On Optimal Inference in the Linear IV Regression Model

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- Weak-IV-robust tests in overidentified regression model.
- If IVs are strong: Moreira's CLR, Kleibergen's K (LM) are optimal.
- If IVs are weak: No uniformly optimal test.
  But,
- Andrews, Moreira, Stock (2006, *Econometrica*, AMS):
  - Derived the power envelope for invariant tests;
  - CLR is numerically on the envelope in a homosk. model.
- For heterosked. models, design tests that behave like CLR in homosked.

This paper

- We find the CLR is in fact not on the power envelope.
- Reason: Consider $H_0 : \beta = \beta_0$ vs $H_1 : \beta = \beta_*$
  - In standard frameworks: Can change β_* or β_0 for power calculations - equivalent.
  - In Weak IV regression: Not equivalent as the variance of reduced-form errors (depends on β_*) can be estimated (known).
  - Changing β_* when the variance of reduced-form errors is fixed changes endogeneity.
    (first pointed out in Davidson and MacKinnon, 2008)

This paper (cont-d)

- Different power results for
  1. Fix β_0 change β_*, or
  2. Fix β_* , change β_0.
-
  1. more standard (used in AMS), but
  2. more appropriate for weak IV regressions with focus on confidence sets (CS) (this paper):
     - Allows for fixed reduced-form variance.
     - Keeps endogeneity constant along the power curve.

Contributions (i)

- Alternative (to AMS) power analysis: $\beta_* =$ fixed and $|\beta_0| \uparrow \infty$.
- Results:
  - CLR is not on the power envelope.
  - Anderson-Rubin (AR) can outperform CLR if endogeneity is low.
  - On average, CLR is still a better test.

Contributions (ii)

- New analytical result: CLR is nearly optimal when endogeneity is nearly perfect.
- Reconcile with AMS: When $\beta_0 = \text{fixed}$ and $\beta_* \to \pm\infty$, endogeneity becomes perfect, and the tests behave as if IVs are strong.

Contributions (iii)

- Power against $\beta_0 \to \pm\infty$ gives the probability of having bounded CSs.
  (Dufour (1997): Under weak ID, valid CSs are unbounded with positive prob.)

Model

$$y_1 = y_2\beta + u,$$
$$y_2 = Z\pi + v_2,$$

- y_1, y_2 are $n \times 1$, endo., $\beta \in \mathbb{R}$.
- Z is $n \times k$, fixed (exog).
- # of IVS: $k > 1$ (over ID).
- Normal homosked. errors:

$$\begin{pmatrix} u_i \\ v_{2i} \end{pmatrix} \sim N\left(\begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} \sigma_u^2 & \rho_{uv}\sigma_u\sigma_v \\ \rho_{uv}\sigma_u\sigma_v & \sigma_v^2 \end{pmatrix}\right),$$

- ρ_{uv} is an unknown endogeneity parameter.

Reduced form

$$y_1 = Z\pi\beta + v_1,$$
$$y_2 = Z\pi + v_2,$$

- Reduced-form errors:

$$\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} u + v_2\beta \\ v_2 \end{pmatrix} \sim N(0, \Omega),$$
$$\Omega = \begin{pmatrix} \omega_1^2 & \omega_{12} \\ \omega_{12} & \sigma_v^2 \end{pmatrix}.$$

- Ω is the var-covar of reduced-form errors,
  - can be estimated consistently regardless of π.
  - Treat Ω as known and fixed.

Problem with standard power calculations

- $H_0 : \beta = \beta_0$ vs. $H_1 : \beta = \beta_*$.
- Consider $\beta = \beta_* \to \pm\infty$ with fixed/known Ω:
- The endogeneity parameter implied by β_* and known Ω:

$$\rho_{uv}(\beta_*, \Omega) = \frac{\omega_{12} - \sigma_2^2 \beta_*}{\left(\omega_1^2 - 2\omega_{12}\beta_* + \sigma_2^2 \beta_*^2\right)^{1/2} \sigma_2} \to \mp 1.$$

- Changing true β_* when Ω is fixed changes endogeneity!
- β_0 and Ω are fixed, $\beta_* \to \pm\infty$: Endogeneity changes with β_* along a power curve.

