Information, Liquidity, and Dynamic Limit Order Markets*

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Abstract

This paper describes price discovery and liquidity provision in a dynamic limit order market with asymmetric information and non-Markovian learning. Investors condition on information in both the current limit order book and also, unlike in previous research, on the prior order history when deciding whether to provide or take liquidity. Numerical examples show that the information content of the prior order history can be substantial. In addition, the information content of arriving orders can be non-monotone in order direction and aggressiveness.

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The aggregation of private information and the dynamics of liquidity supply and demand are closely intertwined in financial markets. In dealer markets, informed and uninformed investors trade via market orders and, thus, take liquidity, while dealers provide liquidity and try to extract information from the arriving order flow (e.g., as in Kyle (1985) and Glosten and Milgrom (1985)). However, in limit order markets — the dominant form of securities market organization today — the relation between who has information and who is trying to learn it and who supplies and demands liquidity is not well understood theoretically.[1] Recent empirical research highlights the role of informed traders not only as liquidity takers but also as liquidity suppliers. O'Hara (2015) argues that fast informed traders use market and limit orders interchangeably and often prefer limit orders to marketable orders. Fleming, Mizrach, and Nguyen (2017) and Brogaard, Hendershott, and Riordan (2016) find that limit orders play a significant empirical role in price discovery.[2]

Our paper presents the first rational expectations model of a dynamic limit order market with asymmetric information and history-dependent Bayesian learning. In particular, learning is not constrained to be Markovian in the limit order book. The model represents a trading day with market opening and closing effects. Our model lets us investigate the information content of different types of market and limit orders, the dynamics of who provides and demands liquidity, and the non-Markovian information content of the order history. In addition, we study how changes in the amount of adverse selection — in terms of both asset-value volatility and the arrival probability of informed investors — affect equilibrium trading strategies, liquidity, price discovery, and welfare. We have four main results:

- Increased adverse selection does not always worsen market liquidity as in Kyle (1985). Liquidity can improve if informed traders with better information trade more aggressively by submitting more limit-orders at the inside quotes rather than by using market orders.

[1] See Jain (2005) about the prevalence of limit order markets. See Parlour and Seppi (2008) for a survey of theoretical models of limit order markets. See Rindi (2008) and Boulatov and George (2013) for models of informed traders as liquidity providers.

[2] Gencay, Mahmoodzadeh, Rojcek, and Tseng (2016) investigate brief episodes of high-intensity/extreme behavior of quotation process in the U.S. equity market (bursts in liquidity provision that happen several hundreds of time a day for actively traded stocks) and find that limit orders during these bursts significantly impact prices.

- The information content of arriving orders can be opposite both order direction and aggressiveness. These patterns happen in markets in which value-shock volatility is small relative to the price grid, and when informed investors have private value shocks as well as information.

- The learning dynamics are non-Markovian in that the order history has information in addition to the current state of the limit order book.[3] In particular, the incremental information content of arriving limit and market orders is history-dependent.

- The conditional price impact of market and limit order flow (as estimated in Hasbrouck VARs) can depend on time, the current standing limit order book, and the prior order history.

Dynamic limit order markets with uninformed investors are studied in a large literature. This includes Foucault (1999), Parlour (1998), Foucault, Kadan, and Kandel (2005), Goettler, Parlour, and Rajan (2005) and Roşu (2009). There is some previous theoretical research that allows informed traders to supply liquidity. Kumar and Seppi (1994) is a static model in which optimizing informed and uninformed investors use profiles of multiple limit and market orders to trade. Kaniel and Liu (2006) extend the Glosten and Milgrom (1985) dealership market to allow informed traders to post limit orders. Aït-Sahalia and Saglam (2013) also allow informed traders to post limit orders, but they do not allow them to choose between limit and market orders. Moreover, the limit orders posted by their informed traders are always at the best bid and ask prices. Goettler, Parlour, and Rajan (2009) allow informed and uninformed traders to post limit or market orders, but their model is stationary and assumes Markovian learning. Roşu (2016b) studies a steady-state limit order market equilibrium in continuous-time also assuming Markovian learning with some additional information-processing restrictions. These last two papers are closest to ours. Our model differs from them in two ways: First, they assume Markovian learning in order to study dynamic trading strategies with order cancellation, whereas we simplify the strategy space (by not allowing dynamic order cancellations and submissions) in order to investigate non-Markovian

[3]To be clear about terminology, we say a stochastic process followed by a set of variables x is *non-Markovian* if the conditional probability distributions $f[x_s|\, x_t, x_{t-1}, \ldots]$ and $f[x_s|\, x_t]$ are different for some times t and $s > t$. If a summary function $g(x_{t-1}, \ldots)$ exists such that $f[(x_s, g(x_{s-1}, \ldots))|\,(x_t, g(x_{t-1}, \ldots)), (x_{t-1}, g(x_{t-2}, \ldots)), \ldots] = f[(x_s, g(x_{s-1}, \ldots))|(x_t, g(x_{t-1}, \ldots)]$, then we say the augmented process (x, g) is Markovian but not that the unaugmented process x is Markovian.

learning (i.e., our model has a larger state space with full order histories). Second, we model a non-stationary trading day with opening and closing effects. Market opens and closes are important daily events in the dynamics of liquidity in financial markets. Bloomfield, O'Hara, and Saar (2005) show in an experimental market analysis that informed traders sometimes provide more liquidity than uninformed traders. Our model provides equilibrium examples of liquidity provision by informed investors.

A growing literature investigates the relation between information and trading speed (e.g., Biais, Foucault, and Moinas (2015); Foucault, Hombert, and Roşu (2016); and Roşu (2016a)). However, these models assume Kyle or Glosten-Milgrom market structures and, thus, cannot consider the roles of informed and uninformed traders as endogenous liquidity providers and demanders. We argue that understanding price discovery dynamics in limit order markets is an essential precursor to understanding speedbumps and cross-market competition given the real-world prevalence of limit order markets.

1 Model

We consider a limit order market in which a risky asset is traded at N discrete times $t_j \in \{t_1, \dots, t_N\}$ over a trading day. The fundamental value of the asset at the end of the day after time t_N is

$$\tilde{v} = v_0 + \Delta = \begin{cases} \bar{v} = v_0 + \delta & with\ Pr(\bar{v}) = \frac{1}{3} \\ v_0 & with\ Pr(v_0) = \frac{1}{3} \\ \underline{v} = v_0 - \delta & with\ Pr(\underline{v}) = \frac{1}{3} \end{cases} \tag{1}$$

where v_0 is the ex ante expected asset value, and Δ is a symmetrically distributed value shock. The limit order market allows for trading through two types of orders: Limit orders are price-contingent orders that are collected in a limit order book. Market orders are executed immediately at the best available price in the limit order book. The limit order book has a price grid with four prices, $P_i \in \{A_2, A_1, B_1, B_2\}$, two each on the ask and bid sides of the market. The tick size is equal to $\kappa > 0$, and the ask prices are $A_1 = v_0 + \frac{\kappa}{2}$, $A_2 = v_0 + 1.5\kappa$, ; and by symmetry the bid prices are

$B_1 = v_0 - \frac{\kappa}{2}$, $B_2 = v_0 - 1.5\kappa$. For simplicity, we normalize the tick size to $\kappa = 1$.

Order execution follows time and price priority. Thus, at each time t_j, seven possible actions x_{t_j} are available to investors: One possibility is to submit a market order $MBA_{i_{t_j}}$ or $MSB_{i_{t_j}}$ to buy or sell immediately at the best available ask $A_{i_{t_j}}$ or bid $B_{i_{t_j}}$ (indexed by i_{t_j}) in the limit order book at time t_j. A subscript $i_{t_j} = 1$ indicates that the best standing quote at time t_j is at an inside price A_1 or B_1, and $i_{t_j} = 2$ means the best quote is at an outside price A_2 or B_2. Alternatively, the investor can submit one of four possible limit orders LBB_i and LSA_i to buy or sell at the different prices on the ask or bid side of the book. A subscript $i = 1$ denotes an aggressive limit order posted at the inside quote, and $i = 2$ is a less aggressive limit order at the outside quotes.[4] Yet another alternative is to do nothing (NT).

Two types of investors trade in the market. The first are a sequence of arriving active traders with potential gains-from-trade due to private information and/or random private values. One active investor arrives at each time t_j. They are risk-neutral and asymmetrically informed. The active investor arriving at time t_j is informed with probability α and uninformed with probability $1 - \alpha$. Informed investors know the realized value shock Δ perfectly. A generic informed investor is denoted as I. When we want to make explicit the specific information known by the informed investor, then we denote the informed investor as $I_{\bar{v}}$ if the value shock is positive ($\Delta = \delta$), as $I_{\underline{v}}$ if the shock is negative ($\Delta = -\delta$), and as I_{v_0} if the shock is zero ($\Delta = 0$). Informed investors arriving at different times during the day all have the identical asset-value information (i.e., there is only one realized Δ). Uninformed investors do not know Δ, so they use Bayes' Rule and their knowledge of the equilibrium to learn about Δ from the observable order history over time. Uninformed investors are denoted as U.

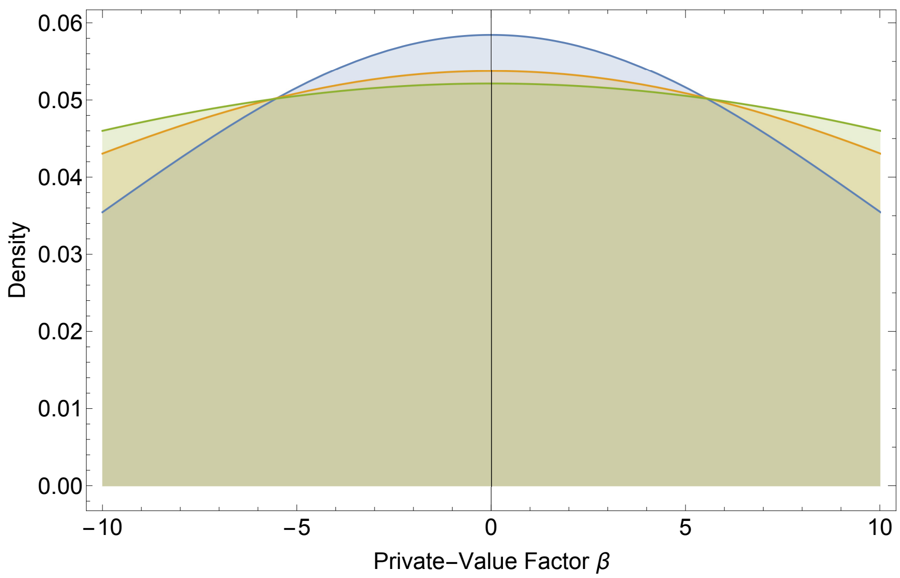
An investor arriving at time t_j may also have an additive random personal private-value trading motive β_{t_j}. Non-informational private-value motives include preference shocks, hedging needs, and taxation. The absence of a non-informational trading motive would lead to the Milgrom and Stokey (1982) no-trade result. In our analysis, the factor β_{t_j} at time t_j is drawn from a truncated-Normal distribution, $Tr[\mathcal{N}(\mu, \sigma^2)]$, with support over the interval $[-10, 10]$, which corresponds to private

[4]For tractability, it is assumed investors cannot post buy limit orders at A_1 and sell limit orders at B_1. This is one way in which the investor action space is simplified in our model.

valuations of up to plus or minus 10 ticks. The mean, $\mu = 0$, is a neutral private factor. The parameter σ determines the dispersion of an investor's private-value factor β_{t_j}, as shown in Figure 1, and, thus, the probability of large private gains-from-trade due to extreme private valuations.

The sequence of arriving active investors is independently and identically distributed in terms of whether investors are informed or uninformed and in terms of their individual private-value factors β_{t_j}. In one specification of our model, only uninformed investors have private valuations, while in a second richer specification both informed and uninformed investors have private valuations.

Figure 1: Distribution of Investor Private-Value Factors - $\beta \sim Tr[\mathcal{N}(\mu, \sigma^2)]$. This figure shows the truncated-Normal probability density function (PDF) of trader private-value factors β_{t_j} with a mean $\mu = 0$ and three different possible values of dispersion σ.



The second type of investors in the market are a group of passive liquidity providers with no active motive to trade. These investors, who we call the *trading crowd*, submit limit orders to provide liquidity. By assumption, the crowd just posts single limit orders at the outside prices A_2 and B_2. The market opens with an initial book submitted by the crowd at time t_0. After the order-submission by the arriving active investor at each time t_j, the crowd replenishes the book at the outside prices, as needed, when either side of the book is empty. Otherwise, if there are limit orders on both sides of the book, the crowd does nothing. The trading crowd effectively establishes a lower bound on the liquidity available in the market.[5]

[5] The trading crowd can endogenized as HFT investors in a Budish, Cramton, and Shim (2015) style model with

For tractability, we make four additional simplifying assumptions. First, limit orders cannot be modified or canceled after submission. Thus, each arriving investor has one and only one opportunity to submit an order. Second, there is no quantity decision. Orders are to buy or sell one share. Third, arriving active investors can only submit one single order. Fourth, limit orders by the active investors have priority over limit orders from the crowd. The focus of our model is on market dynamics involving information and liquidity given the behavior of optimizing informed and uninformed investors. We justify this departure from time priority relative to the crowd in that we want arriving active investors to have a non-trivial choice between aggressive and less aggressive limit orders (as well as between market and limit orders) and because the crowd is simply a modeling device to insure it is always possible for arriving active investors to trade with market orders if they so choose.[6] Taken together, these assumptions let us express the action set for arriving active investors at time t_j as $X_{t_j} = \{MSB_{i_{t_j}}, LSA_1, LSA_2, NT, LBB_2, LBB_1, MBA_{i_{t_j}}\}$, where each of the orders denotes an order for one share.[7]

Our model is intentionally non-stationary over the trading day in order to capture market opening and closing effects and intraday dynamics. When the market opens at t_1, the only standing limit orders in the book are those at prices A_2 and B_2 from the trading crowd.[8] At the end of the day all unexecuted limit orders are cancelled. The state of the limit order book at a generic time t_j during the day is

$$L_{t_j} = [q_{t_j}^{A_2}, q_{t_j}^{A_1}, q_{t_j}^{B_1}, q_{t_j}^{B_2}] \tag{2}$$

where $q_{t_j}^{A_i}$ and $q_{t_j}^{B_i}$ indicate the total depths at prices A_i and B_i at time t_j. The limit order book changes over time due to the arrival of new limit orders (which augment the depth of the book) and market orders (which remove depth from the book) from arriving informed and uninformed

picking-off risk due to immediate public intraday shocks to v_0 that is in addition to the terminal shock Δ that is private information during the day.

[6] In a richer model, we could assume the crowd submits limit orders at prices three ticks from the unconditional common value v_0 and that their limit orders also have time priority.

[7] The action space X_{t_j} of orders that can be submitted at time t_j includes market orders at the standing best bid or offer at time t_j. Our notation $MSB_{i_{t_j}}$ and $MBA_{i_{t_j}}$ reflects the fact that the bid or offer at time t_j is not a fixed number but rather depends on the incoming state of the limit order book. There is no time script in the limit order notation LSA_1, ... because these are just limit orders at particular fixed prices $A_1, \ldots$ in the price grid.

[8] In practice, daily opening limit order books include uncancelled orders from the previous day and new limit orders from opening auctions. For simplicity, we abstract from these interesting features of markets.

investors and due to the submission of limit orders from the crowd. The resulting dynamics are:

$$L_{t_j} = L_{t_{j-1}} + Q_{t_j} + C_{t_j} \quad j = 1, \ldots, N \tag{3}$$

where Q_{t_j} is the change in the book due to an arriving investor's action $x_{t_j} \in X_{t_j}$ at t_j:[9]

$$Q_{t_j} = [Q_{t_j}^{A_2}, Q_{t_j}^{A_1}, Q_{t_j}^{B_1}, Q_{t_j}^{B_2}] = \begin{cases} [-1,0,0,0] & \text{if } x_{t_j} = MBA_2 \\ [0,-1,0,0] & \text{if } x_{t_j} = MBA_1 \\ [+1,0,0,0] & \text{if } x_{t_j} = LSA_2 \\ [0,+1,0,0] & \text{if } x_{t_j} = LSA_1 \\ [0,0,0,0] & \text{if } x_{t_j} = NT \\ [0,0,+1,0] & \text{if } x_{t_j} = LBB_1 \\ [0,0,0,+1] & \text{if } x_{t_j} = LBB_2 \\ [0,0,-1,0] & \text{if } x_{t_j} = MSB_1 \\ [0,0,0,-1] & \text{if } x_{t_j} = MSB_2 \end{cases} \tag{4}$$

where "+1" with a limit order denotes the arrival of an additional order at a particular limit price and "−1" with a market order denotes execution of an earlier BBO limit order and where C_{t_j} is the change in the limit order book due to any limit orders submitted by the crowd

$$C_{t_j} = \begin{cases} [1,0,0,0] & \text{if } q_{t_{j-1}}^{A_2} + Q_{t_j}^{A_2} = 0 \\ [0,0,0,1] & \text{if } q_{t_{j-1}}^{B_2} + Q_{t_j}^{B_2} = 0. \\ [0,0,0,0] & \text{otherwise.} \end{cases} \tag{5}$$

A potentially important source of information at time t_j is the observed history of orders at prior times $t_1, .., t_{j-1}$. In particular, when traders arrive in the market, they observe the history of market activity up through the current standing limit order book at the time they arrive. However, since orders from the crowd have no incremental information beyond that in the arriving investor orders, we exclude them from the notation for the portion of the order-flow history used for informational

[9]There are nine alternatives in (4) because we allow separately for cases in which the best bid and ask for market sells and buys at time t_j are at the inside and outside quotes.

updating of investor beliefs, which we denote by $\mathscr{L}_{t_{j-1}} = \{Q_{t_1}, \ldots, Q_{t_{j-1}}\}$.

Investors trade using optimal order-submission strategies given their information and any private-value motive. If an uninformed investor arrives at time t_j, then his order x_{t_j} is chosen to maximize his expected terminal payoff

$$\begin{aligned} \max_{x \in X_{t_j}} w^U(x \,|\beta_{t_j}, \mathscr{L}_{t_{j-1}}) &= E[(v_0 + \Delta + \beta_{t_j} - p(x))\, f(x)|\beta_{t_j}, \mathscr{L}_{t_{j-1}}] \\ &= \begin{cases} [v_0 + E[\Delta\,|\mathscr{L}_{t_{j-1}}, \theta^x_{t_j}] + \beta_{t_j} - p(x)]\, Pr(\theta^x_{t_j}|\mathscr{L}_{t_{j-1}}) & \text{if } x \text{ is a buy order} \\ [p(x) - (v_0 + E[\Delta\,|\mathscr{L}_{t_{j-1}}, \theta^x_{t_j}] + \beta_{t_j})]\, Pr(\theta^x_{t_j}|\mathscr{L}_{t_{j-1}}) & \text{if } x \text{ is a sell order} \end{cases} \end{aligned} \quad (6)$$

where $p(x)$ is the price at which order x trades, and $f(x)$ denotes the amount of the submitted order that is actually "filled." If x is a market order, then $p(x)$ is the best standing quote on the other side of the market at time t_j, and $f(x) = 1$ for a market buy and $f(x) = -1$ for a market sell (i.e., all of the order is executed). If x is a non-marketable limit order, then the execution price $p(x)$ is its limit price, but the fill amount $f(x)$ is random variable equal to zero if the limit order is never executed and equal to 1 if a limit buy is filled and -1 if a limit sell is filled. If the investor does not trade — either because no order is submitted (NT) or because a limit order is not filled — then $f(x)$ is zero. In the second line of (6), the expression $\theta^x_{t_j}$ denotes the set of future trading states in which an order x submitted at time t_j is executed.[10] This conditioning matters for limit orders because the sequence of subsequent orders in the market, which may or may not result in the execution of a limit order submitted at time t_j, is correlated with the asset value shock Δ. For example, future market buy orders are more likely if the Δ shock is positive (since the average $I_{\overline{v}}$ investors will want to buy but not the average $I_{\underline{v}}$ investor). Uninformed investors rationally take the relation between future orders and Δ into account when forming their expectation $E[\Delta\,|\mathscr{L}_{t_{j-1}}, \theta^x_{t_j}]$ of what the asset will be worth in states in which their limit orders are executed. The second line of (6) also makes clear that uninformed investors use the prior order history $\mathscr{L}_{t_{j-1}}$ in two ways: It affects their beliefs about limit order execution probabilities $Pr(\theta^x_{t_j}|\mathscr{L}_{t_{j-1}})$ and their execution-state-contingent asset-value expectations $E[\Delta\,|\mathscr{L}_{t_{j-1}}, \theta^x_{t_j}]$.

[10] A market orders x_{t_j} is executed immediately at time t_j and so is executed for sure.

An informed investor who arrives at t_j chooses an order x_{t_j} to maximize her expected payoff

$$\max_{x\in X_{t_j}} w^I(x\,|v,\beta_{t_j},\mathscr{L}_{t_{j-1}}) = E[(v_0+\Delta+\beta_{t_j}-p(x))\,f(x)|\beta_{t_j},\mathscr{L}_{t_{j-1}}] \quad (7)$$

$$= \begin{cases} [v_0+\Delta+\beta_{t_j}-p(x)]\,Pr(\theta^x_{t_j}|v,\mathscr{L}_{t_{j-1}}) & \text{if } x \text{ is a buy order} \\ [p(x)-(v_0+\Delta+\beta_{t_j})]\,Pr(\theta^x_{t_j}|v,\mathscr{L}_{t_{j-1}}) & \text{if } x \text{ is a sell order} \end{cases}$$

The only uncertainty for informed investors is about whether any limit orders they submit will be executed. Their belief about order-execution probabilities $Pr(\theta^x_{t_j}|\,v,\mathscr{L}_{t_{j-1}})$ are conditioned on both the trading history up through the current book and on their knowledge about the ending asset value. Thus, informed traders condition on $\mathscr{L}_{t_{j-1}}$, not to learn about the value shock Δ (which they already know) or about future investor private-value factors β_{t_j} (which are i.i.d. over time), but rather because they understand that the trading history is an input in the trading behavior of future uninformed investors (with whom they might trade in the future) and, thus, also in the trading behavior of future informed investors (who will also take history-contingent uninformed-investor learning behavior into account when deciding whether to undercut earlier limit orders). Our analysis considers two model specifications for the informed investors. In the first, informed investors have no private-value motive, so that their β factors are equal to 0. In the second specification, their β factors are random and are independently drawn from the same truncated-Normal distribution $Tr[\mathscr{N}(\mu,\,\sigma^2)]$ as the uninformed investors.

The optimization problem in (6) defines sets of actions $x_{t_j} \in X_{t_j}$ that are optimal for the uninformed investor at different times t_j given different private-value factors β_{t_j} and order histories $\mathscr{L}_{t_{j-1}}$. These optimal orders can be unique, or there may be multiple orders which make the uninformed investor equally well-off. The *optimal order-submission strategy* for the uninformed investor is a probability function $\varphi^U_{t_j}(x|\beta_{t_j},\mathscr{L}_{t_{j-1}})$ that is zero if the order x is suboptimal and equals a mixing probability over optimal orders. If an optimal order x is unique, then $\varphi_{t_j}(x|\beta_{t_j},\mathscr{L}_{t_{j-1}})=1$. Mixed strategies are also allowed. Similarly, the optimization problem in (7) leads to an optimal order-submission strategy $\varphi^I_{t_j}(x|\beta_{t_j},v,\mathscr{L}_{t_{j-1}})$ for informed investors at time t_j given their factor β_{t_j}, their knowledge about the asset value v, and the order history $\mathscr{L}_{t_{j-1}}$.

