now I would like to recall one experimental result given by Loewenstein (1987 [7]), which shows that preferences over the sequences could be incompatible with the independence axiom (and additive separability). Such preferences cannot be representable by the discounting utility models; however, the model suggested here represents this choice structure.

The nature of the experiment is the following. Thirty-seven undergraduates were presented with the following two questionary.

Question 1. Which would you prefer?

Alternativs	This weekend	Next weekend	Two weekends from now	Choices
A	Fancy French	Eat at home	Eat at home	16%
B	Eat at home	Fancy French	Eat at home	84%

Question 2. Which would you prefer?

Alternatives	This weekend	Next weekend	Two weekends from now	Choices
C	Fancy French	Eat at home	Fancy lobster	54%
D	Eat at home	Fancy French	Fancy lobster	46%

From *Question 1* we have that when the third-period option is "eat at home" (less pleasurable alternative in this context), then majority of the students prefer to have a dinner at the Fancy French restaurant in the second period than at the first period. However, when there is a lobster dinner (more pleasurable alternative in this context) at the third period, then the French restaurant is preferable to be chosen at the first period rather than at the second period. Hence, the third-period option which seems has nothing to do with other periods, changes the DM's preference order.

Now let us make sure that the aforementioned model, (1), is consistent with this preference relation and rationalizes the preference reversal. To this end, let u_1, u_2 and u_3 be utility functions for the first, the second and the third periods respectively. Consider two cases. First, assume that there is no lobster dinner at the third period. We will have:

$$U(French, home, home) < U(home, French, home). \tag{2}$$

Applying equation (1) we get

$$\begin{aligned} u_1(French) + u_2(home, French) + u_3(home, French, home) < \\ u_1(home) + u_2(French, home) + u_3(home, home, French). \end{aligned} \tag{3}$$

Second, assume there is a lobster dinner at the third period. In this case we have the following result:

$$U(French, home, lobster) > U(home, French, lobster) \tag{4}$$

Applying (1), we have

$$\begin{aligned} u_1(French) + u_2(home, French) + u_3(lobster, French, home) > \\ u_1(home) + u_2(French, home) + u_3(lobster, home, French). \end{aligned} \tag{5}$$

Note, that in (3) and (5) the utilities u_1 and u_2 are providing identical outcomes. So, in order to have reversed preferences u_3 must satisfy (6), which is mathematically achievable and has a logical background.

$$\begin{aligned} u_3(lobster, French, home) - u_3(lobster, home, French) > \\ u_3(home, French, home) - u_3(home, home, French). \end{aligned} \tag{6}$$

This means that the fancy lobster dinner (comparing to a dinner at home) is more preferable when the previous two activities where the dinner at the fancy French restaurant and the dinner at home than, the dinner at home and the dinner at the fancy French restaurant respectively. This assures that the DM obtain various levels satisfactions, depending on the previous consumptions.

The rest of the paper has the following structure. In the next section, I develop an axiomatic framework for the preference relation on the set of the sequences. In section 3, I define a few basic properties and introduce a representation result. The last section is the conclusion.

2 Preliminaries and Axioms

Let (Z, d) be a nonempty, compact metric space, where Z is the set of prizes, and let X denote the set of all probability distributions on Z. In other words, X is the set of all lotteries on the prize set Z. Hence, X is a connected, compact and metric space (with Prokhorov metric). The typical elements of X, the lotteries, are denoted by $x, x_i, \bar{x}, x', y, etc.$. Let $A_i := X \times ... \times X$ be a set of all ordered sequences of length i $[= 1, ..., n]$[1], endowed with a product metric. For notational convenience I will use A instead of A_n. Let A_i^c be the set of all constant[2] sequences with length i, and A^c the set of all constant sequences with length n. Suppose $(x_1, ..., x_n)$ and $(y_1, ..., y_n)$ are two arbitrary sequences in A, I will define the mixture between them as a sequence of compound lotteries

$$\alpha(x_1, ..., x_n) + (1 - \alpha)(y_1, ..., y_n) := (\alpha x_1 + (1 - \alpha)y_1, ..., \alpha x_n + (1 - \alpha)y_n).$$

for all $\alpha \in [0, 1]$.

[1] n is finite, positive integer.
[2] A sequence is constant if all terms of the sequence are equal to each other.

