EXTRAPOLATIVE ASSET PRICING

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Abstract. We study asset pricing implications of stock return extrapolation by replacing rational expectations with an extrapolative expectation in an otherwise standard Lucas economy. We solve for the equilibrium in closed form. We find that extrapolation generates large deviations in the short rate and stock price from the rational counterparts. However, extrapolation actually exacerbates asset pricing puzzles: the short rate is volatile, the stock return has deficient volatility (lower volatility than the rational counterpart) and lower equity premium. Our results suggest that the extrapolation-based resolutions of these puzzles in recent studies are likely due to other features than extrapolation.

Key words: Extrapolation, riskless rate puzzle, equity premium puzzle, excess volatility, market clearing, momentum, return predictability.

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1. Introduction

Extrapolation bias of investors has been widely documented.[1] Thus, it is important to understand its implications for asset pricing. In this paper, we study a Lucas exchange economy with independent and identically distributed (IID) dividend growth and a representative agent having a constant relative risk aversion coefficient (RRA). Instead of rational expectations, we assume that the agent has an extrapolative expectation in the sense that her expectation about future stock returns is a weighted average of past realized returns, where the stock is the claim to the aggregated consumption. Our objective is not to propose a model that fits data but to isolate the effects of extrapolation on stock prices. Our setup allows us to do this because the equilibrium with either rational expectation or extrapolative expectation can be solved in closed form.

In our model, extrapolation has significant effects on the stock price. The price-consumption ratio decreases with the sentiment (the weighted average of past realized returns). When the sentiment is high, the agent expects both high consumption growth rates (the income effect) and high discount rates (the intertemporal substitution effect). The intertemporal substitution effect dominates the income effect when the RRA is greater than one,[2]; thus, high sentiment leads to a low price-consumption ratio. In contrast, the price-consumption ratio is constant in the rational benchmark, which is defined as the standard Lucas economy with rational expectations.

Extrapolation generates deficient return volatility in the sense that it is lower than consumption volatility in our model. Intuitively, the return volatility is the sum of consumption volatility and price-consumption ratio volatility, and the latter is negative because price-consumption ratio decreases with the sentiment.

Furthermore, return volatility is insensitive to the sentiment. Indeed, price-consumption ratio is an approximately exponential function of the sentiment because both consumption growth rates and discount rates under the subjective measure are linear functions of the sentiment. This implies that price-consumption ratio volatility as well as return volatility is insensitive to the sentiment. Extrapolation has a first order effect on (subjective) expectations (almost by definition), while a second order effect on volatilities.

In our model, the subjective risk premium is small because of deficient volatility and is insensitive to the sentiment because the volatility is insensitive. The physical risk premium is the sum of the subjective risk premium and a sentiment premium

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[1]Vissing-Jorgensen (2004), Bacchetta, Mertens and van Wincoop (2009), Amromin and Sharpe (2014), Greenwood and Shleifer (2014), and Kuchler and Zafar (2019), among others, document that many individual and institutional investors have extrapolative expectations.

[2]We focus on the case when the RRA is greater than one because most asset pricing models study this case. Our analytical results also hold for the case when the RRA is less than one. Some return properties in the two cases are the opposite.

that accounts for the difference between the subjective measure and the physical measure and reflects the market price of sentiment. The sentiment premium is low if RRA is greater than one. Low subjective risk premium and sentiment premium lead to low physical risk premium is low.

The short rate is endogenously determined in our model, and it is volatile. Because the risk premium is low and insensitive to the changes in the sentiment, the short rate is largely equal to the expected stock return, which is specified exogenously by return extrapolation. The short rate is volatile because the expected return is volatile and extrapolation impacts the short rate more than the risk premium. In fact, most features of the expected stock return generated by extrapolation, such as the predictive ability of price-consumption ratio, are reflected in the short rate rather than in the risk premium.

In our model, the stock returns exhibit momentum under the physical measure when the RRA is greater than one. This is because stock returns have momentum under the subjective measure by definition and the sentiment premium is small. This result is different from most studies of extrapolation. On the other hand, when the RRA is less than one, the sentiment premium dominates, which leads to return reversal.

For finite horizons, we show that equilibrium exists with and without extrapolation bias, so extrapolation in itself does not lead to instability. However, with an infinite horizon, equilibrium may not exist due to violation of the transversality condition, which happens even in the rational benchmark. Extrapolation leads to a stricter transversality condition. More importantly, we show that the instability is not caused by the feedback, different from the popular argument in the extrapolation literature.

Barberis, Greenwood, Jin and Shleifer (2015) (BGJS) also studies asset pricing with extrapolative expectations and is closely related to our paper. BGJS assumes that the agents have a constant absolute risk aversion (CARA) utility and the consumption follows a Brownian motion with drift. More importantly, BGJS assumes that the short rate is a constant. BGJS obtains the closed-form solution with two agents, one has extrapolative expectation while the other has rational expectation. Their model generates many asset return features that are consistent with empirical findings.

One key difference between BGJS and our model is in the short rate. Constant short rate is a key assumption of BGJS and is a standard assumption under its setting; however, the consumption goods market does not clear under this assumption.[3] This allows for more freedom to fit data, and likely due to this, their model captures

[3]Loewenstein and Willard (2006) show that market clearing is important to avoid some problematic implications on asset prices, such as arbitrage opportunities.

many return features observed in the data. In our model, the short rate is endogenous and volatile; in fact, the variation in the short rate (not the risk premium) contributes to most variation in the expected return.

In Jin and Sui (2018), the aggregate consumption and the dividend of the aggregate market follow different processes, and extrapolation significantly impacts dividend while it slightly impacts the aggregate consumption. With extra flexibility of recursive utility, they show that their model can explain many features of asset returns observed in the data.

Nagel and Xu (2018) study asset pricing with fading memory. The best forecast of consumption growth is a geometrically-decaying weighted average of past consumption growth, similar to the best forecast of stock return under our model. The decaying rate is small in their model, leading to a less volatile short rate, which fits the data, while the decaying rate is large in the extrapolation literature, leading to a volatile short rate in our paper. More significantly, they model fading memory using loss of information. As a result, the agent's information structure is not a filtration, the law of iterated expectations does not hold, and buy-and-hold valuation differs from resale valuation. This setup is a large and innovative departure from the standard asset pricing framework. They also use a recursive utility and assume that dividend is different from the aggregate consumption. With these assumptions, their model can generate many return features observed in the data.

In summary, return extrapolation has large effects on stock prices but does not help resolve asset pricing puzzles. Rather, it actually exacerbates these puzzles; it leads to volatile short rate, deficient return volatility, and low equity premium (even lower than the rational counterpart). Our results suggest that the extrapolation-based resolutions of these puzzles in recent studies are due to additional assumptions therein.

The remainder of the paper is organized as follows. Section 2 presents our model setup. Section 3 solves for equilibrium. Section 4 studies asset pricing implications of extrapolation. We discuss several effects specific to the extrapolation literature in Section 5. Section 6 concludes. Appendix provides the proofs.

2. Model Setup

In this section, we build a Lucas-type model with return extrapolation. Our objective is to isolate the effects of return extrapolation rather than to fit the real data. Our model is standard except that the rational expectation is replaced by an extrapolative expectation. This enables us to focus on and isolate the effects of extrapolation bias.

Consider a continuous-time Lucas (1978) economy with one consumption good and a representative agent with CRRA preference. There are two assets in the economy. One is a risky asset, the stock market, that is the claim to a continuous aggregate consumption stream. Assume that the consumption follows

$$dC_t = C_t(\mu_c dt + \sigma_c dZ_t), \tag{2.1}$$

where the consumption growth rate μ_c and volatility σ_c are constants, and Z is a standard Brownian motion under the physical measure $\mathbb{P}$. The other asset is a riskless asset with short rate r_f that is determined in equilibrium. The riskless asset is in net zero supply.

The agent maximizes the expected utility over consumption with a subjective discount rate $\rho > 0$ by choosing consumption $\{C_t\}$ and the fraction of wealth invested in the stock $\{\phi_t\}$:

$$\max_{\{C_t,\phi_t\}_{t=0}^T} \mathbb{E}_0^e\left[\int_0^T e^{-\rho t}\frac{C_t^{1-\gamma}}{1-\gamma}dt\right], \tag{2.2}$$

subject to the budget constraint

$$dW_t = \left[W_t\big(r_f + \phi_t(\mu_p^e - r_f)\big) - C_t\right]dt + W_t\phi_t\sigma_p dZ_t^e, \tag{2.3}$$

where $\mathbb{E}_0^e[\cdot]$ is the expectation under the agent's subjective probability measure $\mathbb{P}^e$, which is equivalent to but can be different from the physical measure $\mathbb{P}$. T is horizon. When $T \to \infty$, the economy has an infinite horizon. In this case, we need to impose appropriate transversality condition. Our analytical results hold for both finite and infinite horizons. The constant γ is relative risk aversion coefficient and also is the inverse of the elasticity of intertemporal substitution (EIS), μ_p^e is the expected stock return under $\mathbb{P}^e$, σ_p is stock return volatility, and Z^e is a standard Brownian motion under $\mathbb{P}^e$. When $\gamma = 1$, the utility in (2.2) is replaced by $\ln(C_t)$.

2.1. **Subjective Expectations.**

For CRRA utility, stock price is completely determined by the agent's subjective expectation about consumption growth in consumption-based asset pricing models. The survey data analyzed by Vissing-Jorgensen (2004), Bacchetta et al. (2009), Greenwood and Shleifer (2014), and Adam, Matveev and Nagel (2018), among others, show that investors' expectations about stock returns are inconsistent with rational return expectations. This implies that the agent's subjective expectation about consumption growth is also different from the rational expectation. Especially, many individual and institutional investors have extrapolative expectations: they believe that stock prices will continue rising (falling) after a sequence of high (low) past returns. We focus on extrapolative expectation in this paper. If the deviations from rational expectations are restricted to change of measure, our method can also be used.

2.1.1. *Extrapolative Expectation.*

Motivated by the survey evidence, we assume that the representative agent forms expectations about future stock return (scaled by volatility) by extrapolating past realized returns,

$$\mathbb{E}_t^e\left[\frac{dR_t}{\sigma_{pt}}\right] = (\alpha_0 + \alpha S_t)dt, \tag{2.4}$$

where $dR_t = (dP_t + C_t dt)/P_t$ is the instantaneous stock return, σ_p is return volatility, and S is defined as

$$S_t = \int_{-\infty}^{t} \kappa e^{-\kappa(t-u)} \frac{dR_u}{\sigma_{pu}}, \tag{2.5}$$

a weighted average of stock returns (scaled by volatility) over all historical observations with exponentially decaying weights. We call S sentiment by following the literature. It reflects the agent's belief about expected return. Because the agent's belief depends on equilibrium prices, which are endogenous, the equilibrium pricing problem is more difficult than standard equilibrium pricing problems where agents' beliefs are exogenous.

In (2.4), α_0 measures the optimism/pessimism of the agent in the sense that an increase in α_0 increases the agent's unconditional expected return. When $\alpha = 0$ and $\alpha_0 \neq \mu^r/\sigma_c$, the agent is irrational but her irrationality does not depend on the state of the economy. In this paper, we choose

$$\alpha_0 = (1-\alpha)\mu^r/\sigma_c. \tag{2.6}$$

That is, the agent has a correct unconditional expectation about volatility-scaled return. Under this specification, the degree of irrationality is completely determined by α. For $\alpha = 0$, the agent is fully rational and the economy reduces to the rational benchmark in Subsection 2.1.4. The higher the absolute value of α is, the larger the deviations from the rational benchmark are. Our solution also works for other choices of α_0.

The choice of α_0 in (2.6) also implies that the unconditional mean of S under the subjective measure equals that in the rational benchmark. In fact, we have

$$\begin{aligned} dS_t &= \kappa(\mu_p^e/\sigma_p - S_t)dt + \kappa dZ_t^e \\ &= \kappa\big[\alpha_0 - (1-\alpha)S_t\big]dt + \kappa dZ_t^e \\ &= (1-\alpha)\kappa(\mu^r/\sigma_c - S_t)dt + \kappa dZ_t^e. \end{aligned} \tag{2.7}$$

The first equality follows from the definition of sentiment (2.5), the second equality follows from the definition of extrapolation (2.4), and the third equality follows from (2.6). So S mean-reverts to $\alpha_0/(1-\alpha) = \mu^r/\sigma_c$ under the subjective measure.

In the rational benchmark, if we define S by (2.5), then we have

$$dS_t = \kappa(\mu^r/\sigma_c - S_t)dt + \kappa dZ_t, \tag{2.8}$$

which has the same form as the first equation of (2.7), as expected. The unconditional mean of S under the physical measure also equals μ^r/σ_c. Therefore, the agent also has a correct unconditional expectation about S.

For $\alpha > 0$, the agent believes that returns have momentum and we call her extrapolator in this paper by following the literature (e.g., BGJS). In this case, α measures the level of extrapolation. For $\alpha < 0$, the agent has a contrarian expectation. For $\alpha = 0$, the agent does not extrapolate historical returns.

In this paper, we focus on the case $0 < \alpha \leq 1$ to study extrapolation. Note that S is not stationary when $\alpha > 1$, see (2.7). For $\alpha = 1$, the sentiment becomes a martingale under the subjective measure, which is the case studied in BGJS. In this case, the transversality condition is defined but the unconditional distributions are not. For unconditional distributions to be defined, we need S to have an invariant distribution (S needs to be mean-reverting, i.e., $\alpha < 1$).

In (2.5), $\kappa > 0$ measures the decaying rate of the weights on past returns. An agent with a higher decaying rate relies more heavily on recent versus distant returns when predicting future returns. In particular, for $\kappa \to \infty$, the weights are concentrated on the current return. For $\kappa \to 0$, the agent assigns equal weights on all historical returns.

The assumption that the agent extrapolates the return scaled by volatility in (2.4)-(2.5) is consistent with the finding in Da, Huang and Jin (2018) that higher volatility increases the difficulties for extrapolators to infer a trend. This assumption leads to closed-form solutions. Our main conclusions about the risk premium, short rate, return predictability and excess volatility do not change if we alternatively assume that the agent extrapolates return. In fact, the variation in volatility in our model is small for typical parameters, and hence plays a marginal role, as to be shown later.

BGJS model extrapolation where the agents extrapolate price changes. Instead, we model extrapolation in terms of percentage return. This is in line with the evidence on how investors form expectations about future stock returns documented in Greenwood and Shleifer (2014).

One can also produce closed-form solutions if we alternatively assume that the agent extrapolates excess return. Instead, in this paper, we study extrapolation of gross return to be consistent with the survey evidence, e.g., Greenwood and Shleifer (2014). Although we consider a representative agent (the extrapolator) in the economy, our insights can be extended to a more general setting with both extrapolators and rational investors.

Definition 2.1. An equilibrium is a set of processes $\{P_t, S_t, r_{ft}\}_{t=0}^{T}$ and of consumption and investment policies $\{C_t^*, \phi_t^*\}_{t=0}^{T}$ such that consumption and investment policies solve the dynamic optimization problem (2.2) for the agent, given processes

$\{P_t, S_t, r_{ft}\}_{t=0}^T$, and that the markets for consumption and for both securities clear, that is, $C_t^* = C_t$ and $\phi_t^* = 1$ for $t \in [0, T]$.

Note that we require both asset market and consumption good market clear.

2.1.2. *Subjective Expectation about Consumption Growth.*

Given a subjective expectation about consumption growth μ_c^e, the short rate satisfies

$$r_f = \rho + \gamma\mu_c^e - \frac{\gamma(\gamma+1)\sigma_c^2}{2}, \tag{2.9}$$

where $\gamma\mu_c^e$ represents intertemporal substitution and $-\gamma(\gamma+1)\sigma_c^2/2$ represents precautionary savings. The subjective risk premium satisfies

$$\mu_p^e - r_f = \gamma\sigma_c\sigma_p. \tag{2.10}$$

Then the subjective expected return (the sum of the short rate and subjective risk premium) is given by

$$\mu_p^e = \rho + \gamma\mu_c^e - \frac{\gamma(\gamma+1)\sigma_c^2}{2} + \gamma\sigma_c\sigma_p. \tag{2.11}$$

Note that, in (2.9)–(2.11), we do not specify the subjective expected consumption growth. Therefore, (2.11) provides a generic relationship between the subjective expected return and subjective expected consumption growth under the CRRA utility.

In our model, the subjective expected return is specified as

$$\mu_p^e = \sigma_p(\alpha_0 + \alpha S)$$

by the definition of return extrapolation (2.4)-(2.5). Substituting it into (2.11), we obtain the subjective expected consumption growth rate

$$\mu_c^e = \sigma_p\frac{\alpha_0 + \alpha S}{\gamma} - \frac{\rho}{\gamma} - \sigma_c\sigma_p + \frac{(\gamma+1)\sigma_c^2}{2}. \tag{2.12}$$

It shows that the expected consumption growth depends positively on the sentiment under the subjective measure, rather than a constant as under the physical measure. The extrapolator's biased expectation about consumption growth is due to the fact that the expectations about consumption growth and return determine each other, as generally shown in (2.11).

We rewrite (2.11) as

$$\mu_p^e - \mu^r = \gamma\big[\mu_c^e - \mu_c + \sigma_c(\sigma_p - \sigma_c)\big], \tag{2.13}$$

where μ^r is the expected return under the rational benchmark given by (2.19). Equation (2.13) links models of non-rational expectations with the rational benchmark. It implies that the two economies can be directly compared in terms of (subjective) expected return or expected consumption growth.

The left-hand side of (2.13) depends positively on the sentiment and does not depend directly on γ by the definition of return extrapolation. Therefore, (2.13) implies that the sensitivity of the subjective expected consumption growth μ_c^e to the sentiment becomes smaller as γ increases. This is due to the substitution effect.

