Dissecting Spurious Factors with Cross-Sectional Regressions

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Outline of the talk

1. Motivation and contribution
2. Methodology
3. Generalizations
4. Simulation evidence
5. Conclusion

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  - Risk-premia estimation and inference
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  - Risk versus characteristics
  - Mutual funds/hedge funds applications
  - Corporate finance (cost of capital)

The two-pass methodology (Fama and MacBeth (1973))

- Let asset returns follow a latent factor structure:

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- Second-pass: run one cross-sectional regression by OLS/GLS

$$\bar{R}_i = \gamma_0 + \gamma_1 \hat{\beta}_i + \eta_i, \ 1 \leq i \leq N.$$

  where $\bar{R}_i = \sum_{t=1}^{T} R_{it} / T$ and $\hat{\beta}_i$ is OLS estimator of β_i from first-pass.

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- This gives the estimated risk-premium $\hat{\Gamma} = (\hat{\gamma}_0, \hat{\gamma}_1')'$.

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- However...it might be that $e_i = 0$ and yet the model is wrong: useless factors.

The two-pass methodology: useless factors

- Consider special case when presumed beta-pricing models has two factors A and B. Then we think that:

$$ER_{it} = \gamma_0 + \gamma_{1A}\beta_{iA} + \gamma_{1B}\beta_{iB},$$

but in reality only factor A is priced:

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- So where is the problem?

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- ...second-pass CSR:

$$\begin{pmatrix} \hat{\gamma}_0 \\ \hat{\gamma}_A \\ \hat{\gamma}_B \end{pmatrix} = \hat{\Gamma} = (\hat{X}'\hat{X})^{-1}\hat{X}'\bar{R},$$

where

$$\hat{X} = (1_N, \hat{\beta}_A, \hat{\beta}_B) \approx (1_N, \hat{\beta}_A, 0_N) \text{ when } T \text{ large .}$$

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- 'denominator' of $\hat{\Gamma}$ arbitrarily close to "zero" (that is, $(\hat{X}'\hat{X})$ becomes singular)!
- Similar problem when say the $\beta_{iB} \approx$ constant cross-sectionally (documented when B is market factor).

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  (that is, whether β_{iA} describes entirely the cross-section of expected returns).
- Complicated: it depends on the fraction of assets for which $\beta_{iB} = 0$.

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- when model correctly specified ($E(R_{it}) = \gamma_0$)

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- In particular $\hat{\gamma}_B \approx \sqrt{T}\frac{Z_1' M c}{Z_1' M Z_1}$ where $c = \gamma_0 1_N + e$.

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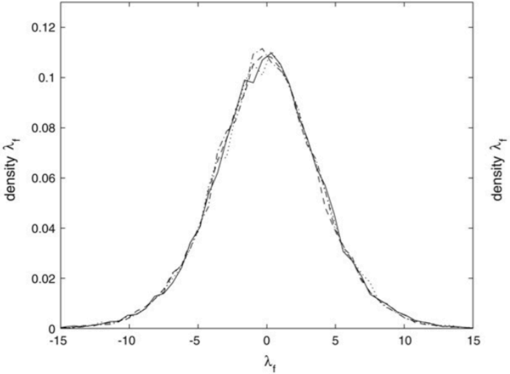
Case $\beta_{iB} = 0$ and correctly specified model



Useless factors (Fig 1.2 from Kleibergen (2009) JOE)

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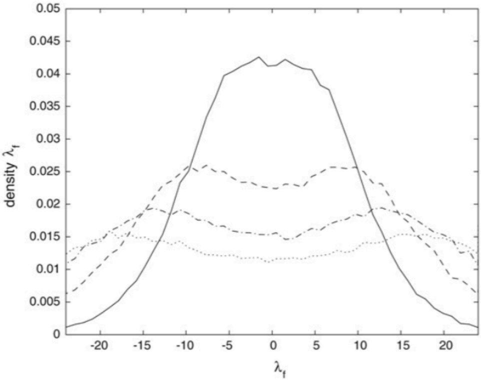
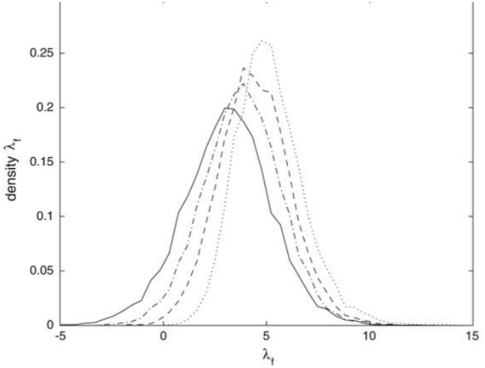


Useless factors (Fig 1.3 from Kleibergen (2009) JOE)

Case $\beta_{1B} \neq 0$, $\beta_{iB} = 0$, $2 \leq i \leq N$ and correctly specified model



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- IN SUMMARY: due useless factors inference on beta-pricing models is corrupted using standard CSRs methods valid for large-T.
- Gospodinov et al. (2017): GMM-tests of asset pricing restriction on SDF parameters have power equal to size when useless factors.

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- Bryzgalova (2016): penalized (LASSO) version of two-pass procedure.
- Anatolyev and Mikusheva (2018): estimation procedure based on sample-splitting instrumental variables regression robust to weak identification (near-zero betas) and omitted weak factors.

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- This paper: completely standard as it just uses the plain OLS CSR estimator.
- This paper: traditional asymptotics (normal and chi-square limiting distributions immune of nuisance-parameters).
- This paper: distinction between lack of identification (zero betas) and weak identification (quasi-zero betas) irrelevant.

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- This sampling scheme empirically motivated as tens of thousands of assets traded every day (individual assets) but only short time-series used in practice (for data availability; for structural breaks; for time-variation of parameters, etc.)
- Our result: OLS CSR is a powerful tool to dissect useless factors in a large-N environment!

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- In particular

$$\hat{\beta}_{ig} = 0_{K_g} + (\tilde{G}'\tilde{G})^{-1} \tilde{G}'\epsilon_i \text{ where } \tilde{G} = G - 1_T \bar{g}'.$$

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Theorem

Under Assumptions 1-5 and correct specification:

(i)

$$\hat{\Gamma}_g - \begin{pmatrix} \gamma_0 \\ 0_{K_g} \end{pmatrix} = O_p\left(\frac{1}{\sqrt{N}}\right).$$

(ii)

$$\sqrt{N}\left(\hat{\Gamma}_g - \begin{pmatrix} \gamma_0 \\ 0_K \end{pmatrix}\right) \xrightarrow{d} \mathcal{N}\left(0_{K+1}, V\right)$$

where

$$V = \begin{pmatrix} \frac{\sigma^2}{T} & 0_K' \\ 0_K & \frac{1}{\sigma^4} C' U_\epsilon C \end{pmatrix}, \quad \text{with} \qquad C = \left(\frac{1_T}{T} \otimes \tilde{G}\right).$$

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- Remark: correctly-sized Wald test for $H_0 : \gamma_g = 0_{K_g}$.