Assume that the DM has a preference relation on A, denoted by $\succeq$, with a symmetric $\sim$ and an asymmetric $\succ$ parts. In the representation result I will require the following four axioms stated on $\succeq$ preference relation. The first two axioms are well known from the classic axiomatic works, they are: Preference Relation and Continuity.

Axiom 1 (Preference Relation)**.** $\succeq$ is a complete and transitive binary relation.

Axiom 2 (Continuity)**.** For all $(x_1, ..., x_n), (y_1, ..., y_n) \in A$, the sets $\{(y_1, ..., y_n) : (y_1, ..., y_n) \succeq (x_1, ..., x_n)\}$ and $\{(y_1, ..., y_n) : (x_1, ..., x_n) \succeq (y_1, ..., y_n)\}$ are closed.

I will skip the interpretation of these two axioms since it is the same as in the common decision theoretic literature. The next two axioms are a bit different than in the common texts and adjusted for this representation.

Axiom 3 (Diagonal Independence)**.** For every $(x, ..., x), (y, ..., y), (z, ..., z) \in A^c$ and every $\alpha \in (0, 1)$,

$$(y, ..., y) \succeq (x, ..., x) \text{ iff } \alpha(y, ..., y)+(1-\alpha)(z, ..., z) \succeq \alpha(x, ..., x)+(1-\alpha)(z, ..., z).$$

Diagonal Independence is a common independence axiom stated merely for the constant sequences. In words, it means that if the DM has to choose only between the constant sequences, that is, sequences that provide the same alternative for all periods, then her preferences should satisfy independence axiom in a classical sense. In the meantime, there is not any restriction on the preference relation out of the set of constant sequences. The relaxation of the independence axiom, in this way, allows us to go along with the DM's "exotic" preferences presented in the introduction.

Axiom 4 (Constant Equivalence)**.** For every $(x_1, ..., x_n) \in A$, there is $(x, ..., x) \in A^c$ such that $(x_1, ..., x_n) \sim (x, ..., x)$.

This axiom is a technical constraint and states that each sequence has a constant equivalence, that is, for each sequence there is a sequence on the diagonal such that the DM is indifferent between those two sequences.

3 History-Based Representation

This work aims to provide a representation result which is synchronised with the phenomena discussed in the introduction. The following additive functional form defines the utility of the sequence as a sum of utilities depending on current and previous choices.

$$U(x_1, ..., x_n) = u_1(x_1) + ... + u_n(x_n, x_1, ..., x_{n-1}) \tag{7}$$

As I have shown with an example, in the introduction, (7) model is a simple and intuitive result which allows to represent preferences which manifest utility from anticipation and memory. Furthermore, (7) represents the preference relation via $\{u_1, ..., u_n\}$ collection of "periodic" utility functions.

Definition 1. $\{u_1, ..., u_n\}$ is called *a collection generated by* U if $\{u_1, ..., u_n\}$ defines U as in (7) and $u_i \geq 0$ for all $i = 1, ..., n$.

Let G_U *be the set of all collections generated by* U.

Definition 2. Given (7) representation, a collection of nonnegative functions, $\{u_1, ..., u_n\}$, is *a maximum splitter* if $\{u_1, ..., u_n\} \in G_U$ and for every $i = 1, ..., n$ there is no collection $\{u_1, ..., u_{i-1}, u'_i, ..., u'_n\} \in G_U$ such that $u'_i(x_i, x_1, ..., x_{i-1}) > u_i(x_i, x_1, ..., x_{i-1})$ for some $(x_1, ..., x_i) \in A_i$.

In words, there is not another collection which is generated by the same U and the first (by index) differ utility function, in the collection, returns higher value for some sequence. Note, that according to (7) model, the DM's utility at a given period depends on consumption at that period and periods before that, and is independent of the future consumptions, i.e., we deal with a one-sided dependence. In this context, the maximum splitter property provides a collection of utility functions which are the best revelation of a given period's utility independently from the future (after the given period) consumptions.

Definition 3. *A history-based additive representation* of $\succeq$ relation is a collection of nonnegative, continuous utility functions $\{u_1, ..., u_n\}$, where $u_i : A_i \to \mathbb{R}$ $i = 1, ..., n$, such that (a) function $U : A \to \mathbb{R}$, defined by (7), is continuous and represents $\succeq$ relation, and (b) $\{u_1, ..., u_n\}$ is a maximum splitter.

