Ordered Reference Dependent Choice

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Abstract

We study how violations of structural assumptions like expected utility and exponential discounting can be connected to reference dependent preferences with set-dependent reference points, even if behavior conforms with these assumptions when the reference is fixed. An axiomatic framework jointly and systematically relaxes general rationality (WARP) and structural assumptions to capture reference dependence across domains. It gives rise to a linear order that determines references points, which in turn determines the preference parameters for a choice problem. This allows us to study risk, time, and social preferences collectively, where seemingly independent anomalies are interconnected through the lens of reference-dependent choice.

1 Introduction

The standard model of choice in economics faces two separate strands of empirical challenges. First, structural assumptions such as the *expected utility form* (Independence) and *exponential discounting* (Stationary) are violated in simple choice experiments, most notably the Allais paradox and present bias. Second, and separately, studies have shown that choices are affected by reference points, resulting in "*non-rational*" behavior that violates the weak axiom of revealed preferences (WARP). With few exceptions,

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these two classes of prominent departures from standard models have been studied separately, and independently for each domain of choice, propelling models that seek to explain one phenomenon in isolation of the others.[1]

In this paper, we propose a unified framework that uses reference dependency to jointly explain failures of WARP and violations of structural assumptions across the risk, time, and social domains. This novel approach allows us to study different types of documented departure from standard models as related to one another, and in doing so suggests new empirical directions.

The intuition comes from a simple observation: If decision makers have preferences (e.g., utility functions, discount factors) that depend on a reference point, then even if they are otherwise standard and maximize exponentially discounted expected utility, they would still violate both WARP and structural assumptions like the Independence and Stationarity axioms from time to time—when reference points change.

Working with choice behavior, we provide the axiomatic foundation for a set of four models—generic choice, risk preference, time preference, and social preference—in which behavioral anomalies are explained through a common channel: changing preferences due to reference dependence. In these models, reference points are endogenously determined by *reference orders*, which rank each alternative by their relevance in affecting the reference point and influencing preferences.

To illustrate, consider a decision maker who exhibits increased risk aversion in the presence of safer options and therefore defies the expected utility theory (EU). This behavior is consistent with a myriad of anomalous choice documented in Herne (1999); Wakker & Deneffe (1996); Andreoni & Sprenger (2011), and prominently Allais (1953)'s paradox.[2] However, choice pattern of this kind can be explained without fully rejecting the expected utility form—that decision makers maximize the expectation of

[1]For reference dependence, see for example Kahneman & Tversky (1979), Kőszegi & Rabin (2006), Masatlioglu & Ok (2005), Masatlioglu & Ok (2013), Ok et al. (2015), and Dean et al. (2017). For models weakening the expected utility form see Quiggin (1982), Bell (1982); Loomes & Sugden (1982), Chew (1983); Fishburn (1983); Dekel (1986), Gul (1991), and Cerreia-Vioglio et al. (2015). For models weakening the discounted utility form see Loewenstein & Prelec (1992), Laibson (1997) and Frederick et al. (2002). For models that use reference dependency to explain violations of structural assumptions, see for example Kőszegi & Rabin (2007), Ortoleva (2010). An exception where both WARP and structural assumptions are relaxed is Bordalo et al. (2012).

[2]In the Allias paradox, a decision maker is drawn to the safe option when it is available, contradicting the irrelevance of common consequence assumption in standard expected utility theory. We will (re)introduce the Allais paradox and discuss the application of our model in Section 3. Herne (1999); Wakker & Deneffe (1996); Andreoni & Sprenger (2011) document other behaviors consistent with our risk model, discussed in Section 3.

some utility function for each choice problem—but by allowing for reference dependent utility functions that varies in concavity. This leads to our model in the risk domain: a decision maker's utility function depends on the safest available alternative, which reflects changing risk aversion. When the safest alternative is fixed, standard expected utility holds. But when reference point changes, then the safer the reference, the more concave the utility function. We then show that the same concept can be applied to time preference and social preference. Hence, we have a "unified framework".

This framework has two persistent components: (i) a complete and transitive binary relation that determines the reference points and (ii) preference parameters (i.e., utility functions, discount rates, utility from sharing) that depend on the reference point. Through our three applications, we demonstrate that our framework allows for a reference point that takes the form of an alternative (risk preference), a feature of an alternative (time preference), or an external index such as the Gini coefficient (social preference).

Our first step is a general representation theorem for choices in a generic domain: *Ordered-Reference Dependent Utility (ORDU)* (Section 2). In this model, the decision maker uses a reference order to identify the reference alternative of a choice problem. In turn, this determines a utility function that she maximizes. Hence, it is as if that alternatives are ranked by their relevance in affecting preferences, and the underlying preference for a given choice problem is determined by the alternative that ranks highest in this order among those that are available.

The key behavioral postulate, *Reference Dependence* (RD), provides a general condition that captures reference dependency in choice. It posits that if we fix the reference point, WARP holds. Since we do not know which alternative is the reference point, we scarcely posit that there is one option in every choice problem such that if we keep said option when taking subsets, WARP holds. To illustrate, consider two choice sets $B \subset A$ such that WARP is violated; for instance, when an alternative is available in both A and B but is only chosen from A. Our axiom RD makes the behavioral assumption that the reference alternative of A is not present in B, causing a change in reference and therefore a WARP violation. Hence for any choice set A, RD demands that choices from subsets of A satisfy WARP as long as a certain (reference) alternative remains present. A formal definition is provided in Section 2. This axiom, along with a standard continuity assumption when X is infinite, characterizes the ORDU representation.

Next, we consider the special case of risk preference in Section 3. Now the postulate becomes: preserving one of the safest alternatives in a choice set preserves WARP *and* the Independence condition. The intuition remains the same: by maintaining the reference point, normative postulates hold, which include Independence in addition to WARP in the case of risk preference. We call this *Risk Reference Dependence*. A second axiom, *Avoidable Risk*, states that if we expand a choice problem, then choices are weakly more risk averse, since an even safer reference increases risk aversion. Together with standard continuity and first order stochastic dominance we obtain the *Avoidable Risk Expected Utility* (AREU) representation, in which a decision maker's utility function depends on the safest alternative available, and safer references lead to more concave utility functions. Once a utility function is determined, standard expected utility maximization follows.

