Disagreement, information quality and asset prices

Costas Xiouros[*]   Fernando Zapatero[†]

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Abstract

We solve analytically a pure exchange general equilibrium model with a continuum of agents that agree to disagree on how they interpret information. Disagreement fluctuates with information quality and the disagreement model is estimated using data on professional forecasts. We find that fluctuations in information quality generate about half the stock price volatility in the data, help explain the equity premium, and explain empirical relations between the forecast dispersion and asset prices. Constant information quality cannot account for the variation in forecast dispersion and in this case, disagreement has almost no effect on the stock return volatility.

Keywords: Asset Prices, Heterogeneous Expectations, Information Quality, Habit-Formation Preferences.

JEL Classification: G10, G12.

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[*]Department of Finance, BI Norwegian Business School, N-0442 Oslo, Norway. E-mail: costas.xiouros@bi.no

[†]Finance Department, Questrom School of Business, Boston University, Boston, MA 02215. E-mail: fzapa@bu.edu

1   Introduction

There is little debate on the prevalence of heterogeneous beliefs or disagreement among market participants and the importance of this fact on asset prices and their dynamics.[1] In parallel, many experimental studies by the psychologists and economists responsible for the inception and growth of behavioral economics, conclude that an essential feature in economic and financial decision-making is the existence and critical role of a reference value––for example, current wealth, wealth of peers, a benchmark, are some of the most frequently used in the literature, among many others– for the decision-maker.[2] In this paper we combine both elements in a parsimonious model that captures not only qualitative properties of asset prices –which other studies also achieve––but also quantifies them and evaluates the relative importance of the features of the model in explaining the data.

The consequence of a reference point in asset prices is straightforward, so we discuss it first: it affects the effective risk-aversion of the agents –as opposed to the parametric value of their risk-aversion–– which in turn can explain the level of the price of risk, an elusive goal for models that rely on standard utility and ignore the reference point. Among the different possible ways to incorporate reference-dependence in preferences, we adapt the model of habit-formation of Campbell and Cochrane (1999) to non-i.i.d. consumption growth, very influential in the mainstream asset pricing literature, as well as in the behavioral economics camp –for example, Rayo and Becker (2007). Further, the persistence in the habit assumed by Campbell and Cochrane (1999) generates variations in the price of risk, which explain the excess stock return predictability and account for a significant portion of the stock return volatility.

Yet, as we show in this paper, equally important for the dynamic properties of asset prices are heterogenous beliefs. However, most of the previous theoretical literature analyzes very stylized models, for example featuring only two agents, and offers just qualitative predictions. An obvious shortcoming is that a purely qualitative analysis can lead to misleading conclusions as to the importance of the effects analyzed. Further, due to data limitations, the empirical evidence is partial and inconclusive.[3] In this paper, we analyze a rich yet tractable

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[1] In a recent study, Giglio, Maggiori, Stroebel and Utkus (2019) survey a large panel of individual retail investors and document that individuals have large and persistent heterogeneity in beliefs and show strong willingness to *agree to disagree*.

[2] The pathbreaking Kahneman and Tversky (1979) that compiles the conclusions of many of these studies is the point of departure of a large literature on reference-dependent preferences.

[3] Several empirical studies have tried to uncover the relation between disagreement and future stock returns. For example, Diether, Malloy and Scherbina (2002) (DMS), Chen, Hong and Stein (2002), Goetzmann and Massa (2005) Yu (2011) find a negative relation, in support of the overvaluation hypothesis of

model that permits a quantitative analysis of the asset pricing implications, with a focus on the stock price volatility, the equity premium, and the risk-free rate. We show that some of the qualitative effects derived in the previous literature are in practice negligible while other consequences of our model not studied in the previous literature are quantitatively important.

Our analysis is based on a model of the cross-section of beliefs about the growth rate of the economy, with a continuum of agents that update their beliefs constantly and yet *agree to disagree*, in line with the evidence provided by Kandel and Pearson (1995) and most of the literature on heterogeneous beliefs. We estimate the model using data from the Survey of Professional Forecasters (SPF) on GDP growth forecasts. There are several notable features in the data. First, the mean forecast is a strong predictor of GDP growth and changes in the mean forecast are weakly correlated with the realized GDP growth. Evidently, forecasters observe useful information about changes in the conditional mean of the economy's growth rate, which we model with an informative common signal. Second, disagreement among forecasters is persistent, volatile, counter-cyclical and negatively correlated with realized GDP growth. We envision that each individual has her own way of perceiving the complex economic system and how shocks and new information propagate and affect it.[4] We further presume that negative economic shocks deteriorate the quality of information. High informational quality could be related, for example, to cases where all economic indicators "point to the same direction", as opposed to cases where the economic indicators give conflicting signals. Meanwhile, if there is good information the scope of differences in perception and disagreement are low. When this information deteriorates everyone has to rely on their own way of perceiving and interpreting new information, and disagreement increases. To incorporate these features, we draw from Scheinkman and Xiong (2003) and assume that

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Miller (1977). However, Johnson (2004) finds that the analyst dispersion used by DMS to measure disagreement proxies for uncertainty and higher uncertainty is what leads to lower future returns for levered firms. Barron, Stanford and Yu (2009) provide further support to the argument in Johnson (2004) and provide opposite evidence to DMS. Avramov, Chordia, Jostova and Philipov (2009) show that the negative relation between analyst dispersion and future stock returns is accounted for by credit risk. Further, Doukas, Kim and Pantzalis (2006), utilizing a different measure of forecast dispersion, find a positive relation with future stock returns.

[4]There is evidence that cultural differences, individual abilities and personal experiences influence differences in perceptions and financial decisions. Grinblatt, Keloharju and Linnainmaa (2012) document that IQ affects trading behavior and performance. Kumar, Page and Spalt (2011) show that religious beliefs affect risk taking behavior, while Malmendier and Nagel (2011) show that personal experiences influence risk taking behavior. Burke and Manz (2014) find that economic literacy improves forecast accuracy. Hoffmann and Post (2017) find that investors' past returns positively impact return expectations and negatively impact their risk perceptions.

agents observe "private" noisy signals. We further assume that the weight the agents place on these signals is inversely proportional to the informativeness of the common signal. This captures the common wisdom that when common information is weak or absent, individuals revert to their "tribal" beliefs. Consequently, when the information quality of the common signal deteriorates, disagreement increases.

With the estimated disagreement model at hand, we calibrate preferences and quantitatively analyze how disagreement affects asset price properties. Our main findings are as follows: (i) In the absence of shocks to information quality, disagreement is constant and has a negligible effect on the volatility in stock prices. (ii) Shocks to information quality generate close to half of the stock price volatility observed in the data, where stock prices increase with information quality. (iii) Shocks to information quality are highly positively correlated with growth shocks and, hence, negatively correlated with the state price; as a result, the stock price volatility that they generate significantly increases the equity premium. (iv) Disagreement is negatively correlated with the market price-dividend ratio and predicts future interest rates, both in the model and in the data. (v) Disagreement increases significantly the price of risk when the utility curvature parameter is low and habit is "strong".[5] In contrast, with standard CRRA preferences the effect is negligible, irrespective of the level of risk-aversion.

The recent market turmoil resulting from the Covid-19 health crisis provides an illustration of the mechanism we analyze in the paper. During the month of March, we saw a drop of about 30% in the S&P500, which over the following weeks quickly regained half of that drop on a steady trend. Overall, the market registered changes of more than 1% for a number of consecutive days that had not been seen since 1930. What do our economic models have to say about such swings? The habit-formation theory explains them as an increase in risk-aversion due to a drop in the level of consumption relative to habit and then a subsequent drop in risk aversion (possibly due to an increase in expected consumption, as the government announces a very large, multi-prong, economic package). The long-run risk theory explains them as very persistent fluctuations in expected consumption growth. That is, a significant drop (and then a significant increase, adjustment triggered by the economic package) in expected consumption growth that will remain in place for a long time. In our model we have both channels. The first is quantitatively important but cannot account for the entire magnitude of the stock market fluctuations we observe. The second is minuscule because the estimated persistence in the shocks to the expected consumption growth is not

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[5]The utility curvature parameter is the CRRA in the case of no habit.

high enough. But we also have a third channel, which is disagreement, that basically says that in addition to a drop in consumption (relative to habit) and expected consumption growth, we had a spike in uncertainty, in a good part driven by very low information quality (scientist and government leaders expressing different opinions and revising them almost daily), which makes long-run predictions more difficult. This naturally induces an increase in disagreement: investors disagree on how the shock translates to stock market impact, and this is a persistent effect. That is, the low information quality is expected to remain for a long time, and this predicts disagreement in the future. The rebound of the stock market can also be explained by subsequent positive shocks to information quality resulting from the approval of the economic package, which mitigates the effect of conflicting predictions among politicians and/or scientists. The overall rise in disagreement causes a small but persistent increase in the real rate–as expected deflation is the consensus–and the risk premium. Additionally, our model predicts a decrease in diversification: anecdotal evidence is consistent with this prediction, as we have seen that investors focus on subsets of the market, like tech companies, and the rise in the stock market is mostly explained by a few (but very large) stocks.

Despite the rich model, we derive the state dynamics in closed form by assuming that the initial priors are drawn from a Gaussian distribution. The cross-section of beliefs is characterized by three variables. The first is the *mean forecast*. The second is the *forecast dispersion*, which refers to the standard deviation across forecasts. The third is the *uncertainty*, which is the same for all agents.[6] We also solve in closed form the stochastic discount factor, which yields analytical expressions for the risk-free rate and the price of risk. Through comparative statics and different model calibrations, we analyze and quantify the various effects. Our findings put several results of the related theoretical literature into perspective.

Most of the existing theory on heterogeneous expectations and asset prices predicts that disagreement generates excess volatility.[7] In the absence of variations in information quality, there are two main channels through which disagreement affects volatility. The first is that disagreement affects the volatility of the mean forecast. This is the focus of the recent study by Atmaz and Basak (2018), who analyze a model with dogmatic beliefs and constant fundamentals. They find that disagreement increases the volatility of the mean forecast through the reallocation of wealth generated by the speculative trading activity. Our analysis

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[6]The amount of information that agents believe they can extract from the observed signals is the same across agents. This is a common assumption for parsimony.

[7]See, for example, Buraschi and Jiltsov (2006), David (2008), Dumas, Kurshev and Uppal (2009), Xiong and Yan (2010), Ehling, Gallmeyer, Heyerdahl-Larsen and Illeditsch (2016) and Atmaz and Basak (2018).

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shows that this is not always the case. In fact, when we consider fluctuations in the mean growth rate, uncertainty, learning and intermediate consumption, disagreement in general generates lower volatility of the mean forecast than a full information case. Regardless, we find that these fluctuations are not quantitatively important for the stock price because the shocks to the mean growth rare are not persistent enough.

The second channel by which disagreement may generate excess volatility is through the fluctuations in the forecast dispersion. According to Dumas, Kurshev and Uppal (2009) this effect generates most of the excess volatility. This channel is also emphasized in Buraschi and Jiltsov (2006) and David (2008), but all three studies analyze two-agent economies. Our model generates an altogether different result. As opposed to the case of a two-agent economy, disagreement with constant information quality in a large economy does not generate any stochastic fluctuations in forecast dispersion. Due to the law of large numbers, changes in dispersion depend neither on the growth rate realizations nor on the signals. Starting from any level, it soon converges deterministically to a steady state. Thus, in our model, shocks to the forecast dispersion can only come from shocks to information quality. This feature enables the identification of the parameters of the information quality process.

Because of the above two results, a disagreement economy with constant information quality is equivalent to a homogeneous agent economy and disagreement has negligible effect on the stock price volatility. The introduction of shocks to information quality makes the forecast dispersion stochastic with asset pricing effects whose magnitudes depend on the preference parameters. The theoretical literature has generated several results that explain how an increase in disagreement affects asset prices. Miller (1977) hypothesizes that disagreement with short sale constraints leads to overvaluation, because prices reflect more the valuation of optimistic investors. Harrison and Kreps (1978) and Scheinkman and Xiong (2003) substantiate this hypothesis with partial equilibrium analyses, assuming, however, risk-neutral agents. Gallmeyer and Hollifield (2008) extend the analysis to general equilibrium with risk-averse agents and they find that disagreement with short-sale constraints may increase or reduce the stock price depending on the investors' intertemporal elasticity of substitution.[8] Similarly, Jouini and Napp (2006b) and Jouini and Napp (2006a) show that in a frictionless setting disagreement may increase or decrease stock prices, because depending on the risk preference parameters disagreement may decrease or increase the risk-free interest rate.[9]

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[8]In a related paper, Osambela (2015) provides a connection between liquidity and speculation in the presence of disagreement and endogenous portfolio constraints.

[9]Varian (1985) shows, in a static analysis, that the impact of heterogeneous beliefs on state prices depends

We also find that forecast dispersion affects the risk-free rate and the effect depends on whether agents are more or less risk-averse than a logarithmic investor (with or without habit). When agents are more risk-averse than a logarithmic investor, an increase in dispersion increases the risk-free rate. In this case, the relative pessimists have a stronger effect on the aggregate demand for savings. As a result, the risk-free rate needs to increase so that the relative optimists can borrow from the relative pessimists. Therefore, a shock to information quality increases future dispersion and either increases or decreases the future risk-free rates, depending on the utility curvature parameter.

We then examine whether disagreement also affects the price of risk. According to David (2008), disagreement explains about half of the observed equity premium when agents have low constant relative risk aversion (CRRA). In such a case, disagreement makes agents take large speculative positions that increase substantially the per-capita consumption risk. Our analytical results, however, show to the contrary that, even though lower CRRA leads to higher speculation and per-capita consumption risk, the net effect of a decrease in CRRA on the price of risk is negative. Ceteris paribus, the price of risk always decreases with a lower CRRA because the decrease due to the lower CRRA is greater than the increase due to higher speculation. As a result, with CRRA preferences, disagreement only has a marginal effect on the price of risk which in absolute terms is quantitatively insignificant, regardless of risk aversion. This prediction changes when we introduce habit formation preferences. When agents are highly risk averse due to habit but with low utility curvature parameter, then an increase in dispersion leads to a significant increase in the price of risk.

The price of risk is also affected when the mean forecast is biased in relation to the true mean growth rate. For example, in David (2008) the high equity premium is generated partly because in his model the mean forecast in on average negatively biased. In our model, the mean forecast is on average unbiased and fluctuations in this bias are, as already discussed, quantitatively unimportant for the stock price.

We calibrate preferences to fit the stock market Sharpe ratio and the volatility of the risk-free rate. As a result, agents are slightly more risk averse than a logarithmic investor with "strong" habit. Hence, higher dispersion leads to increases in both the price of risk and the risk-free rate. Further, according to the estimated disagreement model, the information quality is volatile, persistent and driven by aggregate growth shocks.[10] Consequently, shocks

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on the properties of the utility function.

[10]According to the estimated model, shocks to information quality are almost entirely driven by aggregate growth shocks, whereby a positive shock leads to a decrease in information quality. David (2008) points also that forecast dispersion is counter-cyclical, which his model explains by having the two agents disagree

to information quality generate large fluctuations in stock prices, because they have persistent effects in both the price of risk and the risk-free rate. Also, these stock price fluctuations require a risk premium because such shocks are mainly driven by the aggregate growth rate, which is the only source of priced risk in the economy. The calibrated model explains most of the stock price volatility, half of which is generated by habit and the rest due to variations in information quality. At the same time, the calibrated model explains the risk-free rate in terms of its mean, volatility, autocorrelation and correlation with the stock price. About half of the volatility in the risk-free rate is due to fluctuations in the forecast dispersion and the rest due to fluctuations in the mean forecast. Finally, we find that the dispersion in professional forecasts predicts interest rates, as anticipated by the model.

2 Further related literature

The literature on heterogeneous beliefs and asset prices is long. Detemple and Murthy (1994) analyze a production economy with heterogeneous beliefs on the economy's growth rate and show that asset prices are determined by the wealth-weighted beliefs of the agents. In a pure exchange economy Zapatero (1998) shows that disagreement generates excess volatility, mainly due to the volatility of the wealth shares. In a similar setting, Basak (2000) shows that heterogeneous beliefs over non-fundamentals give rise to additional priced factors and Basak (2005) expresses the equilibrium in the form of a representative agent and derives the price of risk in the case of CRRA preferences. Buraschi and Jiltsov (2006) study the effects on options and draw a link between disagreement, option volumes and option prices.[11] In all these studies, as well as in many others, the main effect on prices come from fluctuations in the expectations of the representative agent, or some other measure of mean forecast. In our analysis we construct a measure of the mean forecast using the consumption weights and together with an appropriate distributional assumption we solve the equilibrium in closed form.[12] However, we find that fluctuations in the mean forecast are not quantitatively important for the stock price.

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more during recessions. He points out that this feature is consistent with evidence that the cross-sectional differences in consumption growth increase during recessions.

[11]Buraschi, Trojani and Vedolin (2014) also study connections between disagreement and implied volatilities and show how disagreement is related to volatility risk premia.

[12]Methodologically, our study is related to Xiouros and Zapatero (2010), Cvitanić and Malamud (2011) and Atmaz and Basak (2018) who utilize an appropriate distributional assumption over a continuum of types to provide analytical results of the market equilibrium.

Atmaz and Basak (2018) also analyze the asset pricing effects of disagreement with a continuum of agents and solve analytically for the equilibrium. Their main result is that dispersion makes the stock price convex in cash-flow news. They also find that dispersion increases both the stock price volatility and trading. Our study differs in many respects. Firstly, they perform a qualitative comparative statics analysis on how the initial disagreement and different cash-flow news affect the stock price. We instead focus on the quantitative predictions of an estimated model of disagreement. Further, we analyze an infinite rather than a finite horizon economy, we introduce fluctuations in the conditional growth rate, intermediate consumption, learning, fluctuating information quality and habit formation preferences. Our model yields a stationary equilibrium which we calibrate to asset prices. In terms of predictions, we find instead that what is important for stock prices is shocks to information quality and not fluctuations in the mean forecast/bias.

Dumas, Kurshev and Uppal (2009) study the "sentiment" risk caused by overconfident investors, which generates excess volatility in stock prices and further study the optimal behavior of the rational investor. Xiong and Yan (2010) and Ehling, Gallmeyer, Heyerdahl-Larsen and Illeditsch (2016) find that heterogeneity in inflation expectations generates excess volatility in bond yields. With our analysis we echo Atmaz and Basak (2018) in the importance of studying models with a continuum of agents, since some of the result in two-agent settings do not carry over to large economies.

Two other related studies are those of Cvitanić, Jouini, Malamud and Napp (2011) and Bhamra and Uppal (2014) who offer general equilibrium analyzes of economies with both preference and belief heterogeneity. Both studies offer a number of analytical results in terms of several asset pricing quantities, as well as results pertaining to agent survival and the long-run properties of those economies. In particular, Bhamra and Uppal (2014) offer analytical expressions of all main asset pricing quantities, including the stock return volatility, as well as the conditions to obtain a stationary equilibrium in a two-agent setting. In our model, the continuum of agents always results in a stationary equilibrium, independently of the model parameters, because it is not tied to the survival of any agent stemming from the fact that all agents have negligible amount of wealth.[13]

Our paper is also related to the literature that tries to explain quantitatively the asset

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[13]Several papers study the agent survival and specifically the market selection hypothesis that conjectures the eventual financial ruin of irrational agents; see, for example, Sandroni (2000), Blume and Easley (2006), Yan (2008) and Fedyk, Heyerdahl-Larsen and Walden (2012). Further, Kogan, Ross, Wang and Westerfield (2007), Kogan, Ross, Wang and Westerfield (2009), Cvitanić and Malamud (2011) and Cvitanić, Jouini, Malamud and Napp (2011) show that non-surviving agents may have long-run effects on equilibrium asset prices and portfolios.

prices on the aggregate, of which the main features are the equity premium and the volatility of valuation ratios. Common examples are the habit-formation models of Campbell and Cochrane (1999) and Wachter (2006), the long-run risk model of Bansal and Yaron (2004) and the rare disaster models of Barro (2006) and Gabaix (2012). We contribute to this literature by incorporating heterogeneous beliefs into a habit-formation model that accounts not only for the equity premium and stock return volatility, but also for several relations between disagreement and asset price quantities. In addition, our model explains the low correlation between the risk-free rate and the stock price-dividend ratio. Effectively, we introduce a new factor, namely information quality in the presence of disagreement, that we find to be important for asset prices. Shocks to information quality significantly affect both the risk-free rate and risk premia. We perform a variance decomposition of the stock price-dividend in the model and we find it to be similar to what we see in the data.

3   A model of disagreement

The model of disagreement is built on the basis of an endowment economy, where agents face the aggregate consumption risk. Agents differ in their opinions about the aggregate risk because they disagree on how they interpret common information, in line with the evidence provided by Kandel and Pearson (1995). We assume that all information is available to all agents so that there is no need to learn from asset prices. This allows us to develop and estimate the disagreement model independently of preferences and asset prices. The model has three distinctive features: the continuum of agents, a common informative signal, unobservable to the econometrician, and the fluctuating degree by which the agents interpret information differently caused by fluctuations in information quality. These features generate rich dynamics for the distribution of beliefs and allow us to estimate the model using data from the Survey of Professional Forecasters (SPF) provided by the Philadelphia Fed.

3.1   Aggregate endowment

The aggregate per-capita consumption, C, is exogenous. Its log-growth rate is conditionally normal with a time-varying conditional mean that follows an autoregressive process;

$$g_{t+1} \sim N(\mu_t, \sigma_c^2), \tag{1}$$

$$\mu_{t+1} = \varphi_\mu \mu_t + (1 - \varphi_\mu)\mu_c + \sigma_\mu \epsilon_{t+1}^\mu, \tag{2}$$

where $g_{t+1} := \ln(C_{t+1}/C_t)$, μ_c is the long-run mean and the shocks ϵ^μ are i.i.d. $N(0,1)$ and independent of g. The fluctuations in the mean growth rate represent changes in the economic fundamentals and the fact that it is unobservable is the source of disagreement.

