Default and Liquidity: A Continuous Time Approach*

(Preliminary and Incomplete)

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Abstract

Classical General Equilibrium models are discrete-time and assume away financial frictions, especially endogenous default and liquidity. We develop a continuous time stochastic macroeconomic model that incorporates both features. We identify the default channel that accelerates the rate of decrease of leverage and consequently investment. We also assess the equilibrium trade-off relationship between money supply and default penalty. We show that there exists an optimal monetary and regulatory mix, which achieves optimal levels of equilibrium welfare.

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1 Introduction

The global financial crisis (2008-10) underscored the importance of endogenous default when studying financial crises. The role of default has now recognised as a necessary ingredient of any attempt to assess the interaction of regulatory and monetary policy (see [Goo+18]). Indeed liquidity and default should be studied contemporaneously since one can not logically study the occurrence of default absent of the liquidity needs, Hence, the introduction of default in liquidity based models is warranted as the recent experience of the global financial crisis demonstrated clearly. A series of continuous time general equilibrium models, such as [B214], [HK11] and [HK13], addressed the entire dynamics of the economy. Furthermore, there are evident differences between the steady economy usually analysed under classical discrete-time macroeconomic literature and the long term behavior of an economy which incorporates financial frictions and aggregate shocks considered in [B214] and [BS16a]. This contrast is primarily attributed to the existence of the endogenous risk that is induced by the economy which shifts the equilibrium dynamics far from the situation where no such risk is accounted for. Continuous time frameworks have the advantage that they grasp this risk fully without losing any second order effects.

Until the global financial crisis, important financial frictions such as liquidity and default, were omitted in the mainstream macro models, which were also constructed on a discrete time structure. There are two principal reasons why most of the DSGE literature fails to incorporate default. First, there is great difficulty in modeling it. Second, when default coexists with other financial frictions, the computational complexity substantially increases. Excluding default, however, causes uncertainty to policy makers and regulators, to estimate the effects of their proposals.[1]. A rational approach mod-

[1]A comprehensive survey of modelling liquidity and default is included in [Goo+18].

elling default would also allow for a strategic default whenever the default cost is lower than the effective benefit.

Until now, there is no investigation regarding the interaction of endogenous default and liquidity in Continuous Time Finance. The advantages of using a continuous time framework are the following. Firstly, as opposed to a discrete time setting, continuous time frameworks (in most cases) guarantee smoothness of functions and variables, leading to simple first order optimality conditions that guarantee tractability. Tractability allows for a closed-form characterization of equilibrium and, implicitly, more analytical solutions prior to numerical optimization and simulations. For example, in our model, we derive explicit closed form expressions for the loan repayment rate. The majority of discrete time settings would require log-linear approximation around the steady state solution, hence obviating the importance of the second moments of distributions. Additionally, it has been widely argued by economists [BS16a] that continuous time representation provides a more accurate depiction of reality. For example, agents do not consume only at the end of the month or the end of the quarter. In a discrete time setting linear time aggregation within a quarter and a non-linear across quarters is implicitly assumed. Consequently, the great advantage of using a continuous time frameworks is the analytical tractability of the system's dynamics and the lack of necessity to log-linearize around the steady state.

The organization of the rest of this paper is as follows. The next subsection discusses the related literature. Section 2 describes the model and defines the equilibrium dynamics. Section 3 solves for the equilibrium and characterizes its dynamics. Section 4 is dedicated to simulations and the results that stem from them and section 5 highlights the comparative statics of the model. Section 6 provides the trade-off between money supply and default penalty, and finally section 7 concludes.

1.1 Related Literature

The model that will be presented below is a modified version of the working paper [RT19], that in turn extends [B214] by introducing default on a nominal bond that is exchanged between two agent. In our model, we introduce a central bank and fiat money as a stipulated mean of exchange as in [DG92], so that to assess the relationship between money supply and the default penalty. In [B214], the utility functions of the agents are risk neutral whereas in [BS16a], the authors consider more general utility functions as well as they allow for precautionary savings and endogenous equity issuance. The above series of papers in continuous time finance can be thought as extensions of [BC98], [HK13] and [HK11] which are the seminal papers in this literature. There is also a series of papers in continuous time macrofinance regarding monetary economics. For instance, in [BS16b] , money is a bubble as in [Sam58] and [Bew76] . In our paper however, the banking sector creates inside money endogenously and there is an interaction between monetary and macroprudential policy. Furthermore, in [Ach+17], an algorithm that solves Bewley models with uninsurable endowment risk in continuous time is presented. Finally, [DSS18] shows how risk premia are affected by monetary policy in a monetary economy were banks are less risk averse.

The last class of models, are the ones that incorporate endogenous default. In these models, every time each agent decides to default (due to a strategic decision or ill fortune), his utility function faces a real penalty proportional to the amount of debt. This idea (which is also called λ default) was first introduced in [Shu73]. [SW77] first introduced a banking sector into a general equilibrium model and investigated the optimal default penalty. They introduced bankruptcy penalties to represent the idea that the banking sector might only imperfectly be able to enforce the repayment of its loans. In this formulation, an agent suffered a loss of utility proportional to the number of dollars of his unpaid bank loans. [DG92] considers a banking sector in a monetary economy and investigates the existence of monetary equilibrium when there are finite bankruptcy penalties. [DGS05] extends the standard model of general equilibrium with incomplete markets to allow for default and punishment by thinking of assets as pools. Their work also captures the adverse selection and signalling phenomena due to the presence of sellers with a proclivity for default having an incentive to sell disproportionately many promises to the pool, thus decreasing the delivery rate. Furthermore, in [Tso+18], Martinez and Tsomocos developed a model that by including agent heterogeneity, liquidity and endogenous default in a pure exchange economy, addressed issues of financial stability as well as suggesting appropriate policy responses. Their results, suggest that liquidity and default in equilibrium should be studied contemporaneously due to their interconnectedness and welfare effects. Finally, [EGT09] shows that, in a monetary exchange economy, assets prices in a complete markets general equilibrium are a function of the supply of liquidity by the Central Bank. An interesting result is also shown, namely that higher spot interest rates increase state prices and sequentially risk neutral probabilities. In our paper, we aim to prove a continuous time version of this result.

2 The Model

We develop a rigorous continuous time framework that can be used to investigate the relationship between endogenous default and liquidity provided by Central Banks. A novel endeavor is made to create an analytical yet tractable continuous time model that incorporates liquidity and endogenous default. Optimal monetary responses will be examined, when the above financial frictions are present, in a formal general equilibrium framework. In particular we consider a stochastic continuous time economy that can be viewed as a combination of the model considered by [RT19], (which is an extension of [B214]) and the model of [DG92]. We examine the relationship of money supply, provided by the central bank, with the bankruptcy penalty that each agents faces in case he opts to default.

2.1 Model Setup

Let as begin with the complete probability space $(\Omega, \mathcal{F}, \mathcal{P})$ which is endowed with a standard Brownian Motion Z_t. We assume that $Z_0 = 0$ almost surely. All economic activity will be assumed to take place in the horizon $[0, \infty)$. Let :

$$\mathcal{F}^Z(t) \triangleq \sigma\{Z_s; 0 \leq s \leq t\}, \forall t \in [0, T]$$

be the filtration generated by $Z(.)$ and let $\mathcal{N}$ denote the $\mathcal{P}$- null subsets of $\mathcal{F}^Z(T)$. We shall use the augmented filtration:

$$\mathcal{F}(t) \triangleq \sigma\{\mathcal{F}^Z(t) \cup \mathcal{N}\}, \forall t \in [0, T]$$

One should interpret the $\sigma-$algebra $\mathcal{F}(t)$ as the information available to agents at time t, in the sense that if $\omega \in \Omega$ is the true state of nature and if $A \in \mathcal{F}(t)$, then all agents will know whether $\omega \in A$. Note that $\mathcal{F}(0)$ contains only sets of measure one and sets of measure zero, so every $\mathcal{F}(0)$ measurable random variable is almost surely constant.

We consider an economy with two agents, agent A and agent B, in which we introduce a strategic "dummy" that we shall call a monetary authority (Central Bank). Both types of agents can own capital and money, but agents of type A are able to use capital in a more productive way.

2.1.1 Technology

The aggregate amount of capital in the economy is denoted by K_t and capital owned by an individual agent i by k_t^i , where $t \in [0, \infty)$ is time. Physical capital k_t^A held by an agent of type A produces output at rate:

$$y_t^A \triangleq \gamma^A k_t^A, \quad \forall t \in [0, \infty) \tag{1}$$

per unit of time, where γ^A is an exogenous productivity parameter. In a world without fiat money, output could be modeled as a numeraire, and its price could be normalized to one. However, in the current model, we will denote the price of output (and as a result the price of the consumption good) as p_t^c which is going to be determined endogenously in equilibrium. Capital owned by agent A, with state space $\mathscr{F} \subseteq \mathbb{R}$ satisfies the following Ito's process:

$$dk_t^A \triangleq (\Lambda(\iota_t^A) - \delta^A) k_t^A dt + \sigma k_t^A dZ_t, \quad \forall t \in [0, \infty) \tag{2}$$

where ι_t^A is the investment rate per unit of capital and dZ_t are exogenous aggregate standard Brownian shocks defined above. We assume that $\mathscr{F}$ is an interval with endpoints $-\infty \leq a < b \leq \infty$ and that k_t^A is regular in (a, b); k_t^A reaches y with positive probability starting at x. for every x and y in (a, b). We shall denote $\mathbb{F} = \mathscr{F}$ the natural filtration of k_t^A. Function Λ, which satisfies $\Lambda(0) = 0$, $\Lambda'(0) = 1$, $\Lambda'(.) > 0$, and $\Lambda''(.) < 0$, represents a standard investment technology with adjustment costs. In case there is no investment, capital managed by agent A depreciates at rate δ^A. The concavity of $\Lambda(\iota)$ represents technological illiquidity, that is adjustment costs of converting output to new capital and vice versa.