We then turn to the time domain in Section 4. The standard model for time preferences is Exponentially Discounted Utility, yet it is routinely challenged in empirical studies as economic agents are less patient for short-term decisions, or *present bias*.[3] In our model, a decision maker uses a discount factor that depends on the earliest availability of a payment from a choice problem, and the availability of a sooner payment makes the decision maker impatient. The key axiom, *Time Reference Dependence*, is the counterpart of RD, where now we require WARP *and* Stationarity to hold only when we preserve the earliest alternative. The reference effect is characterized by the axiom *Present Bias*, which simply posits that symmetrically advancing the options can only increase delay aversion, where the decision maker is less willing to wait. The resulting model explains the well-known violation of dynamic consistency, in which the same delay between consumption is tolerable in the future but not in the present. The model also captures WARP violations that occur in the same spirit, where the availability of an immediate payment tempts the decision maker toward sooner payments, even if the immediate payment is not itself chosen.

The application for social preference is studied in Section 5. Often viewed as a desire to be *fair*, subjects in economics and psychology experiments display increased altruism when a more balanced split of payment is available than when it is not.[4] In our setup, an alternative is an income distribution between the decision maker and another individual. Decision makers agree on what it meas to be *equal*, but disagree on what it means to be *fair*. They use the Gini coefficient to measure equality, and *attainable equality* is therefore the lowest Gini coefficient that can be achieved in a given choice

[3] See for example Laibson (1997), Frederick et al. (2002), and Benhabib et al. (2010).

[4] See for example Ainslie (1992), Rabin (1993), Nelson (2002), Fehr & Schmidt (2006), and Sutter (2007).

problem. When attainable equality is higher (lower Gini), they experience greater (subjective) utility from sharing, which reflects their desire to be fair. This type of behavior is a result of our unified framework adapted to this setting—the presence of certain alternatives, as given by a reference order, affects the underlying preference for sharing. Like before, the main axiom *Equality Reference Dependence* calls for conformity with WARP and quasi-linear preferences when we preserve the most-balanced option. A second axiom, *Fairness*, posits that decision makers are weakly more willing to share when options are added to a choice problem, since this can only increase attainable equality. In addition to capturing changing altruism, the model also explains increased sharing when splitting a fixed pie due to the availability of a more balanced division, as well as increased tendency to forgo a larger pie in favor of sharing a smaller one.

In our three applications, failures of WARP and failures of structural assumptions are inextricably linked. For example in the risk domain, adding WARP to our model immediately implies full compliance with Independence, and vice-versa (Proposition 1). Therefore, our model departs from standard expected utility only when *both* WARP and Independence fail. The same results are obtained for time and social preferences (Proposition 3 and Proposition 4). These findings separate our work from models that weaken structural assumptions but retain WARP—often interpreted as a *stable preference* that simply lacks structural properties. In fact, necessary violation of WARP suggests a behavioral manifestation of *changing preferences* in our models. It provides a new perspective to study classic paradoxes like Allais and present bias using WARP violations in non-binary choice, which we elaborate in their respective sections.

In relation to the existing literature, we first note that reference points are not exogeneously observed in our models. This strikes a fundamental difference in primitives/datasets to prospect theory by Kahneman & Tversky (1979), the endowment effect by Kahneman et al. (1991), and models of status quo bias led by Masatlioglu & Ok (2005).[5] Our models belong to a separate set of literature built on *endogenous reference*, where reference points are neither part of the primitive nor directly observable, such as in Kőszegi & Rabin (2006) and Ok et al. (2015). Unlike these models, our reference alternatives are given by a *reference order*. This added structure allows us to quickly pin down reference points that may not be otherwise observable. Subsection 2.2 discusses the identification of our reference points and the consequent out-of-sample predictions via the reference order. Using this defining feature, Appendix B provides in details

[5]For other models of status quo bias, see Masatlioglu & Ok (2013) and Dean et al. (2017). Ortoleva (2010) extends this idea to preferences under uncertainty.

testable distinctions between ORDU and related non-WARP models (Kőszegi & Rabin (2006), Ok et al. (2015), Manzini & Mariotti (2007), Masatlioglu et al. (2012)).

Our most general model, in which choices are over generic alternatives, is most similar in spirit and concurrent to Kıbrıs et al. (2018) albeit having different axiomatimization. Their paper focuses on choices over generic alternatives and contains no counterpart to our applications in the risk, time, and social domains. Their axiom depicts a conspicuity ranking between any two alternatives: if dropping x in the presence of y results in a WARP violation, then dropping y in the presence of x does not. Our approach is different and more involved, as it requires comparison between multiple choice problems differing by more than one alternative. However, this allows us to accommodate a wide range of behavioral postulates (in addition to WARP), such as the Independence and Stationarity conditions, with which we deliver reference-dependent expected utility and reference-dependent exponential discounting respectively. Moreover, their model is limited to a finite set of alternatives, whereas we allow the set to be any separable metric space. This is not (just) a technical contribution, as the added generality is indispensable for choices over lotteries.

We compare our applications in risk, time, and social preferences to existing models in their respective sections. However our main contribution is, instead of a single model that captures a specific departure from standard theory, a unified framework. The closest work that also takes the form of a unified framework is *salience*, pioneered by Bordalo et al. (2012, 2013), where options are evaluated differently depending on which attribute is salient. We are different in that our framework comprises of a systematic reference dependence approach of weakening normative postulates, with which we apply universally to the risk, time, and social domains. This approach allows us to study reference dependence in risk as related to reference dependence in time, and failure of WARP as related to failure of structural assumptions. Indispensable to this innovation is the use of choice correspondences as opposed to preference relations as primitive *and* foregoing WARP—the conventional "rationality" assumption increasingly scrutinized by empirical evidence. Otherwise, behavior is summarized by binary comparisons, leaving behind useful information about how people make decisions when they face more than two choices at once. This richer scope utilizes behavior from large choice sets to help us further understand anomalies traditionally found in binary choice.