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- Let $\hat{e}_g = \bar{R} - \hat{X}_g \hat{\Gamma}_g$ denote the vector of pricing errors (OLS CSR residuals).

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- The R-squared (where we define $\mathcal{M} = I_N - \frac{1_N 1_N'}{N}$) is:

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- The F-statistic to test whether all the K coefficients except for the intercept are zero is:

$$F_{CSR_g} = \frac{R^2_{CSR} / K}{(1 - R^2_{CSR}) / (N-K-1)} \quad (3)$$

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$$t_{g,k} \xrightarrow{d} \mathcal{N}\left(0, \frac{\frac{1}{T}k_4 + \sigma_4}{\sigma^4}\right)$$

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- R^2 is not inflated (goes to zero, as it should).

Useless factors (base case): misspecified case

- Let $1_N' c/N \to \mu_c$ and $c' M_{1_N} c/N \to \nu_c$ where $ER_{it} = c_i = \gamma_0 + e_i$.

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where V as for correctly-specified case and $W = \begin{pmatrix} 0 & 0_K' \\ 0_K & \frac{\nu_c}{\sigma^2}\tilde{G}'\tilde{G} \end{pmatrix}$.

Useless factors: base case with misspecification

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- Qualitatively, the results do not differ from correctly-specified case.
- All quantities can be consistently estimated for $N \to \infty$:

$$\hat{\mu}_c = 1_N' \bar{R} / N, \hat{\nu}_c = 1_N' \bar{R}^2 / N - \hat{\mu}_c^2.$$

Useless factors (base case): FM t-ratios

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- Let $Z = (Z_0 \cdots Z_K)' \sim N(0_{K+1}, V + W)$ as defined before.
- Let $\frac{\epsilon' 1_N}{\sqrt{N}} \to_d \xi \sim N(0, \sigma^2 I_T)$, $\frac{(\epsilon'\epsilon - N\sigma^2 I_T)}{\sqrt{N}} \to_d \Xi$ with $vec(\Xi) \sim N(0, U_\epsilon)$.

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- Let

$$\Phi_k \equiv \frac{1}{(T-1)^{\frac{1}{2}}} \left(\iota_{k+1,K+1}' (\Sigma_X + \Lambda)^{-1} \left(\begin{pmatrix} \xi' A \xi & \xi' A \Xi \mathcal{P} \\ \mathcal{P}' \Xi A \xi & \mathcal{P}' \Xi A \Xi \mathcal{P} \end{pmatrix} \right) (\Sigma_X + \Lambda)^{-1} \iota_{k+1,K+1} \right)^{\frac{1}{2}}, \quad k = 1, \cdots, K.$$

for $A = I_T - \frac{1_T 1_T'}{T} - \tilde{G}(\tilde{G}'\tilde{G})^{-1}\tilde{G}'$.

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for $A = I_T - \frac{1_T 1_T'}{T} - \tilde{G}(\tilde{G}'\tilde{G})^{-1}\tilde{G}'$.

- These non-conventional quantities characterize the FM t-ratios when N is large.

Useless factors (base case): FM t-ratios

Theorem

Under Assumptions 1-5:

(i) *for the ex-ante risk premia*

$$|t_{FM}(\hat{\gamma}_0)| = \frac{|\hat{\gamma}_0 - \mu_c|}{SE_0^{FM}} \to_p \frac{Z_0}{\Phi_0} \quad \text{and} \quad \sqrt{N}|t_{FM}(\hat{\gamma}_{1k})| = \sqrt{N}\frac{|\hat{\gamma}_{1k}|}{SE_k^{FM}} \to_d \frac{Z_k}{\hat{\sigma}_k^2}.$$

(ii) *for the ex-post risk premia*

$$|t_{FM}(\hat{\gamma}_0)| = \frac{|\hat{\gamma}_0 - \mu_c|}{SE_0^{FM,P}} \to_d \frac{Z_0}{\Phi_0} \quad \text{and} \quad |t_{FM}(\hat{\gamma}_{1k})| = \frac{|\hat{\gamma}_{1k}|}{SE_k^{FM,P}} \to_d \frac{Z_k}{\Phi_k}.$$

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- Same results for correctly-specified (except that $\mu_c = \gamma_0$) and misspecified cases.
- Same results (obviously) for zero-beta rate t-ratios.

Useless factors: useful with useless

- The true model is

$$R_t = \alpha + B_f f_t + 0_{N,K_g} g_t + \epsilon_t = \alpha + B_f f_t + \epsilon_t.$$

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- Estimated risk premia (setting $\tilde{D} = (\tilde{F}, \tilde{G})$)

$$\hat{\Gamma}_{0fg} = (\hat{X}_{fg}' \hat{X}_{fg})^{-1} \hat{X}_{fg}' \bar{R} \text{ where } (\hat{B}_f, \hat{B}_g) = R' \tilde{D} (\tilde{D}' \tilde{D})^{-1}.$$

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$$ER_{it} = \gamma_0 + \gamma_{1f}' \beta_{if}.$$

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- Under correct specification

$$ER_{it} = \gamma_0 + \gamma'_{1f}\beta_{if}.$$

- Under misspecification

$$ER_{it} = \gamma_0 + \gamma'_{1f}\beta_{if} + e_i.$$

Useless factors: useful with useless (G and F orthogonal)

Theorem

When $\tilde{G}'\tilde{F} = 0$ (G and F orthogonal):

(i)

$$\hat{\Gamma}_{fg} - \begin{pmatrix} \gamma_0 + d_0 \\ \gamma_{1_f}^P + d_1 \\ 0_{K_g} \end{pmatrix} = O_p\left(\frac{1}{\sqrt{N}}\right).$$

(ii)

$$\left(\hat{\Gamma}_{fg} - \begin{pmatrix} \gamma_0 + d_0 \\ \gamma_{1_f}^P + d_1 \\ 0_{K_g} \end{pmatrix}\right) \xrightarrow{d} \mathcal{N}\left(0, \left(\Sigma_{X_{fg}} + \Lambda_{fg}\right)^{-1} (V_{fg} + W_{fg}) \left(\Sigma_{X_{fg}} + \Lambda_{fg}\right)^{-1}\right)$$