Agents of type B are less productive. Capital managed by agent B, with state space $\mathscr{F} \subseteq \mathbb{R}$, produces the following output:

$$y_t^B \triangleq \gamma^B k_t^B, \quad \forall t \in [0, \infty) \tag{3}$$

with $\gamma^B \leq \gamma^A$ and evolves according to

$$dk_t^B \triangleq (\Lambda(\iota_t^B) - \delta^B) k_t^B dt + \sigma k_t^B dZ_t, \quad \forall t \in [0, \infty) \tag{4}$$

with $\delta^B > \delta^A$, where ι_t^B is the agent's investment rate per unit of capital. We assume that $\mathscr{F}$ satisfies the same conditions as for agent A above.

2.1.2 Preferences

Agents A and B will have preferences that are generally characterized by the instantaneous utility function $u^i(c_t^i) : \mathbb{R}_+ \to \mathbb{R}$, where $i \in \{A, B\}$, and they also have constant discount factors β^i. The consumption space defined above must also be square-integrable:

$$\int_0^{\infty} \left\| c_t^i \right\| dt < \infty$$

Agents want to maximize their lifetime utility function given by:

$$U(c^i) \triangleq \int_0^{\infty} e^{-\beta^i t} u^i(c_t^i) dt, \quad \forall t \in [0, \infty) \tag{5}$$

The utility function obeys the standard assumption summarized below.

Assumption 1: The utility function $u^i : \mathbb{R}_+ \to \mathbb{R}$, is concave and continuously diffentiable and also

$u'^i \triangleq \frac{\partial u^i}{\partial c_t^i} > 0$ for $i \in \{A, B\}$

Only agent A has access to borrow money from the Central Bank, and she needs to repay the very next moment dt with interest r_t. In this paper, default is modelled as in [DGS05]. The extend of default is determined by the existence of non-pecuniary penalties that are proportional to the amount of the contractual obligations that are not repaid to the Central Bank. Penalties, incurred by default, are subtracted from the utility function of the agents and can be thought as either reputational costs or material sanctions that households suffer by deciding not to fulfill their contractual obligations.

We must now specify the utility U^A, to A, of the outcome (c^A, d^A), where d^A is the percentage of outstanding balance owed to the central banks with respect to the total nominal value of the capital in the economy that is given by $d^A = \mid \frac{(1-v_t)(\mu_t^A)}{p_t K_t} \mid$. In case there is no incentive to return money to the bank, the agent would always choose

$v_t = 0$ and c_t^A and d^A very large. Prices would go to infinity in this case, and hence equilibrium will not obtain. Thus we consider, a penalty equal to $\tau max(0, d^A)$ where τ proxies the severity of the default penalty, u_t is the repayment rate and μ_t^A is the amount that agent A borrowed by the Central Bank. Hence, overall payoff of agent A:

$$U^A(c^A, d^A) \triangleq \int_0^\infty e^{-\beta^i t}(u^A(c_t^A) - \tau d_{t+})dt \tag{6}$$

where $\nu_+ = max\{0,\nu\}$ for any real number ν.

The function U^A incorporates a bankruptcy penalty through the rate τ. The penalty increases in harshness directly with the size of the debt. For different values of this penalty, we can assess different bankruptcy regimes defined in terms of the harshness of the default penalty.

2.1.3 Market for Capital

Agents A and B have the opportunity to trade physical capital in a competitive market. We denote the equilibrium market price of capital in terms of output by p_t with state space $\mathscr{F} \subseteq \mathbb{R}$ and we will also assume that it will satisfy the following Ito's process:

$$dp_t \triangleq \mu_t^p p_t dt + \sigma_t^p p_t dZ_t \tag{7}$$

for some Borel functions $\mu_t^p : \mathscr{F} \to \mathbb{R}$ and $\sigma_t^p : \mathscr{F} \to (0, \infty)$. Using the definition above, capital k_t costs $p_t k_t$. Please note that, in equilibrium p_t is determined endogenously.

2.1.4 Return to Capital

When an agent A buys k_t units of capital at price p_t, by Ito's Lemma the value of this capital evolves according to:

$$\frac{d(k_t^A p_t)}{k_t^A p_t} = (\Lambda(\iota_t) - \delta^A - \mu_t^p + \sigma\sigma_t^p)dt + (\sigma + \sigma_t^p)dZ_t \tag{8}$$

Hance, the total return that experts earn from capital (per unit of wealth invested) is:

$$dr_t^{Ak} = \frac{\gamma^A - \iota_t^A}{p_t}dt + (\Lambda(\iota_t^A) - \delta^A - \mu_t^p + \sigma\sigma_t^p)dt + (\sigma + \sigma_t^p)dZ_t \tag{9}$$

Similarly, the less productive agent B earns the return of

$$dr_t^{Bk} = \frac{\gamma^B - \iota_t^B}{p_t}dt + (\Lambda(\iota_t^B) - \delta^B - \mu_t^p + \sigma\sigma_t^p)dt + (\sigma + \sigma_t^p)dZ_t \tag{10}$$

2.2 Agent's A Optimization Problem

The non-monetary net worth w_t^A of an agent of type A who invests fraction x_t of her wealth in capital, consumes $c_t^A dt$ and sells part of his production to agent B evolves according to:

$$dw_t^A = x_t w_t^A dr_t^{Ak} + (x_t - 1)w_t^A dt - p_t^c c_t^A dt \tag{11}$$

At the beginning of each period, agent A will buy capital from agent B, by using his outside money (m_t^A) together with the moneys he will borrow from the bank. Thus, we will have,

$$(x_t - 1)w_t^A dt \leq (\mu_t e^{-r_t} + m_t^A)dt \tag{12}$$

An the end of each period, he repays his loan to the bank by selling part of the output produced to agent B. Thus we will get:

$$v_t \mu_t dt \leq \frac{b_t(\gamma^A - i^A)p_t^c w_t^A}{p_t} x_t dt \tag{13}$$

Here b_t is the proportion of output that agent A will sell and as a result it will be between 0 and 1.

Finally, agent A will consume the rest of her output. That is,

$$p_t^c c_t^A dt \leq \frac{(1 - b_t)(\gamma^A - i^c)p_t^c w_t^A}{p_t} x_t dt \tag{14}$$

Note that x_t represents the weight of wealth that agent A will invest in capital. In our model, agent A will use leverage ($x_t > 1$), and sequentially she will pay $-(1 - x_t)w_t^A$ to agent B to buy capital.

She buys capital, by using her new monetary endowment (m_t^A), and and borrowed funds from the bank which are approximately equal to $\mu_t^A e^{-r_t} \simeq \mu_t^A/(1 + r_t)$. In case the agent defaults she repays $v_t^A \mu_t^A$ to the bank where v_t^A is the repayment rate which is bounded below by zero and above by one. In case x_t is less than 1, there is no borrowing at period t and hence there is no potential for default and v_t is equal to one.

Formally, each agent type A solves

$$max_{\mu_t \geq 0, x_t \geq 0, c_{t,}^A \geq 0, 0 \leq v_t^A \leq 1, 0 \leq b_t^A \leq 1} E[\int_0^\infty e^{-\beta^A t}(u^A(c_t^A) - \tau_t d_{t+}^A)dt] \tag{15}$$

where $d_{t+}^A = max(0, \frac{(1-v_t)\mu_t}{p_t K_t})$

subject to the solvency constraint $w_t^A \geq 0$, the dynamic budget constraints (11) and to the cash in advance constraints (12) , (13) and (14).

2.3 Agent's B Optimization Problem

Agent of type B behaves as follows. First, she sells capital to A, and thus she receives fiat money equal to $w_t^A(x_t - 1)$. Her goal is to maximize her utility and she can do so by consuming the output that her capital produces as well as by buying consumption goods from agent A. Thus, she will use the amount of money he received from agent A, plus the private monetary endowment to buy consumption goods from A.

Similarly with agent A, the non-monetary net worth w_t^B of B who invests fraction $\bar{x}_t$ of his wealth in capital and consumes $c_t^B dt$ is,

$$dw_t^B = \bar{x}_t w_t^B dr_t^{Bk} + w_t^B(\bar{x}_t - 1)dt - p_t^c c_t^B dt \tag{16}$$

Agent B will also sells part of his capital in order to buy output as $\bar{x}_t < 1$. Therefore,

$$(w_t^B(\bar{x}_t - 1) + s_t^b)dt \leq m_t^b dt \tag{17}$$

Formally, each Agent of type B solves

$$max_{\bar{x}_{t,}, \geq 0, c_{t,}^B \geq 0, s_t^B \geq 0,} E[\int_0^\infty e^{-\beta^B t}(u(c_t^B)dt] \tag{18}$$

subject to the solvency constraint $w_t^B \geq 0$, the dynamic budget constraints (16) and the cash-in-advance constraint (17).

2.4 Equilibrium

Intuitively, an equilibrium is characterized by a map from shock histories $\{Z_S, s \in [0, t]\}$, to prices p_t and p_t^c and asset allocations such that, given rational expectation prices, agents maximize their expected utilities and markets clear. To define an equilibrium formally, we denote the set of agents of type A to be the interval $I = [0, 1]$ and index individual experts by $i \in I$. Similarly, we denote the set of agents of type B by $J = (1, 2]$ with index j.

We now proceed by stating the market clearing conditions and finally defining the equilibrium.

2.5 Market Clearing

The three markets that clear in equilibrium $\forall t \in T$ are the capital, commodity and money markets.

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2.5.1 Capital Market

The total capital held by the firms plus the total capital held by the households should be equal to the total supply of capital i.e.

$$\int_0^1 k_t^i di + \int_1^2 \underline{k}_t^j dj = K_t, \forall t \in [0, \infty) \tag{19}$$

where K_t is the total supply of capital. We hasten to add that the total supply of capital in the model is not fixed since both agents create new capital through their production. In addition, the following equation describes the evolution of the total supply of capital,

$$dK_t \triangleq \left(\int_0^1 \left(\Lambda \left(\iota_t^i \right) - \delta \right) k_t^i di + \int_1^2 \left(\Lambda \left(\underline{\iota}_t^j \right) - \underline{\delta} \right) \underline{k}_t^j dj \right) dt + \sigma K_t dZ_t, \forall t \in [0, \infty) \tag{20}$$

2.5.2 Money Market

Money market equates money supply with money demand. That is:

$$\mu_t e^{-r_t} dt = M_t dt \tag{21}$$

2.5.3 Commodity Market

Finally, if the capital and money market clear then the market for consumption also clears, i.e.,

$$\int_0^1 k_t^i \left(\gamma^A - \iota_t^i \right) di + \int_1^2 \underline{k}_t^j \left(\gamma^B - \underline{\iota}_t^j \right) dj = \int_1^2 \underline{c}_t^j dj + \int_0^1 c_t^i di, \forall t \in [0, \infty) \tag{22}$$

Now, we are in a position to formally define the equilibrium of our economy.