3.2   Agents, information quality and disagreement

The economy is populated with a continuum of agents indexed by $i \in [0,1]$ of unit total mass and constant density of one. Agents can not observe μ_t and hold beliefs according to

$$\mu_t | \mathcal{F}_t \overset{i}{\sim} N(\mu_t^i, v_t^2), \qquad \forall i \in [0,1], \tag{3}$$

where $\overset{i}{\sim}$ denotes the beliefs of agent i and $\mathcal{F}_t$ the common information set in period t. The beliefs about the aggregate consumption growth are, thus, given by $N(\mu_t^i, \sigma_c^2 + v_t^2)$. The uncertainty v_t is the same across agents. This is a result of the assumption, to be formalized later, that the amount of information that the agents believe they can extract is the same for all. As a result, the agents differ only in their forecasts for next period, i.e. $\mu_t^i = \mathbb{E}_t^i(g_{t+1}) = \mathbb{E}_t^i(\mu_t)$, where we use the notation $\mathbb{E}_t^i(\cdot) = \mathbb{E}^i(\cdot|\mathcal{F}_t)$ to denote a conditional expectation.

An agent's type in period t is determined by her beliefs μ_t^i. We further assume that the initial distribution of types is Gaussian, in the sense that the density of agents as a function of beliefs is the normal density function with mean $\bar{\mu}_0$ and standard deviation $\bar{\nu}_0$. We denote with $n_t(\mu)$ the density of agents whose forecast is given by μ and, hence, $n_0(\mu) = \phi(\mu; \bar{\mu}_0, \bar{\nu}_0^2)$, where $\phi(\cdot; \cdot, \cdot)$ denotes the normal density function.[14] In what follows, we will specify how agents update their beliefs. Nevertheless, the distribution of types remains Gaussian; for any period t it is completely described by the average forecast $\bar{\mu}_t$ and the dispersion in forecasts $\bar{\nu}_t$, where

$$\bar{\mu}_t = \int_0^1 \mu_t^i di \qquad \text{and} \qquad \bar{\nu}_t^2 = \int_0^1 (\mu_t^i - \mu_t)^2 di. \tag{4}$$

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[14]To be able to derive our results, we define the distribution of types in some period t as follows:

$$n_t(\mu) := \frac{d}{d\mu} \int_0^1 H(\mu - \mu_t^i) di = \int_0^1 \delta(\mu - \mu_t^i) di,$$

where $H(x)$ is the Heaviside function, equal to 1 for $x \geqslant 0$ and 0 otherwise and whose derivative is the Dirac delta function, $\delta(x)$. The second integral in the above equation is derived using the Leibniz integral rule. Note that the above is a properly defined density function, since integrating over the real line wrt. μ gives one.

Every period, agents observe the realization of the growth rate with which they update their beliefs about the current period's mean growth rate. They then observe signals that contain information about the shock to the mean growth rate and form their beliefs about next period's mean growth rate knowing that it follows the process (2). However, similar to Detemple and Murthy (1994), Zapatero (1998) and Scheinkman and Xiong (2003), the agents disagree about the interpretation of this information.

All agents observe the common signal

$$s_t^m = \rho_t \epsilon_t^\mu + \sqrt{1 - \rho_t^2}\epsilon_t^m, \tag{5}$$

where $\rho_t \in [0, 1]$ represents *information quality* and ϵ^m is i.i.d. $N(0, 1)$. Learning from the realized growth rate and the common signal decreases the dispersion in beliefs. The common signal is unobservable to the econometrician and this assumption is necessary to explain two things: (i) the fact that forecasters' beliefs react to signals other than the observed realized growth rates and (ii) the predictability of the quarter-ahead growth rate by the mean forecast.

Agents also observe a continuum of uninformative signals, indexed by i. Each one of these signals is the common signal s^m distorted by some independent noise,

$$s_t^i = \eta s_t^m + \sqrt{1 - \eta^2}\epsilon_t^i, \qquad \forall i \in [0, 1], \tag{6}$$

where the ϵ^i's are i.i.d. $N(0, 1)$ and $\eta \in [0, 1]$. Nevertheless, each agent believes that one of these signals carries additional information. Specifically, agent i believes that signal i is generated as follows:

$$s_t^i \overset{i}{=} \zeta_t \epsilon_t^\mu + \sqrt{1 - \zeta_t^2}\epsilon_t^i, \qquad \forall i \in [0, 1], \tag{7}$$

for some $\zeta_t \in [0, 1]$, where $\overset{i}{=}$ indicates agent i's beliefs. Agent i correctly considers the rest of the signals as uninformative. For ease of reference we call signals $i \in [0, 1]$ as individual signals, even though all signals are public.[15] The variable ζ_t is common to all agents and controls the differential interpretation of the available information. In reality, this may be due to different underlying assumptions stemming from different education, experiences, observed histories, culture, age and so on. For example, Malmendier and Nagel (2011) find that individual experiences of macroeconomic shocks affect future expectations and financial

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[15]We could also assume that the individual signals are private as long as all agents believe that no other agent possesses additional information, so that they have no reason to learn from prices and attribute different demands for assets to differences in opinions.

risk taking. Essentially, agents have different models in mind with which they interpret the information and ζ_t represents the differential interpretation of common information.

The true correlation between the individual signals and the common signal is given by η, whereas each agent believes that her signal has a correlation with the common signal equal to $\zeta_t\rho_t$. We impose, therefore, a rationality constraint that $\zeta_t\rho_t$ has to equal η at all times. This implies that the differential interpretation of common information is inversely proportional to quality of information, i.e. $\zeta_t = \eta/\rho_t$. As a result, $\eta \in [0,1]$ is the *disagreement* parameter and $\rho_t \in [\eta, 1]$.

Proposition 1 derives the dynamics of the distribution of beliefs and the common uncertainty.

Proposition 1. *The uncertainty about the mean growth rate evolves according to*

$$v_{t+1}^2 = \varphi_\mu^2 \kappa_t v_t^2 + \sigma_\mu^2 A_{v,t+1}.$$

where

$$\kappa_t := \frac{\sigma_c^2}{\sigma_t^2}, \qquad \sigma_t := \sqrt{\sigma_c^2 + v_t^2} \qquad and \qquad A_{v,t} := \frac{(1 - \zeta_t^2)(1 - \rho_t^2)}{1 - \zeta_t^2 \rho_t^2}.$$

Further, if in period t the distribution of beliefs μ_t^i is Gaussian with mean $\bar{\mu}_t$ and variance $\bar{v}_t^2$, then the distribution of beliefs in period $t+1$ is also Gaussian with mean and variance given by

$$\bar{\mu}_{t+1} = (1 - \varphi_\mu)\mu_c + \varphi_\mu \kappa_t \bar{\mu}_t + \varphi_\mu(1 - \kappa_t)g_{t+1} + \sigma_\mu A_{\mu,t+1} s_{t+1}^m,$$

$$\bar{v}_{t+1}^2 = \varphi_\mu^2 \kappa_t^2 \bar{v}_t^2 + \sigma_\mu^2 A_{\nu,t+1}.$$

where

$$A_{\mu,t} := \frac{\eta\zeta_t(1 - \rho_t^2) + \rho_t(1 - \zeta_t^2)}{1 - \zeta_t^2 \rho_t^2} \qquad and \qquad A_{\nu,t} := \frac{\zeta_t^2(1 - \rho_t^2)^2(1 - \eta^2)}{(1 - \zeta_t^2 \rho_t^2)^2}.$$

Since $\zeta_t = \eta \div \rho_t$ then

$$A_{\mu,t} = \rho_t, \qquad A_{\nu,t} = \frac{\eta^2(1 - \rho_t^2)^2}{\rho_t^2(1 - \eta^2)} \qquad and \qquad A_{v,t} = \frac{(\rho_t^2 - \eta^2)(1 - \rho_t^2)}{\rho_t^2(1 - \eta^2)}.$$

The mean forecast $\bar{\mu}$ follows an autoregressive process just like the mean growth rate and it is driven by two shocks. The first is the growth rate g_{t+1} which allows the agents to update their beliefs on μ_t and the sensitivity to this shock depends on the persistence

of and the uncertainty about μ–lower uncertainty implies higher value for κ. The second shock is the common signal s^m and the sensitivity to this shock depends on A_μ, which in turn depends on all three parameters, η, ζ and ρ, that determine the true and perceived informativeness of the signals. For example, when the common signal is fully informative, i.e. $\rho = 1$, then A_μ takes its maximum value of 1. When the common signal is uninformative ($\rho = 0$) then the common signal still affects the average beliefs with a sensitivity of $A_\mu = \eta\zeta$ because η determines the correlation of the individual signals with the common signal and ζ the perceived informativeness of the individual signals. When the differential interpretation is at its maximum ($\zeta = 1$) then $A_\mu = \eta$ due to the fact that the agents only consider their individual signals that have a correlation with the common signal equal to η.

The dispersion in beliefs $\bar{\nu}$ does not depend either on the signals or the growth rate realizations, but it depends on the true and perceived informativeness of the signals. The first term shows how the dispersion decays due to learning, depending on the level of uncertainty and the persistence of the mean growth rate. The second term is the shock to dispersion which is either zero or positive. It is zero when either the common signal is perfectly informative ($\rho = 1$) or all individual signals are perfectly correlated with the common signal ($\eta = 1$). Imposing the restriction $\zeta = \eta/\rho$ then the term A_ν reaches the maximum value of one when the common signal bears no information and information quality is at its minimum, i.e. $\eta = \rho = 0$ and $\zeta = 1$.

The final element of the disagreement model is the process of the information quality ρ, which is essential to capture several features of the data, most importantly the large fluctuations in dispersion. We assume that ρ is generated as follows:

$$\rho_t = \eta + (1 - \eta)\Phi(\chi_t + \sigma_\rho\epsilon_t^\rho) \tag{8}$$

$$\chi_t = \phi_\chi\chi_{t-1} + (1 - \phi_\chi)\mu_\chi + \beta_\chi(g_t - \tilde{\mu}_{t-1}) + \sigma_\chi\epsilon_t^\chi, \tag{9}$$

where ϵ^ρ and ϵ^χ are i.i.d. $N(0, 1)$. $\Phi(\cdot)$ denotes the standard normal cumulative distribution function so that $\rho_t \in [\eta, 1]$. The combination of transitory shocks ϵ^ρ and persistent shocks ϵ^χ is enough to capture well the autocorrelation structure of the forecast dispersion. Further, the dependence on growth rate surprises $(g_t - \tilde{\mu}_{t-1})$ is necessary given that growth is negatively correlated with changes in dispersion. The quantity $\tilde{\mu}$ represents some weighted mean forecast, which is an equilibrium variable and will be derived later. The reason for not using $\bar{\mu}_t$ is because otherwise we would need to have both variables in the state vector, that would make the computations of the equilibrium prices much more demanding. The effect of this

choice on the estimated model parameters and the properties of the disagreement model is negligible.

3.3   Data and estimation

To estimate the model we use aggregate consumption data and data from the Survey of Professional Forecasters (SPF). We use the annual consumption data provided by the National Income and Product Accounts (NIPA) from 1929 to 2017 and compute the per-capita consumption of non-durable goods and services. The annual growth rates where then adjusted for inflation by subtracting the growth in the annual average CPI index–data taken from the Bureau of Labor Statistics (BLS). We simulate the model at quarterly frequency and time-aggregate to compute annual moments.

From the SPF, we use the real GDP forecasts for the current quarter and for the quarter ahead, made by professional forecasters. The data is quarterly and span the period from 1968Q3 to 2016Q4. Every quarter there are between 9 and 83 professional forecasts with an average of 38.5 and standard deviation of 12. From the level data we compute growth rates which then we adjust for population growth (computed from NIPA tables) to obtain real quarterly per-capita growth forecasts. For every quarter we then compute the cross-sectional average forecast ($\bar{\mu}$) and a measure for the dispersion in forecasts. To lessen the effect of outliers we compute the dispersion as the distance between two percentiles, where the percentiles are those of a normal distribution half a standard deviation left and right from the mean. The dispersion measure corresponds, thus, to the cross-sectional standard deviation in the case the distribution of forecasts is Gaussian. We also collect quarterly data for real per-capita GDP growth over the same period as the SPF data. We thus analyze the moments of the average forecast and the dispersion in forecasts in relation to the variable in concern.

To compute model moments we run 1,000 simulations with 400 periods each and a burn-in of 100 periods. The model is estimated using the Simulated Method of Moments, trying to match 16 moments with 10 parameters. Table 1 shows the moments for the data alongside those of the estimated model. The parenthesis next to each data statistic is the standard error estimated using the Newey and West (1987) method and 16 lags. The parenthesis next to each model moment shows the t-statistic of the hypothesis that the data moment is equal to the model value. From the table we see that the differences between the model and the data are statistically insignificant, with the highest t-statistic being 1.28 and the rest less than 1. Table 2 shows the estimated model parameters.

The model matches almost perfectly the mean, volatility and first-lag autocorrelation of annual consumption growth, shown in Panel A. Panel B shows moments related to beliefs. The average forecast $\bar{\mu}$ is quite volatile–the volatility is standardized by the volatility of the related variable (GDP growth in the data and consumption growth in the model)–and relatively persistent; both captured well by the model. The average and the volatility of the forecast dispersion $\bar{\nu}$, standardized by the volatility of the related variable, are both quite high and the model matches them almost perfectly. The model also captures quite well the autocorrelation structure of the forecast dispersion as given by lags 1, 4, 8 and 12. As a result, χ, which is the variable driving ρ (and ζ), is estimated to be quite persistent with $\phi_\chi = 0.965$. The power of the average forecast to predict the quarter ahead growth is shown by the relevant correlation of 0.54 in the data and 0.51 in the model. The model also explains the low correlation between changes in the average forecast and the growth rate. The model explains these two statistics with the common signal s^m and the parameters η and μ_χ. An increase in either of these parameters increases the sensitivity of the average forecast $\bar{\mu}$ to the common signal and, hence, the volatility of $\bar{\mu}$; as a result, the correlation between changes in $\bar{\mu}$ and growth rate realizations decreases. A higher μ_χ also implies higher predictive power for the average forecast because μ_χ determines the average information quality, as seen from equations (8) and (9). On the other hand, an increase in the disagreement parameter η does not increase the predictive power, because it introduces uninformative fluctuations in the mean forecast through higher disagreement and higher correlation among individual signals.

A statistic that is of particular interest is the negative correlation between forecast dispersion changes with the related growth variable, which is -0.20 in the data and -0.17 in the model. The model generates this correlation by having most of the fluctuations in ρ be driven by growth rate surprises. The positively estimated β_χ implies that the information quality increases decreases with positive growth rate surprises, which has important implications for the equity premium.

4   Investor preferences and market equilibrium

So far, the disagreement model was analyzed and estimated irrespective of investor preferences, wealth distribution and market structure. These elements are required to complete the economy and generate asset pricing predictions. We assume that markets are complete so that agents can choose their optimal consumption paths through trading.

4.1 Risk preferences

All agents live forever and exhibit identical power utility preferences with external habit;

$$\mathbb{E}_t^i \sum_{\tau=0}^{\infty} \beta^\tau \frac{(C_{t+\tau}^i - H_{t+\tau})^{1-\gamma} - 1}{1 - \gamma}, \qquad \forall i \in [0, 1], \qquad (10)$$

where C^i denotes the consumption of agent i and H the common external habit. The subjective discount factor is $\beta < 1$ and $\gamma > 0$ is the utility curvature parameter. When $\gamma = 1$ the preferences take the logarithmic form. The "difference" habit preferences we adopt are similar to those of Constantinides (1990) and Campbell and Cochrane (1999). The same analytical results can be provided with "multiplicative" habit preferences, similar to Sundaresan (1989), Abel (1990) and Xiouros and Zapatero (2010), however, the "difference" habit form allows us to explain the high market Sharpe ratio. Specifically, the Arrow-Pratt measure of relative risk aversion in a certain period t is given by $\gamma\, C_t^i/(C_t^i - H_t)$. Thus, lower consumption or higher level of habit implies higher risk aversion.

Following Campbell and Cochrane (1999) we specify the process of the log surplus consumption ratio $s_t := \ln[(C_t - H_t) \div C_t]$, where C represents the per-capita consumption level. The log surplus consumption ratio follows the autoregressive process

$$s_{t+1} = \phi_h s_t + (1 - \phi_h)\bar{s} + \lambda_t (g_{t+1} - \tilde{\mu}_t),$$

where $\bar{s}$ is the unconditional mean and λ_t is the potentially time-varying conditional volatility. Campbell and Cochrane (1999) also assume such a process with the difference that we introduce the quantity $\tilde{\mu}$, which is some cross-sectional weighted mean forecast and to be defined later. The degree of risk aversion depends on how volatile the marginal utility is next period, which is given by $(C_{t+1} - H_{t+1})^{-\gamma} = \exp[-\gamma(s_{t+1} + c_{t+1})]$, and, hence, it depends on the conditional volatilities of s_{t+1} and the log consumption risk. More accurately, the degree of risk aversion is (approximately) given by $\gamma\,(1+\lambda_t)$ and the price of risk is (approximately) given by the degree of risk aversion times the amount of risk, i.e. $\gamma\,(1+\lambda_t)\tilde{\sigma}_t$. The quantity $\tilde{\sigma}_t$ is the aggregate consumption growth volatility under the market beliefs, to be defined later, and is determined in equilibrium together with $\tilde{\mu}$.[16]

For our analysis it is convenient to define the zero mean state variable $\omega_t := (1-\phi_h)(s_t-\bar{s})$,

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[16]The choice of $\tilde{\mu}$ and $\tilde{\sigma}$ was for computational tractability.

which follows the autoregressive process below:

$$\omega_{t+1} = \phi_h \omega_t + (1 - \phi_h)\lambda_t(g_{t+1} - \tilde{\mu}_t). \tag{11}$$

Note that as $\phi \to 1$–when $\phi = 1$ then s follows a random walk–the variance of ω goes to zero. To complete the risk preferences, we need to specify λ_t. Following Campbell and Cochrane (1999), we choose λ_t so that the consumption surplus ratio (and ω) does not affect the risk-free rate. Variations in the consumption surplus ratio also generate variations in the elasticity of intertemporal substitution whereas fluctuations in the risk aversion generate fluctuations in the precautionary savings. Offsetting these two effects yields the specification for λ. To do so we consider a homogeneous agent economy, which is the case with no disagreement ($\zeta = \eta = 0$), where the stochastic discount factor (SDF) is given by the marginal rate of intertemporal substitution of the representative agent that consumes the per-capita consumption:

$$\beta \left(\frac{C_{t+1} - H_{t+1}}{C_t - H_t} \right)^{-\gamma} = \beta \exp \left[-\gamma \tilde{\mu}_t + \gamma \omega_t - \gamma (1 + \lambda_t)(g_{t+1} - \tilde{\mu}_t) \right]. \tag{12}$$

The risk-free rate is then given by the negative logarithm of the conditional expected SDF. The term related to habit is $-\gamma \omega_t + 0.5[\gamma (1 + \lambda_t)\tilde{\sigma}_t]^2$ which we equate to its steady state value. Letting $\lambda_t = \theta$ when $\omega_t = 0$ we thus obtain

$$\lambda_t = \sqrt{(1 + \theta)^2 - 2\frac{\omega_t}{\gamma \tilde{\sigma}_t^2}} - 1. \tag{13}$$

In addition to β and γ, the preferences are thus determined by the habit persistence ϕ_h and $\theta \geqslant 0$. When $\phi_h < 1$ the time-varying λ generates variations in the price of risk. Since our focus is the effects of disagreement, we initially consider the case of $\phi_h = 1$ where ω and λ are constant and equal to zero and θ, respectively.

4.2 The market equilibrium

Agents maximize their utility (10) every period subject to their period budget constraint. Since markets are complete, the period budget constraints can be replaced by the intertemporal budget constraint according to which the present value of an agent's consumption stream should be less or equal to the initial total wealth, which equals the present value of the income stream plus the initial financial wealth. The present value of a stream of cash-

flows can be written using the state price process p, where $p_0(z_t)$ gives the price at time 0 of a unit of consumption in state z_t. Thus, the intertemporal budget constraint is written as

$$\sum_{t=0}^{\infty} \int C_t^i \, p_0(z_t) dz_t \leqslant W_0^i, \tag{14}$$

where W_0^i is the initial total wealth of agent i.

The economy is in equilibrium when agents choose their consumption processes optimally and the financial markets clear. Due to completeness, financial markets clear if and only if the consumption good market clears in all states. The market clearing condition in a given period t is then given by setting the integral of the consumption across agents equal to the per-capita consumption C_t. Equivalently, we can set the integral of the *consumption surplus shares* across agents equal to one,

$$C_t = \int_0^1 C_t^i di \qquad \Leftrightarrow \qquad 1 = \int_0^1 \alpha_t^i di, \qquad \text{where} \qquad \alpha_t^i := \frac{C_t^i - H_t}{C_t - H_t}. \tag{15}$$

Note that the consumption surplus distribution across agents, $(\alpha^i, i \in [0,1])$, has the same properties as a probability density of a continuous variable. Based on this distribution we define another distribution which is what determines asset prices; namely, the consumption surplus share distribution across beliefs, which we denote with $\alpha_t(\mu)$. It is defined similarly to $n_t(\mu)$, but instead of equal weights we use consumption surplus shares.[17] The mean and variance of this distribution are given by,

$$\tilde{\mu}_t := \int_0^1 \alpha_t^i \mu_t^i di \qquad \text{and} \qquad \tilde{\nu}_t^2 := \int_0^1 \alpha_t^i (\mu_t^i - \mu_t)^2 di. \tag{16}$$

We refer to $\tilde{\mu}_t$ as the *weighted mean forecast*, which is endogenously determined in equilibrium and is the quantity we used to specify the process of χ in (9) and the process of the log consumption surplus ratio s. Next, we characterize the equilibrium allocations and state prices with the following proposition.

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[17]Similar to $n_t(\mu)$, we define the consumption surplus distribution of types in some period t as follows:

$$\alpha_t(\mu) := \frac{d}{d\mu} \int_0^1 \alpha_t^i H(\mu - \mu_t^i) di = \int_0^1 \alpha_t^i \delta(\mu - \mu_t^i) di.$$

where $H(x)$ is the Heaviside function, equal to 1 for $x \geqslant 0$ and 0 otherwise and whose derivative is the Dirac delta function, $\delta(x)$.