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(ii)

$$R^2_{CRS_{fg}} = 1 - \frac{\hat{e}_{fg}'\hat{e}_{fg}}{\bar{R}'\mathcal{M}_N\bar{R}} \to 1 - \frac{\frac{\sigma^2}{T} + \gamma_{1_f}^{P'}\sigma^2(\tilde{F}'\tilde{F})^{-1}D^{-1}\tilde{\Sigma}_{\beta_f}\gamma_{1_f}^{P}}{\frac{\sigma^2}{T} + \gamma_{1_f}^{P'}\tilde{\Sigma}_{\beta_f}\gamma_{1_f}^{P}}$$

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When $\tilde{G}'\tilde{F} = 0$ (G and F orthogonal):

(iii) *Let*

$$F_{CSR_{fg}} = \frac{\left(\hat{e}_f^{*'}\hat{e}_f^{*} - \hat{e}_{fg}'\hat{e}_{fg}\right)/K_g}{\hat{e}_{fg}'\hat{e}_{fg}/\left(N - (K_f + K_g + 1)\right)} \tag{4}$$

be the F-statistic to test the null hypotesis $\gamma_{1_g}^P = 0_{K_g}$. Then

$$F_{CSR_{fg}} \xrightarrow{d} \left(Z_1', \quad Z_2'\right) \frac{\mathcal{W}_{fg}/K_g}{\frac{\sigma^2}{T} - d_1'\tilde{\Sigma}_{\beta_f}\gamma_{1_f}^P} \begin{pmatrix} Z_1 \\ Z_2 \end{pmatrix}$$

where $Z_1 \equiv \mathcal{N}\left(0_{T^2}, U_\epsilon\right)$ and $Z_2 \equiv \mathcal{N}\left(0_T, \sigma^2 d_1'\tilde{\Sigma}_{\beta_f} d_1 I_T\right)$ are two normally distributed vectors of dimension $T^2 \times 1$ and $T \times 1$, respectively, and where $\mathcal{W}_{fg}$ suitable matrix.

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- Instead, risk premia for G (useless) converges to zero.
- Results more complicated than previous case but similar spirit: all quantities can be consistently estimated and test with correct size and power be derived.

Useless factors: useful with useless (G and F not orthogonal)

- When G and F not orthogonal:

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- When G and F not orthogonal:
- $$\hat{\Gamma}^P_{fg} \xrightarrow{p} \begin{pmatrix} \gamma_0 - \mu'_{\beta_f}(I_{K_f} - E^{-1}\Sigma_{\beta_f})\gamma^P_{1_f} \\ E^{-1}\Sigma_{\beta_f}\gamma^P_{1_f} \\ AE^{-1}\Sigma_{\beta_f}\gamma^P_{1_f} \end{pmatrix} \quad (5)$$

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- Set

$$\theta_f = E^{-1}\Sigma_{\beta_f}\gamma^P_{1_f} \text{ and } \theta_g = -\frac{D}{\sigma^2}Q'_{fg}E^{-1}\Sigma_{\beta_f}\gamma^P_{1_f}.$$

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- Under the null of useless factors, the following linear restriction holds:

$$H_0 : \theta_g = A\theta_f.$$

for an observed $A = (\tilde{G}'\tilde{G} - \tilde{G}'\tilde{F}(\tilde{F}'\tilde{F})^{-1}\tilde{F}'\tilde{G})(\tilde{F}'\tilde{F})^{-1}\tilde{F}'\tilde{G}D^{-1}$.

Useless factors: useful with useless (G and F not orthogonal)

- When G and F not orthogonal:
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- Using the distribution of part of our theorem, derive the test.
- If F and G orthogonal in sample, then $H_0 : \theta_g = 0$.

Useless factors: useful with useless (G and F not orthogonal)

- When G and F not orthogonal:
- $$\hat{\Gamma}^P_{fg} \quad \xrightarrow{p} \begin{pmatrix} \gamma_0 - \mu'_{\beta_f}(I_{K_f} - E^{-1}\Sigma_{\beta_f})\gamma^P_{1_f} \\ E^{-1}\Sigma_{\beta_f}\gamma^P_{1_f} \\ AE^{-1}\Sigma_{\beta_f}\gamma^P_{1_f} \end{pmatrix} \tag{5}$$
- Set
$$\theta_f = E^{-1}\Sigma_{\beta_f}\gamma^P_{1_f} \text{ and } \theta_g = -\frac{D}{\sigma^2}Q'_{fg}E^{-1}\Sigma_{\beta_f}\gamma^P_{1_f}.$$
- Under the null of useless factors, the following linear restriction holds:
$$H_0 : \theta_g = A\theta_f.$$
for an observed $A = (\tilde{G}'\tilde{G} - \tilde{G}'\tilde{F}(\tilde{F}'\tilde{F})^{-1}\tilde{F}'\tilde{G})(\tilde{F}'\tilde{F})^{-1}\tilde{F}'\tilde{G}D^{-1}$.
- Using the distribution of part of our theorem, derive the test.
- If F and G orthogonal in sample, then $H_0 : \theta_g = 0$.
- Bias-adjusted estimator for $\gamma^P_{1_f}$ can be obtained (not the focus here).

Useless factors: useful with useless under misspecification

- The true model is still

$$R_t = \alpha + B_f f_t + 0_{N,K_g} g_t + \epsilon_t = \alpha + B_f f_t + \epsilon_t.$$

but we estimate

$R_t = \alpha + B_{f1} f_{1t} + B_g g_t + residual$ setting $F = (F_1, F_2)$(misspecification:

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  but we estimate

  $R_t = \alpha + B_{f1} f_{1t} + B_g g_t + residual$ setting $F = (F_1, F_2)$(misspecification:

- As a special case, we could miss out F entirely, so estimated model:

$$R_t = \alpha + B_g g_t + residual.$$

Useless factors: useful with useless and misspecification (G and F orthogonal)

Theorem (i)

$$\hat{\Gamma}_{f_1 g} - \begin{pmatrix} \gamma_0 + \tilde{d}_0 \\ \tilde{d}_{11}\gamma_{1_f}^{P[1]} + \tilde{d}_{12}\gamma_{1_f}^{P[2]} \\ 0_{K_g} \end{pmatrix} = O_p\left(\frac{1}{\sqrt{N}}\right)$$

Useless factors: useful with useless and misspecification (G and F orthogonal)

Theorem

[ii]

$$\sqrt{N}\left(\hat{\Gamma}_{f_1 g} - \begin{pmatrix} \gamma_0 + \tilde{d}_0 \\ \tilde{d}_{11}\gamma_{1_f}^{P[1]} + \tilde{d}_{12}\gamma_{1_f}^{P[2]} \\ 0_{K_g} \end{pmatrix}\right)$$

$$\xrightarrow{d} \mathcal{N}\left(0, \left(\Sigma_{X_{fg}}^{[1]} + \Lambda_{fg}^{[1]}\right)^{-1}\left(V_{fg} + W_{fg}\right)\left(\Sigma_{X_{fg}}^{[1]} + \Lambda_{fg}^{[1]}\right)^{-1}\right)$$

Problem is that acm is function of both F_1 and F_2 so not feasible!