Definition 1. An equilibrium in this economy consists of the stochastic processes of capital and consumption prices $\{p_t, p_t^c, t \geq 0\}$, the interest rate $\{r_t, t \geq 0\}$, the investment, consumption, default and borrowing decisions for both agents i.e. $\{(k_t^i, \iota_t^i, c_t^i, v_t, \mu_t), t \geq 0\}$ for agent A ($i \in [0,1]$) and $\left\{\left(\underline{k}_t^j, \underline{\iota}_t^j, \underline{c}_t^j\right), t \geq 0\right\}$ for agent B ($j \in (1,2]$). These processes should satisfy the following two conditions:

1. Given the price processes p_t, p_t^c and r_t, each agent of type A ($i \in [0,1]$) and each agent of type B ($j \in (1,2]$) maximize their objective functions (15) and (18) respectively with the set of choice variables $\{(k_t^i, \iota_t^i, c_t^i, v_t, \mu_t), t \geq 0\}$ and $\left\{\left(\underline{k}_t^j, \underline{\iota}_t^j, \underline{c}_t^j\right), t \geq 0\right\}$, respectively, subject to their corresponding budget constraints.
2. Capital, consumption and money markets clear.

3 Characterization for Equilibrium

In this section, we will discuss how to find the equilibrium prices p_t and p_t^c , the agents ' consumption decisions, as well as the optimal repayments rates given the history of macro shocks $\{Z_s, 0 \leq s \leq t\}$. Furthermore, we will present a simple proof that establishes the relationship between money supply and the default penalty in the case of logarithmic utilities. For computational simplicity, we assume that all the money in the economy will be homogeneous of degree 1 with respect to the total capital of the economy K_t.. Put differently, we conjecture a map between liquidity and capital formation at every point that allows us a computationally tractable solution that is confirmed in equilibrium.

We first start with some definitions.

Definition 2. The entire non-monetary wealth of agent A and agent B is given, respectively, by summing up their individual wealth, that is:

$$W_t = \int_0^1 w_t^i di \qquad \underline{W}_t = \int_1^2 \underline{w}_t^j dj, \quad \forall t \in [0, \infty)$$

Please observe that clearing conditions (21) become,

$$\int_0^1 x_t^i w_t^i di + \int_1^2 \underline{x}_t^j \underline{w}_t^j dj = p_t K_t, \quad \forall t \in [0, \infty) \tag{23}$$

$$W_t + \underline{W}_t = p_t K_t \tag{24}$$

$$\int_0^1 x_t^i w_t^i \left(\gamma_t^{Ai} - \iota_t^{Ai}\right) di + \int_1^2 \underline{x}_t^j \underline{w}_t^j \left(\underline{\gamma}^{Bj} - \underline{\iota}_t^{Bj}\right) dj = p_t \int_1^2 c_t^j dj + p_t \int_0^1 c_t^i di \tag{25}$$

We now provide the definition of the most important state variable of our economy, that is the proportion of wealth agent of type A possesses. By representing each variable in the model in terms of η_t which is bounded between 0 and 1, we can fully characterize the equilibrium variables.

Definition 3. The proportion of wealth that Agent A possesses is given by,

$$\eta_t = \frac{W_t}{W_t + \underline{W}_t} \tag{26}$$

We further postulate that the dynamics of η_t evolve as follows,

$$\frac{d\eta_t}{\eta_t} = \alpha(\eta_t)dt + \beta(\eta_t)dZ_t \tag{27}$$

By using (25) and (27), we obtain the equilibrium condition,

$$\eta_t = \frac{W_t}{p_t K_t} \tag{28}$$

The entire dynamics of the model are being driven by η_t. Evidently, in equilibrium we have,

$$x_t \leq \frac{1}{\eta_t} \tag{29}$$

which means that the leverage that agent A will use is bounded above, by the maximum leverage she may obtain. The above constraint will be binding. When η_t is large then $x_t = \frac{1}{\eta_t}$ in equilibrium, which implies that agent A is holding all the capital. Note that in [B214] all agents use risk neutral utilities, and thus the optimal leverage would be the maximum possible. Consequently, if she leverages to her maximum, only corner solutions obtain!

3.1 Internal Investment

Hereafter, we proceed by defining the investment function Λ as follows,

$$\Lambda(i_t^j) = \ln(i_t^j + 1), \quad \forall t \in [0, \infty), \quad j \in \{A, B\}$$

which satisfies all standard assumptions discussed in section 2.1.1. In this model we do not allow for disinvestment, thus we also assume that $i_t^j \geq 0$.

3.2 Agent's A optimization Problem

Agent A maximizes her objective function (15) subject to constraints (11), (12), (13) and (14). We now fully characterize her optimal consumption c_t^A, her optimal investment i_t^A, the optimal amount of output that she sells to agent B, the optimal amount of money she borrows μ_t and the optimal repayment rate v_t.

Proposition 1. *Assume that all agents have the same logarithmic utility. Then:*

i) The optimal consumption is given by :

$p_t^c {c^A}_t^* = \frac{p_t K_t}{\tau}$

ii) The equilibrium interest rate r_t^* is given by:

$r_t^* = \ln \frac{\tau(\frac{\gamma^A - \iota_t^A}{p_t})p_t^c + 1/\rho(\Lambda(\iota_t^A) - \delta - \mu_t^p + \sigma\sigma_t^p + 1) - \eta_t 1/\rho(\sigma+\sigma^p)^2 x_t}{\tau}$

iii) The optimal investment rate ι_t^{A*} is given by:

$i_t^{A*} = \frac{p_t}{\tau \rho_1 p_t^c} - 1$

Proof. The Hamiltonian-Jacobian-Bellman (HJB) equation of agent A is given by:

$$\rho_1 V \;=\; max_{b_t, x_t, c_t^A, \mu_t, v_t} \ln c_t^A - \frac{\tau}{p_t K_t}(1 - v_t)\mu_t + \lambda_1(\mu_t e^{-r_t} + m_t^A - (x_t - 1)w_t^A) +$$

$$+\lambda_2(\frac{b_t(\gamma^A - \iota_t^A)p_t^c w_t^A}{p_t} x_t - v_t\mu_t) + \lambda_3(\frac{(1-b_t)(\gamma^A - \iota_t^A)p_t^c w_t^A}{p_t} x_t - p_t^c c_t^A) + (\frac{\gamma^A - \iota_t^A}{p_t} + \Lambda(\iota_t^A) - \delta^A - \mu_t^p + \sigma\sigma_t^p) +$$

$$-p_t^c c_t^A)\frac{dV_t}{dw_t} + \frac{(x_t w_t(\sigma + \sigma_t^p))^2}{2}\frac{d^2 V_t}{d{w_t^A}^2} + \frac{dV}{d\eta_t}\{drift\, of\, \eta_t\} + \frac{d^2 V}{d\eta_t^2}\frac{\{vol\, of\, \eta_t\}}{2}$$

where λ_1, λ_2 and λ_3 are the corresponding langrage multipliers of constraints (12), (13) and (14).

The first order conditions with respect to c_t^A, v_t, and μ_t are given by:

$$\frac{1}{c_t^{A*}} = p_t^c \frac{dV_t}{dw_t^A} + \lambda_3 p_t^c \tag{30}$$

$$\frac{\tau}{p_t K_t} = \lambda_2 \tag{31}$$

$$\frac{\tau(1 - v_t)}{p_t K_t} + \lambda_1 e^{-r_t} + \lambda_2 v_t = 0 \tag{32}$$

Using equation (32) we get:

$$\frac{\tau}{p_t K_t} = \lambda_1 e^{-r_t} \tag{33}$$

From (32) and (34) we observe that the multipliers for constraints (12) and (13) are positive and thus by Kuhn-Tucker the constraints must bind.

Now, by taking the first order condition with respect to b_t (the percentage of output that he sells to Agent B), we obtain:

$$-\frac{dV_t}{dw_t} + \lambda_2 - \lambda_3 = 0 \Longleftrightarrow \tau p_t K_t = \lambda_2 = \frac{dV_t}{dw_t} + \lambda_3 (34)$$

By combining the above result with equation (31) we get that optimal consumption has to be:

$$p_t^c c_t^{A*} = \frac{p_t K_t}{\tau(35)}$$

Now, the first order condition with respect to x_t, becomes:

$$x_t {w_t^A}^2 (\sigma + \sigma_t^p)^2 \frac{d^2 V_t}{d{w_t^A}^2} + ((1 - b_t)\frac{\gamma^A - \iota_t^A}{p_t} + \Lambda(\iota_t^A) - \delta^A - \mu_t^p + \sigma\sigma_t^p + 1) w_t^A \frac{dV_t}{dw_t^A} - \lambda_1 w_t^A +$$

$$+\lambda_2(\frac{b_t(\gamma^A - i_t^A) p_t^c w_t^A}{p_t}) + \lambda_3(\frac{(1 - b_t)(\gamma^A - i_t^A) p_t^c w_t^A}{p_t}) = 0$$

Here, we just have to use equation (36) in order to eliminate of b_t. Consequently, we observe that w_t^A cancels out as well and therefore we solve for the equilibrium interest rate:

$$r_t^* = \ln \frac{\tau(\frac{\gamma^A - \iota_t^A}{p_t}) p_t^c + 1/\rho(\Lambda(\iota_t^A) - \delta - \mu_t^p + \sigma\sigma_t^p + 1) - \eta_t 1/\rho(\sigma + \sigma^p)^2 x_t}{\tau}$$

The final step that concludes the proof is finding the optimal investment rate i_t^{A*}. However, this is just a static problem since i_t^A appears only multiplicatively with λ_2 and the $\frac{dV_t}{dw_t^A}$ terms of the HJB.. Hence, all the other terms of (35) can be safely ignored.