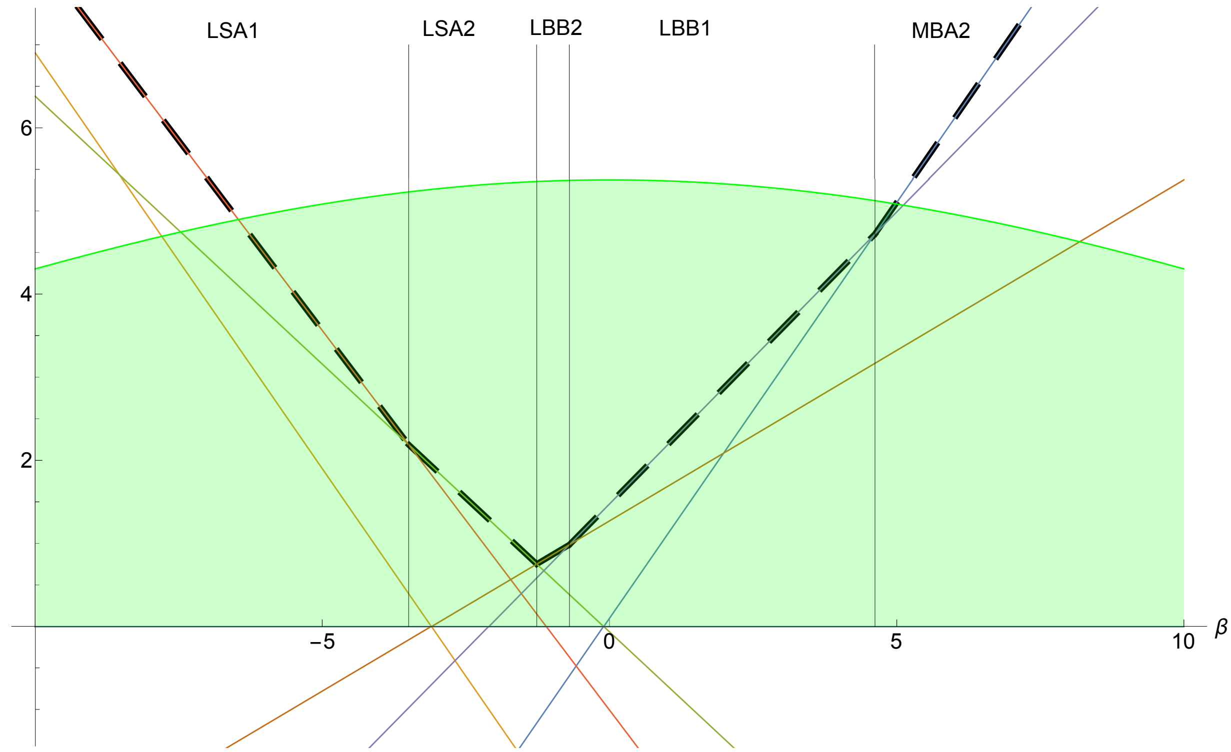
Based on the foregoing, our model has four sources of potential order-flow randomness. First, orders are random due to the random arrival of informed vs uninformed investors. Second, they are random due to the asset-value shock Δ. Third, orders are random due to randomness in investors' personal private values β_{t_j}. This is illustrated in Figure 2 for a numerical example of our model that is considered in detail in Section 2.2 and Appendix A. The plot shows where the order-submission probabilities come from for an informed investor $I_{\overline{v}}$ at time t_1 by superimposing the upper envelope of the expected payoffs for the different optimal orders at time t_1 for the case of good news about a positive value shock δ on the truncated Normal β distribution. It shows how different β subranges correspond to a discrete set of optimal orders delimited by the β thresholds. Similar constructions at other dates for informed investors and also for uninformed investors who must update their asset-value beliefs using Bayes Rule. Fourth and lastly, orders are sometimes random due to possible mixed strategies $\varphi^U_{t_j}$ and $\varphi^I_{t_j}$. However, this only happens when an investor is indifferent between a set of orders.

1.1 Equilibrium

An equilibrium is a set of mutually consistent optimal strategy functions and beliefs for uninformed and informed investors for each time t_j, given each order history $\mathscr{L}_{t_{j-1}}$, private-value factor β_{t_j}, and (for informed traders) private information v. This section explains what "mutually consistent" means and then gives a formal definition of an equilibrium.

A central feature of our model is asymmetric information. The presence of informed investors means that, by observing orders over time, uninformed traders can infer information about the asset value v and use it in their order-submission strategies. More precisely, uninformed traders rationally learn from the trading history about the probability that v will go up, stay constant, or go down. However, investors cannot learn about the private values (β) or information status (I or U) of future traders since, by assumption, these are both i.i.d over time. Informed investors do not need to learn about v since they know it directly. However, they do condition their orders on v (both because v is the final stock value and also because v tells them what type of informed investors I_v will arrive in the future along with the uninformed U traders). Informed investors

Figure 2: β Distribution and Upper Envelope for Informed Investor $I_{\bar{v}}$ at time t_1. This figure shows the private-value factor $\beta \sim Tr[\mathcal{N}(\mu, \sigma^2)]$ distribution superimposed on the plot of the expected payoffs the informed investor $I_{\bar{v}}$ with good news at time t_1 for each equilibrium order type MBA_2, MSB_2, LSA_2, LSA_1, LBB_1, LBB_2, NT, (solid colored lines) when the total book (including crowd limit orders) opens $L_{t_0} = [1\ 0\ 0\ 1]$. The dashed line shows the investor's upper envelope for the optimal orders. The vertical black lines show the β-thresholds at which two adjacent optimal strategies yield the same expected payoffs. For example LSA_1 is the optimal strategy for values of β between 0 and the first vertical black line; LSA_2 is instead the optimal strategy for the values of beta between the first and the second vertical lines; and so forth. The parameters are $\alpha = 0.8$, $\delta = 1.6$, $\mu = 0$, $\sigma = 15$, and $\kappa = 1$.



also condition on the order-flow history $\mathscr{L}_{t-1}$, since $\mathscr{L}_{t-1}$ affects the trading behavior of future investors.[11]

The underlying *economic state* in our model is the realization of the asset value v and a realized sequence of investors who arrive in the market. The investor who arrives at time t_j is described by two characteristics: their status as being informed or uninformed, I or U, and their private-value factor β_{t_j}. The underlying economic state is exogenously chosen over time by Nature. More formally, it follows an exogenous stochastic process described by the model parameters δ, α, μ, and σ. A sequence of arriving investors together with a pair of strategy functions — which we

[11]The order history $\mathscr{L}_{t-1}$ is an input in the uninformed-investor learning problem and, thus, is an input in their order-submission strategy. In addition, since future informed investors know that $\mathscr{L}_{t-1}$ can affect uninformed investor trading behavior, it also enters the order-submission strategies of future informed investors.

denote here as $\Phi = \{\varphi^U_{t_j}(x|\beta_{t_j}, \mathscr{L}_{t_{j-1}}), \varphi^I_{t_j}(x|\beta_{t_j}, v, \mathscr{L}_{t_{j-1}})\}$ — induce a sequence of trading actions x_{t_j} which — together with the predictable actions of the trading crowd — results in a sequence of observable changes in the state L_{t_j} of the limit order book. Thus, the stochastic process generating paths of order histories is induced by the economic state process and the strategy functions. Given the order-path process, several probabilistic quantities can be computed directly: First, we can compute the unconditional probabilities of different paths $Pr(\mathscr{L}_{t_j})$ and the conditional probabilities $Pr(Q_{t_j}|\mathscr{L}_{t_{j-1}})$ of particular order book changes Q_{t_j} due to arriving investors given a prior history $\mathscr{L}_{t_{j-1}}$. Certain paths of orders are *possible* (i.e., have positive probability $Pr(\mathscr{L}_{t_j})$) given the strategy functions $\{\varphi^U_{t_j}(x|\beta, \mathscr{L}_{t_{j-1}}), \varphi^I_{t_j}(x|\beta, v, \mathscr{L}_{t_{j-1}})\}$, and certain paths of orders are not possible (i.e., for which $Pr(\mathscr{L}_{t_j}) = 0$). Second, the endogenous order-path process also determines the order-execution probabilities $Pr(\theta^x_{t_j}|\, v, \mathscr{L}_{t_{j-1}})$ and $Pr(\theta^x_{t_j}|\, \mathscr{L}_{t_{j-1}})$ for informed and uninformed investors for various orders x submitted at time t_j. Computing each of these probabilities is simply a matter of listing all of the possible underlying economic states, mechanically applying the order-submission rules, identifying the relevant outcomes path-by-path, and then taking expectations across paths.

Let ℓ denote the set of all feasible histories $\{\mathscr{L}_{t_j} : j = 1, \ldots, 4\}$ of physically available orders of lengths up to four trading periods. A four-period long history is the longest history a order-submission strategy can depend on in our model. In this context, *feasible* paths are simply sequences of actions from the action choice sets X_{t_j} over time without regard to whether they are *possible* in the sense that they occur with positive probability given the strategy functions Φ. Let $\ell^{\,in,\,\Phi}$ denote the subset of all possible trading paths in ℓ that have positive probability, $Pr(\mathscr{L}_{t_j}) > 0$, given a pair of order strategies Φ. Let $\ell^{\,off,\,\Phi}$ denote the complementary set of trading paths that are *feasible* but *not possible* given Φ. This notation will be useful when discussing "equilibrium" beliefs on order paths that have positive probability and "off equilibrium" beliefs on paths that have zero probability given investor strategies. In our analysis, strategy functions Φ are defined for all feasible paths in ℓ. In particular, this includes all of the possible paths in $\ell^{\,in,\,\Phi}$ given Φ and also the paths in $\ell^{\,off,\,\Phi}$. As a result, the probabilities $Pr(Q_{t_j}|\mathscr{L}_{t_{j-1}})$, $Pr(\theta^x_{t_j}|\, v, \mathscr{L}_{t_{j-1}})$ and $Pr(\theta^x_{t_j}|\, \mathscr{L}_{t_{j-1}})$ are always well-defined, because the continuation trading process going forward — even after an unexpected order-arrival event (i.e., a path $\mathscr{L}_{t_{j-1}} \in \ell^{\,off,\,\Phi}$) — is still well-defined.

The stochastic process for order paths and its relation to the underlying economic state also determine the uninformed-investor expectations $E[v\,|\mathscr{L}_{t_{j-1}}, \theta^x_{t_j}]$ of the terminal asset value given the previous order history ($\mathscr{L}_{t_{j-1}}$) and conditional on future execution of a limit order x submitted at time t_j (denoted here by the set of future states $\theta^x_{t_j}$ in which this happens). In particular, belief and expectation formation for the uninformed investor involve backward conditioning on the prior order history $\mathscr{L}_{t_{j-1}}$ and forward conditioning on the endogenous set of future states $\theta^x_{t_j}$ in which limit orders are executed. These beliefs and expectations are determined as follows:

- Step 1: The conditional probabilities $\pi^v_{t_j} = Pr(v|\mathscr{L}_{t_j})$ of a particular final asset value $v = \bar{v}, v_0$ or $\underline{v}$ given a possible trading history $\mathscr{L}_{t_j} \in \ell^{in,\Phi}$ up through time t_j is given by Bayes' Rule. At time t_1, this probability is

$$\begin{aligned} \pi^v_{t_1} &= \frac{Pr(v, \mathscr{L}_{t_1})}{Pr(\mathscr{L}_{t_1})} = \frac{Pr(\mathscr{L}_{t_1}|v)Pr(v)}{Pr(\mathscr{L}_{t_1})} = \frac{Pr(Q_{t_1}|v)Pr(v)}{Pr(Q_{t_1})} \\ &= \frac{Pr(Q_{t_1}|v, I)Pr(I) + Pr(Q_{t_1}|U)Pr(U)}{Pr(Q_{t_1})} Pr(v) \\ &= \frac{E^\beta[\varphi^I_{t_1}(x_{t_1}|\beta^I_{t_1}, v)|v]\alpha + E^\beta[\varphi^U_{t_1}(x_{t_1}|\beta^U_{t_1})](1-\alpha)}{Pr(Q_{t_1})} \pi^v_{t_0} \end{aligned} \tag{8}$$

where the prior is the unconditional probability $\pi^v_{t_0} = Pr(v)$, x_{t_1} is the order at time t_1 that leads to the order book change Q_{t_1}, and $\beta^I_{t_1}$ and $\beta^U_{t_1}$ are independently distributed private-value β realizations for informed and uninformed investors at time t_1.[12] At times $t_j > t_1$, the history-conditional probabilities are given recursively by[13]

$$\begin{aligned} \pi^v_{t_j} &= \frac{Pr(v, \mathscr{L}_{t_j})}{Pr(\mathscr{L}_{t_j})} = \frac{Pr(v, Q_{t_j}, \mathscr{L}_{t_{j-1}})}{Pr(Q_{t_j}, \mathscr{L}_{t_{j-1}})} = \frac{\left(\begin{array}{c} Pr(Q_{t_j}|v, \mathscr{L}_{t_{j-1}}, I)Pr(I|\mathscr{L}_{t_{j-1}})Pr(v|\mathscr{L}_{t_{j-1}}) \\ + Pr(Q_{t_j}|v, \mathscr{L}_{t_{j-1}}, U)Pr(U|\mathscr{L}_{t_{j-1}})Pr(v|\mathscr{L}_{t_{j-1}}) \end{array} \right)}{Pr(Q_{t_j}|\mathscr{L}_{t_{j-1}})} \\ &= \frac{E^\beta[\varphi^I_{t_j}(x_{t_j}|\beta^I_{t_j}, v, \mathscr{L}_{t_{j-1}})|v, \mathscr{L}_{t_{j-1}}]\ \alpha + E^\beta[\varphi^U_{t_j}(x_{t_j}|\beta^U_{t_j}, \mathscr{L}_{t_{j-1}})|\mathscr{L}_{t_{j-1}}]\ (1-\alpha)}{Pr(Q_{t_j}|\mathscr{L}_{t_{j-1}})} \pi^v_{t_{j-1}} \end{aligned} \tag{9}$$

[12] A trader's information status (I or U) is independent of the asset value v, so $P(I|v) = Pr(I)$ and $Pr(U|v) = Pr(U)$. Furthermore, uninformed traders have no private information about v, so the probability $Pr(Q_{t_1}|U)$ with which they take a trading action Q_{t_1} does not depend on v.

[13] A trader's information status is again independent of v, and it is also independent of the past trading history $\mathscr{L}_{t_1}$. While the probability with which an uninformed trader takes a trading action Q_{t_1} may depend on the past order history $\mathscr{L}_{t_j}$, it does not depend directly on v which uninformed traders do not know.

Given these probabilities, the expected asset value conditional on the order history is

$$E[\tilde{v}|\mathscr{L}_{t_{j-1}}] = \pi^{\bar{v}}_{t_{j-1}}\,\bar{v} + \pi^{v_0}_{t_{j-1}}\,v_0 + \pi^{\underline{v}}_{t_{j-1}}\underline{v} \tag{10}$$

- Step 2: The conditional probabilities $\pi^v_{t_j}$ given a "feasible but not possible in equilibrium" order history $\mathscr{L}_{t_j} \in \ell^{\,\mathit{off},\Phi}$ in which a limit order book change Q_{t_j} that is inconsistent with the strategies Φ at time t_j are set as follows:
  1. If the priors are fully revealing in that $\pi^v_{t_{j-1}} = 1$ for some v, then $\pi^v_{t_j} = \pi^v_{t_{j-1}}$ for all v.
  2. If the priors are not fully revealing at time t_j, then $\pi^v_{t_j} = 0$ for any v for which $\pi^v_{t_{j-1}} = 0$ and the probabilities $\pi^v_{t_j}$ for the remaining v's can be any non-negative numbers such that $\pi^{\bar{v}}_{t_j} + \pi^{v_0}_{t_j} + \pi^{\underline{v}}_{t_j} = 1$.
  3. Thereafter, until any next unexpected trading event, the subsequent probabilities $\pi^v_{t_{j'}}$ for $j' > j$ are updated according to Bayes' Rule as in (9).

- Step 3: The execution-contingent conditional probabilities $\hat{\pi}^v_{t_j} = Pr(v|\mathscr{L}_{t_{j-1}}, \theta^x_{t_j})$ of a final asset value v conditional on a prior path $\mathscr{L}_{t_{j-1}}$ and on execution of a limit order x submitted at time t_j is

$$\begin{aligned} \hat{\pi}^v_{t_j} &= \frac{Pr(\mathscr{L}_{t_{j-1}})Pr(v|\mathscr{L}_{t_{j-1}})\,\Pr(\theta^x_{t_{j-1}}|v,\mathscr{L}_{t_{j-1}})}{Pr(\theta^{x_{t_j}},\mathscr{L}_{t_{j-1}})} \\ &= \frac{\Pr(\theta^x_{t_j}|v,\mathscr{L}_{t_{j-1}})}{Pr(\theta^x_{t_j}|\mathscr{L}_{t_{j-1}})}\pi^v_{t_{j-1}} \end{aligned} \tag{11}$$

This holds when adjusting for a future execution contingency both when the probabilities $\pi^v_{t_{j-1}}$ given the prior history $\mathscr{L}_{t_{j-1}}$ are for possible paths in $\ell^{\,in,\Phi}$ (from (8) and (9) in Step 1) and also for feasible but not possible paths in $\ell^{\,\mathit{off},\Phi}$ (from Step 2). These execution-contingent probabilities $\hat{\pi}^v_{t_j}$ are used to compute the execution-contingent conditional expected value

$$E[\tilde{v}|\mathscr{L}_{t_{j-1}}, \theta^x_{t_j}] = \hat{\pi}^{\bar{v}}_{t_j}\,\bar{v} + \hat{\pi}^{v_0}_{t_j}\,v_0 + \hat{\pi}^{\underline{v}}_{t_j}\,\underline{v} \tag{12}$$

used by uninformed traders to compute expected payoffs for limit orders. In particular, the probabilities in (12) are the execution-contingent probabilities $\hat{\pi}^v_{t_j}$ from (11) rather than the probabilities $\pi^v_{t_j}$ from (9) that just condition on the prior trading history but not on the future states in which the limit order is executed.

Given these updating dynamics, we can now define an equilibrium.

Definition. A *Perfect Bayesian Nash Equilibrium* of the trading game in our model is a collection $\{\varphi^{U,*}_{t_j}(x|\beta_{t_j}, \mathscr{L}_{t_{j-1}}), \varphi^{I,*}_{t_j}(x|\beta_{t_j}, v, \mathscr{L}_{t_{j-1}}), Pr^*(\theta^x_{t_j}|\, v, \mathscr{L}_{t_{j-1}}), Pr^*(\theta^x_{t_j}|\, \mathscr{L}_{t_{j-1}}), E^*[\tilde{v}|\mathscr{L}_{t_{j-1}}, \theta^x_{t_j}]\}_{j=1}^N$ of order-submission strategies, execution-probability functions, and execution-contingent conditional expected asset-value functions such that:

- The equilibrium execution probabilities $Pr^*(\theta^x_{t_j}|\, v, \mathscr{L}_{t_{j-1}})$ and $Pr^*(\theta^x_{t_j}|\, \mathscr{L}_{t_{j-1}})$ are consistent with the equilibrium order-submission strategies $\{\varphi^{U,*}_{t_{j+1}}(x|\beta_{t_{j+1}}, \mathscr{L}_{t_j}), \ldots, \varphi^{U,*}_{t_5}(x|\beta_{t_5}, \mathscr{L}_{t_4})\}$ and $\{\varphi^{I,*}_{t_{j+1}}(x|\beta_{t_{j+1}}, v, \mathscr{L}_{t_j}), \ldots, \varphi^{I,*}_{t_5}(x|\beta_{t_5}, v, \mathscr{L}_{t_4})\}$ after time t_j.

- The execution-contingent conditional expected asset values $E^*[\tilde{v}|\mathscr{L}_{t_{j-1}}, \theta^x_{t_j}]\}$ agree with Bayesian updating equations (8), (9), (11), and (12) in Steps 1 and 3 when the order x is consistent with the equilibrium strategies $\varphi^{U,*}_{t_j}(x|\beta_{t_j}, \mathscr{L}_{t_{j-1}})$ and $\varphi^{I,*}_{t_j}(x|\beta_{t_j}, v, \mathscr{L}_{t_{j-1}})$ at date t_j and, when x is an off-equilibrium action inconsistent with the equilibrium strategies, with the off-equilibrium updating in Step 2.

- The positive-probability supports of the equilibrium strategy functions $\varphi^{U,*}_{t_j}(x|\beta_{t_j}, \mathscr{L}_{t_{j-1}})$ and $\varphi^{I,*}_{t_j}(x|\beta_{t_j}, v, \mathscr{L}_{t_{j-1}})$ (i.e., the orders with positive probability in equilibrium) are subsets of the sets of optimal orders for uninformed and informed investors computed from their optimization problems (6) and (7) and the equilibrium execution probabilities and outcome-contingent conditional asset-value expectation functions $Pr^*(\theta^x_{t_j}|\, v, \mathscr{L}_{t_{j-1}})$, $Pr^*(\theta^x_{t_j}|\, \mathscr{L}_{t_{j-1}})$, and $E^*[\tilde{v}|\mathscr{L}_{t_{j-1}}, \theta^x_{t_j}]$.

Our equilibrium concept differs from the Markov Perfect Bayesian Equilibrium used in Goettler et al. (2009). Beliefs and strategies in our model are path-dependent; that is to say, traders use Bayes Rule given the full prior order history when they arrive in the market. In contrast, Goettler et al. (2009) restricts Bayesian updating to the current state of the limit order book and does not allow for conditioning on the previous order history. Roşu (2016b) also assumes a Markov Perfect Bayesian Equilibrium. The quantitative importance of the order history is considered when we discuss our results in Section 2.

To help with intuition, Appendix A walks through the order-submission and Bayesian updating mechanics for a particular realized equilibrium path in the extensive form of the trading game. Appendix B explains the algorithm used to compute equilibria in our model.

2 Results

This section presents results about how liquidity supply and demand decisions of informed and uninformed traders and the learning process of uninformed traders affect market liquidity, price discovery, and investor welfare. Section 2.1 first considers a model specification in which only uninformed investors have random private-value trading motives. Section 2.2 considers a second specification that generalizes the analysis and shows the robustness of our findings and extends them when informed investors also have private-value motives. Throughout the numerical illustrations, the number of trading rounds is $N = 5$, and the private-value dispersion σ is 15.

We focus on two time windows. The first is when the market opens at time t_1. The second is over the middle of the trading day from times t_2 through t_4. We look at these two windows because our model is non-stationary over the trading day. Much like actual trading days, our model has start-up effects at the beginning of the day and terminal horizon effects at the market close. When the market opens at time t_1, there are time-dependent incentives to provide, rather than to take, liquidity: The opening book is thin (with limit orders only from the crowd), and there is the maximum time for future investors to arrive to hit limit orders from t_1. There are also time-dependent disincentives for limit orders. Information asymmetries are maximal at time t_1, since there has been no learning through the trading process. Also, there is the maximal time for early less aggressive limit orders (at A_2 and B_2) to be undercut by more aggressive later limit orders (at A_1 and B_1). Over the day, information is revealed (lessening adverse selection costs), but the book can also become fuller (i.e., there is competition in liquidity provision from earlier limit orders with time priority at their respective limit prices), and the remaining time for market orders to arrive and execute limit orders becomes shorter. Comparing these two time windows shows how market dynamics change over the day. The market close at t_5 is also important, but trading then is straightforward. At the end of the day, investors only submit market orders (or do not trade), because the execution probability for new limit orders at t_5 is zero given our assumption that unfilled limit orders are canceled once the market closes. Our choice of $N = 5$ trading rounds in a day is computationally tractable while still allowing time for relatively less constrained endogenous choices between market and limit orders at times t_2 through t_4 away from the immediate mechanical effects of the relatively thin book at the market open at t_1 and the end-of-day market orders at t_5.

We use our model to investigate three questions: First, who provides and takes liquidity, and how does the amount of adverse selection affect investor decisions to take and provide liquidity? Second, how does market liquidity vary with different amounts of adverse selection? Third, how does the information content of different types of orders depend on an order's direction, aggressiveness, and on the prior order history?

