Money Mining and Price Dynamics*

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Abstract

We develop a random-matching model to study the price dynamics of monies produced privately according to a time-consuming mining technology. There exists a unique equilibrium where the value of money reaches a steady state. There is also a continuum of perfect-foresight equilibria indexed by the starting value of the currency where the price of money inflates and bursts gradually over time. In the aftermath of its introduction, private money is held for a speculative motive and it acquires a transactional role when it becomes sufficiently abundant. We study divisible, indivisible, fiat and commodity monies, and adopt implementation and equilibrium approaches.

Keywords: Money, Search, Private Monies, Mining
JEL codes: E40, E50

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1 Introduction

Since the creation of Bitcoins a decade ago, hundreds of new crypto-currencies – digital objects with no intrinsic value whose ownership can be transferred through encryption techniques – have been introduced. The total market capitalization of crypto-assets peaked at $830 billion at the beginning of 2018, with an estimated $290 billion for Bitcoins alone (Financial Stability Board, 2018), see the left panel of Figure 1. Like gold, the supply of Bitcoins is finite and is not controlled by a government: gold is mined out of the ground while Bitcoins are mined by solving numerical puzzles. Unlike gold — a commodity with intrinsic usefulness — Bitcoins are intrinsically useless and unbacked. Over the year 2018, Bitcoin lost 70% of its value while other crypto-currencies lost on average 95% of their value, hence the view that crypto-currencies are "*the mother of all bubbles*" (Roubini, 2018).[1] The right panel of Figure 1 illustrates the similarities between the price dynamics of Bitcoin and two others asset price bubbles in history.[2]

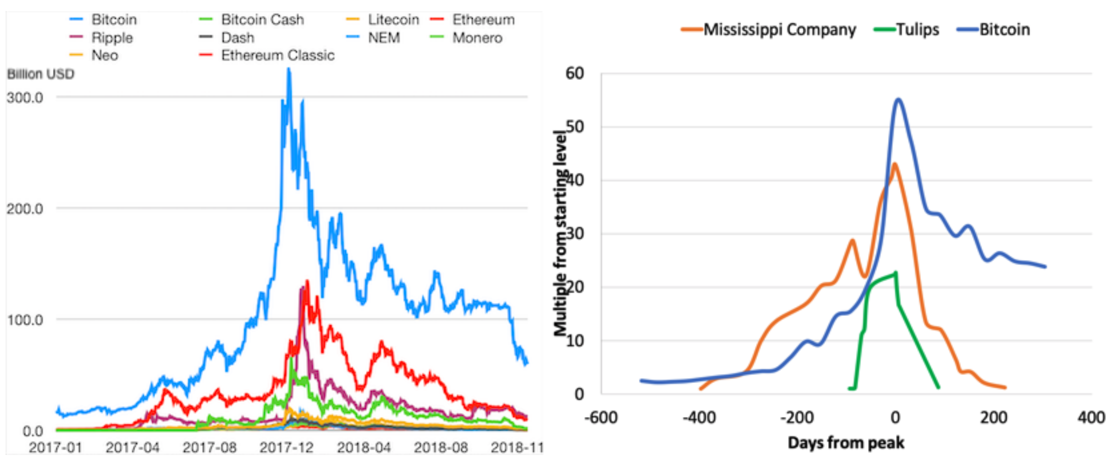

Figure 1: (Left) Market capitalization of various crypto-currencies. (Right) Asset bubbles in history. We plot the value of outstanding bank notes by the Banque Royale during the Mississippi bubble, the price of tulips during the Tulip mania and the price of Bitcoin around 2018.

The phenomenon of crypto-currencies has brought some foundational questions of monetary theory to the forefront. Can privately-produced, intrinsically useless objects serve as media of exchange and have a positive value? How is the initial value of a new money determined and how does its price evolve over time? Is a boom and burst of crypto-currency prices consistent with perfect foresight? Are all privately-produced money bubbles condemned to burst? Is the private production of money socially efficient? Do the dynamics

[1] Testimony of Nouriel Roubini before the United States Senate Banking Committee, Exploring the crypto-currency and Blockchain Ecosystem, Oct 2018.

[2] Data from Velde (2003), Thompson (2007) and Bitcoinity.org.

of prices differ for fiat and commodity monies?

The goal of this paper is to revisit these questions by studying the dynamics of an economy where money is privately produced at some endogenous opportunity cost through mining — a time-consuming occupation. Our theory applies to the mining of commodity monies, e.g., gold and silver, as well as the production of fiat currencies, e.g., Bitcoins (Throughout the paper we use Wallace's (1980) definition of a fiat money as an object that is inconvertible and intrinsically useless).[3] Because the determination of currency prices is better understood in models where there is an essential role for a medium of exchange, we adopt the search-theoretic model of monetary exchange of Shi (1995) and Trejos and Wright (1995). In this environment, trades take place within pairwise meetings that are formed randomly. Heterogeneity in preferences and specialization in production generate a lack-of-double-coincidence-of-wants problem and rule out barter trades. In addition, agents, who are anonymous, cannot finance random consumption opportunities by issuing private debts, hence a role for money (Kocherlakota, 1998). Money is indivisible and there is an upper bound on money holdings. While this assumption was originally made for tractability, it captures the notion that the quantity of liquid assets is scarce and affects the measure of transactions. We also study a version of the model where the scarcity of the liquidity is mitigated by making money perfectly divisible and by allowing agents to adjust their asset holdings in competitive exchanges in continuous time, as in Lagos and Wright (2005).

We add two components to the New Monetarist model. First, we introduce a mining technology. This technology is time consuming and discoveries of new units of money happen at random arrival times that are state-dependent. Second, we add an occupation choice to endogenize the opportunity cost of mining and link it to the private gains from production. Agents can either be producers looking for opportunities to sell goods in pairwise meetings or they can be miners searching for opportunities to dig units of money.[4] Hence, the opportunity cost of mining is the foregone profits of producers, which are endogenous and vary with the quantity of money in the economy. As a result our model incorporates a feed-back loop according to which mining provides liquidity to the trading sector of the economy, which in turn determines the opportunity cost of mining.

[3] Goldberg (2005) discusses the notion of fiat money in monetary economics and disputes the common wisdom that fiat monies defined as inconvertible and intrinsically useless media of exchange ever existed. In that regards, crypto-currencies might be the first creation of fiat monies as defined by monetary theorists.

[4] For completeness, in Online Appendix B we investigate alternative assumptions where agents can look for trading partners while mining.

Some empirical motivation There is plenty of evidence to support the assumption that money mining has an endogenous opportunity cost by diverting input factors from alternative productive uses. The California Gold Rush (1848–1855) is a case in point. The Gold Rush tripled the population in California by bringing approximately 300,000 people from the rest of the world (see Britannica). South Africa offers another example where gold mining had a large impact on the allocation of workers across sectors of the economy (Gilbert, 1933). In the case of crypto-currencies, CoinDesk (11/12/2017) reported that the number of blockchain jobs posted in the U.S. in 2017 increased by 207% and it has increased by 631% since November 2015.[5] Upwork, a large freelancing website, ranked blockchain as the top fastest-growing skill in the first quarter of 2018.[6] This rapid growth is consistent with the increase in the number of crypto-currencies — according to investing.com, there were less than 1600 crypto-currencies in February 2018 and there are 2520 of them in February 2019.

A key mechanism of our model is that the size of the mining sector is endogenous and responds to prices. To test this proposition, we compare deviations from trend in the purchasing power of gold to that in the production of gold. If the production is endogenous, then one expects the deviations of the real value of gold from its trend to lead the corresponding changes in the production of gold.[7] Figure 2 confirms this conjecture for the period 1880-1970. In Online Appendix A we test this causality in a two-variable vector autoregression model and find that the purchasing power of gold Granger-causes gold production in the same time period at the 5-percent significance level.[8]

In contrast to commodity monies, Bitcoin has been designed so that its supply is predictable.[9] The aggregate supply of Bitcoins is controlled by varying the difficulty level of the mathematical puzzles that miners have to solve. If there is a sudden increase in the number of miners, then the difficulty level increases to keep the money supply along a pre-determined path. As a result, one can infer the intensity of the mining activities (e.g., the number of miners and the CPU time they invest into mining) by looking at the difficulty level of the puzzles. We show in the Online Appendix that the growth rate of Bitcoin prices Granger-causes the growth rate of the mining difficulty level at a 1-percent significance level. This is consistent with our

[5] https://www.coindesk.com/blockchains-big-year-competitive-job-market-grows-200

[6] https://www.upwork.com/press/2018/05/01/q1-2018-skills-index/

[7] Bordo (1981) uses a similar idea to show a rising purchasing power of gold induces an increase in the monetary gold stock.

[8] The data on purchasing power and production are from Jastram (2009). A variable x_t Granger-causes another variable y_t if the lags of x_t are able to increase the accuracy of the prediction of y_t with respect to a forecast that considers only past values of y_t.

[9] The creator of Bitcoin, Satoshi Nakamoto, wrote: "The steady addition of a constant amount of new coins is analogous to gold miners expending resources to add gold to circulation. In our case, it is CPU time and electricity that is expended."

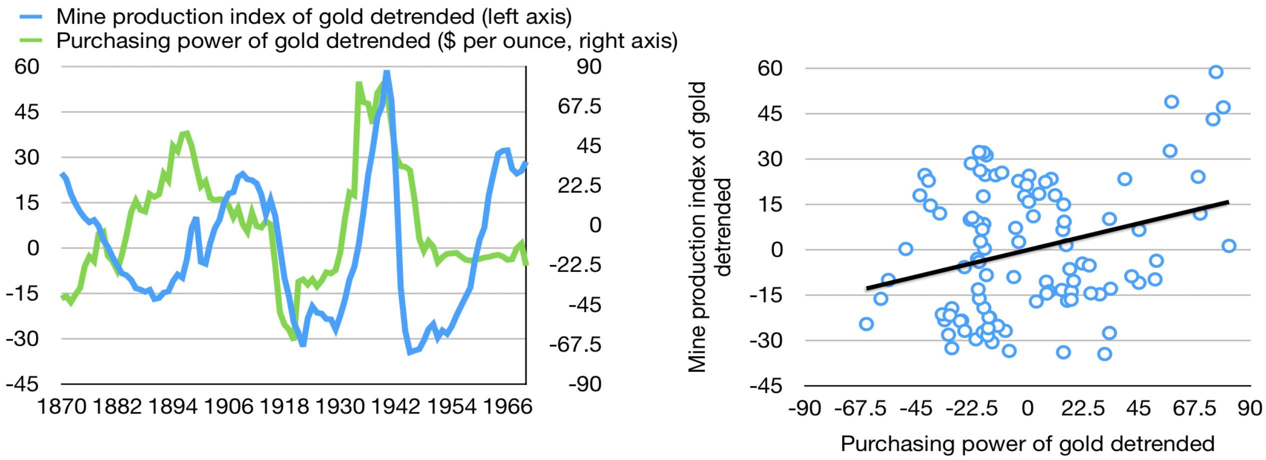

Figure 2: Mine production and price of gold.

assumption that the intensity of Bitcoin mining is driven by the real value of Bitcoins. Relatedly Prat and Walter (2018) use the Bitcoin-to-US dollar exchange rate to predict the computing power of Bitcoin's network. They argue that one third of the seigniorage revenue is spent on electricity consumption.

Succinct preview of our results. In accordance with monetary folk-theorems, a privately-produced fiat money can be valued in the long run only if agents are sufficiently patient and trading frictions are not too severe. In addition, the maximum amount of money that can be mined cannot be too large and money discoveries cannot happen too frequently since otherwise agents would have incentives to mine too many units, thereby prompting a breakdown of the monetary equilibrium.

The initial price of a new private money is indeterminate within a nonempty interval. The largest value in this interval corresponds to the unique equilibrium leading to a positive value of the currency in the long run. For lower initial values, the equilibrium path for the value of money is first increasing and then decreasing, and it reaches zero asymptotically. So unless agents can coordinate on the highest equilibrium – one equilibrium among a continuum of perfect-foresight equilibria — the life cycle of a privately-produced currency is composed of a boom, where agents mine money, and a bust where agents trade a depreciating money. Across equilibria, the peak for the value of money is positively correlated with its initial value. This result shows that the starting value of a new currency is crucial for its long-run viability. In summary, our model can describe the genesis of a bubble, its evolution through time, and its disappearance. We plot the price trajectories predicted by our model (in terms of market capitalization) for different equilibria in the right panel of Figure 3. Interestingly, those trajectories are qualitatively similar to those of actual crypto-currencies in the left panel.

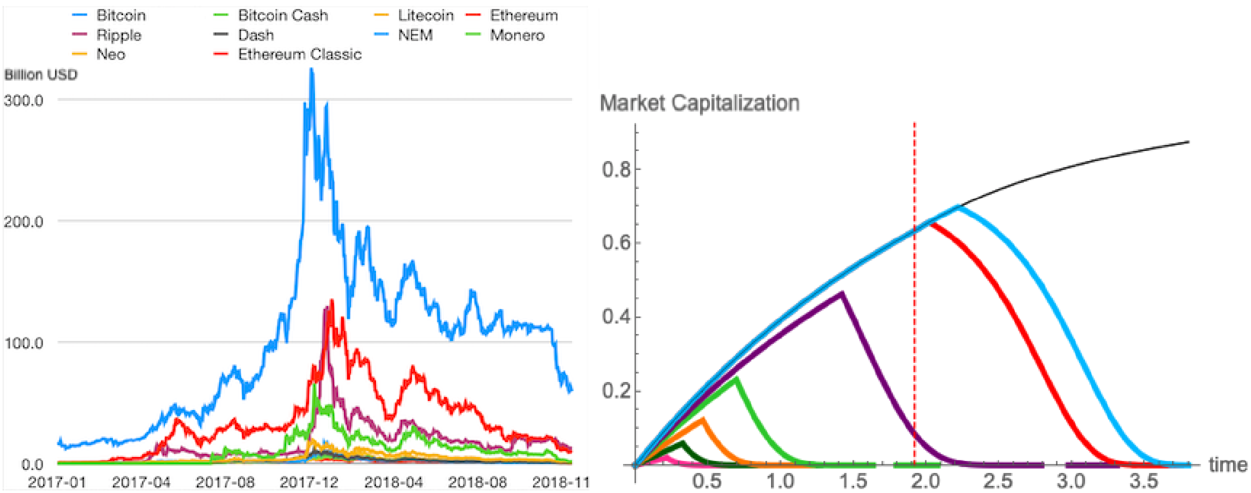

Figure 3: (Left) Market capitalization of various crypto-currencies. (Right) Market capitalization in different equilibria in our model. Each colored curve represents one equilibrium.

Increases in the amount of money that can be privately mined, e.g., through discoveries of new mines or through the introduction of new crypto-currencies, generates price waves. The value of money falls initially and then increases gradually over time. The overall trend for the price of money is downward slopping. The correlation between the quantity of money and its price can change sign in the short and long run: the correlation is positive along the transitional path but negative across steady states.

A critical component to the fundamental value of a new currency is the extent of its transaction role (Tirole, 1985). Our model shows that assessing this role based on the velocity of the currency in the short run can be misleading. Indeed, a new money can have a transactional role in the long run even if does not serve as a means of payment in the short run. In all equilibria of our model, the new money does not circulate initially and this outcome is shown to be constrained-efficient. To an outside observer, the new currency looks like a speculative bubble since it is only held for its capital gains. It is only when money is sufficiently abundant that agents stop hoarding it.

A new (crypto-)currency can be designed so as to keep its value over time. First, inflationary equilibria where the value of money vanishes asymptotically are eliminated if money pays a real (possibly small) dividend. Second, the mining speed and the path for the money supply can be chosen so that the value of money remains constant along the transition path. The money growth rate must decrease gradually over time and approach zero when the money supply is close to its steady-state value. In terms of social efficiency, we provide conditions to implement a constrained-efficient allocation when the mining technology is taken as a primitive and when it is part of the mechanism design problem.

Finally, in the presence of a government-produced money, we show that the government can prevent the private money from emerging by making its own money sufficiently valuable and by taking measures to reduce the acceptability of the private money (or to enhance the acceptability of the government money).

Literature review Our benchmark model builds on the search-theoretic models of monetary exchange of Shi (1995) and Trejos and Wright (1995).[10] Relative to these models, we add an occupation choice and a time-consuming mining activity, thereby allowing the growth of the money supply and the opportunity cost of money production to be endogenous and depend on the aggregate money supply. We study both fiat and commodity monies, steady states and out-of-steady-state dynamics, and we adopt both equilibrium and implementation approaches. Related papers include Burdett, Trejos and Wright (2001) where the quantity of commodity money (cigarettes) is endogenous, Cavalcanti and Wallace (1999) and Williamson (1999) where banks issue inside money, Lotz and Rocheteau (2002) and Lotz (2004) who study the launching and adoption of a new fiat money, Cavalcanti and Nosal (2011) who interpret the production of counterfeited notes as the issuance of a private money that is difficult to monitor, Hendrickson et al. (2016) and Hendrickson and Luther (2017) who study the coexistence of Bitcoin and a regular currency under endogenous matching. A survey of this class of models is provided by Lagos et al. (2017).

We extend the model to have divisible assets in a continuous-time version of the Lagos and Wright (2005) model. The privately-produced asset is either a Lucas tree, as in Geromichalos et al. (2007) and Lagos (2010), or fiat money. The supply of assets is endogenous as in Lagos and Rocheteau (2008), Rocheteau and Rodriguez (2014), and Geromichalos and Herrenbrueck (2018), among others. Relative to these papers, we emphasize the creation of assets through an occupation choice and formalize an explicit, time-consuming mining technology. Branch et al. (2016) also have private provision of liquidity and an occupation choice but the focus is different: the privately-produced asset takes the form of homes produced by Pissarides firms and the occupation choice is made by unemployed workers who can either be in the construction sector or the consumption-good sector. The study of dynamic equilibria in this class of models includes Lagos and Wright

[10] While we adopt the search-theoretic approach to obtain an essential role for media of exchange, there is a related literature on rational bubbles in the context of OLG models, e.g., Wallace (1980) and Tirole (1985), among many others. An application to crypto-currencies is provided by Garratt and Wallace (2017).

(2003) and Rocheteau and Wright (2013). The continuous-time assumption allows us to eliminate some exotic dynamics (such as cycles or chaotic dynamics) as shown by Oberfield and Trachter (2012). Berentsen (2006) is the first paper to study the private provision of fiat currency in the context of a search-theoretic model with divisible money.

Fernandez-Villaverde and Sanches (2018) study currency competition in the Lagos-Wright model extended to have a unit measure of entrepreneurs who can issue distinguishable tokens at some exogenous cost. They show that the shape of the cost function determines whether a monetary equilibrium with stable prices exists. Complementary to their approach, in our model the measure of miners is determined endogenously. There is no direct, exogenous cost of mining but there is an endogenous opportunity cost that depends on the money supply and the allocation of agents across occupations. Our description of the mining technology differs as we model its time dimension explicitly. Mining takes place in continuous time and individual discoveries of money are formalized by a Poisson process with endogenous and state-dependent intensity. Our focus is also different as we emphasize price dynamics starting from the creation of a new currency up to its disappearance.

Our extensions with two competing assets are related to Zhang (2014) and Gomis-Porqueras et al. (2017). Both papers focus on eliminating the indeterminacy of the nominal exchange rate in dual currency economies. Our approach to pin down the exchange rate between privately-produced and government monies is closer to Zhang (2014), which in turn follows Lester et al. (2012). Schilling and Uhlig (2018) consider the coexistence of government money and Bitcoin in a stochastic endowment economy and show the exchange rate between Bitcoin and fiat money is a martingale. Lotz and Vasselin (2019) develop a New Monetarist model to study the coexistence of fiat and E-money.

We adopt an implementation approach to study the constrained-efficient production of money and to achieve price stability. Chiu and Koeppl (2017) study the optimal design of crypto-currencies to overcome double-spending and show that the current Bitcoin scheme generates a large welfare loss. Chiu and Koeppl (2018) determine necessary conditions for blockchain-based settlement to be feasible. Biais et al. (2019) formalize the proof-of-work blockchain protocol as a stochastic game and show it has multiple equilibria, including equilibria with forks and orphaned blocks. They also identify negative externalities implying that equilibrium investment in computing capacity is excessive. In Pagnotta (2018) miners contribute resources that enhance network security and compete for mining rewards in the form of Bitcoins. The equilibrium level of network security and the price of Bitcoins are jointly determined and, among many insights, the price of Bitcoins can vary in a non-monotone fashion with the growth rate of the supply of Bitcoins.

2 Environment

The environment is based on Shi (1995) and Trejos and Wright (1995). Time is continuous and indexed by $t \in \mathbb{R}_+$. The economy is composed of a unit measure of ex ante identical, infinitely-lived agents and a perishable good, q, that comes in different varieties. Each agent can produce a single variety and consumes a subset of all varieties that does not include her own production good, thereby creating a need for trade. The utility of consumption is $u(q)$ with $u(0) = 0$, $u' > 0$, $u'(0) = +\infty$, and $u'' < 0$. There exists a $\bar{q} < +\infty$ such that $u(\bar{q}) = \bar{q}$. The disutility of production is q. Agents discount future utility at rate $r > 0$.

Agents meet bilaterally and at random times according to a Poisson process with arrival rate $\alpha > 0$. All agents participate in this meeting process irrespective of their occupation or asset holdings. We assume that preferences and specialization rule out double-coincidence-of-wants matches. Conditional on meeting a potential producer, the probability of a single-coincidence match is $\sigma \in [0, 1]$.[11] Agents are anonymous (i.e., there is no public record of their trading histories), they lack commitment, and there is no technology to enforce private debt contracts. These frictions create a need for a medium of exchange (Kocherlakota, 1998).

There is an intrinsically useless object, called money, that is perfectly storable and durable. For now it is indivisible and holdings of this object are restricted to $\{0, 1\}$. We relax this assumption in Section 5. The overall fixed quantity of this object is $\bar{A} \in (0, 1)$. The amount held by agents at time t is $A_t \leq \bar{A}$, where A_0 is given. The remaining $\bar{A} - A_t$ has yet to be mined. For now there is a single money in the economy. We will consider explicitly the coexistence of different monies in the second part of the paper.

At any point in time agents without money must select one among two occupations. They can choose to operate their technology to produce a variety of good q or they can enter the money mining sector. The mining technology is such that each miner finds a unit of money according to a Poisson process with arrival rate $\lambda(\bar{A} - A_t)$ where $\lambda > 0$. This specification contains two assumptions. First, the mining rate declines as the quantity of money that has been mined, A_t, increases. Second, the money supply is bounded above. When miners meet other individuals in the economy, they cannot produce for them − an assumption we

[11] For instance, there are $J \geq 3$ varieties of the consumption good. Agents are divided evenly across J types. Agent of type $j \in \{1, ...J\}$ can produce variety j if she chooses to be a producer but she only consumes variety $j+1$ (modulo J). In this case, the probability of single coincidence is $\sigma = 1/J$.

relax in Online Appendix B. Agents can transition back and forth between occupations at no cost.

3 Equilibria

We characterize monetary equilibria where money has a positive value at all dates. In this section and the next we focus on the economy with $\{0,1\}$ asset holdings. In this economy, the quantity of money, A, matters for the number of trade matches (extensive margin), thereby providing a precise notion of an optimal quantity of money. In addition, the model is sufficiently tractable to allow us to solve out-of-steady-state dynamics in closed form.

In the rest of the section, first, we define an equilibrium as a list of Bellman equations, bargaining outcomes, occupation choices, and a law of motion for the money supply. Second, we examine steady-state equilibria where the money supply is constant over time. Third, we describe all dynamic monetary equilibria starting from arbitrary initial conditions for the money supply.