2.1.3. *Market Price of Sentiment.*

In models with biased expectations, the subjective measure (the measure of the agent) differs from the physical measure (the measure of an outside econometrician). As a result, the market prices of risk under the two measures are also different. The difference between them characterizes the "market price of sentiment". More specifically, we define the market price of sentiment η as

$$\eta = \frac{\mu_p - r_f}{\sigma_p} - \frac{\mu_p^e - r_f}{\sigma_p} = \frac{\mu_p - \mu_p^e}{\sigma_p} = \frac{\mu_c - \mu_c^e}{\sigma_c}, \tag{2.14}$$

where μ_p is the physical expected return (the expected return under the physical measure).[4] The last equality is due to the fact that return (consumption) volatilities are the same under both measures.[5] It further leads to the following relationship between the two measures:

$$dZ_t^e = \eta dt + dZ_t. \tag{2.15}$$

Substituting (2.13) into (2.14), we have the following proposition.

Proposition 2.2. *The market price of sentiment is given by*

$$\eta(S,t) = \frac{\mu_c - \mu_c^e}{\sigma_c} = -\frac{\sigma_p(\alpha_0 + \alpha S) - \mu^r}{\gamma\sigma_c} + \sigma_p - \sigma_c. \tag{2.16}$$

It depends negatively on S.

The subjective market price of risk is a constant $\gamma\sigma_c$. However, the physical market price of risk given by $\gamma\sigma_c + \eta$ depends negatively on the sentiment by noting that η depends negatively on the sentiment in Proposition 2.2. Indeed, the deviation of the subjective expectation from the rational expectation leads to an expectation adjustment (a common feature of the equilibrium pricing models with biased expectations). The sentiment price η accounts for the difference between the two measures. When the sentiment is high, the extrapolator expects a high future return and hence a high consumption growth rate. However, under the physical measure, the consumption growth is a constant and the expected return should not be as high as expected by the extrapolator. As a result, although the risk price under the subjective measure is a constant, the price of a dollar in each state under the physical measure depends negatively on the sentiment.

[4]It worth noting that (2.14) holds generally for any biased expectations (about either return or about consumption growth). For example, if the agent instead extrapolates consumption growth as studied in Appendix C, the market price of sentiment η is still governed by (2.14).

[5]It follows from $\mu_p dt + \sigma_p dZ = \mu_p^e dt + \sigma_p dZ^e$ and $\mu_c dt + \sigma_c dZ = \mu_c^e dt + \sigma_c dZ^e$.

Further, the product $\sigma_p\eta$ of the sentiment price and return volatility represents a sentiment premium. Proposition 2.2 shows that it depends negatively on the sentiment. Therefore, an outside econometrician always negatively adjusts the extrapolator's biased expectation about the future return:

$$\mu_p = \mu_p^e + \sigma_p\eta. \tag{2.17}$$

2.1.4. *Rational Benchmark.*

Before studying extrapolative expectation, we first present the rational benchmark in which the agent has rational expectation ($\mathbb{P}^e = \mathbb{P}$). We consider the limiting case $T \to \infty$, through which we can discuss the existence of equilibrium.[6] The following proposition characterizes the equilibrium, where the superscript "r" is an abbreviation for "rational benchmark".

Proposition 2.3. *(Rational benchmark.) Assume that the agent is fully rational and $T \to \infty$.*

(1) The equilibrium stock price P satisfies

$$\frac{dP_t + C_t dt}{P_t} = \mu^r dt + \sigma_p^r dZ_t, \tag{2.18}$$

where

$$\mu^r = \rho + \gamma\Big[\mu_c - \frac{(\gamma-1)\sigma_c^2}{2}\Big], \qquad \sigma_p^r = \sigma_c, \tag{2.19}$$

the short rate is given by

$$r_f^r = \rho + \gamma\Big[\mu_c - \frac{(\gamma+1)\sigma_c^2}{2}\Big], \tag{2.20}$$

and price-consumption ratio is given by

$$\Phi^r = \frac{1}{\rho + (\gamma-1)(\mu_c - \gamma\sigma_c^2/2)}. \tag{2.21}$$

(2) The equilibrium exists if and only if (transversality condition)

$$\rho + (\gamma-1)\Big(\mu_c - \frac{\gamma\sigma_c^2}{2}\Big) > 0. \tag{2.22}$$

The relationship between price-consumption ratio Φ^r and consumption growth μ_c depends on γ. Φ^r decreases with μ_c when $\gamma > 1$, and increases with μ_c when $\gamma < 1$. We will show later that the dependence of Φ^r on expected consumption growth determines the role played by extrapolation and also determines many key features of returns, such as return predictability.

The second part of Proposition 2.3 is a result of the transversality condition. The first γ in (2.22) is the inverse of the agent's EIS, and the term "$\gamma - 1$" reflects the relative strength of the income effect and the intertemporal substitution effect. The

[6] For $T < \infty$, the transversality condition is not needed and equilibrium always exists.

second γ is the risk aversion coefficient, which disappears if there is no uncertainty ($\sigma_c = 0$). The above two statements can be verified explicitly with recursive utility. Equation (2.22) leads to two bounds of γ,

$$\begin{aligned} \overline{\gamma}^r &= \frac{\sigma_c^2/2 + \mu_c + \sqrt{(\sigma_c^2/2 - \mu_c)^2 + 2\rho\sigma_c^2}}{\sigma_c^2}, \quad \text{and} \\ \underline{\gamma}^r &= \frac{\sigma_c^2/2 + \mu_c - \sqrt{(\sigma_c^2/2 - \mu_c)^2 + 2\rho\sigma_c^2}}{\sigma_c^2}, \end{aligned} \tag{2.23}$$

between which the equilibrium exists.

The upper bound $\overline{\gamma}^r$ arises as follows. The expected return has two components, the short rate and the risk premium. Proposition 2.3 shows that the risk premium is given by $\gamma\sigma_c^2$, while the short rate is given by $\rho + \gamma\mu_c - \gamma(\gamma+1)\sigma_c^2/2$. The term $\gamma\mu_c$ represents intertemporal substitution and $-\gamma(\gamma+1)\sigma_c^2/2$ represents precautionary savings. While the risk premium increases linearly with γ, the precautionary savings component decreases quadratically with γ. When γ is large enough, precautionary savings can be very negative, leading to a low discount rate. The transversality condition is violated.

The lower bound $\underline{\gamma}^r$ depends on the relative level of consumption growth rate μ_c to the subjective discount rate ρ. Especially, for $\gamma \to 0$ (infinite EIS), the consumption growth rate must be lower than the subjective discount rate; otherwise, perfect substitution leads to an infinite utility, and hence the nonexistence of equilibrium.

Note that the upper bound always exists, but the lower bound exists if and only if $\mu_c \geq \rho$.

3. Equilibrium

In this section, we derive the equilibrium under extrapolation. The agent's value function solves the following Hamilton-Jacobi-Bellman (HJB) equation (Merton, 1971):

$$\begin{aligned} 0 = \max_{C,\phi} \Big\{ & e^{-\rho t}\frac{C^{1-\gamma}}{1-\gamma} + \frac{\partial J}{\partial t} + \big[W\big(r_f + \phi(\sigma_p(\alpha_0 + \alpha S) - r_f)\big) - C\big]\frac{\partial J}{\partial W} \\ & + \kappa\big[\alpha_0 - (1-\alpha)S\big]\frac{\partial J}{\partial S} + \frac{1}{2}W^2\phi^2\sigma_p^2\frac{\partial^2 J}{\partial W^2} + \kappa W\phi\sigma_p\frac{\partial^2 J}{\partial W \partial S} + \frac{\kappa^2}{2}\frac{\partial^2 J}{\partial S^2}\Big\}, \end{aligned} \tag{3.1}$$

with boundary condition $J_T = 0$. The value function J has the form:

$$J(S, W, t) = e^{-\rho t}\frac{W^{1-\gamma}}{1-\gamma}\big[\Phi(S,t)\big]^{\gamma}. \tag{3.2}$$

Substituting it into (3.1) and using FOC, we obtain

$$C^* = W\Phi^{-1}, \qquad \phi^* = \frac{\sigma_p(\alpha_0 + \alpha S) - r_f}{\gamma\sigma_p^2} + \frac{\kappa}{\sigma_p}\frac{\partial \ln \Phi}{\partial S}. \tag{3.3}$$

Equation (3.3) shows that $\Phi = W/C$ is the price-consumption ratio. Substituting (3.3) into (3.1), we obtain a PDE of Φ:

$$\begin{aligned}&\frac{\partial\Phi}{\partial t} + \frac{\kappa^2}{2}\frac{\partial^2\Phi}{\partial S^2} + \frac{\gamma-1}{2}\kappa^2\Phi^{-1}\Big(\frac{\partial\Phi}{\partial S}\Big)^2 + \kappa\big[\alpha_0 - (1-\alpha)S - (\gamma-1)\sigma_p\big]\frac{\partial\Phi}{\partial S}\\ &-\Big[\Big(1-\frac{1}{\gamma}\Big)\sigma_p(\alpha_0+\alpha S) - \frac{(\gamma-1)\sigma_p^2}{2} + \frac{\rho}{\gamma}\Big]\Phi + 1 = 0,\end{aligned} \tag{3.4}$$

with boundary condition $\Phi(S,T) = 0$. Applying Ito's lemma to $\Phi = P/C$, we obtain

$$dC_t = C_t\Big[\mu_c^e dt + \Big(\sigma_p - \kappa\frac{\partial \ln\Phi}{\partial S}\Big)dZ_t^e\Big], \tag{3.5}$$

where

$$\mu_c^e(S,t) = \frac{\sigma_c}{\gamma}(\alpha_0+\alpha S) - \frac{\rho}{\gamma} + \kappa\frac{\partial\ln\Phi}{\partial S}\Big(\frac{\alpha_0+\alpha S}{\gamma} - \sigma_c\Big) + \frac{(\gamma-1)\sigma_c^2}{2}. \tag{3.6}$$

By matching consumption volatility in (2.1) and (3.5), we have

$$\sigma_p(S,t) = \sigma_c + \kappa\frac{\partial\ln\Phi}{\partial S}. \tag{3.7}$$

Using the market clearing condition, $\phi^* = 1$, we have

$$r_f(S,t) = \sigma_p(\alpha_0 + \alpha S - \gamma\sigma_c). \tag{3.8}$$

It follows from (2.4) and (3.8) that, with a representative agent and a constant consumption volatility, the market price of risk under $\mathbb{P}^e$ is always a constant ($\gamma\sigma_c$).

By substituting (3.7) into (3.4), we obtain

$$\frac{\partial\Phi}{\partial t} + \frac{\kappa^2}{2}\frac{\partial^2\Phi}{\partial S^2} + \frac{\kappa}{\gamma}\big[(\alpha-\gamma)S+\alpha_0\big]\frac{\partial\Phi}{\partial S} - \Big[\Big(1-\frac{1}{\gamma}\Big)\sigma_c(\alpha_0+\alpha S) - \frac{(\gamma-1)\sigma_c^2}{2} + \frac{\rho}{\gamma}\Big]\Phi + 1 = 0, \tag{3.9}$$

with boundary condition $\Phi(S,T) = 0$. Following Liu (2007), we define $\widehat{\Phi}(S,t)$ such that

$$\Phi = \int_t^T \widehat{\Phi}(S,u)du.$$

Then $\widehat{\Phi}$ satisfies

$$\frac{\partial\widehat{\Phi}}{\partial t} + \frac{\kappa^2}{2}\frac{\partial^2\widehat{\Phi}}{\partial S^2} + \frac{\kappa}{\gamma}\big[(\alpha-\gamma)S+\alpha_0\big]\frac{\partial\widehat{\Phi}}{\partial S} - \Big[\Big(1-\frac{1}{\gamma}\Big)\sigma_c(\alpha_0+\alpha S) - \frac{(\gamma-1)\sigma_c^2}{2} + \frac{\rho}{\gamma}\Big]\widehat{\Phi} = 0, \tag{3.10}$$

with boundary condition $\widehat{\Phi}(S,T) = 1$. By conjecturing that

$$\widehat{\Phi}(S,t) = e^{A(t)S+B(t)},$$

and substituting it into (3.10), we obtain

$$\begin{aligned} &A' + \kappa\Big(\frac{\alpha}{\gamma} - 1\Big)A - \alpha\sigma_c\Big(1 - \frac{1}{\gamma}\Big) = 0, \\ &B' + \frac{\kappa^2}{2}A^2 + \frac{\kappa\alpha_0}{\gamma}A - \Big(1 - \frac{1}{\gamma}\Big)\sigma_c\alpha_0 - \frac{\rho}{\gamma} + \frac{(\gamma-1)\sigma_c^2}{2} = 0, \end{aligned} \tag{3.11}$$

with terminal conditions $A(T) = B(T) = 0$, where the operator " $'$ " denotes the derivative with respect to t. Appendix D.1 presents the solution of (3.11). It shows that $A(t) \leq 0$ for $\gamma > 1$ and $A(t) \geq 0$ for $\gamma < 1$. Proposition 3.1 summarizes the equilibrium under the subjective measure, which has closed-form solutions.

Proposition 3.1. *(Return under subjective measure.) The equilibrium stock price under the subjective measure $\mathbb{P}^e$ satisfies*

$$\frac{dP_t + C_t dt}{P_t} = \mu_p^e dt + \sigma_p dZ_t^e,$$

where

$$\mu_p^e = \sigma_p(\alpha_0 + \alpha S), \qquad \sigma_p = \sigma_c + \kappa\frac{\int_t^T A(u)e^{A(u)S+B(u)}du}{\int_t^T e^{A(u)S+B(u)}du}, \tag{3.12}$$

and A and B are deterministic functions governed by (3.11)*. The short rate is given by*

$$r_f = \sigma_p(\alpha_0 + \alpha S - \gamma\sigma_c), \tag{3.13}$$

the subjective risk premium (the risk premium under $\mathbb{P}^e$) is given by

$$\mu_p^e - r_f = \gamma\sigma_c\sigma_p, \tag{3.14}$$

and price-consumption ratio is given by

$$\Phi = \int_t^T e^{A(u)S+B(u)}du. \tag{3.15}$$

Especially for $\alpha = 0$, the agent is fully rational and the equilibrium reduces to that characterized in Proposition 2.3.

Proposition 3.2 provides the condition for the existence of the equilibrium.

Proposition 3.2. *(Existence of equilibrium.)*

(1) *Suppose $T < \infty$. The equilibrium characterized in Proposition 3.1 exists for all $\gamma > 0$.*
(2) *Suppose $T = \infty$. The equilibrium exists if and only if*

$$\gamma \geq \alpha \quad \textit{and} \quad \Delta(\gamma, \alpha) > 0, \tag{3.16}$$

where Δ *is given by* (A.4). *Especially for* $\alpha = 0$, *condition* (3.16) *reduces to* (2.22) *in Proposition 2.3; for* $\alpha = 1$, *condition* (3.16) *becomes*

$$1 \leq \gamma < \sqrt{2\rho}/\sigma_c. \tag{3.17}$$

Using the relationship (2.15) between the two measures, Proposition 3.3 summarizes the equilibrium return under the physical measure.

Proposition 3.3. *(Return under physical measure.) Under the physical measure* $\mathbb{P}$, *the stock price satisfies*

$$\frac{dP_t + C_t dt}{P_t} = \mu_p dt + \sigma_p dZ_t, \tag{3.18}$$

where

$$\mu_p = \sigma_p(\psi_0 + \psi S), \quad \psi = \alpha\Big(1 - \frac{\sigma_p}{\gamma\sigma_c}\Big), \quad \psi_0 = \alpha_0\Big(1 - \frac{\sigma_p}{\gamma\sigma_c}\Big) + \frac{\mu^r}{\gamma\sigma_c} + \sigma_p - \sigma_c, \tag{3.19}$$

and σ_p *follows* (3.12), *and the sentiment follows*

$$dS_t = \kappa\big[\psi_0 - (1-\psi)S_t\big]dt + \kappa dZ_t. \tag{3.20}$$

Note that the change of measure does not affect the short rate and price-consumption ratio that are given by (3.13) and (3.15) respectively.

4. Asset Pricing Implications of Extrapolation

In this section, we study asset pricing implications of return extrapolation. We show that, for $\gamma > 1$, price-consumption ratio decreases with the sentiment and stock return has deficient volatility; return extrapolation generates low and stable risk premium and volatile short rate. The results are both demonstrated analytically and illustrated numerically. Unless specified otherwise, in numerical examples of this paper, we set $\rho = 0.02$, $\mu_c = 0.018$, $\sigma_c = 0.032$ (e.g., Basak and Cuoco, 1998), and $T = 100$ in annual terms. We also set $\kappa = 2.3$ according to Greenwood and Shleifer (2014)[7] and $\alpha = 0.5$ to study extrapolation.

4.1. **Price-Consumption Ratio.**

Proposition 3.1 shows that price-consumption ratio Φ is a decreasing function of the sentiment when $\gamma > 1$. It varies from zero to positive infinity as the sentiment varies from positive infinity to negative infinity. Therefore, the sentiment can lead to large deviations from the rational benchmark where Φ is a constant.

[7]According to equation (3) in Greenwood and Shleifer (2014), the ratio of the weights on returns in quarter t and $t-1$ equals λ, so κ in our paper and λ satisfy $e^{\kappa/4} = 1/\lambda$. We set $\kappa = 2.3$, corresponding to $\lambda = 0.56$, which is the average of the estimates for λ reported in Table 4 of Greenwood and Shleifer (2014).

Intuitively, under extrapolation, high past returns (high sentiment) lead the extrapolator to expect both high discount rate (substitution effect) and high consumption growth (income effect). The substitution effect dominates the income effect when $\gamma > 1$. As a result, high sentiment decreases price-consumption ratio.

Note that the discount rate under the physical measure depends on the market price of sentiment (due to the change of measure (2.15)) and differs from the discount rate under the subjective measure. Under the physical measure, because the expected consumption growth is a constant, the variation of price-consumption ratio is completely due to the variation of the physical expected discount rate.