The amount of adverse selection can change in two ways: The proportion α of informed traders can change, and the magnitude δ of the asset-value shocks can change. We present comparative statics using four different combinations of parameters with high and low informed-investor arrival probabilities ($\alpha = 0.8$ and 0.2) and high and low value-shock volatilities ($\delta = 1.6$ and 0.2). We call markets with $\delta = 0.02$ *low-volatility* markets and markets with $\delta = 1.6$ *high-volatility* markets, because the arriving information is small relative to the $\kappa = 1$ tick size in the former parameterization and larger relative to the tick size in the later. In high-volatility markets, the final asset value v given good or bad news is beyond the outside quotes A_2 or B_2, and so even market orders at the outside prices are profitable for informed traders. However, in low-volatility markets, v is always within the inside quotes A_1 and B_1, and so market orders are never profitable for informed investors. A real-world example are markets for individual stocks where heteroskedastic fluctuation in the daily volatility of arriving information can flip the market for a given stock with a fixed one-penny tick size over time between being a high and low volatility market. Another example is that futures contracts on different underlyings have customized price grids that can be large or small relative to their underlying information flow.

2.1 Uninformed traders with random private-value motives

In our first model specification, only uninformed U traders have random private values β_{t_j}. Informed I traders have fixed neutral private-value factors $\beta_{t_j} = 0$. Thus, as in Kyle (1985), there is a clear differentiation between investors who speculate on private information and those who trade for purely non-informational reasons. Unlike Kyle (1985), informed and uninformed investors here can choose to trade using limit or market orders rather than being restricted to just market orders.

2.1.1 Trading strategies

We begin by investigating who supplies and takes liquidity and how these decisions change with the amount of adverse selection. Our starting point establishes from first principles that different forms of adverse selection affect investors' trading decisions differently.

Proposition 1 Trading strategies are affected differently by changes in adverse selection due to changes in the value-shock size δ vs. changes in the informed-investor arrival probability α.

Proof: Consider first the effect of changes in the value-shock δ on informed-investor order submissions given any fixed α. If the value-shock δ is sufficiently close to zero, then directionally informed $I_{\bar{v}}$ and $I_{\underline{v}}$ investors with good or bad news never use market orders, since the terminal asset value v is always between the inside bid and ask prices A_1 and B_1 given a discrete tick size κ. However, once δ is sufficiently large, investors with good and bad news start to use market orders for their guaranteed execution. Thus, the set of orders used by directionally informed investors can change when δ changes. This is true independently of the informed-investor arrival probability α. In contrast, consider the effect of the informed-investor arrival probability α on informed-investor order submission given a fixed δ. If the value-shocks δ are close to zero, informed investors with good or bad news never use market orders for any informed-investor arrival probability α. They are unwilling to pay a large tick size to trade on their small information. Instead act as liquidity providers using limit orders to supply liquidity asymmetrically depending on the direction of their information. Thus, the set of orders used by directionally informed investors in low-volatility markets never changes to include market orders when α changes. □

Numerical results illustrate other facets of how adverse selection affects investor trading. Table 1 reports results about trading early in the day at time t_1 using a 2×2 format. Each of the four cells corresponds to a different combination of parameters. Comparing cells horizontally shows the effect of a change in the value-shock size δ while holding the arrival probability α for informed traders fixed. Comparing cells vertically shows the effect of a change in the informed-investor arrival probability while holding the value-shock size fixed. In each cell corresponding to a set of parameters, there are four columns reporting conditional results for informed investors with good news, neutral news, and bad news about the asset ($I_{\overline{v}}$, I_{v_0}, $I_{\underline{v}}$) and for an uninformed investor (U) and a fifth column with the unconditional market results ($Uncond$). The table reports the order-submission probabilities and several market-quality metrics. Specifically, we report expected bid-ask spreads conditioning on the three informed-investor types $E[Spread\,|I_v]$ and on the uninformed trader $E[Spread\,|U]$, the unconditional expected market spread $E[Spread]$, and expected depths at the inside prices (A_1 and B_1) and the total at both prices ($A_1 + A_2$ and $B_1 + B_2$) on each side of the market. As we shall see, our results are symmetric for the directionally informed investors $I_{\overline{v}}$ and $I_{\underline{v}}$ on the buy and sell sides of the market. In addition, we report the probability-weighted contributions to the different investors' welfare (i.e., expected gains-from-trade) from limit and market orders respectively, and their total expected welfare.[14] Table B1 in Appendix B provides additional results about conditional and unconditional future execution probabilities for the different orders ($P^{EX}(x_{t_1})$) and also the uninformed investor's updated expected asset value $E[v|x_{t_1}]$ given different types of buy orders x_{t_1} at time t_1.

Table 2 shows average results for times t_2 through t_4 during the day using a similar 2×2 format. The averages are across time and trading histories. Comparing results for time t_1 with the averages for t_2 through t_4 shows intraday variation in the trading process. There is no table for time t_5, because only market orders are used at the market close.

One order-submission property that is important for market-quality and order-informativenes results below is that directionally informed investors $I_{\overline{v}}$ and $I_{\underline{v}}$ tend to trade more aggressively in a

[14] Let $W^U(\beta_{t_1})$ and $W^I(v, \beta_{t_1})$ denote the value functions when (6) and (7) are evaluated at time t_1 using the optimal strategies for the uninformed and informed investors respectively. The total ex ante welfare gain is $E[W^U(\beta_{t_1})]$ for the uninformed investor where the expectation is taken over β_{t_1} and $E[W^I(v, \beta_{t_1})]$ for the informed investor where the expectation is taken over v and β_{t_1}.

high-volatility markets in which value shocks are large relative to the tick size. This is intuitive since larger potential trading gains-from-trade make price improvement less important relative to trade execution. This property can be seen in Table 1 where $I_{\bar{v}}$ and $I_{\underline{v}}$ investors at time t_1 only post limit orders at the less-aggressive outsides quotes A_2 and B_2 in the two low-volatility parameterizations on the right (with $\delta = 0.2$ and $\alpha = 0.2$ or 0.8) but use limit orders with positive probability at both the aggressive inside quotes A_1 and B_1 as well as at the outside quotes in the two high-volatility parameterization cells on the left (with $\delta = 1.6$ and the same two respective αs). This trading-aggressiveness property can also be seen in different ways in the average order-submission probabilities at times t_2 through t_4 in Table 2. In the low-volatility parameterizatons on the right, informed $I_{\bar{v}}$ and $I_{\underline{v}}$ investors supply liquidity via limit orders on both sides of the market with order-submission probabilities that are somewhat skewed at the inside quote in the direction of their small amount of private information. Moving to the high-volatility parameterizations on the left, we see that, when the informed-investor arrival probability α is low (0.2), directionally informed investors increase the probability of using aggressive limit orders at the inside prices to trade in the direction of their information. However, when the informed-investor arrival probability α is high (0.8), the increased trading aggressiveness by informed investors in the high-volatility market takes a different form. Informed $I_{\bar{v}}$ and $I_{\underline{v}}$ investors reduce their use of all types of limit orders and increase their use of market orders at times t_2 to t_4.

Next, consider the neutrally informed I_{v_0} investors and uninformed U investors. These investors respond differently to adverse selection because the informed I_{w_0} investors have an advantage over uninformed U investors: There is no adverse selection risk for the I_{v_0} investors. They know the value shock Δ is 0 and, thus, that the unconditional valuation v_0 is correct. Tables 1 and 2 show that as adverse selection increases (via both larger δs and larger αs), liquidity-provision by the I_{v_0} investors is unchanged at time t_1 and becomes somewhat more aggressive on average in the use of limit orders at the inside prices at times t_2 through t_4. These results are qualitatively consistent with the intuition of Bloomfield, O'Hara and Saar (2005), who find in laboratory experiments that informed investors provide liquidity via limit orders when mispricing is small in a market. In contrast, uninformed U investors become less willing to provide liquidity via aggressive limit orders at the inside quotes as adverse selection worsens. Rather, they increasingly take liquidity via market orders or supply liquidity via less aggressive limit orders at the outside quotes. This reduction in liquidity provision at the inside quotes by uninformed U investors happens at time t_1 (Table 1) and at times t_2 through t_4 (Table 2) and for both larger value shocks δ and higher informed-investor arrival probabilities α.

An equilibrium interaction in investor trading behavior is noteworthy in this context. Uninformed U investors are unwilling to use aggressive limit orders at the inside quotes when the adverse selection risk is sufficiently high as in the upper-left parametrization ($\alpha = 0.8$ and $\delta = 1.6$). This explains the fact that informed $I_{\bar{v}}$ and $I_{\underline{v}}$ investors use aggressive limit orders at the inside quotes with a higher probability at time t_1 in the lower-left intermediate adverse-selection parametrization (0.930 with $\alpha = 0.2$ and $\delta = 1.6$) than in the upper-left high adverse-selection parameterization (0.360). At first glance this might seem counterintuitive since competition from future informed investors (and the possibility of being undercut by later limit orders) is greater when the informed-investor arrival probability is large ($\alpha = 0.8$) than when α is smaller. However, in equilibrium there is camouflage from the uninformed U investor limit orders at the inside quotes in the lower-left parametization, whereas limit orders at the inside quotes are fully revealing in the upper-left parametrization. Table B1 in Appendix B shows that, as a result, the execution probabilities for the fully revealing limit orders at prices that are revealed to be far from the asset's actual value are much lower (0.078) relative to the non-fully revealing limit orders (0.713).

Table 1: Trading Strategies, Liquidity, and Welfare at Time t_1 in an Equilibrium with Informed Traders with $\boldsymbol{\beta = 0}$ and Uninformed Traders with $\boldsymbol{\beta \sim Tr[\mathcal{N}(\mu, \sigma^2)]}$. This table reports results for two different informed-investor arrival probabilities α (0.8 and 0.2) and two different value-shock volatilities δ (1.6 and 0.2). The private-value factor parameters are $\mu = 0$ and $\sigma = 15$, and the tick size is $\kappa = 1$. Each cell corresponding to a set of parameters reports the equilibrium order-submission probabilities, the expected bid-ask spreads and expected depths at the inside prices (A_1 and B_1) and total depths on each side of the market after order submissions at time t_1, and expected welfare of the market participants. The first four columns in each parameter cell are for informed traders with positive, neutral and negative signals, ($I_{\bar{v}}$,I_{v_0},$I_{\underline{v}}$) and for uninformed traders (U). The fifth column ($Uncond.$) reports unconditional results for the market.

		$\delta = 1.6$					$\delta = 0.2$				
		$I_{\bar{v}}$	I_{v_0}	$I_{\underline{v}}$	U	$Uncond.$	$I_{\bar{v}}$	I_{v_0}	$I_{\underline{v}}$	U	$Uncond.$
	LSA_2	0	0.500	0.640	0.145	0.333	0	0.500	1.000	0.052	0.410
	LSA_1	0	0	0.360	0	0.096	0	0	0	0.079	0.016
	LBB_1	0.360	0	0	0	0.096	0	0	0	0.079	0.016
	LBB_2	0.640	0.500	0	0.145	0.333	1.000	0.500	0	0.052	0.410
	MBA_2	0	0	0	0.355	0.071	0	0	0	0.369	0.074
	MBA_1	0	0	0	0	0	0	0	0	0	0
	MSB_1	0	0	0	0	0	0	0	0	0	0
	MSB_2	0	0	0	0.355	0.071	0	0	0	0.369	0.074
	NT	0	0	0	0	0	0	0	0	0	0
$\alpha = 0.8$											
	E[Spread $\vert\cdot$]	2.640	3.000	2.640	3.000	2.808	3.000	3.000	3.000	2.842	2.968
	E[Depth A_2+A_1 $\vert\cdot$]	1.000	1.500	2.000	1.145	1.429	1.000	1.500	2.000	1.131	1.426
	E[Depth A_1 $\vert\cdot$]	0	0	0.360	0	0.096	0	0	0	0.079	0.016
	E[Depth B_1 $\vert\cdot$]	0.360	0	0	0	0.096	0	0	0	0.079	0.016
	E[Depth B_1+B_2 $\vert\cdot$]	2.000	1.500	1.000	1.145	1.429	2.000	1.500	1.000	1.131	1.426
	E[Welfare LO $\vert\cdot$]	0.347	0.596	0.347	0.194	0.383	0.288	0.688	0.288	0.153	0.368
	E[Welfare MO $\vert\cdot$]	0	0	0	3.361	0.672	0	0	0	3.390	0.678
	E[Welfare $\vert\cdot$]	0.347	0.596	0.347	3.554	1.055	0.288	0.688	0.288	3.543	1.046
	LSA_2	0	0.500	0.070	0.065	0.090	0	0.500	1.000	0.063	0.150
	LSA_1	0	0	0.930	0.368	0.356	0	0	0	0.397	0.318
	LBB_1	0.930	0	0	0.368	0.356	0	0	0	0.397	0.318
	LBB_2	0.070	0.500	0	0.065	0.090	1.000	0.500	0	0.063	0.150
	MBA_2	0	0	0	0.068	0.054	0	0	0	0.040	0.032
	MBA_1	0	0	0	0	0	0	0	0	0	0
	MSB_1	0	0	0	0	0	0	0	0	0	0
	MSB_2	0	0	0	0.068	0.054	0	0	0	0.040	0.032
	NT	0	0	0	0	0	0	0	0	0	0
$\alpha = 0.2$											
	E[Spread $\vert\cdot$]	2.070	3.000	2.070	2.265	2.288	3.000	3.000	3.000	2.206	2.365
	E[Depth A_2+A_1 $\vert\cdot$]	1.000	1.500	2.000	1.432	1.446	1.000	1.500	2.000	1.460	1.468
	E[Depth A_1 $\vert\cdot$]	0	0	0.930	0.368	0.356	0	0	0	0.397	0.318
	E[Depth B_1 $\vert\cdot$]	0.930	0	0	0.368	0.356	0	0	0	0.397	0.318
	E[Depth B_1+B_2 $\vert\cdot$]	2.000	1.500	1.000	1.432	1.446	2.000	1.500	1.000	1.460	1.468
	E[Welfare LO $\vert\cdot$]	2.726	1.471	2.726	3.094	2.937	0.809	1.497	0.809	3.595	3.084
	E[Welfare MO $\vert\cdot$]	0	0	0	1.045	0.836	0	0	0	0.642	0.514
	E[Welfare $\vert\cdot$]	2.726	1.471	2.726	4.139	3.773	0.809	1.497	0.809	4.238	3.598

Table 2: Averages for Trading Strategies, Liquidity, and Welfare across Times t_2 through t_4 for Informed Traders with $\beta = 0$ and Uninformed Traders with $\beta \sim Tr[\mathcal{N}(\mu, \sigma^2)]$. This table reports results for two different informed-investor arrival probabilities α (0.8 and 0.2) and for two different asset-value volatilities δ (1.6 and 0.2). The private-value factor parameters are $\mu = 0$ and $\sigma = 15$, and the tick size is $\kappa = 1$. Each cell corresponding to a set of parameters reports the equilibrium order-submission probabilities, the expected bid-ask spreads and expected depths at the inside prices (A_1 and B_1) and total depths on each side of the market after order submissions at times t_2 through t_4, and expected welfare for the market participants. The first four columns in each parameter cell are for informed traders with positive, neutral and negative signals, ($I_{\bar{v}}$,I_{v_0},$I_{\underline{v}}$) and for uninformed traders (U). The fifth column ($Uncond.$) reports unconditional results for the market.

		$\delta = 1.6$					$\delta = 0.2$				
		$I_{\bar{v}}$	I_{v_0}	$I_{\underline{v}}$	U	$Uncond.$	$I_{\bar{v}}$	I_{v_0}	$I_{\underline{v}}$	U	$Uncond.$
	LSA_2	0	0.244	0.049	0.155	0.109	0.399	0.255	0.108	0.026	0.209
	LSA_1	0	0.256	0.253	0.027	0.141	0.192	0.239	0.288	0.064	0.205
	LBB_1	0.253	0.256	0	0.027	0.141	0.288	0.239	0.192	0.064	0.205
	LBB_2	0.049	0.244	0	0.155	0.109	0.108	0.255	0.399	0.026	0.209
	MBA_2	0.491	0	0	0.297	0.190	0	0	0	0.347	0.069
	MBA_1	0.001	0	0	0.018	0.004	0	0	0	0.058	0.012
	MSB_1	0	0	0.001	0.018	0.004	0	0	0	0.058	0.012
	MSB_2	0	0	0.491	0.297	0.190	0	0	0	0.347	0.069
	NT	0.206	0	0.206	0.007	0.111	0.013	0.010	0.013	0.011	0.012
$\alpha = 0.8$											
	E[Spread $\|\cdot$]	2.174	2.276	2.174	2.529	2.272	2.269	2.275	2.269	2.738	2.364
	E[Depth A_2+A_1 $\|\cdot$]	1.048	2.326	2.467	1.755	1.909	2.165	2.300	2.433	1.608	2.161
	E[Depth A_1 $\|\cdot$]	0.001	0.362	0.826	0.235	0.364	0.226	0.362	0.506	0.131	0.318
	E[Depth B_1 $\|\cdot$]	0.826	0.362	0.001	0.235	0.364	0.506	0.362	0.226	0.131	0.318
	E[Depth B_1+B_2 $\|\cdot$]	2.467	2.326	1.048	1.755	1.909	2.433	2.300	2.165	1.608	2.161
	E[Welfare LO $\|\cdot$]	0.092	0.128	0.092	1.075	0.298	0.143	0.133	0.143	0.055	0.123
	E[Welfare MO $\|\cdot$]	0.093	0	0.093	2.960	0.642	0	0	0	3.538	0.708
	E[Welfare $\|\cdot$]	0.185	0.128	0.185	4.036	0.940	0.143	0.133	0.143	3.592	0.830
	LSA_2	0	0.385	0.525	0.101	0.141	0.375	0.389	0.443	0.093	0.155
	LSA_1	0	0.099	0.242	0.058	0.069	0.044	0.096	0.116	0.066	0.070
	LBB_1	0.242	0.099	0	0.058	0.069	0.116	0.096	0.044	0.066	0.070
	LBB_2	0.525	0.385	0	0.101	0.141	0.443	0.389	0.375	0.093	0.155
	MBA_2	0.130	0	0	0.219	0.184	0	0	0	0.218	0.175
	MBA_1	0.093	0	0	0.118	0.101	0	0	0	0.120	0.096
	MSB_1	0	0	0.093	0.118	0.101	0	0	0	0.120	0.096
	MSB_2	0	0	0.130	0.219	0.184	0	0	0	0.218	0.175
	NT	0.010	0.031	0.010	0.006	0.009	0.022	0.030	0.022	0.005	0.009
$\alpha = 0.2$											
	E[Spread $\|\cdot$]	2.160	2.154	2.160	2.402	2.353	2.212	2.173	2.212	2.478	2.422
	E[Depth A_2+A_1 $\|\cdot$]	1.299	2.094	2.513	1.585	1.662	1.932	2.091	2.257	1.576	1.680
	E[Depth A_1 $\|\cdot$]	0.190	0.423	0.727	0.304	0.332	0.346	0.414	0.442	0.262	0.290
	E[Depth B_1 $\|\cdot$]	0.727	0.423	0.190	0.304	0.332	0.442	0.414	0.346	0.262	0.290
	E[Depth B_1+B_2 $\|\cdot$]	2.513	2.094	1.299	1.585	1.662	2.257	2.091	1.932	1.576	1.680
	E[Welfare LO $\|\cdot$]	1.179	0.566	1.179	0.523	0.614	0.596	0.654	0.596	0.500	0.523
	E[Welfare MO $\|\cdot$]	0.177	0	0.177	3.419	2.759	0	0	0	3.417	2.734
	E[Welfare $\|\cdot$]	1.357	0.566	1.357	3.942	3.372	0.596	0.654	0.596	3.917	3.257

2.1.2 Market quality

Market liquidity changes when the amount of adverse selection in a market changes. A standard intuition, as in Kyle (1985), is that liquidity deteriorates given more adverse selection. Roşu (2016b) also finds worse liquidity (a wider bid-ask spread) given higher value volatility in his limit order market. However, we show the standard intuition is not always true when informed investors endogenously choose whether to supply liquidity via limit orders or take liquidity via market orders.

Observation 1 Liquidity can sometimes improve when adverse selection increases.

In particular, markets can become more liquid when, given the tick size, increasing the value-shock volatility flips the value shock δ from being small to being large relative the price grid. In addition, we show how different measures of market liquidity — expected spreads, inside depth, and total depth — can respond differently to changes in adverse selection.

The impact of adverse selection on market liquidity follows directly from the trading strategy effects in Section 2.1.1. Three intuitions are useful in understanding our market liquidity results. First, the most aggressive way to trade (both on directional information and private values) is via market orders, which take liquidity. However, the next most aggressive way to trade is via limit orders at the inside prices. Thus, changes in market conditions (i.e., δ and α) that make directionally informed investors trade more aggressively (i.e., that reduce their use of limit orders at the outside prices A_2 and B_2) can improve liquidity if their stronger trading interest migrates to aggressive limit orders at the inside quotes (A_1 and B_1) rather than to market orders. We call this the *aggressive directional informed liquidity provision effect.* Second, informed investors have a comparative advantage in providing liquidity over uninformed investors since I_{v_0} investors know that the unconditional asset value is correct. We call this the *Bloomfield-O'Hara-Saar effect* since they were the first to discuss liquidity provision by neutrally informed investors. Third, liquidity can change due to composition effects when changes in α change the mix of informed and uninformed investors, since different types of investors affect liquidity differently. Informed I_{v_0} investors with neutral news are natural liquidity providers. Their impact on liquidity comes from whether they supply liquidity at the inside (A_1 and B_1) or outside (A_2 and B_2) prices. In contrast, informed

$I_{\bar{v}}$ and $I_{\underline{v}}$ investors with directional news and uninformed U traders affect liquidity depending on whether they opportunistically take or supply liquidity. All three effects can contribute to overturning the standard intuition about adverse selection and liquidity.

Our main result in this section is that the relation between adverse selection and market liquidity depends on the relative magnitudes of asset-value shocks and the tick size. As measures of liquidity, we focus here on the expected bid-ask spread and on expected depth at the inside prices. In Table 1, liquidity improves at time t_1 when the value-shock volatility δ increases (comparing parameterizations horizontally so that α is kept fixed). This happens, contrary to the standard intuition, because the informed $I_{\bar{v}}$ and $I_{\underline{v}}$ traders submit limit orders at the inside quotes in these high-volatility markets, whereas they only use limit orders at the outside quotes in low-volatility markets. In contrast, liquidity at time t_1 worsens, as predicted by the standard intuition, when the informed-investor arrival probability α increases holding the value-shock size δ fixed at the high level. Thus, the standard intuition is sometimes wrong but can also hold.

The evidence against the standard adverse-selection intuition is even stronger on average at times t_2 through t_4 in Table 2. First, consider the effect of increased information volatility δ. For both high and low proportions α of informed investors, liquidity improves when δ is increased. However, the underlying causes are different. When α is high (0.8), most investors reduce their total use of inside limit orders (i.e., on both sides of the market). Thus, the reason that average liquidity at times t_2 through t_4 is better in the high-volatility market is a carry-over effect from the greater liquidity of the high-volatility market at time t_1. In contrast, when α is low (0.2), high-volatility markets are more liquid due to the increased use of inside limit orders by both the directionally informed investors and the neutrally informed investors (i.e., both the aggressive directional informed liquidity provision effect and the Bloomfield-O'Hara-Saar effect) as well as due to the liquidity carry-over effect from time t_1. Second, consider the effect of a higher arrival probability α for informed investors. For both values of asset-value volatility δ, a higher probability α of informed investors leads neutrally informed I_{v_0} investors to increase their total use of limit orders at the inside prices far more than the other investors reduce their use of these orders. That, together with a composition effect (i.e., with $\alpha = 0.8$ there are more informed investors and informed investors use inside limit orders more than the uninformed investors) and the liquidity carry-over from t_1, is why liquidity improves in this case.