The remainder of the paper is organized as follows. In Section 2, we provide the axioms and the representation theorem for a generic ordered-reference dependent utility representations. Later in that section, we introduce a companion result to incorporate the accommodation of properties other than WARP, and a template for additional structure in the reference order R. Section 3, Section 4, and Section 5 each provides a representation theorem under this unified framework for the risk, time, and social preference settings respectively, discusses the model's implications, as well as compares it to related models in the literature.

2 Ordered-Reference Dependence

We start with most general model, in which a decision maker chooses from generic alternatives.

2.1 Reference Dependent Choice

We introduce a reference-based approach of imposing a standard behavioral postulate. In this section, said postulate is WARP.

Let Y be an arbitrary set of alternatives, $\mathcal{A}$ the set of all finite and nonempty subsets of Y, and $c : \mathcal{A} \to \mathcal{A}$, $c(A) \subseteq A$, a choice correspondence. Recall that c satisfies WARP if for all choices problems A, B such that $B \subset A$, $c(A) \cap B \neq \emptyset$ implies $c(A) \cap B = c(B)$.[6]

Even though choices may violate WARP, it may still be the case that they comply with it among a subset of all choice problems $\mathcal{S} \subset \mathcal{A}$. We define this notion formally.

Definition 1. Let $c : \mathcal{A} \to \mathcal{A}$ be a choice correspondence and $\mathcal{S} \subseteq \mathcal{A}$. We say c satisfies *WARP* over $\mathcal{S}$ if for all $A, B \in \mathcal{S}$,

$$B \subset A,\ c(A) \cap B \neq \emptyset \Rightarrow c(A) \cap B = c(B).$$

WARP is hence equivalent to the statement "c satisfies WARP over $\mathcal{A}$."

Our first axiom is a reference-based generalization of WARP.

Axiom 1 (Reference Dependence (RD))**.** *For every choice problem $A \in \mathcal{A}$, there exists an alternative $x \in A$ such that c satisfies WARP over $\mathcal{S} = \{B \subseteq A : x \in B\}$.*

Note that this axiom generalizes WARP, since "c satisfies WARP over $\mathcal{A}$" implies "c satisfies WARP over $\mathcal{S}$" for any $\mathcal{S} \subseteq \mathcal{A}$.

[6]For an arbitrary $\mathcal{A}$, this definition of WARP is weaker than another popular version: $x \in c(A)$, $y \in c(B)$, and $x, y \in A \cap B$ implies $x \in c(B)$. They are equivalent whenever $\mathcal{A}$ contains all doubletons and tripletons subsets of Y.

We explain the intuition of Axiom 1. Suppose choices between choice problems A and $B\,(\subset A)$ violate WARP; for example, $y \in c\,(A)$ but $y \in B \backslash c\,(B)$. We postulate that this is due to a change in reference point. Specifically, that the reference alternative of A must have been removed when take subset B of A, that is, it is in the set $A \backslash B$. Then, a natural limitation of WARP violations arise: have we *not* removed the reference alternative of A when taking an arbitrary subset B of A, choices would have complied with WARP. To put it differently, suppose that when taking subsets of A, if by preserving some alternative x in this process choices from these subsets comply with WARP. x is hence an endogenous candidate for "the reference alternative of A".[7] Axiom 1 demands that every choice problem contains (at least) one candidate alternative that achieves this.

Next we provide an example of compliance. Consider the following choice correspondence for $Y = \{a, b, c, d\}$, where the notation $\{a, \underline{b}, c, d\}$ means b is chosen from the choice problem $\{a, b, c, d\}$.

$$\{a, \underline{b}, c, d\}$$
$$\{a, \underline{b}, c\} \quad \{a, \underline{b}, d\} \quad \{a, c, \underline{d}\} \quad \{\underline{b}, \underline{c}, d\}$$
$$\{a, \underline{b}\} \quad \{\underline{a}, c\} \quad \{a, \underline{d}\} \quad \{\underline{b}, c\} \quad \{\underline{b}, d\} \quad \{\underline{c}, d\}$$

This choice correspondence does not satisfy WARP globally (there are three instances of WARP violations: (i) between $\{a, \underline{b}, c, d\}$ and $\{\underline{b}, \underline{c}, d\}$, (ii) $\{\underline{b}, \underline{c}, d\}$ and $\{\underline{b}, c\}$, and (iii) $\{a, c, \underline{d}\}$ and $\{\underline{c}, d\}$). Yet WARP is satisfied from choice sets that contain a. To reconcile with Axiom 1, when $S = Y$, a is a candidate reference alternative. This is also true for any choice set S that contains a. Likewise, for $S = \{b, c, d\}$, d is a candidate reference, and this is true for any choice set S that contains d but not a. The only choice set left to be checked is $S = \{b, c\}$, but since the only non-singleton subset of $\{b, c\}$ is itself, WARP is trivial.

Although Axiom 1 allows for WARP violations, it is falsifiable as long as $|Y| \geq 3$ (i.e., as soon as WARP is non-trivial). For example, the following choice correspondence violates Axiom 1.

$$\{a, \underline{b}, c\} \quad \{\underline{a}, b\} \quad \{b, \underline{c}\} \quad \{\underline{a}, c\}$$

In this example, instances of WARP violations are (i) between $\{a, \underline{b}, c\}$ and $\{\underline{a}, b\}$ and (ii) between $\{a, \underline{b}, c\}$ and $\{b, \underline{c}\}$. So when $A = \{a, b, c\}$, a does not preserve WARP since

[7] Using the language in Ok et al. (2015), this alternative can be called a *potential reference alternative* of A.

the first instance is not excluded, b does not preserve WARP since neither instance is excluded, and c does not preserve WARP since the second instance is not excluded. Hence the axiom does not hold.

Another way of "measuring" falsifiability is to count the number of observations (choice problems) required to falsify an axiom. For standard WARP that number is 2: for example, when WARP is violated between $\{a, \underline{b}, c\}$ and $\{\underline{a}, b\}$. Whereas for Axiom 1, a weakening of WARP, that number is 3: for example $\{a, \underline{b}, c\}$, $\{\underline{a}, b\}$, and $\{\underline{a}, c\}$, since the reference of $\{a, b, c\}$ is in $\{a, b\}$ and/or $\{a, c\}$, but WARP is violated both between $\{a, \underline{b}, c\}$, $\{\underline{a}, b\}$ and between $\{a, \underline{b}, c\}$, $\{\underline{c}, b\}$.[8] Thus reference dependence makes Axiom 1 harder to reject relative to WARP by one additional observation.