Useless factors: useful with useless and misspecification (G and F orthogonal)

- In particular V_{fg} equal

$$\sigma^2 \left(\frac{1}{T} + (\tilde{d}_{11}\gamma_{1_f}^{P[1]} + \tilde{d}_{12}\gamma_{1_f}^{P[2]})'(\tilde{F}'\tilde{F})^{-1}(\tilde{d}_{11}\gamma_{1_f}^{P[1]} + \tilde{d}_{12}\gamma_{1_f}^{P[2]}) \right) \Sigma_{X_{fg}}^{[1]}$$
$$+\sigma^2\Omega_{fg},$$

with Ω_{fg} equal to

$$\begin{pmatrix} 0 & 0'_{K_{f_1}} & 0'_{K_g} \\ 0_{K_{f_1}} & \vartheta(\tilde{F}^{[1]'}\tilde{F}^{[1]})^{-1} - \left(\tilde{\Sigma}_{\beta_f}^{[1]}\tilde{d}_1 - \tilde{\Sigma}_{\beta_f}^{[1,2]}\gamma_{1_f}^{P[2]}\right) - \left(\tilde{d}_1'\tilde{\Sigma}_{\beta_f}^{[1]} - \gamma_{1_f}^{P[2]'}\tilde{\Sigma}_{\beta_f}^{[2,1]}\right) & 0_{K_{f_1}\times K_g} \\ 0_{K_g} & 0_{K_g\times K_{f_1}} & \vartheta(\tilde{G}'\tilde{G})^{-1} \end{pmatrix}$$

Useless factors: useful with useless and misspecification (G and F orthogonal)

- $W_{f_1 g}$ equal

$$
\begin{array}{cc}
0'_{K_{f_1}} & 0'_{K_g} \\
(Q_f^{[1,2]'} \otimes P_f^{[1]'}) U_\epsilon (Q_f^{[1,2]} \otimes P_f^{[1]}) & (Q_f^{[1,2]'} \otimes P_f^{[1]'}) U_\epsilon (Q_f^{[1,2]} \otimes P_g) \\
(Q_f^{[1,2]'} \otimes P_g') U_\epsilon (Q_f^{[1,2]} \otimes P_f^{[1]}) & (Q_f^{[1,2]'} \otimes P_g') U_\epsilon (Q_f^{[1,2]} \otimes P_g)
\end{array}
\right)
$$

with

$$
Q_f^{[1,2]} = \left(\frac{1_T}{T} - P_f^{[1]} \tilde{d}_{11} \gamma_{1_f}^{P[1]} - P_f^{[1]} \tilde{d}_{12} \gamma_{1_f}^{P[2]} \right).
$$

Useless factors: useful with useless and misspecification (G and F orthogonal)

- $$t_{f_1 g, k_g} = \frac{\hat{\gamma}_{1_g, k_g}}{s_{f_1 g} \cdot \sqrt{c_{g, k_g k_g}}}$$

  is the t-statistic for the k_g-th regression coefficient ($k_g = 1, ..., K_g$) and $c_{g, k_g k_g}$ is the (k_g, k_g)-th element of the matrix $(\hat{X}'_{f_1 g} \hat{X}_{f_1 g})^{-1}$.

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- $$R^2_{CSR_{f_1 g}} = 1 - \frac{\hat{e}'_{f_1 g} \hat{e}_{f_1 g}}{\bar{R}' \mathcal{M}_N \bar{R}}.$$

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- $$R^2_{CSR_{f_1 g}} = 1 - \frac{\hat{e}'_{f_1 g} \hat{e}_{f_1 g}}{\bar{R}' \mathcal{M}_N \bar{R}}.$$

- $$F_{CSR_{f_1 g}} = \frac{(\hat{e}'^{*}_{f_1} \hat{e}^{*}_{f_1} - \hat{e}'_{f_1 g} \hat{e}_{f_1 g}) / K_g}{\hat{e}'_{f_1 g} \hat{e}_{f_1 g} / (N - K_{f_1} - K_g - 1)},$$

  is the F-statistic to test $\gamma^P_{1_g} = 0_{K_g}$.

Useless factors: useful with useless and misspecification (G and F orthogonal)

Theorem

[i]

$$t_{f_1 g, k_g} \xrightarrow{d} \mathcal{N}\left(0, \frac{\vartheta + \sigma^{-2} W_{[k_g, k_g]}}{\frac{\sigma^2}{T} + \Gamma_{1_f}^{P'} \tilde{\Sigma}_{X_f} \Gamma_{1_f}^{P'}}\right),$$

where $W_{[k_g, k_g]}$ *denotes the* (k_g, k_g)*-th element of the matrix* $(Q_f^{[1,2]'} \otimes \tilde{G}')U_\epsilon(Q_f^{[1,2]} \otimes P_g)$, $\Gamma_{1_f}^P = \left[\gamma_{1_f}^{P[1]'},\ \gamma_{1_f}^{P[2]'}\right]'$ *and*

$$\tilde{\Sigma}_{X_f} = \begin{pmatrix} \tilde{\Sigma}_{\beta_f}^{[1]} - \tilde{\Sigma}_{\beta_f}^{[1]} D^{-1} \tilde{\Sigma}_{\beta_f}^{[1]} & \sigma^2 (\tilde{F}^{[1]'} \tilde{F}^{[1]})^{-1} D^{-1} \tilde{\Sigma}_{\beta_f}^{[1,2]} \\ \sigma^2 \tilde{\Sigma}_{\beta_f}^{[1,2]'} D^{-1} (\tilde{F}^{[1]'} \tilde{F}^{[1]})^{-1} & \tilde{\Sigma}_{\beta_f}^{[2]} - \tilde{\Sigma}_{\beta_f}^{[1,2]'} D^{-1} \tilde{\Sigma}_{\beta_f}^{[1,2]} \end{pmatrix}.$$

Useless factors: useful with useless and misspecification (G and F orthogonal)