Sufficient statistics S and T for β and π

- $Y = [y_1 : y_2]$, $b_0 = (1, -\beta_0)'$, so that $Yb_0 = y_1 - \beta_0 y_2$.
- S measures the violation of H_0:

$$S = (Z'Z)^{-1} Z' Yb_0 \cdot (b_0' \Omega b_0)^{-1/2}$$
$$\sim N(c_\beta(\beta_0, \Omega) \cdot \mu_\pi, I_k),$$

where

$$c_\beta(\beta_0, \Omega) = (\beta - \beta_0) \cdot (b_0' \Omega b_0)^{-1/2},$$
$$\mu_\pi = (Z'Z)^{1/2} \pi.$$

- The distr. of S is independent of π under $H_0 : \beta = \beta_0$.

Sufficient statistics S and T for β and π

- Let $a_0 = (\beta_0, 1)'$, so that $b_0' a_0 = 0$.
- T is independent of S ($b_0' a_0 = 0$):

$$
\begin{aligned}
T &= (Z'Z)^{-1} Z' Y \Omega^{-1} a_0 \cdot (a_0' \Omega^{-1} a_0)^{-1/2} \\
&\sim N(d_\beta(\beta_0, \Omega) \cdot \mu_\pi, I_k),
\end{aligned}
$$

  where

$$
\begin{aligned}
d_\beta(\beta_0, \Omega) &= (1, -\beta) \Omega b_0 \cdot (b_0' \Omega b_0)^{-1/2} \cdot \det(\Omega)^{-1/2}, \\
\mu_\pi &= (Z'Z)^{1/2} \pi.
\end{aligned}
$$

- The distr. of T depends on π.

Invariance to transforms of S and T

- AMS consider similar tests invariant to orthonormal rotations of (S, T).
- AMS maximal invariant statistic:

$$Q = \left[\begin{array}{cc} S'S & S'T \\ S'T & T'T \end{array} \right] = \left[\begin{array}{cc} Q_S & Q_{ST} \\ Q_{ST} & Q_T \end{array} \right].$$

- Q $\sim$Non-central Wishart,
- Distr. of Q depends on π through the concentration parameter:

$$\lambda = \pi' Z' Z \pi = \|\mu_\pi\|^2.$$

- Strong IVs: $\lambda \to \infty$,
- Weak IVs: $\lambda \to \lambda_\infty < \infty$.

Tests statistics

- Anderson-Rubin:
$$AR = Q_S/k.$$
  - χ^2_k/k distr. under H_0.
- Kleibergen's K or LM:
$$LM = Q^2_{ST}/Q_T.$$
  - χ^2_1 distr. under H_0.
- Moreira's CLR:
$$CLR = 0.5(Q_S - Q_T) + 0.5\sqrt{(Q_S - Q_T)^2 + 4Q^2_{ST}}.$$
  - Crit. values simulated/computed numerically conditional on Q_T for every β_0.
- LM and CLR are efficient under strong IVs.

AMS **two-sided point-optimal** invariant statistic:

$$POIS2(Q;\beta_0,\beta_*,\lambda) = \frac{\psi(Q;\beta_0,\beta_*,\lambda)+\psi(Q;\beta_0,\beta_{2*},\lambda_2)}{2\psi_2(Q_T;\beta_0,\beta_*,\lambda)},$$

where for a Bessel function I_v,

$$\psi(Q;\beta_0,\beta,\lambda) = e^{-\lambda(c_\beta^2+d_\beta^2)/2}(\lambda\xi_\beta(Q))^{-(k-2)/4} I_{(k-2)/2}\left(\sqrt{\lambda\xi_\beta(Q)}\right),$$