3.1 Definition of equilibrium

Bellman equations

We denote $V_{a,t}$ the value function of an agent holding $a \in \{0,1\}$ units of money at time t. Throughout most of the paper we keep the time argument as implicit. The lifetime expected discounted utility of an agent with $a = 1$ solves the following Hamilton-Jacobi-Bellman (HJB) equation:

$$rV_1 = \alpha\sigma\left(1 - A - m\right)\left[u(q) + V_0 - V_1\right] + \dot{V}_1, \tag{1}$$

where m denotes the measure of agents engaging in mining and a dot over a variable represents a time derivative. According to the right side of (1) the agent meets another agent at Poisson rate α; this potential trading partner drawn at random from the whole population is a producer with probability $1 - A - m$ (he does not hold money and is not a miner); and she produces a good that the agent likes with probability σ. In that case, a trade takes place where one unit of money buys q units of output. The last term on the right side is the change in the value function over time in nonstationary equilibria.

The value function of an agent with $a = 0$ solves the following HJB equation:

$$rV_0 = \max\left\{\alpha\sigma A\left(-q + V_1 - V_0\right), \lambda\left(\bar{A} - A\right)\left(V_1 - V_0\right)\right\} + \dot{V}_0. \tag{2}$$

At any point in time, unmatched agents without money can choose among two occupations: mining or producing a consumption good. According to the first term in the maximization problem, an agent who chooses to become a producer meets a money holder who likes her good with Poisson arrival rate $\alpha\sigma A$ in which case she produces q at a linear cost in exchange for one unit of money. According to the second term, an agent who chooses to be a miner becomes a money holder at Poisson arrival rate $\lambda\left(\bar{A}-A\right)$ when she successfully digs a unit of money out of the ground. A miner can meet other agents in the economy but cannot produce for them (his occupation is to mine) or buy from them (he has no money), hence those encounters lead to no trade.

Bargaining over the price of money

The quantity q produced in a bilateral match is determined according to the Kalai (1977) bargaining solution that specifies that the buyer receives a constant fraction of the match surplus.[12] Formally,

$$u(q)+V_0-V_1=\theta\left[u(q)-q\right], \tag{3}$$

where $\theta\in[0,1]$ is the buyer's share. Solving for the value of money, V_1-V_0, we obtain:

$$V_1-V_0=\omega(q)\equiv(1-\theta)u(q)+\theta q. \tag{4}$$

Substituting V_1-V_0 by its expression given by (4) into (1) and (2) leads to:

$$rV_1 = \alpha\sigma\left(1-A-m\right)\theta\left[u(q)-q\right]+\dot{V}_1 \tag{5}$$

$$rV_0 = \max\left\{\alpha\sigma A(1-\theta)\left[u(q)-q\right],\lambda\left(\bar{A}-A\right)\omega(q)\right\}+\dot{V}_0. \tag{6}$$

Upon trading the surplus of the buyer is $\theta\left[u(q)-q\right]$ while the surplus of the seller is $(1-\theta)\left[u(q)-q\right]$. Upon digging a unit of money the gain of the miner is $\omega(q)$. Subtracting (6) from (5) and making use of (4) we obtain that q is the solution to the following ODE:

$$r\omega(q)=\alpha\sigma\left(1-A-m\right)\theta\left[u(q)-q\right]-\max\left\{\alpha\sigma A(1-\theta)\left[u(q)-q\right],\lambda\left(\bar{A}-A\right)\omega(q)\right\}+\omega'(q)\dot{q}. \tag{7}$$

Occupation choice

The net instantaneous gain from being a miner rather than a producer is:

$$\Delta(q,A)\equiv\lambda\left(\bar{A}-A\right)\omega(q)-\alpha\sigma A(1-\theta)\left[u(q)-q\right]. \tag{8}$$

[12] We use Kalai bargaining instead of Nash because of its simplicity. For the merits of this solution, see Aruoba et al. (2007).

From (6) or (7) the measure of miners is given by:

$$m \begin{array}{l} = 1-A \\ \in [0, 1-A] \\ = 0 \end{array} \quad \text{if} \quad \Delta(q, A) \begin{array}{c} > \\ = \\ < \end{array} 0. \tag{9}$$

By (8) the indifference condition $\Delta = 0$ can be rewritten as:

$$A = \mu(q) \equiv \frac{\lambda \bar{A} \omega(q)}{\alpha\sigma(1-\theta)\left[u(q) - q\right] + \lambda\omega(q)}. \tag{10}$$

Since $\omega(q)/\left[u(q) - q\right]$ increases in q by the concavity of $u(q)$, so does $\mu(q)$. Therefore, as A increases, so must q for agents to be indifferent across occupations.

Money growth

The law of motion for the supply of money in circulation in the economy is:

$$\dot{A} = m\lambda\left(\bar{A} - A\right). \tag{11}$$

There is a measure m of agents who specialize in mining and each agent extracts one unit of money from the ground at Poisson rate $\lambda\left(\bar{A} - A\right)$. We are now in position to define an equilibrium.

Definition 1 *An equilibrium is a pair of value functions, $V_{0,t}$ and $V_{1,t}$, the quantity traded in each match, q_t, the measure of miners, m_t, and the quantity of money in circulation, A_t, that solve: (5)-(6), (7), (9), (11), and the initial condition A_0.*

3.2 Steady states

We first describe steady-state equilibria where q and A are constant over time and $m = 0$. We focus on the steady state with the lowest A as it is the one that will be reached from the initial condition $A_0 = 0$. From (7):

$$r\omega(q) = \alpha\sigma\left(\theta - A\right)\left[u(q) - q\right]. \tag{12}$$

Substituting $\omega(q)$ by its expression given by (4) and rearranging,

$$rq = \left\{\alpha\sigma(\theta - A) - r(1-\theta)\right\}\left[u(q) - q\right]. \tag{13}$$

There is a unique $q > 0$ solution to (13) provided that $r < \alpha\sigma(\theta - A)/(1-\theta)$. Hence, a necessary (but not sufficient) condition for a monetary equilibrium to exist is $\theta > A$. Moreover, $\partial q/\partial A < 0$, i.e., an increase in the money supply reduces the purchasing power of money.

The condition for $m = 0$, $\Delta(q, A) \leq 0$, holds if $A \geq \mu(q)$, which from (10) and (12) can be reexpressed as

$$rA(1-\theta) \geq \lambda \left(\bar{A} - A\right)(\theta - A). \tag{14}$$

We represent inequality (14) in Figure 4. The left side is linear in A while the right side is quadratic with two roots, $A = \bar{A}$ and $A = \theta$. They intersect for two values, $A_1 < \min\{\bar{A}, \theta\}$ and $A_2 > \max\{\bar{A}, \theta\}$. The left side is located above the right side for all $A \in (A_1, A_2)$. Since A cannot be greater than θ for a monetary equilibrium to exist, we must have $A < \min\{\bar{A}, \theta\}$. So a steady-state monetary equilibrium exists for all A in the half-closed interval $\left[A_1, \min\{\bar{A}, \theta\}\right)$. In the following we focus on the steady state $A^s = A_1$.

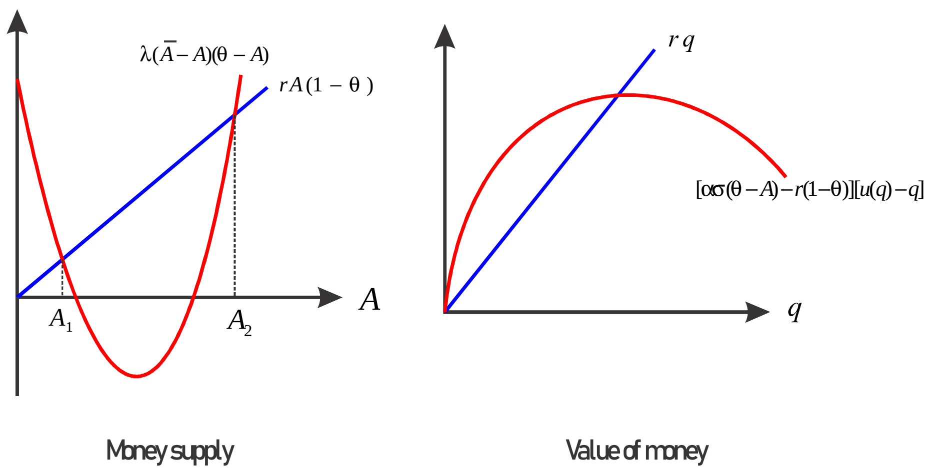

Figure 4: Steady states

The steady-state equilibrium is determined recursively. First, A^s is obtained as the smallest solution to (14). Given $A = A^s$, q^s exists if and only if $r < \alpha\sigma(\theta - A^s)/(1-\theta)$ by (13) or, equivalently,

$$A^s < \theta - \frac{r(1-\theta)}{\alpha\sigma}. \tag{15}$$

Figure 4 provides a graphical representation of the determination of the equilibrium.

Proposition 1 ***(Steady-state monetary equilibria)*** *There exists a unique steady-state monetary equilibrium (where $\Delta(q, A) = 0$) if and only if*

$$r < \frac{\alpha\sigma}{1-\theta}\left[\frac{\theta\alpha\sigma + \lambda\left(\theta - \bar{A}\right)}{\alpha\sigma + \lambda}\right] \tag{16}$$

where the steady-state money supply is

$$A^s = \frac{\lambda\theta + \lambda\bar{A} + r(1-\theta)}{2\lambda} - \sqrt{\left(\frac{\lambda\theta + \lambda\bar{A} + r(1-\theta)}{2\lambda}\right)^2 - \bar{A}\theta}. \tag{17}$$

Comparative statics are summarized in the following table:

	$\partial\lambda$	$\partial\bar{A}$	$\partial\theta$	∂r	$\partial(\alpha\sigma)$
$\partial A^s/$	$+$	$+$	$+$	$-$	0
$\partial q^s/$	$-$	$-$	$\pm$	$-$	$+$
$corr(q^s, A^s)$	$-$	$-$	$\pm$	$+$	0

Table 1: Comparative statics

Under which conditions can fiat money be privately produced and maintain a positive value in the long run? According to (16), the existence of a monetary equilibrium requires agents to be sufficiently patient — a standard "folk theorem" in monetary theory. The threshold for r below which money is valued decreases in the two parameters that affect the private production of money, i.e., $\bar{A}$ and λ. Higher $\bar{A}$ or λ are associated with higher incentives to mine, and hence a higher money supply at the steady state. But a higher A^s reduces buyers' trading opportunities, thereby making it harder to sustain a positive value for money.

Comparative statics in Table 1 have implications for the correlation between the endogenous money supply and price level. The sign of this correlation (the bottom row in Table 1) depends on which fundamentals drive the movements of A^s and q^s. If λ or $\bar{A}$ increases, then A^s increases and q^s decreases. In this case there is a positive correlation between money supply and price level ($1/q^s$), which is consistent with the quantity theory. However, if r increases, then both A^s and q^s decrease. There is now a negative correlation between the money supply and the price level. A change in θ generates a non-monotone relation between A^s and q^s: numerical examples show that for low θ, there is a positive correlation between A^s and q^s while for high θ there is a negative correlation. Finally, an increase in α or σ does not affect incentives to mine and the money supply but it raises q^s.

3.3 Dynamics

We now turn to transitional dynamics from an arbitrary initial condition, A_0. Without loss of generality we will focus on $A_0 = 0$ because the equilibrium is time consistent, i.e., if the equilibrium reaches A_t at time t from $A_0 = 0$, then the path onward is the same as the one obtained from the initial condition A_t.

From (7) and (11) (q, A, m) solve the following system of differential equations:

$$\begin{aligned} \omega'(q)\dot{q} &= \left[r + \lambda\left(\bar{A} - A\right)\right]\omega(q) - \alpha\sigma\left(1 - A - m\right)\theta\left[u(q) - q\right] && (18) \\ \dot{A} &= m\lambda\left(\bar{A} - A\right) && (19) \\ m &\leq 1 - A \text{ “}=\text{” if } \Delta(q, A) > 0. && (20) \end{aligned}$$

Equation (18) is an asset pricing equation for the value of money. The first term on the right side is the appreciation of the value of money over time if it does not provide transactional services: ω grows at rate $r + \lambda\left(\bar{A} - A\right)$, which compensates the buyer for her rate of time preference and the foregone opportunities of mining. The second term on the right side corresponds to the liquidity services that money provides to a buyer as measured by the expected surplus from a trade match. These liquidity services constitute a non-pecuniary return that reduces the appreciation rate of money. Equation (19) is the law of motion of the stock of money. Equation (20) is the optimality condition for the occupation choice between being a producer or a miner.

We distinguish two regimes. In the first regime all agents without money (the potential producers) engage in mining, namely $m = 1 - A$. Then by (18) and (19):

$$\begin{aligned} \dot{q} &= \left[r + \lambda\left(\bar{A} - A\right)\right]\frac{\omega(q)}{\omega'(q)} && (21) \\ \dot{A} &= (1 - A)\lambda\left(\bar{A} - A\right). && (22) \end{aligned}$$

Note from (21) that the value of money, ω, grows at a rate larger than r because it provides no liquidity services yet. Along the equilibrium path, by virtue of (21) and (22), the relation between q and A is given by

$$\left.\frac{\partial q}{\partial A}\right|_{m=1-A} = \frac{\omega(q)}{\omega'(q)}\frac{r + \lambda\left(\bar{A} - A\right)}{(1 - A)\lambda\left(\bar{A} - A\right)}. \tag{23}$$

The path is upward sloping in the (A, q) space.

Consider next the regime where miners and producers coexist, $m \in (0, 1 - A)$. In that case $A = \mu(q)$ and, from (19), the measure of miners is

$$m = \frac{\mu'(q)}{\lambda\left[\bar{A} - \mu(q)\right]}\dot{q}. \tag{24}$$

The measure of miners increases with the capital gain $\dot{q}$. The next proposition characterizes the unique dynamic equilibrium that converges to (q^s, A^s).

Proposition 2 ***(Transitional Dynamics to Steady-State Monetary Equilibrium)*** *Suppose (16) holds and $A_0 = 0$. There exists a unique monetary equilibrium such that (q_t, A_t) converges to $(q^s, A^s) > 0$. Along the equilibrium path q_t and A_t increase over time. Moreover:*

1. *There exists $t_0 > 0$, such that for all $t < t_0$, $m_t = 1 - A_t$, and*

$$A_t = \frac{\bar{A}\left[1 - e^{-\lambda(1-\bar{A})t}\right]}{1 - \bar{A}e^{-\lambda(1-\bar{A})t}} \tag{25}$$

$$\omega(q_t) = e^{rt}\omega_0\left[\frac{1 - \bar{A}e^{-\lambda(1-\bar{A})t}}{1 - \bar{A}}\right]. \tag{26}$$

2. *If*

$$\frac{\mu'(q^s)/\mu(q^s)}{\omega'(q^s)/\omega(q^s)} > \frac{1-\theta}{\theta}, \tag{27}$$

then $m_t < 1 - A_t$ in the neighborhood of the steady state and convergence to (q^s, A^s) is asymptotic. Otherwise, $m_t = 1 - A_t$ until the steady state is reached in finite time.

Proposition 2 proves the existence and uniqueness of a dynamic equilibrium leading to (q^s, A^s) starting from an initial condition $A_0 = 0$. It allows us to answer the following question: how does the supply of privately-produced money and its price covary over time? The equilibrium features monotone trajectories for q and A over time. As the money supply increases, the price level falls, and quantities traded in pairwise meetings increase.[13] This result seems in contradiction with the quantity theory according to which the price level increases with the money supply and the long-run comparative statics in Table 1 where an increase in $\bar{A}$ reduces q. Intuitively, the value of money must appreciate over time in order to induce agents to mine because as A increases the mining speed decreases but the frequency of trading opportunities in the production sector increases.

Proposition 2 also answers the question: can money be valued if it does not serve as a medium of exchange? Early on, when A is close to 0, all agents without money choose to be miners and all agents with money hoard it because they have no opportunities to use it as a medium of exchange. From the viewpoint of an outside observer, money resembles a pure speculative bubble: it does not play any role in exchange, and hence it should not have any liquidity premium, but its value grows at a rate larger than r. This path for the value of money is sustainable because in finite time money starts being used as a medium of exchange.

[13] As shown in Online Appendix D, we can obtain less dramatic results with alternative matching functions, i.e., agents trade at all dates, but the insight that market tightness measured by the ratio of producers to buyers increases over time is robust.

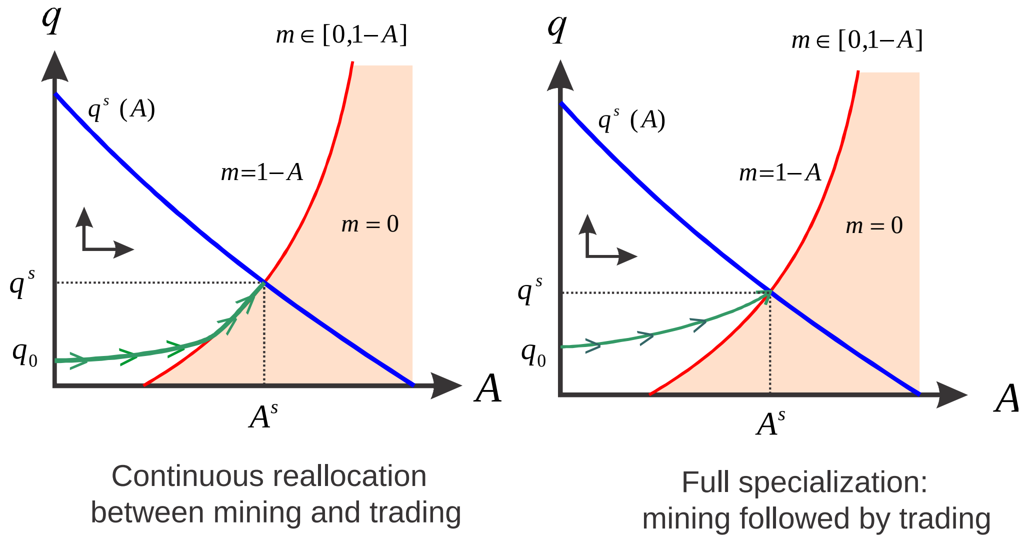

Figure 5: Dynamic equilibria with mining

Can the government prevent the emergence of a private money? The government can remove the incentives to produce money by supplying $A_0 > A^s$. If the initial quantity of money is sufficiently abundant, then the benefits from being in the production sector outweigh the gain from mining. As shown by Table 1, however, the larger λ and $\bar{A}$, the larger A_0 has to be to prevent money mining. In Sections 5.2 we will revisit this question in a version of the model where agents can hold a portfolio of monies.

Finally, we showcase the tractability of the model by solving the equilibrium path in closed form in (25)-(26). This result follows from the observation that the law of motion for A, (19), when $m = 1 - A$, is a Riccati equation that admits an analytical solution (see Section 2.15 in Ince (1956) for details).

In the neighborhood of the steady state we distinguish two types of equilibria illustrated in Figure 5. There are equilibria where occupations coexist, i.e., the economy features both miners and producers. In this case the steady state is only reached asymptotically. There is another type of equilibrium where all agents without money choose to mine until the steady-state money supply has been reached, which happens in finite time. These regimes have implications for the transaction velocity of money measured by $\mathcal{V}_t \equiv \alpha\sigma\,(1 - A_t - m_t)$. In the initial phase, $\mathcal{V}_t = 0$ since all potential producers prefer to mine. If (27) holds, then $\mathcal{V}_t$ becomes positive for some $t > 0$ and it increases as m_t decreases toward its steady-state value. If (27) does not hold, then $\mathcal{V}_t = 0$ until the steady state is reached, at which point $\mathcal{V} = \alpha\sigma\,(1 - A^s)$. Along the equilibrium path there is a positive correlation between the velocity of money, its price, and its supply.[14] The next lemma

[14]In Online Appendix D we describe equilibria assuming that only buyers (money holders) and producers participate in the

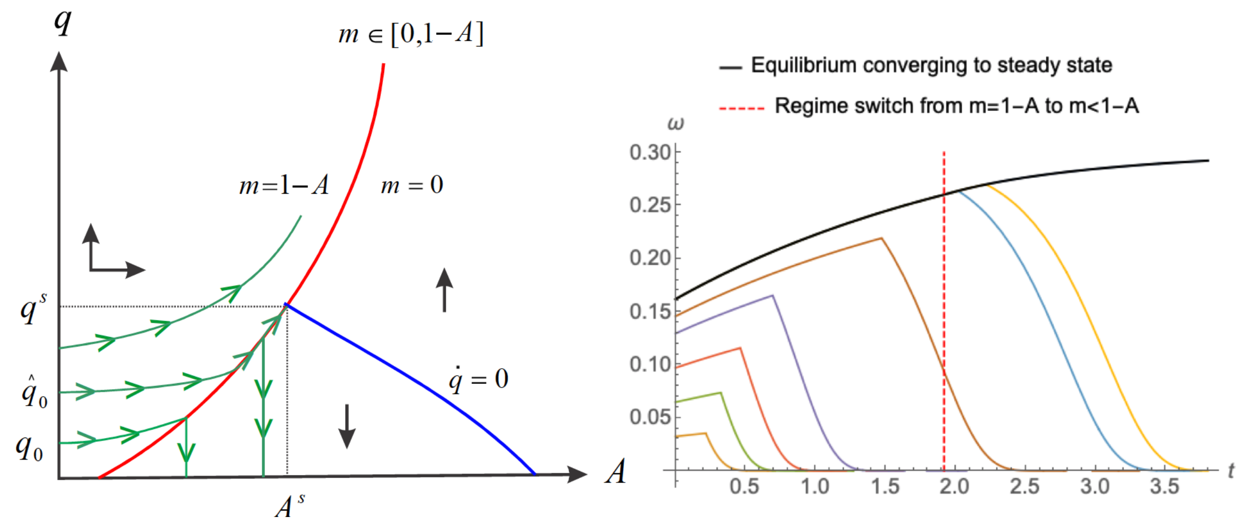

Figure 6: (Left) Phase diagram of different equilibria (Right) The value of money under different equilibria for the same parameters.

provides conditions for mining and production to coexist along the equilibrium path.

Lemma 1 ***(Coexistence of trades and mining)***

1. *If $\epsilon(q) \equiv u'(q)q/u(q)$ is non-increasing in q, then there exists $\lambda^* < +\infty$ and $\kappa^* \in (0, +\infty)$ such that $m_t < 1 - A_t$ in the neighborhood of (A^s, q^s) if and only if $\lambda < \lambda^*$ or $\sigma\alpha \geq \kappa^*$. Moreover, if $\alpha\sigma > \lambda\theta/(1-\theta)$, then the equilibrium features at most one regime switch.*

2. *If $\theta \leq 1/2$ and λ is sufficiently large, then $m_t = 1 - A_t$ for all t such that $A_t < A^s$.*

The condition on the elasticity of $u(q)$ in Lemma 1 is satisfied by $u(q) = q^{1-a}$ or $u(q) = 1 - e^{-aq}$. Part 1 of Lemma 1 establishes that if the efficiency of mining is low and the matching rate is high, then mining and trades coexist close to the steady state. Part 2 of Lemma 1 provides a global characterization of the occupation choice. If the mining technology is sufficiently efficient and if producers have more bargaining power than buyers, then no trade takes place until the supply of money has reached its steady-state level.

Proposition 2 establishes the existence of a unique equilibrium, denoted $(\hat{A}_t, \hat{q}_t)$, that converges to (A^s, q^s). The following proposition aims to address the question of the determination of the initial value of money by characterizing the set of all initial values of the new currency that are consistent with a monetary equilibrium.

matching process according to a constant-returns-to-scale matching function. The matching probability of a buyer is $\alpha(\tau)$, where $\tau = (1 - A - m)/A$ is market tightness expressed as the ratio of sellers to buyers. We assume that $\alpha' > 0$, $\alpha'' < 0$, $\alpha'(0) = +\infty$, $\alpha'(+\infty) = 0$. If $\lim_{\tau \to 0} \alpha(\tau)/\tau = +\infty$, then $\tau > 0$ for all t along the equilibrium path. The velocity of money is $\mathcal{V}_t = \alpha(\tau_t)\sigma$ and it increases along the transitional path leading to the steady state.