Because Φ depends on the sentiment and the sentiment is stochastic in the long-run, the price-consumption ratio does not necessarily converge to the rational counterpart in the long-run. If there are rational investors in our economy, in addition to extrapolators, as the case studied in BGJS, we expect that equilibrium may converge to the rational benchmark.[8]

Price-consumption ratio follows the opposite pattern when $\gamma < 1$.

4.2. **Return Volatility.**

Proposition 4.1 summarizes the properties of return volatility.

Proposition 4.1. *Return volatility σ_p is positive and depends positively on the sentiment S:*

$$\frac{\partial \sigma_p}{\partial S} > 0.$$

It satisfies

$$\begin{cases} \sigma_c > \sigma_p > \frac{(1-\alpha)\gamma}{\gamma-\alpha}\sigma_c, & \textit{for} \quad \gamma > 1, \\ \sigma_p = \sigma_c, & \textit{for} \quad \gamma = 1, \\ \frac{(1-\alpha)\gamma}{\gamma-\alpha}\sigma_c > \sigma_p > \sigma_c, & \textit{for} \quad \gamma < 1. \end{cases} \tag{4.1}$$

Several observations follows from Proposition 4.1. First, return volatility is lower than consumption volatility for $\gamma > 1$. Therefore, extrapolation makes the excess volatility puzzle (e.g., LeRoy and Porter, 1981; Shiller, 1981) even more puzzling.[9]

This is because the extrapolator believes that the sentiment *positively* predicts future returns, leading to a (conditionally) positive correlation of the sentiment and consumption. According to the accounting identity of $P = C\Phi$, return volatility equals the sum of consumption volatility and price-consumption ratio volatility:

[8]When both rational investors and irrational investors have constant beliefs and have the same CRRA utility, Yan (2008) shows that the rational agents will dominate the market in the long-run. However, this result may not hold if the agents have different preferences (e.g., Yan, 2008) or if the agents have recursive preferences (e.g., Borovička, 2018; Dindo, 2019).

[9]In this case, even if we assume dividend and consumption follow different processes, return volatility is still lower than dividend volatility by noting that consumption volatility is lower than dividend volatility in the real data.

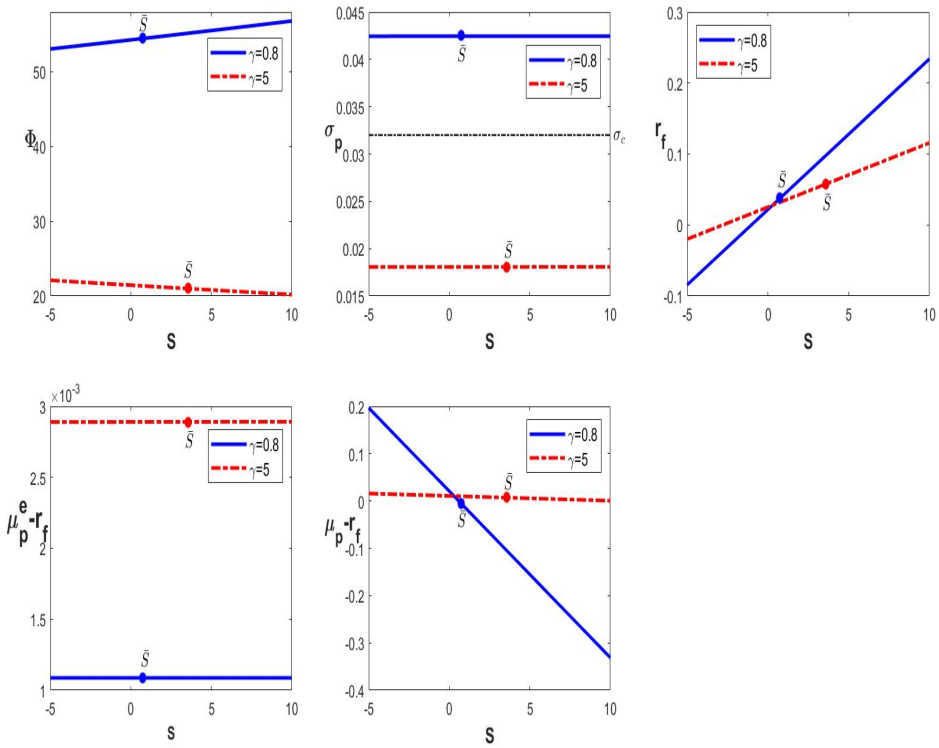

FIGURE 4.1. The impact of S on the return volatility, short rate, subjective risk premium, and physical risk premium. Here $\bar{S}$ denotes the unconditional mean of S under the physical measure.

$\sigma_p = \sigma_c + \kappa \frac{\partial \ln \Phi}{\partial S}$. Price-consumption ratio Φ depends negatively on the sentiment S, and hence is negatively correlated with consumption when $\gamma > 1$. The movement in consumption is partially offset by the movement in Φ, leading to deficient volatility. This is different from some pricing models, in which state variables *negatively* forecast returns and hence produce excess volatility for $\gamma > 1$.[10] In our model, excess volatility occurs for $\gamma < 1$.

Second, return volatility σ_p increases with the sentiment S. However, (4.1) shows that σ_p is bounded from both above and below. With extreme sentiment ($S \to \pm\infty$), σ_p tends to one of the two boundaries. Proposition 4.1 also shows that return volatility is always positive.

The upper panel of Fig. 4.1 shows that the variation of return volatility with respect to the sentiment and hence with respect to time is very small for both $\gamma > 1$ and $\gamma < 1$. As a result, the effect of the sentiment on return volatility is much less

[10]For example, when the agent is a contrarian as discussed in Section 5.6, the sentiment negatively forecast returns, leading to excess volatility for $\gamma > 1$.

than that on the short rate. This is consistent with (4.1), which demonstrates that return volatility σ_p is very stable and close to the rational benchmark level.

It is worth noting that in Fig. 4.1 all equilibrium quantities, including price-consumption ratio, return volatility, the short rate, the risk premiums and expected returns under both measures, are monotonic functions of the sentiment (within, e.g., two standard deviations of its unconditional mean). Appendix A.1 further proves that this result also holds for large sentiment ($S \to \pm\infty$).[11] This property provides a very clear understanding of the effect of the sentiment. When the sentiment is close to its unconditional mean, the equilibrium behavior is close to the rational counterpart. However, the agent becomes more irrational when the sentiment deviates from the unconditional mean, generating large deviations from the rational benchmark.

4.3. **Risk Premium.**

The subjective risk premium is a constant $\gamma\sigma_c\sigma_p$. The physical risk premium given by

$$\mu_p - r_f = \sigma_p\eta + (\mu_p^e - r_f) = -\frac{\sigma_p}{\gamma\sigma_c}\big[\sigma_p(\alpha_0 + \alpha S) - \mu^r\big] + \sigma_p^2 + (\gamma - 1)\sigma_c\sigma_p, \quad (4.2)$$

depends negatively on the sentiment S. Equation (4.2) also shows that the sentiment has marginal impact on both subjective risk premium and physical risk premium for $\gamma > 1$, even though it significantly affects subjective expected return by definition. The sentiment has more significant effect on the physical risk premium for $\gamma < 1$. In this case, although the subjective risk premium is still insensitive to the sentiment, the sentiment premium becomes more sensitive to the sentiment. The results are numerically demonstrated in the lower panel of Fig. 4.1.

Although the subjective risk premium is always positive, the physical risk premium can be negative when the sentiment is high. This holds for both $\gamma > 1$ and $\gamma < 1$. The negative risk premium is due to the large and negative sentiment premium for high sentiment. In words, the short rate increases with the sentiment faster than does the physical expected return because the effect of the sentiment is largely on the short rate. The negative risk premium is also found in the empirical literature, e.g., in Greenwood and Hanson (2013) and Cassella and Gulen (2018).[12]

The subjective risk premium is lower (higher) than the rational counterpart for $\gamma > 1$ ($\gamma < 1$) due to deficient (excess) volatility. More importantly, the risk

[11]Numerical simulations (not reported here) verify that the result is robust to different sets of parameters.

[12]In this case, the agent's myopic demand is negative; while her positive total demand ($\phi^* = 1$) is due to positive intertemporal hedging. In a dynamic asset allocation problem, Li and Liu (2019) show that an investor with CRRA preference and $\gamma > 1$ may long (short) a risky asset with negative (positive) momentum to effectively home-make an asset with return reversal.

premium is much lower (higher) than the rational counterpart when the sentiment is sufficiently high (low) due to the negative sentiment premium.

Now we study the unconditional mean of the risk premium. In this paper, we discuss the unconditional distributions of the equilibrium mainly through the limiting case of large κ, in which the unconditional properties are qualitatively comparable to those for typical κ. This case allows for closed-form solutions. All equilibrium quantities in this case is also consistent with the limiting cases of large sentiment ($S \to \pm\infty$) studied in Appendix A.1. We will also present the results for small κ later.

For unconditional expectations to be defined, we need the sentiment S to have an invariant distribution (S needs to be mean-reverting). So we assume $\alpha < 1$ when studying unconditional distributions in the paper. However, the conditional results is this paper also hold for $\alpha = 1$. Let $\mathbb{E}^e$ and $\mathbb{E}$ denote the unconditional expectations under the subjective and physical measures respectively. Corollary 4.2 presents the unconditional mean of the risk premium for large κ.

Corollary 4.2. *To the leading order of $1/\kappa$, the unconditional means of risk premium under the subjective measure and the physical measure are given, respectively, by*

$$
\begin{aligned}
\mathbb{E}^e[\mu_p^e - r_f] &= \frac{1-\alpha}{\gamma-\alpha}\gamma^2\sigma_c^2, \\
\mathbb{E}[\mu_p - r_f] &= \frac{\alpha(1-\alpha)(\gamma-1)}{\gamma(\gamma-\alpha)}\rho + \frac{\alpha(1-\alpha)(\gamma-1)}{\gamma-\alpha}\mu_c \\
&\quad + \frac{1-\alpha}{\gamma-\alpha}\Big[\gamma^2 - \frac{\alpha(\gamma-1)^2}{2} - \frac{\alpha\gamma(\gamma-1)}{\gamma-\alpha} + \frac{\alpha^2(1-\alpha)\gamma(\gamma-1)}{2(\gamma-\alpha)^2}\Big]\sigma_c^2.
\end{aligned}
\tag{4.3}
$$

In the second equation of (4.3), the unconditional mean of risk premium under the physical measure depends on ρ and μ_c, which, however, do not affect the risk premium in rational equilibrium pricing models. This dependence is caused by the market price of sentiment.

The unconditional mean of risk premium under the subjective measure is close to the rational benchmark level ($\gamma\sigma_c^2$); and under the physical measure it is too small to explain the equity premium puzzle (e.g., Mehra and Prescott, 1985) when $\gamma > 1$.[13] The unconditional mean of risk premium can be very large under the subjective measure but tends to be negative under the physical measure when $\gamma < 1$.

Fig. 4.2 (A) plots the unconditional mean of risk premium under the physical measure against γ for the extrapolation model with large κ. To satisfy the transversality condition, we need $\gamma > \alpha$ ($= 0.5$). For comparison, we also plot the risk premium $\mu^r - r_f^r$ for the rational benchmark. In both cases, the unconditional means of the

[13] It is dominated by the second term, which is less than $\mu_c/4$.

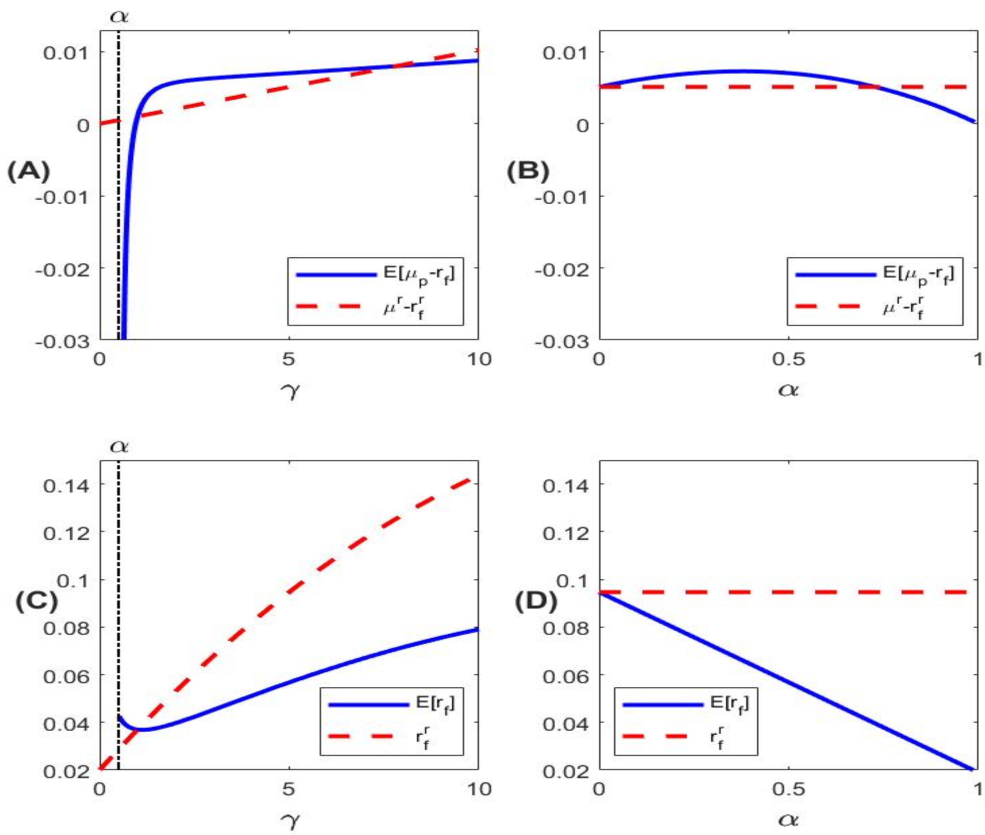

FIGURE 4.2. The impacts of γ and α on the unconditional means of the risk premium and short rate under the physical measure for the extrapolation model (Corollaries 4.2 and 4.3) and the rational benchmark (Proposition 2.3). Here $\alpha = 0.5$ in panels (A) and (C), and $\gamma = 5$ in (B) and (D).

physical risk premium increase with γ, which is intuitive. The unconditional mean in the extrapolation model equals the rational benchmark level for $\gamma = 1$. When γ is low, it decreases fast with γ and becomes negative. It can be very negative when γ approaches α because of very negative sentiment premium. In contrast, the risk premium in the rational benchmark equals $\gamma\sigma_c^2$, which is positive for all $\gamma > 0$.

Due to the deficient volatility for large γ, the unconditional mean of risk premium grows slower with γ and is lower than the rational benchmark level. It means that, to have a high risk premium, we need to set a much higher risk aversion coefficient than in the rational benchmark.

Fig. 4.2 (B) shows that, for $\alpha = 0$, the agent is fully rational and the unconditional mean of risk premium is at the rational benchmark level. For $0 < \alpha < 0.7$, extrapolation slightly amplifies the risk premium (by comparing with the rational benchmark). In this case, extrapolation helps relieve the equity premium puzzle but is far not enough. When α is close to one, extrapolation makes the risk premium puzzle even worse. In this case, stock return volatility converges to zero because consumption volatility is completely offset by extrapolation (deficient volatility) as shown in Corollary 4.2. As a result, the unconditional mean of risk premium tends to zero. Numerical simulations in Table 4.1 show that the unconditional mean of the physical risk premium for typical κ is also low.

In sum, we show that extrapolation alone is unlikely to solve the equity premium puzzle. BGJS also find that extrapolation cannot account for the equity premium puzzle. Our finding is also consistent with the arguments in Cochrane (2011, 2017).

4.4. **Short Rate.**

Extrapolation leads the short rate to depend positively on the sentiment as shown in (2.9). Intertemporal substitution component in the second term depends significantly on the sentiment, but precautionary savings component in the third term is a constant and does not offset the changes in intertemporal substitution component with the sentiment. Therefore, the short rate changes proportionally with the sentiment and is volatile. As the sentiment changes, the short rate varies widely from negative to very positive (see Fig. 4.1).[14]

It is worth noting that the sentiment affects the short rate more than the subjective risk premium. Intuitively, return extrapolation by definition directly specifies the subjective expected return, while it has an indirect impact on the return volatility. The short rate is mainly driven by the subjective expected consumption growth, on which the extrapolation has significant impact; while the subjective risk premium depends only on risk aversion, consumption volatility, and stock volatility, which are either constant or insensitive to extrapolation as shown in Section 4.2.

The sentiment affects the short rate more than the physical risk premium for large γ by noting that the sentiment premium is insensitive to the sentiment as shown in Section 2.1.3. The above results can be also understood from the example of pure change of expectation in Appendix B. In this case, the biased expectation does not change return volatility. The effect of expectation bias is mainly reflected in the short rate rather than the risk premium.

In all, we show that the sentiment significantly affects the equilibrium. Extrapolation in general has a first order effect on the subjective expected return, physical expected return and short rate, while it has a second order effect on the return volatility, subjective risk premium, and physical risk premium. The effect of the sentiment is largely reflected in the short rate, leading to volatile short rate while

[14]This result holds even if the dividend follows a different process from the aggregate consumption because state price density is completely determined by the consumption process in consumption-based asset pricing models.

stable risk premium, when $\gamma > 1$. The effect of the sentiment is mainly on the sentiment premium, leading to negative risk premium, when $\gamma < 1$.

Now we study the unconditional mean of the short rate.

Corollary 4.3. *To the leading order of $1/\kappa$, the unconditional means of the short rate under the subjective measure and the physical measure are given, respectively, by*

$$\mathbb{E}^e[r_f] = \frac{\gamma - \gamma\alpha}{\gamma - \alpha} r_f^r, \qquad \mathbb{E}[r_f] = \Big(1 - \frac{\gamma - 1}{\gamma}\alpha\Big) r_f^r.$$

Corollary 4.3 shows that, when κ is large, the unconditional mean of the short rate under the physical measure $\mathbb{E}[r_f]$ is higher than r_f^r (i.e., the rational counterpart) for $\gamma < 1$, equal to r_f^r for $\gamma = 1$, and lower than r_f^r for $\gamma > 1$. This is also illustrated in Fig. 4.2 (C).