Thus by taking first order conditions we obtain:

$-\frac{\tau p_t^c}{p_t} + \frac{1}{\rho}\Lambda'(\iota_t^A) = 0 \Longleftrightarrow \Lambda'(\iota_t^A) = \frac{\rho\tau p_t^c}{p_t}$

Finally, by choosing $\Lambda(\iota_t^A) = \ln(\iota_t^A + 1)$, the proof is concluded. $\square$

Note that by choosing value function equal to $V(w_t^A, \eta_t) = \frac{ln(p_t K_t)}{\rho} + f(\eta_t)$ we can verify that the HJB is satisfied (i.e. is all w_t^A terms cancel out).

3.2.1 Agent's B Problem

Agent B optimizes her utility function (18) subject to its net worth evolution (16) and her cash-in advance (17). The proposition below, solves B's optimal choices for consumption $\underline{c}_t^B$, leverage $\underline{x}_t$ (expected to be negative), and investment ι_t^B.

Proposition 2. *Agent's B optimal choices for consumption, leverage and investment are given by the following equations*

$$\underline{x}_t = \frac{{}^{E\left[d\underline{r}_t^k\right]}\!/_{dt} + 1}{\left(\sigma + \sigma_t^p\right)^2} \quad \underline{\iota}_t = \frac{p_t}{p_t^c} - 1 \quad c_t^B = \rho_2 \underline{w}_t^B \tag{36}$$

Proof. Let

$$V\left(t,\, \underline{w}_t,\, \eta_t\right) = sup_{\underline{x}_t \geq 0,\, \underline{\iota}_{\underline{t}}^B{}_t \geq 0,\, \underline{\varepsilon}_t \geq 0, s_t \geq 0} E_t \left[\int_t^{+\infty} e^{-\gamma s} \left[\ln\left(\underline{\varepsilon}_t \underline{w}_t^B\right)\right] ds\right] \tag{37}$$

where $\underline{\varepsilon}_t = \frac{\underline{c}_t^B}{\underline{w}_t}$.

We have that

$$\frac{d\eta_t}{\eta_t} = \alpha(\eta_t)dt + \beta(\eta_t)dZ_t$$

.

Consequently, the HJB equation that V must satisfy to be optimized is given by

$$\rho_2 V = sup_{\underline{x}_t \geq 0,\, \underline{\iota}_t \geq 0,\, \underline{\varepsilon}_t \geq 0, s_t \geq 0} \{e^{-\rho_1 t} \ln\left(c_t^B\right) + \frac{dV}{d\underline{w}_t^B} \left\{drift\, of\, \underline{w}_t^B\right\} + \frac{d^2 V}{d\underline{w_t^B}^2} \frac{\left\{vol\, of\, w_t^B\right\}}{2} +$$

$$+\frac{dV}{d\eta_t}\{drift\, of\, \eta_t\}+\frac{d^2V}{d\eta_t^2}\frac{\{vol\, of\, \eta_t\}}{2}\} \tag{38}$$

We conjecture that the solution will satisfy the form $V = e^{-\gamma t}\left(f(\eta_t)+\frac{log\left(\underline{w}_t^B\right)}{\rho_2}\right)$ for some function f. By direct substitution, we obtain:

$$\rho_2 f(\eta_t)+\ln\left(\underline{w}_t^B\right) = sup_{\underline{x}_t\geq 0,\, \underline{\iota}_t\geq 0,\, \underline{\varepsilon}_t\geq 0}\{\ln\left(c_t^B\right)+$$

$$+\frac{1}{\rho_1 \underline{w}_t^B}\left[\underline{x}_t\underline{w}_t\left({}^{E\left[d\underline{r}_t^k\right]}/dt+1\right)-p_t^c c_t^B\right]$$

$$-\frac{1}{\rho_1}\left[\frac{\left(\left(\sigma+\sigma_t^p\right)\underline{w}_t\right)^2}{2}\right]+\eta_t f'(\eta_t)\,\alpha(\eta_t)+\frac{1}{2}\eta_t^2 f''(\eta_t)\,\beta^2(\eta_t)\} \tag{39}$$

Clearly, when maximizing with respect to $\underline{\iota}_t$, we obtain $\Lambda'(\underline{\iota}_t)=\frac{p_t}{p_t^c}$, which similarly to agent's A problem gives

$$\underline{\iota}_t^B=\frac{p_t^c}{p_t}-1$$

Set $\underline{b}_t = {}^{E\left[d\underline{r}_t^k\right]}/dt+1$. Re-arranging equation (39) and taking into account that f should satisfy it for all values of η_t, then

$$\rho_2 f(\eta_t)-\frac{1}{\gamma}-\eta_t f'(\eta_t)\,\alpha(\eta_t)-\frac{1}{2}\eta_t^2 f''(\eta_t)\,\beta^2(\eta_t)-$$

$$-sup_{\underline{x}_t\geq 0}\frac{1}{\rho_2}\left[\underline{b}_t\underline{x}_t-\frac{1}{2}\left(\left(\sigma+\sigma_t^q\right)\underline{x}_t\right)^2\right]-sup_{\underline{\varepsilon}\geq 0_t}\left[-\frac{\underline{\varepsilon}_t}{\rho_2}+\ln\left(\underline{\varepsilon}_t\right)\right]=0 \tag{40}$$

Finally, optimizing with respect to ε gives

$$\underline{\varepsilon}_t=\rho_2 \Longrightarrow c_t^B=\rho_2\underline{w}_t^B$$

Also optimizing $\underline{b}_t \underline{x}_t - \frac{1}{2}\left(\left(\sigma + \sigma_t^p\right)\underline{x}_t\right)^2$, with respect to $\underline{x}_t \geq 0$ we have:

$$\underline{x}_t = \frac{\underline{b}_t}{\left(\sigma + \sigma_t^q\right)^2} = \frac{E\left[d\underline{r}_t^k\right]/dt + 1}{\left(\sigma + \sigma_t^q\right)^2}$$

□

However, to fully characterize the equilibrium, we need to derive the evolution of the state variable η_t. Define the fraction of capital held by agent A by

$$\psi_t = \frac{\int_0^1 k_t^i di}{K_t}$$

This entails that, in equilibrium, $1 - \psi_t = \frac{\int_1^2 k_t^j dj}{K_t}$, the fraction of capital held by agent B, is equal to

Proposition 3. *In equilibrium*

$$x_t = \frac{\psi_t}{\eta_t} \qquad \underline{x}_t = \frac{1 - \psi_t}{1 - \eta_t} \tag{41}$$

Proof. It follows immediately from the clearing condition (24).

□

Proposition 4. *The evolution of the expert's wealth relative to the entire economy is given by*

$$\frac{d\eta_t}{\eta_t} = \frac{\psi_t - \eta_t}{\eta_t}\left(dr_t^k + (1 - \frac{\gamma^A - \iota_t^A}{p_t}p_t^c - \left(\sigma + \sigma_t^q\right)^2 + \left(1 - \psi_t\right)\left(\underline{\delta}^B - \delta^A\right))dt\right) \tag{42}$$

Proof. Aggregating the wealth evolution over all experts and using proposition 3 we get

$$dW_t = \psi_t p_t K_t dr_t^k - \psi_t p_t K_t (\frac{\gamma^A - \iota_t^A}{p_t})p_t^c dt - W_t dt \tag{43}$$

By Ito's quotient rule[2], we have

$$\frac{d\eta_t}{\eta_t} = \frac{dW_t}{W_t} - \frac{d\left(p_t K_t\right)}{p_t K_t} + \left(\frac{d\left(p_t K_t\right)}{p_t K_t}\right)^2 - \frac{dW_t}{W_t}\frac{d\left(p_t K_t\right)}{p_t K_t} \tag{44}$$

In addition, Ito's product rule and the fact that the optimal investment choice is identical for both agents give

$$\frac{d\left(p_t K_t\right)}{p_t K_t} = dr_t^k - \frac{A - \iota_t^A}{q_t} p_t^c dt - \left(1 - \psi_t\right)\left(\underline{\delta} - \delta\right) dt \tag{45}$$

where the first two terms of the right-hand side are the expert capital gains and the third term is the adjustment for the household held capital. Now, from (46) we obtain

$$\frac{d\eta_t}{\eta_t} = \frac{\psi_t - \eta_t}{\eta_t}\left(dr_t^k + (1 - \frac{\gamma^A - \iota_t^A}{p_t} p_t^c - \left(\sigma + \sigma_t^q\right)^2 + \left(1 - \psi_t\right)\left(\underline{\delta}^B - \delta^B\right))dt\right)$$

and this concludes the proof. □

Corollary 1. *The drift and volatility of* η_t *are given by*

$$\alpha\left(\eta_t\right) = \frac{\psi_t - \eta_t}{\eta_t}\left(\frac{E\left[dr_t^k\right]}{dt} + 1 - \frac{\gamma^A - \iota_t^A}{p_t} p_t^c - \left(\sigma + \sigma_t^q\right)^2 + \left(1 - \psi_t\right)\left(\underline{\delta}^B - \delta^A\right)\right) \tag{46}$$

and

$$\beta\left(\eta_t\right) = \frac{\psi_t - \eta_t}{\eta_t}\left(\sigma + \sigma_t^p\right) \tag{47}$$

Proof. It follows immediately from proposition 4 and the fact that dr_t^k has volatility $\sigma + \sigma_t^p$. □

[2] Ito's quotient rule states that $\frac{d(X/Y)}{X/Y} = \frac{dX}{X} - \frac{dY}{Y} + \left(\frac{dY}{Y}\right)^2 - \frac{dX}{X}\frac{dY}{Y}$.

Now by knowing the drift and the volatility of our state variable, we can calculate the the drift and the volatility of the price of capital.