Theorem

[ii]

$$R^2_{CSR_{f_1 g}} \xrightarrow{p} 1 - \frac{\frac{\sigma^2}{T} + \Gamma^{P'}_{1_f} \tilde{\Sigma}_{X_f} \Gamma^{P}_{1_f}}{\frac{\sigma^2}{T} + \Gamma^{P'}_{1_f} \tilde{\Sigma}_{\beta_f} \Gamma^{P}_{1_f}}$$

where $\Gamma^P_{1_f} = \left[\gamma^{P[1]'}_{1_f}, \ \gamma^{P[2]'}_{1_f} \right]'$ *and* $\tilde{\Sigma}_{\beta_f} = \begin{pmatrix} \tilde{\Sigma}^{[1]}_{\beta_f} & \tilde{\Sigma}^{[1,2]}_{\beta_f} \\ \tilde{\Sigma}^{[2,1]}_{\beta_f} & \tilde{\Sigma}^{[2]}_{\beta_f}. \end{pmatrix}.$

$$F_{CSR_{f_1 g}} \xrightarrow{d} \left(Z_1', \quad Z_2' \right) \frac{\mathcal{W}_{fg} / K_g}{\frac{\sigma^2}{T} + \Gamma^{P'}_{1_f} \tilde{\Sigma}_{X_f} \Gamma^{P'}_{1_f}} \begin{pmatrix} Z_1 \\ Z_2 \end{pmatrix},$$

where $Z_1 \equiv \mathcal{N}(0_{T^2}, U_\epsilon)$ *and* $Z_2 \equiv \mathcal{N}(0_T, \vartheta\sigma^2 I_T)$ *are two normally distributed vectors of dimension* $T^2 \times 1$ *and* $T \times 1$.

Useless factors: useful with useless and misspecification

iii. Results extend to G and F not orthogonal.

Useless factors: useful with useless and misspecification

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- Problem: asymptotic distributions depend on F_2 which is not observed. Bounds can be derived but inaccurate for large N.

Useless factors: useful with useless and misspecification

- Results extend to G and F not orthogonal.
- Problem: asymptotic distributions depend on F_2 which is not observed. Bounds can be derived but inaccurate for large N.
- Solution: estimate the useful factors by PCA and derive asymptotics for useless factors based on the PCA distribution (along the idea of Giglio and Xiu (2017)).

Simulation results: base case

- The table reports the percentage bias (Bias) and root mean squared error (RMSE), all in percent, over 10,000 simulated data sets.

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where $\epsilon_t \sim \mathcal{N}(0, \Sigma)$ and where we calibrate γ_0 as $\gamma_0 = \frac{1}{NT} \sum_{i=1}^{N} \sum_{t=1}^{T} R_{it}$.

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- Fitted Model is a One-Factor Model $R_{it} = a_i + b_i' g_t + u_{it}$, where g_t is the excess market return (from Kenneth French's website) from January 2008 to December 2010 for T=36, and the excess market return from January 2008 to December 2013 for T=72.

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- The table also reports the R-squared (R^2) of the fitted model for different cross-sections of $N = 100, 500, 1000, 3000$ stocks.

Table I
Bias and RMSE of the OLS Estimator in a One-Factor Model with a useless factor (Σ scalar)

Statistics	$N = 100$	$N = 500$	$N = 1000$	$N = 3000$
		Panel A: $T = 36$		
Bias($\hat{\gamma}_0$)	0.32%	0.18%	0.12%	0.11%
RMSE($\hat{\gamma}_0$)	0.184	0.083	0.058	0.035
Bias($\hat{\gamma}_1$)	0.000	0.000	0.000	0.000
RMSE($\hat{\gamma}_1$)	0.429	0.191	0.134	0.082
R^2	0.006	0.002	0.001	0.000

Table I
Bias and RMSE of the OLS Estimator in a One-Factor Model with a useless factor (Σ scalar)

Statistics	$N = 100$	$N = 500$	$N = 1000$	$N = 3000$
		Panel B: $T = 72$		
Bias($\hat{\gamma}_0$)	0.05%	0.04%	0.03%	0.04%
RMSE($\hat{\gamma}_0$)	0.146	0.066	0.046	0.028
Bias($\hat{\gamma}_1$)	0.000	0.000	0.000	0.000
RMSE($\hat{\gamma}_1$)	0.379	0.166	0.119	0.072
R^2	0.002	0.002	0.001	0.000

Simulation results: base case (t-test) - Scalar Σ

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- The null hypothesis is that the parameter of interest is equal to its true value.
- $t_{FM}(\cdot)$ denotes the t-statistic associated with the OLS estimator that uses the traditional Fama-MacBeth standard error.
- $t(\cdot)$ denotes the t-statistic associated with the OLS estimator.
- The t-statistics are compared with the critical values from a standard normal distribution.

Simulation results: base case (t-test) - Scalar Σ

Table II
Empirical size of t-tests in a One-Factor Model with a useless factor (Σ Scalar)

Panel A: $T = 36$

N	0.10	0.05	0.01	0.10	0.05	0.01
		$t_{FM}(\hat{\gamma}_0)$			$t_{FM}(\hat{\gamma}_1)$	
100	0.105	0.053	0.010	0.105	0.053	0.013
500	0.108	0.053	0.011	0.108	0.054	0.011
1000	0.105	0.051	0.011	0.103	0.053	0.011
3000	0.101	0.050	0.010	0.101	0.053	0.011
		$t(\hat{\gamma}_0)$			$t(\hat{\gamma}_1)$	
100	0.105	0.053	0.010	0.105	0.055	0.013
500	0.107	0.052	0.011	0.108	0.054	0.011
1000	0.106	0.051	0.011	0.103	0.053	0.011
3000	0.102	0.050	0.010	0.102	0.052	0.011

Simulation results: base case (t-test) - Scalar Σ

Table II
Empirical size of t-tests in a One-Factor Model with a useless factor (Σ Scalar)

Panel A: $T = 72$

N	0.10	0.05	0.01	0.10	0.05	0.01
		$t_{FM}(\hat{\gamma}_0)$			$t_{FM}(\hat{\gamma}_1)$	
100	0.105	0.054	0.011	0.101	0.052	0.011
500	0.105	0.052	0.011	0.0.98	0.049	0.009
1000	0.102	0.052	0.009	0.097	0.051	0.010
3000	0.102	0.051	0.010	0.099	0.050	0.009
		$t(\hat{\gamma}_0)$			$t(\hat{\gamma}_1)$	
100	0.103	0.053	0.009	0.098	0.055	0.011
500	0.103	0.051	0.010	0.098	0.046	0.009
1000	0.101	0.051	0.009	0.096	0.051	0.010
3000	0.101	0.051	0.010	0.099	0.050	0.009

Table III
Empirical size of F-tests in a One-Factor Model with a useless factor (Σ scalar)

The table presents the size properties of F-tests of statistical significance. The F-statistics are compared with the critical values from a $\chi^2_K\left(\frac{\sigma_4}{\sigma^4}/K\right)$.

	Panel A: $T = 36$			Panel A: $T = 72$		
N	0.10	0.05	0.01	0.10	0.05	0.01
100	0.107	0.056	0.012	0.108	0.056	0.012
500	0.101	0.052	0.011	0.104	0.053	0.011
1000	0.101	0.051	0.011	0.101	0.052	0.010
3000	0.100	0.049	0.010	0.101	0.051	0.010

Simulation results: base case (Bias and RMSE) - Diagonal Σ

Table IV
Bias and RMSE of the OLS Estimator One-Factor Model with a useless factor(Σ Diagonal).