Our results for the expected spread and inside depth are driven by limit-order submissions at the inside quotes. However, the effect of adverse selection on total depth at the inside and outside quotes combined can differ from those liquidity measures driven by inside limit orders. For example, total depth at time t_1 increases (in Table 1) when value-shock volatility δ increases when the informed-investor arrival probability α is high (comparing horizontally the top two parametrizations), but decreases in δ when α is low. In contrast, average total depth at times t_2 through t_4 is decreasing (in Table 2) in the value-shock volatility (comparing parameterizations horizontally). This is opposite the effect on the inside depth. Thus, different metrics for liquidity can give different results.

Our results show that the relation between adverse selection and market liquidity in limit order markets is more subtle than the standard intuition. In particular, it is the ability of investors to choose endogenously whether to supply or demand liquidity and at what limit prices that can overturn the standard intuition. Goettler et al. (2009) also investigate a market with informed traders with no private-value motives and uninformed having only private-value motives. In their model, when volatility increases, informed traders reduce their provision of liquidity and increase their demand of liquidity; with the opposite holding for uninformed traders. Our results are more nuanced. Increased value-shock volatility is associated with increased liquidity supply in some cases and decreased liquidity in others. This is because the tick size of the price grid constrains the prices at which liquidity can be supplied and demanded.

2.1.3 Welfare

Tables 1 and 2 also report results about investor welfare. Not surprisingly, the utility of directionally informed investors increases when information volatility δ is higher. Interestingly, more than half of their expected gains-from-trade come from limit-order submissions. Perhaps more surprisingly, uniformed-investor utility is also often higher when δ is larger. This is consistent with the associated increase in liquidity that allows uninformed investors to capture more of their potential gains from trade. The net effect is that total active investor welfare increases in high volatility markets. In contrast, total welfare is less when the arrival probability α of informed investors increases. This is due to the fact that in this model only the uninformed U investors have gains-from-trade.

2.1.4 Information content of orders

Traders in real-world markets and empirical researchers are interested in the information content of different types of orders.[15] A necessary condition for an order to be informative is that informed investors use it. However, the magnitude of order informativeness is determined by the mix of equilibrium probabilities with which informed and uninformed traders use an order. If uninformed traders use the same orders as informed investors, they add noise to the price discovery process, and orders become less informative. In our model, the mix of information– and noise–based orders depends on the underlying proportion α of informed investors and the value-shock volatility δ.

We expect different market and limit orders to have different information content. A natural conjecture is that the sign of the information revision associated with an order should agree with the direction of the order (e.g., buy market and limit orders should lead to positive valuation revisions). Another natural conjecture is that the magnitude of information revisions should be greater for more aggressive orders. However, while the order-sign conjecture is true in our first model specification, the order-aggressiveness conjecture does not always hold here.

Observation 2 Order informativeness is not always increasing in the aggressiveness of an order.

This, at-first-glance surprising, result is another consequence of how informed investors trade on their information. As a result, the relative informativeness of different market and limit orders can flip in high-volatility and low-volatility markets. The result is immediate for market orders versus (less aggressive) limit orders in low-volatility markets in which informed investors avoid market orders (see Table 1). However, this reversed ordering can also hold for aggressive limit orders at the inside quotes (A_1 and B_1) versus less aggressive limit orders at the outside quotes (A_2 and B_2).

Figure 3 shows the informativeness of different types of orders. Each row contains four plots showing the informativeness of particular types of orders submitted at different times during the day

[15] Fleming et al. (2017) extend the VAR estimation approach of Hasbrouck (1991) to estimate the price impacts of limit orders as well as market orders. See also Brogaard et al. (2016).

for the indicated market parameterizations. Informativeness at time t_1 is measured as the Bayesian revision $E[v|\, x_{t_1}] - E[v]$ in the uninformed investor's expectation of the terminal value v after observing different given types of orders x_{t_1} at time t_1. The analogous measure of informativeness at later dates t_2 through t_4 is the Bayesian revision $E[v|\mathscr{L}_{t_{j-1}}, x_{t_j}] - E[v|\mathscr{L}_{t_{j-1}}]$ for different given types of orders x_{t_j} at time t_j relative to the incoming expectation conditional on the preceding order-flow history $\mathscr{L}_{t_{j-1}}$. In particular, the informativeness of a given order may change over time and may differ conditional on different preceding order histories. The vertical heights of the individual dots in the plots indicate the informativeness of given orders at particular times given specific preceding histories.[16] The associated probabilities can differ across the different dots. The rectangles show the range of our informativeness metrics across paths. The vertical height of the blue squares indicate the probability-weighted average informativeness of a given type of order. The figure reports results for market and limit buy orders. The results are symmetric for sell orders.

The results in Figure 3 point to a variety of properties about order informativeness. First, perhaps the most obvious point is the heterogeneity in the information content of a given order at different times during the day and conditional on different prior order-flow histories. For example, plot 3(c) shows the Bayesian revisions for a LBB_1 limit buy order at the inside quotes B_1 in a high volatility market with a high arrival probability of informed investors ($\delta = 1.6$ and $\alpha = 0.8$). At time t_1, an LBB_1 order is fully revealing (and so the Bayesian revision relative to the unconditional expectation is 1.6). This follows from the fact in Table 1 that only informed $I_{\bar{v}}$ investors with good news use LBB_1 orders at time t_1. However, at later dates an LBB_1 limit order has different information content depending on the prior history. For example, in equilibrium an LBB_1 at time t_2 can be preceded by one of four possible equilibrium orders at t_1. If it follows a LSA_2 at t_1 (i.e., from an uninformed U investor which partially lowered prices), then an LBB_1 at t_2, which is fully revealing, leads to a positive Bayesian revision of 2.42 (the high dot). If it is preceded by either a market buy or sell MBA_2 or MSB_2 (at the outside prices) at t_1 (which are uninformative since only uninformed investors use them), then the LBB_1 at t_2 is again fully revealing and is associated

[16] A given sequence of equilibrium orders might be produced by more than one investor-arrival sequence. Thus, individual dots correspond to sets of investor arrival sequences. Note here that the horizontal spacing of the dots in the plots is simply for ease of viewing.

with a positive Bayesian revision of 1.6. Lastly, if the time t_1 order is a LBB_2 limit order (which raise prices somewhat), the LBB_1 order at t_2 is only partially revealing but still produces a smaller upward incremental revision of 0.75. In this context, note that the order histories associated with the different dots can have different probabilities of occurring in equilibrium. For example, in Plot 3(b), we see that a few equilibrium order histories cause a MBA_1 market order at time t_4 to have a large Bayesian revision of almost 3. One way this can happen, for example, is when the proceeding path of orders is $\{LSA_2, MSB_2, LSA_1\}$ which is possible given the right sequence of uninformed investors. Over time the number of equilibrium paths grows by definition, but, in addition, we also see that, in equilibrium, the amount of informational heterogeneity across paths also grows. Moreover, this includes an increasing number of paths with zero Bayesian revisions. One reason this happens is that the number of fully revealing prior order histories is non-decreasing over time.

Second, Figure 3 shows that the aggressiveness conjecture for order informativeness can fail in a variety of ways. One way it can fail is that the average Bayesian revisions for limit orders are frequently larger than for market orders. This is follows immediately from Proposition 1 in low-volatility markets ($\delta = 0.2$). However, the conjecture also fails in high-volatility markets. For example, with $\delta = 1.6$ in the high-informed-investor proportion $\alpha = 0.8$ case, the average revisions for limit orders in Plots 3(c) and 3(d) are always larger than for market orders in Plots 3(a) and 3(b). This is also true in the low-informed investor proportion $\alpha = 0.2$ case in Plots 3(h) through 3(k). We also see that the conjecture can fail for aggressive vs. less-aggressive limit orders. Comparing Plots 3(g) and 3(h), is visually apparent that less-aggressive LBB_2 limit buys at t_1 have larger average Bayesian revisions than the aggressive LBB_1 limit buys. Visually, the differences are smaller in Plots 3(n) and 3(o), but the less-aggressive limit order averages are larger at all dates than for the aggressive limit orders. Having shown that the aggressiveness conjecture can fail, we also note that it does not always fail. For example, the average Bayesian revions for aggressive limit orders at times t_1 through t_3 in Plot 3(c) are larger than for the less-aggressive limit orders in Plot 3(d).

Figure 3: Order Informativeness for the Model with Informed Traders with $\beta = 0$ and Uninformed Traders with $\boldsymbol{\beta \sim Tr[\mathcal{N}(\mu, \sigma^2)]}$. for times t_1 to t_4. This figure shows the path-contingent Bayesian value-forecast revisions $E[v|\mathscr{L}_{t_j-1}, x_{t-j}] - E[v|\mathscr{L}_{t_j-1}]$, which shows the change in the uninformed traders's expected value of the fundamental conditional on the order. We only consider orders when they are equilibrium orders for the trading periods. Each dot indicates an equilibrium revision, the plots indicate the maximum and the minimum. The plots are grouped by their respective market parameterizations (δ, α).

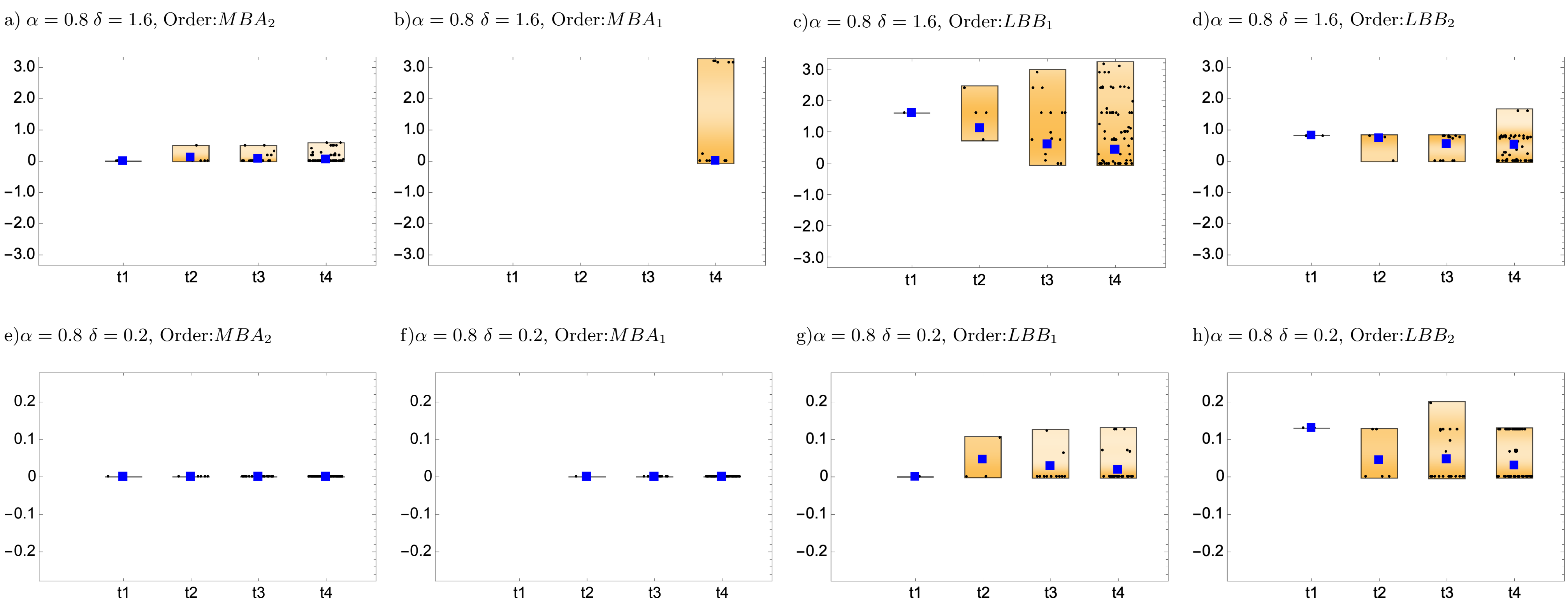
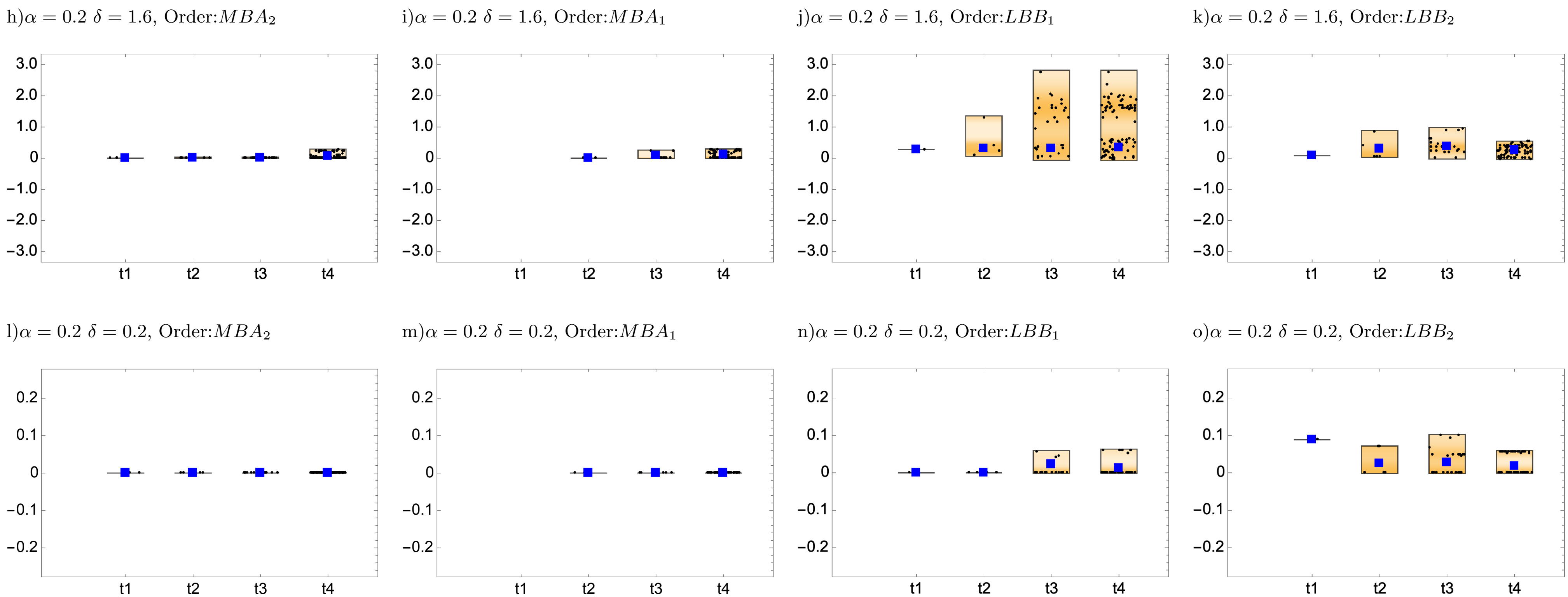


Figure 3 Continued: Order Informativeness for the Model with Informed Traders with $\beta = 0$ and Uninformed Traders with $\boldsymbol{\beta} \sim \boldsymbol{Tr}[\mathscr{N}(\boldsymbol{\mu}, \boldsymbol{\sigma}^2)]$. for times t_1 to t_4.



2.1.5 Non-Markovian learning

This section investigates the role of the order history on Bayesian learning. A major difference between our model and Goettler et al. (2009) and Roşu (2016b) is that they assume learning is Markovian in the sense that the current limit order book L_{t_j} is a sufficient statistic at time $t_j > t_1$ for the information content of the full prior trading history $\mathscr{L}_{t_j}$. Thus, our first question here is whether the prior order history has information about the asset value v in excess of the information in the current limit order book. If it does, then learning is non-Markovian.[17]

The plots in Figure 4 measure the non-Markov information content of order histories by

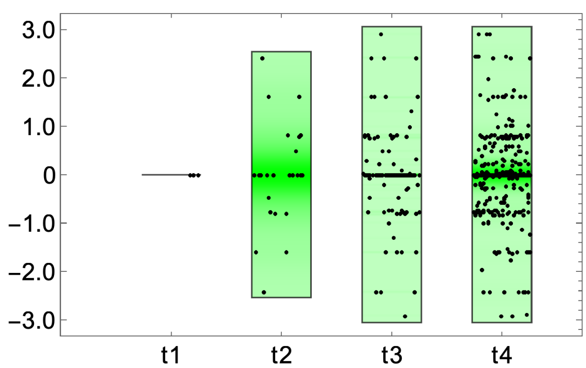
$$E[v|\,\mathscr{L}_{t_j}(L_{t_j})] - E[v|\,L_{t_j}], \tag{13}$$

which is incremental information in the uninformed investors' expected asset value conditional on an order history path $\mathscr{L}_{t_j}(L_{t_j})$ ending with a particular limit order book L_{t_j} at time t_j net of the corresponding expectation conditional on just the ending book L_{t_j}. In particular, we are interested in books L_{t_j} that can be preceded in equilibrium by more than one different prior history. If learning is Markov, then order histories $\mathscr{L}_{t_j}(L_{t_j})$ preceding a book L_{t_j} should convey no additional information beyond L_{t_j}; in which case our metric in (13) should be zero. Individual dots in the plots indicate the incremental information content of particular histories preceding different orders submitted at each of the different dates. Time t_1 is included in the plot because books L_{t_1} at t_1 can potentially be produced by different sequences of investor actions x_{t_1} and crowd responses at t_1. More generally, the book L_{t_j} at each time t_j reflects information due to the path of past active investor actions, but past crowd actions can partially obscure this information (e.g., as when the crowd replenishes the book after active investors deplete the book at the outside prices). Each plot is for a different combination of adverse-selection parameters. For brevity, the plots contain all possible books, rather than having individual plots (as in Figure 3) for each individual order.

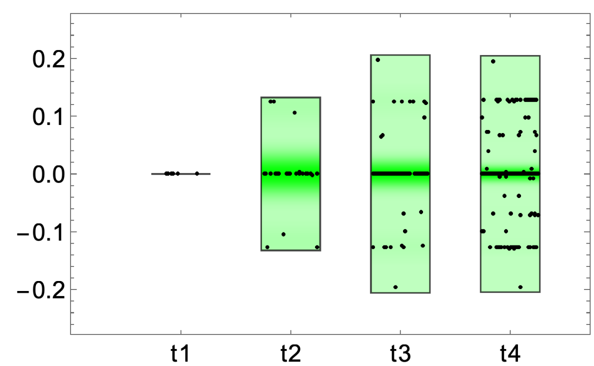
[17]The evidence of path-contingent order informativeness in Figure 3 by itself does not necessarily imply non-Markovian learning. Markovian learning is still possible if the incoming book L_{t_j} at time t_j summarizes the information content of the full order history $\mathscr{L}_{t_j}(L_{t_j})$ preceding book L_{t_j}.

Figure 4: Informativeness of the Order History for the Model with Informed Traders with $\beta = 0$ and Uninformed Traders with $\beta \sim Tr[\mathscr{N}(\mu, \sigma^2)]$ for Times t_1 through t_4. This figure shows the incremental information content of the past order history in excess of the information in the current limit order book observed at the end of time t_j as measured by $E[v|\mathscr{L}_{t_j}(L_{t_j})] - E[v|L_{t_j}]$ where $\mathscr{L}_{t_j}(L_{t_j})$ is a history ending in the limit order book L_{t_j}. We only consider books L_{t_j} when they occur in equilibrium in the different trading periods. The dots indicate values for particular books and paths, and the rectangles show the range of maximum and minimum values.

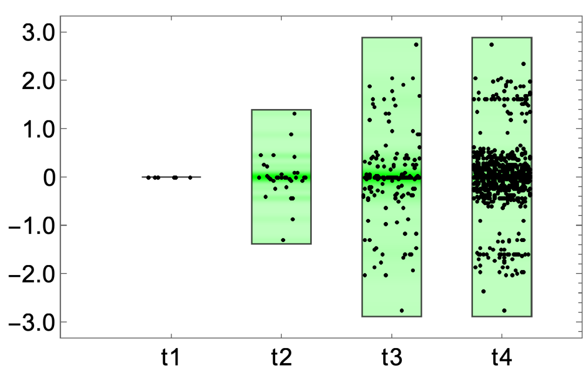
(a) Parameters: $\alpha = 0.8$, $\delta = 1.6$



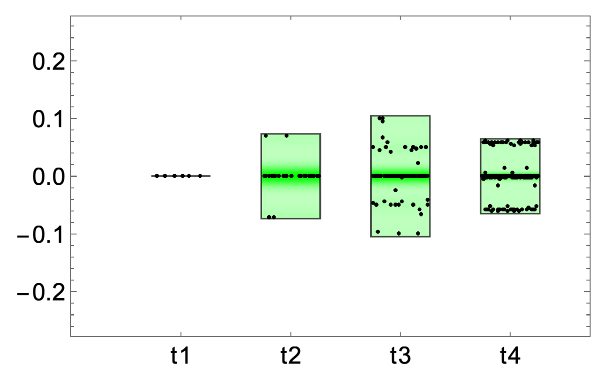
(b) Parameters: $\alpha = 0.8$, $\delta = 0.2$



(c) Parameters: $\alpha = 0.2$, $\delta = 1.6$



(d) Parameters: $\alpha = 0.2$, $\delta = 0.2$

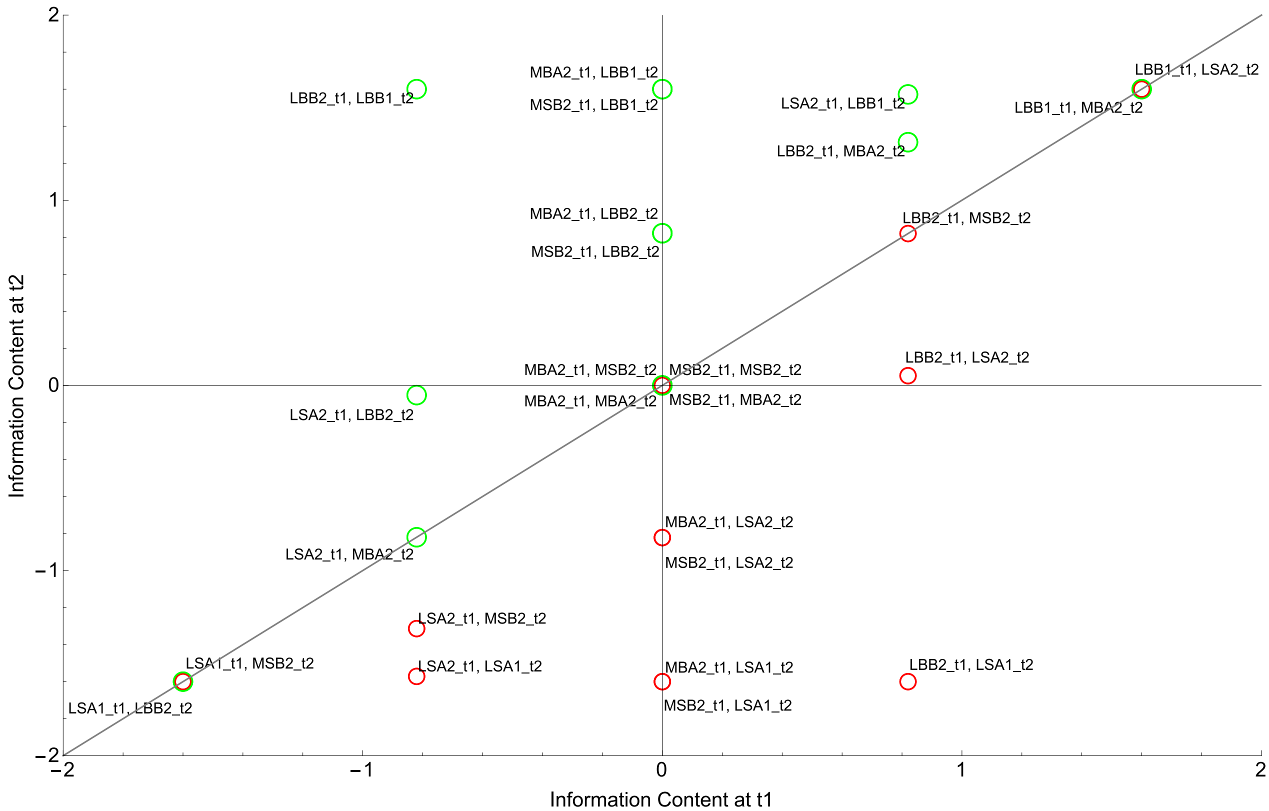


The main result from Figure 4 is that there is substantial incremental information in the preceding order histories after conditioning on the prior limit order book.