Proposition 5. *The drift of the price of capital is given by*

$$\mu_t^p = \frac{\frac{p^{'}(\eta_t)}{p_t}\left(x_t\eta_t - \eta_t\right)\left(\Lambda\left(\iota_t^A\right) - \delta^A + \sigma\sigma_t^p + 1 + \left(\sigma + \sigma_t^p\right)^2 + \eta_t(1 - x_t\eta_t)(\underline{\delta}^B - \delta^A)\right)}{\eta_t + \left(x_t\eta_t - \eta_t\right)\frac{p^{'}(\eta_t)}{p_t}} +$$

$$+ \frac{\frac{p^{''}(\eta_t)}{pt}\beta\left(\eta_t\right)^2\eta_t^2}{\eta_t + \left(x_t\eta_t - \eta_t\right)\frac{p^{'}(\eta_t)}{p_t}}, \tag{48}$$

where $\beta\left(\eta_t\right)$ is given by equation (48) and the volatility of the price of capital, σ_t^p, is given by

$$\sigma_t^p = \frac{\left(x_t\eta_t - \eta_t\right)\frac{p^{'}(\eta_t)}{p_t}\sigma}{1 - \left(x_t\eta_t - \eta_t\right)\frac{p^{'}(\eta_t)}{p_t}} \tag{49}$$

Proof. To prove this, we will use Ito's lemma.[3]

Applying Ito's lemma to the function $p_t = p\left(\eta_t\right)$, we get

$$dp_t = p^{'}\left(\eta_t\right)d\eta_t + \frac{1}{2}p^{''}\left(\eta_t\right)\left(d\eta_t\right)^2$$

Recall that

$$\frac{d\eta_t}{\eta_t} = \alpha(\eta_t)dt + \beta(\eta_t)dZ_t$$

[3]Ito's formula states that if you have a function f of a random process S_t then the evolution of f is given by

$$df\left(S_t\right) = \frac{\vartheta f\left(S_t\right)}{\vartheta S_t}dS_t + \frac{1}{2}\frac{\vartheta^2 f\left(S_t\right)}{\left(\vartheta S_t\right)^2}\left(dS_t\right)^2$$

From this last equation, we obtain that

$$(d\eta_t)^2 = \eta_t^2 \beta(\eta_t)^2 dt$$

and thus, substituting the last two equations into the first one and dividing by p_t, we obtain

$$\frac{dp_t}{p_t} = \left(\eta_t \alpha\left(\eta_t\right) \frac{p^{'}\left(\eta_t\right)}{p_t} + \frac{\eta_t^2 \beta\left(\eta_t\right)^2}{2} \frac{p^{''}\left(\eta_t\right)}{p_t} \right) dt + \eta_t \beta\left(\eta_t\right) \frac{p^{'}\left(\eta_t\right)}{p_t} dZ_t \tag{50}$$

From proposition 4, we have that

$$\alpha\left(\eta_t\right) = \frac{\psi_t - \eta_t}{\eta_t} \left(\frac{E\left[dr_t^k\right]}{dt} + 1 - \frac{\gamma^A - \iota_t^A}{p_t} p_t^c - \left(\sigma + \sigma_t^p\right)^2 + \left(1 - \psi_t\right)\left(\underline{\delta}^B - \delta^A\right) \right) \tag{51}$$

and

$$\beta\left(\eta_t\right) = \frac{x_t \eta_t - \eta_t}{\eta_t} \left(\sigma + \sigma_t^q\right)$$

Thus, we have that

$$\sigma_t^p = \frac{\left(x_t \eta_t - \eta_t\right) \frac{p^{'}(\eta_t)}{p_t} \sigma}{1 - \left(x_t \eta_t - \eta_t\right) \frac{p^{'}(\eta_t)}{p_t}}$$

Finally, by combining the above equation for $\alpha\left(\eta_t\right)$ and corollary 1 of proposition 4 equation (51) obtains. It gives the drift of the price of capital as a function of exogenous variables, the state variable η_t and the leverage x_t. This concludes the proof. $\square$

3.2.2 Optimal Default, Leverage and Borrowing

We derive closed formed solutions for optimal default, optimal amount of output that agent A sells to agent B as well as the optimal borrowing μ_t. From equations (32) and

(34) we observe that the langrange multipliers are positive and so the constraints are binding. The following theorems summarise our results.

Proposition 6. *Assume that all agents have logarithmic utilities. Then:*

i) The optimal borrowing is given by

$\mu_t^* = ((x_t - 1)p_t^c \eta_t - m_t^{A*})e^{r_t}$,

ii) The equilibrium default is given by

$v_t = \frac{M_t + m_t^A + m_t^B}{e^{r_t} M_t}$,

iii) Agent A sells to agent B the following proportion of his output

$b_t = \frac{\tau v_t \mu_t}{p_t + \tau v_t \mu_t}$,

iv) The price of consumption is given by

$p_t^c = \frac{p_t + \tau_t p_t \eta_t x_t - \tau m_t^A \frac{M_t + m_t^A + m_t^B}{M_t}}{\gamma \eta_t x_t - \tau(x_t - 1)\eta_t \frac{M_t + m_t^A + m_t^B}{M_t}}$.

Proof. Since λ_t is positive, the inequality (12) binds. Consequently, $(x_t - 1)w_t^A = (\mu_t e^{-r_t} + m_t^A) \Rightarrow (x_t - 1)w_t^A = (\mu_t^* K_t e^{-r_t} + m_t^{A*} K_t) \Rightarrow \frac{(x_t-1)w_t^A}{K_t} = (\mu_t^* e^{-r_t} + m_t^{A*})$

By using the definition of η_t we get

$\mu_t^* = ((x_t - 1)p_t^c \eta_t - m_t^{A*})e^{r_t}$.

This establishes the first claim. In order to find the optimal default we use equation (22). In equilibrium, we have that money supply is equal to money demand, and hence we have:

$e_t^{r_t} M_t v_t = M_t + m_t^A + m_t^B$

The right hand side shows the money will be repaid to the bank, and the left hand side the amount of money that the agent A repays to the bank. Solving for v_t, we obtain an equation for optimal default.

In order to find the optimal proportion of output that agent A will sell to agent B, we first observe that constraints (13) and (14) are binding, since the corresponding multipliers are not zero. By dividing these equation by parts we obtain:

$$\frac{b_t}{1-b_t} = \frac{\tau v_t \mu_t}{p_t}$$

By adding the both numerators to their corresponding denominator, we obtain:

$$b_t = \frac{\tau v_t \mu_t}{p_t + \tau v_t \mu_t}$$

Finally, we need to find the optimal price of consumption. Using the facts that (14) is binding, we obtain

$$\frac{1}{\tau} = \frac{(1-b_t)(\gamma^A - i_t^A)p_t^c}{p_t} x_t \eta_t \Rightarrow \frac{1}{\tau} = \frac{(1 - \frac{\tau v_t \mu_t}{p_t + \tau v_t \mu_t})(p_t^c \gamma^A - p_t)}{p_t} x_t \eta_t \Rightarrow$$

$$\Rightarrow \frac{p_t^2 +_t ((x_t - 1)p_t^c \eta_t - m_t^{A*})\frac{M + m_t^A + m_t^B}{M_t}}{\tau} = (p_t^c \gamma^A - p_t) x_t \eta_t \Rightarrow$$

$$p_t^c = \frac{p_t^2 + \tau p_t x_t \eta_t - \tau p_t m_t^A \frac{M_t + m_t^A + m_t^B}{M_t}}{\gamma^A x_t \eta_t - \tau p_t (x_t - 1)\eta_t \frac{M_t + m_t^A + m_t^B}{M_t}}$$

.

This concludes the proof.

$\square$

4 Simulations and Results

We solve a numerical example to assess the dynamics of our model.. For our simulation we use the following parametrization: $\tau = 3.5, \gamma^A = 1.2, \gamma^B = 0.8, \delta^A = 0.6, \delta^B, \rho_1 = 0.07, \rho_2 = 0.06, M = 2, m^a = 0.02$ and $m^b = 0.03$. We plot explicitly the interest rate r_t, the repayment rate v_t and the price of capital p_t and the price of consumption ${p^c}_t$ with respect to η_t. But before doing that we need to find the optimal leverage which will be given for the following proposition. Hereafter, for the sake of simplicity, we drop the subscript t from η_t and all exogenous parameters.

Proposition 7. *The optimal leverage x_t is given by:*

$$x_t = \begin{cases} \frac{-b(\eta)+\sqrt{b(\eta)^2-4a(\eta)\gamma(\eta)}}{2a(\eta)} & \textit{if} \quad \frac{1}{\eta} \geq x_t \\ \frac{1}{\eta} & \textit{otherwise,} \end{cases}$$

where

$a(\eta) = \eta^2(M + m^a + m^b)\gamma^A - \eta(\eta(M + m^a)(\tilde{\gamma}^B - \gamma^A) + \eta\gamma^A(M + m^a + m^b)$
$b(\eta) = -\gamma^A\eta^2(M + m^a + m^b) - \gamma^A\rho_1\eta(1-\eta)(M + m^a) - (1/\tau(M + m^a)\tilde{\gamma}^B - \gamma^A\eta(M + m^a + m^b))\eta + \eta(M + m^a)(\tilde{\gamma}^B - \gamma^A\eta\tilde{\gamma}^B(M + m^a + m^b)$
and
$\gamma(\eta) = 1/\tau(M + m^a)\tilde{\gamma}^B - \gamma^A\eta(M + m^a + m^b)$

Proof:

Follows immediately from Proposition 6.

We plot the result of Proposition 7 and obtain a graph depicting the leverage of agent A. For this example, we find that leverage is less than $1/\eta$ and, therefore, a solution exists.