In addition, Fig. 4.2 (D) shows that $\mathbb{E}[r_f]$ decreases with α in the extrapolation model but is independent of α in the rational benchmark. With large α, extrapolation generates low short rate. This is largely due to our choice of α_0. Under our current choice (2.6), $\mathbb{E}^e[r_f]$ is lower than the rational counterpart for $\gamma > 1$ because $\mathbb{E}[\sigma_p] < \sigma_c$. Alternatively, when we choose α_0 such that the unconditional subjective average of μ_p equals μ^r (i.e., the agent has an unbiased belief about the mean of return), $\mathbb{E}^e[r_f]$ is higher than the rational counterpart also because $\mathbb{E}[\sigma_p] < \sigma_c$.[15] In this case, extrapolation still cannot resolve the riskless rate puzzle and actually even exacerbates the puzzle.

Table 4.1 reports the unconditional mean of different variables in the equilibrium. It shows that the unconditional mean of the short rate for typical κ is also high (much higher than the level for the real data).

Table 4.1 also shows that the unconditional mean $\bar{S}$ of the sentiment is higher (lower) than its counterpart in the rational benchmark $S^r = \mu^r/\sigma_c$ for $\gamma > 1$ ($\gamma < 1$).[16] In addition, relative to the rational benchmark, extrapolation decreases the unconditional expected return, short rate and return volatility, while it slightly increases the risk premium for $\gamma > 1$. Untabulated numerical simulations show that the unconditional risk premium is lower than the rational counterpart for large α.

The results for typical κ in Table 4.1 are in line with the results for large κ proved in Corollaries 4.2 and 4.3 and illustrated in Fig. 4.2. All the results consistently show that extrapolation alone cannot resolve the asset pricing puzzles. Our finding is consistent with the survey evidence analyzed in Giglio, Maggiori, Stroebel and Utkus (2019), which suggests that frictionless macro-finance models, whether rational or

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[15] In addition, if we choose α_0 such that $\mathbb{E}^e[\mu_c^e] = \mu_c$, then $\mathbb{E}^e[r_f] = r_f^r$.

[16] The unconditional mean of S under the physical measure can be decomposed into two components: $\bar{S} = S^r + \mathbb{E}\big[\frac{r_f^r(\sigma_c - \sigma_p)}{\gamma\sigma_c^2[1-\alpha(1-\frac{\sigma_p}{\gamma\sigma_c})]}\big]$. It shows that $\bar{S} > S^r$ for $\gamma > 1$ because $\sigma_c > \sigma_p$. Similarly, $\bar{S} < S^r$ for $\gamma < 1$.

Table 4.1. This table reports the unconditional means of different variables in equilibrium, as well as the rational benchmark counterparts. Here the superscript "r" stands for the rational benchmark. The unconditional distribution of S under the physical measure is derived using Fokker-Planck equation, and the unconditional distributions of the other variables that are functions of S are derived based on S's distribution.

		$\gamma = 0.8$	$\gamma = 5$
Sentiment	$\mathbb{E}[S]$	0.73	3.58
	S^r	1.08	3.12
Price-consumption ratio	$\mathbb{E}[\Phi]$	73.35	21.24
	Φ^r	60.67	12.23
Consumption growth	$\mathbb{E}[\mu_c^e]$	2.27×10^{-2}	1.06×10^{-2}
	$\mathbb{E}[\mu_c]$	1.80×10^{-2}	1.80×10^{-2}
Return volatility	$\mathbb{E}[\sigma_p]$	4.25×10^{-2}	1.81×10^{-2}
	σ^r	3.20×10^{-2}	3.20×10^{-2}
Short rate	$\mathbb{E}[r_f]$	3.74×10^{-2}	5.77×10^{-2}
	r_f^r	3.37×10^{-2}	9.46×10^{-2}
Risk premium	$\mathbb{E}[\mu_p - r_f]$	-0.52×10^{-2}	0.71×10^{-2}
	$\mathbb{E}[\mu_p^e - r_f]$	0.10×10^{-2}	0.29×10^{-2}
	$\mu^r - r_f^r$	20.08×10^{-2}	0.51×10^{-2}
Sentiment premium	$\mathbb{E}[\sigma_p \eta]$	-0.62×10^{-2}	0.42×10^{-2}

behavioral, are likely to overstate the power of expectation changes in explaining asset price movements.[17]

In our model, the agent has CRRA utility. We speculate that replacing it by recursive utility cannot ease the riskless rate puzzle either. This is because the expected return is specified by extrapolation, which limits the effect of EIS on intertemporal substitution. This limitation is more clearly seen from the example of pure change of expectations in Appendix B. However, this is different from the rational benchmark with IID consumption growth where recursive utility can help resolve the puzzle. In this case, if the agent has a CRRA utility, an increase in risk aversion increases both expected return and risk premium. Therefore, recursive utility can ease the riskless rate puzzle by adjusting both EIS and risk aversion to obtain proper intertemporal substitution and precautionary savings.

[17]By administering a newly-designed survey to a large panel of retail investors, Giglio et al. (2019) find that the sensitivity of portfolios to beliefs is substantially lower than predicted by frictionless macro-finance models. In addition, their findings on the relationship amongst investors' expectations about cash flows and returns, and price-dividend ratio (corresponding to price-consumption ratio in our model) are consistent with our model for $\gamma > 1$.

The unconditional results in Corollaries 4.2 and 4.3 are for large κ. The following corollary shows that unconditional properties for small κ are comparable to those in the rational benchmark.

Corollary 4.4. *To the leading order of κ, the unconditional means of risk premium and the short rate under the subjective measure and the physical measure are given, respectively, by*

$$\mathbb{E}^e[\mu_p^e - r_f] = \mathbb{E}[\mu_p - r_f] = \gamma\sigma_c^2, \quad \mathbb{E}^e[r_f] = \mathbb{E}[r_f] = r_f^r.$$

The sentiment becomes the historical mean of stock returns and is stable over time for $\kappa \to 0$. So the agent tends to have a constant belief. As a result, the unconditional means of the risk premium and short rate under both measures equal the rational benchmark levels.

The sentiment has an invariant distribution. The short rate and risk premium as functions of the sentiment also have invariant distributions. The following corollary presents the volatilities of the short rate and risk premium under the invariant distributions.

Corollary 4.5. *Under the physical measure, for $\gamma > 1$, the unconditional volatility of the short rate is greater than the unconditional volatility of the risk premium for both small κ and large κ.*

Corollary 4.5 shows that extrapolation causes the short rate more volatile than risk premium in the long-run. This is consistent with the conditional effect of the sentiment. According to (3.13), the instantaneous volatility of the short rate is approximately equal to $\kappa\alpha\sigma_p$. If the decaying rate κ is large as documented in the extrapolation literature, $\kappa\alpha\sigma_p$ is large, leading to nonpersistent sentiment and volatile short rate.

5. Further Discussions

The previous section studies the asset pricing implications of extrapolation. In this section, we discuss some effects specific to the extrapolation literature.

5.1. **Feedback Effect.**

The feedback effect documents the impact of the sentiment on price-consumption ratio. Past returns affect extrapolator's expectation about future returns – *(past) price impact on expectation* (characterized by S). The price impact is always positive in the sense that an extrapolator expects high consumption growth after a sequence of high past returns. On the other hand, an investor's expectation in turn feeds into asset prices in the sense that it determines consumption growth that further determines current price – *expectation impact on (current) price.*

For $\gamma < 1$, the expectation impact is positive. The combined effect of the positive expectation impact and the positive price impact leads to a positive feedback (that is, high past returns increase price-consumption ratio Φ). It is called "infinite feedback loop" in BGJS in the sense that

> *if good cash-flow news pushes the stock market up, this price increase feeds into extrapolators' expectations about future price changes, which then leads them to push the current price up even higher. However, this then further increases extrapolators' expectations about future price changes, leading them to push the current price still higher, and so on.*

With CARA utility, the extant literature shows that an extrapolator always leads to a positive feedback (e.g., De Long, Shleifer, Summers and Waldmann, 1990; Cutler, Poterba and Summers, 1990; Hong and Stein, 1999; BGJS).

However, in our model, the expectation impact is negative for $\gamma > 1$ due to the dominance of the substitution effect over the income effect. It, together with the positive price impact, leads to a *negative* feedback in the sense that Φ decreases in the sentiment. As a result, high past returns tends to "pull down" (rather than "push up") price-consumption ratio. This differs from the popular argument that price-consumption ratio is pulled back by the rational traders (note that we do not have rational traders in the model).[18]

In fact, a high extrapolator's expectation about future stock returns leads her to expect both high discount rates and high consumption growth rates in the future. If the increase in discount rate dominates (substitution effect dominates income effect), then Φ decreases, leading to a negative feedback.[19] This highlights the effect of the discount rate.[20] However, the discussion on feedback by BGJS above only mentions the expectation about future cash flows but not on the expectation about discount rate (equivalently pricing kernel). An increase in the sentiment increases both; the combined effect on current price depends on γ.

The mechanism in generating positive (negative) feedback has been also reflected by the rational benchmark in Section 2.1.4. When the agent is fully rational, Proposition 2.3 shows that high expected consumption growth decreases Φ for $\gamma > 1$.

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[18]In addition, the term "good cash-flow news" may be ambiguous. Good news (a positive shock) under the physical measure could be bad news under the subjective measure, and vice versa.

[19]A sequence of high past returns (high sentiment) increases both price and consumption. For $\gamma > 1$, consumption increases faster than price, amplifying Φ. However, in the rational benchmark, a positive shock increases price and consumption proportionally, leading to a constant Φ that does not depend on market states. Put differently, an agent's optimal demand does not only depend on the expected return. It also depends on the intertemporal hedging, short rate, and volatility.

[20]Indeed, under the physical measure, although the consumption growth is constant, an increase in the sentiment leads to a decrease in the expected return when the substitution effect dominates.

When the agent is an extrapolator, she expects high consumption growth after high realized returns (high sentiment), and hence pulls down Φ.

In addition, the negative expectation impact generates negative price-consumption ratio volatility and hence deficient return volatility.

5.2. **Momentum and Reversal.**

Short-run momentum and long-run reversal in returns are two of the most prominent financial market anomalies, and have been extensively documented in the literature, e.g., Jegadeesh and Titman (1993) and Moskowitz, Ooi and Pedersen (2012) for momentum; De Bondt and Thaler (1985), Fama and French (1988*b*) and Poterba and Summers (1988) for reversal; and Cutler, Poterba and Summers (1991) for both, among others.

5.2.1. *The Effect of γ.*

In our model, momentum and reversal depend on γ. Indeed, as shown in Proposition 2.2, an increase in γ decreases the sensitivity of the market price of sentiment η to the sentiment, due to the substitution effect. To understand the effect of γ, we first examine a special case of $\gamma = 1$. In this case, the agent is myopic and Propositions 3.1 and 3.3 reduce to the following corollary.

Corollary 5.1. *For $\gamma = 1$, the subjective expected return, market price of sentiment, and physical expected return are given, respectively, by*

$$\mu_p^e = \sigma_c(\alpha_0 + \alpha S), \quad \eta = -(\alpha_0 + \alpha S) + \mu^r, \quad \mu_p = \rho + \mu_c,$$

and the return volatility, short rate, and price-consumption ratio are given, respectively, by

$$\sigma_p = \sigma_c, \quad r_f = \sigma_c(\alpha_0 + \alpha S - \sigma_c), \quad \Phi = \rho^{-1}\big[1 - e^{-\rho(T-t)}\big].$$

The extrapolator believes that the sentiment is able to positively predict stock returns. Stock returns turn out to be unpredictable under the physical measure when $\gamma = 1$. In this case, the effect of the sentiment on stock price is completely offset by the expectation adjustment. However, all the effect of extrapolation is reflected in the short rate.

The magnitude of the sentiment price η is low when $\gamma > 1$ by noting that the sensitivity of η to the sentiment decreases with γ. The small sentiment price only partially offsets the effect of the sentiment (a smaller adjustment relative to the case $\gamma = 1$). Thus, high sentiment still predicts high future returns under the physical measure. In contrast, small γ ($\gamma < 1$) produces large sentiment price that can even reverse the effect of the sentiment, leading the sentiment to negatively predict future returns under the physical measure. In this case, the price impact (on expectation) has the opposite patterns under the two measures.

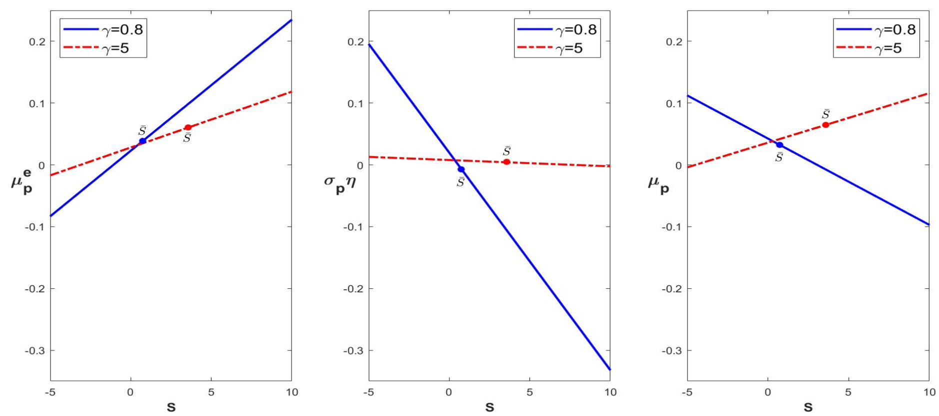

FIGURE 5.1. The impact of the sentiment S on the subjective expected return μ_p^e, sentiment premium $\sigma_p\eta$, and the physical expected return μ_p. Here $\bar{S}$ denotes the unconditional mean of S under the physical measure.

The impacts of γ and the sentiment on the expected returns and sentiment premium are numerically illustrated in Fig. 5.1. The left panel of Fig. 5.1 shows that the subjective expected return depends positively on the sentiment for both $\gamma > 1$ and $\gamma < 1$ by definition. As demonstrated in Proposition 2.2 and illustrated in the middle panel, the sentiment premium depends negatively on the sentiment, and its sensitivity to the sentiment increases as γ decreases. As a result, the physical expected return depends positively (negatively) on the sentiment for $\gamma > 1$ ($\gamma < 1$) as illustrated in the right panel.

Fig. 5.1 also illustrates the unconditional mean $\bar{S}$ of the sentiment under the physical measure. When the sentiment is sufficiently high, the subjective expected return is always higher than the rational counterpart, while the physical expected return is higher (lower) than the rational counterpart for $\gamma > 1$ ($\gamma < 1$).

5.2.2. *Momentum and Reversal.*

The different impacts of γ on the sentiment price determine whether stock has momentum or reversal. Denote by $r_{pt} = \int_{t-\Delta}^{t} \frac{dP_u + C_u du}{P_u}$ the stock return during $[t-\Delta, t]$. The serial correlation of stock returns at lag h is defined as $ACF(h) = cov(r_{p,t+h\Delta}, r_{pt})/var(r_{pt})$.

Corollary 5.2. *To the order of α, the expected stock returns under the subjective measure and the physical measure are given, respectively, by*

$$\mu_p^e = \alpha\sigma_c S_t + \alpha_0\sigma_p, \quad \mu_p = \frac{\gamma - 1}{\gamma}\alpha\sigma_c S_t + a_2(t), \tag{5.1}$$

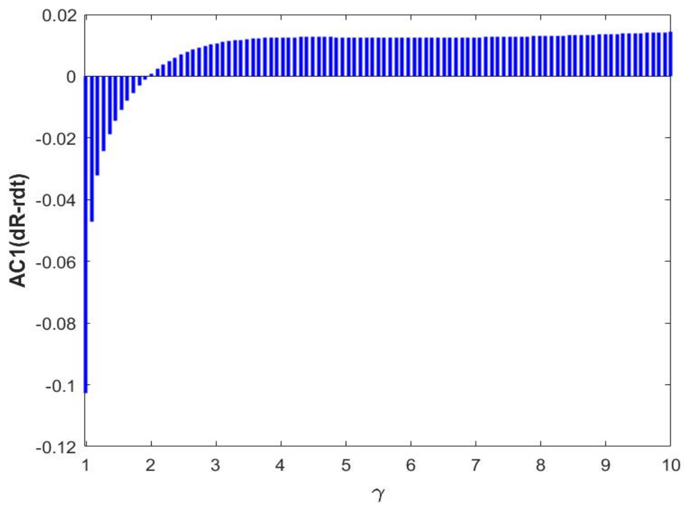

FIGURE 5.2. The first order serial correlations of excess returns for different γ. The results are based on Monte Carlo simulations. Here $\alpha = 0.95$.

where $\sigma_p = \sigma_c[1 - \frac{\gamma-1}{\gamma}\alpha a_1(t)]$, *and both* $a_1(t) \in [0, 1)$ *and* $a_2(t)$ *are deterministic functions of* t *given by* (D.10) *in Appendix D.5. In this case, the serial correlation of gross returns at lag* h *satisfies*

$$ACF(h)\begin{cases} > 0 & if \quad \gamma > 1; \\ = 0 & if \quad \gamma = 1; \\ < 0 & if \quad \gamma < 1, \end{cases} \tag{5.2}$$

for $\alpha > 0$, *and the sign of the serial correlation has the opposite pattern for* $\alpha < 0$. *The absolute value of series correlation decreases with* h.