Observation 4 The price discovery dynamics can be significantly non-Markovian.

As expected, the variation in the incremental information content of the prior order history in Figure 4 is greater when the shock volatility δ is greater (note the difference in vertical scales).

Figure 5: Order Informativeness for the Model with Informed Traders with $\beta = 0$ and Uninformed Traders with $\beta \sim Tr[\mathcal{N}(\mu, \sigma^2)]$ for times t_1 to t_2 and parameters $\alpha = 0.8$, $\delta = 1.6$. The horizontal axis reports $E(v|x_{t_1}) - E(v)$ which shows how the uninformed traders' Bayesian value-forecast changes with respect to the unconditional expected value of the fundamental when uninformed traders observe at t_1 an equilibrium order x_{t_1}. The vertical axis reports $E(v|x_{t_1}, x_{t_2}) - E(v)$ which shows how the uninformed traders' Bayesian value-forecast changes with respect to the unconditional expected value of the fundamental when uninformed traders observe at x_{t_2} at t_2. We consider all the equilibrium strategies at t_1 and t_2 which are symmetrical. Green (red) circles show equilibrium buy (sell) orders at t_2.



Given that learning is non-Markovian, the next question is about how the size of the valuation revisions depends on the prior trading history. In Figure 5, the horizontal axis shows the valuation revision $E(v|x_{t_1}) - E(v)$ given different equilibrium actions x_{t_1} at t_1, and the vertical axis gives the corresponding cumulative valuation revision $E(v|x_{t_1}, x_{t_2}) - E(v)$ as of time t_2 given different sequences of equilibrium actions x_{t_1} at time t_1 followed by different possible equilibrium successor actions x_{t_2} at time t_2. From iterated expectations, the expectation of the two-period revision given the first period action is the first-period revision, which is denoted here by the 45° line.

Consistent with our previous analysis, the size of the valuation revision depends crucially on the informed investors' equilibrium strategies. As informed investors do not use market orders at t_1 (see Table 1), market orders have a zero price impact at t_1 and, thus, the points for pairs of time t_1 and t_2 price-impacts for sequences of a market order at t_1 and then different orders at time t_2 all line up on the vertical axis line. Interestingly, there are no observations in the first and fourth quadrants in our model, which means there are no sign reversals in the direction of the cumulative price impacts. However, there is randomness around the 45° line induced by different successor date-2 actions. The second and third quadrants (which are symmetrical) show the sequences of orders that have a positive and a negative price impact, respectively. One intuitive result about the relation between earlier orders and subsequent valuation-revision dynamics is the following: Conditional on the amount of adverse selection (i.e., the δ and α parameterization), the volatility of the incremental valuation revision at time t_2 relative to time t_1 (i.e., the vertical dispersion around the 45° line) is weakly decreasing in the magnitude of the valuation revision associated with the trading action at time t_1.

2.1.6 Price impact of order flow

A standard empirical measure of price-discovery is the price impact of order flow. The idea is that the price impact of orders can be decomposed into two components: One measures the size of surprises in an arriving order relative to its expectation given the prior history, and the second measures the marginal (per-share) impact of order-flow surprises on the informational component of a security's valuation. Fleming et al. (2017) and Brogaard et al. (2016) extend the Hasbrouck (1991) vector autoregression methodology — a standard empirical technique to estimate this decomposition — to allow for limit orders as well as market orders. Using our notation, their information innovation equation can be written as

$$E[v|x_t, \mathscr{L}_{t-1}] - E[v|\mathscr{L}_{t-1}] = \sum_k \lambda_k [Q^{x_t}_{k,t} - E[Q^{x_t}_{k,t}|\mathscr{L}_{t-1}]] \tag{14}$$

where $Q^{x_t}_{k,t} - E[Q^{x_t}_{k,t}|\mathscr{L}_{t-1}]$ in the innovation in the number of shares $Q^{x_t}_{k,t}$ associated with an order type k (e.g., a particular market or limit order) given the investor action x_t at time t, and λ_k is a constant marginal price impact for order type k.

Our model suggests an extension of the VAR approach that we call the *conditional price impact of order flow.* In particular, the price impact of order flow, rather than being a constant λ_k, can vary over time given different types of conditioning information. In our model, the price impact is a function $\lambda_k(t, \mathscr{L}_{t-1})$ that is conditional on the prior trading history $\mathscr{L}_{t-1}$ and on time t_j. In its most general form, our model would require machine learning techniques to deal with large amounts of transactional data and high dimensional functional relationships. Simpler empirical specifications might look at the effect of conditioning just on time via a function $\lambda_k(t)$ or conditioning just on the standing limit order book L_{t-1} at the time orders arrives via a function $\lambda_k(t, L_{t-1})$.

Figure 6 shows that even our very simple model generates substantial variation in the conditional price impact of orders. Consider an order sequence $\{\mathscr{L}_{t_{j-1}}, x_{t_j}\}$ where sequences $\{\mathscr{L}_{t_{j-1}}, x_{t_j}\}$ and $\{\mathscr{L}_{t_{j-1}}, NT\}$ both have positive probabilities. As a metric for dispersion in the conditional price impact of order flow, we compute

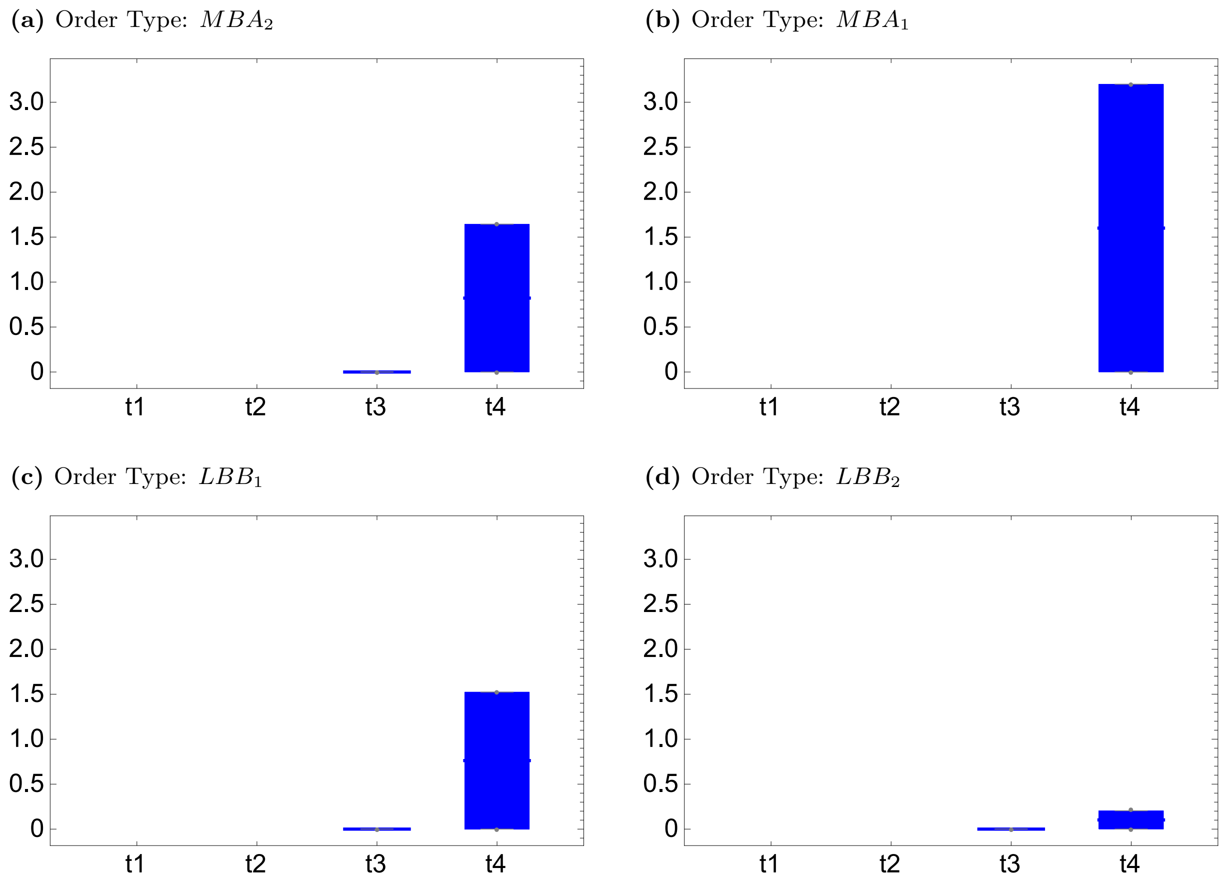
$$\max_{\mathscr{L}_{t_{j-1}}} E[v|\mathscr{L}_{t_{j-1}}, x_{t_j}] - E[v|\mathscr{L}_{t_{j-1}}, NT] - \min_{\mathscr{L}_{t_{j-1}}} E[v|\mathscr{L}_{t_{j-1}}, x_{t_j}] - E[v|\mathscr{L}_{t_{j-1}}, NT] \qquad (15)$$

In words, $E[v|\mathscr{L}_{t_{j-1}}, x_{t_j}] - E[v|\mathscr{L}_{t_{j-1}}, NT]$ is the differential informational impact of a one-unit innovation in order type x_{t_j} relative to NT where differencing controls for expectations given the prior history $\mathscr{L}_{t_{j-1}}$. The metric in (15) is the spread between the maximal and minimum differential informational innovation across all paths $\mathscr{L}_{t_{j-1}}$ such that order x_{t_j} and NT both occur with positive probability following the different paths $\mathscr{L}_{t_{j-1}}$. As can be seen, the amount of cross-path dispersion in the conditional impact of order flow can be substantial.

2.1.7 Summary

The analysis of our first model specification has identified a number of empirically testable predictions. First, liquidity and the relative information content of different orders differ in high-volatility markets (in which value shocks are large relative to the tick size) vs. in low-volatility markets. Second, it is possible for less-aggressive orders to be more informative than more aggressive orders. Third, price discovery is non-Markov, and the price impact of individual orders varies conditional on the prior order-flow history.

Figure 6: Dispersion in the price impact of order flow The plot reports $max_{\mathscr{L}_{t_{j-1}}}(E[v|\mathscr{L}_{t_{j-1}}, x_{t_j}] - E[v|\mathscr{L}_{t_{j-1}}, NT]) - min_{\mathscr{L}_{t_{j-1}}}(E[v|\mathscr{L}_{t_{j-1}}, x_{t_j}] - E[v|\mathscr{L}_{t_{j-1}}, NT])$ at different times, which shows how the prior order history affects the marginal price impact of the surprise in a given order. The parameterization is: $\alpha = 0.8$, $\delta = 1.6$



2.2 Informed and uninformed traders both have private-value motives

Our second model specification generalizes our earlier analysis. Now informed investors also have random private-valuation factors β_{t_j} with the same truncated-Normal distribution $\beta_{t_j} \sim Tr[\mathscr{N}(\mu, \sigma^2)]$ as the uninformed investors. Hence, informed traders not only speculate on their information, but they also have private-value motives to trade. As a result, informed investors with the same signal may end up buying and selling from each other. This combination of trading motives has not been investigated in earlier models of dynamic limit order markets. We use our second model specification to show the robustness of the results in Section 2.1 and to extend them.

Table 3: Trading Strategies, Liquidity, and Welfare at Time t_1 in an Equilibrium with Informed and Uninformed Traders both with $\beta \sim Tr[\mathcal{N}(\mu, \sigma^2)]$. This table reports results for two different informed-investor arrival probabilities α (0.8 and 0.2) and two different value-shock volatilities δ (1.6 and 0.2). The private-value factor parameters are $\mu = 0$ and $\sigma = 15$, and the tick size is $\kappa = 1$. Each cell corresponding to a set of parameters reports the equilibrium order-submission probabilities, the expected bid-ask spreads and expected depths at the inside prices (A_1 and B_1) and total depths on each side of the market after order submisions at time t_1, and the expected welfare of the market participants. The first four columns in each parameter cell are for informed traders with positive, neutral and negative signals, ($I_{\bar{v}}$,I_{v_0},$I_{\underline{v}}$) and for uninformed traders (U). The fifth column ($Uncond.$) reports unconditional results for the market.

		$\delta = 1.6$					$\delta = 0.2$				
		$I_{\bar{v}}$	I_{v_0}	$I_{\underline{v}}$	U	$Uncond.$	$I_{\bar{v}}$	I_{v_0}	$I_{\underline{v}}$	U	$Uncond.$
$\alpha = 0.8$	LSA_2	0.118	0.054	0.031	0.064	0.067	0.054	0.048	0.042	0.048	0.048
	LSA_1	0.314	0.446	0.282	0.426	0.363	0.438	0.452	0.466	0.452	0.452
	LBB_1	0.282	0.446	0.314	0.426	0.363	0.466	0.452	0.438	0.452	0.452
	LBB_2	0.031	0.054	0.118	0.064	0.067	0.042	0.048	0.054	0.048	0.048
	MBA_2	0.256	0	0	0.009	0.070	0	0	0	0	0
	MBA_1	0	0	0	0	0	0	0	0	0	0
	MSB_1	0	0	0	0	0	0	0	0	0	0
	MSB_2	0	0	0.256	0.009	0.070	0	0	0	0	0
	NT	0	0	0	0	0	0	0	0	0	0
	E[Spread $\vert\cdot$]	2.404	2.109	2.404	2.147	2.274	2.096	2.096	2.096	2.096	2.096
	E[Depth A_2+A_1 $\vert\cdot$]	1.432	1.500	1.312	1.491	1.430	1.492	1.500	1.508	1.500	1.500
	E[Depth A_1 $\vert\cdot$]	0.314	0.446	0.282	0.426	0.363	0.438	0.452	0.466	0.452	0.452
	E[Depth B_1 $\vert\cdot$]	0.282	0.446	0.314	0.426	0.363	0.466	0.452	0.438	0.452	0.452
	E[Depth B_1+B_2 $\vert\cdot$]	1.312	1.500	1.432	1.491	1.430	1.508	1.500	1.492	1.500	1.500
	E[Welfare LO $\vert\cdot$]	2.589	4.452	2.589	4.098	3.388	4.462	4.465	4.462	4.461	4.462
	E[Welfare MO $\vert\cdot$]	1.874	0	1.874	0.155	1.030	0	0	0	0	0
	E[Welfare $\vert\cdot$]	4.463	4.452	4.463	4.253	4.418	4.462	4.465	4.462	4.461	4.462
$\alpha = 0.2$	LSA_2	0.063	0.051	0.042	0.051	0.051	0.049	0.048	0.046	0.048	0.048
	LSA_1	0.356	0.449	0.476	0.449	0.445	0.441	0.452	0.464	0.452	0.452
	LBB_1	0.476	0.449	0.356	0.449	0.445	0.464	0.452	0.441	0.452	0.452
	LBB_2	0.042	0.051	0.063	0.051	0.051	0.046	0.048	0.049	0.048	0.048
	MBA_2	0.063	0	0	0	0.004	0	0	0	0	0
	MBA_1	0	0	0	0	0	0	0	0	0	0
	MSB_1	0	0	0	0	0	0	0	0	0	0
	MSB_2	0	0	0.063	0	0.004	0	0	0	0	0
	NT	0	0	0	0	0	0	0	0	0	0
	E[Spread $\vert\cdot$]	2.168	2.103	2.168	2.102	2.111	2.096	2.096	2.096	2.096	2.096
	E[Depth A_2+A_1 $\vert\cdot$]	1.419	1.500	1.518	1.500	1.496	1.490	1.500	1.510	1.500	1.500
	E[Depth A_1 $\vert\cdot$]	0.356	0.449	0.476	0.449	0.445	0.441	0.452	0.464	0.452	0.452
	E[Depth B_1 $\vert\cdot$]	0.476	0.449	0.356	0.449	0.445	0.464	0.452	0.441	0.452	0.452
	E[Depth B_1+B_2 $\vert\cdot$]	1.518	1.500	1.419	1.500	1.496	1.510	1.500	1.490	1.500	1.500
	E[Welfare LO $\vert\cdot$]	3.943	4.448	3.943	4.424	4.362	4.466	4.465	4.466	4.465	4.465
	E[Welfare MO $\vert\cdot$]	0.591	0	0.591	0	0.079	0	0	0	0	0
	E[Welfare $\vert\cdot$]	4.535	4.448	4.535	4.424	4.440	4.466	4.465	4.466	4.465	4.465

Table 4: Averages for Trading Strategies, Liquidity, and Welfare across Times t_2 through t_4 for Informed and Uninformed Traders both with $\boldsymbol{\beta \sim Tr[\mathcal{N}(\mu, \sigma^2)]}$. This table reports results for two different informed-investor arrival probabilities α (0.8 and 0.2) and for two different asset-value volatilities δ (1.6 and 0.2). The private-value factor parameters are $\mu = 0$ and $\sigma = 15$, and the tick size is $\kappa = 1$. Each cell corresponding to a set of parameters reports the equilibrium order-submission probabilities, the expected bid-ask spreads and expected depths at the inside prices (A_1 and B_1) and total depths on each side of the market after order submissions at times t_2 through t_4, and the expected welfare of the market participants. The first four columns in each parameter cell are for informed traders with positive, neutral and negative signals, ($I_{\bar{v}}$,I_{v_0},$I_{\underline{v}}$) and for uninformed traders (U). The fifth column ($Uncond.$) reports unconditional results for the market.

		$\delta = 1.6$					$\delta = 0.2$				
		$I_{\bar{v}}$	I_{v_0}	$I_{\underline{v}}$	U	$Uncond.$	$I_{\bar{v}}$	I_{v_0}	$I_{\underline{v}}$	U	$Uncond.$
	LSA_2	0.140	0.121	0.090	0.114	0.117	0.127	0.123	0.119	0.123	0.123
	LSA_1	0.108	0.058	0.050	0.067	0.071	0.057	0.053	0.048	0.053	0.053
	LBB_1	0.050	0.058	0.108	0.067	0.071	0.048	0.053	0.057	0.053	0.053
	LBB_2	0.090	0.121	0.140	0.114	0.117	0.119	0.123	0.127	0.123	0.123
	MBA_2	0.275	0.192	0.113	0.195	0.194	0.207	0.194	0.181	0.194	0.194
	MBA_1	0.158	0.127	0.062	0.122	0.117	0.133	0.128	0.124	0.129	0.128
	MSB_1	0.062	0.127	0.158	0.122	0.117	0.124	0.128	0.133	0.129	0.128
	MSB_2	0.113	0.192	0.275	0.195	0.194	0.181	0.194	0.207	0.194	0.194
	NT	0.003	0.003	0.003	0.005	0.004	0.004	0.003	0.004	0.004	0.004
$\alpha = 0.8$											
	E[Spread $\|\cdot$]	2.365	2.325	2.365	2.375	2.356	2.336	2.337	2.336	2.337	2.336
	E[Depth A_2+A_1 $\|\cdot$]	1.599	1.600	1.537	1.563	1.576	1.590	1.593	1.596	1.593	1.593
	E[Depth A_1 $\|\cdot$]	0.301	0.339	0.338	0.314	0.324	0.324	0.333	0.344	0.333	0.334
	E[Depth B_1 $\|\cdot$]	0.338	0.339	0.301	0.314	0.324	0.344	0.333	0.324	0.333	0.334
	E[Depth B_1+B_2 $\|\cdot$]	1.537	1.600	1.599	1.563	1.576	1.596	1.593	1.590	1.593	1.593
	E[Welfare LO $\|\cdot$]	0.892	0.709	0.892	0.723	0.809	0.674	0.671	0.674	0.670	0.672
	E[Welfare MO $\|\cdot$]	3.285	3.324	3.285	3.315	3.301	3.357	3.357	3.357	3.358	3.357
	E[Welfare $\|\cdot$]	4.177	4.033	4.177	4.038	4.110	4.031	4.028	4.031	4.028	4.029
	LSA_2	0.131	0.123	0.114	0.122	0.122	0.124	0.123	0.122	0.123	0.123
	LSA_1	0.059	0.054	0.049	0.053	0.054	0.053	0.053	0.052	0.053	0.053
	LBB_1	0.049	0.054	0.059	0.053	0.054	0.052	0.053	0.053	0.053	0.053
	LBB_2	0.114	0.123	0.131	0.122	0.122	0.122	0.123	0.124	0.123	0.123
	MBA_2	0.257	0.194	0.137	0.196	0.196	0.202	0.194	0.186	0.194	0.194
	MBA_1	0.160	0.127	0.090	0.127	0.127	0.133	0.128	0.124	0.128	0.128
	MSB_1	0.090	0.127	0.160	0.127	0.127	0.124	0.128	0.133	0.128	0.128
	MSB_2	0.137	0.194	0.257	0.196	0.196	0.186	0.194	0.202	0.194	0.194
	NT	0.004	0.003	0.004	0.004	0.004	0.004	0.003	0.004	0.004	0.004
$\alpha = 0.2$											
	E[Spread $\|\cdot$]	2.337	2.335	2.337	2.340	2.339	2.337	2.337	2.337	2.337	2.337
	E[Depth A_2+A_1 $\|\cdot$]	1.547	1.595	1.636	1.591	1.591	1.587	1.593	1.599	1.592	1.593
	E[Depth A_1 $\|\cdot$]	0.288	0.334	0.378	0.332	0.332	0.327	0.333	0.339	0.333	0.333
	E[Depth B_1 $\|\cdot$]	0.378	0.334	0.288	0.332	0.332	0.339	0.333	0.327	0.333	0.333
	E[Depth B_1+B_2 $\|\cdot$]	1.636	1.595	1.547	1.591	1.591	1.599	1.593	1.587	1.592	1.593
	E[Welfare LO $\|\cdot$]	0.682	0.685	0.682	0.668	0.671	0.671	0.671	0.671	0.671	0.671
	E[Welfare MO $\|\cdot$]	3.481	3.345	3.481	3.355	3.371	3.359	3.357	3.359	3.357	3.358
	E[Welfare $\|\cdot$]	4.163	4.029	4.163	4.022	4.042	4.030	4.028	4.030	4.028	4.029

2.2.1 Trading strategies

Tables 3 and 4 report order-submission probabilities and other statistics for our second model specification for time t_1 and for averages over times t_2 through t_4. There are a few differences relative to Tables 1 and 2 for the simpler model in Section 2.1. First, since all investors have private-value motives to trade, all investors use all of the possible limit orders in both time windows. In addition, now informed investors sometimes use market orders at t_1 as well as over times t_2 through t_4 and also sometimes (during times t_2 through t_4) even when asset volatility δ is small (0.2). Second, directionally informed investors sometimes now trade opposite their asset-value information. In particular, we say an $I_{\bar{v}}$ or $I_{\underline{v}}$ investor is trading *with* their information when they are buying (selling) given good (bad) news. Trading *opposite* their information is doing the reverse. Investors trade opposite their information when their random private-value motive overwhelms their speculative motive. In particular, note that often in both tables limit orders are used more by investors trading opposite their information than with their information. That will have important implications for the information content (considered below) of such limit orders. Third, informed I_{v_0} investors with neutral news no longer just provide liquidity using limit orders. Rather, due to their private-value motive, they sometimes also take liquidity via market orders both at time t_1 when δ is large (in Table 3) and later at times t_2 through t_4 even when δ is small (in Table 4).

Consider next the impact of adverse selection on trading behavior. The effect of higher δ and higher α on the trading behavior of informed traders $I_{\bar{v}}$ and $I_{\underline{v}}$ with directional news differs when they are trading with or opposite their information. For investors trading with their information, we see the aggressiveness effect again, similar to the results in Section 2.1. In particular, for these investors, increased adverse selection leads to a reduction in the use of less-aggressive outside limit orders trading with directional good and bad news and an increase in the use of more aggressive orders. The net effect on aggressive limit orders at inside prices is ambiguous in these cases due to in-migration of probability from the reduced use of the outside limit orders but possible out-migration of probability to market orders. For example, comparing the upper two parameterizations in Table 3 shows that when δ is increased with α fixed at 0.8, the $I_{\bar{v}}$ investors with good news at time t_1 reduce the strategy probability for LBB_2 orders from 0.042 to 0.031 and increase the probability for MBA_2 orders from 0 to 0.256, and reduce the use of LBB_1 limit orders from 0.466 to 0.282.