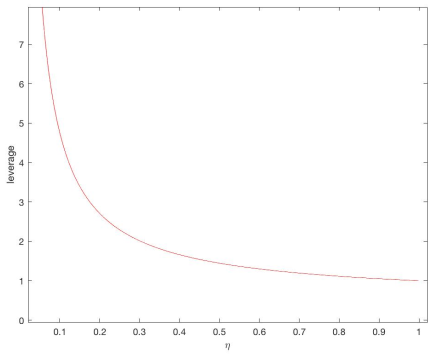
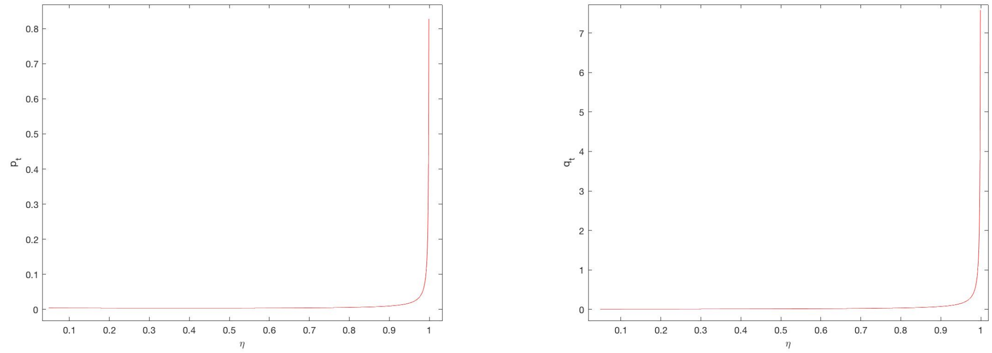


Figure 1: The leverage of Agent A

By using this x_t, we can use equations $i)$ and $iv)$ of proposition 6 and we can solve for the price of consumption p_t^c and for the price of capital p_t.



(a) Price of Consumption (b) Price of Capital

Figure 2: Prices in the model

The price of capital increases with the proportion of wealth that Agent A holds. This is expected since as the productive agent A becomes richer, he demands more capital, hence the price the capital increases. The same applies to the price of consumption.

Using the equations for prices and we can explicitly find the optimal investment i_t which we plot with respect to η in the following figure:

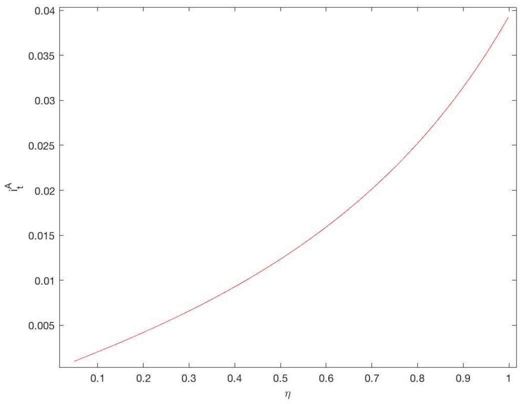
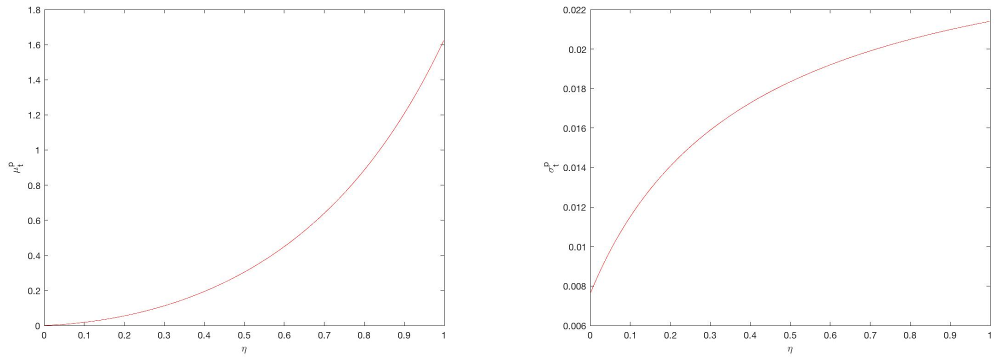


Figure 3: Optimal Internal Investment

With the above in hand, we can calculate the drift μ^p and the volatility σ^p using the expression of Proposition 5. The result we obtain are:

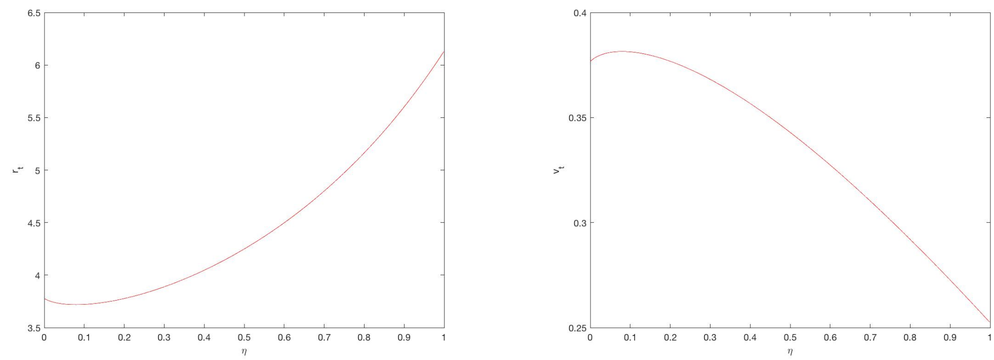


(a) Drift

(b) Volatility

Figure 4: Drift and Volatility of Price of Capital

Finally, now we have all the necessary endogenous variables to obtain the endogenous interest rate r_t as well as the repayment rate v_t. This is done by just replacing these endogenous variables in equations ii) of Proposition 1. The results we obtain are shown below:



(a) Endogenous Interest Rate

(b) Repayment Rate

Figure 5

We observe that as the proportion of wealth of Agent A increases it becomes easier for this agent to default strategically

as high wealth indicates tolerance to not repaying the loan back. As a result the repayment rate decreases. This, in turn, will be reflected to a default premium in the endogenous interest rate which increases. The relative dynamics of these two variables can be shown below.

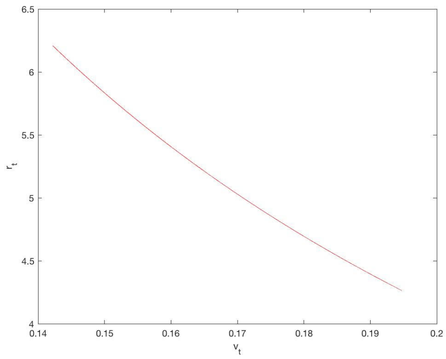


Figure 6: Endogenous Interest Rate vs Repayment Rate

The last result we plot is the response of b_t which is the proportion of output that Agent A sell to Agent B in order to finance his loan. We observe that Agent A sells a lower proportion of output when his proportion of wealth is higher. This is because in this case Agent A buys less capital from Agent B, as a result he needs to repay less, thus he will need to sell lower output. This can be shown in the following diagram.

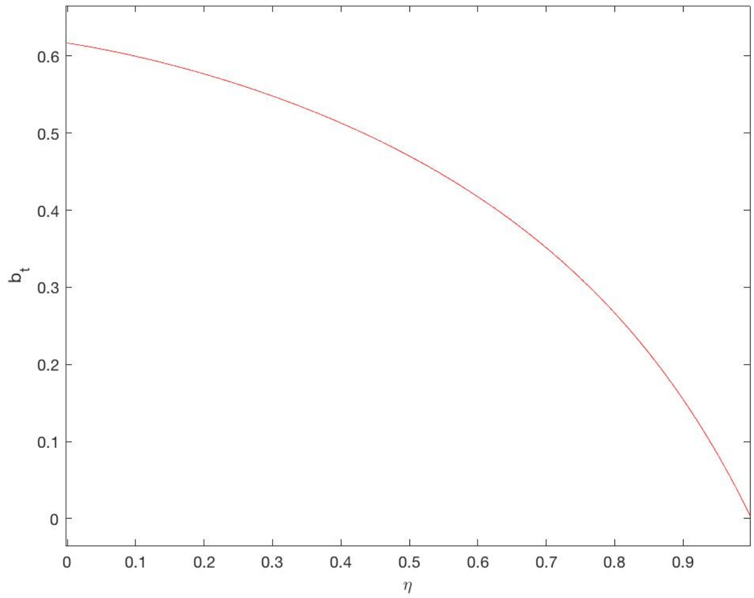


Figure 7: Proportion of Agent's A output sold

5 Comparative Statics

In this section, we investigate the relationship of our basic endogenous quantities with the default penalty τ. In particular, the following propositions show how leverage x_t, the price of capital p_t, the price of consumption p_t^c, the proportion of output sold b_t and the interest rate r_t are related to τ.

Proposition 8. *The optimal leverage x_t increases with τ.*

Proof. Follows immediately from considering the definition of optimal leverage using Proposition 7 and taking the first order conditions with respect to τ. It follows that $\frac{\partial x_t}{\partial \tau} > 0$. $\square$

In the following graph we plot the optimal leverage for different values of τ. We observe the outcome of proposition 8 in practice. The maximum leverage possible is given by $\frac{1}{\eta_t}$ and for higher τ, we obtain larger x_t. The reason for that is that as we increase τ, Agent A has to default less (otherwise he will incur a higher penalty), and thus the repayment rate will be increased as well. As a result, the default premium will go down, leading to a greater capital acquisition from agent A.

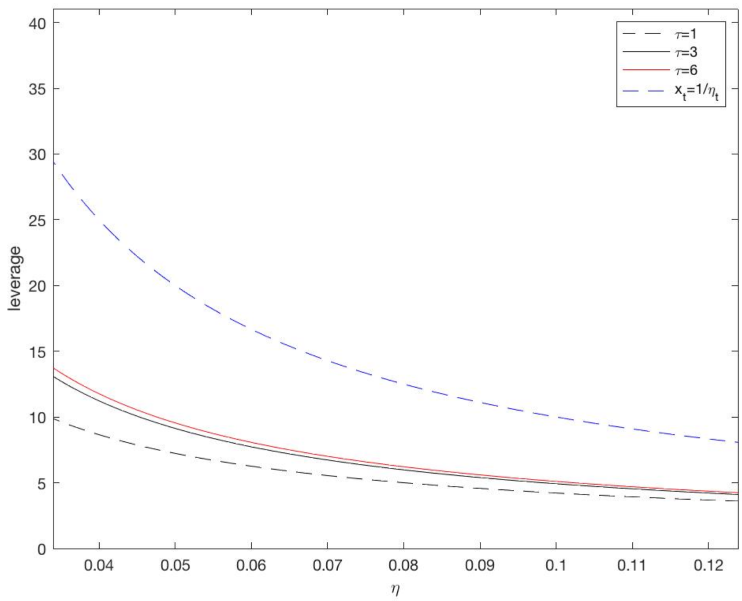


Figure 8: Leverage for different values of τ

Then relation of the price of capital and the price of consumption are given in the following proposition.