The history-based additive representation is the fundamental concept of this work. It provides an alternative way to model preferences over sequences as a sum of utilities assigned to each period consumption. In this model, each period's utility is a state-dependent, and each state is a vector with the previous consumptions. The last allows us to be more flexible to represent preferences which are not consistent with the changes in other periods' choices. The following theorem gives necessary and sufficient conditions for this representation; also, it proves that under some sufficient conditions the representation has a particular uniqueness property.

Theorem 1. *A. The preference relation* $\succeq$ *defined on* A *has a history-based additive representation if and only if it satisfies Preference Relation and Continuity.*

B. Moreover, if $\succeq$ *satisfies Diagonal Independence and Constant Equivalence as well then the functions* u_i $(i = 1, ..., n)$ *have the following uniqueness properties:*

Given that $\{u_1, ..., u_n\}$ *collection is a history based additive representation of* $\succeq$, $\{u'_1, ..., u'_n\}$ *collection also will be a history based additive representation of* $\succeq$ *if and only if there are* $a > 0$, $b > 0$, *such that* $u'_1 = au_1 + b$ *and* $u'_i = au_i$ *for all* $i \in \overline{2, n}$.

This theorem has two principal contributions. The first enrichment of the model is the additive representation of the preference relation over the sequences, through the collection of the continuous functions which has maximum splitter property. This functional form allows us to represent the utility of a sequence as a sum of utilities assigned to each period. Each utility is defined by the state dependent utility function and each state is a sequence of consumptions made before that period. The second enrichment relates to the second part of the theorem. This lets us to have a cardinal ranking over the set of sequences without requiring the independence axiom on all set of sequences. Consequently it gives a uniqueness property for the collection of utility functions which represents the preference relation.

The formal proof of the theorem is given in the appendix. However here I will give the intuition and the sketch of the proof.

The A part of the theorem shows the existence of the history-based additive representation. The necessity of the axioms are obvious. For the sufficiency result first I will show that according to Debreu's theorem there is a continuous utility function which represents $\succeq$ relation. Then based on that function I will generate a collection of utility functions through OSMS method (Appendix, Step 2). The rest of the prove is to show that the generated collection is a history-based additive representation.

The proof of the B part is mainly based on the three lemmata. In addition the first lemma is quite interesting and important result per se. It states that the preference relation which satisfies axioms 1-4 has is representable with a function which is unique up to affine transformation, i.e., it provides a cardinal ranking of the preference relation. The second lemma gives the uniqueness of the maximum splitter collection and the third lemma states one to one correspondence between history-based additive representation collection and the collection derived by the OSMS method. The proof of the theorem part B is following from these three results.

4 Conclusion

Utility from anticipation, preferences for improvement, preferences for happy endings and memorable consumptions are empirically and/or experimentally tested behavioral phenomena, which show a significant gap between the conventional economic models and the reality. This work rationalizes those behavioral manifestations via history-based representation model. Based on this model, the utility over the sequences is represented by the additive form of state-dependent utility functions, where each utility function depends on the consumption at that period and the consumptions before that period, a state. The state-dependent structure let us consider all previous consumptions, which form a reference point for the DM preferences, such that, her current consumption could be strongly affected by that reference point. This work improves the existing literature in this manner and provides the behavioral foundation, which fills the gap between the theory and the empirical ground. Thanks to the simple structure and realistic background, the history-based representation has a sound potential to be applied in various fields.

Appendix

Proof of Theorem A; Existence

The necessity of axioms 1 and 2 is quite straightforward, so I will skip it. I will construct the proof of the sufficiency step by step.

Step 1. Existence of U.

Since X is connected, compact and metric then the finite product space, $A = X \times ... \times X$, will be connected, compact and metric as well. Hence A is a separable space too.

Since A is a connected and separable, then from axioms 1 and 2 follow that there is a continuous, ordinal utility function U, defined on A, which represents $\succeq$ preference relation (Debreu 1954 [2]).

Step 2. The One Side Maximum Separation Method - OSMS Method.