When Y is infinite, we also assume Continuity. Say (Y, d) is a metric space.

Axiom 2 (Continuity)**.** *We say $c : \mathcal{A} \to \mathcal{A}$ satisfies Continuity if it has a closed-graph (with respect to the Hausdorff distance): $x_n \to_d x$, $A_n \to_H A$, and $x_n \in c(A_n)$ for every $n = 1, 2, ...$ implies $x \in c(A)$.*[9]

2.2 Representation theorem

Let R be a complete and transitive binary relation, $\arg\max_{x \in A} R$ denotes the set $\{x \in A : xRy \, \forall y \in A\}$.

Definition 2 (Ordered-Reference Dependent Utility)**.** c admits an Ordered-Reference Dependent Utility (ORDU) representation if there exist a complete, transitive, and antisymmetric reference order R on Y and a set of reference-indexed utility functions $\{u_x : Y \to \mathbb{R}\}_{x \in A}$ such that

$$c(A) = \arg\max_{y \in A} u_{r(A)}(y),$$

where $r(A) = \arg\max_{x \in A} R$.

Theorem 1.

1. *Let Y be a finite set. c satisfies RD if and only if it admits an ORDU representation.*

[8]This can be generalized: Axiom 1 is falsified when there are WARP violations between A, B_1 and between A, B_2 such that $B_1 \cup B_2 = A$, where $A, B_1, B_2 \in \mathcal{A}$.

[9]By $\to_H$ we mean convergence in the Hausdorff distance, defined by $d_H(X, Y) = \max\left\{\sup_{x \in X} \inf_{y \in Y} d_2(x, y), \sup_{y \in Y} \inf_{x \in X} d_2(x, y)\right\}$.

2. *Let Y be a separable metric space. c satisfies RD and Continuity if and only if it admits an ORDU representation where $c(A) = \arg\max_{y \in A} u_{r(A)}(y)$ has a closed-graph.*

ORDU represents a special type of context-dependent preferences. A decision maker's preference may change with the choice set, but depends only on its reference alternative, characterized by *reference-dependent utilities*. Reference-dependent utilities are more restrictive than *set-dependent utilities*, where each choice problem has its own utility function.[10] When $|Y|$ is finite, there are at most $|Y|$ distinct utility functions but around $2^{|Y|}$ choice problems, and this difference increases exponentially in $|Y|$. Furthermore, a linear order, called *reference order*, uniquely pins down the reference point for each choice problem.

The reference order has natural interpretations in richer settings, as we demonstrate in the risk, time, and social preference sections. When the setting is choices over generic alternatives, an interpretation of the reference order is a subjective salience ranking of alternatives. The most salient alternative determines the underlying preference used with the problem. In this setting, it is as if that the decision maker's attention is drawn to a certain salient alternative, and her preference ranking depends on that alternative. It is the fact that her attention is not always drawn to the same (reference) alternative that gives rise to WARP violations. But when she has the same reference alternative for a set of choice problems, her choices are consistent with a stable preference ranking.

The suggestion that certain salient component of a choice problem affect choices is not new, for example Bordalo et al. (2012, 2013). In their model, alternatives have attributes, and depending on which alternatives are being compared certain attributes are more salient than others, and weighted differently, from one choice problem to another. This is the source of WARP violations in their model. In ORDU, attributes are not part of the primitive/model, allowing for a different but related characterization of salience when the modeler either does not observe attributes or do not know the relevant attributes that play in role in decision making.

Combining *reference-dependent utilities* with a *reference order* yields out-of-sample predictions. For example, when the reference alternative in choice problem A is present in the choice problem $B \subset A$, that alternative is still the reference, and the preference ranking remains the same. This is a feature of the reference order. So once we have

[10] Set-dependent utilities, that each choice problem A has a utility function $u_A(x)$ that is maximized, puts no restriction on behavior, since we can simply set $u_A(c(A)) = 1$ and $u_A(x) = 0$ for all $x \neq c(A)$.

identified the reference alternative r of A, we know choices from subsets of A that contain r use the same utility function. Due to this defining feature, ORDU neither nests nor is nested by Ok et al. (2015)'s *revealed (p)reference*, Kőszegi & Rabin (2006)'s *reference-dependent preferences* (*personal equilibrium*), Manzini & Mariotti (2007)'s *rational shortlist method*, and Masatlioglu et al. (2012)'s *choice with limited attention*, for which we elaborated and provide testable distinctions in Appendix B.

Furthermore, reference alternatives are uniquely identified whenever we observe WARP violation upon removing them, since it is *only* through changes in references points that WARP violations arise. For example if WARP is violated between A and B, and $B = A \setminus \{x\}$, then x is the reference of A. Hence it is through inconsistency or incoherence in choice with respect to WARP that we infer the presence of reference points, pin them down uniquely, and identify their "effects". Instead, if a decision maker complies with WARP, the idea that preferences are reference dependent cannot be substantiated.

Together, the model allows us to make out-of-sample predictions between two worlds: On one end, the decision maker's reference alternatives are not identifiable with choice data precisely because she satisfies WARP and maximizes a single preference ranking; on the other end, the decision maker's choices are reference-dependent and result in WARP violations, which allows us to identify reference points and subsequently make predictions using the reference order.

2.3 A unified framework for structural anomalies

ORDU has natural applications. The rest of the paper demonstrates it in the risk, time, and social preference domains. In each setting, a domain-specific interpretation is given to the reference order. In the risk setting, the minimum amount of risk the decision maker must take, as measured by the *safest* alternative in a choice problem, may influence risk aversion. In the time setting, the *earliest* available payday creates temptation for immediate consumption, which may reduce patience. In the social preference setting, the *lowest* possible Gini coefficient from a choice problem, which characterizes how equatable a distribution could have been, may induce a greater desire to share. We formally present these models in Section 3, Section 4, and Section 5 respectively.