Statistics	$N = 100$	$N = 500$	$N = 1000$	$N = 3000$
		Panel A: $T = 36$		
Bias($\hat{\gamma}_0$)	-0.15%	0.15%	0.08%	0.02%
RMSE($\hat{\gamma}_0$)	0.923	0.425	0.308	0.190
Bias($\hat{\gamma}_1$)	0.000	0.000	0.000	0.000
RMSE($\hat{\gamma}_1$)	0.764	0.330	0.227	0.135
R^2	0.030	0.006	0.003	0.001

Simulation results: base case (Bias and RMSE) - Diagonal Σ

Table IV
Bias and RMSE of the OLS Estimator One-Factor Model with a useless factor(Σ Diagonal).

Statistics	$N = 100$	$N = 500$	$N = 1000$	$N = 3000$
		Panel B: $T = 72$		
Bias($\hat{\gamma}_0$)	0.07%	0.03%	0.03%	0.02%
RMSE($\hat{\gamma}_0$)	0.400	0.160	0.127	0.075
Bias($\hat{\gamma}_1$)	0.000	0.000	0.000	0.000
RMSE($\hat{\gamma}_1$)	1.070	0.521	0.332	0.208
R^2	0.069	0.018	0.008	0.003

Table V
Empirical size of t-tests in a One-Factor Model with a useless factor (Σ Diagonal)

Panel A: $T = 36$

N	0.10	0.05	0.01	0.10	0.05	0.01
		$t_{FM}(\hat{\gamma}_0)$			$t_{FM}(\hat{\gamma}_1)$	
100	0.103	0.052	0.010	0.113	-0.060	0.015
500	0.101	0.050	0.010	0.101	0.053	0.011
1000	0.101	0.050	0.011	0.103	0.054	0.011
3000	0.100	0.050	0.009	0.102	0.050	0.011
		$t(\hat{\gamma}_0)$			$t(\hat{\gamma}_1)$	
100	0.095	0.048	0.010	0.113	0.059	0.014
500	0.098	0.048	0.009	0.102	0.050	0.011
1000	-0.099	0.050	0.011	0.103	0.052	0.011
3000	0.102	0.050	0.009	0.100	0.050	0.012

Table V
Empirical size of t-tests in a One-Factor Model with a useless factor (Σ Diagonal)

Panel B: $T = 72$

N	0.10	0.05	0.01	0.10	0.05	0.01
		$t_{FM}(\hat{\gamma}_0)$			$t_{FM}(\hat{\gamma}_1)$	
100	0.101	0.047	0.007	0.134	0.077	0.022
500	0.100	0.050	0.010	0.108	0.057	0.013
1000	0.098	0.049	0.010	0.103	0.053	0.011
3000	0.099	0.051	0.010	0.101	0.052	0.011
		$t(\hat{\gamma}_0)$			$t(\hat{\gamma}_1)$	
100	0.087	0.041	0.006	0.122	0.072	0.023
500	0.096	0.050	0.009	0.104	0.057	0.012
1000	0.097	0.047	0.010	0.108	0.053	0.013
3000	0.099	0.051	0.010	0.101	0.051	0.010

Simulation results: base case (F-test) - Diagonal Σ

Table VI
Empirical size of F-tests in a One-Factor Model with a useless factor (Σ Diagonal)

The table presents the size properties of F-tests of statistical significance. The F-s
the critical values from a $\chi^2_K\left(\frac{\sigma_4}{\sigma^4}/K\right)$.

	Panel A: $T = 36$			Panel A: $T = 72$		
N	0.10	0.05	0.01	0.10	0.05	0.01
100	0.113	0.060	0.015	0.134	0.077	0.022
500	0.101	0.053	0.011	0.108	0.057	0.013
1000	0.102	0.052	0.011	0.106	0.053	0.011
3000	0.102	0.050	0.011	0.101	0.052	0.011

Simulation results: base case (Bias and RMSE) - Full Σ $(\delta = 0.5)$

Table VII
Bias and RMSE of the OLS Estimator in a One-Factor Model with a useless factor(Σ Full - $\delta = 0.5$).

Statistics	$N = 100$	$N = 500$	$N = 1000$	$N = 3000$
		Panel A: $T = 36$		
Bias$(\hat{\gamma}_0)$	-0.16%	0.13%	0.06%	0.05%
RMSE$(\hat{\gamma}_0)$	0.923	0.425	0.305	0.189
Bias$(\hat{\gamma}_1)$	0.000	0.000	0.000	0.000
RMSE$(\hat{\gamma}_1)$	1.253	0.474	0.349	0.196
R^2	0.031	0.006	0.003	0.001

Simulation results: base case (Bias and RMSE) - Full Σ ($\delta = 0.5$)

Table VII
Bias and RMSE of the OLS Estimator in a One-Factor Model with a useless factor(Σ Full - $\delta = 0.5$).

Statistics	$N = 100$	$N = 500$	$N = 1000$	$N = 3000$
		Panel B: $T = 72$		
Bias($\hat{\gamma}_0$)	-0.03%	0.02%	0.05%	0.03%
RMSE($\hat{\gamma}_0$)	0.353	0.178	0.118	0.078
Bias($\hat{\gamma}_1$)	0.000	0.000	0.000	0.000
RMSE($\hat{\gamma}_1$)	0.764	0.329	0.230	0.138
R^2	0.090	0.015	0.009	0.003

Simulation results: base case (t-test) - Full Σ ($\delta = 0.5$)

Table VIII
Empirical size of t-tests in a One-Factor Model with a useless factor(Σ Full - $\delta = 0.5$)

Panel A: $T = 36$

N	0.10	0.05	0.01	0.10	0.05	0.01
		$t_{FM}(\hat{\gamma}_0)$			$t_{FM}(\hat{\gamma}_1)$	
100	0.103	0.053	0.010	0.113	0.060	0.015
500	0.102	0.050	0.010	0.101	0.053	0.011
1000	0.102	0.049	0.010	0.103	0.053	0.011
3000	0.099	0.050	0.010	0.101	0.052	0.011
		$t(\hat{\gamma}_0)$			$t(\hat{\gamma}_1)$	
100	0.096	0.048	0.010	0.113	0.059	0.014
500	0.097	0.048	0.010	0.101	0.051	0.011
1000	0.102	0.048	0.010	0.104	0.052	0.011
3000	0.099	0.050	0.010	0.103	0.051	0.010

Simulation results: base case (t-test) - Full Σ ($\delta = 0.5$)