Proposition 2. *In equilibrium, the consumption surplus shares evolve according to*

$$\alpha_{t+1}^i = \alpha_t^i \frac{\phi(g_{t+1}; \mu_t^i, \sigma_t^2)^{1/\gamma}}{\int_0^1 \alpha_t^j \, \phi(g_{t+1}; \mu_t^j, \sigma_t^2)^{1/\gamma} dj}, \qquad i \in [0,1],$$

and the price of state z_{t+1} in period t is given by

$$p_t(z_{t+1}) = \beta \left(\frac{C_{t+1} - H_{t+1}}{C_t - H_t} \right)^{-\gamma} f(z_{t+1}|\mathcal{F}_t, g_{t+1}) \left[\int_0^1 \alpha_t^i \phi(g_{t+1}; \mu_t^i, \sigma_t^2)^{1/\gamma} di \right]^{\gamma},$$

where σ_t is defined in Proposition 1 and $f(z_{t+1}|\mathcal{F}_t, g_{t+1})$ denotes the density of state z_{t+1} conditional on the information at time t and the realization g_{t+1}.

Agents in maximizing their utility attempt to smooth their consumption surplus shares both across states and across time. In the absence of belief heterogeneity there is perfect risk-sharing, in which case the consumption surplus shares are constant. However, disagreement is incompatible with perfect risk sharing because this would imply disagreement about the state prices and, hence, of the prices of the financial securities. Since in equilibrium all agents agree on prices, they deviate from risk sharing and their consumption surplus shares vary over time. Specifically, they increase in states that they consider relatively more likely compared to the rest of the agents, and decrease otherwise.

The final element we need to determine allocations and prices in equilibrium is the initial wealth distribution across agents. To provide analytical results, we use the approach in Xiouros and Zapatero (2010), also employed by Cvitanić and Malamud (2011) in the context of heterogeneous beliefs, whereby we make a convenient assumption about the initial endogenous consumption distribution. Specifically, the equilibrium consumption processes and state prices admit closed form expressions if the initial consumption surplus share distribution across beliefs is Gaussian. We have already assumed that the initial distribution across beliefs, $n_0(\mu)$ is Gaussian with mean $\bar{\mu}_0$ and variance $\bar{\nu}_0^2$. We thus assume that the initial wealth distribution is such that $\alpha_0(\mu)$ is Gaussian with mean $\tilde{\mu}_0$ and variance $\tilde{\nu}_0^2$. This implies that relatively pessimistic agents would need to have a bit more wealth than average to (partially) protect themselves form the expensive states following negative shocks. Alternatively, we can assume that agents begin with a common prior, i.e. $\bar{\nu} = 0$, in which case we need not impose any restriction on the initial wealth allocation.

Proposition 3. *In equilibrium, the consumption surplus share distribution across beliefs $\alpha_t(\mu)$ is Gaussian at all times $t \geqslant 0$ and the law of motion of the mean and variance are*

given by

$$\tilde{\mu}_{t+1} = (1 - \varphi_\mu)\mu_c + \frac{\varphi_\mu \kappa_t}{1 + \xi_t}\tilde{\mu}_t + \varphi_\mu \left(1 - \frac{\kappa_t}{1 + \xi_t}\right) g_{t+1} + \sigma_\mu A_{\mu,t+1} s_{t+1}^m,$$

$$\tilde{\nu}_{t+1}^2 = \frac{\varphi_\mu^2 \kappa_t^2}{1 + \xi_t}\tilde{\nu}_t^2 + \sigma_\mu^2 A_{\nu,t+1},$$

where

$$\xi_t := \frac{\tilde{\nu}_t^2}{\gamma \sigma_t^2}.$$

The quantities κ_t, σ_t, $A_{\mu,t}$ and $A_{\nu,t}$ are defined in Proposition 1. Further, let $\widetilde{\mathbb{P}}$ denote the probability measure under which the distribution of g_{t+1} conditional on $\mathcal{F}_t$ is Gaussian with mean $\tilde{\mu}_t$ and variance $\tilde{\sigma}_t^2 := \sigma_t^2(1+\xi_t)$. Then, the one-period stochastic discount factor (SDF) under the probability measure $\widetilde{\mathbb{P}}$, defined by $M(z_t, z_{t+1})d\widetilde{\mathbb{P}}_t(z_{t+1}) := p_t(z_{t+1})dz_{t+1}$, is given by

$$\ln M(z_t, g_{t+1}) = \ln \beta + \frac{1 - \gamma}{2}\ln(1 + \xi_t) - \gamma\tilde{\mu}_t + \gamma\omega_t - \gamma(1 + \lambda_t)(g_{t+1} - \tilde{\mu}_t),$$

where λ_t is as specified by (13). Finally, the state of the economy is given by $z = (\tilde{\mu}, \tilde{\nu}, \upsilon, \chi, \omega)$ and is driven by the shocks $(g, s^m, \epsilon^\chi, \epsilon^\rho)$.

Proposition 3 together with the law of motion of υ (Proposition 1) and the laws of motion of χ and ω, shown by equations (8), (9) and (11), fully characterize the dynamics of the state of the economy and the SDF that prices financial assets.

4.3 Implications of large vs small economies

Proposition 3 shows some important features of our model, especially in relation to models of small economies with finite numbers of agents. Arguably, the continuum of types is a more realistic representation of financial markets. Apart from this, the important distinction is the following: In our model the state and, hence, asset pricing quantities that are functions of the state, are stationary; whereas the literature has mostly analyzed models with states and asset pricing quantities that are only asymptotically stationary.[18] What this means for our model is that we can study the unconditional asset pricing moments that do not depend on the initial condition. Whereas, the literature in analyzing asymptotically stationary models has produced interesting results that depend on the initial conditions, which are not drawn from the stationary distribution. They effectively study the dynamics of such economies in

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[18]A process is asymptotically stationary if it is stationary only in the limit.

their transitions to their stationary distributions, while the asset pricing moments change with the horizon. Yet, this may not be a serious shortcoming of such models because in many cases it takes a very long time to converge to the stationary distribution. Even so, the implications for asset prices still differ because in an economy with few types the convergence path is stochastic while with infinite types the path is deterministic.

To explain the above comments we first consider the case of constant information quality and we begin from the fact that the relevant state that describes the beliefs in the economy is the endogenous allocation of consumption across beliefs. With n agents there are $2n - 1$ state variables. For example, in a model with two agents the three state variables are the consumption share–for this discussion we omit habit–of one of the agents and the beliefs of the two agents, or three independent functions of these variables.[19] Yan (2008) shows that in a model with a finite number of agents one agent survives and asset prices converge to the valuations of the surviving agent. This means that the stationary distribution is that of one state variable, the beliefs of the surviving agent, and asset prices are those implied by a homogeneous economy. Obviously, the implications of disagreement in such a model only come from the transition to the stationary state, where the initial condition must assume a non-negligible amount of consumption for all agents. This reflects the asymptotic stationarity of these models.

Moving to the behavior of the transition path, consider that, with a finite number of agents, all $2n - 1$ state variables are stochastic and produce non-trivial dynamics for asset prices. For example, Dumas, Kurshev and Uppal (2009) analyze their model of one rational and one overconfident agent and find that, out of the three state variables, the difference in the beliefs of the two agents is the main driver of the excess volatility in asset prices. Our model, however, generates a very different result. The difference in beliefs, which is represented by the variable $\tilde{\nu}$, is non-stochastic because with constant information quality A_ν is also constant and $\tilde{\nu}$ converges deterministically to a constant long-run state. What we infer from this is that, as we increase the number of agents the "stochasticity" of the difference in beliefs diminishes and completely vanishes when the number of agents becomes infinite. Our conclusion, therefore, is that in a world with constant information quality the dispersion in beliefs is probably neither stochastic nor a source of shocks to asset prices. For this reason, we only focus on the stationary dynamics of our model, which are "trivial" in the case of constant information quality.

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[19]Interestingly, the consumption distribution across beliefs in our model is represented by only two variables, mean and standard deviation, instead of $2n - 1$. This facilitates considerably the study of the quantitative implications of the model as well as the qualitative analysis.

The only stochastic element of beliefs in our model with constant information quality is the consumption (surplus) weighted mean forecast $\tilde{\mu}$. This leads us to the study of Atmaz and Basak (2018) who also analyse a model with a continuum of agents where the implications come from the weighted mean forecast. Despite this, unlike ours, their model is only asymptotically stationary, because the dispersion converges deterministically to zero, even though at a very slow rate. Nevertheless, their analysis and conclusions are quite general because their focus is on how the transition dynamics of the stock price depend on the initial level of dispersion and not the unconditional asset pricing moments. Namely, how the average expectation behaves and affects the stock price for different initial values of dispersion. Many of their qualitative results hold true in our model where we do not have a vanishing dispersion.

4.4 Speculation and state dynamics

Here we show that ξ_t is a measure of the speculative activity, which is central to all the effects of disagreement, and then see how it affects the dynamics of the state of the economy. But first, we point out that when there is no disagreement ($\zeta_t = \eta = 0$) there is no dispersion in beliefs ($\nu_t = 0$), since $A_{\nu,t} = 0$, and the consumption surplus shares are constant. Further, in this case $\tilde{\mu}_t$ equals $\bar{\mu}_t$ and $\tilde{\sigma}_t$ equals σ_t at all times. The dynamics of the homogeneous agent economy are, thus, driven by the weighted mean forecast $\tilde{\mu}_t$, the uncertainty v_t (if $\rho_t > 0$) and λ_t which determines the degree of risk aversion.

The introduction of disagreement ($\eta > 0$) changes the dynamics of the economy because of speculation. By speculation we refer to the fact that agents invest in financial assets to deviate from perfect risk sharing in a way that is optimal according to their beliefs. As already discussed after Proposition 2, agents increase their consumption surplus shares in states that they consider more likely relative to the rest of the market. This consumption reallocation changes the probability measure $\widetilde{\mathbb{P}}$, through changing both the weighted mean forecast $\tilde{\mu}$ and the weighted dispersion $\tilde{\nu}$. Before we see how this depends on speculation we provide a measure for the speculative activity.

Lemma 1. *Let the speculative activity in period t be defined as the cross-sectional variance of the sensitivity of the log growth in the consumption surplus to the log aggregate consumption growth at $g_{t+1} = \tilde{\mu}_t$. That is, speculation is defined as the cross-sectional variance of*

$$\frac{\partial}{\partial g_{t+1}} \log \left(\frac{C_{t+1}^i - H_{t+1}}{C_t^i - H_t} \right) \Bigg|_{g_{t+1} = \tilde{\mu}_t}$$

weighted by the consumption surplus shares. The speculative activity is then given by ξ_t.

Naturally, the speculative activity is zero when there is no disagreement, increases with dispersion $\tilde{\nu}$, which itself decreases with information quality, and decreases with the utility curvature parameter γ. Note that we define the speculative activity in terms of the growth in consumption surpluses so that it is independent of habit. We do so to separate the various effects, because the cross-sectional variation in consumption growth depends also on λ_t.

We can now examine how the speculative activity ξ_t affects the dynamics of the distribution of beliefs starting with $\tilde{\mu}_t$. As shown in Proposition 3, the weighted mean forecast $\tilde{\mu}$ is driven by the aggregate consumption growth and the common signal, just like $\bar{\mu}$, but with a difference in the coefficient of g. We note first that the coefficient of the common signal $A_{\mu,t}$ is always equal to the information quality ρ_t and independent of disagreement η (Proposition 1). What this means is that for any given path of ρ_t the level of disagreement, determined by η, does not affect the way that the common signal s^m affects $\tilde{\mu}_t$. Nevertheless, speculation increases the volatility of $\tilde{\mu}$ by making it more sensitive to shocks to aggregate consumption growth, as a result of wealth reallocation. Since agents bet on states that they consider more likely, every period wealth reallocates from agents whose beliefs μ_t^i happen to be far away from the realization g_{t+1} to agents whose beliefs happen to be close to the realized value. This effect can be clearly seen by the fact that the coefficient of g_{t+1}, given by $1 - \kappa_t \div (1 + \xi_t)$, increases with ξ_t. When we compare this coefficient with the equivalent coefficient in the law of motion of $\bar{\mu}$ (Proposition 1), given by $1 - \kappa_t$, we deduce that approximately $\kappa_t \xi_t$ represents the reallocation of consumption surplus shares across types.[20] Naturally, lower uncertainty (higher κ_t) increases the effect of speculation.

Turning to the dispersion in beliefs we point out again that $\tilde{\nu}$ is independent of both g and s^m and in the absence of shocks to information quality (constant A_ν) it follows a deterministic path toward a long-run steady state. The reason lies with the law of large numbers which makes all the factors that affect beliefs to have predictable effects on dispersion. At the steady state these factors offset each other. The same holds true for the uncertainty, which means that with constant information quality (A_μ and A_v are also constant) dispersion, uncertainty and speculation become constant once they reach their corresponding steady states.

The dispersion $\tilde{\nu}$ is driven by three factors: wealth reallocation, learning and information quality. Its law of motion (Proposition 3) has two terms. The first term shows how $\tilde{\nu}^2$ "decays"–the decay factor being $\varphi_\mu^2 \kappa_t^2 \div (1 + \xi_t)$–due to wealth reallocation and learning.

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[20]Using a first-order approximation around $\xi_t = 0$ we obtain that $1 - \kappa_t \div (1 + \xi_t) \approx 1 - \kappa_t + \kappa_t \xi_t$.

The factor $1 \div (1 + \xi_t)$ represents the decrease in dispersion due to the reallocation of wealth and the factor κ_t^2 represents the decrease in dispersion due to learning from the realization of g. Both higher speculation and greater prior uncertainty (low κ_t) lead to greater decrease in dispersion. One might think that speculation would make the decrease in dispersion depended on the realization of g, but this is not the case for the same reason that the decrease in the uncertainty is independent of g. The second term, $\sigma_\mu^2 A_{\nu,t+1}$, does not allow the dispersion to vanish in the presence of disagreement ($\eta > 0$). Every period, the dispersion receives a positive shock whose size depends on disagreement, since A_ν increases with η, but also limited by the information quality, since A_ν decreases with ρ. For example, if the common signal is fully informative ($\rho = 1$) then A_ν equals zero.

4.5 Speculation and state prices

We close this section with an analysis of the equilibrium SDF that will give us the necessary intuition to understand how disagreement affects asset prices in our model. The probability measure $\widetilde{\mathbb{P}}$ under which we express the SDF was chosen because it highlights the various effects. Under this probability measure, the aggregate consumption growth has a mean equal to the weighted average expectation. An increase in this mean implies that the economy as a whole expects higher consumption growth in the future, which decreases the discount rates, but also expects higher cash-flow growth for the risky assets. Under $\widetilde{\mathbb{P}}$ the amount of aggregate risk is equal to the risk as perceived by all agents, σ_t, increased by the speculative activity as follows: $\tilde{\sigma}_t = \sigma_t \sqrt{1 + \xi_t}$. The intuition is that $\tilde{\sigma}$ represents some "average" individual consumption risk which increases by speculation. In fact, in the absence of habit $\tilde{\sigma}_t^2$ is approximately the cross-sectional average of the individual consumption growth variance, using consumption shares as weights.[21] What this implies is that the economy, because of speculation, prices assets as if the aggregate consumption risk were higher than it is.

The final element is the factor $(1 + \xi_t)^{0.5\,(1-\gamma)}$. It shows the effect of disagreement and speculation on the risk-free discount rate, since it does not depend on the realization of g_{t+1}. Given that $\xi_t \geqslant 0$, it states that disagreement affects the discount rate either downwards or upwards, depending on whether agents are more or less risk averse than a logarithmic investor.[22] This effect comes from the fact that when $\gamma > 1$ pessimists have a stronger effect

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[21]See proof of Lemma 1.

[22]Jouini and Napp (2006a, 2007) calls this factor the *aggregation bias* and provide an analysis using a more general utility specification. An earlier result connected with this, is provided by Varian (1985) who shows that each state price is affected either upwards or downwards by heterogeneity in beliefs depending on certain properties of the utility function. The aggregation bias is the common component of the effect on

on prices than optimists and the other way round for $\gamma < 1$. To see why, consider the price in period t of a state g_{t+1}, in relation to the beliefs of agent i about the particular state,

$$p_t(g_{t+1}) = \beta \mathbb{P}_t^i(g_{t+1}) \left(\frac{\alpha_{t+1}^i}{\alpha_t^i} \cdot \frac{C_{t+1} - H_{t+1}}{C_t - H_t} \right)^{-\gamma},$$

with a slight abuse of notation where $\mathbb{P}_t^i(g_{t+1})$ denotes the conditional "probability" of state g_{t+1} according to agent i. Then, suppose that agent i becomes more "optimistic" about the state in the sense that $\mathbb{P}_t^i(g_{t+1})$ increases. Given that the price of the state does not change, the agent will want to increase her consumption surplus share a_{t+1}^i. The direction of the effect is given by the first derivative of a_{t+1}^i with respect to $\mathbb{P}_t^i(g_{t+1})$. Now consider the strength of this effect in relation to the probability $\mathbb{P}_t^i(g_{t+1})$, that is, the second derivative which depends on γ:

$$\frac{\partial^2 a_{t+1}^i}{\partial \mathbb{P}_t^i(g_{t+1})^2} = \frac{\alpha_{t+1}^i}{\gamma \, \mathbb{P}_t^i(g_{t+1})^2} \left(\frac{1}{\gamma} - 1 \right) \lesseqgtr 0 \qquad \text{when} \qquad \gamma \gtreqless 1.$$

Finally, consider a case where the dispersion in beliefs about the particular state increases, in the sense that the relative optimists become more optimistic and the relative pessimists become more pessimistic by the same "amount". Then, what the above relation implies for $\gamma > 1$ is that the pessimists will want to decrease their consumption surplus shares by more than what the optimists want to increase theirs. For the equilibrium to be restored, the state price needs to adjust. Obviously, the pessimists will not decrease their consumption surplus shares as much and the optimists will increase theirs more than they would if the state price remained the same. As a result, the state price will adjust downwards. The opposite happens when $\gamma < 1$, in which case the optimists want to react more strongly and for the pessimists to be able to supply the additional consumption desired by the optimists, the price of the state needs to increase.

The dispersion in beliefs affects all state prices, some more and some less. The factor $(1 + \xi_t)^{0.5\,(1-\gamma)}$ represents the "average" effect across states. The variation of the effect across states is reflected in $\tilde{\sigma}_t$, that is, how the speculative activity augments the variance of g_{t+1} under $\widetilde{\mathbb{P}}$.

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all state prices. In our paper we reserve the term *bias* to refer to the difference between the weighted mean forecast $\tilde{\mu}$ and the true conditional mean μ.

5 Representative agent and asset prices

In this section we study some key asset pricing effects of heterogeneity in expectations. We start by deriving an equivalent representative agent economy which shows how we can obtain equivalent effects from different elements. Specifically, the heterogeneous expectations economy with stochastic information quality is equivalent to a representative or homogeneous agent economy with time-varying subjective discount factor and a modified uncertainty process. If information quality is constant, then the equivalent representative agent has constant but different subjective discount factor and uncertainty. Then, we derive the risk-free rate and the price of risk and conduct a comparative statics analysis of the model with constant information quality.

The equilibrium admits a representative agent representation, in the sense that there is a homogeneous or representative agent economy with certain beliefs and preferences that generates the same asset prices. Suppose asset j is the claim to the cash-flow stream $(D^j_{t+\tau}, \tau > 0)$. Assuming the usual transversality condition to exclude asset bubbles, the price of the asset is given by the present value of its future cash-flows,

$$P^j_t = \widetilde{\mathbb{E}}_t \left[M(z_t, g_{t+1}) \left(P^j_{t+1} + D^j_{t+1} \right) \right] = \widetilde{\mathbb{E}}_t \left[\sum_{\tau=1}^{\infty} M_{t,t+\tau} D^j_{t+\tau} \right], \tag{17}$$

where $\widetilde{\mathbb{E}}$ denotes the expectation under the probability measure $\widetilde{\mathbb{P}}$ and the stochastic discount factor (SDF) between periods t and $t + \tau$ is denoted with $M_{t,t+\tau} = \prod_{s=t}^{\tau-1} M(z_s, g_{s+1})$. An economy is, thus, equivalent in its pricing implications if it has the same SDF under the probability measure $\widetilde{\mathbb{P}}$.

Proposition 4. *Consider a representative (or homogeneous) agent economy, where the representative (or each) agent has the same preferences as the agents in the heterogeneous agent economy but with a subjective discount factor given by $\widehat{\beta}_t := \beta \left(1 + \xi_t \right)^{\frac{1-\gamma}{2}}$. Further, the representative (or each) agent believes that the conditional volatility of μ_t and the informativeness of the common signal are given by*

$$\widehat{\sigma}_{\mu,t} := \sigma_\mu \sqrt{A^2_{\mu,t} + A_{v,t} + \frac{A_{\nu,t}}{\gamma}} \qquad and \qquad \widehat{\rho}_t := \frac{A_{\mu,t}}{\sqrt{A^2_{\mu,t} + A_{v,t} + \frac{A_{\nu,t}}{\gamma}}},$$

respectively. Then, this economy is equivalent in its pricing implications to the heterogeneous

agent economy. Since $\zeta_t = \eta \div \rho_t$ then

$$A_{\mu,t}^2 + A_{v,t} + \frac{A_{\nu,t}}{\gamma} = 1 + \left(\frac{1}{\gamma} - 1\right) A_{\nu,t} \lessgtr 1 \qquad when \qquad \gamma \gtrless 1.$$

The adjustment of the subjective discount factor shows that the first effect of disagreement on asset prices is similar to higher (lower) impatience or pessimism (optimism) if $\gamma > 1$ ($\gamma < 1$), in the sense that disagreement pushes all prices down (up), since $\hat{\beta}_t < \beta$ ($\hat{\beta}_t > \beta$). The adjustment in relation to the conditional volatility of μ and the informativeness of the common signal are required so that the uncertainty of the representative agent is equal to $v_t^2 + \tilde{\nu}_t^2 \div \gamma$. This indicates that in relation to consumption risk and its effect on asset prices, uncertainty and speculative activity are equivalent. The last two adjustments are also required so that the effect of s^m on the expectation of the representative agent is the same as the effect on $\tilde{\mu}$.