The effect of adverse selection is different from above when $I_{\bar{v}}$ and $I_{\underline{v}}$ investors trade opposite their directional information. Increased adverse selection causes informed investors trading opposite their information to increase their use of less-aggressive limit orders at the outside prices. In particular, when δ increases, informed investors with good news $\bar{v}$ (bad news $\underline{v}$) know the security is worth more (less) and require a higher (lower) price when selling. However, when α increases, the reason is a supply/demand effect: The demand for buying (selling) increases since now more investors know the good (bad) news, and, thus, informed investors willing to sell (buy) can increase the price of the liquidity they provide.

The effects of higher volatility on uninformed U traders slightly differs at t_1 as opposed to times t_2 through t_4. At t_1 uninformed traders post slightly more aggressive orders when they demand liquidity (the strategy probabilities for MBA_2 and MSB_2 increase from 0 to 0.009), and more patient orders when they supply liquidity (the strategy probabilities for LBB_2 and LSA_2 increase slightly from 0.048 to 0.064). This change in order-submission strategies is the consequence of uninformed traders facing higher adverse selection costs. They feel safer hitting the trading crowd at A_2 and B_2 and offering liquidity at more profitable price levels to make up for the increased adverse selection costs. In later periods t_1 through t_4, as uninformed traders learn about the fundamental value of the asset, they still take liquidity at the outside quotes (the probabilities of MBA_2 and MSB_2 increase slightly to 0.195 in Table 4), but move to the inside quotes to supply liquidity (LSA_1 and LBB_1 increase to 0.067 for times t_2 through t_4). As they learn about the future value of the asset, uninformed traders perceive less adverse selection costs and can afford to offer liquidity at more aggressive quotes. In contrast, the effect of increased value-shock volatility on the trading behavior of I_{v_0} investors with neutral news is relatively modest both at time t_1 and at times t_2 through t_4.

2.2.2 Market quality

Market quality — as measured by both expected spreads and inside depth in Tables 3 and 4 — is almost always decreasing in adverse selection in this second model. This is a notable difference from our first model. However, this is not surprising given the generally greater use of market orders due to the potentially large range of private values. In particular, when the gains-from-trade are large, order execution is more important than price improvement.

2.2.3 Information content of orders

Figure 7 shows the distribution of Bayesian revisions for the different orders at different times and conditional on different prior order-flow paths. The format is the same as in Figure 3. Once again, there is heterogeneity in the information content of orders over time and conditional on the preceding history. Not surprisingly, the amount of heterogeneity is less since there is substantially less price discovery in this second model specification given that informed investor orders are now affected by noise from private values as well as information. In addition, we still see violations of the order aggressiveness conjecture. Consider, for example, the high adverse-selection parameterization with high value-shock volatility and a high informed-investor arrival probability. The most informative orders at t_1 and t_2 are the market orders. However, the less-aggressive LBB_2 limit orders are more informative than the aggressive LBB_1 limit orders at t_1 and also, less obviously visually, at t_2.

A new finding in this second model is that, surprisingly, the order-sign conjecture need not hold:

Observation 5 The Bayesian value revision can be opposite the direction of an order.

This is to say that the direction of orders is sometimes different from the sign of their information content. For example, in the high δ/high α parameterization, LBB_2 limit buys at t_1 reveal bad news (rather than good news as one might expect given that they are buy orders). The same is true of LBB_1 limit buys at t_2 through t_4. This is because, in our second model, these limit buys are used more frequently by directionally informed investor to trade opposite (rather than with) their information (i.e., due to their private-values β_{t_j}).

2.2.4 Non-Markovian learning

Figure 8 shows once again the variation in the incremental information $E[v|\mathscr{L}_{t_j}(L_{t_j})] - E[v|L_{t_j}]$ in the prior order histories $\mathscr{L}_{t_j}(L_{t_j})$ preceding different books L_{t_j}. The plots here confirm our earlier results about non-Markovian learning.

Figure 7: Order Informativeness for the Model with Informed Traders and Uninformed Traders both with $\beta \sim Tr[\mathcal{N}(\mu, \sigma^2)]$. for times t_1 to t_4. This figure shows the path-contingent Bayesian value-forecast revisions $E[v|\mathcal{L}_{t_j-1}, x_{t-j}] - E[v|\mathcal{L}_{t_j-1}]$, which shows the change in the uninformed traders's expected value of the fundamental conditional on the order. Plots a,c,e and g show graphs for the parametrization with $\alpha = 0.8$ and $\delta = 1.6$. Plots b,d,f and h show graphs for the parametrization with $\alpha = 0.8$ and $\delta = 0.2$. Plots i,k,m and o show graphs for the parametrization with $\alpha = 0.2$ and $\delta = 1.6$. Plots j,l,n and p show graphs for the parametrization with $\alpha = 0.2$ and $\delta = 0.2$. We only consider orders when they are equilibrium orders for the trading periods. Each dot indicates an equilibrium revision, the plots indicate the maximum and the minimum.

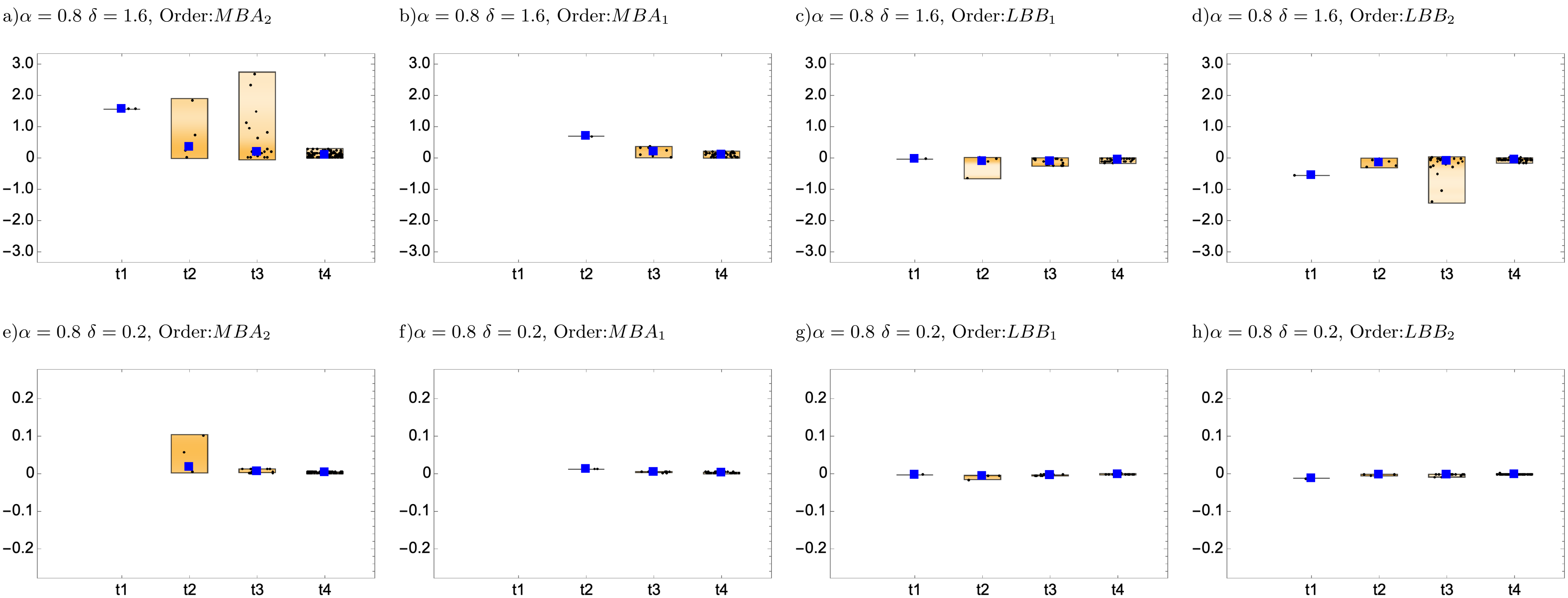


Figure 7 Continued: Order Informativeness for the Model with Informed Traders and Uninformed Traders both with $\beta \sim Tr[\mathcal{N}(\mu, \sigma^2)]$. for times t_1 to t_4.

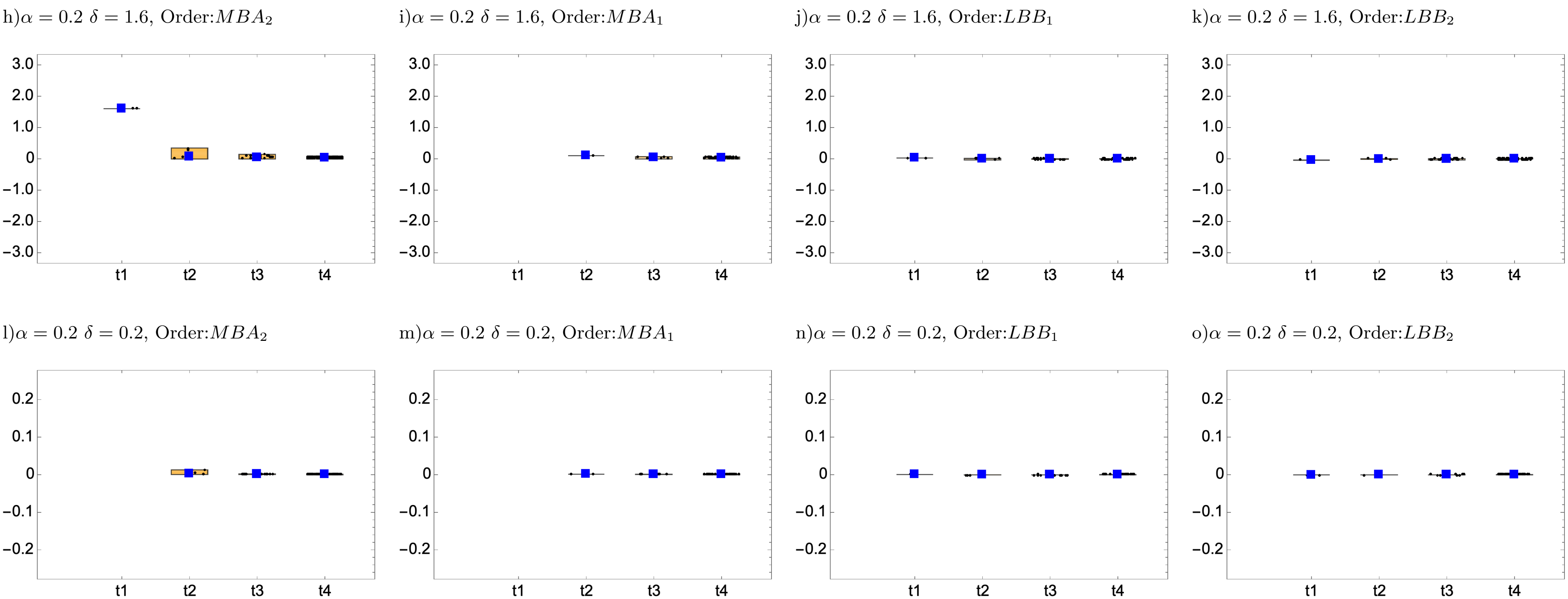
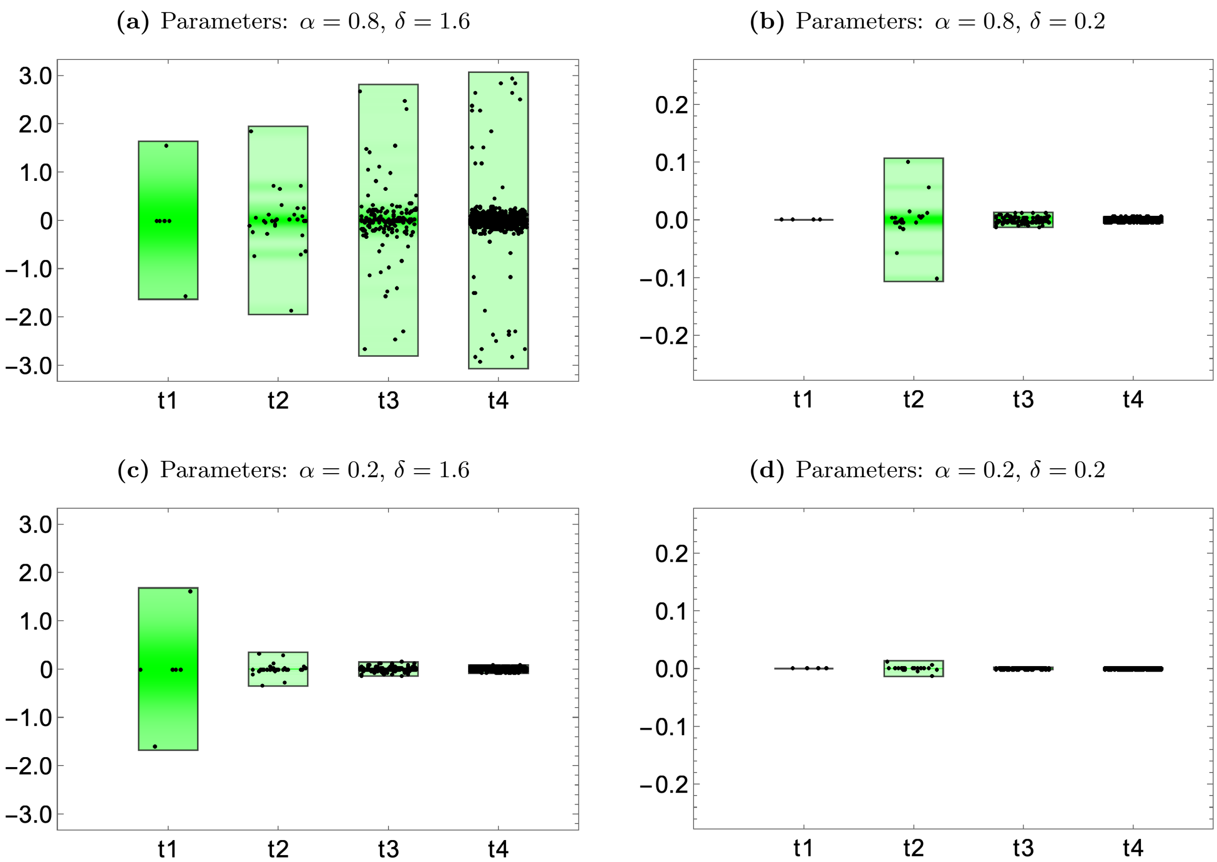


Figure 9 plots the cumulative valuation revisions up though time t_2 against the corresponding revisions along that path through time t_1 for the high adverse-selection (high δ and high α) parameterization. The relationship is more complicated than in Figure 5 due to the violation of the order-sign property in our second model. In particular, a LSA_2 sell limit order at time t_1 is associated with good news (rather than bad news) due to the opposite-side effect. The volatility of the incremental revision at time t_2 is large due to the possibility of a market buy order MBA_2 at t_2 (which would reveal further good news) or a market sell order MSB_2 (which would reveal bad news resulting in a negative cumulative revision up through time t_2). Note also that the distribution of the incremental revision at time t_2 is very skewed following a market buy order MBA_2 at time t_1. Most of the revisions are clustered near the 45° line, but there is a small equilibrium probability of a market sell order MSB_2 leading to a very negative downward revision in the lower-right quadrant.

2.3 Summary

The results for our second model specification — with the richer specification of the informed investors' trading motives — confirm and extend the analysis from our first model specification. First, increased adverse selection affects informed-investor trading behavior differently when directionally informed investors trade with their information versus (because of private-value shocks) against their information. Second, it is again possible for the informativeness of orders to be opposite the order aggressiveness and now also opposite the order direction. Third, information content of arriving orders is again history-dependent.

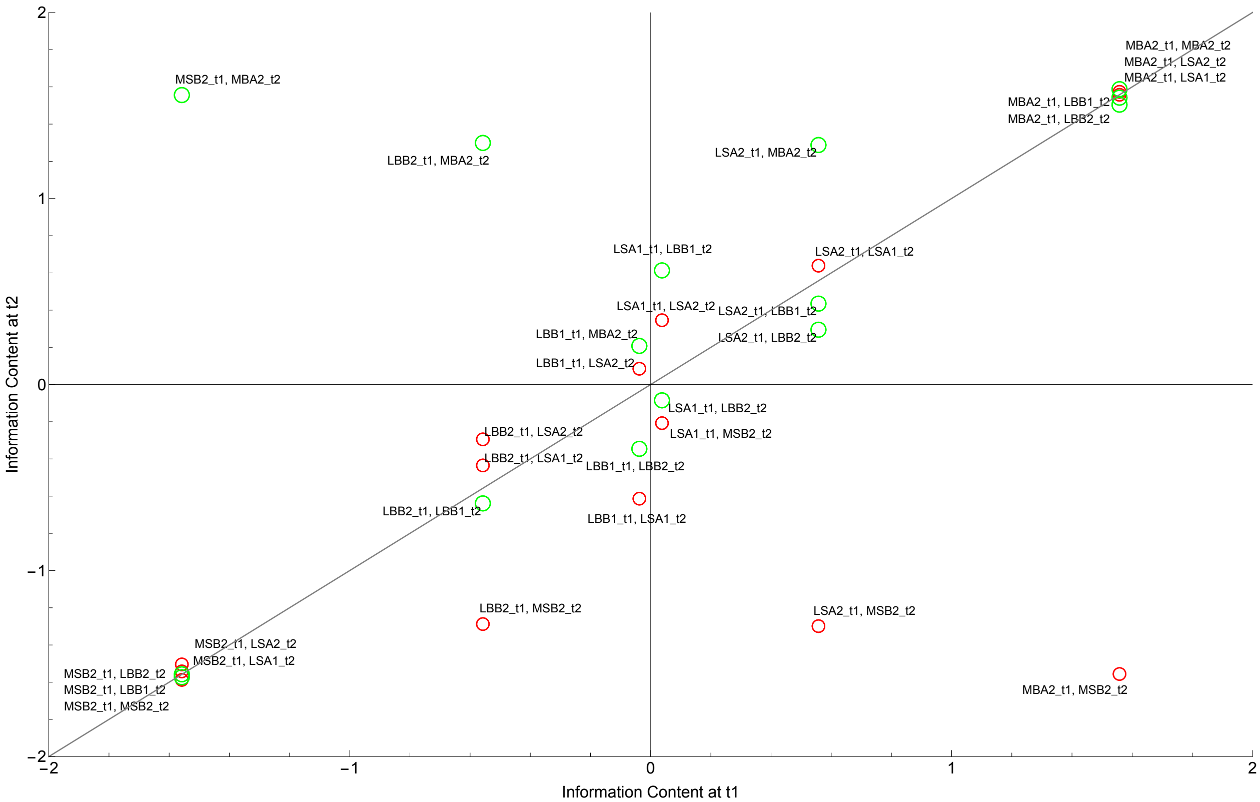
Figure 8: History Informativeness for Informed and Uninformed Traders both with $\beta \sim Tr[\mathcal{N}(\mu, \sigma^2)]$ for times t_1 through t_4. This Figure shows the incremental information content of the past order history in excess of the information in the current limit order book observed at the end of time t_j as measured by $E[v|\mathscr{L}_{t_j}(L_{t_j})] - E[v|L_{t_j}]$ where $\mathscr{L}_{t_j}(L_{t_j})$ is a history ending in the limit order book L_{t_j}. We only consider books L_{t_j} when they occur in equilibrium in the different trading periods. The candlesticks indicate for each of these two metrics the maximum, the minimum, the median and the 75^{th} (and 25^{th}) percentile respectively as the top (bottom) of the bar.



3 Robustness

Our analysis makes a number of simplifying assumptions for tractability, but we conjecture that our qualitative results are robust to relaxing these assumptions. We consider two of these assumptions here. First, our model of the trading day only has five periods. Relatedly, our analysis abstracts

Figure 9: Order Informativeness for Informed and Uninformed both with $\beta \sim Tr[\mathcal{N}(\mu, \sigma^2)]$ for times t_1 to t_2 and parameters $\alpha = 0.8$, $\delta = 1.6$. The horizontal axis reports $E(v|x_{t_1}) - E(v)$ which shows how the uninformed traders' Bayesian value-forecast changes with respect to the unconditional expected value of the fundamental when uninformed traders observe at t_1 an equilibrium order x_{t_1}. The vertical axis reports $E(v|x_{t_2}, x_{t_1}) - E(v)$ which shows how the uninformed traders' Bayesian value-forecast changes with respect to the unconditional expected value of the fundamental when uninformed traders observe at x_{t_2} at t_2. We consider all the equilibrium strategies at t_1 and t_2 which are symmetrical. Green (red) circles show equilibrium buy (sell) orders at t_2.



from limit orders being carried over from one day to the next. However, our results about the impact of adverse selection on investor trading strategies and about order informativeness are driven in large part by the relative size of information shocks and the tick size rather than by the number of rounds of trading. In addition, increasing the trading horizon leads to longer histories that are potentially even more informative. Second, arriving investors are only allowed to submit single orders that cannot be cancelled or modified subsequently. However, it seems likely that order-flow histories will still be informative if orders at different points in time are correlated due to correlated actions of returning investors.

4 Conclusions

This paper has studied information aggregation and liquidity provision in dynamic limit order markets. We show a number of notable theoretical properties in our model. First, informed investors switch between endogenously demanding liquidity via market orders and supplying liquidity via limit orders. Second, the information content/price impact of orders can be non-monotone in the direction of the order and in the aggressiveness of their orders. Third, the information aggregation process is non-Markovian. In particular, the prior order history has information content beyond that in the current limit order book.

Our model suggests several directions for future research. Most importantly, our analysis provides a framework for empirical research about the changing price impacts of order flow conditional on order-flow history and time of day. There are also promising directions for future theory. First, the model can be enriched by allowing investors to trade dynamically over time (rather than just submitting an order one time) and to face quantity decisions and to use multiple orders. Second, the model could be extended to allow for trading in multiple fragmented limit order markets. This would be a realistic representation of current equity trading in the US. Third, the model could be used to study high frequency trading in limit order markets and the effect of different investors being able to process and trade on different types of information at different latencies.

5 Appendix A: Illustration of order paths and Bayesian updating

This appendix uses an excerpt of the extensive form of the trading game in our model to illustrate order-submission and trading dynamics and the associated Bayesian updating process. The particular trading history path in Figure 10 is from the equilibrium for a model specification in which informed and uninformed investors both have random private-value motives. The model is considered in detail in Section 2.2. There are $N = 5$ rounds of trade, and the parameter values are $\kappa = 1$, $\sigma = 15$, $\alpha = 0.8$, and $\delta = 1.6$. This is a market with a relatively high informed-investor arrival probability and large value shocks. In this example, Nature has chosen an economic state in which there is good news ($\overline{v}$) about the asset, and the realized sequence of arriving traders over time is $\{I, U, U, I, I\}$. At each node shown here, Figure 10 reports the total book L_{t_j} of limit orders from both arriving investors and the crowd. Trading starts at t_1 with a book $[1, 0, 0, 1]$ consisting of no orders from informed and uninformed investors (since none have arrived yet) plus the additional limit orders from the trading crowd (i.e., 1 each at the outside prices A_2 and B_2). For simplicity, our discussion here only reports a few nodes of the trading game with their associated equilibrium strategies. For example, we do not include NT at the end of t_1, since Section 2.2 will show that NT is not an equilibrium action at t_1 for these parameters.