Table VIII
Empirical size of t-tests in a One-Factor Model with a useless factor(Σ Full - $\delta = 0.5$)

Panel B: $T = 72$

N	0.10	0.05	0.01	0.10	0.05	0.01
		$t_{FM}(\hat{\gamma}_0)$			$t_{FM}(\hat{\gamma}_1)$	
100	0.102	0.046	0.005	0.138	0.080	0.030
500	0.107	0.053	0.009	0.106	0.055	0.011
1000	0.099	0.045	0.009	0.101	0.052	0.014
3000	0.101	0.049	0.011	0.097	0.049	0.012
		$t(\hat{\gamma}_0)$			$t(\hat{\gamma}_1)$	
100	0.087	0.044	0.008	0.129	0.073	0.022
500	0.102	0.050	0.009	0.105	0.052	0.011
1000	0.096	0.047	0.009	0.100	0.050	0.013
3000	0.100	0.049	0.010	0.097	0.049	0.011

Simulation results: base case (F-test) - Full Σ ($\delta = 0.5$)

Table IX
Empirical size of F-tests in a One-Factor Model with a useless factor (Σ Full - $\delta = 0.5$)

	Panel A: $T = 36$			Panel A: $T = 72$		
N	0.10	0.05	0.01	0.10	0.05	0.01
100	0.113	0.060	0.014	0.138	0.080	0.030
500	0.101	0.053	0.011	0.106	0.055	0.011
1000	0.103	0.053	0.011	0.101	0.052	0.014
3000	0.101	0.051	0.011	0.097	0.049	0.012

Simulation results: useful plus useless - scalar Σ

- DGP is

$$R_t = \gamma_0 1_N + B_f(\gamma_1 + f_t - E[f]) + \epsilon_t,$$

where $\epsilon_t \sim \mathcal{N}(0, \sigma^2 I_T)$ and where we calibrate γ_0 and γ_1 as the OLS estimates from the one factor model (CAPM).

- Fitted Model is a Two-Factor Model"

$$R_t = \alpha + B_f f_t + B_g g_t + \epsilon_t,$$

where g_t is an orthogonal (useless) factor to f_t.

- All factors are orthogonalized to each other such that $\tilde{F}'\tilde{G} = 0_{K_f \times K_g}$

Simulation results: useful plus useless (Bias and RMSE) - scalar Σ

Table XIII
Bias and RMSE of the OLS Estimator in a correctly specified model with useful and useless factors (Σ scalar).

Statistics	$N = 100$	$N = 500$	$N = 1000$	$N = 3000$
		Panel A: $T = 36$		
Bias($\hat{\gamma}_0$)	0.78%	0.06%	-0.15%	0.10%
RMSE($\hat{\gamma}_0$)	0.291	0.132	0.071	0.047
Bias($\hat{\gamma}_{1_f}$)	0.43%	0.07%	0.08%	-0.03%
RMSE($\hat{\gamma}_{1_f}$)	0.211	0.102	0.053	0.039
Bias($\hat{\gamma}_{1_g}$)	0.000	0.000	0.000	0.000
RMSE($\hat{\gamma}_{1_g}$)	1.769	0.766	0.543	0.326
Bias(R^2)	4.66%	1.62%	0.38%	0.22%

Simulation results: useful plus useless (Bias and RMSE) - scalar Σ

Table XIII
Bias and RMSE of the OLS Estimator in a correctly specified model with useful and useless factors (Σ scalar).

Statistics	$N = 100$	$N = 500$	$N = 1000$	$N = 3000$
		Panel B: $T = 72$		
Bias$(\hat{\gamma}_0)$	0.10%	0.05%	0.06%	0.07%
RMSE$(\hat{\gamma}_0)$	0.582	0.195	0.079	0.055
Bias$(\hat{\gamma}_{1_f})$	0.27%	0.08%	0.11%	0.03%
RMSE$(\hat{\gamma}_{1_f})$	0.278	0.125	0.052	0.033
Bias$(\hat{\gamma}_{1_g})$	0.000	0.000	0.000	0.000
RMSE$(\hat{\gamma}_{1_g})$	1.215	0.540	0.376	0.227
Bias(R^2)	4.00%	1.57%	0.65%	0.39%

Simulation results: useful plus useless (t-test) - scalar Σ

Table XIV
Empirical Size of t-tests in a correctly specified model with useful and useless factors (Σ Scalar)

Panel A: $T = 36$

N	0.10	0.05	0.01	0.10	0.05	0.01	0.10	0.05	0.01
		$t(\hat{\gamma}_0)$			$t(\hat{\gamma}_{1_f})$			$t(\hat{\gamma}_{1_g})$	
100	0.102	0.049	0.011	0.099	0.053	0.009	0.100	0.053	0.012
500	0.102	0.052	0.009	0.098	0.048	0.008	0.099	0.051	0.012
1000	0.100	0.051	0.011	0.098	0.048	0.009	0.100	0.051	0.010
3000	0.099	0.050	0.010	0.101	0.052	0.010	0.099	0.049	0.010

Table XIV
Empirical Size of t-tests in a correctly specified model with useful and useless factors (Σ Scalar)

Panel B: $T = 72$

N	0.10	0.05	0.01	0.10	0.05	0.01	0.10	0.05	0.01
		$t(\hat{\gamma}_0)$			$t(\hat{\gamma}_{1_f})$			$t(\hat{\gamma}_{1_g})$	
100	0.097	0.049	0.009	0.101	0.048	0.010	0.110	0.054	0.012
500	0.103	0.051	0.013	0.096	0.048	0.009	0.098	0.048	0.010
1000	0.099	0.052	0.010	0.095	0.049	0.010	0.098	0.048	0.011
3000	0.101	0.052	0.010	0.102	0.052	0.010	0.100	0.049	0.009

Simulation results: useful plus useless (Bias and RMSE) - diagonal Σ

Table XV
Bias and RMSE of the OLS Estimator in a correctly specified model with useful and useless factors (Σ Diagonal).

Statistics	$N = 100$	$N = 500$	$N = 1000$	$N = 3000$
		Panel A: $T = 36$		
Bias($\hat{\gamma}_0$)	0.09%	0.06%	0.02%	0.04%
RMSE($\hat{\gamma}_0$)	0.052	0.034	0.029	0.021
Bias($\hat{\gamma}_{1_f}$)	0.11%	0.05%	0.03%	0.02%
RMSE($\hat{\gamma}_{1_f}$)	0.038	0.029	0.025	0.017
Bias($\hat{\gamma}_{1_g}$)	0.000	0.000	0.000	0.000
RMSE($\hat{\gamma}_{1_g}$)	1.820	1.203	0.958	0.543
Bias(R^2)	2.30%	0.90%	0.23%	0.18%

Simulation results: useful plus useless (Bias and RMSE) - diagonal Σ

Table XV
Bias and RMSE of the OLS Estimator in a correctly specified model with useful and useless factors (Σ Diagonal).