Corollary 5.2 highlights the impact of γ on momentum and reversal. Indeed, the agent reacts to past returns and then affects prices in two *dimensions*. The first is extrapolation that forms the agent's expectation about future returns, and the second is γ, which determines the degree of response. Small γ leads to a large response that produces overreaction to past returns (sentiment) and subsequent return reversal. Reversal is typically found in the extrapolation literature. However, we show that large γ can also lead to underreaction to past returns and produce momentum, which is different from the extant literature that exploits only the first dimension (i.e., extrapolation) but overlooks the second (i.e., γ).

Corollary 5.3. *The serial correlations of excess returns are positive when both* γ *and* α *are large.*

The mechanism in generating momentum and reversal in gross returns also produces momentum and reversal in excess returns accordingly. However, the short rate depends positively on the sentiment, leading to smaller series correlations of excess returns relative to those of gross returns. As proved in Corollary 5.3 and verified by Fig. 5.2, excess returns exhibit momentum with small bias adjustment (large γ) and reversal with large bias adjustment (small γ).

In our model, return reversal is not caused by either the counteraction of rational investors or overvaluation generated from extrapolation as argued in most extrapolation literature. On the one hand, there is no rational investor who uses the physical measure in our economy. On the other hand, there is no time-series overvaluation in our economy. In fact, the stock price is determined by a full equilibrium, which has taken into account all future effects of a current consumption change, and is correct by definition. Therefore, there is no adjustment in the long-run, and the momentum in both gross returns and excess returns does not reverse in the long-run.

BGJS find that their model can generate momentum by instead putting some delays in the reaction to past prices when extrapolators are forming expectations. We show that our model can generate momentum for large γ even without the assumption of delayed reaction. Our mechanism in generating momentum is also different from that of underreaction proposed in Hong and Stein (1999), where the underreaction is caused by the gradual diffusion of information. However, there is no underreaction to information in our model because price is determined by a full equilibrium where a consumption shock (under the physical measure) is immediately and correctly priced in our economy without any requirements for future adjustments. Our results are related to the findings in Peng and Wang (2019) that the positive feedback trading of mutual funds generates momentum because of its price pressure.

5.3. **Is Extrapolation Destabilizing?**

Proposition 3.2 shows that the equilibrium always exists for finite horizons. So extrapolation in itself does not lead to instability. For an infinite-horizon economy ($T = \infty$), the equilibrium exists under condition (3.16) due to the transversality condition. Therefore, condition (3.16) does not depend on the level of the sentiment S. The transversality condition is affected by extrapolation (α). When the transversality condition is violated, the price explodes (price goes to infinity).

Fig. 5.3 shows that, for an infinite-horizon economy, there are two bounds of γ, between which equilibrium exists. Due to the transversality condition, the economy may not have equilibrium even if there is no extrapolation ($\alpha = 0$), as discussed in Section 2.1.4 for the rational benchmark.

There is a positive feedback for $\gamma < 1$. It may or may not lead to "explosion" in the sense that there is no equilibrium because of extrapolation. When $\underline{\gamma}^r < \gamma \leq \underline{\gamma}$,

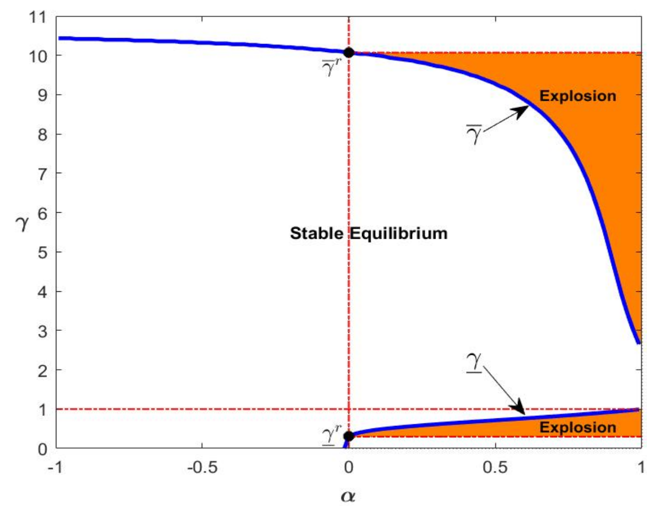

FIGURE 5.3. This figure illustrates the upper boundary ($\overline{\gamma}$) and lower boundary ($\underline{\gamma}$) for γ, between which equilibrium exists. For $\alpha = 0$, $\overline{\gamma}^r$ and $\underline{\gamma}^r$ are the upper and lower boundaries for γ in the rational benchmark. Here $\mu_c = 0.03$ ($> \rho$).

there is explosion caused by extrapolation, as illustrated by the shaded area in Fig. 5.3.[21] When $\underline{\gamma} < \gamma < 1$, there is no explosion and the positive feedback does not destabilize the equilibrium.

There is a negative feedback for $\gamma > 1$. However, interestingly, extrapolation may also destabilize the equilibrium. This happens when $\overline{\gamma}^r < \gamma \leq \overline{\gamma}$ (the shaded area in Fig. 5.3). In this case, the implied discount rate is higher than implied consumption growth rate.

When α increases, the extrapolation effect increases as can be seen from smaller interval of γ for stable equilibrium. The reduced region for stable equilibrium is essentially caused by the time-varying consumption growth resulting from extrapolation rather than caused by the positive (or negative) feedback.[22] This is different from the popular argument in the extrapolation literature that relates the nonexistence of equilibrium to the positive feedback. As shown above, the former is due to violation of the transversality condition while the latter depends on the relative strength of the substitution effect and the income effect.

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[21]We set $\rho < \mu_c$ in Fig. 5.3. In this case, both the upper bound $\overline{\gamma}^r$ and the lower bound $\underline{\gamma}^r$ exist in the rational benchmark, see Proposition 2.3. If we set $\rho > \mu_c$, the lower bound ($\underline{\gamma}^r$) does not exist. The interval of γ in which positive feedback causes explosion becomes $0 < \gamma \leq \underline{\gamma}$.

[22]For example, let's consider a rational Lucas-type model where the expected consumption growth is linear in an Ornstein-Uhlenbeck process that has the same innovation as consumption, and a representative agent has a CRRA utility. The interval of γ for stable equilibrium also reduces when the coefficient of the Ornstein-Uhlenbeck process in the expected consumption growth increases, even if there is no such feedback defined above. The calculations are available upon request.

Proposition 2.3 shows that γ needs to be greater than the level of extrapolation α. Especially, for $\alpha = 1$, the equilibrium exists if and only if $1 \leq \gamma < \sqrt{2\rho}/\sigma_c$. Therefore, small subjective discount rate ρ or large consumption volatility σ_c tends to destabilize the equilibrium, consistent with the rational benchmark (2.22). More importantly, the equilibrium exists only for $\gamma \geq 1$. In this case, most return properties follow the oppositive patterns to BGJS, who choose $\alpha = 1$ in their paper.

Indeed, the nonexistence of equilibrium in our model, which leads to price divergence, is different from the nonexistence of equilibrium in BGJS, which may not lead to price divergence. By clearing only the risky asset market, the nonexistence of equilibrium in BGJS is not due to the transversality condition, but defined as the nonexistence of such solution that stock price is linear in state variables and the short rate is a constant. Such solution may be not consistent with the state price density.

Further, the assumption of a constant short rate in BGJS also reduces the region in which the equilibrium exists. If we instead consider that the short rate is endogenously determined in the equilibrium, then the short rate increases with the sentiment, decreasing extrapolators' demand for the risky asset when the sentiment is high. This tends to prevent their expected utility from approaching infinity, and tends to preserve the equilibrium.

When γ approaches its upper or lower boundary, price-consumption ratio Φ goes to infinity. Although return volatility depends on Φ, it is bounded from both above and below and does not go to infinity as γ approaches the boundary as proved in Proposition 4.1. In addition, the short rate and expected return do not go to infinity either, as shown in Proposition A.1.

In contrast, a contrarian ($\alpha < 0$) plays a stabilizing role and tends to preserve equilibrium. There can be even no lower bound for γ when $\alpha < 0$.

5.4. **Return Predictability.**

The extrapolator thinks that the sentiment predicts future returns and that consumption growth depends on the sentiment. This gives rise to the dependence of price-consumption ratio Φ on the sentiment in (3.15), and further relates Φ to future returns. However, in the rational benchmark, because the consumption growth is IID, Φ does not vary and return is not predictable.

Corollary 5.4. *Future return is negatively predicted by Φ for $\gamma \neq 1$, but is not predictable for $\gamma = 1$.*

Price-consumption ratio is negatively related to the sentiment and the sentiment is positively related to the physical expected return (due to the small sentiment premium) when $\gamma > 1$. In this case, Φ negatively predicts future return. Price-consumption ratio still negatively predicts future return when $\gamma < 1$ because it is positively related to the sentiment, and the sentiment is negatively related to future return (large sentiment premium). The negative predictability of Φ has been documented by Campbell and Shiller (1988) and Fama and French (1988*a*), among others.

However, because most variation of total expected return is due to the short rate, price-consumption ratio Φ forecasts mostly the short rate rather than the risk premium, especially for the case of $\gamma > 1$.

Return predictability is related to excess volatility: the negative relationship between price-consumption ratio and the sentiment leads to the excess volatility. However, excess volatility is not always consistent with the predictability of returns. For $\gamma > 1$, Φ negatively predicts future returns, but return volatility is lower than consumption volatility, showing that return predictability can be easily generated by time-variation of the risk premium that may not necessarily lead to excess volatility. The positive predictability of returns still exists for contrarian expectations ($\alpha < 0$) as shown in Section 5.6. This differs from the argument that excess volatility of prices is exactly the same phenomenon as the predictability of returns, e.g., Cochrane (2017).

5.5. **Correspondence with Rational Benchmark.**

When the sentiment is around its unconditional mean, Proposition 3.1 shows that Φ decreases (increases) with extrapolation measured by α when $\gamma > 1$ ($\gamma < 1$).[23] In this case, extrapolation leads to "undervaluation" ("overvaluation") *relative to the rational benchmark* for $\gamma > 1$ ($\gamma < 1$) in the sense that the price level decreases (increases) with extrapolation α given a consumption process.

However, there is no rationale for the relative undervaluation/overvaluation to the rational benchmark to be corrected in the long-run in our model. As discussed before, there is no undervaluation/overvaluation *in time series* in our economy. Therefore, Φ under extrapolation does not converge to its rational benchmark level in the long-run, even though the agent is assumed to have a correct unconditional expectation about return. It further implies that it is inappropriate to attribute price features produced by extrapolation to the overvaluation (either in time series or relative to the rational benchmark) as did in the literature. Numerical simulations in Table 4.1 further confirm that the unconditional distributions of different variables under extrapolation do not converge to the rational benchmark levels.

5.6. **Contrarian Expectations.**

When $\alpha < 0$, the agent has contrarian expectations. Table 5.1 summarizes the

[23]Proposition 3.1 shows that $\Phi = \int_t^T e^{A(u)S+B(u)}du$. For $\gamma > 1$ ($\gamma < 1$), (D.1) shows that both A and B decrease (increase) with α. Here we consider a fixed α_0 so that a change in α affects only extrapolation. In addition, large κ decreases (increases) price level for $\gamma > 1$ ($\gamma < 1$).

effects of the sentiment on equilibrium under both extrapolative expectation ($\alpha > 0$) and contrarian expectation ($\alpha < 0$).

TABLE 5.1. Extrapolative expectation versus contrarian expectation.

Panel	$\alpha > 0$ & $\gamma > 1$	$\alpha > 0$ & $\gamma < 1$	$\alpha < 0$ & $\gamma > 1$	$\alpha < 0$ & $\gamma < 1$
A	$\partial\mu_p/\partial S > 0$ $\partial\Phi/\partial S < 0$ $\sigma_c > \sigma_p$	$\partial\mu_p/\partial S < 0$ $\partial\Phi/\partial S > 0$ $\sigma_c < \sigma_p$	$\partial\mu_p/\partial S < 0$ $\partial\Phi/\partial S > 0$ $\sigma_c < \sigma_p$	$\partial\mu_p/\partial S > 0$ $\partial\Phi/\partial S < 0$ $\sigma_c > \sigma_p$
B	$\partial r_f/\partial S > 0$ $\partial\eta/\partial S < 0$ $\partial(\mu_p - r_f)/\partial S < 0$ destabilizing	$\partial r_f/\partial S > 0$ $\partial\eta/\partial S < 0$ $\partial(\mu_p - r_f)/\partial S < 0$ destabilizing	$\partial r_f/\partial S < 0$ $\partial\eta/\partial S > 0$ $\partial(\mu_p - r_f)/\partial S > 0$ stabilizing	$\partial r_f/\partial S < 0$ $\partial\eta/\partial S > 0$ $\partial(\mu_p - r_f)/\partial S > 0$ stabilizing
C	negative feedback	positive feedback	negative feedback	positive feedback
D	$\partial\mu_p/\partial\Phi < 0$	$\partial\mu_p/\partial\Phi < 0$	$\partial\mu_p/\partial\Phi < 0$	$\partial\mu_p/\partial\Phi < 0$

The results on contrarian expectation become straightforward given the analysis on extrapolative expectation. First, some price properties, including momentum/reversal (the sign of $\partial\mu_p/\partial S$), excess volatility, and the dependence of Φ on the sentiment (the sign of $\partial\Phi/\partial S$) depends on both α and γ as shown in Panel A. Changing the sign of α or the sign of $\gamma - 1$ can qualitatively change these properties.

Second, Panel B shows that α significantly affects the following properties: the effect of the sentiment on the short rate (the sign of $\partial r_f/\partial S$), sentiment premium (the sign of $\partial\eta/\partial S$), and physical risk premium (the sign of $\partial(\mu_p - r_f)/\partial S$), and the transversality condition (a contrarian tends to preserve the equilibrium). They are qualitatively different when the sign of α is changed. But changing the sign of $\gamma - 1$ does not qualitatively affect these properties.

Third, Panel C shows that feedback is determined by γ. There is negative feedback for $\gamma > 1$ and positive feedback for $\gamma < 1$, regardless of the sign of α.

Fourth, Panel D shows that price-consumption ratio Φ negatively predicts future return (i.e., $\partial\mu_p/\partial\Phi < 0$) for both $\gamma > 1$ and $\gamma < 1$ and for both $\alpha > 0$ and $\alpha < 0$.

5.7. **CARA Utility.**

Many papers on extrapolation use a setup with normally distributed consumption, an exogenous short rate, and CARA utility, e.g., De Long, Shleifer, Summers and Waldmann (1990), Cutler, Poterba and Summers (1990), Hong and Stein (1999), and BGJS, among others. We provide the rational benchmark for CARA preference in this subsection for comparison. Assume consumption growth is IID: $dC_t = \mu_c dt +$

$\sigma_c dZ_t$, the short rate r_f is exogenously given, and utility is given by

$$\mathbb{E}_0\Big[-\int_0^\infty \frac{e^{-\rho t - AC_t}}{A} dt\Big],$$

where A is the absolute risk aversion coefficient. BGJS show that the price of the risky asset is given by

$$P_t = \frac{\mu_c}{r_f^2} - \frac{A\sigma_c^2 Q}{r_f^2} + \frac{C_t}{r_f}, \tag{5.3}$$

where Q is a constant per-capita supply of the risky asset.

The riskless asset is in perfectly elastic supply and the risky asset has unlimited liability. This setup cannot define percentage returns, while the empirical literature studies percentage returns, e.g., Greenwood and Shleifer (2014). In contrast, under our setup, the stock price is positive and percentage returns are well defined, allowing for a direct correspondence with the empirical studies. For example, extrapolation in our paper is defined over returns, as in the survey data, instead of over price changes as in many existing models.[24]

Price-consumption ratio is measured by $P - C/r_f$ in BGJS, and (5.3) shows that it always increases with consumption growth rate μ_c, like the case of $\gamma < 1$ in our rational benchmark with CRRA preference. However, we also show that it decreases with μ_c in our benchmark for $\gamma > 1$. This difference between the two preferences will be preserved under extrapolation. Under extrapolation, how historical returns (the sentiment) affect future return and price-consumption ratio through the expected consumption growth depends on γ. In contrast, $P - C/r_f$ in BGJS depends positively on the consumption growth rate under the subjective measure, as in their rational benchmark. As a result, many return properties are similar to those in our model with $\gamma < 1$.

More importantly, the assumption of a non-clearing good market in these models has a significant effect on price behavior. In fact, the effect of extrapolation on the short rate is absorbed by the difference between optimal consumption and aggregate consumption.[25] In our setup, we show that extrapolation leads to large time variations of the short rate and hence the short rate is not a constant under

[24]The setup with normally distributed consumption and CARA utility also ignores the wealth effect. However, Giglio et al. (2019) find that the sensitivity of portfolios to beliefs is significantly affected by investor wealth.

[25]The short rate and return volatility are constants in BGJS's model with CARA utility, due to non-clearing markets. There are three markets in the economy studied by BGJS, namely, the markets for the risky asset, for the riskless asset, and for the consumption good. The risky asset market is assumed to clear, while the remaining two markets do not clear. If we set the rational-benchmark short rate in (5.3) as $r_f = \rho + A\mu_c - \frac{A^2\sigma_c^2}{2}$, as determined by the state price density $e^{-\rho t - AC_t}$, then all markets clear. In this special case with rational agents only, the short rate happens to be a constant as assumed in BGJS. But their results for the general case in the presence of extrapolation cannot be consistent with the equilibrium determined by the state price density, which implies that the short rate depends on the sentiment and cannot be a constant anymore. We refer readers to

extrapolation. In fact, many return features are mainly driven by the short rate, rather than the risk premium. However, it is difficult to differentiate short rate and risk premium in most extrapolation models with an exogenous short rate.

6. Conclusion

We find that extrapolative expectation, in an otherwise standard asset pricing model, has large effects on the short rate and stock prices. However, extrapolation actually exacerbates asset pricing puzzles; it leads to volatile interest rates, deficient volatility (lower volatility than the rational counterpart), and low equity premium (even lower than the rational counterpart).