Investors in our equilibrium choose from a discrete number of possible orders given their respective information and any private-value trading motives. Along the particular equilibrium path considered in this example, the optimal strategies do not involve any randomization. Optimal orders are unique given the inputs. However, orders are random after conditioning on the arriving investor's informational type (I_v or U) due to randomness in investors' private factor β. Figure 10 shows below each order type at each time the probabilities with which the different orders are submitted by the trader who arrived. For example, if an informed investor $I_{\overline{v}}$ arrives at t_1, she chooses a limit order LSA_2 to sell at A_2 with probability 0.118. Each of these unique optimal orders is associated with a different range of β types (for both informed and uninformed investors) and value signals (for informed investors). Figure 2 in the main body of the paper shows an example of how order-submission probabilities are determined. At each trading time, as the trading game progresses along this path, traders submit orders (or do not trade) following their equilibrium order-submission strategies. The equilibrium execution probabilities of their orders depend on the order-submission decisions of future traders, which, in turn, depend on their trading strategies and the input information (i.e., their β realizations, any private knowledge about v, and the order history path when they arrive). At time t_1, the initial trader has rational-expectation beliefs that the execution probability of her LSA_2 order posted at t_1 is 0.644.[18] This equilibrium execution probability depends on all of the possible future trading paths proceeding from submission time t_1 up through time t_5. For example, one possibility is that the LSA_2 order will be hit by an investor

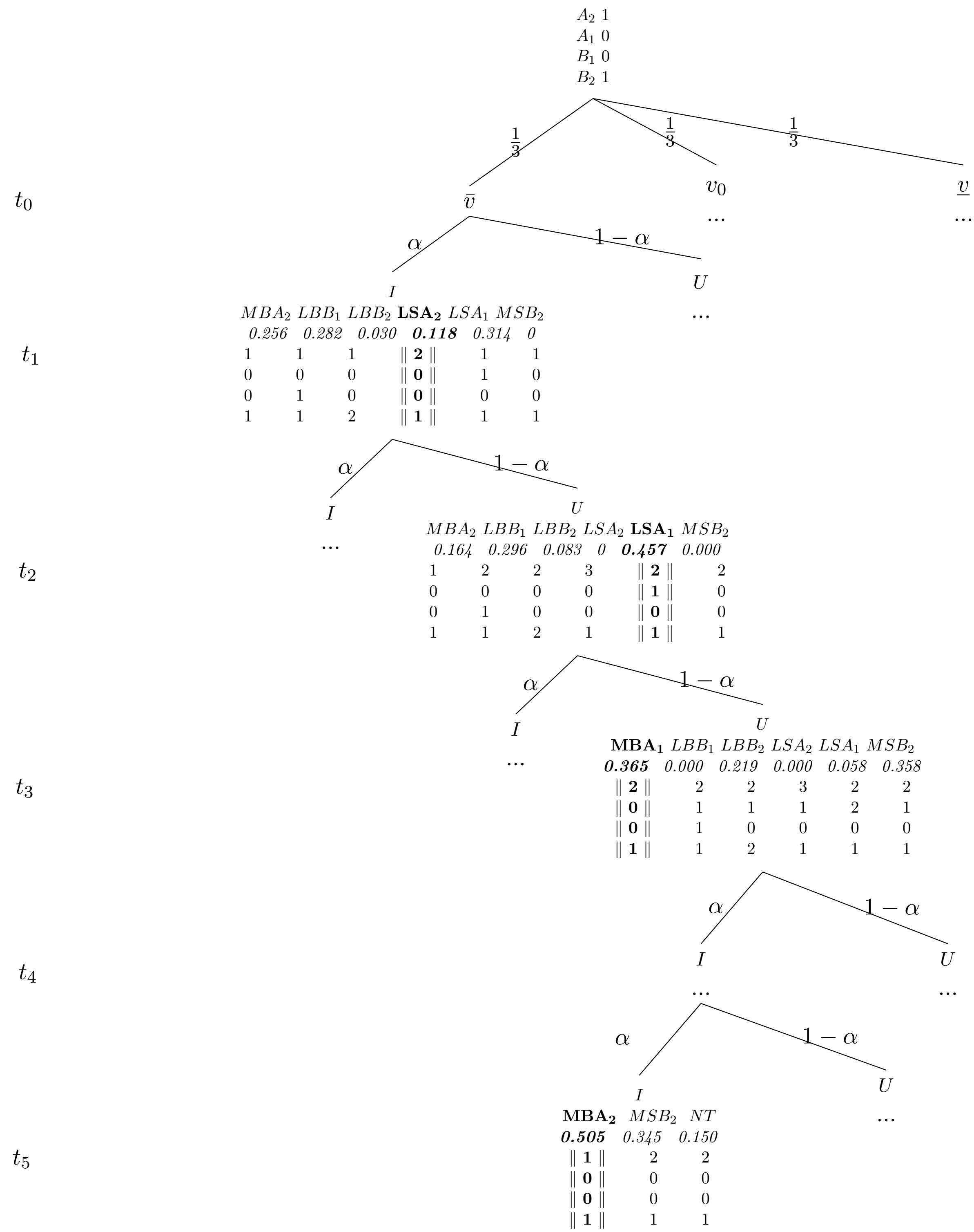
[18]Some of the numerical values discussed here are from equilibrium calculations reported in more detail in Tables 3 and 4 and Table B2 in Appendix B. Others are unreported calculations available from the authors upon request.

arriving at time t_2 who submits a market order. Another possibility (which is what happens along this particular path) is that an uninformed trader will arrive at t_2 and post a limit order LSA_1 to sell at A_1, thereby undercutting the earlier LSA_2 order — so that the book at the end of t_2 is $[2, 1, 0, 1]$). In this scenario, the initial LSA_2 order from t_1 will only be executed provided that the LSA_1 order submitted at t_2 is executed first. For example, the probability of a market order MBA_1 hitting the limit order at A_1 at t_3 is 0.365, and then the probability of another market order hitting the initial limit sell at A_2 is 0.423 at t_4 and 0.505 at t_5.[19] Therefore, there is a chance that the LSA_2 order from t_1 will still be executed even after it is undercut by the order LSA_1 at t_2.

The path in Figure 10 also illustrates Bayesian updating in the model. After the investor at t_1 has been observed submitting a limit order LSA_2, the uninformed trader who arrives in this example at time t_2 — who just knows the submitted order at time t_1 but not the identity or information of the trader at time t_1 — updates his equilibrium conditional valuation to be $E[\tilde{v}|LSA_2] = 10.558$ and his execution-contingent expectation given his limit order LSA_1 at time t_2 to be $E[\tilde{v}|LSA_2, \theta_{t_2}^{LSA_1}] = 10.639$. In subsequent periods, later investors observe additional realized orders and then further update their beliefs.

[19]Due to space constraints, we do not include the t_4 node in Figure 10.

Figure 10: Excerpt of the Extensive Form of the Trading Game. This figure shows one possible trading path of the trading game with parameters $\alpha = 0.8$, $\delta = 1.6$, $\mu = 10$, $\sigma = 15$, $\kappa = 1$, and 5 time periods. Before trading starts at time t_1, the incoming book $[1, 0, 0, 1]$ from time t_0 consists of just the initial limit orders from the crowd at A_2 and B_2. Nature selects a realized final value $v = \{\bar{v}, v_0, \underline{v}\}$ with probabilities $\{\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\}$. At each trading period nature also selects an informed trader (I) with probability α and an uninformed trader (U) with probability $1 - \alpha$. Arriving traders choose the optimal order at each period which may potentially include limit orders LSA_i (LBB_i) or market orders at the best ask, $MBA_{i,t}$, or at the best bid, $MSB_{i,t}$. Below each optimal trading strategy we report in italics its equilibrium order-submission probability. Boldfaced equilibrium strategies and associated states of the book (within double vertical bar) indicate the states of the book that we consider at each node of the chosen trading path.



6 Appendix B: Algorithm for computing equilibrium

The computational problem to solve for a Perfect Bayesian Nash equilibrium in our model (as defined in Section 1.1) is complex. Given investors' equilibrium beliefs, the optimal order-submission problems in (6) and (7) require computing limit-order execution probabilities and stock-value expectations that are conditional on both the past order history and on future state-contingent limit-order execution at each time t_j at each node of the trading game. For an informed trader (who knows the asset value v), there is no uncertainty about the payoff of a market order. In contrast, the payoff of a market order for an uninformed trader entails uncertainty about the future asset value and, therefore, computing the optimal order requires computing the expected stock value $E[v|\mathscr{L}_{t_{j-1}}]$ conditional on the prior trading history up to time t_j. For limit orders, the expected payoff depends on the future limit-order execution probabilities, $Pr(\theta^x_{t_j}|v, \mathscr{L}_{t_{j-1}})$ and $Pr(\theta^x_{t_j}|\mathscr{L}_{t_{j-1}})$, for informed and uninformed investors, which depend, in turn, on the optimal order-submission probabilities of future informed and uninformed investors. In addition, the uninformed investors' learning problem for limit orders requires uninformed investors to extract information about the expected future stock value $E[v|\mathscr{L}_{t_{j-1}}, \theta^x_{t_j}]$ from both the past trading history and also from state-contingent future order execution given that the future states in which limit orders are executed are correlated with the stock value. Thus, optimal actions at each time t_j depend on past information and future order-flow contingencies where future orders also depend on the then-prior histories at future dates (which include the action at time t_j) as traders dynamically update their equilibrium beliefs as the trading process unfolds. Thus, the learning problem for limit order beliefs is both backward- and forward-looking. Lastly, rational expectations (RE) involves finding a fixed point so that the equilibrium beliefs underlying the optimal order-submission strategies are consistent with the execution probabilities and value expectations that the endogenous optimal strategies produce in equilibrium.

Our numerical algorithm uses backward induction to solve for optimal order strategies given a set of asset-value beliefs for all dates and nodes in the trading game and uses an iterative recursion to solve for RE equilibrium asset-value and order-execution beliefs. The backward induction makes order-execution probabilities consistent with optimal future behavior by later-arriving investors. It also takes future state-contingent execution into account in the uninformed investors' beliefs. Given a set of history-contingent asset-value probability beliefs, we start at time t_5 — when traders only use market orders which allows us to compute the execution probabilities of limit orders at t_4 — and recursively solve the model for optimal order strategies back to time t_1. We then embed the optimal order strategy calculation in an iterative recursion to solve for a fixed point for the RE asset-value beliefs. For a generic round r in this recursion, the outgoing asset-value probabilities $\pi_{t_j}^{v,r-1}$ from round $r-1$ are used iteratively as incoming asset-value beliefs in round r. In particular, these beliefs are used in the learning problem of the uninformed investor to extract information about the ending asset value v from the prior trading histories. They also affect the behavior of informed investors whose order-execution probability beliefs depend in part on the behavior of uninformed traders. Thus, the recursion for a generic round r involves solving by backward induction for optimal strategies for buyers

$$\max_{x \in X_{t_j}} w^{I,r}(x \,|\, v, \mathscr{L}_{t_{j-1}}) \quad = \quad [v_0 + \Delta \, + \beta_{t_j} - p(x)] \, Pr^r(\theta_{t_j}^x \,|\, v, \mathscr{L}_{t_{j-1}}) \tag{16}$$

and

$$\max_{x \in X_{t_j}} w^{U,r}(x \,|\mathscr{L}_{t_{j-1}}) \quad = \quad [v_0 + E^r[\Delta \,|\mathscr{L}_{t_{j-1}}, \theta_{t_j}^x] \, + \beta_{t_j} - p(x)] \, Pr^r(\theta_{t_j}^x|\mathscr{L}_{t_{j-1}}) \tag{17}$$

where

$$E^r[\Delta|\mathscr{L}_{t_{j-1}}, \theta_{t_j}^x] = (\hat{\pi}_{t_j}^{\bar{v},r} \, \bar{v} + \hat{\pi}_{t_j}^{v_0,r} \, v_0 + \hat{\pi}_{t_j}^{\underline{v},r} \underline{v}) - v_0 \tag{18}$$

$$\hat{\pi}_{t_j}^{v,r} \quad = \quad \frac{\Pr^r(\theta_{t_j}^x|v, \mathscr{L}_{t_j})}{Pr^r(\theta_{t_j}^x|\mathscr{L}_{t_j})} \pi_{t_j}^{v,r-1} \tag{19}$$

and where the calculations for sellers are symmetric. Note that at each time t_j the backward induction has already determined the future contingencies $\theta_{t_j}^x$ for limit order executions at times $t > t_j$. Thus, the order-execution probabilities $Pr^r(\theta_{t_j}^x \,|\, v, \mathscr{L}_{t_{j-1}})$ and $Pr^r(\theta_{t_j}^x|\mathscr{L}_{t_{j-1}})$, and the history- and execution-contingent probabilities $\hat{\pi}_{t_j}^{v,r}$ and associated asset-value expectations $E^r[\Delta|\mathscr{L}_{t_{j-1}}, \theta_{t_j}^x]$ are "mongrel" moments in that they are computed using the outgoing history-contingent asset value beliefs $\pi_{t_j}^{v,r-1}$ from round $r-1$ and then updated given the order-execution contingencies computed by backward induction in round r using the round $r-1$ asset-value beliefs. At the end of round r, we then compute updated outgoing asset-value beliefs $\pi_{t_j}^{v,r}$ for round r, which are used as incoming beliefs for the next round $r+1$. The recursion is iterated to find a RE fixed point $\pi_{t_j}^{v}$ in the uninformed investor beliefs.

The fixed-point recursion is started in round $r=1$ by setting the initial asset-value beliefs $\pi_{t_j}^{v,0}$ of uninformed traders at each time t_j in the backward induction to be the unconditional priors $Pr(v)$ in (1). In particular, the algorithm starts by ignoring conditioning on history in the initial round $r=1$. Hence the traders' optimization problems in (17) and (16) in round $r=1$ simplify to:

$$\max_{x \in X_{t_j}} w^{I,r=1}(x \,|\, v, \mathscr{L}_{t_{j-1}}) \quad = \quad [v_0 + \Delta \, + \beta_{t_j} - p(x)] \, Pr^1(\theta_{t_j}^x \,|\, v) \tag{20}$$

$$\max_{x \in X_{t_j}} w^{U,r=1}(x \,|\mathscr{L}_{t_{j-1}}) \quad = \quad [v_0 + E^1[\Delta|\theta_{t_j}^x] \, + \beta_{t_j} - p(x)] \, Pr^1(\theta_{t_j}^x) \tag{21}$$

The order-execution contingencies in round r are modeled as follows: In each round r given the asset-value beliefs $\pi_{t_j}^{v,r-1}$ in that round, we solve for investors' optimal trading strategies by backward induction. Starting at t_5, the execution probability for new limit orders is zero, and therefore optimal order-submission strategies only use market orders. Given the linearity of the expected payoffs in the private-value factor β in (16) and (17), the optimal orders for an informed trader at t_5 are[20]

$$x_{t_5}^{I,r}(\beta|\mathscr{L}_{t_4}, v) = \begin{cases} MSB_{i,t_5} & if \;\; \beta \in [0, \beta^{MSB_{i,t_5}^{I,r}, NT_{t_5}^{I,r}}) \\ NT & if \;\; \beta \in [\beta^{MOB_{i,t_5}^{I,r}, NT_{t_5}^{I,r}}, \, \beta^{NT_{t_5}^{I,r}, MBA_{i,t_5}^{I,r}}) \\ MBA_{i,t_5} & if \;\; \beta \in [\beta^{NT_{t_5}^{I,r}, MBA_{i,t_5}^{I,r}}, \, 2] \end{cases} \tag{22}$$

[20]For instance, an informed trader would post a MBA_1 only if the payoff is positive and thus outperforms the NT payoff of zero, i.e, $\beta v + \Delta - A_1 > 0$ or $\beta > \frac{A_1 - \Delta}{v}$.

where for each possible combination of $MSB_{i,t_5} = MSB_1, MSB_2$ and $MBA_{i,t_5} = MBA_1, MBA_2$

$$\begin{aligned} \beta^{MSB^{I,r}_{i,t_5}, NT^{I,r}_{t_5}} &= \frac{B_{i,t_5} - \Delta}{v} \\ \beta^{NT^{I,r}_{t_5}, MBA^{I,r}_{i,t_5}} &= \frac{A_{i,t_5} - \Delta}{v} \end{aligned} \tag{23}$$

are the critical thresholds that solve $w^{I,r}(MSB_{i,t_5}|v, \mathscr{L}_{t_4}) = w^{I,r}(NT|v, \mathscr{L}_{t_4})$ and $w^{I,r}(NT|v, \mathscr{L}_{t_4}) = w^{I,r}(MBA_{i,t_5}|v, \mathscr{L}_{t_4})$, respectively. Our notation here for market orders differs slightly from the notation in the body of the paper because we need to denote both different possible price levels and the time at which different possible orders are being compared. The optimal trading strategies and β thresholds for an uninformed traders are similar but the conditioning set does not include the asset value v:

$$x^{U,r}_{t_5}(\beta|\mathscr{L}_{t_4}) = \begin{cases} MSB_{i,t_5} & if\ \beta \in [0, \beta^{MSB^{U,r}_{i,t_5}, NT^{U,r}_{t_5}}) \\ NT & if\ \beta \in [\beta^{MSB^{U,r}_{i,t_5}, NT^{U,r}_{t_5}},\ \beta^{NT^{U,r}_{t_5}, MBA^{U,r}_{i,t_5}}) \\ MBA_{i,t_5} & if\ \beta \in [\beta^{NT^{U,r}_{t_5}, MBA^{U,r}_{i,t_5}},\ 2] \end{cases} \tag{24}$$

where

$$\begin{aligned} \beta^{MSB^{U,r}_{i,t_5}, NT^{U,r}_{t_5}} &= \frac{B_{i,t_5} - E^{r-1}[\Delta|\mathscr{L}_{t_4}]}{v} \\ \beta^{NT^{U,r}_{t_5}, MBA^{U,r}_{i,t_5}} &= \frac{A_{i,t_5} - E^{r-1}[\Delta|\mathscr{L}_{t_4}]}{v} \end{aligned} \tag{25}$$

Given the β ranges associated with each possible action at t_5, we compute the submission probabilities associated with each optimal order at t_5 using the truncated-Normal density $\mathfrak{n}(\cdot)$ for the private factor β.[21] At time t_4 these are the execution probabilities for new limit orders by an informed investor at the different possible best bids and asks, B_{i,t_4} and A_{i,t_4} respectively at time

[21]The discussion here is for the case where both informed and uninformed investors have random private factors β.

t_5:

$$Pr^r(\theta_{t_4}^{LBB_i}|\mathscr{L}_{t_3}, v) = \begin{cases} \alpha\Big[\int_0^{\beta^{MSB_{i,t_5}^{I,r},NT_{t_5}^{I,r}}} \mathfrak{n}(\beta)\, d\beta\Big] + \Big(1-\alpha\Big)\Big[\int_0^{\beta^{MSB_{i,t_5}^{U,r},NT_{t_5}^{U,r}}} \mathfrak{n}(\beta) d\beta\Big] \\ 0 \qquad \text{otherwise} \end{cases} \tag{26}$$

$$Pr^r(\theta_{t_4}^{LSA_i}|\mathscr{L}_{t_3}, v) = \begin{cases} \alpha\Big[\int_{\beta^{NT_{t_5}^{I,r},MBA_{i,t_5}^{I,r}}}^2 \mathfrak{n}(\beta)\, d\beta\Big] + \Big(1-\alpha\Big)\Big[\int_{\beta^{NT_{t_5}^{U,r},MBA_{i,t_5}^{U,r}}}^2 \mathfrak{n}(\beta)\, d\beta\Big] \\ 0 \qquad \text{otherwise} \end{cases} \tag{27}$$

where the book is either empty at A_1 and/or B_1 (but may have non-crowd limit orders at the outside prices) or is empty except for just crowd orders at A_2 and B_2. The analogous execution probabilities for an uninformed investor arriving at time t_4 are:

$$Pr^r(\theta_{t_4}^{LBB_i}|\mathscr{L}_{t_3}) = \begin{cases} \alpha\Big[\sum_{v\in\{\overline{v},v_0,\underline{v}\}} \hat{\pi}_{t_4}^{v,r} \int_0^{\beta_{t_5}^{MSB_{i,t_5}^{Iv,r},NT_{t_5}^{Iv,r}}} \mathfrak{n}(\beta)\, d\beta\Big] + \Big(1-\alpha\Big)\Big[\int_0^{\beta_{t_5}^{MSB_{i,t_5}^{U,r},NT_{t_5}^{U,r}}} \mathfrak{n}(\beta)\, d\beta\Big] \\ 0 \qquad \text{otherwise} \end{cases} \tag{28}$$

$$Pr^r(\theta_{t_4}^{LSA_i}|\mathscr{L}_{t_3}) = \begin{cases} \alpha\Big[\sum_{v\in\{\overline{v},v_0,\underline{v}\}} \hat{\pi}_{t_4}^{v,r} \int_{\beta^{NT_{t_5}^{Iv,r},MBA_{i,t_5}^{Iv,r}}}^2 \mathfrak{n}(\beta)\, d\beta\Big] + \Big(1-\alpha\Big)\Big[\int_{\beta^{NT_{t_5}^{U,r},MBA_{i,t_5}^{U,r}}}^2 \mathfrak{n}(\beta)\, d\beta\Big] \\ 0 \qquad \text{otherwise} \end{cases} \tag{29}$$

At t_4 there is only one period before the end of the trading game. Thus, the execution probability of a limit order is positive if and only if the order is posted at the best price on its own side of the market (A_{i,t_4} or B_{i,t_4}), and if there are no non-crowd limit orders already standing in the limit order book at that price at the time the new limit order is posted.

Having obtained the execution probabilities in (26) – (29) for the different limit orders at t_4, we next derive the optimal order-submission strategies at t_4. The incoming book can be configured in many different ways at t_4 depending on the different possible prior order paths $\mathscr{L}_{t_3}$ in the trading game up through time t_3. As the payoffs of both limit and market orders are functions of β, we rank all the payoffs of adjacent optimal strategies in terms of β and equate them to determine the

β thresholds at time t_4.[22] Consider, for example, an order path such that t_4 has only crowd orders in the book, so that new limit and market orders are both potentially optimal orders at t_4. For an informed trader, the the optimal orders are given by:

$$x_{t_4}^{I,r}(\beta|\mathscr{L}_{t_3}, v) = \begin{cases} MSB_2 & if\ \beta \in [0,\ \beta^{MSB_{2,t_4}^{I,r}, LSA_{1,t_4}^{I,r}}) \\ LSA_1 & if\ \beta \in [\beta^{MSB_{2,t_4}^{I,r}, LSA_{1,t_4}^{I,r}},\ \beta^{LSA_{1,t_4}^{I,r}, LSA_{2,t_4}^{I,r}}) \\ LSA_2 & if\ \beta \in [\beta^{LSA_{1,t_4}^{I,r}, LSA_{2,t_4}^{I,r}},\ \beta^{LSA_{2,t_4}^{I,r}, NT_{t_4}^{I,r}}) \\ NT & if\ \beta \in [\beta^{LSA_{2,t_4}^{I,r}, NT_{t_4}^{I,r}},\ \beta^{NT_{t_4}^{I,r}, LBB_{2,t_4}^{I,r}}) \\ LBB_2 & if\ \beta \in [\beta^{NT_{t_4}^{I,r}, LBB_{2,t_4}^{I,r}},\ \beta^{LBB_{2,t_4}^{I,r}, LBB_{1,t_4}^{I,r}}) \\ LBB_1 & if\ \beta \in [\beta^{LBB_{2,t_4}^{I,r}, LBB_{1,t_4}^{I,r}},\ \beta^{LBB_1^{I,r}, MBA_{2,t_4}^{I,r}}) \\ MBA_2 & if\ \beta \in [\beta^{LBB_{1,t_4}^{I,r}, MBA_{2,t_4}^{I,r}},\ 2] \end{cases} \tag{30}$$

and for an uninformed trader the optimal strategies are qualitatively similar but with different values for the β thresholds given the uninformed investor's different information.[23] As the payoffs of both limit and market orders are functions of β, we can rank all the payoffs of adjacent optimal strategies in terms of β and equate them to determine the β thresholds at t_4. For example, for the first β threshold we have:

$$\beta_{t_4}^{MSB_{2,t_4}^{I,r}, LSA_{1,t_4}^{I,r}} = \beta \in \mathbb{R} \text{ s.t. } w_{t_4}^{I,r}\left(MSB_2 \,|\, v, \beta, \mathscr{L}_{t_3}\right) = w_{t_4}^{I,r}\left(LSA_1 \,|\, v, \beta, \mathscr{L}_{t_3}\right) \tag{31}$$

and we obtain the other thresholds similarly.