It shows that money can be valued and privately produced even it is anticipated that it will disappear in the long run.

Proposition 3 ***(Boom/Bust equilibria)***

1. *For all* $q_0 \in (0, \hat{q}_0)$*, there exist* $0 < T_0 \leq T_1 < +\infty$ *such that a monetary equilibrium exists with the following properties:*

   (a) *Boom phase: For all* $t \leq T_0$*,* $m_t = 1 - A_t$ *and* $\dot{\omega}/\omega = r + \lambda\left(\bar{A} - A\right) > 0$*.*

   (b) *Bust phase: For* $t > T_1$*,* $m_t = 0$*,* $\dot{\omega}_t = r\omega_t - \alpha\sigma\left(\theta - A_{T_1}\right)[u(q_t) - q_t] < 0$*, and* $\lim_{t \to +\infty} \omega_t = 0$*.*

2. *If (27) holds, then there is a continuum of monetary equilibria indexed by* $T \in \{t \in \mathbb{R}_+ : \hat{A}_t = \mu(\hat{q}_t)\}$ *such that* $q_0 = \hat{q}_0$ *and:*

   (a) *Boom phase: For all* $t \leq T$*,* $(A_t, q_t) = (\hat{A}_t, \hat{\omega}_t)$*.*

   (b) *Bust phase: For* $t > T$*,* $m_t = 0$*,* $A_t = \hat{A}_T$*,* $\dot{\omega}_t = r\omega_t - \alpha\sigma\left(\theta - \hat{A}_T\right)[u(q_t) - q_t] < 0$*, and* $\lim_{t \to +\infty} \omega_t = 0$*.*

There is a continuum of monetary equilibria featuring a boom and a bust of the currency price. Those equilibria can be indexed by the initial value of money in the interval $(0, \hat{q}_0)$. If the initial beliefs are not optimistic enough to bootstrap the value of money to $\hat{q}_0$, then a boom/bust equilibrium exists.[15] Along the equilibrium path the value of money first increases at a rate larger than r. It reaches a maximum at which point agents stop mining. Even though the money supply remains constant afterwards, the value of money declines and converges to 0 asymptotically. In the phase diagram of Figure 6, the equilibrium path is upward sloping until it reaches the locus $A = \mu(q)$. At that point it becomes vertical since the money supply remains constant with arrows of motion oriented toward the horizontal axis since money loses its value over time.[16]

There can also be boom/bust equilibria where $q_0 = \hat{q}_0$. Such equilibria occur when agents are indifferent between mining or producing in the neighborhood of the steady state, i.e., (27) holds, and they are indexed

[15] Such equilibria capture the idea that new currencies might be likely to fail in the absence of coordination mechanisms. See Selgin (1994) for historical examples.

[16] The result according to which the initial value of a new currency is indeterminate was acknowledged by earlier adopters of Bitcoins. Luther (2018) reports the following post on bitcoin-list in January 2009, the month when Bitcoin was first introduced:

> "One immediate problem with any new currency is how to value it. Even ignoring the practical problem that virtually no one will accept it at first, there is still a difficulty in coming up with a reasonable argument in favor of a particular non-zero value for the coins."

by the time T at which the value of money starts falling. Such an equilibrium path is represented in the left panel of Figure 6 by a trajectory starting at $q_0 = \hat{q}_0$. The trajectory is upward sloping and follows the $A = \mu(q)$ locus for a while until it becomes vertical and oriented toward the horizontal axis. From the viewpoint of an outside observer it would be impossible to tell whether the currency will be successful until the time T at which the currency starts declining has been reached.

The next Proposition shows that among the continuum of equilibria described above a single equilibrium survives if we endow money with a commodity value, $d > 0$, and consider the limit as d goes to 0.

Proposition 4 ***(Interest-bearing money)*** *Suppose money pays an arbitrarily small interest $d > 0$. Then, there exists a unique equilibrium and it is such that $(A_t, q_t) \rightarrow (A^s, q^s)$ as $t \rightarrow \infty$.*

If money generates a flow dividend $d > 0$ (e.g., a utility flow from a commodity money or a real interest payment), then any equilibrium path must be such that the value of holding money is bounded below by the discounted sum of dividends, d/r.[17] This observation rules out the continuum of boom-and-bust equilibria where the value of money vanishes asymptotically. The important insight from Proposition 4 is that d does not need to be large to prevent the collapse of the currency. As long as $d > 0$, crypto-currencies can be priced above their fundamental value, d/r, without losing their value in the long run.

3.4 Implementing price stability

So far the mining technology specifies an individual mining rate, $\lambda(\bar{A} - A)$, as a function of the quantity of money left to be mined. The aggregate mining rate is then the product of the individual mining rate and the measure of miners, $m\lambda(\bar{A} - A)$, which then determines the dynamics of the money supply according to (11). In the context of crypto-currencies the logic is reversed: the designer of the currency chooses a path for the money supply, $\dot{A}_t = \pi_t A_t$, where π_t is the rate of money creation at time t. The seigniorage revenue, $\pi_t A_t$, is exogenous and independent of the measure of miners. This revenue is divided among miners by adjusting the individual mining rate, denoted λ_t, so that aggregate mining is equal to seigniorage, i.e., $\lambda_t m_t = \pi_t A_t$. Hence, the individual mining rate is $\lambda_t = \pi_t A_t / m_t$, which is increasing with the rate of money creation but decreasing with the measure of miners. An equilibrium is a bounded solution, $\{\omega_t, A_t, m_t\}$, to the following

[17] This commodity-money refinement is often used to reduce the equilibrium set of pure currency economies. See. e.g., Wallace and Zhu (2004) or Garratt and Wallace (2018).

system:

$$\dot{\omega} = \left(r + \frac{\pi A}{m}\right)\omega - \alpha\sigma\left(1 - A - m\right)\theta\left[u(q) - q\right], \tag{28}$$

$$\dot{A} = \pi_t A \tag{29}$$

$$m = \min\left\{\frac{\pi\omega}{\alpha\sigma(1-\theta)\left[u(q) - q\right]}, 1 - A\right\}. \tag{30}$$

Obviously, any equilibrium characterized in Proposition 2 is also an equilibrium of the economy where A_t is chosen by the currency designer.

Suppose the currency designer chooses the rate of money creation, π_t. Can he choose it so as to achieve price stability? Let $\omega = \omega^s(\bar{A})$ be the targeted value of money where the corresponding long-run money supply is $\bar{A}$. The next proposition derives the path for π_t such that q_t and ω_t are constant over time.

Proposition 5 ***(Stabilizing the value of money)***

1. *In any equilibrium where the value of money is constant, $\omega = \omega^s(\bar{A})$, the rate of money creation evolves according to:*

$$\pi_t^* = \pi^*(A_t) \equiv \frac{1-\theta}{\theta}\left[\alpha\sigma\left(\theta - A_t\right)\frac{u(q^s) - q^s}{\omega^s} - r\right]. \tag{31}$$

2. *There exists a unique equilibrium and it features price stability if money pays an arbitrarily small dividend and the money growth rate is determined by the following rule:*

$$\pi(\omega, A) = \begin{cases} \pi^*(A) & \omega = \omega^s \\ 0 \quad\quad \text{if} & \omega < \omega^s \\ \frac{\alpha\sigma(1-\theta)[u(q)-q](1-A)}{\omega} & \omega > \omega^s. \end{cases} \tag{32}$$

Proposition 5 shows that it is possible to choose a path for the money supply that implements price stability. From (31) the money growth path is such that π decreases over time and it approaches $\pi = 0$ as A approaches $\bar{A}$. The equilibrium measure of miners corresponding to this path decreases over time and converges to 0.

The second part of Proposition 5 provides a policy rule to implement price stability as a unique equilibrium. The rule, (32), is such that the money growth rate is lowered to 0 if ω falls below its steady-state value, ω^s, and it is raised to a level such that $m = 1 - A$ if ω rises above ω^s. As a result, the only path consistent with an equilibrium is the one where $\omega_t = \omega^s$ for all t.

3.5 Price waves

Historically, the world supplies of silver and gold have increased through sequential discoveries of new mining sites, e.g., South America during the 16-17th centuries, South Africa and Australia during the 19-20th centuries. In the context of crypto-currencies, one can interpret mine discoveries as the introduction of new currencies. In order to capture such discoveries and their impact on price dynamics, we describe a sequence of unanticipated shocks on $\bar{A}$ starting from a steady state. Initially, $\bar{A} = \bar{A}_0$ and the economy is at a stationary equilibrium (q_0^s, A_0^s). At time 0, the maximum amount of money agents can mine, $\bar{A}$, increases from $\bar{A}_0$ to $\bar{A}_1$. This could correspond to a new estimate of the gold resources of the planet or a coordinated change in the supply of Bitcoins. After the economy reaches a new steady state, (q_1^s, A_1^s), another discovery happens that raises the potential money supply from $\bar{A}_1$ to $\bar{A}_2$. And so on.

In the phase diagram of Figure 7 the locus $A = \mu(q)$ shifts to the right. The new steady state is such that the money supply increases, $A_1^s > A_0^s$, and money loses some value, $q_1^s < q_0^s$. At time 0^+, q falls below q_1^s so that the value of money overshoots its steady state. Along the transition to the new steady state the value of money increases. The sequence of unanticipated increases in $\bar{A}$ generates fluctuations in the value of money around a downward trend. We summarize these results in the following proposition.

Proposition 6 ***(Unanticipated mine discoveries)*** *Consider a sequence of unanticipated increases in $\bar{A}$ (i.e., mine discoveries). Every mine discovery triggers a downward jump in q followed by a gradual recovery. Over time q exhibits a downward trend.*

The response to an unanticipated increase in the mining intensity, λ, are similar to those of an increase in $\bar{A}$: the value of money falls on impact and increases afterwards to reach a new steady state with a lower q and higher A.

4 Efficient mining

We now ask whether the decentralized private production of money can generate a socially efficient outcome. We describe the problem of a social planner who is subject to the mining technology and the matching technology between asset holders and producers. Implicit in the latter constraint is the requirement that all trades take the form of one unit of money for some q, i.e., trades are quid pro quo. The planner chooses agents' occupation and output in pairwise meetings in order to maximize the discounted sum of all agents'

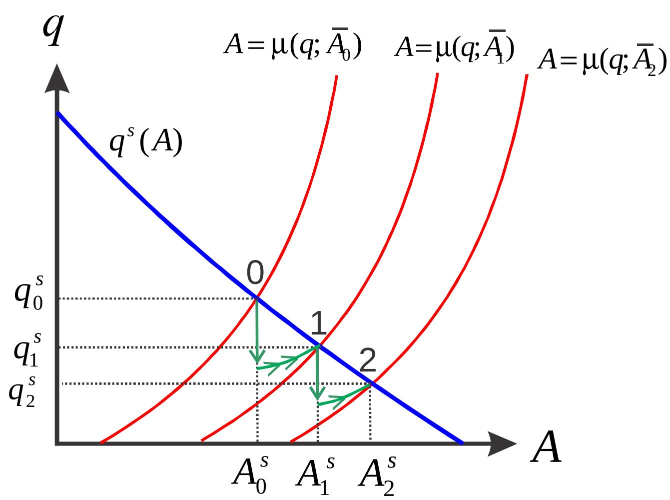

Figure 7: Mine discoveries: Unanticipated increases of $\bar{A}$

utilities (The planner's problem is written explicitly in Lemma 2 in the Appendix). We then provide an incentive-feasible mechanism to implement such constrained-efficient allocations. In the following recall that q^* is the solution to $u'(q^*) = 1$ and it is the efficient level of production in a trade meeting.

Proposition 7 ***(Constrained-efficient allocation)*** *Assume $A_0 = 0$.*

1 Efficient allocations are such that $q_t = q^*$ for all t and $m_t = 1 - A_t$ for all $t < T^*$ where $T^* > 0$ is the time it takes to mine A^* where

$$A^* = \frac{1}{4}\left[\left(2\bar{A} + 1 + \frac{r}{\lambda}\right) - \sqrt{\left(2\bar{A} + 1 + \frac{r}{\lambda}\right)^2 - 8\bar{A}}\right]. \tag{33}$$

For all $t \geq T^*$, $m_t = 0$ and $A_t = A^*$.

2 Implementation. If

$$r \leq \frac{\alpha\sigma(1 - A^*)\left[u(q^*) - q^*\right]}{q^*} \tag{34}$$

$$\lambda\left(\bar{A} - A^*\right) \leq \frac{\alpha\sigma A^*(1 - A^*)\left[u(q^*) - q^*\right] - rq^*}{(1 - A^*)u(q^*) + A^*q^*}, \tag{35}$$

then the constrained-efficient allocation is implementable with

$$\begin{aligned} \theta_t &= 1 \text{ if } t < T^* \\ &= \theta^* \equiv \frac{rq^* + (\alpha\sigma A^* + r)\left[u(q^*) - q^*\right]}{\left[u(q^*) - q^*\right](\alpha\sigma + r)} \quad \text{if } t \geq T^*. \end{aligned} \tag{36}$$

The planner chooses q^* in all trade matches and it assigns all non-asset holders to mining until the efficient quantity of money has been produced. We show in the proof of Proposition 7 that along the optimal path the shadow value of money ξ (i.e. the co-state variable associated with A_t) satisfies

$$\frac{\dot{\xi}}{\xi} = r + \lambda(1 + \bar{A} - 2A).$$

If we compare with the equilibrium ODE, (18), when $m = 1 - A$,

$$\frac{\dot{\omega}}{\omega} = r + \lambda\left(\bar{A} - A\right),$$

we see that the rate of growth of ξ is larger than the rate of growth of ω by a term equal to $\lambda(1 - A)$. This additional term reflects the fact that the planner internalizes the scarcity effect of digging new coins on future miners: as more money is taken out of the ground, it becomes harder for future miners to find new units of money. As a result, the growth in the shadow value of money must compensate for that increased scarcity. The optimal quantity of money, A^*, is less than 1/2, which is the quantity that would maximize the measure of trades. As agents become infinitely patient, $\lim_{r \to 0} A^* = \min\{1/2, \bar{A}\}$. By comparing (33) and (17) we obtain the following corollary regarding the efficiency of the private production of money:

Corollary 1 *$A^s > A^*$ if $\theta > 1/2$ and $A^s < A^*$ if $\theta < 1/2$.*

There is over-production of money in the decentralized equilibrium if buyers receive a larger share of the match surplus than producers. Even if $\theta = 1/2$ so that $A^s = A^*$, the output produced in the decentralized equilibrium is different from the first best.

In the first part of Proposition 7 we restricted trades to take place between money holders and producers but we did not impose that the actual trades in pairwise meetings satisfy individual rationality constraints. In the second part of Proposition 7 we propose an incentive-feasible trading mechanism that implements the constrained-efficient allocation. The mechanism is incentive feasible if it satisfies the individual rationality constraints of the buyer, $u(q) + V_0 - V_1 \geq 0$, and the producer, $-q + V_1 - V_0 \geq 0$, in a pairwise meeting. Any incentive-feasible trading mechanism can be described by a sequence of time-varying bargaining shares, θ_t. From (36) the incentive-feasible trading mechanism that implements the constrained-efficient allocation is such that buyers have all the bargaining power until the efficient quantity of money, A^*, has been dug at time T^*. Giving no bargaining power to producers initially guarantees that agents without money choose to be miners rather than producers. Following T^* the buyer's bargaining power is $\theta^* > 0$, which is the value that implements q^* in all pairwise meetings.

Condition (34) is the standard implementation condition of the first best in monetary search models (see, e.g., Wright 1999). It requires that the opportunity cost of holding money as measured by rq^* is smaller than the expected surplus from holding money assuming the buyer has all the bargaining power, $\alpha\sigma(1 - A^*)\left[u(q^*) - q^*\right]$. A key difference from the existing literature is that the money supply here is endogenous and depends on fundamentals. As r approaches 0, A^* tends to $\min\{\bar{A}, 1/2\}$. Hence the implementation condition is satisfied for r sufficiently small.

Condition (35) is new and guarantees that agents have no incentive to over-produce money. Assuming that the buyer's bargaining share is θ^*, it requires that the expected gain from mining, $\lambda\left(\bar{A} - A^*\right)\omega^*$ where $\omega^* \equiv (1-\theta^*)u(q^*)+\theta^* q^*$, is no greater than the expected gain from being a producer $\alpha\sigma(1-\theta^*)\left[u(q^*) - q^*\right]$. If $\bar{A} < 1/2$, then this condition holds for r sufficiently close to 0.

Finally, suppose that the planner can design the mining technology by choosing both λ and $\bar{A}$. If the planner can enforce both q_t and m_t then welfare is non-decreasing in λ and $\bar{A}$. From (33), as λ goes to infinity then A^* tends to $\min\{1/2, \bar{A}\}$. Hence, it is optimal to increase the money supply to $A^* = 1/2$ as fast as possible—the model does not provide a rational for a gradual increase of the money supply in the absence of technological limitations. The efficient allocation can be implemented if (34) holds. In order to guarantee that (35) holds, one can set $\bar{A} = 1/2$. We summarize these results in the following proposition.

Proposition 8 ***(Optimal currency design)*** *Suppose the planner can choose λ and $\bar{A}$. If (34) holds, then it is optimal to set $\lambda = +\infty$ and $\bar{A} = 1/2$ and $q = q^*$ is implementable.*

5 Divisible assets and centralized exchanges

So far, we have described an economy with indivisible assets and a shortage of liquidity. We now study price dynamics when all assets are perfectly divisible and individual asset holdings are unrestricted, $a \in \mathbb{R}_+$. This model, which is a continuous-time version of the New-Monetarist model of Lagos and Wright (2005), will be useful to check the robustness of our earlier results and to study the competition between a privately-produced money and government money.[18]

[18] Choi and Rocheteau (2019) provides a detailed description of the New Monetarist model in continuous time and its solution methods.

In order to keep the model tractable, we add a centralized market (CM) where price-taking agents can trade continuously a good, distinct from the one traded in pairwise meetings, for money. The purpose of these CMs is to allow agents to readjust their money holdings to some targeted level in-between pairwise meetings, so as to keep the distribution of money holdings degenerate. In reality, the CMs could correspond to the several exchanges where individuals trade crypto-currencies for different government-supplied currencies using credit or debit cards (e.g., Coinbase, Coinmama, Luno...). In the following we take the CM good as the numeraire. Agents have the technology to produce h units of the numeraire good at a linear cost h ($h < 0$ is interpreted as consumption). Hence, agents' discounted lifetime utility in-between pairwise meetings is $-\int_0^{+\infty} e^{-rt} dH(t)$ where $H(t)$ is a measure of the cumulative production of the numeraire good (net of its consumption) up to t. This formulation allows agents to produce or consume the numeraire good in flows (in which case $H(t)$ admits a density $h(t)$) or in discrete amounts (in which case $H(t^+)-H(t^-) \neq 0$). Preferences during pairwise meetings are as before. Money takes the form of a Lucas tree that pays a dividend flow $d \geq 0$. The case $d = 0$ corresponds to fiat money. The CM price of the asset is denoted ϕ_t.

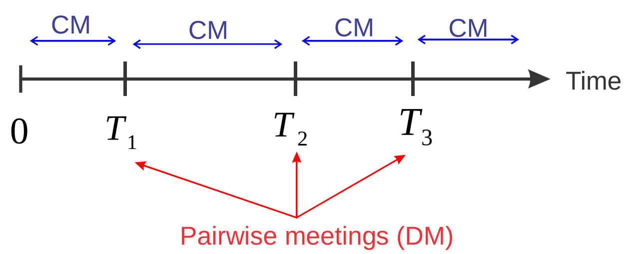

Figure 8: Timing of the model with divisible assets

5.1 Equilibrium

Let $V(a)$ denote the value function of an agent with a units of assets expressed in terms of the numeraire. At any point in time between pairwise meetings, an agent can readjust her asset holdings by consuming or producing the numeraire good. Formally,

$$V(a) = \max_h \{-h + V(a+h)\} = a + \max_{a^* \geq 0} \{-a^* + V(a^*)\},$$

where h is the production of the numeraire and a^* is the agent's targeted asset holdings (expressed in terms of the numeraire). The value function, $V(a)$, is linear in a.

We now turn the bargaining problem in a pairwise meeting between a buyer holding a^b units of assets and a seller holding a^s units of asset. The outcome of the negotiation is a pair $(q,p) \in \mathbb{R}_+ \times [-a^s, a^b]$ where q is the amount of goods produced by the seller for the buyer and p is the transfer of assets from the buyer to the seller. Feasibility requires that $-a^s \leq p \leq a^b$. By the linearity of $V(a)$ the buyer's surplus is $u(q) + V(a^b - p) - V(a^b) = u(q) - p$ and the seller's surplus is $-q + V(a^s + p) - V(a^s) = -q + p$. According to the Kalai proportional solution, the buyer's surplus is equal to a fraction θ of the total surplus of the match, i.e., $u(q) - p = \theta\,[u(q) - q]$. Moreover, the solution is pairwise Pareto efficient, which implies that $q \leq q^*$ with an equality if $p \leq a^b$ does not bind. Using the notation $\omega(q)$ from (4), the buyer's consumption as a function of her asset holdings, $q(a^b)$, is such that $q(a^b) = q^*$ if $a^b \geq \omega(q^*)$ and $\omega(q) = a^b$ otherwise.

We can now write the lifetime expected utility of the agent holding her targeted asset holdings, $V(a^*)$. In Choi and Rocheteau (2019) we show it solves the following HJB equation that is reminiscent to (5) and (6) combined:

$$\begin{aligned} rV(a^*) &= \rho a^* + \alpha(1-m)\sigma\theta\left\{u\left[q(a^*)\right] - q(a^*)\right\} \\ &+ \max\left\{\alpha\sigma(1-\theta)\left\{u\left[q(\bar{a})\right] - q(\bar{a})\right\}, \lambda(\bar{A} - A)\phi\right\} + \dot{V}(a^*), \end{aligned} \tag{37}$$

where the rate of return of assets is

$$\rho = \frac{d + \dot{\phi}}{\phi}. \tag{38}$$

The first term on the right side of (37) is the flow return of the asset. The second term is analogous to the first term on the right side of the HJB equation for V_1, (5). The agent receives an opportunity to consume at Poisson arrival rate $\alpha\sigma$. The partner can produce if she is not a miner, with probability $1 - m$. The third term on the right side of (37) is analogous to the right side of the HJB equation for V_0 in (6). It corresponds to the occupational choice according to which agents can choose to be producers and enjoy the flow payoff $\alpha\sigma(1-\theta)\left\{u\left[q(\bar{a})\right] - q(\bar{a})\right\}$ or miners and enjoy $\lambda(\bar{A} - A)\phi$. The term $\bar{a}$ represents asset holdings of other agents in the economy. The expected gain from mining describes the assumption that at Poisson arrival rate $\lambda(\bar{A} - A)$ the miner digs a unit of money which is worth ϕ units of numeraire. The last term is the change in the value function for a given asset position, $\dot{V}(a) = \partial V_t(a)/\partial t$.

The envelope condition associated with (37) together with $V'(a^*) = 1$ gives

$$r - \rho = \alpha(1-m)\sigma\theta\left\{\frac{u'\left[q(a^*)\right] - 1}{(1-\theta)u'\left[q(a^*)\right] + \theta}\right\}, \tag{39}$$

where we have used that $q'(a) = 1/\omega'(q)$ if $a < \omega(q^*)$ and $\partial^2 V(a)/\partial a \partial t = 0$ because $V'(a^*) = 1$ at all t. The opportunity cost of holding the asset on the left side of (39) is the difference between the rate of time preference and the real rate of return of the asset. The right side is the marginal value of an asset if a consumption opportunity arises.