5.1 The risk-free rate

The continuously compounded risk-free rate is given by $r_t^f = -\ln \tilde{\mathbb{E}}_t(M_{t+1})$.

Lemma 2. *The continuously compounded risk-free discount rate is given by,*

$$r_t^f = -\ln(\beta) + \gamma \tilde{\mu}_t + \frac{\gamma - 1}{2} \ln(1 + \xi_t) - \frac{1}{2}\gamma^2(1 + \theta)^2 \sigma_t^2 (1 + \xi_t).$$

The effects of disagreement are clear. The risk-free rate is driven by the weighted mean forecast $\tilde{\mu}_t$, the speculative activity ξ_t, which depends on the weighted dispersion $\tilde{\nu}$, and the uncertainty v, which is part of σ_t. The third term represents the effect already discussed in the previous section, whereby dispersion alongside the speculative positions taken by the agents affect the discount rate depending on whether γ is lower or higher than one. Specifically, with $\gamma > 1$, the "pessimists" of each state have a stronger influence causing all state prices to drop and, as a result, the risk-free rate increases. The fourth term represents the precautionary savings motive, which becomes stronger with speculation and depends on two things: Firstly, it depends on the average risk aversion $\gamma(1 + \theta)$, where $1 + \theta$ is the amplification generated by habit. Secondly, it depends on the cross-sectional "average" consumption risk as given by $\sigma_t\sqrt{1 + \xi_t}$. The last two terms go in opposite directions when $\gamma > 1$. Typically the third term is larger in magnitude, which implies that when $\gamma > 1$ the risk-free rate increases with ξ. In the opposite case, higher dispersion always pushes the risk-free rate down.

Disagreement affects both the average level and the volatility of the risk-free rate. Lower average information quality implies higher average speculative activity ξ and more volatile $\tilde{\mu}$. Further, more volatile information quality generates more volatile speculative activity. In the case of constant information quality, the risk-free rate is driven only by $\tilde{\mu}$.

5.2 The price or risk

In an economy with heterogeneous beliefs, the equilibrium relation between risk and return is not unique because it depends on the probability measure. For this reason we move in two steps, starting with deriving an expression for returns. Consider an asset with a conditionally log-normal payoff equal to e^g, whose one-period log-return is denoted with r and its excess log-return is given by,[23]

$$r_{t+1} - r_t^f = g_{t+1} - \tilde{\mu}_t - \frac{1}{2}(1 + \xi_t)\sigma_t^2 + \gamma(1 + \lambda_t)(1 + \xi_t)\sigma_t^2. \tag{18}$$

Clearly, the expected excess return of the asset depends on beliefs and the heterogeneity in beliefs about g translates into heterogeneity in beliefs about asset returns. For agent i, the expected excess return can be approximated using the log-normal return as follows:

$$\mathbb{E}_t^i(r_{t+1} - r_t^f) + \frac{1}{2}\sigma_t^2 = \left[\frac{\mu_t^i - \tilde{\mu}_t}{\sigma_t} + \gamma(1 + \lambda_t)(1 + \xi_t)\sigma_t - \frac{1}{2}\xi_t\sigma_t \right]\sigma_t.$$

In the above expression we have used not only agent i's expectation but also the aggregate risk as perceived by all agents, σ_t. Thus, the expression in the bracket shows the price of risk for agent i, so that the expected excess return is given by the price of risk times the amount of risk. The price of risk is then composed of three terms. The second term is the price of risk in the homogeneous agent case, i.e. $\gamma(1 + \lambda_t)\sigma_t$, augmented by the amount of speculation as it is multiplied by $(1 + \xi_t)$. The third term is a Jensen's inequality term that arises when there is speculation and works in the opposite direction; that is, the partial effect of speculation through this Jensen's inequality term is to decrease the price of risk. Finally, the first term shows the relative optimism of agent i which implies that the relative optimists will have higher price of risk. Larger optimism is accompanied with larger exposure to the

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[23]To derive the return of this asset we first derive the expression for its price which is given by $\widetilde{\mathbb{E}}_t[M_{t+1}e^{g_{t+1}}]$. The log-return is then equal to the log-payoff minus the log-price. The log-return for a general log-normal payoff is derived in Lemma 12 in the Appendix.

aggregate risk, as shown by the logarithmic growth in individual consumption surplus:

$$\log\left(\frac{C_{t+1}^i - H_{t+1}}{C_t^i - H_t}\right) = \frac{\mu_t^i - \tilde{\mu}_t}{\gamma\sigma_t^2}(g_{t+1} - \tilde{\mu}_t) - \frac{(g_{t+1} - \tilde{\mu}_t)^2 + (\tilde{\mu}_t - \mu_t^i)^2}{2\gamma\sigma_t^2} + \mathcal{L}(z_t, g_{t+1}).$$

The above expression is derived from Proposition 2 and $\mathcal{L}(z_t, g_{t+1})$ is agent independent.

We now define the price of risk, denoted with q_t, as the price of risk of the agent that happens to have the correct beliefs, meaning that the particular agent's expectation is given by the true (unobservable) conditional mean.

Lemma 3. *The price of risk, defined as $q_t := \mathbb{E}_t(r_{t+1} - r_t^f) \div \sigma_t + 0.5\sigma_t$, where r_{t+1} refers to the return of the asset whose payoff is $e^{g_{t+1}}$ shown in equation (18), is given by*

$$q_t = -\frac{\tilde{\mu}_t - \mu_t}{\sigma_t} + \gamma(1 + \lambda_t)(1 + \xi_t)\sigma_t - \frac{1}{2}\xi_t\sigma_t.$$

There are three main factors that drive the price of risk: The first is the habit relative to consumption, where an increase in the level of habit (lower ω) increases λ_t. Note that in the case where $\phi_h = 1$, λ_t is constant and equal to θ. In this case, the price of risk is driven by the other two factors. The second factor is the amount of speculation whose net effect is determined by the sign of the quantity $2\gamma(1 + \lambda_t) - 1$. In the typical case where agents have a degree of risk aversion higher than one-half an increase in speculation increases the price of risk. The first term shows the third factor, which is the bias of the economy in relation to the true growth rate. It states that the true expected excess returns of risky assets fluctuate as the representative agent's forecast fluctuates around the true conditional mean. When the economy exhibits a positive bias ($\tilde{\mu}_t > \mu_t$) risky assets are overpriced relative to the risk-free rate and thus under the true probability measure are expected to "underperform".

5.3 Comparative statics with constant information quality

Our results and qualitative analysis so far has shown how disagreement affects both the level and variation of the risk-free rate and the price of risk. However, armed with an estimated disagreement model for a large economy we can focus on the quantitative implications. As a first step we consider an economy with constant information quality and study the speculative activity, the price of risk and the risk-free rate through a comparative statics analysis where we vary the information quality and the utility curvature parameter. The rest of the parameters take their estimated values, shown in Table 2. For each parameter

set, we analyze the steady state. The convergence to the steady state is very fast, where the various quantities reach the steady state within a distance of 1% in less than 10 quarters.

For a given information quality, where $\zeta = \eta \div \rho$, the coefficients A_μ, A_v and A_ν are as given in Proposition 1. Panel A of Figure 1 shows these coefficients as functions of ζ, that ranges between η and 1, $\zeta = \eta$ corresponding to perfect information quality ($\rho = 1$) and $\zeta = 1$ corresponding to the lowest information quality ($\rho = \eta$). Naturally, A_ν which determines the level of the consumption surplus weighted dispersion $\tilde{\nu}$ increases with ζ. The coefficient A_μ determines the effect of the common signal on the mean forecast which is equal to the information quality ρ. The steady states of the uncertainty and $\tilde{\nu}$ are given by the solution to the following system of equations:

$$\tilde{\nu}^2 = \varphi_\mu^2 \frac{\gamma \sigma_c^2}{\gamma(\sigma_c^2 + v^2) + \tilde{\nu}^2} \cdot \frac{\sigma_c^2}{\sigma_c^2 + v^2} \tilde{\nu}^2 + \sigma_\mu^2 A_\nu, \qquad v^2 = \varphi_\mu^2 \frac{\sigma_c^2 v^2}{\sigma_c^2 + v^2} + \sigma_\mu^2 A_v. \qquad (19)$$

The steady state of the dispersion $\bar{\nu}$ is computed in a similar fashion. Panel B of Figure 1 shows the plots of v, $\bar{\nu}$ and $\tilde{\nu}$ for two values of γ, 0.5 and 5. Clearly, the dispersion measures decrease with information quality. We also see that $\tilde{\nu}$ increases with γ, which we formally prove below, as well as that $\tilde{\nu}$ converges to $\bar{\nu}$ as γ tends to infinity. The reason is that, more risk averse agents speculate less and, thus, less wealth is reallocated to those agents whose beliefs are more correct in hindsight.

The coefficient A_v (Panel A), which determines the level of the uncertainty, and the uncertainty v (Panel B) are hump-shaped, because at the upper bound of information quality the common signal is fully informative and at the lower bound the individual signals are perceived to be fully informative. In Panel C we plot the speculative activity, given by $\tilde{\nu}^2 \div (\gamma v^2)$, for three different values of γ. We see that it increases exponentially with ζ, especially when the utility curvature parameter is low. In Panel D we show the individual consumption risk $\tilde{\sigma}$, which increases with both disagreement and uncertainty. When γ is sufficiently low, the per-capita consumption risk monotonically decreases with information quality, as indicated by the case of $\gamma = 0.5$. In this case, even though the uncertainty decreases beyond a certain value of ζ, the increase in $\tilde{\nu}^2 \div \gamma$ in larger. Otherwise, the per-capita consumption risk is hump-shaped.

Given that the speculative activity is sensitive to the utility curvature parameter when this is low, the next question is whether for sufficiently low values of γ the speculative activity can help explain the high price of risk observed in the data. In the next lemma we show first that the price of risk always increases with the utility curvature parameter.

Lemma 4. *Let ζ be constant and let $(v, \tilde{\nu})$ be the solution to the system of equations (19). Then the following hold:*

$$\lim_{\gamma \to \infty} \tilde{\nu} = \bar{\nu}, \qquad \lim_{\gamma \to 0} \tilde{\nu} = \sigma_\mu \sqrt{A_\nu}, \qquad \frac{\partial \tilde{\nu}}{\partial \gamma} > 0,$$

$$\lim_{\gamma \to \infty} \xi = 0, \qquad \lim_{\gamma \to 0} \xi = \infty, \qquad \frac{\partial \xi}{\partial \gamma} < 0, \qquad \frac{\partial (\gamma \xi)}{\partial \gamma} = \frac{1}{\sigma^2} \frac{\partial \tilde{\nu}^2}{\partial \gamma} > 0.$$

Therefore, we also have that,

$$\frac{\partial q_t}{\partial \gamma} = -\frac{1}{2} \frac{\partial \xi}{\partial \gamma} \sigma + (1 + \lambda_t) \left(1 + \xi + \gamma \frac{\partial \xi}{\partial \gamma} \right) \sigma > 0,$$

where $\sigma = \sqrt{\sigma_c^2 + v^2}$.

To understand the result on the price of risk, consider the case where there is no habit ($\lambda_t = 0$) and no bias ($\tilde{\mu}_t - \mu_t$), so that the price of risk is given by $q = \gamma(1 + \xi)\sigma - \frac{1}{2}\xi\sigma$. In the absence of disagreement, the price of risk becomes $\gamma\sigma$, that is, it is given by the product of the consumption risk and the coefficient of relative risk aversion. The introduction of disagreement generates speculation and increases the "individual" consumption risk, $\tilde{\sigma} = \sigma\sqrt{1 + \xi}$. In fact, when γ tends to zero the individual consumption risk tends to become infinite.[24] Nevertheless, the rate at which the "individual" consumption risk increases is never as high as the rate at which γ decreases. As a result, the price of risk is always smaller when agents are less risk-averse, independently of how large is the dispersion in beliefs.

The next question is then whether speculation can help generate a high price of risk. The answer is negative in the absence of habit, because speculation can only marginally increase the price of risk for large values of γ. From Lemma 3 we know that in the absence of a bias, i.e. $\tilde{\mu}_t = \mu_t$, and with no habit ($\theta = 0$) speculation increases the price of risk by $(1 + 0.5\xi_t)$. Looking at Panel C of Figure 1 we infer that for $\gamma = 5.0$ the maximum possible increase in the price of risk is less than 10%. However, we would need a much higher value for γ with CRRA preferences to come close to explaining the observed price of risk. At such higher values for γ, the impact of speculation is negligible. The only case where speculation can significantly increase the price of risk is if γ is low and the risk aversion coming from habit, i.e. θ is high.

In Figure 2 we plot the risk-free rate and the price of risk against ζ, for three different values of γ: 0.5, 1.5 and 5. In Panels A and C, the value of θ depends on γ, so that the degree

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[24]In our model the utility curvature parameter cannot be zero, otherwise speculation is infinite and all asset prices collapse to zero.

of risk aversion $\gamma(1 + \theta)$ is constant and equal to 25. We choose this value, because about this much degree of risk aversion is needed to explain the risk-return relation in the data. These plots are useful to understand how these quantities vary across time when information quality is stochastic.

In Panel C, by keeping the degree of risk aversion constant across parameter sets, all three models generate the same price of risk for maximum information quality. We then see that the price of risk is much more sensitive to information quality when γ is equal to 0.5, because with lower γ the speculative activity increases more with ζ. For $\gamma = 5$, the price of risk is hump-shaped because the speculative activity is weak and the effect of the uncertainty is stronger.

The risk-free rate plotted in Panel A shows that it becomes more sensitive to information quality as γ deviates from 1. The risk-free rate varies considerably with information quality even when $\gamma = 1.5$, in which case it varies from around 0% to around 10%. These plots indicate that the risk-free rate in a model with time-varying information quality will fluctuate considerably, when γ deviates from 1.

In Panels B and D we keep the value of θ constant and only vary γ across the three graphs. These plots are useful to understand how the average price of risk and risk-free rate depend on γ across models.

In Panel B we notice that the risk-free rate corresponding to $\gamma = 5$ is not monotonic in information quality. Compared to Panel A, the value of θ is bigger implying stronger precautionary savings motive, which works in the opposite direction to the effect of disagreement– see discussion after Lemma 2.[25] Up until a value around $\zeta = 0.4$, a decrease in information quality makes the precautionary savings motive even stronger, which in turn makes the risk-free rate even smaller. Eventually, the effect of disagreement prevails and beyond $\zeta = 0.4$ the risk-free rate becomes decreasing in information quality.

Finally, in Panel D we see the result of Lemma 4, according to which the price of risk is always increasing in γ, independently of the level of information quality.

5.4   Speculation, the equity premium and the equity volatility

A fundamental question is whether speculation helps to explain the high equity premium and the high stock return volatility. The equity premium is (approximately) the product of the price of risk, the stock return volatility and the correlation between stock returns

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[25]See the expression of the risk-free rate in Lemma 2. The last term represents the precautionary savings motive and the third term represents the effect of disagreement.

and aggregate consumption growth. We have already analyzed how disagreement affects the price of risk, with the main conclusion being that disagreement can significantly affect the price of risk if γ is low and θ is high. We now need to examine how disagreement affects the other two factors that determine the equity premium.

There are two main ways in which disagreement affects the equity volatility. We know that stock prices increase either due to a decrease in discount rates or an increase in expected cash-flow growth. The first effect comes from the fact that the weighted mean forecast $\tilde{\mu}$ increases cash-flow growth and decreases the risk-free discount rate. If the net effect is non-zero then speculation may increase the volatility of stock prices if it increases the volatility of $\tilde{\mu}$. In Figure 3 we plot the ratio of the unconditional volatility of $\tilde{\mu}$ to the unconditional volatility of μ for four different values of γ ranging from 0.1 to 5.0. We note that in all these cases the highest volatility is obtained with full information ($\rho = 1$ and $\zeta = \eta$). We then note that in some cases, e.g. for $\gamma = 5.0$ and $\gamma = 1.5$, the volatility of $\tilde{\mu}$ decreases monotonically as the information quality decreases, while in the other two cases at the beginning the volatility decreases with ζ and then increases. Therefore, only in certain cases an increase in disagreement through lower information quality will generate more volatile mean forecast, where the opposite is typically true.

The second effect is present only when information quality is stochastic. We have seen that speculation increases both the risk-free rate and the price of risk for $\gamma > 1$. Therefore, a decrease in information quality increases future speculation and, thus, increases the discount rates and decreases stock prices. Thus, the more volatile is information quality the more volatile is the stock price and the magnitude of the effect depends mainly on three parameters: (i) risk aversion γ where lower γ increases speculation, (ii) risk aversion coming from habit θ, where higher θ implies larger fluctuations in the price of risk, and (iii) the persistence of the shocks to information quality, where more persistence implies larger effects on stock prices.

Therefore, higher and more volatile information quality leads to higher equity volatility. This excess volatility leads to an increase in the equity premium if such fluctuations are correlated with the aggregate consumption growth. This is true for the estimated disagreement model where shocks to information quality are almost exclusively driven by consumption growth shocks. Specifically, positive shocks lead to increased information quality and with a $\gamma > 1$ an increase in information quality leads to an increase in stock prices and, thus a positive correlation with consumption growth.

The conclusion of the above analysis is that disagreement may significantly affect the equity premium if γ is low, θ is high and information quality is stochastic. David (2008)

provides a disagreement model with two agents, constant information quality and CRRA preferences, which accounts for about half of the equity premium when γ is lower than one. David (2008) attributes this principally to the significant increase in the "individual" consumption risk. His model generates an apparently contradictory result, because of two elements that are absent in our model. The first element is that in his model one of the two agents is pessimistic who, given the beliefs of the other agent, on average survives over 30 years of simulated data. As a result, the economy exhibits significant negative bias which generates a significant average price of risk. Lemma 3 shows how the price of risk increases when the bias becomes negative. In our model the average bias is zero because the data do not show any indications of an average bias. The average professional forecast in our sample period (1961Q3:2016Q4) is 0.39% growth per quarter, compared to the average growth rate of 0.30% over the same period and 0.40% over the entire sample period (1947Q1:2016Q4). In addition, the average professional forecast is a strong predictor of the quarter ahead growth, as shown in Table 1, with a correlation of 0.54.

The average bias is unrelated to risk aversion but it is needed for risk aversion to have an effect on the equity premium through the return volatility. In his model, risk aversion affects the quantity of risk through fluctuations in the cross-sectional dispersion. As we have already discussed, in a two agent model the dispersion is stochastic which, as shown by Dumas, Kurshev and Uppal (2009), generates significant excess volatility. This excess volatility is higher when agents are less risk averse because the increased speculation makes the consumption shares more volatile. However, we have already shown that in a large economy with constant information quality the dispersion is non-stochastic. Thus, the argument that disagreement may significantly contribute to explaining the observed equity premium with CRRA preferences and constant information quality does not carry over to our model.

6 Asset prices in a calibrated economy

Given the estimated model of disagreement, we can now calibrate the preferences and study the quantitative implications for asset prices. We study the price properties of three assets; the aggregate consumption claim, the one-period risk-free rate and the claim to the stock market dividend. Given that the stock market dividend process differs substantially from the aggregate consumption process, we specify the following log dividend growth process:

$$g_{t+1}^d = \mu_c + \beta_d \cdot (g_{t+1} - \mu_c) + \sigma_d \epsilon_{t+1}^d,$$

where ϵ^d is $iid.$ standard normal. The exposure to the aggregate log consumption growth is set to $\beta_d = 2.15$, to match the correlation between g and g^d, and the idiosyncratic volatility is set to $\sigma_d = 0.0575$, to match the volatility of g^d in the data.

The asset price data is quarterly and cover the period between 1927Q1 to 2016Q4. The risk-free rate in the data is taken to be the 3-month nominal T-bill rate, obtained from CRSP, adjusted for quarter ahead expected inflation. The expected inflation is estimated using a fitted AR(1) process on the quarterly inflation series obtained from the Bureau of Labor Statistics, where inflation is estimated as the change in the CPI index. For the stock market return we use the CRSP value-weighted return including distributions. In addition, we impute the price-to-dividend ratio for the stock market using also the value-weighted returns excluding dividends. Because dividends are highly seasonal, we compute the price-to-dividend ratio as the price at the beginning of a quarter divided by the average quarterly aggregate dividend over the following year. For the same reason, we calibrate the model's dividend process to the annual dividend growth series in the data, by time-aggregating the quarterly series simulated with the model.

To calibrate the model we need to choose four parameters, whose values are shown in Table 3: (i) The subjective discount factor β is chosen to match the average risk-free rate. (ii) The utility curvature parameter γ is chosen to match the volatility of the risk-free rate. (iii) The volatility of the log surplus consumption ratio at the average state θ allows us to match the stock market Sharpe ratio (SR_m). (iv) The habit persistence ϕ_h is chosen to match the first-lag autocorrelation of the stock market log price-to-dividend ratio (pd).

Table 5 shows various statistics in relation to asset prices for the data and for the model. Model 1 refers to the full model with both time-varying information quality and time-varying risk aversion. The parentheses next to the data estimates show standard errors. The parenthesis next to a model statistic shows the $t-$statistic of the hypothesis that the data estimate is equal to the value given by the model. The model matches well the quantities that is calibrated to, namely the autocorrelation of pd with a $t-$statistic of 0.88, the mean and volatility of r^f with $t-$statistics of 0.16 and 0.10, respectively, and SR_m with a $t-$statistic of 0.11. The model dividend process captures well the corresponding statistics in the data, matching almost exactly the volatility of the aggregate log annual dividend growth and its correlation with the aggregate log annual consumption growth.

The model also captures well the means and volatilities of the stock market return (r^m) and the equity premium ($r^m - r^f$). Further, it matches the autocorrelation of the risk-free rate, which is equal to 0.65, and generates a low correlation between the risk-free rate and the

log price-to-dividend ratio, which is border line statistically different from the data estimate, with a t-statistic of 1.96.

Moving on to the pd-ratio we see that the model generates a quite high volatility of 0.31, yet it is low compared to our data estimate of 0.44. Other studies, however, report lower data estimates that are similar to our model's prediction.[26] Using similar data, Boudoukh, Michaely, Richardson and Michael (2007) report a volatility of 0.41 excluding share repurchases and between 0.29 and 0.33 including share repurchases, depending on how the repurchases are measured. As for the average pd-ratio, for our model it is 4.34, which is a bit low compared to our data estimate of 4.71. Boudoukh, Michaely, Richardson and Michael (2007) report estimates between 4.54 and 4.66.[27] The low model average is mostly due to the fact that the average market return in the model (1.82%) is higher than in the data (1.59%). The higher average market return is due to the higher market return volatility generated by the model.