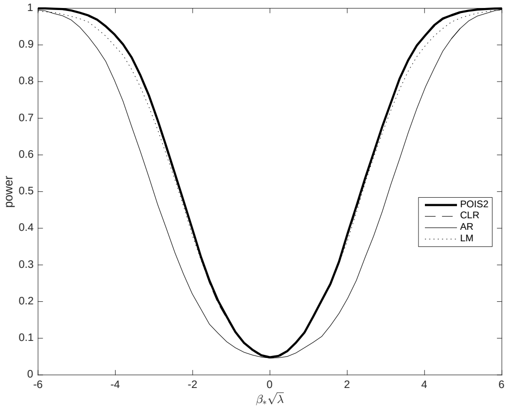
$$\psi_2(Q;\beta_0,\beta,\lambda) = e^{-\lambda d_\beta^2/2}(\lambda d_\beta^2 Q_T)^{-(k-2)/4} I_{(k-2)/2}\left(\sqrt{\lambda d_\beta^2 Q_T}\right),$$

$$\xi_\beta(Q) = c_\beta^2 Q_S + 2c_\beta d_\beta Q_{ST} + d_\beta^2 Q_T.$$

- POIS2 has optimal average-power against (β_*,λ) and (β_{2*},λ_2).
- (β_{2*},λ_2) is chosen so that POIS2 is **efficient under strong IVs**.
- Crit. values simulated conditionally on Q_T for every β_0,β_*,λ.

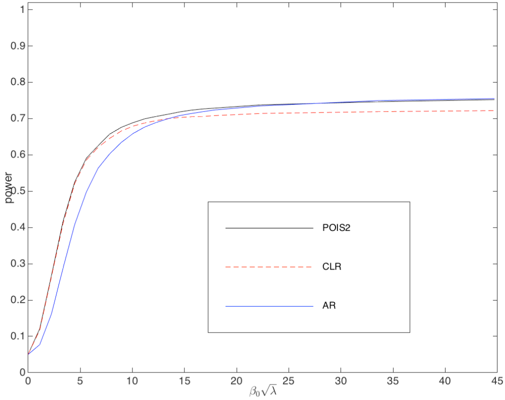
AMS Power curves: $\beta_0 = 0$ fixed, $\beta_* \to \pm\infty$

- $\omega_{12} = 0$.
- $\lambda = 20$, $k = 10$.
- Nearly equal power for CLR and POIS2.
- CLR strongly outperforms AR.
- power $\to 1$ as $\beta_* \to \pm\infty$.



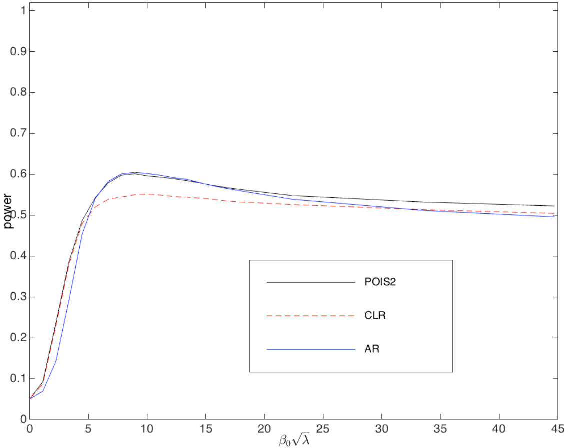
Power curves: $\beta_* = 0$ fixed, $\beta_0 \to \infty$

- $\rho_{uv} = 0$, $\lambda = 15$, $k = 10$.
- power $\nrightarrow 1$ as $\beta_0 \to \pm\infty$ (height depends on λ).
- AR outperforms CLR for large $\beta_* - \beta_0$.
- AR $\approx$ POIS2 as $\beta_0 \to \infty$.
- CLR $\approx$ POIS2 for small $\beta_0 - \beta_*$.