Since now agents can carry money and mine at the same time, the measure of miners solves

$$m \begin{cases} = 1 \\ \in [0,1] \\ = 0 \end{cases} \quad \text{if} \;\; \lambda(\bar{A}-A)\phi \begin{cases} > \\ = \\ < \end{cases} \alpha\sigma(1-\theta)\left[u\left(q\right)-q\right]. \tag{40}$$

By market clearing:

$$a^* = \phi A. \tag{41}$$

The supply of assets evolves according to:

$$\dot{A} = \lambda m(\bar{A}-A). \tag{42}$$

An equilibrium is a list, $\langle a_t^*, m_t, \phi_t, A_t \rangle$, that solves (39), (40), (41), and (42).

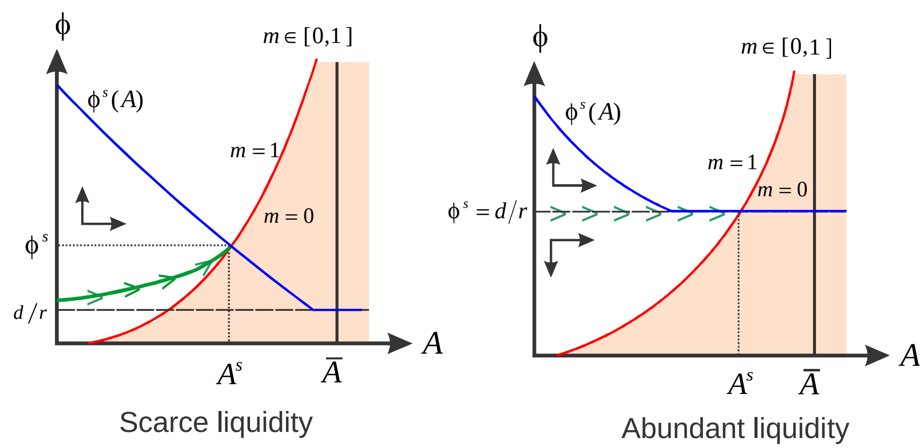

Figure 9: Phase diagram with divisible assets

From (40) and (41) the locus of pairs (A, ϕ) such that agents are indifferent between mining and producing is given by:

$$\lambda(\bar{A}-A)\phi = \alpha\sigma(1-\theta)S(A\phi), \tag{43}$$

where $S(A\phi) = u\left(q\right) - q$ is the total surplus from a pairwise trade where quantities solve $\omega(q) = \min\left\{\omega(q^*), \phi A\right\}$. For all $A > \lambda\bar{A}/(\alpha\sigma+\lambda)$, there is a unique $\phi > 0$ solution to (43). In the (A, ϕ) space, this solution is upward sloping and it has a vertical asymptote, $A = \bar{A}$. To the left of this locus all agents choose to mine, $m = 1$.

The locus of stationary equilibria where $m = 0$ and $\dot{\phi} = 0$ is given by

$$r = \frac{d}{\phi} + \alpha\sigma\theta \left\{ \frac{u'(q) - 1}{(1-\theta)u'(q) + \theta} \right\}, \tag{44}$$

where q is an implicit function of ϕA. It is a standard asset pricing equation where the effective rate of return of the asset is composed of the pecuniary return, d/ϕ, and the liquidity return. It gives a negative relationship between ϕ and A with $\phi = d/r$ for all A such that $A \geq r\omega(q^*)/d$. The two loci, (43) and (44), represented in the phase diagram of Figure 9, allow us to characterize stationary and non-stationary equilibria.

Proposition 9 ***(Mining divisible assets)*** *Suppose* $A_0 = 0$.

1. ***(Abundant liquidity)*** *If*

$$\frac{\bar{A}d}{r} \geq \frac{\alpha\sigma(1-\theta)\left[u(q^*) - q^*\right]}{\lambda} + \omega(q^*), \tag{45}$$

then there exists a unique equilibrium and it is such that $\phi_t = d/r$, $q_t = q^*$ *for all* t, *and* $m_t = \mathbf{1}_{\{t<T\}}$ *where*

$$T = \ln\left\{ \frac{\lambda d\bar{A}}{r\alpha\sigma(1-\theta)\left[u(q^*) - q^*\right]} \right\}^{\frac{1}{\lambda}} < +\infty.$$

Moreover,

$$\begin{aligned} A_t &= \bar{A}\left(1 - e^{-\lambda t}\right) \quad \textit{for all } t < T \\ A_t &= A^s = \bar{A} - \frac{r\alpha\sigma(1-\theta)\left[u(q^*) - q^*\right]}{\lambda d} \quad \textit{for all } t \geq T. \end{aligned}$$

2. ***(Scarce liquidity)*** *If* $d > 0$ *and (45) does not hold, then there exists a unique equilibrium and it is such that* $\phi_t > d/r$ *and* $q_t < q^*$ *for all* t. *Moreover,* $m_t = 1$ *for all* t *sufficiently small and the economy transitions to* $m_t < 1$ *before reaching the steady state if*

$$\frac{\theta}{1-\theta} > \frac{\nu(A\phi) + A/(\bar{A} - A)}{\nu(A\phi)\left[1 - \nu(A\phi)\right]} \tag{46}$$

where $\nu(x) \equiv S'(x)x/S(x)$.

3. ***(Fiat money)*** *If* $d = 0$, *then there exists a unique equilibrium leading to a monetary steady state if* $r < \alpha\sigma\theta/(1-\theta)$. *The equilibrium is such that* $q_t < q^*$ *and* $\dot{\phi}_t \geq 0$ *for all* t *and the steady state solves*

$$r = \alpha\sigma\theta \left\{ \frac{u'(q^s) - 1}{(1-\theta)u'(q^s) + \theta} \right\} \tag{47}$$

$$\frac{A^s}{\bar{A}} = \frac{\lambda\omega(q^s)}{\alpha\sigma(1-\theta)\left[u(q^s) - q^s\right] + \lambda\omega(q^s)}. \tag{48}$$

There is a continuum of boom-burst equilibria indexed by ϕ_0 such that: $\dot{\phi}_t > 0$ for all $t < T(\phi_0)$ where $T(\phi_0) > 0$; $\dot{\phi}_t < 0$ for all $t > T(\phi_0)$; $\lim_{t \to +\infty} \phi_t = 0$.

Proposition 9 provides two main insights relative to the model with indivisible money and no centralized exchanges. First, there is a regime where the asset supply at the steady state is abundant enough to satiate agents' liquidity needs and to allow agents to trade q^* in all matches. In such equilibria, the asset is priced at its fundamental value at all dates. A necessary (but not sufficient) condition is that the potential asset supply when valued at its fundamental price, $\bar{A}d/r$, is larger than agents' liquidity needs, $\omega(q^*)$. It is the standard condition in the literature for abundant liquidity since Geromichalos et al. (2007), except that it applies to the potential asset supply, $\bar{A}$, and not the actual asset supply, A, which is endogenous. Condition (45) has an extra term that captures agents' incentives to stop mining. This term decreases with λ so that liquidity is more likely to be abundant when the mining speed is large. Maybe surprisingly, $q_t = q^*$ even before market capitalization has reached $\omega(q^*)$. The reason for this result is that as long as the steady-state asset supply has not been reached, all agents mine and hence there is no demand for liquidity. When the steady state is reached, in finite time, then agents start trading.

The second insight is that there is a regime with scarce liquidity that is qualitatively similar to the equilibria of the model with indivisible money. The price of the asset is above its fundamental value at all dates and it keeps increasing over time until it reaches a steady state.[19]

In the case where the asset is a pure fiat money, $d = 0$, then q^s is independent of $\bar{A}$ and λ, and the endogenous money supply is proportional to the potential supply. As in the model with indivisible money, there are a continuum of boom/burst monetary equilibria indexed by the initial value of money where the value of money increases and then decreases and goes to zero asymptotically. The next corollary shows changes in $\bar{A}$ are neutral in the long run but have real effects along the transitional path.

Corollary 2 ***(Money neutrality)*** *Assume $d = 0$. An increase in $\bar{A}$ leads to a proportional increase in A^s in the long run and no real effects. In the short run, aggregate real balances and output fall. During the transition to the new steady state, the inflation rate is negative.*

[19] If agents can search for trading partners while mining and $r < \alpha\sigma\theta/(1-\theta)$, then the unique equilibrium leading to the steady state is such that ϕ decreases over time. The phase diagram is analogous to the middle panel in Figure 13 in Online Appendix B. The logic for the price dynamics is now consistent with the quantity theory. The value of money is high initially because the supply is low. As the supply increases, the value of money decreases. A key difference with respect to the model with indivisible studied earlier is that all agents irrespective of their money holdings receive the revenue from mining.

The condition for a regime switch, (46), in the case of fiat money ($d = 0$) can be rewritten as:

$$\frac{r(1-\theta)+\theta\lambda}{r\theta} < \frac{\alpha\sigma\theta\left[u(q^s)-q^s\right]-r\omega(q^s)}{\alpha\sigma\theta\left[u(q^s)-q^s\right]}.$$

It does not hold when r is close to 0, i.e., agents specialize in mining until the steady state is reached, and it reduces to $\theta\lambda < r(2\theta-1)$ when $\alpha\sigma$ goes to $+\infty$. The fact that this condition depends on λ shows that even though λ does not affect the steady-state equilibrium, it does matter for the transition leading to it.

5.2 Competing monies

The model with unrestricted asset portfolios described in this section is useful to study competition among currencies. In the following we investigate the competition between a fiat money supplied by the government and privately-produced fiat money. We ask under which conditions the two monies coexist and whether the government can choose monetary policy to prevent the production of the private money.[20]

Suppose now that there are two fiat monies: the privately produced money (labelled b) and the government-produced money (labelled g). The two monies differ by their acceptabilities in different meetings (e.g., Lester et al., 2012). There is fraction χ_b of meetings where only privately-produced money is acceptable. For instance, the private money is accepted as means of payment for illegal transactions, e.g., on online black markets such as Silk Road, whereas government money is not.[21] By symmetry, there is a fraction χ_g of meetings where only government-produced money is acceptable, e.g., in transactions that involve the government.[22] In the remaining fraction, χ_2, both types of monies are acceptable. We denote $\alpha_j = \alpha\chi_j$ for $j \in \{b, g, 2\}$. By the same reasoning as above the demand for the two monies satisfies the following Euler equations for $j = b, g$:

$$r-\rho_j = \alpha_j(1-m)\sigma\theta\left\{\frac{u'(q_j)-1}{(1-\theta)u'(q_j)+\theta}\right\} + \alpha_2(1-m)\sigma\theta\left\{\frac{u'(q_2)-1}{(1-\theta)u'(q_2)+\theta}\right\} \tag{49}$$

where q_j indicates output in a match of type $j \in \{b, g, 2\}$.[23] The term on the left side, $r-\rho_j$, is the cost of holding money j. The terms on the right side represent the liquidity services that money j provides at

[20] In Online Appendix F we consider two privately-produced monies and ask how miners allocate their efforts between the two monies. We provide conditions for the two monies to coexist.

[21] Foley et al. (2018) found that approximately one-quarter of Bitcoin users are involved in illegal activity and estimate that around \$76 billion of illegal activity per year involves Bitcoin (46% of Bitcoin transactions), which is close to the scale of the US and European markets for illegal drugs.

[22] In the search-theoretic model of Aiyagari and Wallace (1997) and Li and Wright (1998) the government is a positive measure of agents who participate in the matching process with private agents and whose trading strategies, e.g., which objects they accept in payments, are part of the government policy.

[23] This description with three types of meetings where assets have different acceptabilities is analogous to the model with money and bonds of Rocheteau et al. (2017).

the margin in different types of matches. The second term is common to both monies while the first term is specific to each money.

We assume that monetary policy aims at keeping q_g constant by varying ρ_g — a form of price level targeting. As a result, aggregate real balances supplied by the government, $\omega(q_g)$, are constant. It follows that the quantity traded in type-b and type-2 matches solve:

$$\omega(q_2) = \min\left\{\omega(q_g) + \phi_b A_b, \omega(q^*)\right\} \tag{50}$$

$$\omega(q_b) = \phi_b A_b. \tag{51}$$

If the total market capitalization of the two monies (in terms of the numeraire) is larger than $\omega(q^*)$, then agents trade q^* in type-2 matches and spend only a fraction of their money holdings. Otherwise, they spend all their money, both private and public. The indifference condition between occupations becomes

$$\lambda(\bar{A}_b - A_b)\phi_b = \sum_{j \in \{b,g,2\}} \alpha_j \sigma(1-\theta)\left[u\left(q_j\right) - q_j\right]. \tag{52}$$

As before, this condition gives a positive relationship between ϕ_b and A_b. The law of motion for A_b is

$$\dot{A}_b = \lambda(\bar{A}_b - A_b)m. \tag{53}$$

An equilibrium is a list $(\rho_b, \rho_g, \phi_b, q_2, q_b, m, A_b)$ that solves (49), (50), (51), (52), (53) and $\rho_b = \dot{\phi}_b/\phi_b$.

We represent equilibria on the phase diagram in Figure 10. The indifference locus between occupations, (52), is represented by an upward-sloping red curve. The isocline for ϕ_b, from (49), is such that

$$r = \sigma\theta\left\{\alpha_b\left[\frac{u'\left(q_b\right) - 1}{(1-\theta)u'\left(q_b\right) + \theta}\right] + \alpha_2\left[\frac{u'\left(q_2\right) - 1}{(1-\theta)u'\left(q_2\right) + \theta}\right]\right\}, \tag{54}$$

where q_b and q_2 are given by (50)-(51). It is represented by a downward-sloping blue curve. The dynamics are qualitatively similar to the ones studied earlier. In particular, ϕ_b and A_b increase over time until they reach a steady state. Since there is a stock of government money, some agents might choose to become producers even when $A_b = 0$, as illustrated on the left panel of Figure 10. On the right panel, all agents are miners initially and a fraction start producing when the quantity of private money is sufficiently abundant. Note that when $m = 1$ the rate of return of the government money must be $\rho_g = r$ for agents to be willing to hoard it until it can serve as means of payment.

At the steady state $m = 0$ and $\rho_b = 0$, which pins down q_b and q_2. The indifference condition between

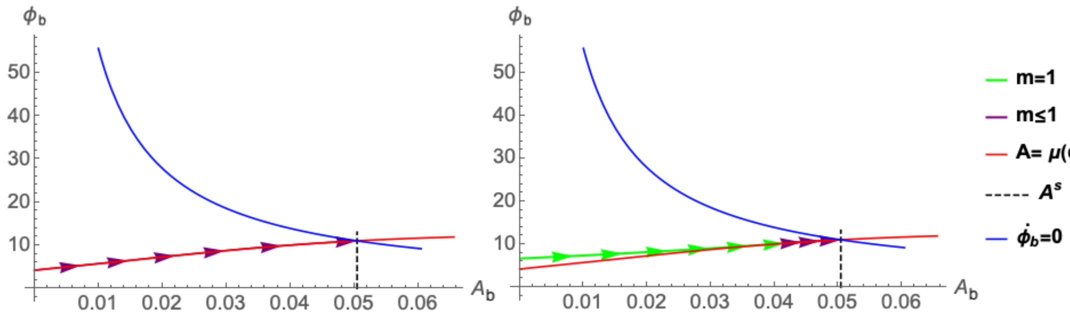

Figure 10: Two numerical examples of equilibrium. (Left) Agents are indifferent between producing and mining even when $A_b = 0$. (Right) Agents specialize in mining early on.

occupations gives ϕ_b. We can now determine a condition under which the government can prevent the emergence of the private money.

Proposition 10 ***(Preventing the emergence of a private money)*** *There is no equilibrium with production of private money if*

$$\frac{\alpha_b \sigma \theta}{1-\theta} + \alpha_2 \sigma \theta \left[\frac{u'(q_g) - 1}{(1-\theta)u'(q_g) + \theta} \right] < r. \tag{55}$$

The proof based on (54) is straightfoward and is therefore omitted. By raising q_g the government reduces the liquidity shortage, and hence incentives to produce the private money. However, if $\alpha_b \sigma \theta > r(1-\theta)$, even the Friedman rule ($q_g = q^*$) is not enough to eliminate the private money. The government must reduce the fraction of matches where the private money is the only means of payment.

6 Conclusion

This paper was motivated by the recent multiplication of crypto-currencies and the questions raised about the dynamics of their prices and their usefulness for transaction. In order to address these questions we studied the dynamics of a privately-produced money and its price in a continuous-time, random-matching economy. We studied indivisible and divisible monies, commodity and fiat monies, single and dual currency systems, steady states and out-of-steady-state dynamics, equilibria and social optima. Our model is versatile and can potentially address a wide array of issues related to private monies. Unanswered questions include the determination of the relative prices of competing private monies, the formalization of the competition among currency designers, the role of reputation and public monitoring for the private provision of monies, and the role of miners of crypto-currencies to authenticate and validate transactions.

References

[1] Aiyagari, Rao, Neil Wallace, and Randall Wright (1996). "Coexistence of money and interest-bearing securities." Journal of Monetary Economics 37, 397-419.

[2] Aiyagari, Rao, and Neil Wallace (1997). "Government transaction policy, the medium of exchange, and welfare." Journal of Economic Theory 74, 1-18.

[3] Aruoba, Boragan, Christopher Waller, and Randall Wright (2011). "Money and capital." Journal of Monetary Economics 58, no. 2, 98-116.

[4] Berentsen, Aleksander (2006). "On the private provision of fiat currency." European Economic Review, 50, 1683-1698.

[5] Biais, Bruno, Bisière, Christophe, Bouvard, Matthieu, and Casamatta, Catherine (2019). "The blockchain folk theorem." Review of Financial Studies 32, no. 5, 1662-1715.

[6] Bordo, Michael (1981). "The classical gold standard: Some lessons for today." Federal Reserve Bank of St. Louis Review May: 2-17.

[7] Branch, William A., Nicolas Petrosky-Nadeau, and Guillaume Rocheteau (2016). "Financial frictions, the housing market, and unemployment." Journal of Economic Theory 164, 101-135.

[8] Burdett, Kenneth, Alberto Trejos, and Randall Wright (2001). "Cigarette money." Journal of Economic Theory 99, 117-142.

[9] Cavalcanti, Ricardo, and Neil Wallace (1999). "Inside and outside money as alternative media of exchange." Journal of Money, Credit and Banking, 443-457.

[10] Cavalcanti, Ricardo, and Ed Nosal (2011). "Counterfeiting as private money in mechanism design." Journal of Money, Credit, and Banking 43, 625-636.

[11] Chiu, Jonathan, and Thorsten Koeppl (2017). "The economics of crypto-currencies: Bitcoin and beyond." Working Paper of the Bank of Canada

[12] Chiu, Jonathan, and Thorsten Koeppl (2018). "Blockchain-based settlement for asset trading." Review of Financial Studies 32, no. 5, 1716-1753.

[13] Choi, Michael, and Guillaume Rocheteau (2019). "New Monetarism in continuous time: Methods and applications," Working Paper.

[14] Fernandez-Villaverde, Jesus, and Daniel Sanches (2018). "Can currency competition work?" Working Paper, University of Pennsylvania.

[15] Financial Stability Board (2018). "Crypto-asset markets: Potential channels for future financial stability implications."

[16] Foley, Sean, Jonathan Karlsen and Talis Putnins (2018). "Sex, drugs, and Bitcoin: How much illegal activity is financed through crypto-currencies?" Review of Financial Studies, 32, no. 5, 1798-1853.

[17] Geromichalos, Athanasios, and Lucas Herrenbrueck (2018). "The strategic determination of the supply of liquid assets." Working Paper UC Davis.

[18] Geromichalos, Athanasios, Juan Manuel Licari, and José Suárez-Lledó (2007). "Monetary policy and asset prices." Review of Economic Dynamics 10, no. 4: 761-779.

[19] Garratt, Rodney and Wallace, Neil (2018). "Bitcoin 1, Bitcoin 2, ...: An experiment in privately issued outside monies." Economic Inquiry 56(3), 1887-1897.

[20] Gilbert, Donald (1933). "The economic effects of the gold discoveries upon South Africa: 1886-1910." Quarterly Journal of Economics 47(4), 553-597.

[21] Goldberg, Dror (2005). "Famous myths of 'fiat money'." Journal of Money, Credit, and Banking 37, 957-967

[22] Gomis-Porqueras, Pedro, Kam, Timothy, and Christopher Waller (2017), "Nominal exchange rate determinacy under the threat of currency counterfeiting." American Economic Journal: Macroeconomics, 9, 256-273.

[23] Hendrickson, Joshua, Thomas Hogan, and William Luther (2016). "The political economy of Bitcoin." Economic Inquiry 54, 925-939.

[24] Hendrickson, Joshua, and William Luther (2017). "Banning Bitcoin." Journal of Economic Behavior and Organization 141, 188-195.

[25] Ince, Edward (1956). "Differential equations." Dover Publications.

[26] Jastram, Roy (2009). "The golden constant." Edward Elgar Publishing.

[27] Kalai, Ehud (1977). "Proportional solutions to bargaining situations: interpersonal utility comparisons." Econometrica 45, no. 7: 1623-1630.

[28] Kocherlakota, Narayana (1998). "Money is memory." Journal of Economic Theory 81, no. 2, 232-251.

[29] Lagos, Ricardo (2010). "Asset prices and liquidity in an exchange economy." Journal of Monetary Economics 57, no. 8: 913-930.

[30] Lagos, Ricardo, and Randall Wright (2003). "Dynamics, cycles, and sunspot equilibria in a genuinely dynamic, fundamentally disaggregative model of money." Journal of Economic Theory 109, 156-171.

[31] Lagos, Ricardo, and Randall Wright (2005). "A unified framework for monetary theory and policy analysis." Journal of Political Economy 113, no. 3: 463-484.

[32] Lagos, Ricardo, and Guillaume Rocheteau (2008). "Money and capital as competing media of exchange." Journal of Economic Theory 142, no. 1, 247-258.

[33] Lagos, Ricardo, Guillaume Rocheteau, and Randall Wright (2017). "Liquidity: A new monetarist perspective." Journal of Economic Literature 55, no. 2: 371-440.

[34] Lester, Benjamin, Andrew Postlewaite, and Randall Wright (2012). "Information, liquidity, asset prices, and monetary policy." Review of Economic Studies 79, no. 3: 1209-1238.

[35] Li, Yiting, and Randall Wright (1998). "Government transaction policy, media of exchange, and prices," Journal of Economic Theory 81, 290-313.

[36] Lotz, Sebastien (2004). "Introducing a new currency: Government policy and prices." European Economic Review 48, 959-982.

[37] Lotz, Sebastien, and Guillaume Rocheteau (2002). "On the launching of a new currency." Journal of Money, Credit, and Banking, 34, no. 3: 563-588 .

[38] Lotz, Sebastien, and Françoise Vasselin (2019). "A New Monetarist model of fiat and e-money." Economic Inquiry 57, 498-514.

[39] Luther, William (2018). "Getting off the ground: The case of Bitcoin." Journal of Institutional Economics, 1-17.

[40] Pagnotta, Emiliano (2018). "Bitcoin as decentralized money: prices, mining rewards, and network security." Working Paper.

[41] Prat, Julien, and Benjamin Walter (2018). "An equilibrium model of the market for bitcoin mining." Working Paper.