The utility curvature parameter γ was calibrated to 1.17. With similar preferences, Campbell and Cochrane (1999) and Wachter (2006) use a value of 2.0. In our case, the value for this parameter is pinned down by the volatility of the risk-free rate. As we have already discussed, given the fluctuations in the speculative activity, a higher value for γ would imply a higher volatility for the risk-free rate, as it can also be inferred from Panel A of Figure 2. Also, higher γ implies lower elasticity of inter-temporal substitution and, thus, larger fluctuations coming from changes in the weighted mean forecast $\tilde{\mu}$. Our estimate of the risk-free rate volatility is conservative given that we adjust for the expected rather than the realized inflation. With a higher γ, the model can generate even higher volatility for the pd-ratio. For example, with $\gamma = 1.4$ the model explains entirely our estimate of the volatility in the pd-ratio, but with twice as much volatility in the risk-free rate. Overall, the benchmark calibration captures very well the main moments of the stock market prices and returns, of the risk-free rate and the correlation between them.

In addition, the model explains the strong negative relation between the stock market and the dispersion in forecasts. In the data this correlation is -0.57 while the model generates a stronger negative correlation of -0.76. The difference is statistically significant, but the model was not calibrated to fit this moment.

Finally, we also report moments of the log price-to-consumption ratio (pc) and the returns

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[26]Wachter (2006) reports a volatility of the log price-to-dividend ratio of 0.31, Campbell and Cochrane (1999) report 0.27, Gabaix (2012) reports 0.33 and Bansal and Yaron (2004) report 0.29.

[27]Boudoukh, Michaely, Richardson and Michael (2007) report annual log dividend yield statistics which we transform to quarterly pd-ratio statistics by taking the negative value and adding $\ln 4$.

of the aggregate consumption claim (r^c). Since we lack a data counterpart, we compare these moments to those of the stock market. The moments of the pc-ratio are comparable to those of the pd-ratio, though it is less volatile. The average returns of the consumption claim are very similar to the market returns in the data, but with lower volatility, due to the lower volatility in cash-flows. Despite this, the equity premium of the consumption claim is very close to the stock market equity premium in the data. Thus, the Sharpe ratio of the consumption claim is higher, close to 0.20, compared with the stock market Sharpe ratio of 0.14. This is due to the fact that the stock market in the model carries some idiosyncratic volatility, mostly due to the idiosyncratic dividend shocks.

6.1 Quantitative analysis

Next we quantify the various effects and analyze how the model explains the main moments of the stock market price, stock returns and the risk-free rate. The analysis supports the view that the time-varying information quality is an important determinant of asset prices. It generates significant fluctuations in stock prices, it helps explain the equity premium and the low correlation between the risk-free rate and stock prices.

We start the analysis by first seeing how the asset pricing moments change when we turn off the time-varying risk-aversion and the time varying information quality. For Model 2 we turn off the time-varying risk aversion by setting $\phi_h = 1$, which results in a constant risk aversion equal to $\gamma \times (1 + \theta)$. The rest of the model parameters (see Table 3) are kept the same. For Model 3, we additionally set σ_ρ, β_χ and σ_χ to zero to obtain constant information quality. The rest of the parameters are kept the same, apart from an adjustment to β so that Model 3 also matches the average risk-free rate. The asset pricing moments of the various models are shown in Table 5 while Table 4 shows several moments of state variables.

The first thing we note is that the volatility of the log price-dividend ratio falls from 0.31 to close to zero (0.02) when we remove both the time-varying risk aversion and the time-varying information quality. What this means is that the fluctuations of the weighted mean forecast ($\tilde{\mu}$), which is the only state variable in Model 1, account for a negligible amount of stock price volatility. The volatility of the stock market excess return in Model 3 of 6.54% is almost entirely generated by dividend growth, which has an annual volatility of 11.10%. Looking at the consumption claim, we see that the volatility of the log price-consumption ratio is zero with precision of two decimal places.

To see why the mean forecast has a negligible effect on stock prices, we derive an approx-

imate expression of the sensitivity of the pd-ratio to changes in the mean forecast,

$$\frac{\partial pd_t}{\partial \tilde{\mu}_t} \approx \frac{\beta_d - \gamma}{1 - \phi_\mu Z_t}, \tag{20}$$

where Z_t is the price-dividend ratio of the one-period dividend claim, which is very close to one and at the steady state it is equal to 0.994. We see that a change in $\tilde{\mu}$ has the usual discount rate effect given by γ, the inverse of the EIS, and the cash-flow effect given by β_d. In our model the cash-flow effect is stronger, since $\beta_d = 2.15$ and $\gamma = 1.17$. The value of the above sensitivity is 4.23, which together with the volatility of $\tilde{\mu}$ (0.56% shown in Table 4) gives a volatility for the pd-ratio of only 0.023. We can compare this result with the model of Bansal and Yaron (2004) where fluctuations in growth expectations are important.[28] The main difference is that they assume ϕ_μ to be equal to 0.979, which is much higher than our estimate of 0.773. In our model, the parameter ϕ_μ is jointly estimated and largely reflects the persistency in the mean forecast. Thus, according to our model disagreement and speculation cannot have a significant effect on the stock price volatility through fluctuations in $\tilde{\mu}$.

Next, we introduce time-varying information quality in Model 2, which increases the pd-ratio volatility to 0.16 and the pc-ratio volatility to 0.14. Comparing these moments across the three models we see that the time-varying information quality accounts for about half of the stock price volatility generated by the model and it is as important as the time-varying risk aversion. We have already explained in Section 5.4 that with a low γ, high θ and persistent shocks to information quality ($\phi_\chi = 0.965$), a negative shock to information quality leads to a persistent increase in discount rates and a significant drop in stock prices. We also note that the stock market Sharpe ratio (SR_m) increases from 0.08 in Model 1 to 0.14 in Model 2. The reason is because the stochastic information quality is almost exclusively driven by aggregate consumption growth shocks, where a positive growth shocks leads to a increase in information quality. This implies that the stock market fluctuations coming from the stochastic information quality require a positive risk premium.[29]

Introducing further the time-varying risk aversion increases even more the stock price and stock return volatility, as well as the equity premium, and leaves all other moments virtually unaltered, including the stock market Sharpe ratio. The stock price fluctuations generated by the time-varying risk aversion also require a risk premium because the ω is

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[28]Bansal and Yaron (2004) consider a model where the expected growth rate is observable and, thus, $\tilde{\mu} = \mu$. They use the following parameters: $\beta_d = 3$, $1/EIS = 1/1.5$, $\phi_\mu = 0.979$ that give a sensitivity of close to 90. Together with a volatility for $\tilde{\mu}$ of 0.17% gives a volatility for the pd-ratio of around 0.15.

[29]Given the estimated process for χ, the term $\beta_\chi(g_{t+1} - \tilde{\mu}_t)$ accounts for more than 99% of the variance of the shocks to χ.

driven by aggregate consumption growth shocks.

In the data, the dispersion in forecasts is strongly negatively correlated with the pd-ratio. The model replicates this behavior firstly because information quality is an important driver of the stock price and has a high correlation with dispersion $\bar{\nu}$. The high correlation is evident by the fact that χ, which drives information quality, has a correlation of -0.75 with $\bar{\nu}$, as shown in Table 4. Secondly, information quality and risk aversion are strongly correlated, since the correlation between χ and ω is 0.90, which makes ν also highly correlated with ω and risk aversion.

Next, we look at the volatility of the risk-free rate and note that in Model 1, the fluctuations of $\tilde{\mu}$ are an important driver of the risk-free rate, since they generate a volatility of 0.61. The introduction of stochastic information quality in Model 2 increases the volatility to 0.85. Looking at the expression of the risk-free in Lemma 2 we see that the other important driver is the speculative activity ξ. Table 4 shows that ξ is weakly negatively correlated with $\tilde{\mu}$, with a correlation of -0.24. What these numbers imply is that the speculative activity is as important for the risk-free rate as the mean forecast. The weak negative correlation between these two factors and the fact that both increase the risk-free rate also explains why the autocorrelation of the risk-free rate decreases from 0.76 to 0.65. The introduction of time-varying risk aversion in Model 3 does not affect the risk-free rate.

In addition to the above analysis, we plot the risk-free rate and the log price-to-dividend ratio against all the state variables in Figure 4. For each plot, we vary one state variable while the rest are kept at their steady states. The gray area in each plot shows the unconditional distribution of the corresponding state variable. Starting with Panels A and B, we see that $\tilde{\mu}$ has a negligible effect on the pd-ratio but a significant effect on the risk-free rate. The same holds true for $\tilde{\nu}$ and v, where from Panels C and D we see that they have very small effects on the stock price but significantly affect the risk-free rate. The explanation is that, even though $\tilde{\mu}$, $\tilde{\nu}$ and v affect the risk-free rate, what is important for the long-lived assets as the stock is persistent shocks to the discount rates. However, neither of these three state variables is particularly persistent as shown from Table 4.

The important factors for stock prices are shown in Panels E and F, where we plot the pd-ratio against χ, that drives information quality, and ω, that drives risk aversion, respectively. These two state variables do not affect the risk-free rate. Both of these state variables are quite persistent with autocorrelations of 0.953 for χ and 0.968 for ω. Either an increase in χ (increase in information quality) or in ω (decrease in risk aversion) leads to a persistent decrease in discount rates and an increase in stock prices. The variable ω affects only the

price of risk, whereas information quality through its effect on speculation affects both the price of risk and the risk free rate.

We close this section by noting that the low correlation between the pd-ratio and the risk-free rate in our model is firstly due to the fact that different state variables matter for these two quantities. Secondly, the state variables χ and ω that are important for the pd-ratio are weakly correlated with $\tilde{\mu}$ and v (see Table 4) that are significant drivers of the risk-free rate.

7   Additional empirical evidence

The main result of our analysis is that information quality with disagreement is an important driver of stock prices, through the fact that shocks to information quality cause persistent shocks to discount rates. For relevant empirical evidence we need to find some variable that is highly correlated with information quality and see whether it predicts future interest rates. The only available quantity is the cross-sectional dispersion in macroeconomic forecasts, which in our model is denoted with $\bar{\nu}$. In addition to this variable we construct a measure for ξ by estimating with the model a quadratic relation between ξ and $\bar{\nu}$. The estimated model has a regression R^2 of 0.99. We then run univariate predictive regressions of cumulative future risk-free rates over various horizons, from 1 quarter up to 28 quarters, using as predictors either $1 \div (1 + \xi)$ or $\bar{\nu}$. The results are shown in Table 6.

In Panel A we have the estimated coefficients of the constructed predictive variable $(1 + \xi)^{-1}$ for the data and Model 1.[30] We see that all coefficients estimated from the data are negative and increasing in magnitude. For the first few horizons the coefficients are statistically significant at the 10% level and beyond the 12-quarter horizon the coefficients become statistically significant at the 5% level. Running the same regressions on simulated data from the model yields very similar results. The coefficients are all negative, increasing in size and very close to those estimated from the data. For horizons longer than 8 quarters, the difference between the data coefficients and the model coefficients are statistically insignificant.

In Panel B we show the regression coefficients when we use as predictor the macroeconomic forecast dispersion $\bar{\nu}$. The data coefficients are all positive and increasing with horizon. Also, almost all of them are statistically significant at the 5% level, except for quarter 1. The model yields very similar results where all coefficients estimated from the

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[30]Model 2 generates identical results.

data are statistically indistinguishable from those estimated for the model.

Finally, to see how well the model is able to explain the stock market fluctuations, we perform Campbell-Shiller variance decompositions of the pd-ratio. We decompose its variance to fluctuations in cumulative dividend growth, stock market returns and the pd-ratio 60 quarters ahead (pd_{60}). The stock market returns are further decomposed into excess returns and the risk-free rate. In the data we see that 12% of the fluctuations are due to fluctuations in dividend growth, 70% due to fluctuations in returns and 16% due to fluctuations in pd_{60}. For Model 1, fluctuations in pd_{60} have very little explanatory power of the pd-ratio volatility with 2%, which implies that the model does not generate some more persistent effects found in the data. Fluctuations in future returns are associated with 109% of the fluctuations in the pd-ratio, out of which 91% is due to excess returns and 19% due to the risk-free rate. Dividend growth offsets 10% of the variations, even though there is positive dependence of stock prices on cash-flow growth. The reason is that dividend growth is positively correlated with interest rates, and high future interest rates are associated with low stock prices.

Overall, the model generates similar patterns to what we see in the data with most of the stock price fluctuations being associated with future stock excess returns. However, with the exception of the coefficient associated with excess returns for all other coefficients the difference between the data and the model is statistically significant. What is interesting though is that the model coefficients vary a lot across simulations. In Model 2, there is stronger dependence on fluctuations in the risk-free rates, not surprisingly since there are no fluctuations in risk-aversion. In Model 3, the principal driver of stock prices is dividend growth, but as we have discussed earlier, it explains very little of the total variations.

8 Conclusions

In this study we develop and solve a model of heterogeneous expectations in a large economy, where the conditional growth rate changes every period and agents update their beliefs through common information they observe. In doing so, they disagree in how they interpret this information depending on the information quality. We find that information quality fluctuates over time and has important asset pricing effects. According to the calibrated model, the fluctuating information quality with disagreement explains one-third to one-half of the stock price fluctuations and about the same portion of the equity premium.

We start our analysis by estimating the model of disagreement using data on professional

forecasts. According to the estimated model, information quality is volatile, persistent and almost exclusively driven by aggregate growth shocks, where a positive shock increases information quality. We then add power utility preferences with external habit to the model, derive the equilibrium and the representative agent formulation and study the asset pricing implications.

We first show that some of the predictions generated by small economies with two or finite number of agents extend to a large economy with an infinite number of agents. The first prediction that holds in both cases is that heterogeneity in beliefs affects the risk-free discount rate either upwards or downwards, depending on whether the utility curvature parameter (the constant relative risk aversion in the case of no habit) is higher or lower than one. The second prediction that also holds in small economies is that the speculative activity increases the individual consumption risks, which increases the price of risk. Both of these effects depend on the dispersion in forecasts. However, some other predictions that are important in small economies do not extend to our model. Specifically, in small economies when information quality is constant the forecast dispersion is stochastic, which makes the effects on the risk-free rate and the price of risk stochastic. These stochastic effects generate excess volatility in asset prices. However in a large economy, by virtue of the law of large numbers, the forecast dispersion evolves deterministically and converges rapidly to a constant steady state, making the heterogeneous agent economy observationally equivalent to a homogeneous agent economy and having a negligible effect on the stock price volatility.

The negligible stock price volatility effect comes from how disagreement affects the volatility of the mean forecast. First, contrary to several earlier studies that analyze more stylized models,we find that an increase in disagreement does not necessarily increase the volatility of the mean forecast. In fact, for a typical parameter set, a decrease in information quality that increases disagreement leads to a decrease in the volatility of the mean forecast. Regardless, because of the fact that the estimated persistence in the conditional mean growth rate is not high enough, fluctuations in the mean forecast, while important for the risk-free rate, are not quantitatively important for the stock price.

The model explains the main moments of stock returns and the risk-free rate as well as the low correlation between them. In addition, the model explains the strong negative correlation between the dispersion in macroeconomic forecasts and the stock price, as well as the fact that the dispersion in forecasts predicts risk-free rates. We hereby provide a channel through which information quality affects asset prices. The quantitative implications make this factor fundamental in understanding asset prices on the aggregate.

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Appendix A Proofs

In the following results $\varphi(x; \mu, \sigma^2)$ denotes the normal density with mean μ and standard deviation σ. The first three lemmas are required for the subsequent results.

Lemma 5. *The following relation holds*

$$\varphi(x; y, \sigma_x^2)\varphi(y; \mu_y, \sigma_y^2) = \varphi(x; \mu_y, \sigma_y^2 + \sigma_x^2)\varphi\left(y; \frac{x\sigma_y^2 + \mu_y\sigma_x^2}{\sigma_x^2 + \sigma_y^2}, \frac{\sigma_x^2\sigma_y^2}{\sigma_x^2 + \sigma_y^2}\right)$$

and therefore

$$\int_{-\infty}^{+\infty} \varphi(x; y, \sigma_x^2)\varphi(y; \mu_y, \sigma_y^2)dy = \varphi(x; \mu_y, \sigma_y^2 + \sigma_x^2).$$

Proof. It is equivalent to stating that if $x = y + \epsilon$ where $y \sim N(\mu_y, \sigma_y^2)$ and $\epsilon \sim N(0, \sigma_x^2)$ and if y and ϵ are independent then $x \sim N(\mu_y, \sigma_y^2 + \sigma_x^2)$. $\qquad\square$

Lemma 6. *Suppose that $z = ax + by$ where $x \sim N(\mu_x, \sigma_x^2)$, $y \sim N(\mu_y, \sigma_y^2)$ and x and y are independent. Then we have that*

$$x|z \sim N\left(a(z - b\mu_y)\frac{\sigma_x^2}{a^2\sigma_x^2 + b^2\sigma_y^2} + \mu_x\frac{b^2\sigma_y^2}{a^2\sigma_x^2 + b^2\sigma_y^2}, \frac{\sigma_x^2 b^2\sigma_y^2}{a^2\sigma_x^2 + b^2\sigma_y^2}\right).$$

Proof. This is a straight-forward application of the Bayes' theorem with densities:

$$f(x|z) = \frac{f(z|x)f(x)}{f(z)} = \frac{\varphi(z; ax + b\mu_y, b^2\sigma_y^2)\varphi(x; \mu_x, \sigma_x^2)}{\varphi(z; a\mu_x + b\mu_y, a^2\sigma_x^2 + b^2\sigma_y^2)}$$

where f denotes density. $\qquad\square$

Lemma 7. *Let $h(i)$ for $i \in [0, 1]$ be a density function. Let also ϵ_i's, one for each agent, be i.i.d. random (continuous) variables with probability distribution $\mathbb{P}$. Then, the mass of i in a set $B \in [0, 1]$, for which the realization of ϵ_i is in a set A is equal to $\mathbb{P}(A)$ times the mass of B, with probability one, that is*

$$\int_0^1 h(i)\mathbb{1}(i \in B)\mathbb{1}(\epsilon_i \in A)di = \mathbb{P}(A)\int_B h(i)di.$$

Further, let $A = (-\infty, g(x)]$, then:

$$\int_B h(i)\delta[g(x) - \epsilon_i]di = \frac{d\mathbb{P}[\epsilon_i \leqslant g(x)]}{dx} = f_\epsilon[g(x)]g'(x)\int_B h(i)di.$$

Proof. Observe first that the above holds in expectation.

$$\mathbb{E}\left[\int_0^1 h(i)\mathbb{1}(i \in B)\mathbb{1}(\epsilon_i \in A)di\right] = \int_0^1 h(i)\mathbb{1}(i \in B)\mathbb{E}\left[\mathbb{1}(\epsilon_i \in A)\right]di = \mathbb{P}(A)\int_B h(i)di,$$

because of the independence of $h(i)$ and ϵ_i and the fact that the ϵ_i's are i.i.d. Next observe that the variance of the integral is zero, since the expectation of the integral squared equals the expectation squared,

$$\begin{aligned}
\mathbb{E}\left[\left(\int_0^1 h(i)\mathbb{1}(i \in B)\mathbb{1}(\epsilon_i \in A)di\right)^2\right] &= \mathbb{E}\left[\int_B \int_B h(i)h(j)\mathbb{1}(\epsilon_i \in A)\mathbb{1}(\epsilon_j \in A)didj\right] \\
&= \mathbb{P}(A)^2 \int_B \int_B h(i)h(j)\mathbb{1}(j \neq i)didj + \mathbb{P}(A)\int_B h(i)^2(di)^2 \\
&= \mathbb{P}(A)^2 \left[\int_B h(i)di\right]^2 + \left[\mathbb{P}(A) - \mathbb{P}(A)^2\right]\int_B h(i)^2(di)^2 \\
&= \mathbb{P}(A)^2 \left[\int_B h(i)di\right]^2.
\end{aligned}$$

Therefore, the integral is equal to its expectation with probability one. The second result uses the first result to derive the density function of the distribution of agents with the same signals, e.g. being equal to $g(x)$. First, for $A = (-\infty, g(x)]$ we obtain

$$\int_B h(i)\mathbb{1}[\epsilon_i \leqslant g(x)]di = \int_B h(i)H[g(x) - \epsilon_i]di = \mathbb{P}[\epsilon_i \leqslant g(x)]\int_B h(i)di,.$$

Then, differentiate both sides with respect to x, using the Leibniz integral rule, to obtain the result:

$$\begin{aligned}
&\frac{d}{dx}\int_B h(i)H[g(x) - \epsilon_i]di = \frac{d}{dx}\mathbb{P}[\epsilon_i \leqslant g(x)]\int_B h(i)di, \\
\therefore \quad &\int_B h(i)\frac{\partial}{\partial x}H[g(x) - \epsilon_i]di = f_\epsilon[g(x)]g'(x)\int_B h(i)di, \\
\therefore \quad &\int_B h(i)\delta[g(x) - \epsilon_i]di = f_\epsilon[g(x)]g'(x)\int_B h(i)di,
\end{aligned}$$

where $f_\epsilon(\cdot)$ is the density function of the ϵ_i's. □

When the growth rate of the economy is realized in some period, all agents update their beliefs about the mean growth rate in a Bayesian way as shown by the next lemma.

Lemma 8. *Let the prior beliefs of agent i about μ_t be given by $N(\mu_t^i, v_t^2)$. Then, once g_{t+1} is observed, the posterior beliefs about μ_t are formed as follows:*

$$\mu_t | \mathcal{F}_{t+1} \stackrel{i}{\sim} N\left(\kappa_t \mu_t^i + (1 - \kappa_t) g_{t+1}, \kappa_t v_t^2\right), \qquad \forall i \in [0, 1], \tag{A21}$$

where $\kappa_t := \sigma_c^2 / (\sigma_c^2 + v_t^2)$.