Reference Dependence (Axiom 1) weakened WARP by demanding that WARP is satisfied among choice problems that share a reference alternative (as opposed to all choice problems). This method of generalizing an axiom is applicable not only to WARP, but also a wide range of behavioral properties defined on choice behavior. For example, we can call for compliance with the Independence condition in a similar way, where Independence is not necessarily satisfied between every two choices, but is complied with whenever the choices come from choice problems that have the same reference point. We therefore have a *reference dependence approach* of weakening an arbitrary set of postulates, which we apply to different choice domains.

In their respective sections, we adapt Axiom 1 to postulates of the form "For every choice problem A, there exists an alternative $x \in A$ such that c satisfies $\mathcal{T}$ over $\mathcal{S} = \{B \subseteq A : x \in B\}$". $\mathcal{T}$ is "WARP and Independence" for the risk domain, "WARP and Stationarity" for the time domain, and "WARP and Quasi-linearity" for the social domain.

The result is anticipated—ordered-reference expected utility, ordered-reference exponentially discounted utility, and ordered-reference quasi-linear utility. In fact, the representation theorems for all four models in the present paper start with a quintessential result in Appendix A, Lemma 2, which demonstrates the wide applicability of our approach by accommodating a class of behavioral postulates we call *finite properties*, for which WARP, Independence, Stationarity, Quasi-linearity, Transitivity, Convexity, Monotonicity, Stochastic dominance are examples. Then, complemented with additional structure on reference orders, we obtain the reference-dependent versions of the corresponding utility representations.

The next three sections are applications of this approach.

3 Risk Preference

We now turn to an application in the domain of risk, where we provide a utility representation, with axiomatic foundation, that explains increased risk aversion when safer options are present than when they are not. Consider a decision maker whose willingness to take risk depends on how much of it is avoidable, as measured by the safest alternative among those that are available. This depends on the underlying choice set: Sometimes, we have the option to fully avoid risk by keeping our asset in cash or by buying an insurance policy, and so the safest option is quite safe. In other situations, all options are risky and we are forced to take some risk, and so the safest option is quite risky. The premise of our model, in the risk setting, is that a decision maker's risk aversion may differ between these two types of choice problems in a particular way:

she could be more risk averse when risk is avoidable than when it is not.

Suggestive evidence for this behavior is present in the literature. In the well-known paradox introduced by Allais (1953), when one choice problem contains a safe option and the other does not, subjects tend to chose the safer option in the former. This observation is consistent with increased risk aversion when safer options are present. We provide a quick recap of the Allais paradox and its relevance as pertain to our model when we discuss applications. We will also present a result that shows that Allais-type behavior is the consequence of changing utility functions by concave transformations, which characterizes greater risk aversion under the expected utility form.

In a separate setting meant to test for the *compromise effect*, Herne (1999) showed that the presence of a safer option results in WARP violations in the direction of more risk averse behavior. Wakker & Deneffe (1996) introduced the *tradeoff method* to elicit risk aversion without using a sure prize and showed that the estimated utility functions are in general less concave relative to the standard certainty equivalent / probability equivalent methods.[11] Andreoni & Sprenger (2011) reinforces this observation when the safest option is close to certainty.

3.1 Preliminaries

Consider a finite set of prizes $X \subset \mathbb{R}$. Let $Y = \Delta(X)$ be the set of all lotteries over X endowed with the Euclidean metric d_2. Let $\mathcal{A}$ be the set of all finite and nonempty subsets of $\Delta(X)$. We call $A \in \mathcal{A}$ a choice problem. We take as primitive a choice correspondence $c : \mathcal{A} \to \mathcal{A}$ that gives, for each choice problem A, a subset $c(A) \subseteq A$.

We assume throughout that c satisfies first order stochastic dominance.

Axiom 3 (FOSD)**.** *For any $p, q \in \Delta(X)$ such that $p \neq q$, if p first order stochastically dominates q, then $p \in A$ implies $q \notin c(A)$.*

Notations: Per convention, δ_x denotes the lottery that gives prize $x \in X$ with probability 1. For $p, q \in \Delta(X)$ and $\alpha \in [0,1]$, we denote by $p^\alpha q$ the convex combination $\alpha p \oplus (1-\alpha) q \in \Delta(X)$. Let $b := \max_{\geq} X$ and $w := \min_{\geq} X$ denote the highest and lowest prizes respectively. Finally, we denote by $p(x)$ the probability that lottery p gives prize $x \in X$, where $\sum_{x \in X} p(x) = 1$.

[11] *Certainty equivalent method* finds the value of a sure prize such that a subject is indifferent to a fixed lottery. *Probability equivalent metho*d fixes the sure prize and alters the probability of a lottery until the subject is indifferent. *Tradeoff method* finds the indifferent point between two lotteries by varying one of the prizes.

3.2 Risk Reference Dependence

Recall that in Section 2 we defined what it means for WARP to hold on an arbitrary set of choice problems. We now do the same for *Independence.*

Definition 3. Let $c : \mathcal{A} \to \mathcal{A}$ be a choice correspondence and $\mathcal{S} \subseteq \mathcal{A}$. We say c satisfies *Independence* over $\mathcal{S}$ if for all $A, B \in \mathcal{S}$ and $\alpha \in (0,1)$,

1. $p \in c(A)$, $q \in A$, $q^\alpha s \in c(B)$ and $p^\alpha s \in B \Rightarrow p^\alpha s \in c(B)$, and
2. $p^\alpha s \in c(A)$, $q^\alpha s \in A$, $q \in c(B)$ and $p \in B \Rightarrow p \in c(B)$.

In standard expected utility, c satisfies WARP and Independence over $\mathcal{A}$.

We depart from standard expected utility and focus on behavior where risk aversion depends on the *safest available alternatives* even though WARP and Independence are complied with whenever a collection of choice problems share the same safest available alternative.

First we define what the *safest available alternativ*e means. We do so using two partial orders. A *mean-preserving spread* (MPS) is clearly not safest, this is our first order. However, mean-preserving spread is a (very) incomplete order, where many lotteries are left unranked. This makes it hard to predict when should WARP and Independence hold.