We show that extrapolation can generate a negative feedback and momentum (when the risk aversion is greater than one), in addition to a positive feedback and reversal (when the risk aversion is less than one). Extrapolation leads to overvaluation relative to the rational benchmark and the overvaluation is never corrected. We also find that extrapolation changes the transversality condition, which is a destabilizing feature of the extrapolation.

Our analytical results suggest that non-rational expectation alone cannot resolve the asset pricing puzzles.

Dudley and Montmarquette (1984), Bénassy (1993, 2006), and Loewenstein and Willard (2006) for the effects of non-clearing markets.

Appendix A. A Limiting Case with Infinite Horizon

For an infinite-horizon economy ($T \to \infty$), the HJB equation (3.1) still holds except that the boundary condition becomes $\mathbb{E}_t^e[J_T] \to 0$ as $T \to \infty$. The value function J has the form (Liu, 2007):

$$J(S, W, t) = e^{-\rho t} \frac{W^{1-\gamma}}{1-\gamma} \big[\Phi(S)\big]^{\gamma},$$

where Φ is not an explicit function of time t anymore. Therefore, equations (3.3)-(2.12) still hold, but PDE (3.9) becomes an ODE

$$\frac{\kappa^2}{2} \frac{\partial^2 \Phi}{\partial S^2} + \frac{\kappa}{\gamma} \big[(\alpha - \gamma)S + \alpha_0\big] \frac{\partial \Phi}{\partial S} - \left[\left(1 - \frac{1}{\gamma}\right)\sigma_c(\alpha_0 + \alpha S) - \frac{(\gamma-1)\sigma_c^2}{2} + \frac{\rho}{\gamma}\right]\Phi + 1 = 0.$$

We define $\hat{\Phi}(S, t)$ such that

$$\Phi = \int_0^\infty \hat{\Phi}(S, u) du. \tag{A.1}$$

Then $\hat{\Phi}$ satisfies

$$-\frac{\partial \hat{\Phi}}{\partial t} + \frac{\kappa^2}{2} \frac{\partial^2 \hat{\Phi}}{\partial S^2} + \frac{\kappa}{\gamma} \big[(\alpha - \gamma)S + \alpha_0\big] \frac{\partial \hat{\Phi}}{\partial S} - \left[\left(1 - \frac{1}{\gamma}\right)\sigma_c(\alpha_0 + \alpha S) - \frac{(\gamma-1)\sigma_c^2}{2} + \frac{\rho}{\gamma}\right]\hat{\Phi} = 0, \tag{A.2}$$

with $\hat{\Phi}(S, 0) = 1$ and $\hat{\Phi}(S, \infty) = 0$.

It can be verified that (3.16) is the condition for the existence of equilibrium. Under this condition, the solution to (A.2) is given by

$$\hat{\Phi}(S, t) = e^{a(t)S + b(t)}, \tag{A.3}$$

where, for $\alpha \neq \gamma$, a and b are given by

$$\begin{aligned}
a(t) &= -(\gamma - 1)\alpha\sigma_c \frac{1 - e^{-\kappa(\gamma-\alpha)t/\gamma}}{\kappa(\gamma - \alpha)}, \\
b(t) &= c_1\Big[1 - e^{-\kappa(\gamma-\alpha)t/\gamma}\Big] + c_2\Big[1 - e^{-2\kappa(\gamma-\alpha)t/\gamma}\Big] - \Delta t, \\
c_1 &= -\frac{\gamma(\gamma-1)\alpha\sigma_c}{\kappa(\gamma-\alpha)^2}\left[\frac{(\gamma-1)\alpha\sigma_c}{\gamma - \alpha} - \frac{\alpha_0}{\gamma}\right], \qquad c_2 = \frac{\gamma(\gamma-1)^2\alpha^2\sigma_c^2}{4\kappa(\gamma-\alpha)^3}, \\
\Delta &= \frac{\rho}{\gamma} + \frac{(\gamma-1)\alpha_0\sigma_c}{\gamma - \alpha} - \frac{\gamma(\gamma-1)(\alpha^2 - 2\alpha + \gamma)\sigma_c^2}{2(\gamma-\alpha)^2};
\end{aligned} \tag{A.4}$$

and, for $\alpha = \gamma$,

$$\begin{aligned}
a(t) &= \sigma_c(1-\gamma)t, \\
b(t) &= \frac{\kappa^2\sigma_c^2(1-\gamma)^2}{6}t^3 + \frac{\kappa\alpha_0\sigma_c(1-\gamma)}{2\gamma}t^2 + \left(\frac{1-\gamma}{\gamma}\sigma_c\alpha_0 - \frac{\rho}{\gamma} - \frac{1-\gamma}{2}\sigma_c^2\right)t.
\end{aligned} \tag{A.5}$$

Equations (A.1) and (A.3) show that

$$\frac{\partial \Phi}{\partial S} = \int_0^\infty a(u) e^{a(u)S+b(u)} du \begin{cases} < 0, & \text{if} \quad \gamma > 1, \\ > 0, & \text{if} \quad \gamma < 1. \end{cases} \tag{A.6}$$

Proposition A.1 summarizes the equilibrium, which has the same form as that in Proposition 3.3 except that the integral in Φ is from 0 to infinity.

Proposition A.1. *Under the physical measure $\mathbb{P}$, the stock return satisfies*

$$dR_t = \mu_p dt + \sigma_p dZ_t, \quad \mu_p = \sigma_p(\psi_0 + \psi S), \quad \sigma_p = \sigma_c + \kappa \frac{\int_0^\infty a(u) e^{a(u)S+b(u)} du}{\int_0^\infty e^{a(u)S+b(u)} du},$$

where

$$\psi(\sigma_p) = \alpha\Big(1 - \frac{\sigma_p}{\gamma\sigma_c}\Big), \quad \psi_0(\sigma_p) = \alpha_0\Big(1 - \frac{\sigma_p}{\gamma\sigma_c}\Big) + \frac{\mu^r}{\gamma\sigma_c} + \sigma_p - \sigma_c,$$

and a, b are given by (A.4)*. The short rate is given by*

$$r_f = \sigma_p(\alpha_0 + \alpha S - \gamma\sigma_c),$$

Φ is given by

$$\Phi = \int_0^\infty e^{a(u)S+b(u)} du,$$

and sentiment follows

$$dS_t = \kappa\big[\psi_0 - (1-\psi)S_t\big]dt + \kappa dZ_t. \tag{A.7}$$

A.1. **Large $S \to \pm\infty$ Limits.**

In this subsection, we present the equilibrium for large $S \to \pm\infty$ limits. The limiting cases provide more insights into and clear understanding of the equilibrium: we will see that all variables in equilibrium are monotonic functions of S, a very nice property. Especially, the short rate, market price of sentiment, expected returns under both measures are linear in S. The results are consistent with those to the leading order of $1/\kappa$ (or κ) studied in Appendix D.4.

Suppose that $\gamma < 1$ and $S \to +\infty$. In this case, $a(t)$ increases monotonically from 0 to $\frac{(1-\gamma)\alpha\sigma_c}{\kappa(\gamma-\alpha)}$, therefore, when $S \to +\infty$,

$$e^{a(t)S+b(t)} \to e^{\frac{(1-\gamma)\alpha\sigma_c}{\kappa(\gamma-\alpha)}S} e^{c_1+c_2} e^{-\frac{(1-\gamma)\alpha\sigma_c}{\kappa(\gamma-\alpha)}S e^{-\frac{\kappa(\gamma-\alpha)}{\gamma}t} - \Delta t}.$$

So

$$\Phi \to e^{\frac{(1-\gamma)\alpha\sigma_c}{\kappa(\gamma-\alpha)}S} e^{c_1+c_2} \int_0^\infty e^{-\frac{(1-\gamma)\alpha\sigma_c}{\kappa(\gamma-\alpha)}S e^{-\frac{\kappa(\gamma-\alpha)}{\gamma}t} - \Delta t} dt$$

which equals to

$$e^{\frac{(1-\gamma)\alpha\sigma_c}{\kappa(\gamma-\alpha)}S} e^{c_1+c_2} \left(\frac{(1-\gamma)\alpha\sigma_c}{\kappa(\gamma-\alpha)}S\right)^{-\frac{\Delta\gamma}{\kappa(\gamma-\alpha)}} \int_0^{\frac{(1-\gamma)\sigma\sigma_c}{\kappa(\gamma-\alpha)}S} e^{-u} u^{\frac{\Delta\gamma}{\kappa(\gamma-\alpha)}} \frac{du}{u},$$

due to a change of variable $u = \frac{(1-\gamma)\alpha\sigma_c}{\kappa(\gamma-\alpha)} S e^{-\frac{\kappa(\gamma-\alpha)}{\gamma}t}$. Note that the integral is finite with S goes to infinity

$$\int_0^{\frac{(1-\gamma)\sigma\sigma_c}{\kappa(\gamma-\alpha)}S} e^{-u} u^{\frac{\Delta\gamma}{\kappa(\gamma-\alpha)}} \frac{du}{u} \to \int_0^{+\infty} e^{-u} u^{\frac{\Delta\gamma}{\kappa(\gamma-\alpha)}} \frac{du}{u} = \Gamma(\frac{\Delta\gamma}{\kappa(\gamma-\alpha)}),$$

where $\Gamma(\cdot)$ is the Gamma function. Finally, the asymptotic behavior of Φ when $S \to +\infty$ (for $\gamma < 1$) is

$$\Phi \to e^{c_1+c_2}\Gamma(\frac{\Delta\gamma}{\kappa(\gamma-\alpha)}) e^{\frac{(1-\gamma)\alpha\sigma_c}{\kappa(\gamma-\alpha)}S} \left(\frac{(1-\gamma)\alpha\sigma_c}{\kappa(\gamma-\alpha)}S\right)^{-\frac{\Delta\gamma}{\kappa(\gamma-\alpha)}}.$$

Therefore,

$$\sigma_p \to \sigma_c + (\Phi'/\Phi)\kappa = \sigma_c + \frac{(1-\gamma)\alpha\sigma_c}{(\gamma-\alpha)} = \frac{(1-\alpha)\gamma}{\gamma-\alpha}\sigma_c. \tag{A.8}$$

For $\gamma > 1$ and $S \to -\infty$, we can use the same method and get the same result (A.8). Note that (A.8) is independent of S and is the same as the result to the leading order of $1/\kappa$ to be studied in Appendix D.4.

Now suppose that $\gamma < 1$ and $S \to -\infty$. In this case, only $e^{a(t)S}$ has highest value at $t = 0$. So $\Phi \approx \int_0^\infty e^{a(t)S+b(0)}dt = \int_0^\infty e^{a(t)S}dt$ and $a(t) \approx \frac{(1-\gamma)\alpha\sigma_c}{\gamma}t$. Therefore,

$$\Phi \approx \int_0^\infty e^{\frac{(1-\gamma)\alpha\sigma_c}{\gamma}tS}dt = -\frac{\gamma}{(1-\gamma)\alpha\sigma_c S}.$$

Note that this integral is positive because $S \to -\infty$. Therefore,

$$\sigma_p = \sigma_c + \kappa\Phi'/\Phi = \sigma_c - \frac{\kappa}{S},$$

which tends to σ_c as $S \to -\infty$. Note that this is independent of S and is same as the $\kappa \to 0$ limit to be studied in Appendix D.4. For $\gamma > 1$ and $S \to +\infty$, we can use the same method and get the same result.

Now we study the asymptotic behavior of other quantities. When $\gamma > 1$ and $S \to +\infty$ or when $\gamma < 1$ and $S \to -\infty$, the riskfree rate r_f is

$$r_f = \sigma_p(\alpha_0 + \alpha S - \gamma\sigma_c) \to \sigma_c(\alpha_0 + \alpha S - \gamma\sigma_c),$$

and the market price of sentiment η is

$$\eta = -\frac{1}{\sigma_c}\frac{\sigma_p(\alpha_0+\alpha S) - \mu^r}{\gamma} + (\sigma_p - \sigma_c) \to -\frac{1}{\sigma_c}\frac{\sigma_c(\alpha_0+\alpha S) - \mu^r}{\gamma}.$$

The subjective expected return is

$$\mu_p^e = \sigma_p(\alpha_0 + \alpha S) \to \sigma_c(\alpha_0 + \alpha S).$$

The objective expected return is

$$\mu_p = \sigma_p(\alpha_0+\alpha S)+\sigma_p\eta \to \sigma_c(\alpha_0+\alpha S) - \frac{\sigma_c(\alpha_0+\alpha S) - \mu^r}{\gamma} = (1-\frac{1}{\gamma})\sigma_c(\alpha_0+\alpha S) + \frac{\mu^r}{\gamma}.$$

The subjective risk premium is

$$\mu_p^e - r_f = \gamma\sigma_p\sigma_c \to \gamma\sigma_c^2.$$

The objective risk premium is

$$\mu_p - r_f = \mu_p^e - r_f + \sigma_p\eta \to \gamma\sigma_c^2 - \frac{\sigma_c(\alpha_0 + \alpha S) - \mu^r}{\gamma} = -\frac{1}{\gamma}\sigma_c(\alpha_0 + \alpha S) + \frac{\mu^r}{\gamma} + \gamma\sigma_c^2.$$

When $\gamma > 1$ and $S \to +\infty$ or when $\gamma < 1$ and $S \to -\infty$, the riskfree rate r_f is

$$r_f = \sigma_p(\alpha_0 + \alpha S - \gamma\sigma_c) \to \frac{(1-\alpha)\gamma}{\gamma - \alpha}\sigma_c(\alpha_0 + \alpha S - \gamma\sigma_c).$$

The market price of sentiment η is

$$\eta = -\frac{1}{\sigma_c}\frac{\sigma_p(\alpha_0 + \alpha S) - \mu^r}{\gamma} + (\sigma_p - \sigma_c) \to -\frac{1}{\sigma_c}\frac{\frac{(1-\alpha)\gamma}{\gamma-\alpha}\sigma_c(\alpha_0 + \alpha S) - \mu^r}{\gamma} + (\frac{(1-\alpha)\gamma}{\gamma - \alpha} - 1)\sigma_c.$$

The subjective expected return is

$$\mu_p^e = \sigma_p(\alpha_0 + \alpha S) \to \frac{(1-\alpha)\gamma}{\gamma - \alpha}\sigma_c(\alpha_0 + \alpha S).$$

The objective expected return is

$$\mu_p = \sigma_p(\alpha_0 + \alpha S) + \sigma_p\eta \to \sigma_c(\alpha_0 + \alpha S) - \frac{\sigma_c(\alpha_0 + \alpha S) - \mu^r}{\gamma} = (1 - \frac{1}{\gamma})\sigma_c(\alpha_0 + \alpha S) + \frac{\mu^r}{\gamma}.$$

The subjective risk premium is

$$\mu_p^e - r_f = \gamma\sigma_p\sigma_c \to \frac{(1-\alpha)\gamma}{\gamma - \alpha}\gamma\sigma_c^2.$$

The objective risk premium is

$$\mu_p - r_f = \mu_p^e - r_f + \sigma_p\eta \to \gamma\sigma_c^2 - \frac{\sigma_c(\alpha_0 + \alpha S) - \mu^r}{\gamma} = -\frac{1}{\gamma}\sigma_c(\alpha_0 + \alpha S) + \frac{\mu^r}{\gamma} + \gamma\sigma_c^2.$$

Appendix B. An Example of Pure Change of Expectation

In this subsection, we provide an example of pure change of expectation to illustrate the impact of the deviations from rational expectations. Here we use "pure" to mean that the expectation bias does not change return volatility.

Suppose that the agent has an incorrect but constant belief about stock return

$$dR_t = \hat{\mu}_p dt + \sigma_p d\hat{Z}_t,$$

where $\hat{\mu}_p$ is a constant. The aggregate consumption follows (2.1) under the physical measure. In this case, the short rate, the risk premiums under the subjective measure and the physical measure are given, respectively, by

$$r_f = \hat{\mu}_p - \gamma\sigma_c^2, \qquad \hat{\mu}_p - r_f = \gamma\sigma_c^2, \qquad \mu_p - r_f = \gamma\sigma_c^2 - \frac{1}{\gamma}(\hat{\mu}_p - \mu^r),$$

where μ^r is the expected return in the rational benchmark given by (2.19). The return volatility and Φ are given, respectively, by

$$\sigma_p = \sigma_c, \quad \Phi = \frac{1}{\rho + (\gamma - 1)(\hat{\mu}_c - \gamma\sigma_c^2/2)}, \quad \text{where} \quad \hat{\mu}_c = (\hat{\mu}_p - \rho)/\gamma + (\gamma - 1)\sigma_c^2/2.$$

There are several observations that will help understand the effect of extrapolation on equilibrium in our full model. First, the subjective risk premium is not affected by the expectation bias $\hat{\mu}_p$, the physical risk premium (the risk premium under the physical measure) depends negatively on the bias, and the short rate depends positively on the bias. For $\gamma > 1$, a typical range for the CRRA coefficient used in the literature, biased expectation has more significant effect on the short rate than on both the subjective risk premium and physical risk premium.

Second, the short rate equals the difference of the subjective expected return and the subjective risk premium. The subjective expected return is (exogenously) specified by the biased expectation and hence does not depend on risk aversion γ, and γ impacts the short rate only through the subjective risk premium. Therefore, the (constant) biased expectation cannot resolve the riskless rate puzzle, even using recursive utility in which the effect of EIS is limited by the specification of subjective expected return.

Third, we consider $\hat{\mu}_c > \mu_c$. Relative to the rational benchmark, the (optimistic) bias decreases price (Φ), and accordingly, increases subjective expected return. Physical expected return is also higher than its rational-benchmark level for $\gamma > 1$, but is lower for $\gamma < 1$ by noting that

$$\mu_p = \Big(1 - \frac{1}{\gamma}\Big)\hat{\mu}_p + \frac{1}{\gamma}\mu^r.$$

Note that the constant biased expectation has no effect on return volatility, leading to a *pure change of expectation (drift).* For state-dependent change of expectation (such as the extrapolative expectation studied in our paper), high/low sentiment leads to different $\hat{\mu}_p$, so it also simultaneously changes return volatility.