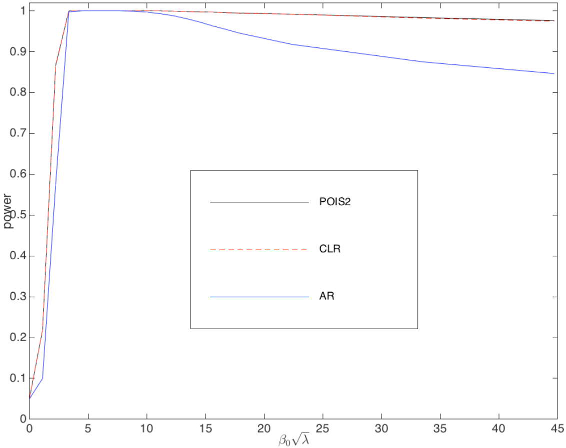
Power curves: $\beta_* = 0$ fixed, $\beta_0 \to \infty$

- $\rho_{uv} = 0.5$, $\lambda = 15$, $k = 40$.
- CLR is below POIS2 and AR for interm. values of $\beta_* - \beta_0$.
- CLR outperforms AR for $\beta_0 \to \infty$.



Power curves: $\beta_* = 0$ fixed, $\beta_0 \to \infty$

- $\rho_{uv} = 0.9$, $\lambda = 15$, $k = 10$.
- CLR $\approx$ POIS2.
- CLR outperforms AR.



Questions

1. Why is there such a diff. btwn. the two scenarios:
   1) $\beta_0 =$ fixed, $\beta_* \to \pm\infty$ and 2) $\beta_* =$ fixed, $\beta_0 \to \pm\infty$?
2. Is CLR optimal or close to optimal in any sense?
3. Should we be interested in $\beta_0 \to \pm\infty$?

Power as $\rho_{uv} \to 1$

- Since the implied endogeneity parameter

$$|\rho_{uv}(\beta_*, \Omega)| \to 1 \text{ as } |\beta_*| \to \infty,$$

consider first

$$\rho_{uv} \to 1.$$

Power under weak IVs as $\rho_{uv} \to 1$

$$T \sim N(d_{\beta_*}(\beta_0, \Omega) \cdot \lambda^{1/2}\Upsilon, I_k),$$

where $\|\Upsilon\| = 1$.

- Strong IV behavior is modeled by $\lambda \to \infty$ (standard).
- We keep $0 < \lambda < \infty$ fixed (weak IVs).
- As $\rho_{uv} \to 1$, reduced-form become errors perfectly corr-ed.:

$$v_1 = u + v_2\beta,$$

$$\rho_\Omega = Cov(v_1, v_2)/\sqrt{Var(v_1)Var(v_2)} \to 1.$$

- $d_{\beta_*}(\beta_0, \Omega) = (1, -\beta_*)\Omega b_0 \cdot (b_0'\Omega b_0)^{-1/2} \cdot \det(\Omega)^{-1/2} \to \infty.$
  - The mean of T goes to ∞ as in the case of strong IVs, while IVs are weak!
  - Due to d_β instead of λ.

Power under weak IVs as $\rho_{uv} \to 1$

- $S \sim N(c_{\beta_*}(\beta_0, \Omega) \cdot \lambda^{1/2}, I_k)$,
- When β_*, β_0, σ_u, σ_v, and λ are fixed,

$$\begin{aligned} c_{\beta_*}(\beta_0, \Omega) &= (\beta_* - \beta_0) \cdot (b_0' \Omega b_0)^{-1/2} \\ &\to \frac{\beta_* - \beta_0}{|\sigma_u + (\beta_* - \beta_0)\sigma_v|} \\ &\equiv c_\infty. \end{aligned}$$

- c_∞ depends on $\beta_* - \beta_0$.

Power under weak IVs as $\rho_{uv} \to 1$

Suppose i) $\lambda =$ fixed (weak IVs), ii) $d^2_{\beta_*} \to \infty$, iii) $c_{\beta_*} \to c_\infty$:

$$P_{\beta_*,\beta_0,\lambda,\Omega}\left(POIS2 > \kappa_{POIS2,\beta_0,\lambda,\alpha}(Q_T)\right) \to P\left(\chi^2_1(\lambda \cdot c^2_\infty) > \chi^2_{1,1-\alpha}\right),$$
$$P_{\beta_*,\beta_0,\lambda,\Omega}\left(CLR > \kappa_{CLR,\alpha}(Q_T)\right) \to P\left(\chi^2_1(\lambda \cdot c^2_\infty) > \chi^2_{1,1-\alpha}\right),$$
$$P_{\beta_*,\beta_0,\lambda,\Omega}\left(LM > \chi^2_{1,1-\alpha}\right) \to P\left(\chi^2_1(\lambda \cdot c^2_\infty) > \chi^2_{1,1-\alpha}\right).$$

Go to proofs

- $P\left(\chi^2_1(\lambda \cdot c^2_\infty) > \chi^2_{1,1-\alpha}\right)$ is the efficiency bound.
- Attained by CLR & LM: they are nearly optimal!