To account for this limitation, we also deem riskier any lottery that is an *extreme spread*, our second risk order, which we now define. We call p an extreme spread of q (denoted by $p\text{ES}q$) if $p = \beta q + (1-\beta)(\alpha(\delta_b) + (1-\alpha)(\delta_w))$ for some $\beta \in [0,1)$ and $\alpha \in (q(b), 1-q(w))$. This notion captures lotteries that assign more probability to extreme prizes while being proportionally identical for intermediate prizes. Extreme spread shares the intuition of Aumann & Serrano (2008)'s risk index (which in their paper only applies to gain-loss prospects), where lotteries are deemed safer in the "economics sense"—decision makers who are more risk averse always prefer them whenever decision makers who are less risk averse do.[12]

[12]For every q, the set of extreme spreads of q is small and lives entirely within the probability triangle that contains q, δ_b, and δ_w. In this probability triangle, it consists of all lotteries such that a more risk loving decision maker would prefer (to q) whenever a more risk averse one does, under the framework of standard expected utility. In particular, it is a superset of mean-preserving spreads in this triangle. The intuition behind this notion is that, when probabilities are allocated to the most extreme prizes, even if mean is not preserved, we should still deem the resulting lottery riskier. Note that an extreme spread need not be a mean-preserving spread, and vice versa.

Although the two risk orders never contradict each other, they are also not nested. Let $MPS(A) = \{p \in A : \exists q \in A \text{ s.t. } p\text{MPS}q\}$ and $ES(A) = \{p \in A : \exists q \in A \text{ s.t. } p\text{ES}q\}$ denote the mean-preserving spreads and extreme spreads in A respectively. The remaining lotteries in A are therefore *least risky* in the sense that no other lotteries in A are safer than them according to our risk orders.

Definition. For each $A \in \mathcal{A}$, define by $\Psi(A) := A \backslash (MPS(A) \cup ES(A))$ the set of *least risky lotteries* in A.

We now replace Reference Dependence from Section 2 with a stronger axiom that demands (i) reference-dependent compliance with both WARP *and* Independence and that (ii) the reference is a least risky lottery.

Axiom 4 (Risk Reference Dependence)**.** *For every choice problem $A \in \mathcal{A}$, there exists $p \in \Psi(A)$ such that c satisfies WARP and Independence over $\mathcal{S} = \{B \subseteq A : p \in B\}$.*

Axiom 4 identifies a candidate reference for choice problem A. If $\Psi(A) = \{p\}$, and B_1 and B_2 are subsets of A containing p, then neither a violation of WARP nor a violation of Independence is produced between $c(B_1)$ and $c(B_2)$.

Like Reference Dependence (Axiom 1 in Section 2), Risk Reference Dependence postulates that there is a reference point, in the sense that WARP holds in its presence. But it additionally postulates that Independence also holds, and that this reference point is in $\Psi(A)$—a least risky lottery.[13]

Clearly, Axiom 4 weakens the axioms of standard expected utility, which demands compliance of WARP and Independence over the entire $\mathcal{A}$.

3.3 Avoidable Risk

Our next axiom captures the behavioral property that reflects increased risk aversion when more options are available. Consider the following axiom.

Axiom 5 (Avoidable Risk)**.** *For any choice problems $A, B \in \mathcal{A}$ such that $B \subset A$,*

$$\delta^\alpha r \in c(B), \; p^\alpha r \in B, p^\beta q \in c(A), \; \delta^\beta q \in A \;\Rightarrow\; \delta^\beta q \in c(A),$$

[13]This is where the decision maker's subjectivity enters the model: For two lotteries not ranked by objective notions of risk, one individual may deem one lottery riskier, whereas another individual disagrees. The axiom demands that a reference point exists and is a least risky alternative, but in instances where $|\Psi(A)| > 1$, the decision maker's choices subjectively determine which lottery in $\Psi(A)$ is the reference.

where $p, q, r \in \Delta(X)$, δ is a degenerate lottery, and $\alpha, \beta \in [0,1]$.

It is standard that a preference relation $\succsim_1$ is deemed *more risk averse* than another $\succsim_2$ if for any degenerate alternative δ and lottery p, $\delta \succsim_2 p \Rightarrow \delta \succsim_1 p$. We extend this definition to lotteries that are not entirely riskless, but differ by a degenerate and (possibly) non-degenerate components: $\delta^\alpha q$ and $p^\alpha q$ (where δ is a degenerate alternative).[14] Under standard expected utility, the two notions coincide, and this extension is without loss.[15] It is precisely because we depart from the standard expected utility model that we require this extended definition—the choice between δ and p does not pin down the choice between $\delta^\alpha q$ and $p^\alpha q$ due to changing risk aversion.

Axiom 5 postulates that a decision maker is *not more risk loving* when a choice problem expands. When new alternatives are added to an existing choice problem, they can only increase the amount of risk that is avoidable, and therefore a decision maker may view risk less favorably and become more risk averse.

Standard expected utility satisfies this axiom trivially. An expected utility maximizer can neither be more risk loving nor more risk averse between any two choice problems, a consequence of the Independence axiom. Therefore, our departure is in fact very limited; of the various ways in which Independence can fail, we only permit a specific kind of failure: increased risk aversion.

The final two axioms are standard: that choice is continuous (defined in Section 2) and abides by first order stochastic dominance.

3.4 Representation theorem

We now introduce the utility representation.

Definition. We say an order R is risk-consistent if, whenever (i) p is a mean-preserving spread of q or (ii) p is an extreme spread of q (or both), we have qRp.

Definition 4. c admits an Avoidable Risk Expected Utility (AREU) representation if there exist (i) a complete, transitive, and antisymmetric reference order R on $\Delta(X)$

[14] $p^\alpha q$ is obtained from $\delta^\alpha q$ by moving probabilities from one prize to one or more prizes. We hence deem $\delta^\alpha q$ safer than $p^\alpha q$, and say that a *more risk averse* decision maker prefers $\delta^\beta q$ to $p^\beta q$ whenever a less risk averse decision maker prefers $\delta^\alpha r$ to $p^\alpha r$.