Proof. This is a straightforward application of Lemma 6. □

Given, the beliefs about the informativeness of the private and public signals, agent i filters the information about the shock ϵ^μ to the mean growth rate from the signals s^i and s^m.

Lemma 9. *Given the beliefs about the informativeness of the signals s^m and s^i, as given by equations (5) and (7), respectively, agent i forms beliefs about the new innovation ϵ_t^μ as follows:*

$$\epsilon_t^\mu | s_t^i, s_t^m \stackrel{i}{\sim} N\left(\frac{s_t^i \zeta_t (1 - \rho_t^2) + s_t^m \rho_t (1 - \zeta_t^2)}{1 - \zeta_t^2 \rho_t^2}, \frac{(1 - \zeta_t^2)(1 - \rho_t^2)}{1 - \zeta_t^2 \rho_t^2}\right), \qquad \forall t, i \in [0, 1]$$

Proof. We use Lemma 6 to obtain the conditional distribution $\epsilon_t^\mu | s_t^m$ given assumption (5). We apply again Lemma 6 to obtain the conditional distribution $\epsilon_t^\mu | s_t^m, s_t^i$, given assumption (7) and the distribution of $\epsilon_t^\mu | s_t^m$. Alternatively, the result is obtained from the following (omitting time subscripts): $f(\epsilon^\mu | s^m, s^i) \propto f(s^m | \epsilon^\mu) f(s^i | \epsilon^\mu) f(\epsilon^\mu)$. □

We close the beliefs process of an agent by combining the individual posterior beliefs $(\mu_t | \mathcal{F}_{t+1})$ and the individual filtering $(\epsilon_{t+1}^\mu | s_{t+1}^i, s_{t+1}^m)$.

Lemma 10. *Agent i's beliefs about the state of the economy in period $t + 1$ are given by $\mu_{t+1} | \mathcal{F}_{t+1} \stackrel{i}{\sim} N\left(\mu_{t+1}^i, v_{t+1}^2\right)$, where*

$$\mu_{t+1}^i = (1 - \varphi_\mu)\mu_c + \varphi_\mu \kappa_t \mu_t^i + \varphi_\mu (1 - \kappa_t) g_{t+1} + \sigma_\mu \frac{s_{t+1}^i \zeta_{t+1} (1 - \rho_{t+1}^2) + s_{t+1}^m \rho_{t+1} (1 - \zeta_{t+1}^2)}{1 - \zeta_{t+1}^2 \rho_{t+1}^2}$$

$$v_{t+1}^2 = \varphi_\mu^2 \kappa_t v_t^2 + \sigma_\mu^2 A_{v,t+1}.$$

where

$$A_{v,t+1} := \frac{(1 - \zeta_{t+1}^2)(1 - \rho_{t+1}^2)}{1 - \zeta_{t+1}^2 \rho_{t+1}^2}.$$

Proof. From the law of motion of μ, given by (2), we know that the conditional (on $\mathcal{F}_{t+1}$) distribution of μ_{t+1} is normal given that the conditional distributions of μ_t and ϵ_{t+1}^μ are

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normal. To obtain the conditional moments of μ_{t+1} we use Lemma 5 noting that the beliefs about ϵ^μ_{t+1} are independent of those about μ_t. $\qquad\square$

Proof of Proposition 1. From Lemma 10, we know that

$$\mu^i_{t+1} = (1 - \varphi_\mu)\mu_c + \varphi_\mu\kappa_t\mu^i_t + \varphi_\mu(1 - \kappa_t)g_{t+1} + \sigma_\mu\frac{s^i_{t+1}\zeta_{t+1}(1 - \rho^2_{t+1}) + s^m_{t+1}\rho_{t+1}(1 - \zeta^2_{t+1})}{1 - \zeta^2_{t+1}\rho^2_{t+1}},$$

where $s^i_{t+1} = \eta s^m_{t+1} + \sqrt{1 - \eta^2}\epsilon^i_{t+1}$, as specified in (6). We substitute in s^i_{t+1} and for ease of notation we skip time subscripts and write the above equation as follows: $\mu'_i = a_0 + a_1\mu_i + a_2\epsilon_i$, where μ_i denotes beliefs in period t and μ'_i those in period $t + 1$. The distribution of beliefs in a given period t defined by:

$$n_t(\mu) := \int_0^1 \delta(\mu - \mu^i_t)di,$$

where $\delta(\cdot)$ is the Dirac delta function. Then, the distribution of beliefs in period $t + 1$ is given by

$$n_{t+1}(\mu') = \int_0^1 \delta(\mu' - \mu'_i)di = \int_0^1 \delta(\mu' - a_0 - a_1\mu_i - a_2\epsilon_i)di.$$

In the above, the Dirac delta function indicates when $\mu'_i = \mu'$. We then integrate over all possible values of μ_i and use Lemma 7

$$n_{t+1}(\mu') = \int_0^1 \int \delta(\mu' - a_0 - a_1\mu - a_2\epsilon_i)\delta(\mu - \mu_i)d\mu\, di,$$

$$= \int \int_0^1 \delta(\mu' - a_0 - a_1\mu - a_2\epsilon_i)\delta(\mu - \mu_i)di\, d\mu,$$

$$= \int \int_0^1 \delta\left(\frac{\mu' - a_0 - a_1\mu}{a_2} - \epsilon_i\right)di \int_0^1 \delta(\mu - \mu_i)di\, d\mu,$$

$$= \int \frac{1}{a_2}\phi\left(\frac{\mu' - a_0 - a_1\mu}{a_2}; 0, 1\right)n_t(\mu)d\mu = \int \phi(\mu'; a_0 + a_1\mu, a_2)n_t(\mu)d\mu.$$

We next use the fact that $n_t(\mu) = \phi(\mu; \bar\mu_t, \bar\nu^2_t)$ and use Lemma 5 to obtain:

$$n_{t+1}(\mu') = \int \phi(\mu'; a_0 + a_1\mu, a_2)\phi(\mu; \bar\mu_t, \bar\nu^2_t)d\mu = \phi(\mu'; a_0 + a_1\bar\mu_t, a_1^2\bar\nu^2_t + a_2^2).$$

Therefore, substituting back the coefficients a_0, a_1 and a_2, we obtain

$$\bar{\mu}_{t+1} = (1 - \varphi_\mu)\mu_c + \varphi_\mu \kappa_t \bar{\mu}_t + \varphi_\mu(1 - \kappa_t)g_{t+1} + \sigma_\mu \frac{\eta\zeta_{t+1}(1 - \rho_{t+1}^2) + \rho_{t+1}(1 - \zeta_{t+1}^2)}{1 - \zeta_{t+1}^2 \rho_{t+1}^2} s_{t+1}^m,$$

$$\bar{\nu}_{t+1}^2 = (\varphi\kappa_t)^2 \bar{\nu}_t^2 + \left(\sigma_\mu \frac{\zeta_{t+1}(1 - \rho_{t+1}^2)\sqrt{1 - \eta^2}}{1 - \zeta_{t+1}^2 \rho_{t+1}^2} \right)^2,$$

which completes the proof. $\qquad\square$

Proof of Proposition 2. Agents maximize their utility (10) subject to their budget constraint (14). Fixing a state z_t, we can derive the optimality condition of consumption in that state. The set of first-order conditions of the agent's optimization problem is characterized by

$$\beta^t \left[\alpha_t^i (C_t - H_t) \right]^{-\gamma} d\mathbb{P}_0^i(z_t) = p_0(z_t)dz_t \tag{A22}$$

In the above optimality condition it is implicit that α_t^i, C_t and H_t are functions of the state z_t. Let $p_t(z_{t+1}) = p_0(z_{t+1})/p_0(z_t)$ be the price of consumption in state z_{t+1} as of period t. Dividing equation (A22) for state z_{t+1} with the same condition in period t we obtain the optimal marginal rate of intertemporal substitution,

$$\beta \left(\frac{\alpha_{t+1}^i (C_{t+1} - H_{t+1})}{\alpha_t^i (C_t - H_t)} \right)^{-\gamma} d\mathbb{P}_t^i(z_{t+1}) = p_t(z_{t+1})dz_{t+1} \qquad \forall i \in [0, 1], \tag{A23}$$

where $\mathbb{P}_t^i(z_{t+1}) = \mathbb{P}^i(z_{t+1}|\mathcal{F}_t)$ represents the beliefs of agent i in period t for the state z_{t+1} next period. Solving for α_{t+1}^i we obtain

$$\alpha_{t+1}^i = \alpha_t^i \frac{C_t - H_t}{C_{t+1} - H_{t+1}} \left(\frac{\beta f_t^i(z_{t+1})}{p_t(z_{t+1})} \right)^{\frac{1}{\gamma}},$$

where $f_t^i(z_{t+1})dz_{t+1} = d\mathbb{P}_t^i(z_{t+1})$ is the conditional density function according to the beliefs of agent i in period t for state z_{t+1} next period. Since agents hold heterogeneous beliefs only in relation to g_{t+1} we can express the conditional density function as follows: $f_t^i(z_{t+1}) = f_t^i(g_{t+1})f(z_{t+1}|\mathcal{F}_t, g_{t+1})$. We substitute this into the above expression of α_{t+1}^i, integrate over the set of agents, apply the market clearing condition (15) and solve for $p_t(z_{t+1})$:

$$p_t(z_{t+1}) = \beta \left(\frac{C_{t+1} - H_{t+1}}{C_t - H_t} \right)^{-\gamma} f(z_{t+1}|\mathcal{F}_t, g_{t+1}) \left[\int_0^1 \alpha_t^i f_t^i(g_{t+1})^{1/\gamma} di \right]^\gamma.$$

We know from (1) and (3) that $f_t^i(g_{t+1}) = \phi(g_{t+1}; \mu_t^i, \sigma_c^2 + v_t^2)$ and substituting in the above expression we obtain the equilibrium condition for $p_t(z_{t+1})$. Finally, we substitute the expression for $p_t(z_{t+1})$ into (A23) and use the decomposition of $f_t^i(z_{t+1})$ to obtain the result. $\qquad\square$

In the following lemma, we express the integral that appears in Proposition 2 in terms of normal density function.

Lemma 11. *Omitting time subscripts, let the consumption surplus distribution across beliefs* $\alpha(\mu)$ *be Gaussian with mean* $\tilde{\mu}$ *and variance* $\tilde{\nu}^2$*, while the agents' beliefs are given by* $f^i(g) = \phi(\mu^i; \sigma_c^2 + v^2)$*. Then, we have that*

$$\int_0^1 \alpha^i \phi(g; \mu^i, \sigma_c^2 + v^2)^{1/\gamma} di = \phi\left(g; \tilde{\mu}, \sigma_c^2 + v^2 + \frac{\nu^2}{\gamma}\right)^{1/\gamma} \left(\sqrt{\frac{\sigma_c^2 + v^2}{\sigma_c^2 + v^2 + \tilde{\nu}^2/\gamma}}\right)^{1-1/\gamma}.$$

Proof. We substitute this into the integral of the above equation and solve the integral by using the consumption distribution over beliefs $\alpha_t(\mu) = \phi(\mu; \tilde{\mu}_t, \tilde{\nu}_t^2)$. We do this below, skipping the time subscripts.

$$
\begin{aligned}
\int_0^1 \alpha^i \phi(g; \mu^i, \sigma_c^2 + v^2)^{1/\gamma} di &= \int_0^1 \int_{-\infty}^{+\infty} \alpha^i \phi(g; \mu, \sigma_c^2 + v^2)^{1/\gamma} \delta(\mu - \mu^i) d\mu di \\
&= \int_{-\infty}^{+\infty} \phi(g; \mu, \sigma_c^2 + v^2)^{1/\gamma} \int_0^1 \alpha^i \, \delta(\mu - \mu^i) di d\mu \\
&= \int_{-\infty}^{+\infty} \phi(g; \mu, \sigma_c^2 + v^2)^{1/\gamma} \alpha(\mu) d\mu \\
&= \int_{-\infty}^{+\infty} \phi(g; \mu, \sigma_c^2 + v^2)^{1/\gamma} \phi(\mu; \tilde{\mu}, \tilde{\nu}^2) d\mu \\
&= \sqrt{\gamma} \left(\sqrt{2\pi(\sigma_c^2 + v^2)}\right)^{1-1/\gamma} \int_{-\infty}^{+\infty} \phi\left(g; \mu, \gamma(\sigma_c^2 + v^2)\right) \phi(\mu; \tilde{\mu}, \tilde{\nu}^2) d\mu \\
&= \sqrt{\gamma} \left(\sqrt{2\pi(\sigma_c^2 + v^2)}\right)^{1-1/\gamma} \phi(g; \tilde{\mu}, \gamma(\sigma_c^2 + v^2) + \tilde{\nu}^2),
\end{aligned}
$$

where the last equation is obtained using Lemma (5). Next, note that

$$\phi(g; \tilde{\mu}, \gamma(\sigma_c^2 + v^2) + \nu^2) = \phi\left(g; \tilde{\mu}, \sigma_c^2 + v^2 + \frac{\nu^2}{\gamma}\right)^{1/\gamma} \left[\sqrt{\gamma} \left(\sqrt{2\pi\left(\sigma_c^2 + v^2 + \frac{\tilde{\nu}^2}{\gamma}\right)}\right)^{1-1/\gamma}\right]^{-1}$$

Combining the above results and rearranging we obtain the result,

$$\int_0^1 \alpha^i \phi(g; \mu^i, \sigma_c^2 + v^2)^{1/\gamma} di = \phi\left(g; \tilde{\mu}, \sigma_c^2 + v^2 + \frac{\nu^2}{\gamma}\right)^{1/\gamma} \left(\sqrt{\frac{\sigma_c^2 + v^2}{\sigma_c^2 + v^2 + \tilde{\nu}^2/\gamma}}\right)^{1-1/\gamma}.$$

$\square$

Proof of Proposition 3. We first derive the equilibrium stochastic discount factor (SDF) under the probability measure $\widetilde{\mathbb{P}}$, under which $g_{t+1}|\mathcal{F}_t \sim N(\tilde{\mu}_t, \tilde{\sigma}_t^2)$, where $\tilde{\sigma}_t^2 := \sigma_t^2 + \tilde{\nu}_t^2/\gamma$. Plugging in the result of Lemma 11 into the expression for $p_t(z_{t+1})$ derived in Proposition 2 we obtain

$$p_t(z_{t+1}) = \beta \left(\frac{C_{t+1} - H_{t+1}}{C_t - H_t}\right)^{-\gamma} (1 + \xi_t)^{\frac{1-\gamma}{2}} f(z_{t+1}|\mathcal{F}_t, g_{t+1}) \phi\left(g_{t+1}; \tilde{\mu}, \tilde{\sigma}_t^2\right),$$

where

$$\xi_t := \frac{\tilde{\nu}_t^2}{\gamma \sigma_t^2} \qquad \text{and} \qquad \tilde{\sigma}_t^2 = \sigma_t^2(1 + \xi_t).$$

By the definition of the probability measure $\widetilde{\mathbb{P}}$ we have that

$$d\widetilde{\mathbb{P}}(z_{t+1}|\mathcal{F}_t) = f(z_{t+1}|\mathcal{F}_t, g_{t+1}) \phi\left(g_{t+1}; \tilde{\mu}, \tilde{\sigma}_t^2\right) dz_{t+1}$$

where $f(z_{t+1}|\mathcal{F}_t, g_{t+1})$ denotes the objective conditional probability density function of the state z. Under this probability measure, the price of a state z_{t+1} in period t is given by

$$p_t(z_{t+1})dz_{t+1} = M(z_t, z_{t+1})d\widetilde{\mathbb{P}}(z_{t+1}|\mathcal{F}_t)$$

where M is the SDF under $\widetilde{\mathbb{P}}$. Finally, we solve for M, where $\beta\left[(C_{t+1} - H_{t+1})/(C_t - H_t)\right]^{-\gamma}$ is given by (12) and (13), and note that the conditional variation of the SDF depends only on g_{t+1}; that is, the only priced risk is the aggregate consumption risk.

We next show that the consumption surplus share distribution across beliefs $\alpha_t(\mu)$ is Gaussian at all times, given that the initial distribution $\alpha_0(\mu)$ is Gaussian, and derive the law of motion of the mean and variance of this distribution. The first step is to show that, if in period $\alpha_t(\mu)$ is Gaussian with mean $\tilde{\mu}_t$ and variance $\tilde{\nu}_t^2$), then after the realization of g_{t+1} and before any updating of beliefs the consumption surplus share distribution over the beliefs held in period t is also Gaussian with mean $\tilde{\mu}_{t+1} = \frac{1}{1+\xi_t}\tilde{\mu}_t + \frac{\xi_t}{1+\xi_t}g_{t+1}$ and variance $\tilde{\nu}_t^2 = \tilde{\nu}_t^2 \div (1 + \xi_t)$. Plugging in the result of Lemma 11 into the law of motion of the

consumption surplus shares derived in Proposition 2 we obtain

$$
\alpha_{t+1}^i = \alpha_t^i \left(\frac{\phi(g_{t+1}; \mu_t^i, \sigma_t^2)}{\phi(g_{t+1}; \tilde{\mu}_t, \tilde{\sigma}_t^2)} \right)^{1/\gamma} (1 + \xi_t)^{\frac{\gamma-1}{2\gamma}} = \alpha_t^i \frac{\phi\left(g_{t+1}; \mu_t^i, \gamma\, \sigma_t^2\right)}{\phi\left(g_{t+1}; \tilde{\mu}_t, \gamma\, \tilde{\sigma}_t^2\right)}.
$$

The consumption surplus share distribution in period $t + 1$ over the beliefs in period t is then given by,

$$
\begin{aligned}
\int_0^1 \alpha_{t+1}^i \delta(\mu - \mu_t^i) di &= \int_0^1 \int \alpha_{t+1}^i \delta(x - \mu_t^i) \delta(\mu - x) dx\, di \\
&= \int \int_0^1 \alpha_t^i \frac{\phi\left(g_{t+1}; \mu_t^i, \gamma\, \sigma_t^2\right))}{\phi\left(g_{t+1}; \tilde{\mu}_t, \gamma\, \tilde{\sigma}_t^2\right)} \delta(x - \mu_t^i) \delta(\mu - x) di\, dx \\
&= \int \frac{\delta(\mu - x)}{\phi\left(g_{t+1}; \tilde{\mu}_t, \gamma\, \tilde{\sigma}_t^2\right)} \int_0^1 \phi\left(g_{t+1}; \mu_t^i, \gamma\, \sigma_t^2\right) \alpha_t^i \delta(x - \mu_t^i) di\, dx \\
&= \frac{1}{\phi\left(g_{t+1}; \tilde{\mu}_t, \gamma\, \tilde{\sigma}_t^2\right)} \int \delta(\mu - x) \phi\left(g_{t+1}; x, \gamma\, \sigma_t^2\right) \alpha_t(x) dx \\
&= \frac{\phi\left(g_{t+1}; \mu, \gamma\, \sigma_t^2\right) \alpha_t(\mu)}{\phi\left(g_{t+1}; \tilde{\mu}_t, \gamma\, \tilde{\sigma}_t^2\right)}
\end{aligned}
$$

Since $\alpha_t(\mu) = \phi(\mu; \tilde{\mu}_t, \tilde{\nu}_t^2)$ and from the first result of Lemma 5 we have that

$$
\phi\left(g_{t+1}; \mu, \gamma\, \sigma_t^2\right) \cdot \phi(\mu; \tilde{\mu}_t, \tilde{\nu}_t^2) = \phi\left(g_{t+1}; \tilde{\mu}_t, \gamma\, \tilde{\sigma}_t^2\right) \cdot \phi\left(\mu; \frac{\tilde{\mu}_t \gamma\, \sigma_t^2 + g_{t+1} \tilde{\nu}_t^2}{\gamma\, \tilde{\sigma}_t^2}, \frac{\sigma_t^2\, \tilde{\nu}_t^2}{\tilde{\sigma}_t^2} \right)
$$

Using the above and the definition of ξ_t we obtain the result,

$$
\int_0^1 \alpha_{t+1}^i \delta(\mu - \mu_t^i) di = \phi\left(\mu; \frac{1}{1 + \xi_t} \tilde{\mu}_t + g_{t+1} \frac{\xi_t}{1 + \xi_t}, \frac{\tilde{\nu}_t^2}{1 + \xi_t} \right).
$$

which is the result.