The next step is to use the β thresholds together with the truncated Normal cumulative distribution $\mathbb{N}(\cdot)$ for β to derive the probabilities of the optimal order-submission strategies at each possible node of the extensive form of the game at t_4. For example, the submission probability of $LSA_1^{I,r}$ is:

$$Pr^r[LSA_1^{I,r} \,|\, \mathscr{L}_{t_3}, v] = \mathbb{N}(\beta^{LSA_1^{I,r}, LSA_2^{I,r}} \,|\, \mathscr{L}_{t_3}, v) - \mathbb{N}(\beta^{MSB_2^{I,r}, LSA_1^{I,r}} \,|\, \mathscr{L}_{t_3}, v) \tag{32}$$

[22] Recall that the upper envelope only includes strategies that are optimal.

[23] If the incoming book from t_3 has non-crowd orders on any level of the book, the equilibrium strategies would be different. For example, if the book has a LSA_1 limit order, then new limit orders on the ask side cannot be equilibrium orders since their execution probability would be zero.

and the submission probabilities of the equilibrium strategies can be obtained in a similar way. Next, given the market-order submission probabilities at t_4 — which together with the execution probabilities at t_5 determine the execution probabilities for new limit orders at time t_3 — we can solve the optimal orders at t_3 and then recursively continue to solve the model by backward induction in this fashion back to time t_1.

Off-equilibrium beliefs: At each time t_j, round r of the recursion needs history-contingent asset-value beliefs $\pi_{t_j}^{v,r-1} = Pr^{r-1}(v|\mathscr{L}_{t_j})$ from round $r-1$ for all feasible paths that traders may use. Beliefs for paths that occur with positive probability in round $r-1$ are computed using Bayes' rule to update the probability $Pr^{r-1}(v|\mathscr{L}_{t_{j-1}})$ of the time-t_{j-1} sub-path $\mathscr{L}_{t_{j-1}}$ that path $\mathscr{L}_{t_j}$ extends. In contrast, Bayes' Rule cannot be used to update probabilities of paths that involve orders that are not used with positive probability in round $r-1$. Our algorithm deals with this by setting $Pr^{r-1}(v|\mathscr{L}_{t_j})$ to be $Pr^{r-1}(v|\mathscr{L}_t)$ where $\mathscr{L}_t$ is the longest positive-probability sub-path from t_0 to some time $t < t_k$ in round $r-1$ that is contained in path $\mathscr{L}_{t_j}$. For example, consider a path $\{MBA_2, MSB_2, LSA_1\}$ at time t_3 where orders $\{MBA_2, MSB_2\}$ are used with positive probability at times t_1 and t_2 in round $r-1$, but LSA_1 is not used at time t_3 after the first two orders in round $r-1$. Our recursion algorithm sets the round $r-1$ belief uninformed traders use for path $\{MBA_2, MSB_2, LSA_1\}$ to be their round $r-1$ belief for the positive-probability sub-path $\{MBA_2, MSB_2\}$. If instead MSB_2 is not a positive-probability order at t_2 in round $r-1$, then we assume that uninformed traders use their belief at t_1 conditional on the shorter sub-path $\{MBA_2\}$. Finally, if MBA_2 is also not a positive-probability order at t_1 in round $r-1$, then we assume that traders use their unconditional prior belief $Pr(v)$.

Mixed strategies: We allow for both pure and mixed strategies in our Perfect Bayesian Nash equilibrium. When different orders have equal expected payoffs, we assume that traders randomize with equal probabilities across all such optimal orders. By construction, the expected payoffs of two different strategies are the same in correspondence of the β thresholds; however because we are considering single points in the support of the β distribution, the probability associated with any strategy that corresponds to those specific points is equal to zero. This means that mixed strategies that emerge in correspondence of the β thresholds, although feasible, have zero probability. Mixed strategies may also emerge in the framework in which informed traders have a fixed neutral private-value factor $\beta = 0$ (section 2.1). More specifically it may happen that the payoffs of two perfectly symmetrical strategies of I_{v_0} are the same, and in this case I_{v_0} randomizes between these two strategies.

In the setting of our model where informed traders have fixed neutral private-value factors $\beta = 0$, it may happen that both informed and uninformed traders switch their strategies back and forth from one round to the next. When this happens, to reach an equilibrium we assume that the informed traders play mixed strategies and at each subsequent round strategically reduce the probability with which they choose the most profitable strategy until the equilibrium is reached. As an example at t_1 informed traders with positive news, $I_{\bar{v}}$, play LBB_2 in round $r = 1$. However, in round $r = 2$ in the subsequent periods uninformed traders do not send market orders to sell at B_2 and in round $r = 3$, informed traders react by changing their strategy to LBB_1. However, in the subsequent periods uninformed traders do not send market orders to sell, this time at B_1. To find an equilibrium, we assume that at each round informed traders play mixed strategies and assign a greater weight to the most profitable strategy. In this case we assume they start playing LBB_2 with probability 0.99 and LBB_1 with probability 0.01. If these mixed strategies do not lead to an equilibrium outcome, in the subsequent round we assume that the informed traders play LBB_2 with probability 0.98 and LBB_1 with probability 0.02. We proceed by lowering the probability with which informed traders choose the most profitable strategy until we reach an equilibrium set of strategies.

Convergence: RE beliefs for a Perfect Bayesian Nash equilibrium are obtained by solving the model recursively for multiple rounds. In particular, the asset-value probabilities $\pi_{t_j}^{v,1}$ from round $r = 1$ from above are used as the priors to solve the model in round $r = 2$ (i.e., the round 1 probabilities are used in place of the unconditional priors used in round 1).[24] The asset-value probabilities $\pi_{t_j}^{v,2}$ from round $r = 2$ are then used as the priors in round $r = 3$ and so on. The recursive iteration is continued until the updating process converges to a fixed point, which are the

[24] In the second round of solutions we again solve the full 5-period model.

RE beliefs. In particular, the recursive process has converged to the RE beliefs when uninformed traders no longer revise their asset-value beliefs. Operationally, we consider convergence to the RE beliefs to have occurred when the probabilities $\pi_{t_j}^{\overline{v},r}$, $\pi_{t_j}^{v_0,r}$ and $\pi_{t_j}^{\underline{v},r}$ in round r are "close enough" to the corresponding probabilities from round $r-1$:

$$
\begin{aligned}
&\pi_{t_j}^{\overline{v},*} \text{ when } \left|\pi_{t_j}^{\overline{v},r} - \pi_{t_j}^{\overline{v},r-1}\right| < 10^{-7} \\
&\pi_{t_j}^{v_0,*} \text{ when } \left|\pi_{t_j}^{v_0,r} - \pi_{t_j}^{v_0,r-1}\right| < 10^{-7} \\
&\pi_{t_j}^{\underline{v},*} \text{ when } \left|\pi_{t_j}^{\underline{v},r} - \pi_{t_j}^{\underline{v},r-1}\right| < 10^{-7}
\end{aligned}
\tag{33}
$$

A fixed-point solution to this recursive algorithm is an equilibrium in our model.

7 Appendix C: Additional numerical results

The tables is this section provide additional information on the execution probabilities of limit orders for informed investor with positive, neutral and negative signals, ($I_{\bar{v}}$,I_{v_0},$I_{\underline{v}}$) and for uninformed traders. The tables also report the asset value expectations of the uninformed investor at time t_2 after observing all the possible buy orders submissions at time t_1. The expectations for sell orders are symmetric with respect to 1. Table B1 reports results for our first model specification in which only uninformed traders have a random private value factor. Table B2 reports results for our second model in which both the informed and uniformed traders have private-value motives.

Table B1: Order Execution Probabilities and Asset-Value Expectation for Informed Traders with $\boldsymbol{\beta = 0}$ and Uninformed Traders with $\boldsymbol{\beta \sim Tr[\mathcal{N}(\mu,\, \sigma^2)]}$. This table reports results for two different values of the informed-investor arrival probability α (0.8 and 0.2) and for two different values of the asset-value volatility δ (1.6 and 0.2). $\sigma = 15$. For each set of parameters, the first four columns report the equilibrium limit order probabilities of executions for informed traders with positive, neutral and negative signals, ($I_{\bar{v}}$,I_{v_0},$I_{\underline{v}}$) and for uninformed traders (U). The fifth column (*Uncond.*) reports the unconditional order-execution probabilities in the market. Next, the columns report conditional and unconditional future order execution probabilities and the asset-value expectations of an uniformed investor at time t_2 after observing different order submissions at time t_1.

		$\delta = 1.6$					$\delta = 0.2$				
		$I_{\bar{v}}$	I_{v_0}	$I_{\underline{v}}$	U	*Uncond.*	$I_{\bar{v}}$	I_{v_0}	$I_{\underline{v}}$	U	*Uncond.*
	$P^{EX}(LSA_2\|\cdot)$	0.940	0.199	0.059	0.399	0.399	0.180	0.229	0.170	0.193	0.193
	$P^{EX}(LSA_1\|\cdot)$	0.988	0.134	0.078	0.400	0.400	0.323	0.323	0.323	0.323	0.323
	$P^{EX}(LBB_1\|\cdot)$	0.078	0.134	0.988	0.400	0.400	0.323	0.323	0.323	0.323	0.323
	$P^{EX}(LBB_2\|\cdot)$	0.059	0.199	0.940	0.399	0.399	0.170	0.229	0.180	0.193	0.193
$\alpha = 0.8$											
	E[v|LBB_1 |·]					11.600					10.000
	E[v|LBB_2 |·]					10.820					10.130
	E[v|MBA_1 |·]										
	E[v|MBA_2 |·]					10.000					10.000
	$P^{EX}(LSA_2\|\cdot)$	0.656	0.490	0.396	0.514	0.514	0.514	0.499	0.476	0.496	0.496
	$P^{EX}(LSA_1\|\cdot)$	0.886	0.763	0.713	0.787	0.787	0.792	0.792	0.790	0.791	0.791
	$P^{EX}(LBB_1\|\cdot)$	0.713	0.763	0.886	0.787	0.787	0.790	0.792	0.792	0.791	0.791
	$P^{EX}(LBB_2\|\cdot)$	0.396	0.490	0.656	0.514	0.514	0.476	0.499	0.514	0.496	0.496
$\alpha = 0.2$											
	E[v|LBB_1 |·]					10.278					10.000
	E[v|LBB_2 |·]					10.083					10.089
	E[v|MBA_1 |·]										
	E[v|MBA_2 |·]					10.000					10.000

Table B2: Order Execution Probabilities and Asset-Value Expectation for Informed and Uninformed Traders both with $\boldsymbol{\beta \sim Tr[\mathcal{N}(\mu,\, \sigma^2)]}$. This table reports results for two different values of the informed-investor arrival probability α (0.8 and 0.2) and for two different values of the asset-value volatility δ (1.6 and 0.2). $\sigma = 15$. For each set of parameters, the first four columns report the equilibrium limit order probabilities of executions for informed traders with positive, neutral and negative signals, ($I_{\bar{v}}$,I_{v_0},$I_{\underline{v}}$) and for uninformed traders (U). The fifth column ($Uncond.$) reports the unconditional order-execution probabilities in the market. Next, the columns report conditional and unconditional future order execution probabilities and the asset-value expectations of an uniformed investor at time t_2 after observing different order submissions at time t_1.

		$\delta = 1.6$					$\delta = 0.2$				
		$I_{\bar{v}}$	I_{v_0}	$I_{\underline{v}}$	U	$Uncond.$	$I_{\bar{v}}$	I_{v_0}	$I_{\underline{v}}$	U	$Uncond.$
	$P^{EX}(LSA_2\|\cdot)$	0.644	0.502	0.410	0.519	0.519	0.502	0.487	0.472	0.487	0.487
	$P^{EX}(LSA_1\|\cdot)$	0.913	0.834	0.702	0.817	0.817	0.849	0.837	0.824	0.836	0.836
	$P^{EX}(LBB_1\|\cdot)$	0.702	0.834	0.913	0.817	0.817	0.824	0.837	0.849	0.836	0.836
	$P^{EX}(LBB_2\|\cdot)$	0.410	0.502	0.644	0.519	0.519	0.472	0.487	0.502	0.487	0.487
$\alpha = 0.8$											
	E[v|LBB_1 |$\cdot$]					9.962					10.003
	E[v|LBB_2 |$\cdot$]					9.442					9.988
	E[v|MBA_1 |$\cdot$]										
	E[v|MBA_2 |$\cdot$]					11.558					
	$P^{EX}(LSA_1\|\cdot)$	0.525	0.494	0.470	0.496	0.496	0.490	0.487	0.483	0.487	0.487
	$P^{EX}(LSA_1\|\cdot)$	0.853	0.833	0.813	0.833	0.833	0.839	0.837	0.834	0.837	0.837
	$P^{EX}(LBB_1\|\cdot)$	0.813	0.833	0.853	0.833	0.833	0.834	0.837	0.839	0.837	0.837
	$P^{EX}(LBB_2\|\cdot)$	0.470	0.494	0.525	0.496	0.496	0.483	0.487	0.490	0.487	0.487
$\alpha = 0.2$											
	E[v|LBB_1 |$\cdot$]					10.029					10.001
	E[v|LBB_2 |$\cdot$]					9.957					9.999
	E[v|MBA_1 |$\cdot$]										
	E[v|MBA_2 |$\cdot$]					11.600					

Table 5: Trading Strategies, Liquidity, and Welfare at Time t_1 in an Equilibrium with Informed Traders with $\boldsymbol{\beta = 0}$ and Uninformed Traders with $\boldsymbol{\beta \sim Tr[\mathcal{N}(\mu, \sigma^2)]}$. This table reports results for two different informed-investor arrival probabilities α (0.8 and 0.2) and two different value-shock volatilities δ (1.6 and 0.2). The private-value factor parameters are $\mu = 10$ and $\sigma = 10$, and the tick size is $\kappa = 1$. Each cell corresponding to a set of parameters reports the equilibrium order-submission probabilities, the expected bid-ask spreads and expected depths at the inside prices (A_1 and B_1) and total depths on each side of the market after order submissions at time t_1, and expected welfare of the market participants. The first four columns in each parameter cell are for informed traders with positive, neutral and negative signals, ($I_{\bar{v}}$,I_{v_0},$I_{\underline{v}}$) and for uninformed traders (U). The fifth column (*Uncond.*) reports unconditional results for the market.

		$\delta = 1.6$					$\delta = 0.2$				
		$I_{\bar{v}}$	I_{v_0}	$I_{\underline{v}}$	U	*Uncond.*	$I_{\bar{v}}$	I_{v_0}	$I_{\underline{v}}$	U	*Uncond.*
$\alpha = 0.8$	LSA_2	0	0.500	0.620	0.155	0.330	0.010	0.500	0.990	0.054	0.411
	LSA_1	0	0	0.380	0	0.101	0	0	0	0.088	0.018
	LBB_1	0.380	0	0	0	0.101	0	0	0	0.088	0.018
	LBB_2	0.620	0.500	0	0.155	0.330	0.990	0.500	0.010	0.054	0.411
	MBA_2	0	0	0	0.345	0.069	0	0	0	0.358	0.072
	MBA_1	0	0	0	0	0	0	0	0	0	0
	MSB_1	0	0	0	0	0	0	0	0	0	0
	MSB_2	0	0	0	0.345	0.069	0	0	0	0.358	0.072
	NT	0	0	0	0	0	0	0	0	0	0
	E[Spread $\vert\cdot$]	2.620	3.000	2.620	3.000	2.797	3.000	3.000	3.000	2.824	2.965
	E[Depth A_2+A_1 $\vert\cdot$]	1.000	1.500	2.000	1.155	1.431	1.010	1.500	1.990	1.142	1.428
	E[Depth A_1 $\vert\cdot$]	0	0	0.380	0	0.101	0	0	0	0.088	0.018
	E[Depth B_1 $\vert\cdot$]	0.380	0	0	0	0.101	0	0	0	0.088	0.018
	E[Depth B_1+B_2 $\vert\cdot$]	2.000	1.500	1.000	1.155	1.431	1.990	1.500	1.010	1.142	1.428
	E[Welfare LO $\vert\cdot$]	0.338	0.581	0.338	0.204	0.376	0.513	0.672	0.513	0.163	0.485
	E[Welfare MO $\vert\cdot$]	0	0	0	3.149	0.630	0	0	0	3.179	0.636
	E[Welfare $\vert\cdot$]	0.338	0.581	0.338	3.353	1.006	0.513	0.672	0.513	3.342	1.121
$\alpha = 0.2$	LSA_2	0	0.500	0.180	0.058	0.091	0	0.500	1.000	0.061	0.149
	LSA_1	0	0	0.820	0.383	0.361	0	0	0	0.395	0.316
	LBB_1	0.820	0	0	0.383	0.361	0	0	0	0.395	0.316
	LBB_2	0.180	0.500	0	0.058	0.091	1.000	0.500	0	0.061	0.149
	MBA_2	0	0	0	0.060	0.048	0	0	0	0.044	0.035
	MBA_1	0	0	0	0	0	0	0	0	0	0
	MSB_1	0	0	0	0	0	0	0	0	0	0
	MSB_2	0	0	0	0.060	0.048	0	0	0	0.044	0.035
	NT	0	0	0	0	0	0	0	0	0	0
	E[Spread $\vert\cdot$]	2.180	3.000	2.180	2.235	2.278	3.000	3.000	3.000	2.211	2.369
	E[Depth A_2+A_1 $\vert\cdot$]	1.000	1.500	2.000	1.440	1.452	1.000	1.500	2.000	1.456	1.465
	E[Depth A_1 $\vert\cdot$]	0	0	0.820	0.383	0.361	0	0	0	0.395	0.316
	E[Depth B_1 $\vert\cdot$]	0.820	0	0	0.383	0.361	0	0	0	0.395	0.316
	E[Depth B_1+B_2 $\vert\cdot$]	2.000	1.500	1.000	1.440	1.452	2.000	1.500	1.000	1.456	1.465
	E[Welfare LO $\vert\cdot$]	2.659	1.373	2.659	3.034	2.873	0.780	1.444	0.780	3.341	2.873
	E[Welfare MO $\vert\cdot$]	0	0	0	0.920	0.736	0	0	0.000	0.695	0.556
	E[Welfare $\vert\cdot$]	2.659	1.373	2.659	3.954	3.610	0.780	1.444	0.780	4.036	3.429

Table 6: Averages for Trading Strategies, Liquidity, and Welfare across Times t_2 through t_4 for Informed Traders with $\beta = 0$ and Uninformed Traders with $\beta \sim Tr[\mathcal{N}(\mu, \sigma^2)]$. This table reports results for two different informed-investor arrival probabilities α (0.8 and 0.2) and for two different asset-value volatilities δ (1.6 and 0.2). The private-value factor parameters are $\mu = 10$ and $\sigma = 10$, and the tick size is $\kappa = 1$. Each cell corresponding to a set of parameters reports the equilibrium order-submission probabilities, the expected bid-ask spreads and expected depths at the inside prices (A_1 and B_1) and total depths on each side of the market after order submissions at times t_2 through t_4, and expected welfare for the market participants. The first four columns in each parameter cell are for informed traders with positive, neutral and negative signals, ($I_{\bar{v}}$,I_{v_0},$I_{\underline{v}}$) and for uninformed traders (U). The fifth column ($Uncond.$) reports unconditional results for the market.

		$\delta = 1.6$					$\delta = 0.2$				
		$I_{\bar{v}}$	I_{v_0}	$I_{\underline{v}}$	U	$Uncond.$	$I_{\bar{v}}$	I_{v_0}	$I_{\underline{v}}$	U	$Uncond.$
$\alpha = 0.8$	LSA_2	0	0.243	0.047	0.157	0.109	0.394	0.254	0.106	0.027	0.206
	LSA_1	0	0.257	0.244	0.029	0.139	0.194	0.240	0.289	0.069	0.207
	LBB_1	0.244	0.257	0	0.029	0.139	0.289	0.240	0.194	0.069	0.207
	LBB_2	0.047	0.243	0	0.157	0.109	0.106	0.254	0.394	0.027	0.206
	MBA_2	0.488	0	0	0.293	0.189	0	0	0	0.338	0.068
	MBA_1	0.001	0	0	0.018	0.004	0	0	0	0.060	0.012
	MSB_1	0	0	0.001	0.018	0.004	0	0	0	0.060	0.012
	MSB_2	0	0	0.488	0.293	0.189	0	0	0	0.338	0.068
	NT	0.220	0	0.220	0.007	0.119	0.017	0.011	0.017	0.012	0.015
	E[Spread $\lvert\cdot$]	2.177	2.270	2.177	2.521	2.271	2.252	2.266	2.252	2.720	2.349
	E[Depth A_2+A_1 $\lvert\cdot$]	1.050	2.331	2.446	1.760	1.906	2.168	2.304	2.434	1.623	2.166
	E[Depth A_1 $\lvert\cdot$]	0.001	0.365	0.823	0.239	0.365	0.234	0.367	0.514	0.140	0.325
	E[Depth B_1 $\lvert\cdot$]	0.823	0.365	0.001	0.239	0.365	0.514	0.367	0.234	0.140	0.325
	E[Depth B_1+B_2 $\lvert\cdot$]	2.446	2.331	1.050	1.760	1.906	2.434	2.304	2.168	1.623	2.166
	E[Welfare LO $\lvert\cdot$]	0.089	0.125	0.089	1.038	0.288	0.235	0.130	0.235	0.058	0.172
	E[Welfare MO $\lvert\cdot$]	0.094	0	0.094	2.796	0.609	0	0	0	3.333	0.667
	E[Welfare $\lvert\cdot$]	0.183	0.125	0.183	3.834	0.898	0.235	0.130	0.235	3.390	0.838
$\alpha = 0.2$	LSA_2	0	0.380	0.472	0.106	0.142	0.374	0.387	0.439	0.095	0.156
	LSA_1	0	0.104	0.275	0.058	0.072	0.045	0.097	0.118	0.072	0.075
	LBB_1	0.275	0.104	0	0.058	0.072	0.118	0.097	0.045	0.072	0.075
	LBB_2	0.472	0.380	0	0.106	0.142	0.439	0.387	0.374	0.095	0.156
	MBA_2	0.133	0	0	0.214	0.180	0	0	0	0.210	0.168
	MBA_1	0.093	0	0	0.119	0.101	0	0	0	0.120	0.096
	MSB_1	0	0	0.093	0.119	0.101	0	0	0	0.120	0.096
	MSB_2	0	0	0.133	0.214	0.180	0	0	0	0.210	0.168
	NT	0.027	0.033	0.027	0.007	0.011	0.023	0.031	0.023	0.006	0.010
	E[Spread $\lvert\cdot$]	2.151	2.131	2.151	2.400	2.349	2.197	2.157	2.197	2.454	2.400
	E[Depth A_2+A_1 $\lvert\cdot$]	1.307	2.103	2.500	1.599	1.673	1.940	2.098	2.264	1.594	1.695
	E[Depth A_1 $\lvert\cdot$]	0.194	0.435	0.759	0.307	0.338	0.354	0.422	0.451	0.274	0.301
	E[Depth B_1 $\lvert\cdot$]	0.759	0.435	0.194	0.307	0.338	0.451	0.422	0.354	0.274	0.301
	E[Depth B_1+B_2 $\lvert\cdot$]	2.500	2.103	1.307	1.599	1.673	2.264	2.098	1.940	1.594	1.695
	E[Welfare LO $\lvert\cdot$]	1.116	0.551	1.116	0.507	0.592	0.581	0.637	0.581	0.505	0.524
	E[Welfare MO $\lvert\cdot$]	0.230	0	0.230	3.229	2.614	0	0	0	3.213	2.570
	E[Welfare $\lvert\cdot$]	1.346	0.551	1.346	3.736	3.205	0.581	0.637	0.581	3.718	3.094

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