[42] Oberfield, Ezra, and Nicholas Trachter (2012). "Commodity money with frequent search." Journal of Economic Theory 147, 2332-2356

[43] Rocheteau, Guillaume, Pierre-Olivier Weill, and Tsz-Nga Wong (2018). "A tractable model of monetary exchange with ex post heterogeneity." Theoretical Economics 13, no. 3: 1369-1423.

[44] Rocheteau, Guillaume, and Antonio Rodriguez-Lopez (2014). "Liquidity provision, interest rates, and unemployment." Journal of Monetary Economics 65: 80-101.

[45] Rocheteau, Guillaume, and Randall Wright (2013). "Liquidity and asset-market dynamics." Journal of Monetary Economics 60, no. 2: 275-294.

[46] Rockoff, Hugh (1984). "Some evidence on the real price of gold, its costs of production, and commodity prices." A Retrospective on the Classical Gold Standard, 1821-1931, pp. 613–650. University of Chicago Press.

[47] Schilling, Linda, and Harald Uhlig (2018). "Some simple Bitcoin economics." No. w24483. National Bureau of Economic Research.

[48] Seierstad, Atle and Knut Sydsaeter (1986). "Optimal control theory with economic applications." Elsevier North-Holland, Inc.

[49] Selgin, George (1994). "On ensuring the acceptability of a new fiat money." Journal of Money, Credit and Banking 26, no. 4: 808-826.

[50] Shi, Shouyong (1995). "Money and prices: A model of search and bargaining." Journal of Economic Theory 67, 467-496.

[51] Tennenbaum, Morris and Harry Pollard (1985). "Ordinary differential equations: an elementary textbook for students of mathematics, engineering, and the sciences," Dover Publications New York.

[52] Thompson, Earl (2007). "The tulipmania: Fact or artifact?" Public Choice 130, no. 1-2: 99-114.

[53] Tirole, Jean (1985). "Asset bubbles and overlapping generations." Econometrica 53, no. 6: 1499-1528.

[54] Trejos, Alberto, and Randall Wright (1995). "Search, bargaining, money and prices." Journal of Political Economy 103, 118-141.

[55] Velde, François, Weber, Warren, and Randall Wright (1999). "A model of commodity money, with applications to Gresham's law and the debasement puzzle." Review of Economic Dynamics 2, 291-323.

[56] Velde, François (2003). "Government equity and money: John Law's system in 1720 France." Working paper.

[57] Wallace, Neil (1980) "The overlapping generations model of fiat money." J Kareken, N Wallace (Eds.), Models of Monetary Economies, Federal Reserve Bank of Minneapolis, 49-82.

[58] Wallace, Neil, and Tao Zhu (2004). "A commodity-money refinement in matching models." Journal of Economic Theory 117, no. 2: 246-258.

[59] Weiss, Thomas J (1989). "Economic growth before 1860: revised conjectures." NBER Working Paper.

[60] Williamson, Stephen (1999). "Private money." Journal of Money, Credit and Banking 31, 469-491.

[61] Wright, Randall (1999). "Comment on: Inside and outside money as alternative media of exchange." Journal of Money, Credit, and Banking 31, 461-68.

[62] Zhang, Cathy (2014), "An information-based theory of international currency." Journal of International Economics, 93, 286-301.

Appendix: Omitted Proofs

Proof of Proposition 1. The steady-state money supply is the lowest root of (14) at equality, i.e.,

$$A^2 - \left[\frac{\left(\theta + \bar{A}\right)\lambda + r(1-\theta)}{\lambda}\right] A + \bar{A}\theta = 0. \tag{56}$$

The lowest root of (56) is (17). If $\theta > 0$, then $A^s > 0$. From (13), $q^s > 0$ is a solution to

$$\Gamma(q) \equiv \{\alpha\sigma(\theta - A^s) - r(1-\theta)\}\,[u(q) - q] - rq = 0. \tag{57}$$

If $\alpha\sigma(\theta - A^s) - r(1-\theta) > 0$ then $\Gamma(q)$ is strictly concave with $\Gamma'(0) = +\infty$ and $\lim_{q\to\infty} \Gamma(q) = -\infty$. Hence, there exists a unique $q^s > 0$ solution to $\Gamma(q^s) = 0$. If $\alpha\sigma(\theta - A^s) - r(1-\theta) \leq 0$ then $\Gamma(q)$ is decreasing with $\Gamma(0) = 0$. Hence, there is no $q^s > 0$ solution to $\Gamma(q^s) = 0$. Comparative statics are straightforward. As an example, consider the effects of changes in λ. Differentiating (56) with respect to A and λ we obtain:

$$\frac{\partial A^s}{\partial \lambda} = \frac{\left(\bar{A} - A^s\right)\left(A^s - \theta\right)}{2\lambda A^s - \left[\left(\theta + \bar{A}\right)\lambda + r(1-\theta)\right]}.$$

Using that A^s is the lowest root of (56), it follows that the denominator is negative (graphically, the slope of the parabola is negative when it intersects the horizontal axis at A^s). Using that $A^s < \min\{\bar{A}, \theta\}$, the numerator is negative and $\partial A^s/\partial\lambda > 0$. From (57) $\Gamma(q)$ decreases with A^s for all q such that $u(q) - q > 0$. Hence, $\partial q^s/\partial\lambda > 0$. As another example, consider a change in r. From (56),

$$\frac{\partial A^s}{\partial r} = \frac{(1-\theta)A^s}{2\lambda A^s - \left[\left(\theta + \bar{A}\right)\lambda + r(1-\theta)\right]} < 0.$$

Since agents are indifferent between mining or producing at A^s, namely $\Delta(q^s, A^s) = 0$, by (8) we have

$$\frac{u(q^s) - q^s}{\omega(q^s)} = \left\{1 - \theta + \frac{q^s}{u(q^s) - q^s}\right\}^{-1} = \frac{\lambda\left(\bar{A} - A^s\right)}{\alpha\sigma A^s(1-\theta)}.$$

The left side falls in q^s and the right side falls in A^s. Consequently, q^s and A^s co-move when r changes and $\partial q^s/\partial r < 0$. ■

Proof of Proposition 2. Parts 1 and 2. We first provide a condition that determines which regime, $m = 1 - A$ or $m \in (0, 1 - A)$, is relevant along the equilibrium trajectory. We use the condition to create two ODEs that fully characterize the dynamics of q_t and A_t. Then we will use a standard result for systems of ODEs to prove the existence and uniqueness of q_t and A_t.

Suppose the economy is at (A', q') where $A' < A^s$ and consider the trajectory as we move backward in time. If $\mu(q') > A'$, then the equilibrium path cannot follow (10) because the solution path is continuous. To see this, note that the path of A_t is continuous by (11) since A_t cannot jump and $\left|\dot{A}\right| \leq \lambda \bar{A}$. The value of money, $\omega(q_t) \equiv V_{1,t} - V_{0,t}$, is continuous over time because the continuation values $V_{1,t}$ and $V_{0,t}$ are integrals of payoffs that arrive randomly according to Poisson processes. As a result $m = 1 - A'$ when $\mu(q') > A'$. In this case it is optimal for agents to mine because $\Delta(q', A')$ in (8) is strictly positive when $\mu(q') > A'$.

Next suppose $\mu(q') = A'$. The equilibrium is in the regime with $m = 1 - A$ if and only if

$$\left.\frac{\partial q}{\partial A}\right|_{m=1-A} \leq \left.\frac{\partial q}{\partial A}\right|_{m\in(0,1-A)} \tag{58}$$

where the first derivative is defined by (23) and the second is obtained by differentiating (10) with respect to A, namely $\partial q/\partial A = 1/\mu'(q)$. If (58) is binding, then both regimes imply $m = 1 - A$ and thus they imply the same trajectory. If (58) holds strictly, then the trajectory defined by (21) and (22) converges to (A', q') from the left of the line $\mu(q) = A$, and thus it is optimal for all agents without money to mine, $m = 1 - A$. The trajectory $\mu(q) = A$ is not an equilibrium near (A', q') because by (18) and (19)

$$\frac{\partial q}{\partial A} = \frac{m^{-1}\{\left[r + \lambda\left(\bar{A} - A\right)\right]\omega(q) - \alpha\sigma\left(1 - A\right)\theta\left[u(q) - q\right]\} + \alpha\sigma\theta\left[u(q) - q\right]}{\lambda(\bar{A} - A)\omega'(q)}. \tag{59}$$

For any (A, q) such $A = \mu(q)$, the right side rises in m because the expression in the braces is negative by (14) and (10). Using that the left side of (58) coincides with (59) when $m = 1 - A$, it follows that if (58) holds strictly at (q', A'), then the measure of miners m implied by the trajectory $A = \mu(q)$ must strictly exceed $1 - A'$, and thus it cannot be an equilibrium path.

If (58) does not hold, then the trajectory defined by (21) and (22) converges to (q', A') from below $A = \mu(q)$ and hence no agent has an incentive to mine. In this case the equilibrium path is in the regime where $m \in (0, 1 - A)$ and the measure of miners m implied by $A = \mu(q)$ satisfies $m < 1 - A$ by (59).

We are now ready to characterize the equilibrium by a system of ODEs in backward time. Let $y \equiv A^s - A$. Then we can define q as a function of y along the equilibrium path. From (23) and (10), the two sides of the inequality (58) can be expressed as:

$$\left.\frac{\partial q}{\partial y}\right|_{m=1-A} = g_1(y, q) \equiv -\frac{\omega(q)}{\omega'(q)}\frac{\left[r + \lambda\left(\bar{A} - A^s + y\right)\right]}{(1 - A^s + y)\lambda\left(\bar{A} - A^s + y\right)} \tag{60}$$

$$\left.\frac{\partial q}{\partial y}\right|_{m\in(0,1-A)} = g_2(y, q) \equiv -\frac{1}{\mu'(q)}. \tag{61}$$

By the discussion above, the equilibrium path $q(y)$ solves the following ODE:

$$\frac{\partial q}{\partial y} = f(y,q) \equiv \max\{g_1(y,q), g_2(y,q)\}\mathbf{1}_{\{\mu(q)\leq A^s - y\}} + g_1(y,q)\mathbf{1}_{\{\mu(q)>A^s-y\}} \tag{62}$$

with the initial condition $q(0) = q^s$, where $\mathbf{1}_{\{\cdot\}}$ is an indicator function. It is easy to check that $f(y,q)$ is bounded and continuous for $y \in [0, A^s]$ and $q \in (0, \bar{q}]$ where $\bar{q} > 0$ is the solution to $u(\bar{q}) - \bar{q} = 0$.

Next we show that the equilibrium eventually enters the regime with $m = 1 - A$ as y increases. As q tends to 0, then $\omega(q)\mu'(q)/\omega'(q) \to 0$ and thus there exists $\underline{q} > 0$ such that (60) exceeds (61) for all $q < \underline{q}$. Therefore the equilibrium stays in the regime $m = 1-A$ for all $q < \underline{q}$ as shown in Figure 11. Let $\underline{y} = A^s - \mu(\underline{q})$ so that the equilibrium has $m = 1 - A$ for all $y \geq \underline{y}$.

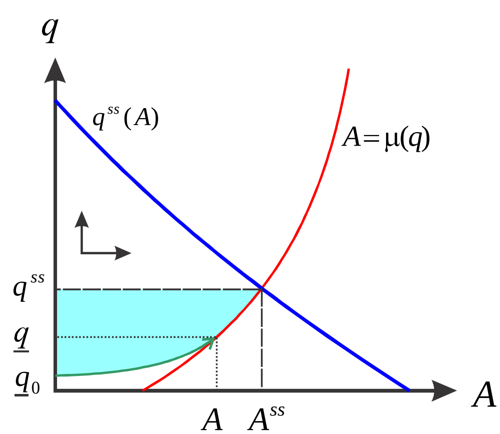

Figure 11: The equilibrium path is contained in the shaded area.

Consider the existence and uniqueness of equilibrium for $q \in [\underline{q}, q^s]$ and $y \in [0, \underline{y}]$. By Theorem 58.5 in Tennenbaum and Pollard (1985), there is a unique solution for $q(y)$ in (62) provided that $f(y,q)$ is Lipschitz continuous, namely there is a real constant $K > 0$ such that

$$|f(y,q') - f(y,q'')| \leq K|q' - q''|$$

for every $y \in [0, \underline{y}]$ and $q', q'' \in [\underline{q}, q^s]$. For any $y \in [0, \underline{y}]$, one can check that the slope of f with respect to q is bounded provided that u'' is bounded. Therefore, f is Lipschitz continuous.

Next we express m as a function of y. By using the solution of $q(y)$ and (59),

$$m(y) = \frac{\alpha\sigma\left(1 - A^s + y\right)\theta\{u[q(y)] - q(y)\} - \left[r + \lambda\left(\bar{A} - A^s + y\right)\right]\omega[q(y)]}{\lambda(\bar{A} - A^s + y)\omega'(q)\partial q/\partial y + \alpha\sigma\theta\{u[q(y)] - q(y)\}}. \tag{63}$$

By (19) the ODE that determines y_t is

$$\dot{y}_t = -\lambda m(y_t)(\bar{A} - A^s + y_t).$$

One can check that the slope of the right side with respected to y is bounded for all $y \leq \underline{y}$. Therefore the right side is Lipschitz continuous and the solution for y_t is unique for any given initial condition. Since $q(y)$ and y_t exist and are unique, q_t and A_t exist and also are unique.

For $A < A^s - \underline{y}$, the equilibrium is in the regime with $m = 1 - A$. The ODE for A, (22), is a Riccati equation, which has a closed form solution (see Section 2.15 in Ince (1956)):

$$A_t = \frac{\bar{A} - \bar{A}e^{-\lambda(1-\bar{A})t}}{1 - \bar{A}e^{-\lambda(1-\bar{A})t}}, \tag{64}$$

where $A_0 = 0$ is the initial condition. By (64) and (21) we can solve for the value of money in closed form.

$$\omega(q_t) = \omega_0 e^{rt} \left\{ 1 + \frac{\bar{A}\left[1 - e^{-\lambda(1-\bar{A})t}\right]}{(1-\bar{A})} \right\}. \tag{65}$$

where ω_0 is the initial value of $\omega(q_t)$ at $t = 0$. We can solve for ω_0 by first solving t_0 in $A_{t_0} = A^s - \underline{y}$ where A_t is given by (64). Then we derive ω_0 by solving $\omega(q_{t_0}) = \omega(\underline{q})$ where $\omega(q_{t_0})$ is derived by evaluating (65) at $t = t_0$. Since $\underline{q} > 0$ as discussed above, $\omega_0 > 0$ by (65).

Part 3. By (23) the inequality (58) is equivalent to

$$\frac{\omega(q)}{\omega'(q)} \frac{r + \lambda\left(\bar{A} - A\right)}{(1-A)\lambda\left(\bar{A} - A\right)} \leq \frac{1}{\mu'(q)}. \tag{66}$$

Suppose $A = A^s$ and $q = q^s$. Since $A^s = \mu(q^s)$ and A^s solves the equality (14), the inequality above is equivalent to

$$\frac{\mu'(q^s)}{\mu(q^s)} \frac{\omega(q^s)}{\omega'(q^s)} \leq \frac{1-\theta}{\theta}.$$

This proves the claim concerning (27) in the proposition. Finally, if $m = 1 - A$ near the neighborhood of the steady state, then $\dot{A}$ in (11) is strictly positive because $\bar{A} - A_t > 0$ and $1 - A_t > 0$ near the steady state. Therefore A_t converges to A^s in finite time near the steady state. When $m < 1 - A$, $m \to 0$ as $(q, A) \to (q^s, A^s)$ because the denominator in (63) is positive by (10) and (14), and it vanishes as $(q, A) \to (q^s, A^s)$. Moreover, since $\partial q / \partial y = -1/\mu'[q(y)]$ when $m < 1 - A$, the numerator in (63) can be written as

$$-\lambda(\bar{A} - A^s + y)\frac{\omega'[q(y)]}{\mu'[q(y)]} + \alpha\sigma\theta\{u[q(y)] - q(y)\} = (1-\theta)\alpha\sigma\{u[q(y)] - q(y)\}\left[\frac{\theta}{1-\theta} - \frac{\omega'[q(y)]/\omega[q(y)]}{\mu'[q(y)]/\mu[q(y)]}\right]. \tag{67}$$

The equation is true because when $m < 1 - A$ we have $A = \mu(q)$, and thus by (10) and $A = A^s - y$

$$\lambda\left(\bar{A} - A^s + y\right) = \frac{(1-\theta)\alpha\sigma\mu[q(y)]\left\{u[q(y)] - q(y)\right\}}{\omega(q)}. \tag{68}$$

As $q(y) \to q^s$, the right side of (67) converges to a strictly positive value by (27) and therefore m in (63) vanishes as $(q, A) \to (q^s, A^s)$. It follows that $\dot{A}$ vanishes by (19) and thus $A_t \to A^s$ only asymptotically. ■

Proof of Lemma 1. Parts 1 and 2. By Proposition 2 mining and trades coexist near the steady state if

$$\frac{\mu'(q^s)/\mu(q^s)}{\omega'(q^s)/\omega(q^s)} > \frac{1-\theta}{\theta}.$$

By differentiating (10),

$$\frac{\mu'(q^s)/\mu(q^s)}{\omega'(q^s)/\omega(q^s)} = \frac{\left(\bar{A} - A^s\right)[1 - \epsilon(q^s)]}{\bar{A}\omega'(q^s)[1 - q^s/u(q^s)]} \tag{69}$$

Hence, (27) can be rewritten as:

$$\frac{\left(\bar{A} - A^s\right)[1 - \epsilon(q^s)]}{\bar{A}\omega'(q^s)[1 - q^s/u(q^s)]} > \frac{1-\theta}{\theta}. \tag{70}$$

The left side rises in q^s by $\epsilon'(q) \le 0$ and the concavity of $u(q)$. As λ rises, A^s rises and q^s falls by Proposition 1, and thus the left side of (70) falls. As $\lambda \to \infty$, $A \to \bar{A}$ and the right side of (70) vanishes, so λ^* is finite.

Next, by Proposition 1, as $\sigma\alpha$ increases, A^s remains constant but q^s rises. Hence, the left side of (70) rises. When $\alpha\sigma$ is sufficiently small, q is arbitrarily close to 0. As $q \to 0$, the left side of (70) vanishes by L'Hospital's Rule and thus the inequality fails. From (13), as $\alpha\sigma \to \infty$, $q^s/u(q^s) \to 1$ and the left side of (70) goes to infinity. This proves that (27) holds if $\alpha\sigma$ is sufficiently large and thus $\kappa^* \in (0, \infty)$.

Part 3. We show $m \in (0, 1 - A)$ is impossible when $\theta < 1/2$. As discussed before (67), the denominator in the right side of (63) is strictly negative for all $q < q^s$ and $A < A^s$. Suppose $m \in (0, 1 - A)$, then the numerator can be written as (67) and it is negative if and only if

$$\frac{\omega(q)}{\omega'(q)}\frac{\mu'(q)}{\mu(q)} > \frac{1-\theta}{\theta} \iff 1 - \frac{\omega(q)\{\alpha\sigma(1-\theta)[u'(q) - 1] + \lambda\omega'(q)\}}{\omega'(q)\{\alpha\sigma(1-\theta)[u(q) - q] + \lambda\omega(q)\}} > \frac{1-\theta}{\theta}.$$

The second inequality uses the definition of μ in (10). If λ is sufficiently large, then $q^s < q^*$ by Proposition 1. In this case $u'(q) - 1 > 0$ for all $q \le q^s$ and thus the left side is less than 1. If $\theta < 1/2$, then $(1-\theta)/\theta > 1$ and therefore this condition always fails. As a result $m < 0$ and thus it is impossible for the equilibrium path to be in the regime with $m \in (0, 1 - A)$.

Part 4. The transition from the regime with $m = 1 - A$ to one with $m \in (0, 1 - A)$ must occur on the line $A = \mu(q)$ and the slope of the two trajectories must be the same, namely

$$\left.\frac{\partial q}{\partial A}\right|_{m=1-A} = \left.\frac{\partial q}{\partial A}\right|_{m\in(0,1-A)}.$$

The equality holds because if the right side is strictly larger, then the measure of miner implied by the trajectory $A = \mu(q)$ strictly exceeds $1 - A$ and thus it cannot be an equilibrium. If the left side is strictly larger, then the trajectory in the $m = 1 - A$ regime will cut the line $A = \mu(q)$ from below, but in this case no agents has incentive to mine and thus it is impossible to have $m = 1 - A$. Therefore the slope of the trajectories must be the same at the transition point. From (23) and (10), the displayed equation is the same as

$$\frac{\omega(q)}{\omega'(q)} \frac{r + \lambda\left(\bar{A} - A\right)}{(1 - A)\lambda\left(\bar{A} - A\right)} = \frac{1}{\mu'(q)}.$$

By the definition of μ in (10) and $A = \mu(q)$, the equation can be rewritten as

$$\left[1 - \frac{\omega(q)\{\alpha\sigma(1-\theta)\left[u'(q) - 1\right] + \lambda\omega'(q)\}}{\omega'(q)\{\alpha\sigma(1-\theta)\left[u(q) - q\right] + \lambda\omega(q)\}}\right] \frac{A[r + \lambda\left(\bar{A} - A\right)]}{(1 - A)\lambda\left(\bar{A} - A\right)} = 1$$

$$\Longleftrightarrow \frac{\alpha\sigma(1-\theta)[1 - \epsilon(q)]}{\omega'(q)} \left[\frac{u(q)}{\alpha\sigma(1-\theta)\left[u(q) - q\right] + \lambda\omega(q)}\right] \frac{A[r + \lambda\left(\bar{A} - A\right)]}{(1 - A)\lambda\left(\bar{A} - A\right)} = 1.$$

Now we argue that the left side rises monotonically as we move along the line $A = \mu(q)$ in the (A, q) space. As we move along $A = \mu(q)$, both q and A increase. The fraction $1 - \epsilon(q)$ rises in q provided that u has decreasing elasticity. The fraction $1/\omega'(q)$ rises in q by the concavity of u. The fraction in the large bracket rises in q when $\alpha\sigma(1-\theta) > \lambda\theta$. The last fraction in the left side rises in A. Altogether the left side increases as the trajectory move along $A = \mu(q)$. It follows that there can be at most one transition from the $m = 1 - A$ regime to the one with $m \in (0, 1 - A)$.

Finally, once the equilibrium enters the regime with $m \in (0, 1 - A)$, it cannot switch regime again. For suppose it switches regime, then the equilibrium must stay in the regime with $m = 1 - A$ as explained above. But this implies the equilibrium trajectory cannot converge to (A^s, q^s) because it cannot intersect the line $A = \mu(q)$ anymore. ∎

Proof of Proposition 3. **Part 1** From the proof of Proposition 2, for any (A', q') where $A' < A^s$ and $A' \leq \mu(q')$, there is a unique q_0 such that $(A_t, q_t) = (A', q')$ for some $t > 0$ only if it starts at $(A_0, q_0) = (0, q_0)$. As a result if two equilibrium trajectories have different initial values for q, then they will not intersect in the (A, q) space. It follows that any equilibrium trajectory with $q_0 < q_0^*$ is located below the trajectory with $q_0 = q_0^*$ that converges to (A^s, q^s) as illustrated in the left panel of Figure 12. Since $A_0 = 0 \leq \mu(q_0)$, the trajectory is in the regime with $m = 1 - A_t$ in the beginning. From (25) in this regime the trajectory solves:

$$\begin{aligned} A_t &= \frac{\bar{A}\left[1 - e^{-\lambda(1-\bar{A})t}\right]}{1 - \bar{A}e^{-\lambda(1-\bar{A})t}} \\ \omega(q_t) &= e^{rt}\omega(q_0)\left[\frac{1 - \bar{A}e^{-\lambda(1-\bar{A})t}}{1 - \bar{A}}\right]. \end{aligned}$$

Since the trajectory cannot intersect with the one that converges to steady state, it must cross the locus $A = \mu(q)$ in the (A, q) space at some $A < A^s$. Let T_0 be the first time the trajectory satisfies $A_t = \mu(q_t)$. Since $A_{T_0} < A^s$ and $m = 1 - A_t$ for all $t < T_0$, the value of A_t reaches A_{T_0} in finite time, thus $T_0 < +\infty$. This proves Part 1(a).