High endogeneity: $\rho_{uv} \approx \pm 1$.

- The usual t or Wald tests have largest size distor. when $\rho_{uv} \approx \pm 1$.
- More reasons to use the robust (AR, CLR, LM) tests when endogeneity is high.
- In those cases, CLR and LM are nearly efficient, but not AR.

AMS Power as $\beta_* \to \infty$

- Suppose Ω, β_0, and λ are fixed:

$$\lim_{\beta_* \to \infty} d^2_{\beta_*}(\beta_0, \Omega) = \lim_{\beta_* \to \infty} \frac{((1, -\beta_*)\Omega b_0)^2}{(b_0' \Omega b_0) \cdot \det(\Omega)} = \infty.$$

$$c^2_\infty = \lim_{\beta_* \to \infty} (\beta_* - \beta_0)^2 / (b_0' \Omega b_0) = \infty.$$

- Power $\to 1$.
- Strong-IV-like behavior for distant alternatives.
- But $|\rho_{uv}| \to 1$ along power curves.

Alternative power calculations: As $\beta_0 \to \pm\infty$

- Suppose β_*, σ_u, σ_v, ρ_{uv}, and λ are fixed:

$$\lim_{\beta_0 \to \pm\infty} d_{\beta_*}(\beta_0, \Omega) = \mp \frac{1}{\sigma_v} \cdot \frac{\rho_{uv}}{(1-\rho_{uv}^2)^{1/2}} \equiv \mp \frac{1}{\sigma_v} \cdot r_{uv},$$

$$\lim_{\beta_0 \to \pm\infty} c_{\beta_*}(\beta_0, \Omega) = \mp \frac{1}{\sigma_v}.$$

- Power $\nrightarrow 1$.
- Under H_1 as $\beta_0 \to \pm\infty$:

$$S \sim N\left(\mp \frac{1}{\sigma_v} \cdot \mu_\pi, I_k\right),$$

$$T \sim N\left(\mp \frac{r_{uv}}{\sigma_v} \cdot \mu_\pi, I_k\right).$$

Can be used to describe the power of AR, CLR, LM as $\beta_0 \to \pm\infty$.

Power bound as $\beta_0 \to \pm\infty$

The limit of AMS POIS2, for $\lambda_v = \lambda/\sigma_v^2$,

$$POIS2_\infty(Q; \infty, r_{uv}, \lambda_v) = \frac{\psi(Q; r_{uv}, \lambda_v) + \psi(Q; -r_{uv}, \lambda_v)}{2\psi_2(Q_T; r_{uv}, \lambda_v)},$$

where

$$\psi(Q; r_{uv}, \lambda_v) = e^{-\frac{\lambda_v(1+r_{uv}^2)}{2}} (\lambda_v \xi(Q; r_{uv}))^{-\frac{(k-2)}{4}} I_{\frac{k-2}{2}}\left(\sqrt{\lambda_v \xi(Q; r_{uv})}\right),$$

$$\psi_2(Q; r_{uv}, \lambda_v) = e^{-\frac{\lambda_v r_{uv}^2}{2}} (\lambda_v r_{uv}^2 Q_T)^{-\frac{(k-2)}{4}} I_{\frac{k-2}{2}}\left(\sqrt{\lambda_v r_{uv}^2 Q_T}\right),$$

$$\xi(Q; r_{uv}) = Q_S + 2 r_{uv} Q_{ST} + r_{uv}^2 Q_T.$$

- The distribution depends only on λ_v, r_{uv}, and k.