The final step, which is similar to the proof of Proposition 1, is to incorporate the updating of beliefs derived in Lemma 10. The consumption surplus share distribution over beliefs in period $t + 1$ is given by

$$
\alpha_{t+1}(\mu') = \int_0^1 \alpha_{t+1}^i \delta(\mu' - \mu_i') di = \int_0^1 \alpha_{t+1}^i \delta(\mu' - a_0 - a_1 \mu_i - a_2 \epsilon_i) di,
$$

where μ_i' stands for μ_{t+1}^i, μ_i stands for μ_t^i and the equation $\mu_i' = a_0 + a_1 \mu_i + a_2 \epsilon_i$ stands for

$$
\mu_{t+1}^i = (1 - \varphi_\mu)\mu_c + \varphi_\mu \kappa_t \mu_t^i + \varphi_\mu (1 - \kappa_t) g_{t+1} + \sigma_\mu A_{\mu,t+1} s_{t+1}^m + \sigma_\mu \sqrt{A_{\nu,t+1}} \epsilon_{t+1}^i.
$$

Therefore, we obtain that

$$
\begin{aligned}
\alpha_{t+1}(\mu') &= \int \phi(\mu'; a_0 + a_1\mu, a_2) \int_0^1 \alpha_{t+1}^i \delta(\mu - \mu_i) di\, d\mu \\
&= \int \phi(\mu'; a_0 + a_1\mu, a_2) \phi\left(\mu; \frac{1}{1+\xi_t}\tilde{\mu}_t + \frac{\xi_t}{1+\xi_t}g_{t+1}, \frac{\tilde{\nu}_t^2}{1+\xi_t}\right) d\mu \\
&= \phi\left(\mu'; a_0 + a_1\left(\frac{1}{1+\xi_t}\tilde{\mu}_t + \frac{\xi_t}{1+\xi_t}g_{t+1}\right), a_1^2\frac{\tilde{\nu}_t^2}{1+\xi_t} + a_2^2\right)
\end{aligned}
$$

Substituting back the coefficients a_0, a_1 and a_2, we obtain

$$
\begin{aligned}
\tilde{\mu}_{t+1} &= (1 - \varphi_\mu)\mu_c + \varphi_\mu\frac{\kappa_t}{1+\xi_t}\tilde{\mu}_t + \varphi_\mu\left(1 - \frac{\kappa_t}{1+\xi_t}\right)g_{t+1} + \sigma_\mu A_{\mu,t+1}\, s_{t+1}^m, \\
\tilde{\nu}_{t+1}^2 &= \frac{(\varphi\kappa_t)^2}{1+\xi_t}\tilde{\nu}_t^2 + \sigma_\mu^2 A_{\nu,t+1}.
\end{aligned}
$$

Finally, we note that the SDF is a function of $\tilde{\mu}$, $\tilde{\nu}$, v and ω. In addition, χ is required for the values of ρ and ζ. The shocks g and s^m drive the law of motion of $\tilde{\mu}$ and ω, while the shocks ϵ^χ and ϵ^ρ determine the other three state variables. $\qquad\square$

Proof of Lemma 1. We know that

$$
\log\left(\frac{C_{t+1}^i - H_{t+1}}{C_t^i - H_t}\right) = \log\left(\frac{\alpha_{t+1}^i(C_{t+1} - H_{t+1})}{\alpha_t^i(C_t - H_t)}\right)
$$

and using Proposition 2 we can express the above as

$$
\begin{aligned}
\log\left(\frac{C_{t+1}^i - H_{t+1}}{C_t^i - H_t}\right) &= \log\left(\frac{C_{t+1} - H_{t+1}}{C_t - H_t}\right) + \log\left(\frac{\phi(g_{t+1}; \mu_t^i, \sigma_t^2)^{1/\gamma}}{\int_0^1 \alpha_t^j \phi(g_{t+1}; \mu_t^j, \sigma_t^2)^{1/\gamma}dj}\right) \\
&= \log\left(\frac{C_{t+1} - H_{t+1}}{C_t - H_t}\right) + \frac{1}{2\gamma}\left[\frac{(g_{t+1} - \tilde{\mu}_t)^2}{\tilde{\sigma}_t^2} - \frac{(g_{t+1} - \mu_t^i)^2}{\sigma_t^2}\right] + \frac{1}{\gamma}\log\frac{\tilde{\sigma}_t}{\sigma_t} \\
&= l(z_t, g_{t+1}) - \frac{(g_{t+1} - \mu_t^i)^2}{2\gamma\sigma_t^2},
\end{aligned}
$$

where $l(z_t, g_{t+1})$ is agent independent. The first derivative of the above with respect to g_{t+1} evaluated at $g_{t+1} = \tilde{\mu}_t$ is given by

$$
\left.\frac{\partial}{\partial g_{t+1}}\log\left(\frac{C_{t+1}^i - H_{t+1}}{C_t^i - H_t}\right)\right|_{g_{t+1}=\tilde{\mu}_t} = \frac{\mu_t^i - \tilde{\mu}_t}{\gamma\sigma_t^2} + \left.\frac{\partial l(z_t, g_{t+1})}{\partial g_{t+1}}\right|_{g_{t+1}=\tilde{\mu}_t}.
$$

Taking the cross-sectional variance of the above derivative and using the consumption surplus shares α_t^i as weights yields that the speculative activity is given by $\xi_t = \tilde{\nu}_t^2 \div (\gamma \sigma_t^2)$. $\qquad \square$

Proof of Proposition 4. By construction, the marginal rate of intertemporal substitution is given by M as derived in Proposition 3. Therefore, we only need to show that the beliefs of the representative agent are given by $\widetilde{\mathbb{P}}$. We use Lemma 10 to derive the evolution of the beliefs of the representative agent. Her beliefs about μ_t are represented by $\mu_t | \mathcal{F}_t \sim N(\widehat{\mu}_t, \widehat{v}_t^2)$. We set $\zeta_{t+1} = 0$, $\sigma_\mu = \widehat{\sigma}_{\mu,t+1}$ and $\rho_{t+1} = \widehat{\rho}_{t+1}$ to obtain

$$\widehat{\mu}_{t+1} = (1 - \varphi_\mu)\mu_c + \varphi_\mu \widehat{\kappa}_t \widehat{\mu}_t + \varphi_\mu(1 - \widehat{\kappa}_t)g_{t+1} + \widehat{\sigma}_{\mu,t+1}\widehat{\rho}_{t+1}s_{t+1}^m,$$
$$\widehat{v}_{t+1}^2 = \varphi_\mu^2 \widehat{\kappa}_t \widehat{v}_t^2 + \widehat{\sigma}_{\mu,t+1}^2(1 - \widehat{\rho}_{t+1}^2),$$

where $\widehat{\kappa}_t := \sigma_c^2/(\sigma_c^2 + \widehat{v}_t^2)$. The beliefs of the representative agent are given by $g_{t+1} | \mathcal{F}_t \sim N(\widehat{\mu}_t, \sigma_c^2 + \widehat{v}_t^2)$, so it suffices to show that $\widehat{\mu}_t = \tilde{\mu}_t$ and $\widehat{v}_t^2 = v_t^2 + \tilde{\nu}_t^2/\gamma$ in all periods. We suppose that these hold true for $t = 0$, then we need to show that if these relations are true for period t then they also hold true for period $t + 1$. The initial condition is not restrictive, since the equivalence holds true asymptotically for any initial condition.

Using the laws of motion of v, given in Proposition 1, and of $\tilde{\nu}$, given in Proposition 3, and noting that $\kappa_t \div (1 + \xi_t) = \widehat{\kappa}_t$ we have that,

$$v_{t+1}^2 + \tilde{\nu}_{t+1}^2/\gamma = \varphi_\mu^2 \kappa_t v_t^2 + \sigma_\mu^2 A_{v,t+1} + \frac{1}{\gamma}\left(\varphi_\mu^2 \frac{\kappa_t^2}{1 + \xi_t}\tilde{\nu}_t^2 + \sigma_\mu^2 A_{\nu,t+1}\right),$$
$$= \varphi_\mu^2 \frac{\kappa_t}{1 + \xi_t}\left((1 + \xi_t)v_t^2 + \kappa_t \frac{\tilde{\nu}_t^2}{\gamma}\right) + \sigma_\mu^2\left(A_{v,t+1} + \frac{A_{\nu,t+1}}{\gamma}\right),$$
$$= \varphi_\mu^2 \widehat{\kappa}_t\left[\left(1 + \frac{\tilde{\nu}_t^2}{\gamma(\sigma_c^2 + v_t^2)}\right)v_t^2 + \frac{\sigma_c^2}{\sigma_c^2 + v_t^2}\frac{\tilde{\nu}_t^2}{\gamma}\right]$$
$$\qquad + \sigma_\mu^2\left(A_{\mu,t+1}^2 + A_{v,t+1} + \frac{A_{\nu,t+1}}{\gamma} - A_{\mu,t+1}^2\right),$$
$$= \varphi_\mu^2 \widehat{\kappa}_t\left(v_t^2 + \frac{\tilde{\nu}_t^2}{\gamma}\right) + \widehat{\sigma}_{\mu,t}^2 - \sigma_\mu^2 A_{\mu,t+1}^2,$$
$$= \varphi_\mu^2 \widehat{\kappa}_t \widehat{v}_t^2 + \widehat{\sigma}_{\mu,t+1}^2 - \widehat{\sigma}_{\mu,t+1}^2 \widehat{\rho}_{t+1}^2 = \widehat{v}_{t+1}^2,$$

where we have used the definitions of $\widehat{\sigma}_{\mu,t}$ and $\widehat{\rho}_t$. Finally, it is also true that $\widehat{\mu}_{t+1} = \tilde{\mu}_{t+1}$ since $\widehat{\kappa}_t = \kappa_t \div (1 + \xi_t)$ and, by construction, $\widehat{\sigma}_{\mu,t+1}\widehat{\rho}_{t+1} = \sigma_\mu A_{\mu,t+1}$. Finally, if $\gamma = 1$ and $\zeta_t = \eta \div \rho_t$ then $A_{\mu,t+1}^2 + A_{v,t+1} + A_{\nu,t+1} \div \gamma = 1$, using the expressions derived in Proposition 1. $\qquad \square$

Proof of Lemma 2. From Proposition 3, which gives the equilibrium stochastic discount factor and the probability measure to price all assets, it follows that

$$
r_t^f = -\ln \widetilde{\mathbb{E}}_t(M_{t+1}) = -\ln(\beta) + \frac{\gamma-1}{2}\ln(1+\xi_t) + \gamma(\tilde{\mu}_t - \omega_t) - \ln \widetilde{\mathbb{E}}_t \left(e^{-\gamma(1+\lambda_t)(g_{t+1}-\tilde{\mu}_t)} \right)
$$

$$
= -\ln(\beta) + \frac{\gamma-1}{2}\ln(1+\xi_t) + \gamma(\tilde{\mu}_t - \omega_t) - \frac{1}{2}\gamma^2(1+\lambda_t^2)\tilde{\sigma}_t^2,
$$

where $\tilde{\mu}_t$ and $\tilde{\sigma}_t^2$ are the conditional mean and variance of g_{t+1}, respectively, under the probability measure $\widetilde{\mathbb{P}}$. Further, by Proposition 3, we have $\tilde{\sigma}_t^2 = \sigma_t^2(1+\xi_t)$. Finally, we use the definition of λ_t shown in equation (13) to obtain the result. $\qquad\square$

Lemma 12. *Suppose that the one-period log payoff of an asset j is conditionally normally distributed and given by*

$$
c_{0,t}^j + c_{1,t}^j(g_{t+1} - \tilde{\mu}_t) + c_{2,t}^j \epsilon_{t+1}^j
$$

where ϵ^j is an idiosyncratic and standard normally distributed risk. Then, the excess log return is given by

$$
r_{t+1}^j - r_t^f + \frac{1}{2}\sigma_{i,t}^2 = c_{1,t}^j(g_{t+1} - \tilde{\mu}_t) + c_{2,t}^j \epsilon_{t+1}^j + \gamma c_{1,t}^j(1+\lambda_t)(1+\xi_t)\sigma_t^2 - \frac{1}{2}(c_{1,t}^j)^2 \xi_t \sigma_t^2
$$

Proof. We first compute the equilibrium price of asset j in period t, p_t^j, using the result of Proposition 3 that gives the equilibrium stochastic discount factor and the probability measure to price all assets, as follows:

$$
p_t^j = \widetilde{\mathbb{E}}_t \left(M_{t+1}\, e^{c_{0,t}^j + c_{1,t}^j(g_{t+1}-\tilde{\mu}_t) + c_{2,t}^j \epsilon_{t+1}^j} \right)
$$

$$
= \beta(1+\xi_t)^{\frac{1-\gamma}{2}} e^{-\gamma(\tilde{\mu}_t - \omega_t) + c_{0,t}^j} \widetilde{\mathbb{E}}_t \left(e^{c_{2,t}^j u_{t+1}^j} \right) \widetilde{\mathbb{E}}_t \left(e^{\left[c_{1,t}^j - \gamma(1+\lambda_t)\right](g_{t+1}-\tilde{\mu}_t)} \right)
$$

$$
= \beta(1+\xi_t)^{\frac{1-\gamma}{2}} \exp \left\{ -\gamma(\tilde{\mu}_t - \omega_t) + c_{0,t}^j + \frac{1}{2}(c_{2,t}^j)^2 + \frac{1}{2}\left[c_{1,t}^j - \gamma(1+\lambda_t)\right]^2 \tilde{\sigma}_t^2 \right\}
$$

$$
= \exp \left\{ -r_t^f + c_{0,t}^j + \frac{1}{2}\left[(c_{2,t}^j)^2 + (c_{1,t}^j)^2 \tilde{\sigma}_t^2 \right] - \gamma c_{1,t}^j(1+\lambda_t)\tilde{\sigma}_t^2 \right\},
$$

since by Proposition 3, under $\widetilde{\mathbb{P}}$ we have $g_{t+1}|\mathcal{F}_t \sim N(\tilde{\mu}_t, \tilde{\sigma}_t^2)$, and r_t^f is given in Lemma 2. The excess return is then given by

$$
r_{t+1}^j - r_t^f = c_{0,t}^j + c_{1,t}^j(g_{t+1} - \tilde{\mu}_t) + c_{2,t}^j \epsilon_{t+1}^j - \ln(p_t^j) - r_t^f,
$$

which gives the result once we substitute in the expression for p_t^j and $\tilde{\sigma}_t^2$. $\qquad\square$

Proof of Lemma 3. From Lemma 12 we know that the return on the asset whose payoff is given by $e^{g_{t+1}}$ is given by

$$r_{t+1} - r_t^f + \frac{1}{2}\sigma_t^2 = (g_{t+1} - \tilde{\mu}_t) + \gamma(1 + \lambda_t)(1 + \xi_t)\sigma_t^2 - \frac{1}{2}\xi_t\sigma_t^2.$$

The price of risk is defined as

$$q_t := \frac{\mathbb{E}\left[r_{t+1} - r_t^f\right] + 0.5\sigma_t^2}{\sigma_t}$$

and applying expectations to the return equation above, where $\mathbb{E}(g_{t+1}) = \mu_t$, gives the result. $\square$
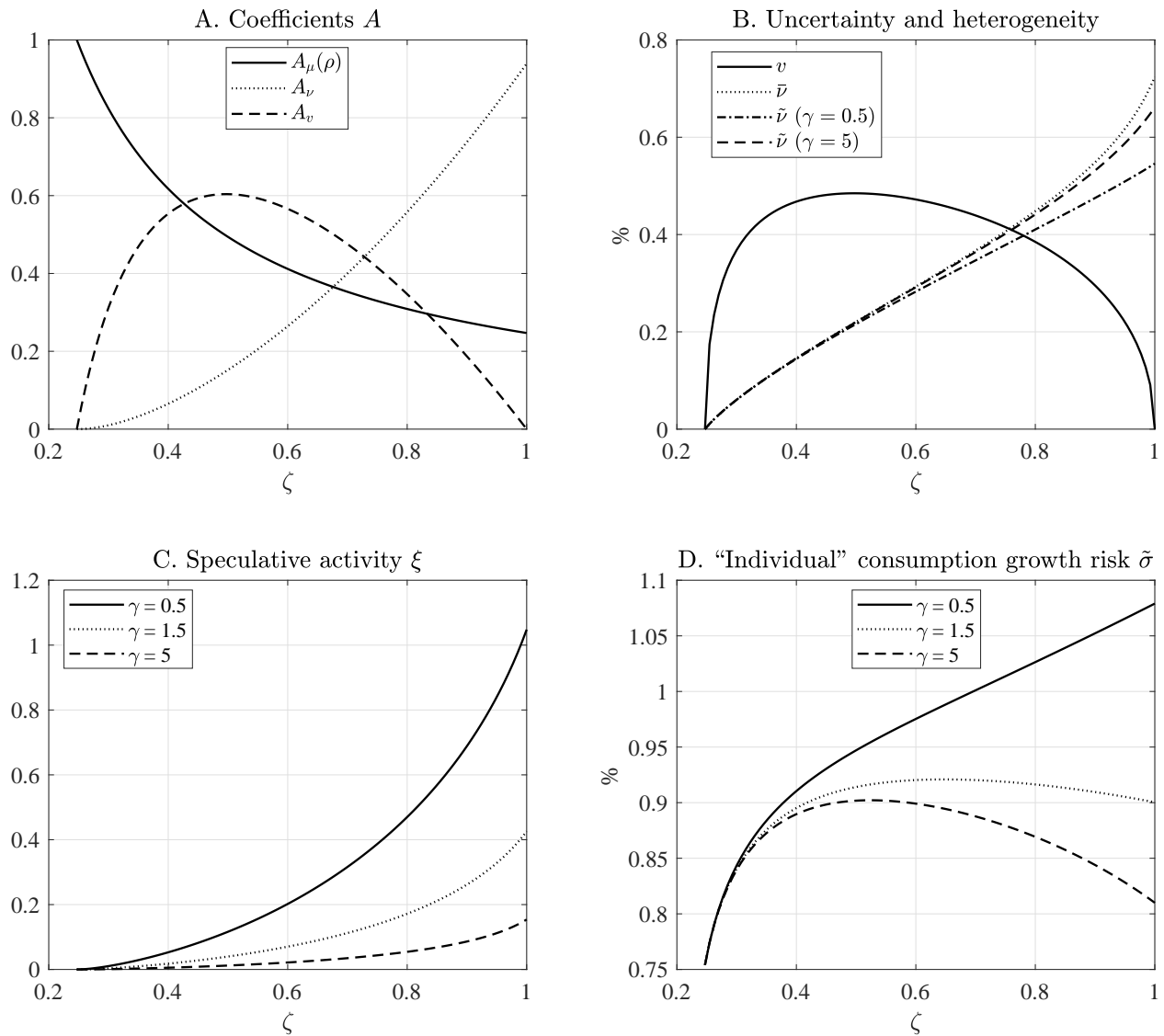
The graphs show several quantities of the model with constant information quality (ρ) as functions of $\zeta = \eta/\rho$ and for several values of the utility curvature parameter (γ). The rest of the parameters are as in Table 2. The parameter ζ ranges from η to 1. Panel A shows the A coefficients as given in Proposition 1 for the case $\zeta = \eta \div \rho$. Panel B shows the steady state values for v, $\bar{\nu}$ and $\tilde{\nu}$. Panel C shows the speculative activity (ξ) as given by Lemma 1 and Panel D the representative agent consumption risk $\tilde{\sigma}$ as given in Proposition 3.

Figure 1: Static analysis ($\eta = 0.247$) – Quantities related to beliefs

Table 1: Disagreement model estimation

The data is quarterly. The real aggregate consumption growth series (g) was computed from the per-capita consumption of non-durable goods and services (NIPA) deflated by the annual CPI index (BEA) and cover the period from 1947Q2 to 2015Q3. The beliefs data, where $\bar{\mu}$ denotes the mean forecast and $\bar{\nu}$ the standard deviation of forecasts, were obtained from the Survey of Professional Forecasters, span the period from 1968Q3 to 2015Q3 and correspond to the real GDP quarter-ahead forecasts. The data standard errors ($s.e.$) were Newey and West (1987) estimated using 16 lags. The model statistics are the averages across 1,000 simulations of 400 quarters each (with a burn-in of 100 quarters). The model t-statistics ($t-stat.$) correspond to the differences between the model and the data statistics. For a variable x, $\mu(x)$ denotes the sample mean, $\sigma(x)$ the standard deviation, $ac_k(x)$ its k-lag autocorrelation and $corr(x, y)$ the correlation with variable y. Each model was estimated using the Simulated Method of Moments, matching all moments shown, except the time-series mean of the cross-section average forecast $\mu(\bar{\mu})$ because this is given by μ_c, the unconditional mean of g. The estimated parameters are shown in Table 2.

Statistic	Description	Data est.	Data s.e.	Model avg.	Model t − st
A. Annual (time-aggregated) consumption growth (data: 1930 - 2017)
$\mu(g^a)$	Mean consumption growth	1.74	(0.27)	1.74	(0.01)
$\sigma(g^a)$	Volatility of consumption growth	2.70	(0.67)	2.70	(0.00)
$ac_1(g^a)$	Autocorrelation of consumption growth	0.48	(0.05)	0.50	(0.49)
B. Quarterly mean forecast, forecast dispersion and growth (data: 1968Q3 - 2016Q4)
$\mu(\bar{\mu})$	Average of mean forecast	0.39	(0.05)	0.44	(0.90)
$\sigma(\bar{\mu})/\sigma(g)$	Standardized volatility of mean forecast	0.47	(0.03)	0.49	(0.51)
$ac_1(\bar{\mu})$	Autocorrelation of mean forecast	0.80	(0.03)	0.77	(0.92)
$\mu(\bar{\nu})/\sigma(g)$	Standardized average of forecast dispersion	0.27	(0.03)	0.28	(0.08)
$\sigma(\bar{\nu})/\sigma(g)$	Standardized volatility of forecast dispersion	0.15	(0.01)	0.15	(0.11)
$ac_1(\bar{\nu})$	Lag 1 autocorrelation of forecast dispersion	0.71	(0.05)	0.77	(1.28)
$ac_4(\bar{\nu})$	Lag 4 autocorrelation of forecast dispersion	0.58	(0.08)	0.52	(0.72)
$ac_8(\bar{\nu})$	Lag 8 autocorrelation of forecast dispersion	0.43	(0.05)	0.42	(0.34)
$ac_{12}(\bar{\nu})$	Lag 12 autocorrelation of forecast dispersion	0.32	(0.05)	0.34	(0.34)
$corr(\bar{\mu}_t, g_{t+1})$	Growth predictability by average forecast	0.54	(0.05)	0.51	(0.52)
$corr(\Delta\bar{\mu}, g)$	Correlation of average forecast changes with growth	0.13	(0.07)	0.16	(0.34)
$corr(\bar{\nu}_t, g_{t+1})$	Growth predictability by forecast dispersion	-0.06	(0.11)	-0.11	(0.41)
$corr(\Delta\bar{\nu}, g)$	Correlation of forecast dispersion changes with growth	-0.20	(0.07)	-0.17	(0.39)
$corr(\bar{\mu}, \bar{\nu})$	Correlation between forecast mean and dispersion	-0.28	(0.17)	-0.20	(0.48)

Table 2: Estimated model parameters - quarterly frequency

Description	Parameter	Value
Unconditional mean of consumption growth	μ_c	0.436%
Volatility of consumption growth shocks	σ_c	0.754%
Volatility of shocks to consumption growth conditional mean	σ_μ	0.474%
Persistence of consumption growth conditional mean	φ_μ	0.773
Correlation between common and individual signals	η	0.247
Volatility of common signal informativeness idiosyncratic shocks	σ_ρ	1.11
Mean of common signal informativeness factor	μ_χ	-0.366
Persistence of common signal informativeness factor	φ_χ	0.965
Sensitivity of common signal informativeness factor to growth shocks	β_χ	30.0
Volatility of common signal informativeness factor idiosyncratic shocks	σ_χ	0.257%

Table 3: Model configurations - quarterly frequency

Model 1 is the full model, Model 2 has constant risk aversion and Model 3 has constant risk aversion and constant information quality. Model 1 parameters were chosen to fit the average risk-free rate, the volatility of the risk-free rate, the stock market Sharpe ratio and the persistence of the Stock market price-to-dividend. For Model 3 we adjust the β parameter to fit the average risk-free rate.