Next, we argue that the trajectory only crosses the locus $A = \mu(q)$ once. Suppose $A_t > \mu(q_t)$, then $m_t = 0$ because no agent wants to mine. Therefore A_t remains constant. The value of money $\omega_t = \omega(q_t)$ solves (7) with $m = 0$, i.e.,

$$\dot{\omega} = r\omega - \alpha\sigma\left(\theta - A_{T_1}\right)\left[u(q) - q\right]. \tag{71}$$

Using that $\dot{\omega} = 0$ when $(A_t, q_t) = (A^s, q^s)$ and $\dot{\omega}/\omega$ increases in q, it follows that $\dot{\omega} < 0$. See right panel of Figure 12. It follows that the trajectory (A_t, q_t) falls vertically whenever (A_t, q_t) lies below the locus $A = \mu(q)$ and $A_t < A^s$. Since the trajectory must and can only cross $A = \mu(q)$ once, there is $T_1(q_0) \in (T_0(q_0), +\infty)$ such that $A_{T_1} = \mu(q_{T_1})$ and $A_t > \mu(q_t)$ for all $t > T_1$. For all $t < T_1$, $m_t > 0$ and $\dot{q}_t > 0$ since otherwise the equilibrium trajectory would fall permanently below the locus $A = \mu(q)$. These properties are also illustrated in the left panel of the Figure 12.

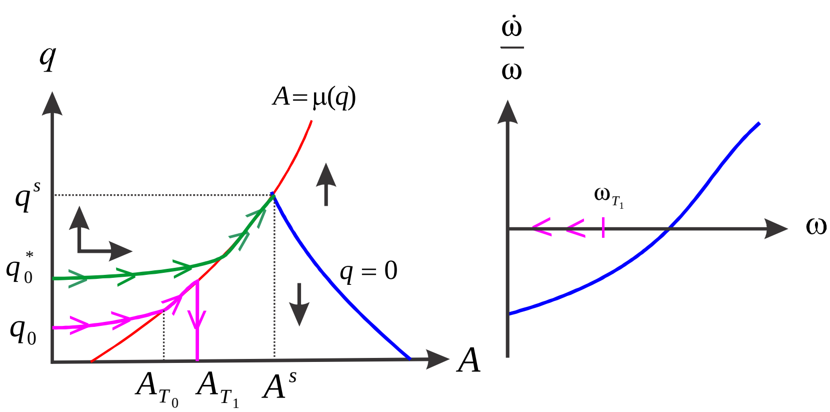

Figure 12: Failing currency equilibria

For all $t > T_1$, $m_t = 0$, $A_t = A_{T_1}$, and $\dot{\omega} < 0$ as discussed above. See right panel of Figure 12. By (71) and the L'Hospital's Rule

$$\lim_{q\searrow 0}\frac{\dot{\omega}}{\omega}=r-\frac{\alpha\sigma\left(\theta-A_{T_1}\right)}{u'(0)/[u'(0)-1]-\theta}.$$

Since the right side is constant for $t \geq T_1$, ω_t falls at a constant percentage rate when $q \approx 0$. Therefore ω_t converges to 0 asymptotically. This proves Part 1(b).

Part 2 If (27) holds, then there is a $\underline{T}< +\infty$ such that for all $t \geq\underline{T}$, $A_t = \mu(q_t)$ along the unique equilibrium, (A_t^*, q_t^*), leading to (A^s, q^s). For all $T \geq\underline{T}$, we can construct an equilibrium such that $(A_t, q_t) = (A_t^*, q_t^*)$ for all $t \leq T$ and $m_t = 0$ for all $t \geq T$. The trajectory up to T is the solution to the system of ODEs in backward time characterized in the proof of Proposition 2. Since $A_T = \mu(q_T)$, at time T agents are indifferent between mining or not. We select $m_T = 0$. As a result, $\dot{A}_T = 0$ and

$$\dot{q}_T=\frac{r\omega(q_T)-\alpha\sigma\left(\theta-A_T\right)\left[u(q_T)-q_T\right]}{\omega'(q_T)}<0.$$

Since $\dot{q}_T < 0$, the trajectory falls below the locus $A = \mu(q)$. As a result for all $t > T$, $m_t = \dot{A}_t = 0$, and $\dot{q}_t < 0$. The rest of the argument is similar to the proof of Part 1 of Proposition 3. ■

Proof of Proposition 4. For $d > 0$, the value functions and equilibrium conditions are detailed in Online Appendix E. Suppose there is a $T > 0$ such that $A_t > \mu(q_t)$ for all $t > T$. Then, $m_t = 0$ and $A_t = A_T$ for all $t > T$. The law of motion of $\omega_t = \omega(q_t)$ is given by

$$\dot{\omega}_t = r\omega_t - d - \alpha\sigma\left(\theta - A_T\right)S(\omega_t).$$

where $S(\omega_t) = u(q_t) - q_t$, $S'(\omega_t) = \left[u'(q_t) - 1\right] / \left[(1-\theta)u'(q_t) + \theta\right]$, and $S''(\omega_t) < 0$. Therefore, the right side is convex in ω_t, it approaches $-d$ as ω_t goes to 0 and $+\infty$ as ω_t grows large. Consider a trajectory where ω_t approaches 0. Then, $\dot{\omega}_t \rightarrow -d$, which implies that ω_t becomes negative for some finite t, which is inconsistent with $\omega(q_t) \geq 0$. This rules out equilibria where $A_t > \mu(q_t)$. The rest of the proof is similar to the one of Proposition 2. ■

Proof of Proposition 5.

Part 1**.** A necessary condition for $\dot{\omega} = 0$ is $m < 1 - A$. Substituting m by its expression from (30) into (28) and setting $\dot{\omega} = 0$ we obtain (31).

Part 2**.** Consider an outcome (ω, A) such that $\omega < \omega^s$. From (32), $\pi = 0$ and from (30) $m = 0$. From (28),

$$\dot{\omega}/\omega \leq r - \alpha\sigma\left(\theta - A\right)\frac{[u(q)-q]}{\omega},$$

where we used $\pi/m \leq \alpha\sigma(1-\theta)\left[u(q)-q\right]/\omega$, this inequality is true because agents prefer not to mine. Using $r = \alpha\sigma\left(\theta - A\right)\left[u(q^s)-q^s\right]/\omega^s$ and the fact that the right side is decreasing in ω^s, it follows that $\dot{\omega}/\omega < 0$. Moreover, $\dot{\omega}/\omega$ decreases with ω so that ω converges to 0. By Proposition 4, if money pays an arbitrarily small interest, such trajectories are not part of an equilibrium. Consider next an outcome (ω, A) such that $\omega > \omega^s$. From (32) and (30) $m = 1 - A$. From (28)

$$\frac{\dot{\omega}}{\omega} = r + \frac{\alpha\sigma(1-\theta)\left[u(q)-q\right]A}{\omega} > 0.$$

The value of money grows at a rate higher than r which violates the transversality condition, $\lim_{t\to+\infty} e^{-rt}\omega_t = 0$. Therefore, the only trajectory consistent with an equilibrium is $\omega_t = \omega^s$. ■

Lemma 2 *There exists a pair of* (m_t, A_t) *that solves the planner's problem (72)-(74), provided that* $q_t = q^*$.

Proof. The planner's problem is given by:

$$\max_{q_t, m_t, A_t} \int_0^{+\infty} e^{-rt}\alpha\sigma A_t(1-A_t-m_t)[u(q_t)-q_t]dt \tag{72}$$

$$\text{s.t. } \dot{A} = m_t\lambda(\bar{A} - A_t), \tag{73}$$

$$m_t \leq 1 - A_t \text{ and } A(0) = A_0. \tag{74}$$

The objective is the discounted sum of all trade match surpluses where the aggregate measure of trade matches between a money holder and a producer is $\alpha\sigma A\left(1-A-m\right)$. The state variable is the money supply which increases with the measure of miners who successfully did money from the ground. There are two control variables, the measure of miners and output in a match. The measure of miners has an upper bound given the measure of agents without money. If the planner can dictate the output traded in each match, she will choose $q_t = q^*$ for all t. One can rewrite (73) as

$$\dot{A} = \min\{m_t, 1-A_t\}\lambda(\bar{A} - A_t). \tag{75}$$

This reformulation is useful because it ensures the planner never chooses $m_t > 1 - A_t$ even when it is feasible. As a result we could drop the constraint $m_t \leq 1 - A_t$ and only impose $m_t \in [0, 1]$. Next we apply a standard result to show the existence of solution for an infinite horizon optimization problem. By Theorem 15 in Seierstad and Sydsaeter (1986) there exists a (A_t, m_t) that solves the new planner's problem if

1. The right side of (75) and the integrand in (72) are continuous in m_t and A_t.

2. There exists a function $\phi(t)$ such that $\phi(t) \geq |e^{-rt}\alpha\sigma A_t(1 - A_t - m_t)[u(q^*) - q^*]|$ for all admissible (m_t, A_t) and $\int_0^\infty \phi(t)dt < \infty$.

3. There exists non-negative function $a(t)$ and $b(t)$ such that

$$\min\{m_t, 1 - A_t\}\lambda(\bar{A} - A_t) \leq a(t)A_t + b(t)$$

for all $A_t \in [0, \bar{A}]$ and $m_t \in [0, 1]$.

4. The set

$$N(A, t) = \{(e^{-rt}\alpha\sigma A_t(1 - A_t - m_t)[u(q^*) - q^*] + \gamma, \min\{m_t, 1 - A_t\}\lambda(\bar{A} - A_t))|m_t \in [0, 1], \gamma \leq 0\}$$

is convex for all A_t and t.

It is easy to see condition (1) is satisfied. Condition (2) is satisfied by assuming $\phi(t) = e^{-rt}\alpha\sigma\bar{A}[u(q^*)-q^*]$. Condition (3) is satisfied because the right side of (75) is bounded above for all A_t and $m_t$by $\lambda\bar{A}$. The last condition is satisfied because the first component of $N(A, t)$ is linear in m_t and γ and the second component is concave in m_t and constant in γ. It follows that there is a pair of (A_t, m_t) that solves the planner's problem provided that $q_t = q^*$. ■

Proof of Proposition 7. Part 1 It is obvious that the optimal output is $q_t = q^*$ provided that trade happens. By Lemma 2 there is a solution to the planner's provided that $q_t = q^*$. Now we characterize this solution and argue it is unique. The current value Hamiltonian corresponding to (72)-(74) is:

$$\mathcal{H}(A, m, \xi, \nu) = \alpha\sigma A (1 - A - m) [u(q^*) - q^*] + \xi m\lambda \left(\bar{A} - A\right) + \nu(1 - A - m),$$

where ξ is the co-state variable associated with A, and ν is the Lagrange multiplier associated with $m \leq 1-A$. The FOC with respect to m is:

$$m \begin{array}{c} = 0 \\ \in [0, 1 - A] \end{array} \text{ if } -\alpha\sigma A [u(q^*) - q^*] + \xi\lambda \left(\bar{A} - A\right) - \nu \begin{array}{c} < 0 \\ = 0 \end{array}, \tag{76}$$

together with the complementary slackness condition, $\nu(1 - A - m) = 0$. The co-state variable satisfies the following ODE:

$$r\xi = \alpha\sigma (1 - 2A - m) [u(q^*) - q^*] - \xi m\lambda - \nu + \dot{\xi}. \tag{77}$$

The stationary solutions to (73) and (77), $\dot{A} = \dot{\xi} = 0$, are such that $m = 0$ and

$$r\xi = \alpha\sigma(1-2A)[u(q^*) - q^*] \tag{78}$$

$$\xi\lambda(\bar{A} - A) \leq \alpha\sigma A[u(q^*) - q^*]. \tag{79}$$

We denote A^* the lowest value of A that satisfies (79). It is the lowest root of the following quadratic equation,

$$2A^2 - \left(1 + 2\bar{A} + \frac{r}{\lambda}\right) A + \bar{A} = 0.$$

In closed form:

$$A^* = \frac{\left(1 + 2\bar{A} + r/\lambda\right) - \sqrt{\left(1 + 2\bar{A} + r/\lambda\right)^2 - 8\bar{A}}}{4}.$$

It is easy to check that $A^* < \min\{1/2, \bar{A}\}$. We denote

$$\xi^* = \alpha\sigma(1 - 2A^*)[u(q^*) - q^*]/r.$$

Now we argue that A_t converges to A^*. Since A_t is continuous, non-decreasing and bounded above by $\bar{A}$, eventually it converges and m vanishes. The process A_t cannot converge to any $A' < A^*$. Suppose it does. Since A^* is the smallest solution to (78) and (79), for all $A' < A^*$ we have

$$-\alpha\sigma A'[u(q^*) - q^*] + \xi\lambda(\bar{A} - A') > 0.$$

This implies $\nu > 0$ by (76) and thus $m = 1 - A$ by the complementary slackness condition. Since $A' < 1$, $m = 1 - A' > 0$ and thus A_t cannot converge to A'. The process A_t also cannot converge to any $A > A^*$ because when A_t goes above A^* the inequality (79) holds strictly and thus $m = 0$ by (76). It follows that the optimal solution can only converge to A^*.

We conjecture and then verify that the solution to the planner's problem is such that for all $A < A^*$, $m = 1 - A$. Then, the ODEs (73) and (77) can be rewritten as

$$\dot{A} = \lambda(1-A)(\bar{A} - A) \tag{80}$$

$$\dot{\xi} = \left[r + (1 + \bar{A} - 2A)\lambda\right]\xi. \tag{81}$$

The ODE for A, (80), is a Riccati equation which can be solved in closed form. See Section 2.15 in Ince (1956) for details. The solutions are

$$A_t = \frac{\bar{A}\left[1 - e^{-\lambda(1-\bar{A})t}\right]}{1 - \bar{A}e^{-\lambda(1-\bar{A})t}} \quad \text{and} \quad \xi_t = \xi_0 e^{[r+\lambda(1-\bar{A})]t}\left(\frac{1 - \bar{A}e^{-\lambda(1-\bar{A})t}}{1 - \bar{A}^2}\right)^2,$$

where we used that $A_0 = 0$. Hence, there is a unique solution to (80)-(81). By the formula for A^* and A_t, one can solve for the time T^*. We denote the path defined by (80)-(81) by $\xi = \xi^p(A)$. From (80)-(81) the slope of $\xi = \xi^p(A)$ is

$$\xi^{p\prime}(A) = \frac{\dot{\xi}}{\dot{A}} = \frac{\left[r + (1 + \bar{A} - 2A)\lambda\right] \xi}{\lambda(1 - A)\left(\bar{A} - A\right)}.$$

From (76) $m = 1 - A$ is optimal only if

$$\xi^p(A) \geq \Omega(A) \equiv \frac{\alpha\sigma A\left[u\left(q^*\right) - q^*\right]}{\lambda\left(\bar{A} - A\right)} \quad \text{for all } A < A^*.$$

We now show that whenever $\xi^p(A) = \Omega(A)$ then $0 < \xi^{p\prime}(A) < \Omega'(A)$. To see this, we evaluate $\xi^{p\prime}(A)$ at $\xi = \Omega(A)$:

$$\begin{aligned} \xi^{p\prime}(A)\big|_{\xi=\Omega(A)} &= \frac{A\left[r + (1 + \bar{A} - 2A)\lambda\right]}{\lambda(1 - A)\bar{A}} \frac{\alpha\sigma\bar{A}\left[u\left(q^*\right) - q^*\right]}{\lambda\left(\bar{A} - A\right)^2} \\ &< \Omega'(A) = \frac{\alpha\sigma\bar{A}\left[u\left(q^*\right) - q^*\right]}{\lambda\left(\bar{A} - A\right)^2} \quad \text{for all } A < A^*. \end{aligned}$$

Given that $\xi^p(A^*) = \Omega(A^*)$, there is no other solution $A < A^*$ to $\xi^p(A) = \Omega(A)$, and thus $\xi^p(A) \geq \Omega(A)$ for all $A < A^*$.

Finally we argue $m \in (0, 1 - A)$ cannot be optimal. Suppose $m \in (0, 1 - \tilde{A})$ at certain $(\tilde{A}, \tilde{\xi})$ where $\tilde{A} < A^*$. Then $\tilde{\xi} = \Omega(\tilde{A})$ by (76). By the ODE (73) and (77),

$$\xi^{p\prime}(\tilde{A}) = \frac{\dot{\xi}}{\dot{A}} = \frac{\tilde{\xi}}{\lambda(\bar{A} - \tilde{A})}\left[\frac{1}{m}\left(r + \frac{\lambda(\bar{A} - \tilde{A})(2\tilde{A} - 1)}{\tilde{A}}\right) + \lambda + \frac{\alpha\sigma[u(q^*) - q^*]}{\tilde{\xi}}\right]. \tag{82}$$

Since $rA + \lambda(\bar{A} - A)(2A - 1) < 0$ for all $A < A^*$ by (78) and (79), the right side of (82) strictly increases in m. As discussed above, if $m = 1 - \tilde{A}$, then $\xi^{p\prime}(\tilde{A}) < \Omega'(\tilde{A})$ and thus $\xi^p(A)$ cuts $\Omega(A)$ from above at $(\tilde{A}, \tilde{\xi})$. By a similar argument, $\xi^p(A)$ must be lower than $\Omega(A)$ for all $A \in (\tilde{A}, A^*)$. But then it is impossible for $\xi(A)$ to reach ξ^* as $A \to A^*$ because $m = 0$ when $\xi^p(A) < \Omega(A)$ by (76). Therefore $m \in (0, 1 - A)$ is sub optimal.

Part 2 In order to guarantee that $m_t = 1 - A_t$ for all $t < T^*$ we set $\theta(t) = 1$ for all $t < T^*$ so that producers receive no gains from trade. From (13) the buyer's bargaining power that implements q^* when T^* has been reached is θ^* defined in (36). Incentive feasibility means $\theta^* \in [0, 1]$, which holds if and only if (34) holds. From (14) agents stop mining when A^* is reached if

$$r \leq \frac{(1 - A^*)\left[u(q^*) - q^*\right]}{q^*}\left[\alpha\sigma A^* - \lambda\left(\bar{A} - A^*\right)\right] - \lambda\left(\bar{A} - A^*\right).$$

This inequality can be rearranged to give (35). ■

Proof of Proposition 9. Part 1. Consider a steady state such that $q^s = q^*$ and $\phi^s = d/r$. Such a steady state requires that $\omega(q^*) \leq \phi^s A^s$. Such an equilibrium exists if the curve representing (43) is located below the curve representing (44) when $A = r\omega(q^*)/d$. Alternatively, the left side of (43) is greater than the right side when $\phi = d/r$ and $A = r\omega(q^*)/d$, i.e., (45) holds. From (39)

$$\frac{\dot{\phi}}{\phi} = \frac{r\left(\phi - \phi^s\right)}{\phi} - \alpha(1-m)\sigma\theta\left[\frac{u'\left(q_t\right)-1}{(1-\theta)u'\left(q_t\right)+\theta}\right].$$

In the neighborhood of the steady state $q_t = q^*$ and the second term on the right side is equal to 0. The only solution to $\dot{\phi} = r\left(\phi - \phi^s\right)$ is such that $\phi_t = \phi^s$. Any other path violates $\lim_{t\to+\infty} e^{-rt}\phi_t = 0$ or $\phi_t \geq d/r$. Solving $\dot{\phi} = r\left(\phi - \phi^s\right)$ in backward time leads to $\phi_t = d/r$ for all t. Using that $\lambda(\bar{A} - A^s)\phi = \alpha\sigma(1-\theta)\left[u\left(q^*\right) - q^*\right]$ at the steady state, it follows that $\lambda(\bar{A} - A_t)\phi > \alpha\sigma(1-\theta)\left[u\left(q^*\right) - q^*\right]$ for all t such that $A_t < A^s$, and hence $m_t = 1$ for all t such that $A_t < A^s$. See the phase diagram in the right panel of Figure 9.

Part 2. If (45) does not hold then the unique steady state features $q^s < q^*$ and $\phi^s > d/r$. By the same reasoning as in the proof of Proposition 2, there is a unique equilibrium leading to the steady state. By (39) and (42) the slope of the trajectory in the (A, ϕ) space is

$$\frac{\partial\phi}{\partial A} = \frac{\dot{\phi}}{\dot{A}} = \frac{\frac{1}{m}\left[r\phi - d - \left(\frac{\alpha\sigma\theta[u'(q)-1]}{(1-\theta)u'(q)+\theta}\right)\phi\right] + \frac{\alpha\sigma\theta[u'(q)-1]}{(1-\theta)u'(q)+\theta}\phi}{\lambda(\bar{A}-A)}. \tag{83}$$

Since q_t increases over time and $q^s < q^*$, the bargaining solution implies

$$\omega(q) = (1-\theta)u(q) + \theta q = \phi A. \tag{84}$$

By the implicit function theorem $\partial q/\partial(\phi A) = 1/\omega'(q)$ and thus $S'(\phi A) = [u'(q)-1]/\omega'(q)$. By (43), (84) and the definition of S, the slope $\partial\phi/\partial A$ can be written as

$$\frac{\partial\phi/\phi}{\partial A/A} = \frac{\alpha\sigma(1-\theta)\frac{[u'(q)-1]}{\omega'(q)} + \lambda}{\alpha\sigma(1-\theta)\left[1 - \frac{\omega(q)[u'(q)-1]}{\omega'(q)[u(q)-q]}\right]\frac{[u(q)-q]}{\omega(q)}}.$$

Let $m_I(q,\phi)$ be the measure of miners implied by the indifference locus. To solve for $m_I(q,\phi)$ replace $\partial\phi/\partial A$ in the above equation by (83). Then by (43) and (84)

$$\begin{aligned}&\left\{1 - \frac{\omega(q)[u'(q)-1]}{\omega'(q)[u(q)-q]}\right\}\left\{\frac{r - d/\phi - \frac{\alpha\sigma\theta[u'(q)-1]}{\omega'(q)}}{m_I(q,\phi)} + \frac{\alpha\sigma\theta[u'(q)-1]}{\omega'(q)}\right\} \\ &= \alpha\sigma(1-\theta)\frac{[u'(q)-1]}{\omega'(q)} + \lambda.\end{aligned}$$

The right side falls strictly in q by the concavity of u. The left side rises in ϕ because $d \geq 0$. The first term in the large square bracket is strictly negative when $q < q^s$ and $\phi < \phi^s$ by the definition of the steady state. Therefore, the left side rises strictly in m_I. If the elasticity of $u(q)$ falls in q then the expression in the large bracket rises in q, and thus the left side rises in q. Altogether as q and ϕ increase, $m_I(q, \phi)$ must fall strictly to balance the equation.

The rest of the proof is similar to that of Proposition 2. Suppose the dynamic system starts from the steady state and goes backward in time. By the proof logic leading to Proposition 2, the economy lies in the regime with $m \in [0, 1]$ if and only if $m_I(q_t, \phi_t) \leq 1$. Since q_t and ϕ_t increase over time and $m_I(q, \phi)$ falls as q and ϕ increase, the system has at most one regime switch. Since $m_t = 1$ when A is sufficiently small, as t increases from 0 the equilibrium either stays in the regime with $m_t = 1$ or has exactly one regime switch from $m_t = 1$ to $m_t < 1$. The economy has $m_t = 1$ until it reaches the steady state if and only if the slope of the trajectory when $m = 1$,

$$\frac{\partial \phi / \phi}{\partial A / A} = \alpha\sigma\theta \frac{S(A\phi)}{\lambda(\bar{A} - A)\phi} \nu(A\phi),$$

is larger than the slope of the locus where agents are indifferent between occupations, i.e.,

$$\frac{\partial \phi / \phi}{\partial A / A} = \frac{\nu(A\phi) + A/(\bar{A} - A)}{1 - \nu(A\phi)}.$$

This happens when (46) does not hold. Otherwise, the equilibrium features $m_t < 1$ in the neighborhood of the steady state.