Statistics	$N = 100$	$N = 500$	$N = 1000$	$N = 3000$
	Panel B: $T = 72$			
Bias($\hat{\gamma}_0$)	0.09%	0.03%	0.02%	0.02%
RMSE($\hat{\gamma}_0$)	0.036	0.032	0.029	0.023
Bias($\hat{\gamma}_{1_f}$)	0.13%	0.03%	0.03%	0.03%
RMSE($\hat{\gamma}_{1_f}$)	0.032	0.023	0.019	0.017
Bias($\hat{\gamma}_{1_g}$)	0.001	0.001	0.000	0.000
RMSE($\hat{\gamma}_{1_g}$)	1.807	0.922	0.653	0.392
Bias(R^2)	3.13%	1.07%	0.19%	0.17%

Table XVI

Empirical Size of t-tests in a correctly specified model with useful and useless factors (Σ Diagonal)

Panel A: $T = 36$

N	0.10	0.05	0.01	0.10	0.05	0.01	0.10	0.05	0.01
		$t(\hat{\gamma}_0)$			$t(\hat{\gamma}_{1_f})$			$t(\hat{\gamma}_{1_g})$	
100	0.101	0.051	0.011	0.109	0.068	0.016	0.112	0.056	0.016
500	0.101	0.051	0.011	0.084	0.045	0.009	0.106	0.049	0.009
1000	0.102	0.052	0.010	0.099	0.051	0.010	0.103	0.054	0.011
3000	0.099	0.049	0.010	0.099	0.051	0.010	0.099	0.049	0.009

Simulation results: useful plus useless (t-test) - diagonal Σ

Table XVI
Empirical Size of t-tests in a correctly specified model with useful and useless factors (Σ Diagonal)

Panel B: $T = 72$

N	0.10	0.05	0.01	0.10	0.05	0.01	0.10	0.05	0.01
		$t(\hat{\gamma}_0)$			$t(\hat{\gamma}_{1_f})$			$t(\hat{\gamma}_{1_g})$	
100	0.079	0.045	0.007	0.073	0.042	0.003	0.106	0.056	0.014
500	0.105	0.054	0.010	0.089	0.045	0.008	0.098	0.053	0.010
1000	0.097	0.049	0.009	0.097	0.048	0.010	0.098	0.049	0.010
3000	0.099	0.049	0.010	0.098	0.049	0.010	0.100	0.049	0.010

Simulation results: useful plus useless (Bias and RMSE) - Full Σ ($\delta = 0.5$)

Table XVII
Bias and RMSE of the OLS Estimator in a correctly specified model with useful and useless factors (Σ Full, $\delta = 0.5$).

Statistics	$N = 100$	$N = 500$	$N = 1000$	$N = 3000$
	Panel A: $T = 36$			
Bias($\hat{\gamma}_0$)	**0.42%**	**0.38%**	**0.09%**	**0.06%**
RMSE($\hat{\gamma}_0$)	**0.086**	**0.042**	**0.035**	**0.021**
Bias($\hat{\gamma}_{1_f}$)	**0.06%**	**0.03%**	**0.04%**	**0.01%**
RMSE($\hat{\gamma}_{1_f}$)	**0.066**	**0.023**	**0.020**	**0.017**
$\hat{\gamma}_{1_g}$	**0.000**	**0.000**	**0.000**	**0.000**
RMSE($\hat{\gamma}_{1_g}$)	**1.211**	**0.916**	**0.903**	**0.543**
R^2	**2.90%**	**1.42%**	**0.49%**	**0.38%**

Simulation results: useful plus useless (Bias and RMSE) - Full Σ ($\delta = 0.5$)

Table XVII
Bias and RMSE of the OLS Estimator in a correctly specified model with useful and useless factors (Σ Full, $\delta = 0.5$).

Statistics	$N = 100$	$N = 500$	$N = 1000$	$N = 3000$
	Panel B: $T = 72$			
Bias$(\hat{\gamma}_0)$	**0.02%**	**0.04%**	**0.01%**	**0.02%**
RMSE$(\hat{\gamma}_0)$	**0.052**	**0.044**	**0.030**	**0.023**
Bias$(\hat{\gamma}_{1_f})$	**0.04%**	**0.07%**	**0.02%**	**0.02%**
RMSE$(\hat{\gamma}_{1_f})$	**0.056**	**0.036**	**0.028**	**0.021**
$\hat{\gamma}_{1_g}$	**0.000**	**0.000**	**0.000**	**0.000**
RMSE$(\hat{\gamma}_{1_g})$	**1.872**	**0.864**	**0.653**	**0.393**
R^2	**2.19%**	**1.29%**	**0.46%**	**0.29%**

Simulation results: useful plus useless (t-test) - Full Σ ($\delta = 0.5$)

Table XVIII
Empirical Size of t-tests in a correctly specified model with useful and useless factors (Σ Full - $\delta = 0.5$)

Panel A: $T = 36$

N	0.10	0.05	0.01	0.10	0.05	0.01	0.10	0.05	0.01
		$t(\hat{\gamma}_0)$			$t(\hat{\gamma}_{1_f})$			$t(\hat{\gamma}_{1_g})$	
100	0.127	0.069	0.016	0.126	0.072	0.019	0.102	0.053	0.010
500	0.107	0.055	0.014	0.110	0.054	0.013	0.102	0.052	0.009
1000	0.104	0.052	0.012	0.103	0.049	0.008	0.100	0.050	0.010
3000	0.099	0.048	0.010	0.099	0.051	0.010	0.100	0.050	0.010

Panel B: $T = 72$

N	0.10	0.05	0.01	0.10	0.05	0.01	0.10	0.05	0.01
		$t(\hat{\gamma}_0)$			$t(\hat{\gamma}_{1_f})$			$t(\hat{\gamma}_{1_g})$	
100	0.076	0.035	0.007	0.065	0.036	0.006	0.104	0.056	0.013
500	0.088	0.040	0.009	0.088	0.044	0.009	0.102	0.051	0.010
1000	0.095	0.047	0.009	0.096	0.046	0.008	0.102	0.051	0.010
3000	0.099	0.048	0.010	0.099	0.049	0.010	0.099	0.048	0.010

Conclusion

- Framework for testing useless factors within the context of beta-pricing models.

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- Designed for when N is large and T is fixed, possibly very small ($T > K$ is enough).
- Unlike the large-T methods, our approach is simple (based simply on the OLS CSR).
- Unlike the large-T methods, our results do NOT depend on degree of misspecification.
- Our results lead to conventional asymptotic distributions of OLS CSR estimator and test statistics.

THANKS