Description	Parameter	Model 1	Model 2	Model 3
Subjective discount factor	β	0.982	0.982	0.9744
Utility curvature parameter	γ	1.17	1.17	1.17
Conditional volatility of ω at steady state	θ	23	23	23
Degree of risk aversion at the average state (at $\omega = 0$)	$\gamma \cdot (1 + \theta)$	26.91	26.91	26.91
Log consumption surplus ratio persistency	ϕ_h	0.985	1.0	1.0
Constant information quality ($\sigma_\rho = \beta_\chi = \sigma_\chi = 0$)	ζ	no	no	yes

Table 4: State variable moments - Model 1

The table shows the correlations, means, standard deviations (stdev), first-lag autocorrelation (ac_1) of the state variable, for Model 1. They are computed as the averages across 1,000 simulations of 400 quarters each (with a burn-in of 100 quarters).

	Correlations								mean	stdev	ac_1
	$\tilde{\mu}$	$\tilde{\nu}$	v	χ	ω	$\bar{\mu}$	$\bar{\nu}$	ξ			
$\tilde{\mu}$	1.00	-0.23	0.06	0.36	0.27	0.98	-0.23	-0.24	0.435	0.557	0.763
$\tilde{\nu}$		1.00	-0.18	-0.73	-0.67	-0.20	0.99	0.95	0.272	0.151	0.710
v			1.00	0.06	0.13	0.02	-0.20	-0.42	0.358	0.085	0.567
χ				1.00	0.90	0.30	-0.75	-0.69	-0.368	0.954	0.953
ω					1.00	0.22	-0.70	-0.66	0.000	0.015	0.968
$\bar{\mu}$						1.00	-0.20	-0.19	0.439	0.515	0.770
$\bar{\nu}$							1.00	0.95	0.291	0.163	0.777
ξ								1.00	0.149	0.145	0.674
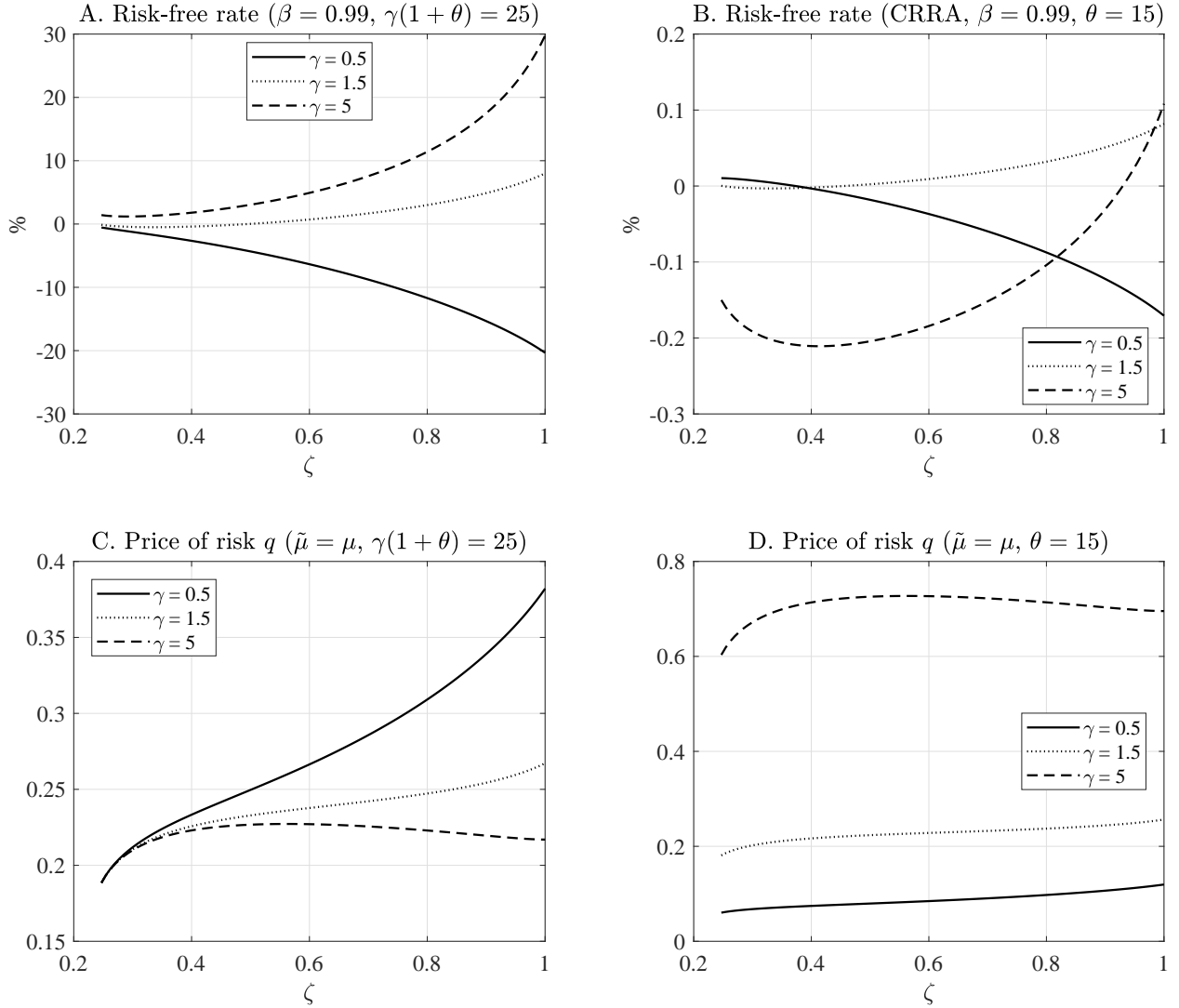
The graphs show the risk-free rate, as given by Lemma 2, and the price of risk, as given by Lemma 3, of the model with constant information quality (ρ) as functions of $\zeta = \eta/\rho$ and for several values of the utility curvature parameter (γ). The rest of the parameters are as in Table 2. The parameter ζ ranges from η to 1. Panels A and B show the risk-free rate for the state $\tilde{\mu} = \mu_c$ and for $\beta = 0.99$. Panels C and D show the price of risk for the state $\tilde{\mu} = \mu$. For Panels A and C the parameter θ depends on γ and given by $\theta = 25 \div \gamma - 1$. For Panels B and D the θ is equal to 15..

Figure 2: Static analysis ($\eta = 0.247$) – Risk-free rate and price of risk
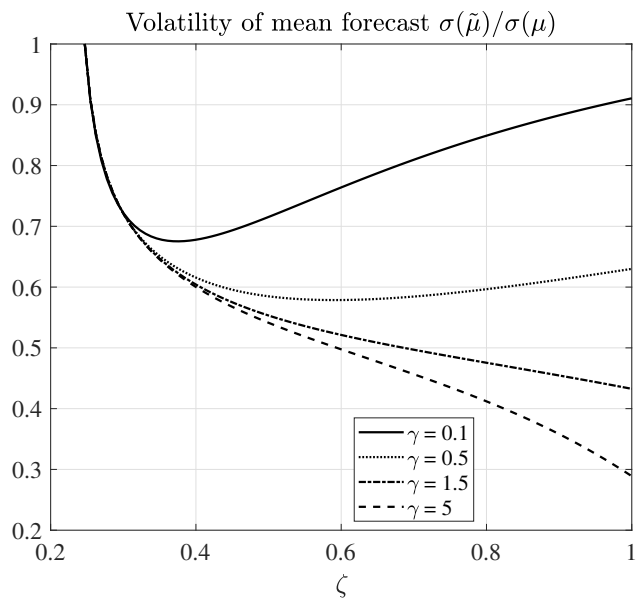
The graph shows the ratio of the volatility of the mean forecast to the volatility of the conditional mean for several values of the utility curvature parameter (γ). The rest of the parameters are as in Table 2. The inverse of the information quality ranges from η to 1.

Figure 3: Static analysis ($\eta = 0.247$) – Volatility of mean forecast

Table 5: Asset pricing statistics – Data and models

The table shows a number of asset pricing moments for the data and three different model configurations. Model 1 refers to the full model, Model 2 refers to the model with constant risk aversion and Model 3 refers to the model with both constant risk aversion and constant information quality–see parameters in Tables 2 and 3. For a variable x, $\mu(x)$ denotes the sample mean, $\sigma(x)$ the standard deviation, $ac_k(x)$ its k-lag autocorrelation and $corr(x,y)$ the correlation with variable y. The notation is as follows: pd denotes the log price-to-dividend ratio for the stock market dividend claim, r^f the log risk-free rate, r^m the log stock market return, SR_m the stock market Sharpe ratio using log returns, g^d is the log growth rate of the stock market dividend claim, pc the log price-to-consumption ratio of the aggregate consumption claim, and r^c the log return of the consumption claim. The standard errors for the data estimates $s.e.$ are estimated using the Newey and West (1987) method with 16 lags. For each model we report the averages of 1,000 simulations of 400 quarters each (with a burn-in of 100 quarters). The $t-$statistics for the models $t-st$ refer to the hypotheses that the data estimates are equal to the model averages.

	Data		Model 1		Model 2		Model 3	
	Data	s.e.	avg.	$t-st$	avg.	$t-st$	avg.	$t-st$
$\mu(pd)$	4.71	(0.09)	4.34	(4.10)	4.64	(0.77)	5.91	(13.43)
$\sigma(pd)$	0.44	(0.05)	0.31	(2.38)	0.16	(5.33)	0.02	(7.82)
$ac_1(pd)$	0.97	(0.01)	0.96	(0.88)	0.94	(2.22)	0.76	(14.17)
$\mu(r^f)$	0.14	(0.13)	0.16	(0.16)	0.16	(0.17)	0.16	(0.17)
$\sigma(r^f)$	0.84	(0.09)	0.85	(0.10)	0.85	(0.11)	0.61	(2.44)
$ac_1(r^f)$	0.65	(0.05)	0.65	(0.03)	0.65	(0.03)	0.76	(2.24)
$\mu(r^m)$	1.59	(0.47)	1.82	(0.50)	1.41	(0.37)	0.71	(1.87)
$\sigma(r^m)$	10.52	(1.55)	11.66	(0.74)	8.99	(0.99)	6.56	(2.55)
$ac_1(r^m)$	-0.03	(0.09)	0.01	(0.54)	0.02	(0.58)	0.03	(0.69)
$\mu(r^m-r^f)$	1.44	(0.49)	1.66	(0.44)	1.25	(0.39)	0.54	(1.83)
$\sigma(r^m-r^f)$	10.53	(1.55)	11.61	(0.70)	8.94	(1.02)	6.54	(2.57)
$ac_1(r^m-r^f)$	-0.03	(0.09)	-0.00	(0.33)	-0.00	(0.34)	-0.00	(0.33)
SR_m	0.14	(0.06)	0.14	(0.11)	0.14	(0.06)	0.08	(0.91)
$corr(pd,\bar{\nu})$	-0.57	(0.07)	-0.76	(2.45)	-0.77	(2.70)	–	
$corr(r^f,\bar{\nu})$	0.26	(0.11)	0.51	(2.35)	0.51	(2.33)	–	
$corr(pd,r^f)$	0.08	(0.15)	-0.20	(1.96)	-0.14	(1.57)	1.00	(6.28)
$\mu(g^d)$	1.45	(0.97)	1.74	(0.30)	1.74	(0.30)	1.74	(0.30)
$\sigma(g^d)$	11.10	(2.87)	11.10	(0.00)	11.10	(0.00)	11.10	(0.00)
$ac_1(g^d)$	0.18	(0.09)	0.29	(1.18)	0.29	(1.18)	0.29	(1.18)
$corr(g,g^d)$	0.52	(0.15)	0.52	(0.01)	0.52	(0.01)	0.52	(0.01)
$\mu(pc)$	4.71	(0.09)	4.54	(1.85)	4.87	(1.85)	7.17	(43.94)
$\sigma(pc)$	0.44	(0.05)	0.25	(3.47)	0.14	(5.56)	0.00	(8.17)
$ac_1(pc)$	0.97	(0.01)	0.96	(0.47)	0.95	(1.07)	0.76	(14.17)
$\mu(r^c)$	1.59	(0.47)	1.54	(0.09)	1.21	(0.81)	0.35	(2.63)
$\sigma(r^c)$	10.52	(1.55)	7.13	(2.18)	4.87	(3.64)	0.98	(6.15)
$ac_1(r^c)$	-0.03	(0.09)	0.02	(0.64)	0.03	(0.74)	0.44	(5.32)
$\mu(r^c-r^f)$	1.44	(0.49)	1.38	(0.13)	1.04	(0.81)	0.19	(2.54)
$\sigma(r^c-r^f)$	10.53	(1.55)	7.06	(2.23)	4.78	(3.69)	0.77	(6.28)
$ac_1(r^c-r^f)$	-0.03	(0.09)	-0.00	(0.33)	-0.00	(0.33)	-0.00	(0.34)

Table 6: Predicting risk-free rate

This table reports the β coefficients of the univariate predictive regressions

$$r^f_{t,t+T} = \alpha + \beta x_t + \varepsilon_{t,t+T},$$

where $r^f_{t,t+T}$ denotes the cumulative log risk-free rate, that is $r^f_{t,t+T} = \sum_{t=1}^{T} r^f_{t+\tau-1}$, where r^f_t denotes the log risk-free rate between period t and $t+1$. In Panel A the predictive variable x is $1 \div (1+\xi)$, whose values in the data is estimated based on Model 1. In Panel B the predictive variable x is $\bar{\nu}$ whose values in the data is given by the cross-sectional dispersion in professional forecasts. The $t-$statistics for the data $t - st$ are estimated using the Newey and West (1987) method with 16 lags. Model 1 refers to the full model and Model 2 refers to the model with constant risk aversion. We report the averages of 1,000 simulations of 400 quarters each (with a burn-in of 100 quarters). The $t-$statistics for the models $t - st$ refer to the hypotheses that the data estimates are equal to the model averages.

	$T=1$	$T=2$	$T=4$	$T=8$	$T=12$	$T=16$	$T=20$	$T=24$	$T=28$
A. $r^f_{t,t+T} = \alpha + \beta(1+\xi_t)^{-1} + \varepsilon_{t,t+T}$									
Data	-0.01	-0.03	-0.06	-0.13	-0.21	-0.29	-0.38	-0.45	-0.52
$t-st$	(1.71)	(1.84)	(1.84)	(1.86)	(1.93)	(2.05)	(2.16)	(2.25)	(2.39)
Models 1–2	-0.05	-0.08	-0.13	-0.23	-0.32	-0.40	-0.46	-0.51	-0.55
$st.dev.$	0.01	0.02	0.04	0.08	0.12	0.16	0.19	0.23	0.26
$t-st$	(3.78)	(3.06)	(2.01)	(1.33)	(0.94)	(0.71)	(0.48)	(0.32)	(0.14)
B. $r^f_{t,t+T} = \alpha + \beta\bar{\nu}_t + \varepsilon_{t,t+T}$									
Data	1.40	2.78	5.70	11.84	18.47	25.28	32.31	38.52	45.21
$t-st$	(1.91)	(2.00)	(1.98)	(1.98)	(2.02)	(2.13)	(2.26)	(2.35)	(2.49)
Models 1–2	2.68	4.50	7.59	13.48	18.86	23.49	27.38	30.62	33.27
$st.dev.$	0.73	1.40	2.74	5.31	7.70	9.97	12.12	14.13	16.04
$t-st$	(1.75)	(1.24)	(0.65)	(0.27)	(0.04)	(0.15)	(0.34)	(0.48)	(0.66)

Table 7: Campbell-Shiller variance decomposition

This table reports the Campbell-Shiller variance decomposition of the stock market log price-to-dividend ratio, pd, into fluctuations in cash-flow growth g^d, the risk-free rate r_t^f, the stock market excess return $r_{t+1}^e = r_{t+1}^m - r_t^f$ and the stock market price dividend ratio 60 quarters ahead pd_{60}. r_{t+1}^m and r_t^f denote the log stock market return and the log risk-free rate, respectively, between period t and $t+1$. The standard errors for the data estimates $s.e.$ are estimated using the Newey and West (1987) method with 16 lags. Model 1 refers to the full model, Model 2 refers to the model with constant risk aversion and Model 3 refers to the model with both constant risk aversion and constant information quality. For each model we report the averages of 1,000 simulations of 400 quarters each (with a burn-in of 100 quarters). The $t-$statistics for the models $t-st$ refer to the hypotheses that the data estimates are equal to the model averages.

	g^d	$-r^m$	$-r^e$	$-r^f$	pd_{60}	$\sigma(r^e)$	$\sigma(r^f)$	$\sigma(pd)$
Data	0.12	0.70	0.79	-0.09	0.16	10.53	0.84	0.44
$s.e.$	(0.08)	(0.12)	(0.21)	(0.12)	(0.06)	(1.55)	(0.09)	(0.05)
Model 1	-0.10	1.09	0.91	0.19	0.02	11.61	0.85	0.31
$st.dev.$	0.37	0.43	0.43	0.11	0.12	0.01	0.00	0.06
$t-st$	(2.74)	(3.25)	(0.57)	(2.39)	(2.26)	(0.70)	(0.10)	(2.38)
Model 2	-0.11	1.13	0.71	0.42	-0.01	8.94	0.85	0.16
$st.dev.$	0.73	0.78	0.80	0.24	0.12	0.00	0.00	0.03
$t-st$	(2.80)	(3.49)	(0.37)	(4.38)	(2.74)	(1.02)	(0.11)	(5.33)
Model 3	1.34	-0.33	0.56	-0.89	-0.01	6.54	0.61	0.02
$st.dev.$	3.00	3.01	2.70	0.63	0.09	0.00	0.00	0.00
$t-st$	(9.87)	(6.67)	(1.02)	(6.79)	(2.75)	(2.57)	(2.44)	(7.82)
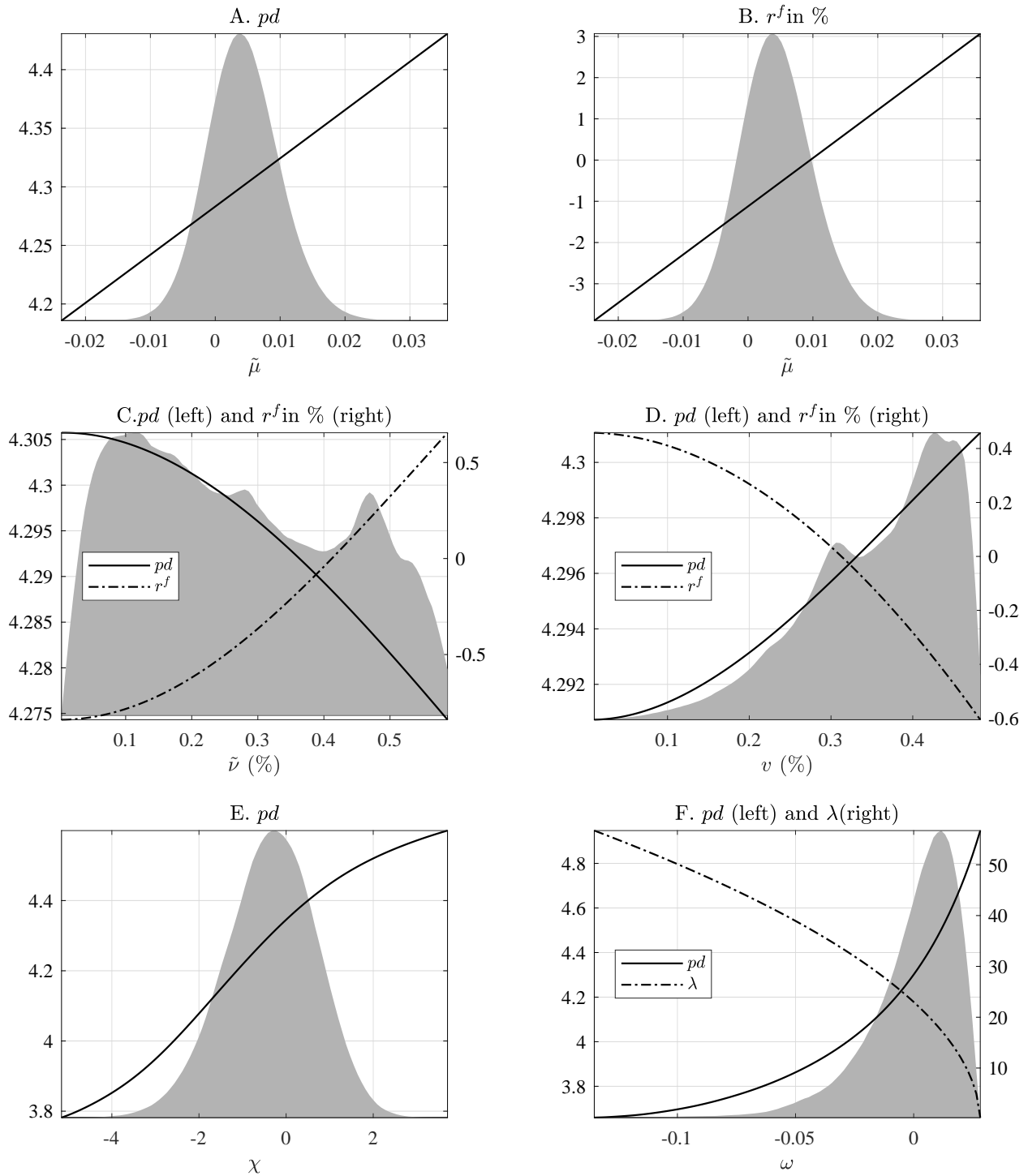
The graphs show the *pd*-ratio and the risk-free rate (r^f) as functions of the state variables in Model 1. For each graph, we vary the corresponding state variable, while the rest of the variables are kept at their steady states. The gray areas show the unconditional distributions of the state variables. Where we plot two quantities, the "left" corresponds to the left axis and "right" to the right axis.

Figure 4: The *pd*-ratio and the risk-free rate (r^f)–Model 1