Proposition 9. *p_t and p_t^c decrease with τ. On the other hand b_t will increase and it approach 1 for very large values of τ.*

Proof. Since we have assume that all money in the economy are homogeneous of degree one with respect to capital k_t, the cash in advance constraint (12) can be re-written as : $p_t(x_t - 1)\eta_t = M^* + m^{a*} \Leftrightarrow p_t = \frac{M^*+m^{a*}}{(x_t-1)\eta_t}$. From proposition 8 we know that x_t increases with τ, thus p_t will decrease. Similar for p_t^c. Now, b_t in equilibrium will be given by: $b_t = \frac{\tau(M*+m^a+m^b)}{p_t+\tau(M^*+m^a+m^b)}$. As τ increases, p_t decreases and the (same) terms $\tau(M^* + m^a + m^b)$ in the numerator and the denominator will dominate. Thus b_t will approach (but never become) 1. □

The main lesson of proposition 9 is that higher τ will force Agent A to sell more output in order to receive more cash to repay his outstanding debt so that he can avoid penalization. We can now proceed to the interest rate r_t and the repayment rate v_t.

Proposition 10. *r_t will decrease with τ and v_t will increase.*

Proof. Equilibrium r_t is given by proposition 1. With direct calculations we obtain that $\frac{\partial r_t}{\partial \tau} < 0$. From the money market we also have the clearing condition $v_t(1 + r_t)M_t = M_t + m^a + m^b$, we secure the direct inverse relationship between the interest rate and the repayment rate. □

Proposition 10 highlights what we expected. With higher default penalty, we force Agent A to repay more debt and thus he defaults less. This forces the repayment rate to increase, and the default probability to go down. As a result, the default premium reflected in the interest rate will decrease and , thus, the interest rate will decrease.

6 Money Supply and Default Penalty Trade-off

In this section we investigate the equilibrium trade-off between Money Supply M and the default penalty τ. In order to do that we need to find the equilibrium welfare. We do this in the following proposition.

Proposition 11. *The welfare function is given by:*

$$W = E[\int_0^\infty e^{-\beta^A t}(\ln(c_t^A) - {}^A_{t+})dt + \int_0^\infty e^{-\beta^B t}\ln(c_t^B)dt]$$

$$= E[\int_0^\infty e^{-\beta^A t}(ln(\frac{M+m^a}{\eta_t(x_t-1)p_t^c\tau}) - \frac{\tau(x_t-1)\eta(r_t+1)M}{M+m^a}$$

$$+ \frac{\tau(M+m^a)\eta(x_t-1)}{M+m^a}dt + \int_0^\infty e^{-\beta^B t}ln(\frac{\beta^A(1-\eta)M}{\eta_t(x_t-1)p_t^c}dt] \tag{52}$$

Proof: From first order condition for consumption of Proposition 1, we obtain that $c_t^A = \frac{p_t K_t}{p_t^c \tau} = \frac{w_t^A}{\eta_t p_t^c \tau}$. However we know that w_t^A can be captured from the cash in advance constraint (12) which binds, thus $w_t^A = \frac{M+m^A}{x_t-1}$. As a result we get $c_t^A = \frac{M+m^A}{(x_t-1)\eta_t p_t^c \tau}$ Consumption of agent B is given by $c_t^B = \rho^B w_t^B = \rho^B(1-\eta_t)p_t K_t = \frac{\rho^B(1-\eta_t)(M+m^A)}{(x_t-1)\eta_t p_t^c}$. The penalty term is given by $\frac{\tau(1-v_t)]\mu_t}{p_t K_t} = \frac{\tau\mu_t}{p_t K_t} - \frac{\tau v_t \mu_t}{p_t K_t} = \frac{\tau(M+m^a)\eta(x_t-1)}{M+m^a} - \frac{\tau(x_t-1)\eta(r_t+1)M}{M+m^a}$

Using Proposition 11, we can plot the welfare function W for different values of τ and M_t, for given η_t. For example, for $\eta_t = 0.2$ we obtain the following graph.

From figure 9, it becomes evident that as there is no monotonic relationship between money supply and default penalty and welfare. Instead, we see that there is a combination of money supply and default penalty which maximizes welfare. If, in an economy, we are able to capture this combination, then we would clearly have welfare improving effects. In particular, as the penalty decreases, 2 things will happen in the economy.Firstly, the lower the penalty, the more agent A will default. Thus agent A will

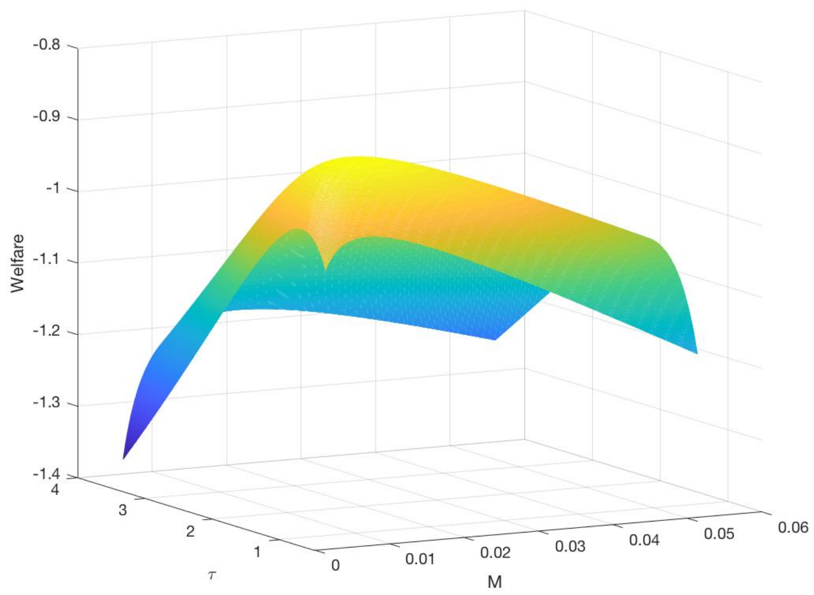


Figure 9: Welfare function as a function of Money Supply and Default Penalty

reduce his output for sale (and thus consumption) to agent B, since he will decide he doesn't need to get money from agent B to repay his debt. This will decrease the payoff of agent B. Thus, increasing default penalty decreases the utility of agent A and increases utility of agent B. Thus there must be a point in between which maximizes welfare. Please not that more default, generates higher default premia, making agent's A ability to borrow money harder. This can be clearly seen in figure 10 where the relationship of the interest rate with different τ and M_t is presented.

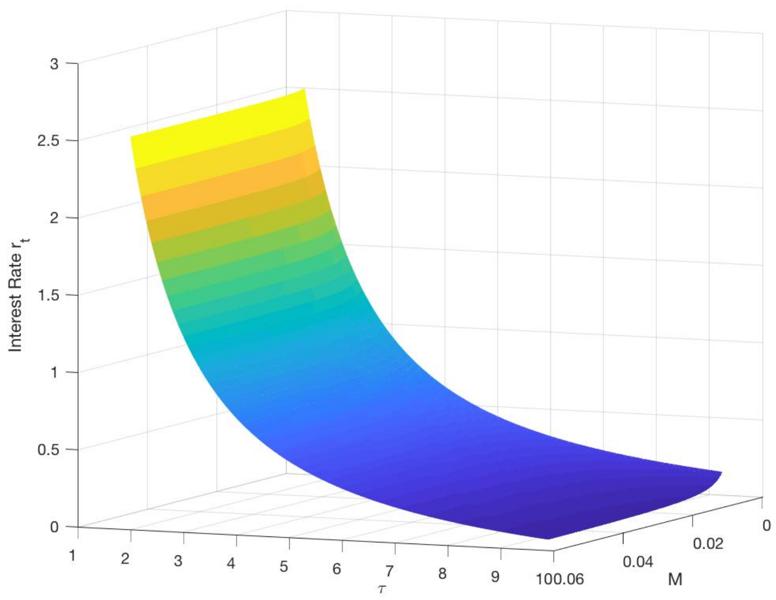


Figure 10: Interest Rate as a function of Money Supply and Default Penalty

The repayment rate, which is shown in the next figure, clearly shows that the more harsh the default penalty is, the less agent A decide to default (for $\tau > 8$ agent does decides to repay fully). Furthermore, increasing the money supply will decrease the repayment rate up to a point, and then the effect of any further increase will negligible. Thus providing agent A will more money increases the chances of him not repaying his obligations.

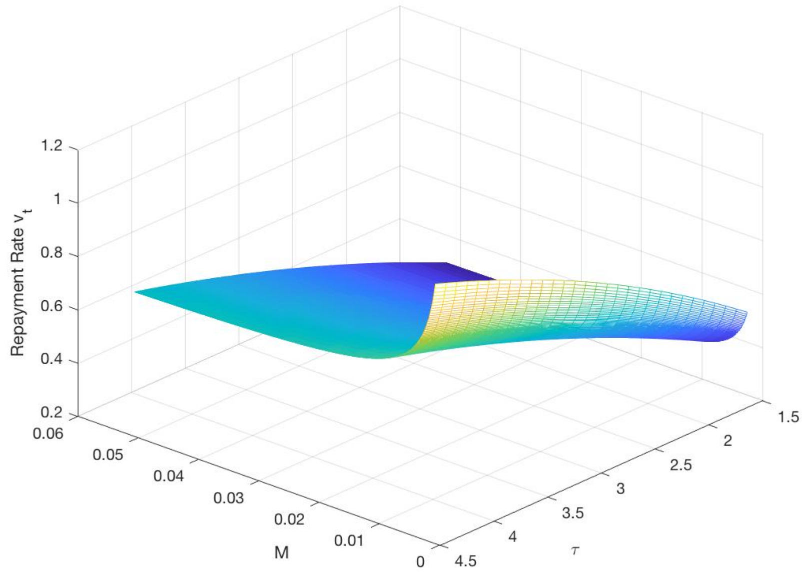


Figure 11: Repayment Rate as a function of Money Supply and Default Penalty

A plausible question to ask, regarding the effect of the penalty on welfare discussed above, is that for a given level a default penalty, say τ^*, what is the money supply that optimizes welfare? Does it exist, and if yes, is it unique? The problem we are interested in can be formulated as:

$$max_M W(\tau^*, M) \tag{53}$$

Figure 12 illustrates that for every positive $\tau < \tau^*$ there exists a money supply which maximizes welfare. The set of these points is shown with the dark colour. From the figure we observe that there is a point (τ^*, M^*) such that a deviation of M^* from this point will reduce welfare. The reason for that is the following: in the presence of default, two things will happen. The first one is that Agent A will incur a penalty from not repaying, thus there will be a welfare decrease. The second one is that Agent A will reduce the output for sale to Agent B and he will consume the proceedings. Since we consider logarithmic utilities, the logarithm of a small number decreases dramatically to $-\infty$. Thus welfare will go further down through the reduction of the utility of Agent B. A deviation from the point (τ^*, M^*) through a decrease of M^* will decrease welfare since smaller M^* will reduce the total output (agent A will buy less capital), and thus the total consumption of agents will decrease reducing welfare. On the other hand, an increase of M^* will cause higher output accumulation from Agent A, reducing dramatically the utility of agent B, causing a reduction of welfare. Changing the default penalty accordingly would have welfare improving effects.