Part 3. From (44) if $d = 0$ then q^s solves

$$r = \alpha\sigma\theta \left\{ \frac{u'(q) - 1}{(1 - \theta)u'(q) + \theta} \right\}.$$

A unique solution exists if $r < \alpha\sigma\theta/(1 - \theta)$. The characterization of the unique equilibrium leading to the steady state is similar to Part 2. ■

Online Appendix for "Money Mining and Price Dynamics"

February 2019

A Granger test

In this section we test whether the prices of gold and Bitcoin affect their production or mining intensity.

Gold: We use the historical mine production index and purchasing power of gold from Jastram (2009). This is an annual data covering 1870-1970, see Figure 2. Consider the two-variable VAR

$$\begin{bmatrix}\text{production}_t \\ \text{price}_t\end{bmatrix} = \mathbf{b_0} + \mathbf{B_1}\begin{bmatrix}\text{production}_{t-1} \\ \text{price}_{t-1}\end{bmatrix} + \cdots + \mathbf{B_k}\begin{bmatrix}\text{production}_{t-k} \\ \text{price}_{t-k}\end{bmatrix} + \begin{bmatrix}\varepsilon_{1,t} \\ \varepsilon_{2,t}\end{bmatrix}$$

where $\mathbf{b_0}$ is a vector of intercept terms and each of $\mathbf{B_1}$ to $\mathbf{B_k}$ is a matrix of coefficients. The lag length $k = 3$ is recommended by the likelihood ratio test, final prediction error and Akaike's information criterion.

We use the Granger test to test the null hypothesis that all coefficients on lags of the price in the production equation are equal to zero, against the alternative that at least one is not non-zero. The p-value is 0.02 and thus we conclude that the real price of gold Granger-causes the production at the 5% level.

Bitcoin: We use the monthly data on mining difficulty and Bitcoin price from the web site Bitcoinity, covering the period Aug 2010 to Oct 2018. We consider the following VAR model

$$\begin{bmatrix}\text{growth of diff level}_t \\ \text{growth of price}_t\end{bmatrix} = \mathbf{b_0} + \mathbf{B_1}\begin{bmatrix}\text{growth of diff level}_{t-1} \\ \text{growth of price}_{t-1}\end{bmatrix} + \cdots + \mathbf{B_k}\begin{bmatrix}\text{growth of diff level}_{t-k} \\ \text{growth of price}_{t-k}\end{bmatrix} + \begin{bmatrix}\varepsilon_{1,t} \\ \varepsilon_{2,t}\end{bmatrix}.$$

The recommended lag is $k = 2$ and the p-value of the causality test is 0.0004. Hence we conclude that the growth rate of prices Granger-causes the growth rate of the difficulty level at the 1% level.

B Search while mining

In the main text we described the decision to mine money as an occupation choice and the cost to mine as the foregone opportunities in the trading sector. In the following we allow miners to search for trading opportunities. We will show that this possibility can change the dynamics of prices depending on the efficiency of the mining technology.

Suppose now that agents who choose to mine can still receive opportunities to produce. Specifically, miners meet a money holder at Poisson arrival rate $\alpha\eta A$ where $\eta \leq 1$ while non-miners meet money holders at rate αA. It follows that a money holder meets someone who can produce at rate

$$\frac{m\alpha\eta A + (1-A-m)\alpha A}{A} = \alpha\left[1 - A - m(1-\eta)\right], \tag{85}$$

and she likes the good offered with probability σ. The case studied so far was $\eta = 0$. Another polar case is $\eta = 1$ where agents can engage in mining without forgiving any trading opportunity. In that case, the buyer's matching rate is simply $\alpha(1-A)$.

Agents' value functions solve:

$$rV_1 = \alpha\sigma\left[1 - A - m(1-\rho)\right]\theta\left[u(q) - q\right] + \dot{V}_1 \tag{86}$$

$$rV_0 = \alpha\sigma A\rho(1-\theta)\left[u(q)-q\right] + \max\left\{\alpha\sigma A(1-\rho)(1-\theta)\left[u(q)-q\right], \lambda\left(\bar{A}-A\right)\omega(q)\right\} + \dot{V}_0. \tag{87}$$

The key novelty in (87) is that the opportunity cost of mining has been multiplied by $1-\rho$. In particular, if $\rho = 1$ there is no opportunity cost of mining and all agents without money mine. Subtracting (87) from (86) the value of money solves:

$$\begin{aligned} r\omega(q) = & \left\{1 - \left[1 + \rho\left(\frac{1-\theta}{\theta}\right)\right]A - m(1-\rho)\right\}\theta\alpha\sigma\left[u(q)-q\right] \\ & - \max\left\{\alpha\sigma A(1-\rho)(1-\theta)\left[u(q)-q\right], \lambda\left(\bar{A}-A\right)\omega(q)\right\} + \omega'(q)\dot{q}. \end{aligned} \tag{88}$$

The law of motion for A is:

$$\dot{A} = m\lambda\left(\bar{A}-A\right). \tag{89}$$

The locus of pairs (A, q) such that agents are indifferent between mining or not is given by:

$$A = \mu(q) \equiv \frac{\lambda\bar{A}\omega(q)}{\alpha\sigma(1-\rho)(1-\theta)\left[u(q)-q\right] + \lambda\omega(q)}$$

The μ-locus shifts to the right as ρ increases and it becomes vertical at $A = \bar{A}$ when $\rho = 1$. Let us start first with steady-state equilibria.

Proposition 11 ***(Search while mining)*** *There exists a steady-state monetary equilibrium iff*

$$r < \frac{\alpha\sigma}{1-\theta}\left[\frac{\alpha\sigma\theta(1-\eta)+\lambda\left(\theta-\bar{A}\right)}{\alpha\sigma(1-\eta)+\lambda}\right]. \tag{90}$$

The steady-state money supply, A^s, increases with η while the value of money, q^s, decreases with η.

Suppose $\eta = 1$. There exists a monetary equilibrium if $r < \alpha\sigma\left(\theta-\bar{A}\right)/(1-\theta)$ and it is such that A^s tends to $\bar{A} < \theta$. For all $A_0 < \bar{A}$ the unique equilibrium leading to the steady state is such that: A increases over time until it reaches $\bar{A}$; q increases over time if $\lambda > r/\left(\theta-\bar{A}\right)$, decreases if $\lambda < r/\left(\theta-\bar{A}\right)$, and remains constant if $\lambda = r/\left(\theta-\bar{A}\right)$.

Proof. By the same reasoning as in Section 3.2, q^s solves (12),

$$r\omega(q) = (\theta - A)\,\alpha\sigma\,[u(q)-q]\,,$$

and A^s is the smallest root to

$$\lambda\left(\bar{A}-A\right)(\theta-A) - A(1-\eta)(1-\theta)r = 0. \tag{91}$$

It is easy to check that A^s increases with ρ while q^s decreases with η. Moreover, as η approaches to 1, A^s approaches to $\min\{\theta, \bar{A}\}$. By the same reasoning as in the proof of Proposition 1 there exists a steady-state monetary equilibrium iff

$$\lim_{q\to 0}\left\{r\omega(q) - [\theta-\mu(q)]\,\alpha\sigma\,[u(q)-q]\right\} < 0.$$

Dividing by $\omega(q) > 0$ this condition can be rewritten as:

$$\lim_{q\to 0}\left\{r - \alpha\sigma\frac{[\theta-\mu(q)]\,[u(q)-q]}{\omega(q)}\right\} < 0.$$

Using that $\lim_{q\to 0}\left\{[u(q)-q]/\omega(q)\right\} = 1/(1-\theta)$ and $\lim_{q\to 0}\mu(q) = \lambda\bar{A}/\left[\alpha\sigma(1-\eta)+\lambda\right]$ the condition above can be rewritten as (90). In particular, when $\eta = 1$,

$$r < \frac{\alpha\sigma}{1-\theta}\left(\theta-\bar{A}\right).$$

In that case a necessary condition for a steady-state monetary equilibrium is $\bar{A} < \theta$. Hence, $A^s = \theta < \bar{A}$.

The condition $\alpha\sigma(\theta-\bar{A}) > r(1-\theta)$ guarantees the existence of a steady-state monetary equilibrium when $\eta = 1$. The system of ODEs, (88) and (89), becomes:

$$\begin{aligned}
\omega'(q)\dot{q} &= \left[r+\lambda\left(\bar{A}-A\right)\right]\omega(q) - (\theta-A)\,\alpha\sigma\,[u(q)-q] \\
\dot{A} &= \lambda(1-A)\left(\bar{A}-A\right)
\end{aligned}$$

Linearizing the system around the steady state we obtain:

$$\left(\begin{array}{c}\dot{q}\\ \dot{A}\end{array}\right)=\left(\begin{array}{cc}\frac{r\omega'(q^s)-\left(\theta-\bar{A}\right)\alpha\sigma\left[u'(q^s)-1\right]}{\omega'(q^s)} & \frac{-\lambda\omega(q^s)+\alpha\sigma[u(q^s)-q^s]}{\omega'(q^s)}\\ 0 & -\lambda(1-\bar{A})\end{array}\right)\left(\begin{array}{c}q-q^s\\ A-A^s\end{array}\right).$$

If $\left(\theta-\bar{A}\right)\alpha\sigma > r(1-\theta)$ then $r\omega'(q^s) > \left(\theta-\bar{A}\right)\alpha\sigma\left[u'(q^s)-1\right]$. It follows that the determinant of the Jacobian matrix is negative, i.e., the steady state is a saddle point. The negative eigenvalue is $e_1=-\lambda(1-\bar{A})$ and the associated eigenvector is

$$\overrightarrow{v}_1=\left(\begin{array}{c}\frac{\left[\lambda-r/\left(\theta-\bar{A}\right)\right]\omega(q^s)}{\left[r+\lambda(1-\bar{A})\right]\omega'(q^s)-\left(\theta-\bar{A}\right)\alpha\sigma[u'(q^s)-1]}\\ 1\end{array}\right)$$

where we used that $r\omega(q^s)=\left(\theta-\bar{A}\right)\alpha\sigma\left[u(q^s)-q^s\right]$. The first component of $\overrightarrow{v}_1$ is of the same sign as $\lambda-r/\left(\theta-\bar{A}\right)$. The solution to the linearized system is

$$\left(\begin{array}{c}q-q^s\\ A-A^s\end{array}\right)=Ce^{-\lambda(1-\bar{A})t}\overrightarrow{v}_1,$$

where C is some constant. Hence, in the neighborhood of the steady state,

$$\frac{\partial q}{\partial A}=\frac{\left[\lambda-r/\left(\theta-\bar{A}\right)\right]\omega(q^s)}{\left[r+\lambda(1-\bar{A})\right]\omega'(q^s)-\left(\theta-\bar{A}\right)\alpha\sigma\left[u'(q^s)-1\right]},$$

which is of the same sign as $\lambda-r/\left(\theta-\bar{A}\right)$. If $\lambda>r/\left(\theta-\bar{A}\right)$, then the saddle path in the neighborhood of the steady state is upward sloping, i.e., q and A increase over time. We can show that this result holds globally since the equation of the q-isocline is:

$$\frac{\omega(q)}{u(q)-q}=\frac{\left(\theta-A\right)\alpha\sigma}{r+\lambda\left(\bar{A}-A\right)}.$$

The q-isocline is upward sloping when $\lambda>r/\left(\theta-\bar{A}\right)$. See left panel of Figure 13. By the same reasoning, if $\lambda<r/\left(\theta-\bar{A}\right)$, then the saddle path is downward sloping and along the equilibrium path, q decreases while A increases. See middle panel of Figure 13. Finally, if $\lambda=r/\left(\theta-\bar{A}\right)$, then the q-isocline is horizontal. In that case q is constant over time. See right panel of Figure 13. ■

According to (90) the set of parameter values for which a steady-state monetary equilibrium exists shrinks as η increases. If agents can meet trading partners more frequently while mining, then the opportunity cost of mining is lower and the incentives to mine are greater, which leads to a higher supply of money. But for a monetary equilibrium to exist, the money supply cannot be too large. A higher η also reduces the value of money. In the limiting case where $\eta=1$, then there is no opportunity cost to engage in mining and all

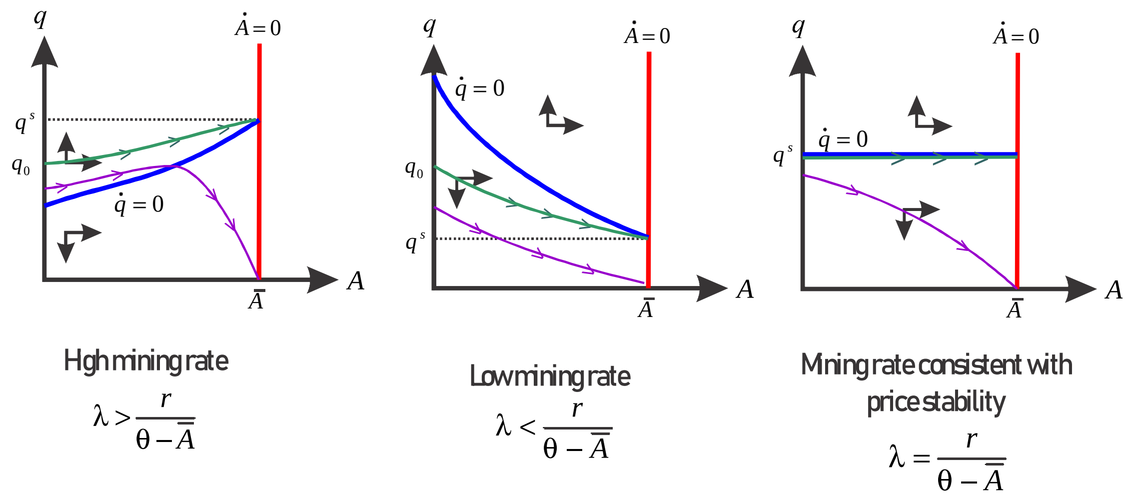

Figure 13: Phase diagrams when agents can mine while searching for trading partners ($\eta = 1$).

agents without money mine, $m = 1 - A$. At the steady state the money supply is equal to the maximum stock of money that could be mined, $\bar{A}$. We now turn to the transition dynamics for this special case.

Proposition 11 shows that when there is no opportunity cost of mining, the correlation between the value of money and the money stock along the transitional path depends on the efficiency of the mining technology.[24] If the mining intensity is high, the value of money increases with the money supply. If the mining intensity is low, then the opposite correlation prevails and the value of money decreases as the money supply increases. Finally, there is a mining rate such that the price level is constant, the value of money is independent of the money stock.

Finally, the path for q_t can be non-monotone when η is close but less than 1. As η falls below one, the q-isocline shifts downward. Hence, the steady state, (A^s, q^s), is now located above the q-isocline, which implies that close to the steady state q increases. If λ is small, as in the middle panel of panel of Figure 13, the q-isocline is downward sloping and the upward trajectory of $[A_t, q_t]$ must cross it. As a result, for low values of A, q is decreases and it increases as A approaches the steady state.

[24] While Proposition 11 focuses on the unique equilibrium leading to the steady state, there is also a continuum of equilibria where the value of money vanishes asymptotically. In the left panel of Figure 13, when λ is high, the value of money increases first and then decreases. In the middle and right panels, when λ is low, the value of money is monotone decreasing in time.

C Endogenous mining intensity

Suppose that agents do not have to give up their occupation to engage in mining, $\rho = 1$, but they have to suffer a disutility cost. Moreover, we let agents choose the intensity with which they mine, λ. The flow cost of the mining technology is $c(\lambda)$ and for simplicity we assume it is quadratic, $c(\lambda) = \bar{c}\lambda^2/2$ with $\bar{c} > 0$.

An agent with one unit of asset will not engage in mining since money holdings have a unit upper bound. Therefore, V_1 solves the Bellman equation (1). The lifetime expected utility of an agent without money solves

$$rV_0 = \alpha\sigma A\left(-q + V_1 - V_0\right) + \max_{\lambda \geq 0}\left\{-c(\lambda) + \lambda\left(\bar{A} - A\right)\left(V_1 - V_0\right)\right\} + \dot{V}_0. \tag{92}$$

An agent without money receives an opportunity to produce at Poisson arrival rate $\alpha\sigma A$. While searching for trading partners, the agent engages in mining by choosing λ. The optimal mining intensity solves

$$\bar{c}\lambda = \left(\bar{A} - A\right)\left(V_1 - V_0\right). \tag{93}$$

The marginal cost of mining on the right side is equalized to the marginal gain on the right side. The mining intensity increases with the amount of asset that is still in the ground and the value of money.

In order to determine terms of trade in pairwise meetings we assume $\theta = 1$, buyers have all the bargaining power. Then $q = V_1 - V_0$ and from (93),

$$\lambda = \frac{\left(\bar{A} - A\right)q}{\bar{c}}. \tag{94}$$

From (1) and (92) the value of money solves the following ODE:

$$rq = \alpha\sigma\left(1 - A\right)\left[u(q) - q\right] - \frac{\left[\left(\bar{A} - A\right)q\right]^2}{2\bar{c}} + \dot{q}. \tag{95}$$

After substituting λ by its expression given by (94) the law of motion for the money supply is:

$$\dot{A} = \frac{\left(1 - A\right)\left(\bar{A} - A\right)^2 q}{\bar{c}}. \tag{96}$$

An equilibrium is a pair of time paths, (q_t, A_t), that satisfies (95), (96), and the initial condition $A(0) = A_0$.

Proposition 12 ***(Endogenous mining intensity)*** *Assume $\bar{A} < 1$. There is a unique monetary steady state and it is such that*

$$A^s = \bar{A} \tag{97}$$

$$rq^s = \alpha\sigma\left(1 - \bar{A}\right)\left[u(q^s) - q^s\right]. \tag{98}$$

For any $A_0 < \bar{A}$ there exists a unique equilibrium and it is such that A increases over time while q decreases over time.

Proof. We select the steady state with the lowest value for A. Setting $\dot{A} = \dot{q} = 0$ in (95) and (96) we obtain (97)-(98). Let us turn to non-stationary equilibria. In order to determine the global solution we represent the phase diagram of (95)-(96). The A-isoclines are $A = \bar{A}$ and $A = 1$. The equation of the q-isocline is

$$rq + \frac{\left(\bar{A} - A\right)^2 q^2}{2\bar{c}} = \alpha\sigma\left(1 - A\right)\left[u(q) - q\right].$$

For all $A < \bar{A}$, the left side is strictly convex while the right side is strictly concave in q. Moreover, for any $A < \bar{A}$, there is a unique positive solution and it is such that q decreases with A. The phase diagram is represented in Figure 14. The steady state is a saddle point and there is unique (saddle) path leading to it.

■

At the steady state all the money in the ground has been dug out and the value of money is independent of the mining technology.

In contrast to the model with mining as an occupation choice, here the value of money decreases over time, i.e., there is a negative correlation between the value of money and its stock.

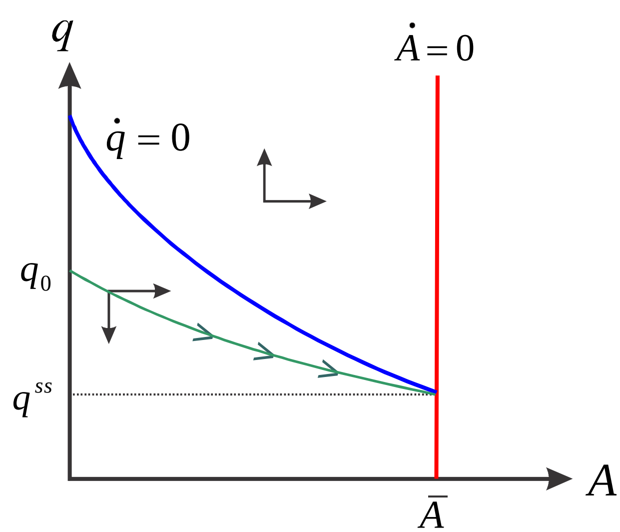

Figure 14: Costly mining: Phase diagram

D General matching function

Suppose now that only buyers (money holders) and producers participate in the matching process according to a constant returns to scale matching function.

D.1 Indivisible money

The matching probability of a buyer is $\alpha(\tau)$ where $\tau = (1 - A - m)/A$ is market tightness expressed as the ratio of sellers to buyers. As is standard, we assume that $\alpha' > 0$, $\alpha'' < 0$, $\alpha'(0) = +\infty$, $\alpha'(+\infty) = 0$. A matching function that satisfies these properties is the Cobb Douglas matching function.

The HJB equations of agents with and without money are:

$$\begin{aligned} rV_1 &= \alpha(\tau)\sigma\theta\left[u(q) - q\right] + \dot{V}_1 && (99) \\ rV_0 &= \max\left\{\frac{\alpha(\tau)}{\tau}\sigma(1-\theta)\left[u(q) - q\right], \lambda\left(\bar{A} - A\right)\omega(q)\right\} + \dot{V}_0. && (100) \end{aligned}$$

The novelty is that the matching rate of a buyer is $\alpha(\tau)$ while the matching rate of a seller is $\alpha(\tau)/\tau$. Using that $\lim_{\tau\to 0}\alpha(\tau)/\tau = +\infty$, it follows that $\tau > 0$ in equilibrium, i.e., $m < 1 - A$. The goods market is always active and

$$\max\left\{\frac{\alpha(\tau)}{\tau}\sigma(1-\theta)\left[u(q) - q\right], \lambda\left(\bar{A} - A\right)\omega(q)\right\} = \frac{\alpha(\tau)}{\tau}\sigma(1-\theta)\left[u(q) - q\right]. \tag{101}$$

Subtracting (100) from (99) the value of money solves:

$$r\omega(q) = \left[\alpha(\tau)\sigma\theta - \frac{\alpha(\tau)}{\tau}\sigma(1-\theta)\right]\left[u(q) - q\right] + \omega'(q)\dot{q}. \tag{102}$$

From (101) market tightness in the goods market solves:

$$\frac{\alpha(\tau)}{\tau}\sigma(1-\theta)\left[u(q) - q\right] \geq \lambda\left(\bar{A} - A\right)\omega(q), \quad " = " \text{ if } \tau < \frac{1-A}{A}.$$

Solving for τ we obtain:

$$\tau(\omega, A) = \min\left\{g^{-1}\left[\frac{\lambda\left(\bar{A} - A\right)\omega}{\sigma(1-\theta)S(\omega)}\right], \frac{1-A}{A}\right\}. \tag{103}$$

where $S(\omega) \equiv u\left[q(\omega)\right] - q(\omega)$ and $g(\tau) \equiv \alpha(\tau)/\tau$. For all (ω, A) such that $\frac{\lambda(\bar{A}-A)\omega}{\sigma(1-\theta)S(\omega)} \geq g\left(\frac{1-A}{A}\right)$, $m > 0$ and $\tau(\omega, A)$ is decreasing in ω and increasing in A. Moreover, $\tau(+\infty, A) = 0$ and $\tau(0, A) > 0$. The money supply evolves according to

$$\dot{A} = \left[1 - A\left(1 + \tau\right)\right]\lambda\left(\bar{A} - A\right), \tag{104}$$

where we used that $1 - A(1+\tau) = m$.