OSMS is a method which design a methodology of construction of a collection $\{u_1, ..., u_n\}$, based on function U. According to OSMS method, u_1 is derived from U by the following equation.

$$u_1(x_1) = \min_{x_2,...,x_n \in X} U(x_1, ..., x_n).$$

For each $i = 2, ..., n-1$, u_i is derived in a recursive way

$$u_i(x_i, x_1, ..., x_{i-1}) = \min_{x_{i+1},...,x_n \in X} U(x_1, ..., x_n) - u_1(x_1) - ... - u_{i-1}(x_{i-1}, x_1, ..., x_{i-2}).$$

And u_n is determined by

$$u_n(x_n, x_1, ..., x_{n-1}) = U(x_1, ..., x_n) - u_1(x_1) - ... - u_{n-1}(x_{n-1}, x_1, ..., x_{n-2}).$$

Step 3. Each u_i is a nonnegative function.

Each u_i constructed by OSMS method is a continuous function on A_i. From the functional forms of u_i and u_{i-1} as given in the last step, we can write $u_i(x_i, x_1, ..., x_{i-1}) = \min_{x_{i+1},...,x_n \in X} U(x_1, ..., x_n) - \min_{x_i,...,x_n \in X} U(x_1, ..., x_n)$. Notice that $\min_{x_{i+1},...,x_n \in X} U(x_1, ..., x_n) \geq \min_{x_i,...,x_n \in X} U(x_1, ..., x_n)$ for all $(x_1, ..., x_n) \in A$ and for all nonnegative U[3]. As a result $u_i(x_i, x_1, ..., x_{i-1}) \geq 0$ on the domain.

Step 4. Continuity of u_i.

Each u_i constructed by OSMS method is a continuous function on A_i. First I will show that $w_i(x_1, ..., x_i) := \min_{x_{i+1},...,x_n \in X} U(x_1, ..., x_n)$ is a continuous function on A_i. For this, I need to show that w_i is a continuous function on all vectors on A_i. Let $(\bar{x}_1, ..., \bar{x}_i)$ is an arbitrary vector in A_i. Assume that $U(\bar{x}_1, ..., \bar{x}_i, x_{i+1}, ..., x_n)$ takes its minimum value on $(\bar{x}_1, ..., \bar{x}_n) \in A_i$

[3] U is a nonnegative because it is a utility function, step 1.

point, i.e. $w_i(\bar{x}_1, ..., \bar{x}_i) = U(\bar{x}_1, ..., \bar{x}_n)$. Similarly, assume that $w_i(\tilde{x}_1, ..., \tilde{x}_i) = U(\tilde{x}_1, ..., \tilde{x}_n)$ for an arbitrary $(\tilde{x}_1, ..., \tilde{x}_n) \in A_i$. From the continuity of U we have that for all $\epsilon > 0$ there is a δ-neighbourhood of $(\bar{x}_1, ..., \bar{x}_i)$, such that $|U(\bar{x}_1, ..., \bar{x}_n) - U(\tilde{x}_1, ..., \tilde{x}_i, \bar{x}_{i+1}, ..., \bar{x}_n)| < \epsilon$, whenever $(\tilde{x}_1, ..., \tilde{x}_i, \bar{x}_{i+1}, ..., \bar{x}_n)$ is in that neighbourhood. Since $(\tilde{x}_1, ..., \tilde{x}_n)$ is the vector where U is minimal, then $U(\tilde{x}_1, ..., \tilde{x}_i, \bar{x}_{i+1}, ..., \bar{x}_n) \geq U(\tilde{x}_1, ..., \tilde{x}_n)$ always holds. Without loss of generality assume that $U(\bar{x}_1, ..., \bar{x}_n) < U(\tilde{x}_1, ..., \tilde{x}_n)$. Since U is a positive function then, it is easy to see that $|U(\bar{x}_1, ..., \bar{x}_n) - U(\tilde{x}_1, ..., \tilde{x}_n)| < \epsilon$ or equivalently $|w_i(\bar{x}_1, ..., \bar{x}_i) - w_i(\tilde{x}_1, ..., \tilde{x}_i)| < \epsilon$ whenever $(\tilde{x}_1, ..., \tilde{x}_i, \bar{x}_{i+1}, ..., \bar{x}_n)$ belongs to δ-neighbourhood of $(\bar{x}_1, ..., \bar{x}_n)$.

For all $i = 1, ..., n, u_i$ will be continues function, since it is a finite sum (difference) of continuous functions.