Optimality of AR when $\beta_0 \to \pm\infty$ and $\rho_{uv} = 0$

- $POIS2_\infty$ depends on

$$\xi(Q; r_{uv}) = Q_S + 2r_{uv}Q_{ST} + r_{uv}^2 Q_T.$$

- When $\rho_{uv} = 0$,

$$r_{uv} = \frac{\rho_{uv}}{(1-\rho_{uv}^2)^{1/2}} = 0 \quad \Longrightarrow$$
$$\xi(Q; r_{uv}) = Q_S = AR.$$

- $POIS2_\infty$ becomes equivalent to $AR \iff Q_{ST}$ is not used.

Intuition

- $Q_{ST} = S'T$.
- Under H_0, $S \sim N(0, I_k)$.
- Under H_1 as $\beta_0 \to \pm\infty$,

$$\begin{aligned} T &\sim N(\mp r_{uv} \cdot \mu_\pi/\sigma_v, I_k). \\ &= N(0, I_k) \quad \text{when } \rho_{uv} = 0. \end{aligned}$$

- $S'T$ has mean zero under H_0 and H_1 (as $\beta_0 \to \pm\infty$).

Why consider $\beta_0 \to \pm\infty$?

- Dufour (1997): valid CSs are unbounded with positive prob. when IVs are weak.
  - CLR, AR, LM, POIS2 based CSs are unbounded with positive prob.
- New result: Power against $\beta_0 \to \pm\infty$ = Prob. of bounded CSs.

Proof

- Test statistic: $\mathcal{T}(Q_{\beta_0}(Y))$.
- Test: $\phi(Q_{\beta_0}(Y)) = 1\{\mathcal{T}(Q_{\beta_0}(Y)) > cv(Q_{T,\beta_0}(Y))\}$.
- CS: $CS_\phi(Y) = \{\beta_0 : \phi(Q_{\beta_0}(Y)) = 0\}$.
- $CS_\phi(Y)$ has right infinite length, $RLength(CS_\phi(Y)) = \infty$, if:

$$\exists K(Y) < \infty \text{ s.t. } \beta \in CS_\phi(Y) \text{ for all } \beta \geq K(Y).$$

- Can similarly define left infinite length.

Proof

1.

$$\begin{aligned}
&1\{RLength(CS_\phi(Y)) = \infty\}\\
=&1\{\mathcal{T}(Q_{\beta_0}(Y)) \le cv(Q_{T,\beta_0}(Y))\ \forall \beta_0 \ge K(Y) \text{ for some } K(Y) < \infty\}\\
=&\lim_{\beta_0 \to \infty} 1\{\mathcal{T}(Q_{\beta_0}(Y)) \le cv(Q_{T,\beta_0}(Y))\}.
\end{aligned}$$

2. Take $E_{\beta_*,\lambda,\Omega}$, apply the Dominated Convergence Theorem:

$$\begin{aligned}
P(RLength(CS_\phi(Y)) = \infty) &= \lim_{\beta_0 \to \infty} P(\mathcal{T}(Q_{\beta_0}(Y)) \le cv(Q_{T,\beta_0}(Y)))\\
&= 1 - \lim_{\beta_0 \to \infty} P(\mathcal{T}(Q_{\beta_0}(Y)) > cv(Q_{T,\beta_0}(Y))).
\end{aligned}$$

Probability of infinite-length CSs

ρ_{uv}	# of IVs	λ	$POIS2_{\infty}$	$CLR - POIS2_{\infty}$	$AR - POIS2_{\infty}$
.0	5	10	.323	.031	.000
.0	40	20	.394	.049	.000
.3	40	20	.380	.029	.012
.5	40	20	.321	.013	.069
.7	40	20	.186	.009	.204
.9	40	20	.038	.000	.350

- AR performs better for small ρ_{uv} (by $\approx$ 5%).
- CLR performs better for large ρ_{uv} (by $\approx$ 40%).
- Approaching strong-IV-like behavior as $\rho_{uv} \to 1$.