We summarize the equilibrium by a system of two ODEs in ω and A:

$$\dot{\omega} = r\omega - \{\alpha[\tau(\omega, A)]\sigma\theta - g[\tau(\omega, A)]\sigma(1-\theta)\} S(\omega) \tag{105}$$

$$\dot{A} = \{1 - A[1 + \tau(\omega, A)]\} \lambda(\bar{A} - A) \tag{106}$$

The locus of the points such that $\dot{A} = 0$ corresponds to all pairs (ω, A) such that $\tau(\omega, A) = (1-A)/A$. From (103) it is given by:

$$\frac{\lambda(\bar{A} - A)\omega}{\sigma(1-\theta)S(\omega)} \leq g\left(\frac{1-A}{A}\right). \tag{107}$$

Condition (107) at equality gives a positive relationship between ω and A. As ω approaches 0, A tends to the solution to $\lambda(\bar{A} - A) = \sigma g\left(\frac{1-A}{A}\right)$. As ω tends to $+\infty$, A tends to $\bar{A}$. This locus is represented by a red upward-sloping curve in Figure 15.

The locus of the points such that $\dot{\omega} = 0$ and $\dot{A} > 0$ is such that

$$r\frac{\omega}{S(\omega)} = \{\alpha[\tau(\omega, A)]\sigma\theta - g[\tau(\omega, A)]\sigma(1-\theta)\}. \tag{108}$$

The left side is increasing in ω while the right side is decreasing in ω but increasing in A. For given A there is a unique ω solution to (108) provided that

$$r(1-\theta) < \{\alpha[\tau(0, A)]\sigma\theta - g[\tau(0, A)]\sigma(1-\theta)\},$$

where $\tau(0, A)$ is the solution to $g(\tau) = \lambda(\bar{A} - A)/\sigma$. If this condition holds for $A = 0$, then it holds for all A. Hence, we assume

$$r(1-\theta) < [\alpha(\tau_0)\sigma\theta - g(\tau_0)\sigma(1-\theta)] \text{ where } \tau_0 = g^{-1}\left[\lambda(\bar{A} - A)/\sigma\right]. \tag{109}$$

Assuming this condition is satisfied, the ω-isocline is upward sloping as illustrated in Figure 15. As A goes to zero, ω tends to a positive value.

There is a unique steady state such that agents are indifferent between mining and not mining and it solves

$$g(\tau) = g\left(\frac{1-A}{A}\right) = \frac{\lambda(\bar{A} - A)\omega}{\sigma(1-\theta)S(\omega)} \tag{110}$$

$$r\frac{\omega}{S(\omega)} = [\alpha(\tau)\sigma\theta - g(\tau)\sigma(1-\theta)]. \tag{111}$$

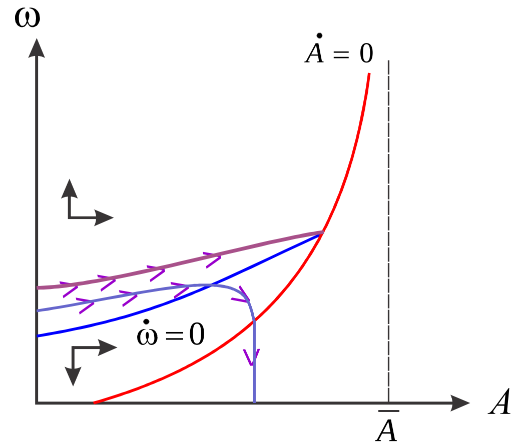

Figure 15: Phase diagram under matching function satisfying Inada conditions

Equation (110) specifies the market tightness such that agents are indifferent between mining or participating in the goods market. Equation (111) gives the value of money given market tightness. Combining (110) and (111), steady-state market tightness solves:

$$\sigma(1-\theta)\left[\frac{r}{\lambda\left(\bar{A}-\frac{1}{1+\tau}\right)}+1\right]g\left(\tau\right)=\alpha\left(\tau\right)\sigma\theta. \tag{112}$$

It is easy to check that there is a unique $\tau^s \in \left(0, \frac{1-\bar{A}}{\bar{A}}\right)$ solution to this equation. The supply of money at the steady state is then $A^s = 1/(1+\tau^s)$. The equilibrium is monetary if (109) holds. The existence of a unique steady state guarantees that the A-isocline and ω-isocline only intersect once, i.e., the ω-isocline is located above the A-isocline as illustrated in Figure 15.

In Figure 15 we represent the phase diagram of the dynamic system (105)-(106) and its arrows of motion. It can easily be checked that the steady state is a saddle path and given the initial condition $A_0 = 0$ there is a unique path leading to it. Along that path the value of money increases over time. There is also a continuum of other equilibria where the value of money vanishes asymptotically. We summarize our results in the following proposition.

In order to characterize the path for market tightness, we can rewrite (108) as

$$r\frac{\omega\left(\tau,A\right)}{S\left[\omega\left(\tau,A\right)\right]}=\left[\alpha\left(\tau\right)\sigma\theta-g\left(\tau\right)\sigma(1-\theta)\right],$$

where $\omega\left(\tau,A\right)$ is defined implicitly by $\tau=\tau\left(\omega,A\right)$. Assuming $m>0$, ω is a decreasing function of τ and an increasing function of A. Hence, the τ-isocline is upward sloping. The A-isocline becomes $A=1/(1+\tau)$.

By the same reasoning as above, the saddle path is upward sloping, which means that τ increases over time.

D.2 Divisible assets

We now study the version of the model with divisible assets. We generalize the matching function in order to obtain interior solutions for the occupation choice and we describe how the matching technology can affect price dynamics. Suppose that each agent receives an opportunity to consume at Poisson arrival rate $\alpha(\tau)\sigma$ where τ is the measure of producers per consumer, i.e., the tightness of the goods market. Because the measure of consumers is one while the measure of producers is $1-m$, tightness is simply $\tau = 1-m$. Each of the $1-m$ producers is matched with a consumer at Poisson arrival rate $\alpha(1-m)\sigma/(1-m)$. The matching function used so far is $\alpha(\tau) = \tau$. In order to guarantee that $m_t < 1$ throughout the equilibrium path, we impose $\alpha'(0) = +\infty$. Hence, the matching rate of a producer as m approaches 1 is $\lim_{m \nearrow 1} \alpha(1-m)\sigma/(1-m) = \sigma\alpha'(0) = +\infty$. Provided that $\theta < 1$, it is always optimal for some agents to choose the production sector over mining.

The measure of agents in the mining sector solves:

$$\frac{\alpha(1-m)}{1-m} \geq \frac{\lambda\left(\bar{A}-A\right)\phi}{\sigma(1-\theta)S(\phi A)}, \quad "=" \text{ if } m > 0,$$

where $S(\phi A) = u(q) - q$ with $\omega(q) = \min\{\omega(q^*), \phi A\}$. We denote $m(\phi, A)$ the solution to this equation. For all (ϕ, A) such that $\lambda\left(\bar{A}-A\right)\phi > \alpha(1)\sigma(1-\theta)S(\phi A)$, $m(\phi, A) > 0$ is an increasing function of ϕ and a decreasing function of A. Otherwise,

$$m(\phi, A) = 0 \;\text{ if }\; \alpha(1)\sigma(1-\theta)S(\phi A) \geq \lambda\left(\bar{A}-A\right)\phi.$$

The money supply evolves according to

$$\dot{A} = \lambda m(\phi, A)\left(\bar{A}-A\right).$$

Hence, $\dot{A} = 0$ if

$$\frac{\alpha(1)\sigma(1-\theta)}{\lambda} \geq \frac{\left(\bar{A}-A\right)\phi}{S(\phi A)}.$$

The frontier of this region in the (A, ϕ) space is upward sloping, it has $A = \bar{A}$ as a vertical asymptote, and it goes through the origin.

The ODE for the value of the asset is

$$r\phi = d + \alpha\left[1 - m(\phi, A)\right]\sigma\theta S'(\phi A)\phi + \dot{\phi},$$

where $q(\phi A)$ solves $\omega(q) = \min\{\omega(q^*), \phi A\}$. The ϕ-isocline is the locus of the pairs, (A, ϕ), such that $\dot{\phi} = 0$, i.e.,

$$r = \frac{d}{\phi} + \alpha \left[1 - m(\phi, A)\right] \sigma\theta S'(\phi A).$$

The ϕ-isocline needs not be monotone. To see this, note that when A is small, $m(\phi, A) \approx 1$, which gives a positive relationship between ϕ and A. When $\phi A \approx \omega(q^*)$, then the isocline is downward-sloping.

Suppose the stationary equilibrium with $m = 0$ is such that $\phi A \geq \omega(q^*)$. The asset is priced at its fundamental value, $\phi = d/r$, and the asset supply is

$$A = \frac{\lambda d\bar{A} - r\alpha(1)\sigma(1-\theta)\left[u(q^*) - q^*\right]}{\lambda d}.$$

This equilibrium exists if

$$\lambda d\bar{A} \geq \lambda r\omega(q^*) + r\alpha(1)\sigma(1-\theta)\left[u(q^*) - q^*\right].$$

In the neighborhood the ϕ-isocline is horizontal and when ϕA is slightly less than $\omega(q^*)$ is is downward sloping. It implies that the price of the asset is larger than the fundamental value initially and it reaches the fundamental value when the asset supply becomes sufficiently abundant.

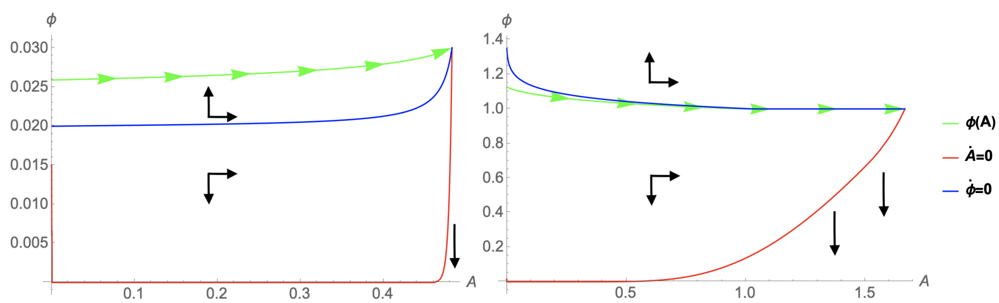

Figure 16: Two numerical examples: (Left) Scarce liquidity. (Right) Abundant liquidity.

In Figure 16 we represent the phase diagrams for two numerical examples. The blue curve is the ϕ-isocline such that $\dot{\phi} = 0$. The red curve is the A-isocline such that $\dot{A} = 0$. The green curve with arrows corresponds to the saddle path leading to the steady state. In the left panel, liquidity is scarce at the steady state. The asset price, ϕ_t, is strictly above its fundamental value, d/r, and it rises over time. These dynamics are similar to the ones described in part 2 of Proposition 9 except that there is production of the consumption good throughout the equilibrium path. In the right panel, liquidity is abundant at the steady state. Now the asset price, ϕ_t, falls over time and it converges to its fundamental value. These dynamics are new and illustrate how the matching technology matters for the time path of asset prices.

E Interest-bearing/commodity monies

Suppose the money is either a commodity that provides some direct utility, e.g., gold or silver, or it is a financial asset that pays interest. We denote $d > 0$ the dividend flow enjoyed by each money holder. The Bellman equation of an agent with one unit of money becomes:

$$rV_1 = d + \alpha\sigma(1 - A - m)\,\theta\,[u(q) - q] + \dot{V}_1. \tag{113}$$

The only novelty is the first term on the right side representing the dividend flow. The Bellman equation for an agent without money is unchanged. It follows that the dynamic equation for the value of money is:

$$\begin{aligned} r\omega(q) &= d + \alpha\sigma(1 - A - m)\,\theta\,[u(q) - q] \qquad (114) \\ &\quad - \max\left\{\alpha\sigma A(1-\theta)\,[u(q) - q]\,, \lambda\left(\bar{A} - A\right)\omega(q)\right\} + \omega'(q)\dot{q}. \end{aligned}$$

A steady-state equilibrium, (q^s, A^s), solves:

$$\begin{aligned} r\omega(q) &= d + \alpha\sigma(\theta - A)\,[u(q) - q] \qquad (115) \\ A &= \frac{\lambda\bar{A}\omega(q)}{\alpha\sigma(1-\theta)\,[u(q) - q] + \lambda\omega(q)} \end{aligned}$$

The first equation gives a negative relationship between q and A while the second equation gives a positive relationship between A and q. So there is a unique steady state and $\partial q^s/\partial d > 0$ and $\partial A^s/\partial d > 0$.

Let us turn to transitional dynamics. Suppose $m < 1 - A$, the trajectory follows $A = \mu(q)$ as in the baseline model. Suppose $m = 1 - A$. Then:

$$\dot{q} = \frac{\left[r + \lambda\left(\bar{A} - A\right)\right]\omega(q) - d}{\omega'(q)} \tag{116}$$

$$\dot{A} = (1 - A)\lambda\left(\bar{A} - A\right). \tag{117}$$

The slope $\partial q/\partial A = \dot{q}/\dot{A}$ falls in d for any given (A, q), but one can show that $\dot{q} > 0$ in equilibrium. If $\dot{q} = 0$ at certain time t, then $\dot{q} < 0$ for after t by (116). The equilibrium cannot change regime after t as a regime switch requires both trajectories to have the same slope and the locus $A = \mu(q)$ is always upward sloping. In the regime $m = 1 - A$, $\dot{q} = \alpha\sigma(1 - A)\theta\,[u(q) - q]\,/\omega'(q) > 0$ when $q \approx q^s$ by (114) and (115). Therefore $\dot{q} = 0$ is impossible at all time.

By the proof of Proposition 2, mining and production co-exist near the steady state if only if

$$\left.\frac{\partial q}{\partial A}\right|_{m=1-A} > \left.\frac{\partial q}{\partial A}\right|_{m\in(0,1-A)} \iff \frac{\mu'(q^s)/\mu(q^s)}{\omega'(q^s)/\omega(q^s)} > \frac{1-\theta}{\theta}.$$

As d increases there are two opposing effects. Since $\partial q/\partial A|_{m=1-A}$ falls in d for any given (A, q), it is more likely that $m = 1 - A$ near the steady state when d is large. On the other hand q^s and A^s increase in d and therefore agents have less incentive to mine around the steady state. The net effect is ambiguous in general.

F Competing private monies

Suppose that there are two assets that can serve as means of payment, silver (A^g) and gold (A^u).[25] We normalize asset supplies so that both assets yield the same dividend, $d > 0$. Potential asset supplies are $\bar{A}^g$ and $\bar{A}^u$ with $\bar{A} = \bar{A}^g + \bar{A}^u$. Mining rates are given by λ^g and λ^u. With no loss in generality, we assume that $\lambda^g \bar{A}^g \geq \lambda^u \bar{A}^u$, i.e., it is easier early on to mine silver rather than gold. Because these assets are perfect substitutes as means of payment, their common price is ϕ. The occupation choice of an agent is now given by

$$\max\left\{\alpha\sigma(1-\theta)\left[u(q)-q\right], \lambda^g(\bar{A}^g - A^g)\phi, \lambda^u(\bar{A}^u - A^u)\phi\right\},$$

where A^g and A^u are the amounts of silver and gold in circulation with $A = A^g + A^u$. In the following m^g is the measure of silver miners, m^u is the measure of gold miners, and $m = m^g + m^u$.

Under the assumption $\lambda^g \bar{A}^g \geq \lambda^u \bar{A}^u$, when A^g and A^u are close to 0, then only silver is mined, $A = A^g$. The indifference condition between occupations is

$$\alpha\sigma(1-\theta)\left[u(q)-q\right] = \lambda^g(\bar{A}^g - A)\phi, \tag{118}$$

and the law of motion of the total asset supply is

$$\dot{A} = m\lambda^g(\bar{A}^g - A). \tag{119}$$

When the supply of silver is sufficiently large, agents have incentives to mine gold as well. Whenever the two assets are mined, $\lambda^g(\bar{A}^g - A^g) = \lambda^u(\bar{A}^u - A^u)$, which implies

$$\bar{A}^g - A^g = \frac{\lambda^u}{\lambda^g + \lambda^u}(\bar{A} - A).$$

The fraction of silver remaining in the ground is a constant fraction of the total quantity of undug assets. The indifference condition between production and mining can be rewritten as

$$\alpha\sigma(1-\theta)\left[u(q)-q\right] = \frac{\lambda^g\lambda^u}{\lambda^g + \lambda^u}(\bar{A} - A)\phi. \tag{120}$$

Note that this condition is identical to the one in the one-asset economy where the effective mining rate is $\lambda = \lambda^g\lambda^u/(\lambda^g + \lambda^u)$. Agents will remain indifferent between mining silver or gold if $\lambda^g \dot{A}^g = \lambda^u \dot{A}^u$, which

[25] The periodic symbols of silver and gold are Ag and Au respectively.

implies $\lambda^g m^g = \lambda^u m^u$, where m^g is the measure of silver miners, m^u is the measure of gold miners, and $m = m^g + m^u$. The growth of the total supply of assets, $\dot{A} = m^g \lambda^g (\bar{A}^g - A^g) + m^u \lambda^u (\bar{A}^u - A^u)$, is equal to

$$\dot{A} = \frac{\lambda^u \lambda^g}{\lambda^g + \lambda^u} m(\bar{A} - A), \tag{121}$$

where we used that $m^g = \lambda^u m / (\lambda^g + \lambda^u)$.

The phase diagram corresponding to this dual asset economy is represented in Figure 17. The indifference condition, (118), corresponds to the upward-sloping purple curve labeled MG. The indifference condition, (120), corresponds to the red upward-sloping curve labeled $M2$. For an equilibrium starting from $A^u = A^g = 0$, the relevant indifference condition is the frontier of the yellow area where $m = 1$, i.e.,

$$\alpha\sigma(1-\theta)\left[u(q) - q\right] = \max\left\{\lambda^g(\bar{A}^g - A)\phi, \frac{\lambda^u \lambda^g}{\lambda^g + \lambda^u}(\bar{A} - A)\phi\right\}.$$

The two indifference loci, MG and $M2$, intersect for a positive ϕ, as illustrated in Figure 17, only if $\lambda^g \bar{A}^g > \lambda^u \bar{A}^u (\alpha\sigma + \lambda^g)/\alpha\sigma$. Otherwise, $M2$ is located to the right of MG. The level of assets at which agents transition from mining silver only to mining both gold and silver is $\hat{A} \equiv \left(\lambda^g \bar{A}^g - \lambda^u \bar{A}^u\right)/\lambda^g$.

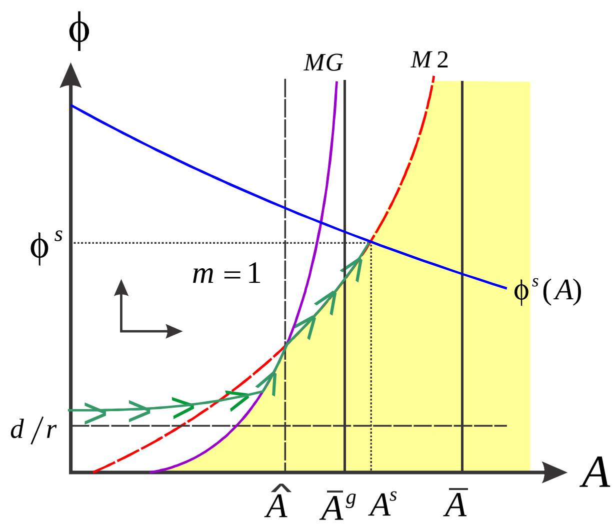

Figure 17: Phase diagram for a dual asset economy

Proposition 13 ***(Dual asset equilibrium)*** *Assume $A_0^u = A_0^g = 0$. For all $\bar{A}^u > 0$, there is a $\kappa_0 > 0$ independent of $\{\lambda^u, \lambda^g, \bar{A}^u, \bar{A}^g\}$ such that if*

$$\frac{\lambda^g \bar{A}^g - \lambda^u \bar{A}^u}{\lambda^g \lambda^u \bar{A}^u} < \kappa_0, \tag{122}$$

then the long-run equilibrium features two competing assets as media of exchange. For all t such that $A_t < \hat{A} \equiv \left(\lambda^g \bar{A}^g - \lambda^u \bar{A}^u\right)/\lambda^g$, the economy has a single means of payment (monometallism), $m_t^u = A_t^u = 0$; For all t such that $A_t > \hat{A}$, the economy has two means of payment (bimetallism) with

$$A_t^u = \frac{\lambda^g}{\lambda^u}\left(A_t^g - \hat{A}\right) > 0 \tag{123}$$

$$m_t^u = \frac{\lambda^g}{\lambda^g + \lambda^u} m_t > 0. \tag{124}$$

Proof of Proposition 13. There is production of both gold and silver along the equilibrium path if $\hat{A} < A^s$. Graphically, this condition is satisfied if the intersection of the loci MG and $M2$, $(\hat{A}, \hat{\phi})$, is located below the locus $\phi^s(A)$. See Figure 17. Let $\hat{\ell} \equiv \hat{\phi}\hat{A}$ denote the aggregate liquidity at the intersection of MG and $M2$. At the intersection $A^u = 0$ and the agents are indifferent between producing or mining gold, therefore $\alpha\sigma(1-\theta)S(\hat{\ell}) = \hat{\ell}\lambda^u \bar{A}^u/\hat{A}$. By the definition of $\hat{A}$, $\hat{\ell}$ is the largest solution of

$$\alpha\sigma(1-\theta)S(\hat{\ell}) = \hat{\ell}\frac{\lambda^g \lambda^u \bar{A}^u}{\lambda^g \bar{A}^g - \lambda^u \bar{A}^u}.$$

This solution is strictly positive if $\lambda^g \bar{A}^g > \lambda^u \bar{A}^u \left(\alpha\sigma + \lambda^g\right)/\alpha\sigma$ and $\hat{\ell} = 0$ otherwise. Moreover, $\hat{\ell}$ increases with $(\lambda^g \bar{A}^g - \lambda^u \bar{A}^u)/\lambda^g \lambda^u \bar{A}^u$. By (44) the condition $\hat{A} < A^s$ is equivalent to:

$$r < \frac{d}{\hat{\phi}} + \alpha\sigma\theta S'(\hat{\ell}) \iff r < \frac{d\lambda^u \bar{A}^u}{\alpha\sigma(1-\theta)S(\hat{\ell})} + \alpha\sigma\theta S'(\hat{\ell}).$$

The right side is decreasing in $\hat{\ell}$ and it tends to infinity as $\hat{\ell}$ goes to 0. Hence, this condition is satisfied provided that $(\lambda^g \bar{A}^g - \lambda^u \bar{A}^u)/\lambda^g \lambda^u \bar{A}^u$ is sufficiently low, i.e.,

$$\frac{\lambda^g \bar{A}^g - \lambda^u \bar{A}^u}{\lambda^g \lambda^u \bar{A}^u} < \kappa_0$$

for $\kappa_0 > 0$. The threshold κ_0 is endogenous but by construction it does not depend on λ^g and $\bar{A}^g$. This implies that, fixing all parameters except $\bar{A}^g$, the inequality is satisfied as $\bar{A}^g \to \lambda^u \bar{A}^u/\lambda^g$ from above. Similarly the inequality is satisfied when λ^g is sufficiently small. The construction of the equilibrium going backward from the steady state is as described in earlier proposition and is therefore omitted. ■

According to (122), if the initial mining rates of the two assets are not too far apart, then the economy transitions from using a single asset as medium of exchange to using two assets. The transition takes place when the supply of the first money (silver) reaches the threshold $\hat{A}$, at which point the mining rates of the two monies are equalized. According to (124) when the two monies coexist, then the measure of miners is allocated across the two monies according to the relative mining speeds, i.e., the money that is easier to mine receives more miners. From (124) the stock of gold is proportional to the quantity of silver above $\hat{A}$ throughout the transition, where the coefficient of proportionality is the relative mining speed of the two monies.