Appendix C. Extrapolation of Fundamentals

The expectation formation literature also studies extrapolation of fundamentals (e.g., Fuster, Hebert and Laibson, 2011; Choi and Mertens, 2013, and Hirshleifer, Li and Yu, 2015, among others), in addition to extrapolation of returns (as modelled in our paper). In this section, we compare both types of extrapolations.

Alternative to extrapolation of returns in (2.4)-(2.5), we assume that the representative CRRA agent extrapolates fundamentals:

$$\mathbb{E}_t^e\Big[\frac{dC_t}{\sigma_c C_t}\Big] = (\alpha_0 + \alpha S_t)dt, \qquad S_t = \int_{-\infty}^t \kappa e^{-\kappa(t-u)}\frac{dC_u}{\sigma_c C_u}, \tag{C.1}$$

while the consumption process follows (2.1) under the physical measure. It follows from (C.1) that the subjective expected consumption growth follows an Ornstein-Uhlenbeck process. A routine calculation leads to the following equilibrium.

Proposition C.1. *When the agent extrapolates fundamentals, the equilibrium short rate, subjective risk premium, market price of sentiment, return volatility, and price-consumption ratio have the same forms as those in our model with extrapolation of returns:*

$$r_f = \rho + \gamma\mu_c^e - \frac{\gamma(1+\gamma)\sigma_c^2}{2}, \qquad \mu_p^e - r_f = \gamma\sigma_c\sigma_p, \qquad \eta = \frac{1}{\sigma_c}(\mu_c - \mu_c^e),$$
$$\sigma_p = \sigma_c + \kappa\frac{\int_0^\infty \hat{a}(u)e^{\hat{a}(u)S_t+\hat{b}(u)}du}{\int_0^\infty e^{\hat{a}(u)S_t+\hat{b}(u)}du}, \qquad \Phi = \int_0^\infty e^{\hat{a}(u)S_t+\hat{b}(u)}du, \tag{C.2}$$

where the subjective consumption growth rate is given by

$$\mu_c^e = \sigma_c(\alpha_0 + \alpha S), \tag{C.3}$$

and $\hat{a}$ and $\hat{b}$ are deterministic functions governed by

$$\hat{a}(u) = (1-\gamma)\alpha\sigma_c\frac{1-e^{-\kappa(1-\alpha)u}}{\kappa(1-\alpha)},$$
$$\hat{b}(u) = \frac{\alpha^2\sigma_c^2(1-\gamma)^2}{2(1-\alpha)^2}\frac{1-e^{-2\kappa(1-\alpha)u}}{2\kappa(1-\alpha)} - \alpha\left[\frac{\sigma_c^2(1-\gamma)^2}{(1-\alpha)^2} + \frac{\alpha_0\sigma_c(1-\gamma)}{1-\alpha}\right]\frac{1-e^{-\kappa(1-\alpha)u}}{\kappa(1-\alpha)} - \hat{\Delta}u,$$
$$\hat{\Delta} = -\frac{\alpha^2-2\alpha+2}{2(1-\alpha)^2}\sigma_c^2(1-\gamma)^2 - \frac{\alpha_0\sigma_c(1-\gamma)}{1-\alpha} + \frac{\gamma(1-\gamma)\sigma_c^2}{2} + \rho. \tag{C.4}$$

Proposition C.1 shows that all the formulas in (C.2) have exactly the same forms as those in our model with extrapolation of returns. So the only difference between extrapolation of returns and extrapolation of fundamentals lies in the subjective consumption growth and the coefficients $\hat{a}$ and $\hat{b}$. In our model with extrapolation of returns, a, which corresponds to $\hat{a}$ in the model of fundamental extrapolation, is given by $a(u) = (1-\gamma)\alpha\sigma_c\frac{1-e^{-\kappa(1-\alpha/\gamma)u}}{\kappa(\gamma-\alpha)}$, on which parameters have the same qualitative impact as in (C.4). In addition, stronger extrapolation in this case also leads to stricter transversality condition, as in our model of return extrapolation.

In our model with extrapolation of returns, the subjective consumption growth is given by $\mu_c^e = \sigma_p\frac{\alpha_0+\alpha S}{\gamma} - \frac{\rho}{\gamma} - \sigma_c\sigma_p + \frac{(\gamma+1)\sigma_c^2}{2}$. By comparing with (C.3), S has a larger impact when extrapolating returns than extrapolating fundamentals for small γ, but the reverse occurs for large γ. Indeed, as discussed in Section 2.1.3, the substitution effect impacts the market price of sentiment when the biased belief is defined over return (such as the return extrapolation) but will not impact the market price of sentiment when the biased belief is defined over consumption (such as the fundamentals extrapolation). As a result, small γ amplifies the effect of S when the biased belief is defined over return.

Because excess volatility and return predictability are caused by time-varying consumption growth in consumption-based models, our conclusions about the short rate, risk premium, and return volatility still hold here, unless we add other inputs that significantly increase return volatility to amplify the impact of the sentiment.

Overall, we show that extrapolation of returns and extrapolation of fundamentals qualitatively have similar impact on equilibrium. However, extrapolation of fundamentals is much easier to study.

Appendix D. Proofs

D.1. **The solution of** (3.11)**.**

For $\alpha \neq \gamma$, the solution of (3.11) is given by

$$\begin{aligned} A(t) =& -(\gamma-1)\alpha\sigma_c \frac{1-e^{-\kappa(\gamma-\alpha)(T-t)/\gamma}}{\kappa(\gamma-\alpha)}, \\ B(t) =& c_1\Big[1-e^{-\kappa(\gamma-\alpha)(T-t)/\gamma}\Big] + c_2\Big[1-e^{-2\kappa(\gamma-\alpha)(T-t)/\gamma}\Big] - \Delta(T-t), \end{aligned} \tag{D.1}$$

where c_1, c_2 and Δ are given by (A.4). For $\alpha = \gamma$, the solution of (3.11) is given by

$$\begin{aligned} A(t) =& \sigma_c(1-\gamma)(T-t), \\ B(t) =& \frac{\kappa^2\sigma_c^2(1-\gamma)^2}{6}(T-t)^3 + \frac{\kappa\alpha_0\sigma_c(1-\gamma)}{2\gamma}(T-t)^2 + \Big(\frac{1-\gamma}{\gamma}\sigma_c\alpha_0 - \frac{\rho}{\gamma} - \frac{1-\gamma}{2}\sigma_c^2\Big)(T-t). \end{aligned}$$

Unless specified otherwise, we focus on the general case $\alpha \neq \gamma$ in most analyses of the paper. Our main conclusions also hold for the case $\alpha = \gamma$.

Equation (D.1) shows that, for $\gamma > 1$, coefficient $A(t)$ in (3.15) is negative and bounded:

$$-\frac{(\gamma-1)\alpha\sigma_c}{(\gamma-\alpha)\kappa} < A(t) \leq 0, \tag{D.2}$$

For $\gamma < 1$, $A(t)$ becomes positive, and satisfies

$$0 \leq A(t) \leq A(0), \quad \text{where} \quad A(0) = (1-\gamma)\alpha\sigma_c \frac{1-e^{-\kappa(\gamma-\alpha)/\gamma T}}{\kappa(\gamma-\alpha)}. \tag{D.3}$$

$A(0)$ is bounded for $\gamma \in (\alpha, 1)$; while $A(0)$ can be unbounded for $\gamma \leq \alpha$ given that T can be unbounded. This will affect the existence of equilibrium for an infinite-horizon economy, as detailed in Section 5.

D.2. **Proof of Proposition 3.2.**

The equilibrium does not exist for $\gamma = 0$. In this case, state price density become $e^{-\rho t}$, and hence both risk-free rate and expected stock return are constant:

$$\mu_p = r_f = \rho, \tag{D.4}$$

implying that there is no equilibrium if the agent believes a time-varying expected return, such as (2.4). In fact, S_t is a weighted average of historical returns and cannot be a constant, so (D.4) cannot hold. In addition, this can be also proved using (D.1), where $A(t)$, $B(t)$ and Δ approaches infinity when $\gamma = 0$.

Now we prove that, when $T = \infty$, (3.16) is the necessary and sufficient condition for the existence of equilibrium.

On the one hand, if the equilibrium exists, then Φ governed by (A.1) and (A.3) has to be finite. We first look at the case $\alpha \neq \gamma$. To guarantee $\Phi < \infty$, (A.4) implies that $\gamma > \alpha$; otherwise, $b(t) \to \infty$, $|a(t)| \to \infty$ and hence $\Phi \to \infty$ as $t \to \infty$. When $\gamma > \alpha$, $a(t)$ and the first two terms of $b(t)$ in (A.4) satisfy

$$a(t) \to \frac{(1-\gamma)\alpha\sigma_c}{\kappa(\gamma-\alpha)}, \quad c_1\Big[1 - e^{-\kappa(\gamma-\alpha)t/\gamma}\Big] \to c_1, \quad c_2\Big[1 - e^{-2\kappa(\gamma-\alpha)t/\gamma}\Big] \to c_2, \quad \text{when } t \to \infty, \tag{D.5}$$

and

$$|a(t)| < \frac{|\gamma-1|\alpha\sigma_c}{\kappa(\gamma-\alpha)} \equiv \bar{a}, \quad \Big|c_1\big[1 - e^{-\kappa(\gamma-\alpha)t/\gamma}\big]\Big| < |c_1|, \quad \Big|c_2\big[1 - e^{-2\kappa(\gamma-\alpha)t/\gamma}\big]\Big| < c_2.$$

If $\Delta \leq 0$, then (D.5) implies that $e^{a(u)S+b(u)} > 0$ is bounded from below for any u; thus the integral (A.1) is infinite. So Δ should be positive. We now look at the case $\alpha = \gamma$. To guarantee $\Phi < \infty$, (A.5) implies that $\alpha = \gamma = 1$. We prove the necessity.

On the other hand, if condition (3.16) is satisfied, then we have

$$0 < \Phi \leq \int_0^\infty e^{|a(u)S+b(u)|} du < \int_0^\infty e^{\bar{a}S+|c_1|+c_2-\Delta u} du = \frac{1}{\Delta} e^{\bar{a}S+|c_1|+c_2},$$

showing that Φ is finite and positive. So (3.16) is also a sufficient condition for the existence of equilibrium.

In sum, we prove that (3.16) is the necessary and sufficient condition for the existence of equilibrium. This can also be proved by studying $T \to \infty$ in (D.1). It is easy to verify that condition (3.16) reduces to (2.22) when $\alpha = 0$. When $\alpha = 1$, $\Delta > 0$ is equivalent to $\gamma^2 < 2\rho/\sigma_c^2$, which, together with $\gamma > \alpha$, leads to (3.17).

Furthermore, condition (3.16) leads to some intervals of γ, in which the equilibrium exists. When γ approaches the boundaries of these intervals from inside, although Φ approaches infinity, return volatility does not. In fact, in this case, either

$$\gamma = \alpha \quad \text{and} \quad \Delta \geq 0, \tag{D.6}$$

or

$$\Delta = 0 \quad \text{and} \quad \gamma > \alpha, \tag{D.7}$$

occurs. However, (D.6) cannot occur for $\alpha \neq 1$. In fact, notice that α is one lower bound of γ. When $\gamma \to \alpha+$ (i.e., $\gamma > \alpha$ and $\gamma \to \alpha$), (D.1) shows that the last term of Δ dominates and approaches $-\frac{\gamma^2(\gamma-1)^2\sigma_c^2}{2(\gamma-\alpha)^2} \to -\infty$ for $\alpha \neq 1$, implying that $\Delta < 0$

when γ is in a right neighborhood of α. So (D.6) cannot happen for $\alpha \neq 1$. If $\alpha = 1$ and (D.6), then $\sigma_p = \sigma_c < \infty$. If (D.7) occurs, then we have

$$\sigma_p < \sigma_d + \kappa \frac{\int_0^\infty \bar{a} e^{a(u)S+b(u)} du}{\int_0^\infty e^{a(u)S+b(u)} du} = \sigma_d + \kappa \bar{a} < \infty.$$

Therefore, return volatility has a finite limit when γ approaches the boundaries of these intervals from inside.

In addition, the above discussion also shows that, if $\alpha \neq 1$, then $\Delta > 0$ implies $\gamma > \alpha$. So condition (3.16) is equivalent to

$$\begin{cases} \Delta(\gamma, \alpha) > 0, & \text{for} \quad \alpha \neq 1, \\ 1 \leq \gamma < \sqrt{2\rho}/\sigma_c, & \text{for} \quad \alpha = 1. \end{cases}$$

D.3. **Proof of Corollary 4.1.**

Equation (D.1) implies that if $\gamma < 1$, then $A(t) > 0$ and $\sigma_p > \sigma_c > 0$; if $\gamma = 1$, then $A(t) = 0$ and $\sigma_p = \sigma_c > 0$; and if $\gamma > 1$, then $A(t) < 0$ and $\sigma_p < \sigma_c$.

First, we show that $\sigma_p > 0$. If $\gamma > 1$, to prove $\sigma_p > 0$, we need to show that

$$\int_t^T e^{A(u)S+B(u)} du > \alpha(\gamma - 1) \int_t^T \frac{1 - e^{-\kappa(\gamma-\alpha)/\gamma(T-u)}}{\gamma - \alpha} e^{A(u)S+B(u)} du.$$

It suffice to show that

$$1 > \alpha(\gamma - 1) \frac{1 - e^{-\kappa(\gamma-\alpha)/\gamma(T-u)}}{\gamma - \alpha}. \tag{D.8}$$

In fact, $\alpha \leq 1$ leads to $\frac{\alpha(\gamma-1)}{\gamma-\alpha} \leq 1$, which further leads to (D.8).

In addition, it follows from (3.7) that

$$\begin{aligned} \frac{\partial \sigma_p}{\partial S} &= \kappa \Phi^{-2} \Big[\frac{\partial^2 \Phi}{\partial S^2} \Phi - \Big(\frac{\partial \Phi}{\partial S} \Big)^2 \Big] \\ &= \kappa \Phi^{-2} \Big[\int_t^T A(u)^2 e^{A(u)S+B(u)} du \int_t^T e^{A(u)S+B(u)} du - \Big(\int_t^T A(u) e^{A(u)S+B(u)} du \Big)^2 \Big] > 0. \end{aligned}$$

So σ_p increases with S for both $\gamma < 1$ and $\gamma > 1$.

Appendix A.1, which is for the case $T \to \infty$, has computed the limits of σ_p when S approaches infinity. The case with finite T can be also studied using the same method. It shows that when $\gamma < 1$ and $S \to +\infty$ or when $\gamma > 1$ and $S \to -\infty$, $\sigma_p \to \frac{(1-\alpha)\gamma}{\gamma-\alpha} \sigma_c$; when $\gamma < 1$ and $S \to -\infty$ or when $\gamma > 1$ and $S \to +\infty$, $\sigma_p \to \sigma_c$. Because σ_p in an increasing function of S, the two limits of σ_p above are the lower and upper boundaries of σ_p.

D.4. **Proof of Corollaries 4.5, 4.2, 4.3 and 4.4.**

To the leading order of $1/\kappa$, a and b in (A.4) become $a(t)=0$, $b(t)=-\Delta t$, and

$$\sigma_p = \frac{\gamma(1-\alpha)}{\gamma-\alpha}\sigma_c, \quad \mu_p^e = \frac{\gamma(1-\alpha)}{\gamma-\alpha}\sigma_c(\alpha_0+\alpha S), \quad r_f = \frac{\gamma(1-\alpha)}{\gamma-\alpha}\sigma_c(\alpha_0+\alpha S-\gamma\sigma_c),$$

$$\mu_p = \frac{\gamma(1-\alpha)}{\gamma-\alpha}\sigma_c\Big[\frac{1-\alpha)\gamma^2+\alpha(\gamma-1)}{\gamma(\gamma-\alpha)}\frac{\mu^r}{\sigma_c} - \frac{\alpha(\gamma-1)}{\gamma-\alpha}\sigma_c + \frac{\alpha(\gamma-1)}{\gamma-\alpha}S\Big], \quad \Phi = \frac{1}{\Delta}.$$

Therefore, the unconditional means and variances under the subjective measure and the physical measure are given by

$$\begin{aligned}
&\mathbb{E}^e[S] = \frac{\mu^r}{\sigma_c}, \quad \mathbb{V}ar^e[S] = \frac{\kappa}{2(1-\alpha)},\\
&\mathbb{E}^e[r_f] = \frac{\gamma(1-\alpha)}{\gamma-\alpha}r_f^r, \quad \mathbb{V}ar^e[r_f] = \frac{\kappa\alpha^2(1-\alpha)\gamma^2\sigma_c^2}{2(\gamma-\alpha)^2},\\
&\mathbb{E}^e[\mu_p^e - r_f] = \frac{\gamma^2(1-\alpha)}{\gamma-\alpha}\sigma_c^2, \quad \mathbb{V}ar^e[\mu_p^e - r_f] = 0,
\end{aligned}$$

and

$$\begin{aligned}
&\mathbb{E}[S] = \Big[1-\frac{\alpha(\gamma-1)}{\gamma^2(1-\alpha)}\Big]\frac{\mu^r}{\sigma_c} - \frac{\alpha(\gamma-1)}{\gamma(1-\alpha)}\sigma_c, \quad \mathbb{V}ar[S] = \frac{\gamma-\alpha}{\gamma}\frac{\kappa}{2(1-\alpha)},\\
&\mathbb{E}[r_f] = \frac{\gamma^2(1-\alpha)+\alpha^2(\gamma-1)}{\gamma(\gamma-\alpha)}r_f^r, \quad \mathbb{V}ar[r_f] = \frac{\kappa\alpha^2(1-\alpha)\gamma\sigma_c^2}{2(\gamma-\alpha)},\\
&\mathbb{E}[\mu_p - r_f] = \frac{\alpha(1-\alpha)(\gamma-1)}{\gamma(\gamma-\alpha)}\rho + \frac{\alpha(1-\alpha)(\gamma-1)}{\gamma-\alpha}\mu_c\\
&\qquad + \frac{1-\alpha}{\gamma-\alpha}\Big[\gamma^2 - \frac{\alpha(\gamma-1)^2}{2} - \frac{\alpha\gamma(\gamma-1)}{\gamma-\alpha} + \frac{\alpha^2(1-\alpha)\gamma(\gamma-1)}{2(\gamma-\alpha)^2}\Big]\sigma_c^2,\\
&\mathbb{V}ar[\mu_p - r_f] = \frac{\kappa\alpha^2(1-\alpha)^3\gamma\sigma_c^2}{2(\gamma-\alpha)^3}.
\end{aligned}$$