Power differences between POIS2 and CLR: fixed $\beta_* = 0$, varying β_0

- Max and avg. power differences over λ and β_0 for $k = 40$:

ρ_{uv}	λ_{max}	$\beta_{0,\text{max}}$	max diff	avg diff
.0	22	-50	.059	.016
.3	22	4.00	.061	.014
.5	15	1.75	.050	.012
.7	15	1.50	.050	.008
.9	5	1.25	.040	.004

- λ_{max} =the value of λ maximizing the diff.
- $\beta_{0,\text{max}}$ =the value of $\beta_{0,\text{max}}$ maximizing the diff.

Conclusion

- CLR is not on the power envelope.
- Optimality of CLR & LM as $\rho_{uv} \to \pm\infty$.
- Power for $\beta_0 \to \pm\infty$ gives the prob. of bounded CSs.
- AR has better prob. of bounded CSs when $\rho_{uv} \approx 0$.
- Overall, CLR is still the recommended test.

Proof for LM: $P_{\beta_*,\beta_0,\lambda,\Omega}\left(LM > \chi^2_{1,1-\alpha}\right) \to P\left(\chi^2_1(\lambda \cdot c^2_\infty) > \chi^2_{1,1-\alpha}\right)$

- $d_{\beta_*} \to \infty$, $c_{\beta_*} \to c_\infty$, $\lambda < \infty$ and fixed.
- $\Upsilon = \mu_\pi / \lambda^{1/2}$, $\|\Upsilon\| = 1$.
- $S = c_{\beta_*} \cdot \mu_\pi + Z_S = \lambda^{1/2} c_{\beta_*} \cdot \Upsilon + Z_S$.
- $T = d_{\beta_*} \cdot \mu_\pi + Z_T = \lambda^{1/2} d_{\beta_*} \cdot \Upsilon + Z_T$.
- $LM = Q^2_{ST} / Q_T$.

Proof for LM

- $d_{\beta_*} \to \infty$, $c_{\beta_*} \to c_\infty$, $\lambda < \infty$ and fixed.

$$\begin{aligned}
\frac{Q_{ST}}{Q_T^{1/2}} &= \frac{\left(\lambda^{1/2} c_{\beta_*} \cdot \Upsilon + Z_S\right)' \left(\lambda^{1/2} d_{\beta_*} \cdot \Upsilon + Z_T\right)}{\left\| \lambda^{1/2} d_{\beta_*} \cdot \Upsilon + Z_T \right\|} \\
&= \frac{\left(\lambda^{1/2} c_{\beta_*} \cdot \Upsilon + Z_S\right)' \left(\lambda^{1/2} d_{\beta_*} \cdot \Upsilon + Z_T\right)}{\lambda^{1/2} d_{\beta_*} (1 + o_{a.s.}(1))} \\
&= \left(\lambda^{1/2} c_{\beta_*} + \Upsilon' Z_S\right) (1 + o_{a.s.}(1)) \\
&\to N\left(\lambda^{1/2} c_\infty, 1\right).
\end{aligned}$$

Sketch of the proof for CLR

- $d_{\beta_*} \to \infty \Rightarrow Q_T \to \infty$.
- $CLR = 0.5(Q_S - Q_T) + 0.5\sqrt{(Q_S - Q_T)^2 + 4Q_{ST}^2}$.

$$\sqrt{(Q_S - Q_T)^2 + 4Q_{ST}^2} \approx \sqrt{(Q_S - Q_T)^2} + \frac{4Q_{ST}^2}{2\sqrt{(Q_S - Q_T)^2}}.$$

$$CLR \approx \frac{Q_{ST}^2}{Q_T(1 + o_{a.s.}(1))} = \frac{LM}{1 + o_{a.s.}(1)}.$$

Sketch of the proof for POIS2

For large d_β and Q_T, POIS2 approximately depends on:

$$\begin{aligned}
&\sqrt{c_\beta^2 Q_S + 2c_\beta d_\beta Q_{ST} + d_\beta^2 Q_T} - \sqrt{d_\beta^2 Q_T} \\
\approx &\frac{c_\beta^2 Q_S + 2c_\beta d_\beta Q_{ST}}{2\sqrt{d_\beta^2 Q_T}} \\
\sim &\frac{Q_{ST}}{Q_T^{1/2}}.
\end{aligned}$$

Go back