[15] This is the consequence of the Independence axiom of standard expected utility, in which $\delta^\alpha q$ is chosen over $p^\alpha q$ if and only if $\delta^\beta r$ is chosen over $p^\beta r$ if and only if δ is chosen over p.

and (ii) a set of strictly increasing utility functions $\{u_p : X \to [0,1]\}_{p\in\Delta(X)}$, such that

$$c(A) = \arg\max_{p\in A} \mathbb{E}_p u_{r(A)}(x),$$

where

- $r(A) = \arg\max_{q\in A} R$,
- R is risk-consistent,
- qRp implies $u_q = f \circ u_p$ for some concave f:$[0,1] \to [0,1]$,
- $\arg\max_{p\in A} \mathbb{E}_p \left[u_{r(A)}(x)\right]$ has a closed-graph.

Theorem 2. *Let $c : \mathcal{A} \to \mathcal{A}$ be a choice correspondence. The following are equivalent:*

1. *c satisfies Risk Reference Dependence, Avoidable Risk, FOSD and Continuity.*
2. *c admits an AREU representation.*

Furthermore in every AREU representation, given R, u_p is unique for all $p \neq (\delta_b)^{\alpha}(\delta_w)$.

When choices admit an AREU representation, it is as if the decision maker goes through the following decision making process: Facing a choice problem, she first looks for the safest alternative using R, which is risk consistent—it ranks safer alternatives higher. This determines the (Bernoulli) utility function for the choice problem and she proceeds to choose the option that maximizes expected utility. Moreover, the safer the reference, a more concave utility function is used, resulting in weakly more risk averse choices. This generalizes the standard model where a decision maker chooses the option that maximizes expected utility using a single utility function. It departs from standard expected utility by allowing greater risk aversion when alternatives are added to a choice set, but prohibits any other types of preference changes.

Note that utility functions in AREU are generically unique (up to an affine transformation). This property guarantees that their relationships by concave transformations are not arbitrary, and choices manifest changing risk aversions. Here, each utility function u_p is used to evaluate options for a set of choice problems that deem p as the safest alternative. When $p \neq (\delta_b)^{\alpha}(\delta_w)$, there are many of these choice problems in which p is not the chosen alternative, making u_p non-arbitrary.

3.5 Applications of AREU

First, we show that AREU provides a natural explanation for the Allais paradox and proposes new form of Allais paradox manifested in the form of a WARP violation.

In experimental settings, subjects tend to choose the degenerate lottery $p_1 = \delta_{3000}$ over the lottery $p_2 = 0.8\delta_{4000} + 0.2\delta_0$, but choose $q_2 = 0.2\delta_{4000} + 0.8\delta_0$ over $q_1 = 0.25\delta_{3000} + 0.75\delta_0$. This is called the Allais *common ratio effect*, a prominent "anomaly" in the study of choices under uncertainty.[16]

AREU explains this behavior using changes in risk aversion. Given a reference order R that deems the safest alternative in the first choice problem—in which a sure prize is available—as safer than the safest alternative in the second choice problem, a decision maker is more risk averse and prefers p_1 over p_2 but not q_1 over q_2. That is, where $A = \{p_1, p_2\}$ and $B = \{q_1, q_2\}$, we have $u_{r(A)} = f \circ u_{r(B)}$ for some concave transform f.[17] It is because of this change in utility function characterizing increased risk aversion that makes p_1, the safe option, appealing in the first choice problem. For the same reason, violation of expected utility theory in the opposite direction—choices of p_2 and q_1—is ruled out, making increased risk aversion the only form of expected utility violations permitted in AREU.[18] Moreover, it captures both the *common ratio effect* and the *common consequence effect*, and the lotteries involved can be generalized.[19]

When a decision maker faces more than two choices at once, AREU predicts WARP violations that resemble the Allais paradox. Consider a decision maker who prefers $p_1 = \delta_{3000}$ to $p_2 = 0.5\delta_{4000} + 0.5\delta_0$, $q_2 = 0.4\delta_{4000} + 0.3\delta_{3000} + 0.3\delta_0$ to $q_1 = 0.2\delta_{4000} + 0.7\delta_{3000} + 0.1\delta_0$, and q_1 to p_1. The first two choices correspond to the standard Allais

[16]This paradox is introduced by Allais (1953). This example is taken from Starmer (2000). Note that the second pair of options are derived from the first pair using a common mixture, $q_1 = 0.2p_1 + 0.8\delta_0$ and $q_2 = 0.2p_2 + 0.8\delta_0$. Under expected utility theory, those who prefer p_1 to p_2 should prefer q_1 to q_2, and vice versa. Hence choices of p_1 and q_2 is a direct contradiction of expected utility theory. Camerer (1995) and Starmer (2000) provide an in-depth survey.

[17]After normalization ($u_A(0) = 0$ and $u_A(4000) = 1$), $u_A(3000) > 0.8$ and $u_b(3000) < 0.8$. This come from solving $u_A(3000) > 0.8u_A(4000) + 0.2u_A(\$0)$ and $0.25u_B(3000) + 0.75u_B(0) > 0.2u_B(4000) + 0.8u_B(0)$ for $u_A(3000)$ and $u_B(3000)$. Since $u_A(3000) > u_B(3000)$, we conclude $u_A = f \circ u_B$ for some concave $f : [0,1] \to [0,1]$.

[18]This is consistent with the behavioral postulate referred to as *Negative Certainty Independence* in Dillenberger (2010); Cerreia-Vioglio et al. (2015) and Kahneman & Tversky (1979)'s *certainty effect*.

[19]Consider a degenerate lottery δ and a lottery p such that neither of them first order stochastically dominates another. Consider the lotteries $\delta' = \delta^\alpha q$ and $p' = p^\alpha q$ for some $\alpha \in (0,1)$ and lottery q and suppose $|X| = 3$. If $\delta \in c(\{\delta, p\})$ and $p' \in c(\{\delta', p'\})$, then for all $u_1, u_2 : X \to \mathbb{R}$ such that u_1 explains the first choice and u_2 explains the second choice, it is straightforward to show that $u_1 = f \circ u_2$ for some concave function $f : \mathbb{R} \to \mathbb{R}$. Moreover, the choices admit an AREU representation such that $r(\{\delta, p\}) R r(\{\delta', p'\})$. Conversely, suppose the choices $c(\{\delta, p\})$ and $c(\{\delta', p'\})$ admit an AREU representation. If $p \in c(\{\delta, p\})$, then $p \in c(\{\delta', p'\})$. If $\delta' \in c(\{\delta', p'\})$, then $\delta \in c(\{\delta, p\})$.

common ratio effect. Now we depart from binary choice and consider the choice problem $\{p_1, q_1, q_2\}$. AREU predicts that the availability of p_1 reduces risk aversion and causes the decision maker to prefer q_1 over q_2—the same way it effects the choice of p_1 over p_2. Therefore, q_1 will be chosen from this choice set, generating a WARP violation since q_2 is chosen over q_1 in binary choice. This observation connects the failure of rationality to the Allais paradox. In AREU, the driving force of the Allais paradox can be manifested in the form of a WARP violation provided that the right choice sets are considered, offering new empirical directions.