Step 5. A collection $\{u_1, ..., u_n\}$, induced by OSMS method, is a maximum splitter.

I will prove by contradicting assumption. Let $\{u_1, ..., u_n\}$ collection be induced by OSMS method. From steps 2 and 3 it follows that $\{u_1, ..., u_n\} \in G_U$ and $\{u_1, ..., u_n\}$ are nonnegative respectively. Suppose that there is another collection $\{u_1, ..., u_{i-1}, u'_i, ..., u'_n\} \in G_U$ and $(x'_1, ..., x'_i) \in A_i$ such that $u'_i(x'_i, x'_1, ..., x'_{i-1}) > u_i(x_i, x_1, ..., x_{i-1})$, that is $\{u_1, ..., u_n\}$ is not a maximum splitter. Without loss of generality let U is minimal on $(x'_1, ..., x'_i, \bar{x}_{i+1}, ..., \bar{x}_n)$ vector, i.e., $u_i(x'_i, x'_1, ..., x'_{i-1}) = U(x'_1, ..., x'_i, \bar{x}_{i+1}, ..., \bar{x}_n) - u(x_1) - ... - u_i(x'_{i-1}, x'_1, ..., x'_{i-2})$. Since $u_1, ..., u_n$ functions are nonnegative then it is easy to see that $\{u_1, ..., u_i, u'_{i+1}, ..., u_n\}$ is not generated by U. A contradiction.

Easy to see that these steps are sufficient to state that the collection generated by OSMS is a history-based additive representation of $\succeq$.

Proof of Theorem B; Uniqueness

Lemma 1 (A.1)**.** *If preference relation $\succeq$ satisfies axioms 1-4 then there is a continuous function $U : A \to \mathbb{R}$, unique up to positive affine transformation, which represents $\succeq$ on A.*

Proof. If axioms 1-3 holds then from vNM theorem we have that there is a continuous function defend on A^c which is unique up to affine transformation (Kreps 1988 [5]). From Axiom 4 and existence of (ordinal) continuous utility function on A (Step 1) it follows that there is a continuous function $U : A \to \mathbb{R}$ which is unique up to affine transformation. □

Lemma 2. *For every U there is a unique maximum splitter collection generated by U, $\{u_1, ..., u_n\} \in G_U$.*

Proof. I will proof by contradiction. Assume there are two different collections generated by the same U and both are the maximum splitters. Without loss of generality suppose they are $\{u_1, ..., u_n\}$ and $\{u_1, ...u_{i-1}, u'_1, ..., u'_n\}$. $u'_i \neq u_i$, which means that there is $(x_1, ..., x_i) \in A_i$ such that either $u'_i(x_i, x_1..., x_{i-1}) > u_i(x_i, x_1..., x_{i-1})$ or $u'_i(x_i, x_1..., x_{i-1}) < u_i(x_i, x_1..., x_{i-1})$. From definition the maximum splitter collection it follows that either either $\{u_1, ..., u_n\}$ is not a maximum splitter collection or $\{u_1, ...u_{i-1}, u'_1, ..., u'_n\}$ is not a maximum splitter collection, respectively. A contradiction. □

Lemma 3. *For every U, $\{u_1, ..., u_n\} \in G_U$ is a history-based additive representation collection of $\succeq$ if and only if it is derived by OSMS method.*

Proof. The proof of the if part immediately follows from the proof of Theorem A. Only if part. On one hand, from Lemma 2 we have that for every U there is a unique maximum splitter collection generated by U, thus, for that U there is a unique collection which is a history-based additive representation of $\succeq$. On the other hand, for a given U, a collection derived by OSMS method, $\{u_1, ..., u_n\}$, is a history-based additive representation. Therefore, for a given U, the collection derived by OSMS method is the only history-based additive representation of $\succeq$ relation.

□

Proof of the Theorem B, If part.

Let $\{u_1, ..., u_n\}$ is a history-based additive representation of $\succeq$ and there are $a, b > 0$ such that $\{u'_1, ..., u'_n\}$ is another history based representation of $\succeq$ defend as follows: $u'_1 = au_1 + b$ and $u'_i = au_i, \forall i = 2, ..., n$. In order to prove this I need to show that it $\{u'_1, ..., u'_n\}$ satisfies the properties of a history-based additive representation.