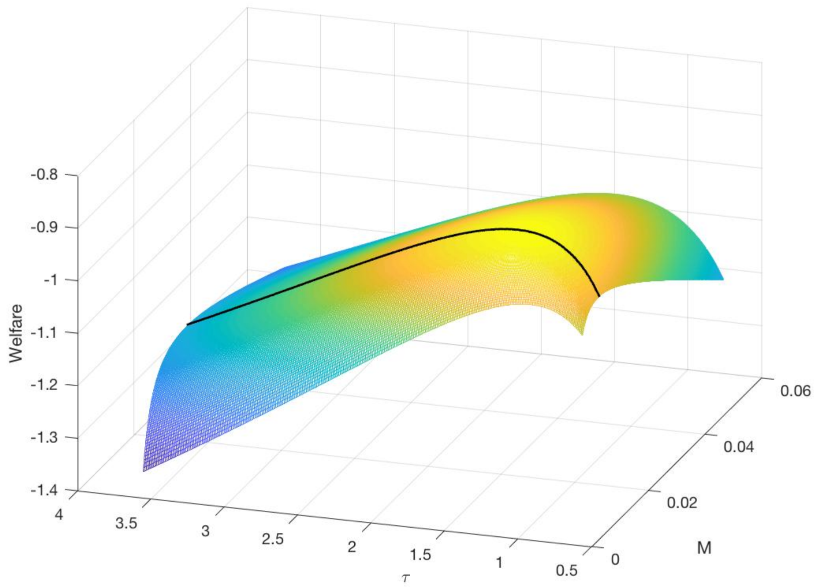


Figure 12: Optimal money supply M_t for given τ

For given η_t we also present the equilibrium trade-off between Money Supply M and the default penalty τ. That is, we find the set of Money Supply and default penalty values that achieve a given and fixed level of welfare. Formally we are interested in all τ and M such that,

$$W(\tau, M) = C \tag{54}$$

where C represents a constant level of welfare.

Figure 13 illustrates the set of points we are interest in:

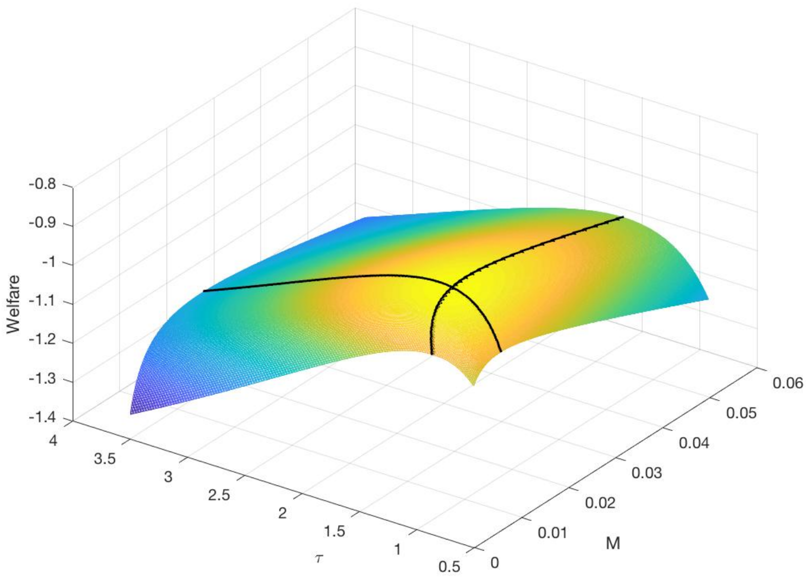


Figure 13: Trade-off between money supply M_t and τ

The above plot, indicates that there are four regions, and thus the combination of monetary and regulatory policy depends on the initial conditions of the money supply and the default penalty. The first region, is the one with low default penalty (so high default region) as well as low money supply. In this region we can perform two policies. First, an increase in money supply has welfare improving effects, since agent A increases the amount of capital he buys from agent B. Furthermore, and increase in default penalty also has welfare improving effects since agent A will be forced to sell more consumption to agent B (and thus her utility increases). Thus an expansionary mix of policies would be appropriate here in order to improve economic welfare. The second region, is the region with low default penalty and high money supply. In this case, agent will buy too much capital from agent B, without selling her back enough output. Here, contractionary monetary policy together with expansionary regulatory policy would be the effective way to restore welfare. The third region is the one with high default penalty and low money supply. High default penalty will reduce the welfare of agent A, since forcing him to repay the loan, results in selling excessive output to agent B. This, in turn, results in lowering agent's A utility which lowers welfare. Increasing the money supply so that agent A could produce more and/or making regulatory policy more lenient, had welfare improving effects. The last region is the one with high initial money supply and high default penalty. With high money supply, agent A will buy excessive capital from agent B, leaving her with very low levels. In this case, we note that, with low levels of default, agent A buys excessive levels of capital from agent B and she needs to sell excessive amounts of output in order to repay the loan. This will lead to decreased levels of welfare for agent A, leading to a welfare reduction. The situation can be inverted by a contractionary mix monetary and regulatory policies.

7 Conclusion and Future Research

In this paper, we presented a monetary continuous time economy with a central bank and endogenous default. The setting of this economy is very similar to that of [RT19] and [DG92]. Our aim is to find a relationship between the default penalty τ and the money supply M that the central bank provides. We showed that there is a combination of monetary and regulatory policy, which achieves optimal welfare. Any deviation from that point will give inferior levels of welfare. Thus, the optimal policy depends heavily on the our starting point of money supply and default penalty. The results of this paper extends [DGS05] which indicates, through an example, that default in GEI could have welfare improving effects. Our paper does not only provides a unified mathematical framework to assess the possible welfare improving effect of default in the economy, but also introduces money into the game. To conclude, introducing a commercial banking sector in the model in order to investigate the interaction between monetary policy and financial stability, will be also a topic for future research.

References

[Ach+17] Yves Achdou et al. *Income and wealth distribution in macroeconomics: A continuous-time approach.* Tech. rep. National Bureau of Economic Research, 2017.

[BC98] Suleyman Basak and Domenico Cuoco. "An equilibrium model with restricted stock market participation". In: *The Review of Financial Studies* 11.2 (1998), pp. 309–341.

[Bew76] Truman Bewley. *The Permanent Income Hypothesis: A Theoretical Formulation.* Tech. rep. Harvard University, 1976.

[BS16a] Markus K Brunnermeier and Yuliy Sannikov. "Macro, money, and finance: A continuous-time approach". In: *Handbook of Macroeconomics* 2 (2016), pp. 1497–1545.

[BS16b] Markus K Brunnermeier and Yuliy Sannikov. *The I theory of money.* Tech. rep. National Bureau of Economic Research, 2016.

[B214] Markus Brunnermeier and Yuliy Sannikov (2014). "A macroeconomic model with a financial sector". In: *American Economic Review* 104.2 (2014), pp. 379–421.

[DSS18] Itamar Drechsler, Alexi Savov, and Philipp Schnabl. "A model of monetary policy and risk premia". In: *The Journal of Finance* 73.1 (2018), pp. 317–373.

[DG92] Pradeep Dubey and John Geanakoplos. "The value of money in a finite horizon economy: a role for banks". In: *Economic Analysis of Market and Games* (1992), pp. 407–444.

[DGS05] Pradeep Dubey, John Geanakoplos, and Martin Shubik. "Default and punishment in general equilibrium". In: *Econometrica* 73.1 (2005), pp. 1–37.

[EGT09] Raphaël A Espinoza, Charles AE Goodhart, and Dimitrios P Tsomocos. "State prices, liquidity, and default". In: *Economic Theory* 39.2 (2009), pp. 177–194.

[Goo+18] Charles Goodhart et al. "Macro-modelling, default and money". In: *Working Paper* (2018).

[HK11] Zhigu He and Arvind Krishnamurthy. "A model of capital and crises". In: *The Review of Economic Studies* 79.2 (2011), pp. 735–777.

[HK13] Zhiguo He and Arvind Krishnamurthy. "Intermediary asset pricing". In: *American Economic Review* 103.2 (2013), pp. 732–70.

[RT19] Nikolaos Romanidis and Dimitrios Tsomocos. "Endogenous Default in Continuous Time Finance". In: *Working Paper* (2019).

[Sam58] Paul A Samuelson. "An exact consumption-loan model of interest with or without the social contrivance of money". In: *Journal of political economy* 66.6 (1958), pp. 467–482.

[Shu73] Martin Shubik. "Commodity money, oligopoly, credit and bankruptcy in a general equilibrium model". In: *Economic Inquiry* 11.1 (1973), pp. 24–38.

[SW77] Martin Shubik and Charles Wilson. "The optimal bankruptcy rule in a trading economy using fiat money". In: *Zeitschrift für Nationalökonomie* 37.3-4 (1977), pp. 337–354.

[Tso+18] Dimitrios P Tsomocos et al. "Liquidity and default in an exchange economy". In: *Journal of Financial Stability* 35 (2018), pp. 192–214.