The unconditional mean of return volatility under both the subjective measure and the physical measure is between those for small κ and large κ. In fact, S is a mean-reverting process. Denote $p(S)$ its unconditional distribution under the physical measure. The unconditional mean of volatility under the physical measure is given by

$$\mathbb{E}[\sigma_p] = \sigma_c + \kappa\int_{-\infty}^{\infty}\frac{\int_0^\infty a(u)e^{a(u)S+B(u)}du}{\int_0^\infty e^{a(u)S+B(u)}du}p(S)dS.$$

Because $-(\gamma-1)\frac{\alpha\sigma_c}{\kappa(\gamma-\alpha)} < a(u) \le 0$, we have

$$\frac{\gamma(1-\alpha)}{\gamma-\alpha}\sigma_c = \sigma_c - \kappa(\gamma-1)\frac{\alpha\sigma_c}{\kappa(\gamma-\alpha)} < \mathbb{E}[\sigma_p] < \sigma_c.$$

To the leading order of κ, a and b in (A.4) become

$$a(t) = -\frac{\gamma-1}{\gamma}\alpha\sigma_c t, \quad b(t) = \frac{(\gamma-1)\alpha\sigma_c}{(\gamma-\alpha)^2}\Big[\frac{\gamma-\alpha}{\gamma}\alpha_0 - \frac{(\gamma-1)\alpha\sigma_c}{2}\Big]t - B^*t,$$

and

$$\sigma_p = \sigma_c, \quad \mu_p^e = \sigma_c(\alpha_0 + \alpha S), \quad \mu_p = \frac{\mu^r}{\gamma} + \frac{\gamma - 1}{\gamma}\sigma_c(\alpha_0 + \alpha S),$$
$$r_f = \sigma_c(\alpha_0 + \alpha S - \gamma\sigma_c), \quad \Phi = \gamma/[\rho - (1-\gamma)\alpha_0\sigma_c + \gamma(1-\gamma)\sigma_c^2/2].$$

Therefore, the unconditional means and variances under the subjective measure and the physical measure are given by

$$\mathbb{E}^e[S] = \frac{\mu^r}{\sigma_c}, \quad \mathbb{V}ar^e[S] = \frac{\kappa}{2(1-\alpha)}, \quad \mathbb{E}[S] = \frac{\mu^r}{\sigma_c}, \quad \mathbb{V}ar[S] = \frac{\kappa}{2[1 - \alpha(\gamma-1)/\gamma]},$$
$$\mathbb{E}^e[r_f] = r_f^r, \quad \mathbb{V}ar^e[r_f] = \frac{\kappa\alpha^2\sigma_c^2}{2(1-\alpha)}, \quad \mathbb{E}[r_f] = r_f^r, \quad \mathbb{V}ar[r_f] = \frac{\kappa\alpha^2\sigma_c^2}{2[1-\alpha(\gamma-1)/\gamma]},$$
$$\mathbb{E}^e[\mu_p^e - r_f] = \gamma\sigma_c^2, \quad \mathbb{V}ar^e[\mu_p^e - r_f] = 0, \quad \mathbb{E}[\mu_p - r_f] = \gamma\sigma_c^2, \quad \mathbb{V}ar[\mu_p - r_f] = \frac{\kappa\alpha^2\sigma_c^2}{2\gamma(\gamma - \gamma\alpha + \alpha)}.$$

D.5. **Proof of Corollary 5.2.**

To the order of α, $A(t)$ and $B(t)$ become

$$A(t) = -\frac{(\gamma-1)\alpha\sigma_c}{\gamma\kappa}\big[1 - e^{-\kappa(T-t)}\big],$$
$$B(t) = -\frac{(\gamma-1)\alpha\alpha_0\sigma_c}{\gamma^2\kappa}\big[1 - e^{-\kappa(T-t)}\big] + \Big[\frac{(\gamma-1)\sigma_c^2}{2} - \frac{(\gamma-1)\alpha_0\sigma_c}{\gamma}\Big(1 + \frac{\alpha}{\gamma}\Big) - \frac{\rho}{\gamma}\Big](T-t). \tag{D.9}$$

Substituting (D.9) into (3.12) and (3.18), σ_p, μ_p^e and μ_p are, to the order of α, given by (5.1), where

$$a_1(t) = 1 - \frac{a_0}{a_0+\kappa}\frac{1 - e^{-(a_0+\kappa)(T-t)}}{1 - e^{-a_0(T-t)}},$$
$$a_2(t) = -\frac{\gamma-1}{\gamma}\alpha\Big[\alpha_0\sigma_c + \mu_c + \frac{\rho}{\gamma} - \frac{(\gamma+1)\sigma_c^2}{2}\Big]a_1(t) + \mu_c + a_0, \tag{D.10}$$
$$a_0 = (\gamma-1)\alpha_0\sigma_c/\gamma - (\gamma-1)\sigma_c^2/2 + \rho/\gamma.$$

Because $a_1(t)' < 0$, we have $0 = a_1(T) \le a_1(t) \le a_1(0) = 1 - \frac{a_0}{a_0+\kappa}\frac{1-e^{-(a_0+\kappa)T}}{1-e^{-a_0T}} < 1$.

To determine the sign of return series correlation, we only need to study $cov(r_{p,t+h\Delta}, r_{pt})$. Because S_t is, to the order of α, given by

$$S_t \approx \int_0^t \kappa e^{-\kappa(1+\frac{1-\gamma}{\gamma}\alpha)(t-u)}\Big[\alpha_0\Big(1 - \frac{\sigma_{pu}}{\gamma\sigma_c}\Big) + \sigma_{pu} + \frac{\mu_c}{\sigma_c} + \frac{\rho}{\gamma\sigma_c} - \frac{(\gamma+1)\sigma_c}{2}\Big]du$$
$$+ S_0 e^{-\kappa(1+\frac{1-\gamma}{\gamma}\alpha)t} + \kappa e^{-\kappa(1+\frac{1-\gamma}{\gamma}\alpha)(t-u)}dZ_u, \tag{D.11}$$

together with (3.18), we have

$$
\begin{aligned}
&cov(r_{p,t+h\Delta}, r_{pt})\\
=&cov\Big(\int_{t+(h-1)\Delta}^{t+h\Delta}\int_0^u \kappa\sigma_{pu}\psi_u e^{-\kappa(1+\frac{1-\gamma}{\gamma}\alpha)(u-v)}dZ_v du, \int_{t-\Delta}^{t}\int_0^u \kappa\sigma_{pu}\psi_u e^{-\kappa(1+\frac{1-\gamma}{\gamma}\alpha)(u-v)}dZ_v du + \sigma_{pu}dZ_u\Big)\\
=&\kappa f(h,t)f_0(t),
\end{aligned}
$$

where

$$
\begin{aligned}
f(h,t) &= \int_{t-\Delta}^{t}\sigma_{p,u+h\Delta}\psi_{u+h\Delta}e^{-\kappa(1+\frac{1-\gamma}{\gamma}\alpha)(u+h\Delta)}du,\\
f_0(t) &= \int_{t-\Delta}^{t}\sigma_{pu}\Big[\kappa\alpha\Big(1-\frac{\sigma_{pu}}{\gamma\sigma_c}\Big)e^{-\kappa(1+\frac{1-\gamma}{\gamma}\alpha)u}\frac{e^{2\kappa(1+\frac{1-\gamma}{\gamma}\alpha)u}-1}{2\kappa(1+\frac{1-\gamma}{\gamma}\alpha)} + e^{\kappa(1+\frac{1-\gamma}{\gamma}\alpha)u}\Big]du.
\end{aligned}
$$

Because $a_1(t) > 0$, we have $1-\frac{\sigma_{pu}}{\gamma\sigma_c}$ when $\gamma > 1$. In addition, it is easy to verify that $\sigma_{pu} > 0$. So $f_0(t) > 0$ when $\gamma > 1$. We rewrite $f_0(t)$ as

$$
f_0(t) = \int_{t-\Delta}^{t}\sigma_{pu}\Big\{\Big[1-\frac{\alpha(\frac{\sigma_{pu}}{\gamma\sigma_c}-1)}{2(1+\frac{1-\gamma}{\gamma}\alpha)}\Big]e^{\kappa(1+\frac{1-\gamma}{\gamma}\alpha)u} + \frac{\alpha(\frac{\sigma_{pu}}{\gamma\sigma_c}-1)}{2(1+\frac{1-\gamma}{\gamma}\alpha)}e^{-\kappa(1+\frac{1-\gamma}{\gamma}\alpha)u}\Big\}du.
$$

If $\gamma < 1$, then

$$
0 < \frac{\alpha(\frac{\sigma_{pu}}{\gamma\sigma_c}-1)}{2(1+\frac{1-\gamma}{\gamma}\alpha)} = \frac{(1-\gamma)\alpha(1+\frac{\alpha}{\gamma}a_1(u))}{2\gamma(1+\frac{1-\gamma}{\gamma}\alpha)} < \frac{(1-\gamma)\alpha}{\gamma+(1-\gamma)\alpha} < 1,
$$

implying that $f_0(t) > 0$. In all, we have $f_0(t) > 0$, and hence the sign of $cov(r_{p,t+h\Delta}, r_{pt})$ is completely determined by $f(h,t)$, whose sign is determined by ψ_u. Notice that $\psi_u = \alpha\frac{\gamma-1}{\gamma}\big(1+\frac{\alpha a_1(u)}{\gamma}\big)$, which is positive if $\gamma > 1$ and negative if $\gamma < 1$. So the series correlations of return are positive for all lag h if $\gamma > 1$ and negative for all h if $\gamma < 1$. If $\gamma = 0$, then both expected return and volatility in (5.1) are deterministic, and hence the series correlations are 0 for all lags.

The series correlation decreases as lag increases because $\partial f(h,t)/\partial h < 0$.

D.6. **Proof of Corollary 5.3.**

It follows from (5.1) that the excess return satisfies

$$
\begin{aligned}
\frac{dP_t + C_t dt}{P_t} - r_{ft}dt = &\Big\{\alpha\sigma_c S_t\Big[-\frac{1}{\gamma}+\Big(1-\frac{1}{\gamma}\Big)\alpha a_1(t)\Big] + a_2(t)\\
&-\sigma_c(\alpha_0-\gamma\sigma_c)\Big[1-\Big(1-\frac{1}{\gamma}\Big)\alpha a_1(t)\Big]\Big\}dt + \sigma_c\Big[1-\Big(1-\frac{1}{\gamma}\Big)\alpha a_1(t)\Big]dZ_t.
\end{aligned} \tag{D.12}
$$

Note that $a_1(t)$ does not depend on α and γ, and that $a_1(t)$ and $a_2(t)$ are deterministic. When both α and γ are sufficiently large, the risk premium in (D.12) depends positively on S, which is positively related to past return innovations according to (D.11), leading to positive serial correlations of excess returns for all lags.

References

Adam, K., Matveev, D. and Nagel, S. (2018), Do survey expectations of stock returns reflect risk-adjustments?, working paper.

Amromin, G. and Sharpe, S. (2014), 'From the horse's mouth: Economic conditions and investor expectations of risk and return', *Management Science* **60**, 845–866.

Bacchetta, P., Mertens, E. and van Wincoop, E. (2009), 'Predictability in financial markets: What do survey expectations tell us?', *Journal of International Money and Finance* **28**, 406–426.

Barberis, N., Greenwood, R., Jin, L. and Shleifer, A. (2015), 'X-CAPM: An extrapolative capital asset pricing model', *Journal of Financial Economics* **115**, 1–24.

Basak, S. and Cuoco, D. (1998), 'An equilibrium model with restricted stock market participation', *Review of Financial Studies* **11**, 309–341.

Bénassy, J. (1993), 'Nonclearing markets: Microeconomic concepts and macroeconomic applications', *Journal of Economic Literature* **31**, 732–761.

Bénassy, J. (2006), Dynamic models with non clearing markets, PSE Working paper.

Borovička, J. (2018), Survival and long-run dynamics with heterogeneous beliefs under recursive preferences, working paper, New York University.

Campbell, J. and Shiller, R. (1988), 'The dividend-price ratio and expectations of future dividends and discount factors', *Review of Financial Studies* **1**, 195–228.

Cassella, S. and Gulen, H. (2018), 'Extrapolation bias and the predictability of stock returns by price-scaled variables', *Review of Financial Studies* **31**, 4345–4397.

Choi, J. and Mertens, T. (2013), Extrapolative expectations and the equity premium, working paper, New York University and Yale University.

Cochrane, J. (2011), 'Discount rates', *Journal of Finance* **66**, 1047–1108.

Cochrane, J. (2017), 'Macro-finance', *Review of Finance* **21**, 945–985.

Cutler, D., Poterba, J. and Summers, L. (1990), 'Speculative dynamics and the role of feedback traders', *American Economic Review Papers and Proceedings* **80**, 63–68.

Cutler, D., Poterba, J. and Summers, L. (1991), 'Speculative dynamics', *Review of Economic Studies* **58**, 529–546.

Da, Z., Huang, X. and Jin, L. (2018), Extrapolative beliefs in the cross-section: What can we learn from the crowds?, working paper.

De Bondt, W. and Thaler, R. (1985), 'Does the stock market overreact?', *Journal of Finance* **40**, 793–808.

De Long, J., Shleifer, A., Summers, L. and Waldmann, R. (1990), 'Positive feedback investment strategies and destabilizing rational speculation', *Journal of Finance* **45**, 379–395.

Dindo, P. (2019), 'Survival in speculative markets', *Journal of Economic Theory* **181**, 1–43.

Dudley, L. and Montmarquette, C. (1984), 'The effects of non-clearing labor markets on the demand for public spending', *Economic Inquiry* **22**, 151–170.

Fama, E. and French, K. (1988*a*), 'Dividend yields and expected stock returns', *Journal of Financial Economics* **22**, 3–25.

Fama, E. and French, K. (1988*b*), 'Permanent and temporary components of stock prices', *Journal of Political Economy* **96**, 246–273.

Fuster, A., Hebert, B. and Laibson, D. (2011), 'Natural expectations, macroeconomic dynamics, and asset pricing', *NBER Macroeconomics Annual* **26**, 1–48.

Giglio, S., Maggiori, M., Stroebel, J. and Utkus, S. (2019), Five facts about beliefs and portfolios, working paper.

Greenwood, R. and Hanson, S. (2013), 'Issuer quality and corporate bond returns', *Review of Financial Studies* **26**, 1483–1525.

Greenwood, R. and Shleifer, A. (2014), 'Expectations of returns and expected returns', *Review of Financial Studies* **27**, 714–746.

Hirshleifer, D., Li, J. and Yu, J. (2015), 'Asset pricing in production economies with extrapolative expectations', *Journal of Monetary Economics* **76**, 87–106.

Hong, H. and Stein, J. (1999), 'A unified theory of underreaction, momentum trading, and overreaction in asset markets', *Journal of Finance* **54**, 2143–2184.

Jegadeesh, N. and Titman, S. (1993), 'Returns to buying winners and selling losers: Implications for stock market efficiency', *Journal of Finance* **48**, 65–91.

Jin, L. and Sui, P. (2018), Asset pricing with return extrapolation, working paper, Caltech.

Kuchler, T. and Zafar, B. (2019), 'Personal experiences and expectations about aggregate outcomes', *Journal of Finance* **74**, 2491–2542.

LeRoy, S. and Porter, R. (1981), 'The present-value relation: Tests based on implied variance bounds', *Econometrica* **49**, 555–574.

Li, K. and Liu, J. (2019), Optimal dynamic momentum strategies, working paper, SSRN. https://ssrn.com/abstract=2746561.

Liu, J. (2007), 'Portfolio selection in stochastic environments', *Review of Financial Studies* **20**, 1–39.

Loewenstein, M. and Willard, G. (2006), 'The limits of investor behavior', *Journal of Finance* **61**, 231–258.

Lucas, R. (1978), 'Asset prices in an exchange economy', *Econometrica* **46**, 1429–1446.

Mehra, R. and Prescott, E. (1985), 'The equity primium: A puzzle', *Journal of Monetary Economics* **15**, 145–161.

Merton, R. (1971), 'Optimum consumption and portfolio rules in a continuous-time model', *Journal of Economic Theory* **3**, 373–413.

Moskowitz, T., Ooi, Y. and Pedersen, L. (2012), 'Time series momentum', *Journal of Financial Economics* **104**, 228–250.

Nagel, S. and Xu, Z. (2018), Asset pricing with fading memory, working paper, University of Chicago and University of Michigan.

Peng, C. and Wang, C. (2019), Positive feedback trading and stock prices: Evidence from mutual funds, working paper, LSE and Yale.

Poterba, J. and Summers, L. (1988), 'Mean reversion in stock prices: Evidence and implications', *Journal of Financial Economics* **22**, 27–59.

Shiller, R. (1981), 'Do stock prices move too much to be justified by subsequent changes in dividends?', *American Economic Review* **71**, 421–436.

Vissing-Jorgensen, A. (2004), 'Perspectives on behavioral finance: Does irrationality disappear with wealth? Evidence from expectations and actions', *NBER Macroeconomics Annual* **18**, 139–194.

Yan, H. (2008), 'Natural selection in financial markets: Does it work?', *Management Science* **54**, 1935–1950.