Since $a, b > 0$ and $u_i \geq 0, \forall i = 1, ..., n$ then $u'_i \geq 0, \forall i = 1, ..., n$. $u'_i \geq 0$ are continuous $\forall i = 1, ..., n$ because $u_i \geq 0$ are continuous $\forall i = 1, ..., n$. Now let me show that U', derived by (7), represents $\succeq$. $U' = u'_1 + u'_2... + u'_n = au_1 + b + au_2 + ... + au_n = aU + b$, so from Lemma 1 it follows that U' represents $\succeq$ as well.

There is left to show that there is U' such that $\{u'_1, ..., u'_n\}$ is a maximum splitter with respect to U'. First I will show that $\{u'_1, ..., u'_n\}$ is derived by OSMS method as well. Note that $u'_1(x_1)$ is possible to represent by the following expression; $u'_1(x_1) = au_1(x_1) + b = a \min_{x_2,...,x_n \in X} U(x_1, ..., x_n) + b = \min_{x_2,...,x_n \in X}(aU(x_1, ..., x_n) + b) = \min_{x_2,...,x_n \in X} U'(x_1, ..., x_n)$. For u'_2 we have; $u'_2(x_2, x_1) = au(x_2, x_1) = a(\min_{x_3,...,x_n \in X} U(x_1, ..., x_n) - u_1(x_1)) = \min_{x_3,...,x_n \in X} aU(x_1, ..., x_n) - a\frac{(u'_1(x_1)-b)}{a} = \min_{x_3,...,x_n \in X}(aU(x_1, ..., x_n) + b) - u'_1(x_1) = \min_{x_3,...,x_n \in X} U'(x_1, ..., x_n) - u'_1(x_1)$. With this intuition we will have, $u'_i(x_i, x_1, ..., x_{i-1}) = \min_{x_{i+1},...,x_n \in X} U'(x_1, ..., x_n) - u'_1(x_1) - ... - u'_{i-1}(x_{i-1}, x_1, ..., x_{i-2})$. So $\{u'_1, ..., u'_n\}$ collection is possible to derive by OSMS method, but from the Theorem A, step 5 we have that the collection derived by OSMS method is a maximum splitter. Hence $\{u'_1, ..., u'_n\}$ collection is a maximum splitter with respect to U'.

Only If part.

I need to show that if $\{u_1, ..., u_n\}$ and $\{u'_1, ..., u'_n\}$ are history-based additive representation collections, then there are $a, b > 0$ such that $u'_1 = au_1 + b$ and $u'_i = au_i$ for all $i = 2, ..., n$.

Having Lemma 2 and Lemma 3 together, we can state that for every U there is a unique collection which is a history-based representation of $\succeq$ and that collection is derived by OSMS method.

Since $\{u_1, ..., u_n\}$ and $\{u'_1, ..., u'_n\}$ are two different history-based additive representing collections, then they should be generated by different U's, that is, $\{u_1, ..., u_n\} \in G_U$ and $\{u'_1, ..., u'_n\} \in GU'$. Both collections are representing the same preference relation, $\succeq$, so U and U' must represent $\succeq$ relation as well. But from Lemma 1 we have that if U and U' represents the same preference relation then $U' = aU + b$ for $a > 0$ and $b \in R$. Since both collections are derived by OSMS method then we have; $u'_1(x_1) = \min_{x_2,...,x_n \in X} U'(x_1, ..., x_n) = \min_{x_2,...,x_n \in X}(aU(x_1, ..., x_n)+b) = a \min_{x_2,...,x_n \in X} U(x_1, ..., x_n)+b = au_1(x_1)+ b$, $u'_2(x_2, x_1) = \min_{x_3,...,x_n \in X} U'(x_1, ..., x_n)-u'_1(x_1) = \min_{x_3,...,x_n \in X}(aU(x_1, ..., x_n)+ b)-(au_1(x_1)+b) = a \min_{x_3,...,x_n \in X} U(x_1, ..., x_n)-au_1(x_1) = au_2(x_2, x_1)$. Recursively we can show $u'_3 = au_3, ..., u'_n = au_n$. Since both collections are history-based additive representations, then $u_i \geq 0$ and $u'_i \geq 0$, so $a \geq 0$ and b is essentially nonnegative.

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