Last we consider another application of AREU. A known phenomenon in behavioral finance is *reaching for yield*, in which investors invest less when the risk-free rate is higher, which is at odds with the standard expected utility model with commonly used specifications such as those that exhibit constant relative risk aversion. Lian et al. (2017) shows that this behavior is at odds with utility functions exhibiting constant or decreasing absolute risk aversion, capturing a large class of utility functions typically used in behavioral finance. The authors also provided evidence of this behavior.

AREU is consistent with this observation, where the addition of a *better* sure prize increases risk aversion. Consider a choice set A that contains a sure prize of \$4 and choice set B obtained from A with an additional option: a sure prize of \$7. Although risk is fully avoidable in both choice problems, the decision maker may find risk less appealing overall since a better sure prize is now available. AREU captures this behavior when $\delta_x R \delta_y$ if $x > y$. Then, the decision maker maximizes expected utility with a more concave utility function when a better sure prize is present. This predicts WARP violation in the direction of increased risk aversion, where a riskier option is chosen over a safer option in A, but the safer option is chosen in B.

3.6 Linkage between WARP violation and Independence violation

Risk Reference Dependence (Axiom 4) assumes WARP and Independence over certain subsets of all choice problems. We now study the consequences of imposing each of these assumptions over the entire set of choice problems $\mathcal{A}$.

It turns out adding any one of WARP and Independence brings us back to standard expected utility. This suggests a formal separation of AREU from a wide range of non-expected utility models in which WARP holds. It also suggests that, in our model, violation of Independence is a matter of *changing preferences*.

As is standard, we say c admits a utility representation if there exists a real valued utility function $U : \Delta(X) \rightarrow \mathbb{R}$ such that $c(A) = \arg\max_{p \in A} U(p)$.

Proposition 1. *Suppose c admits an AREU representation. The following are equivalent:*

1. *c satisfies WARP (over $\mathcal{A}$).*
2. *c satisfies Independence (over $\mathcal{A}$).*
3. *c admits an expected utility representation.*
4. *c admits a utility representation.*

AREU seeks to explain non-EU behavior as a consequence of *changing preferences*, typically symbolized by the failure of WARP. To this end, Proposition 1 shows that AREU leaves no explanatory power in explaining the violations separately. Instead, violation of standard rationality (WARP) and structural violations (Independence) are inextricably linked to one another, and resolving either one will bring us back to standard expected utility.

Similarly, a choice correspondence that admits utility representation is often interpreted as the consequence of a stable preference ranking. Proposition 1 connects AREU's motivation with this interpretation, that expected utility ensues when preferences over lotteries are stable, and failure of expected utility is entirely due to changing preferences.

This also sets us apart from non-EU models that retains WARP. In those cases, a single utility function is maximized, but it need not take the expected utility form. In our case, choices come from utility functions that conform with the expected utility form, but there are multiple of them. Therefore, AREU is the result of a *joint* weakening of both WARP and Independence, the core idea of the framework this paper proposes, and new testable predictions follow. The same results and arguments will hold in the time and social domain (Section 4 and Section 5).

3.7 AREU, Transitivity, Betweenness

Although Proposition 1 provides a strong separation between AREU and many non-EU models, we can more meaningfully recover the extent to which AREU is related to other models by imposing Transitivity *partially.* To this end, we turn our attention to Marschak-Machina triangles (also “probability triangle”) for the next part of our analysis.

We will show that AREU is in fact very close to (but is not sufficient for) Betweenness, a well-known property introduced (on preference relations) by Chew (1983); Fishburn (1983); Dekel (1986).[20] Like expected utility, models of betweenness preferences have the characteristic of linear indifference curves, but indifference curves need not be parallel. Since Betweenness and Transitivity are typically defined on a preference relation, we first proceed to define Betweenness and Transitivity on a choice correspondence.

Definition 5. Let $c : \mathcal{A} \to \mathcal{A}$ be a choice correspondence. We say c satisfies *Betweenness* over $\mathcal{S} \subseteq \mathcal{A}$ if for any $\{p, q\}, \{p, p^\alpha q\}, \{p^\alpha q, q\} \in \mathcal{S}$ and $\alpha \in (0, 1)$,

1. $c(\{p, q\}) = \{p\} \Rightarrow c(\{p, p^\alpha q\}) = \{p\}$ and $c(\{p^\alpha q, q\}) = \{p^\alpha q\}$,
2. $c(\{p, q\}) = \{p, q\} \Rightarrow c(\{p, p^\alpha q\}) = \{p, p^\alpha q\}$ and $c(\{p^\alpha q, q\}) = \{p^\alpha q, q\}$.

Definition 6. Let $c : \mathcal{A} \to \mathcal{A}$ be a choice correspondence. We say c satisfies *Transitivity* over $\mathcal{S} \subseteq \mathcal{A}$ if for any $\{p, q\}, \{q, s\}, \{s, p\} \in \mathcal{S}$,

$$p \in c(\{p, q\}) \text{ and } q \in c(\{q, s\}) \Rightarrow p \in c(\{q, s\}).$$

Our next result uses the following terminology. We will focus our analysis on Marschak-Machina triangles. For any three prizes $\{a, b, c\} \subseteq X$, consider the set of all lotteries induced by them, $\Delta(\{a, b, c\})$. Let $\mathcal{B}_{a,b,c}$ denote the set of all finite and nonempty subsets of $\Delta(\{a, b, c\})$, and therefore $\mathcal{B}_{a,b,c} \subseteq \mathcal{A}$. Going forward, we omit subscripts and use the notation $\mathcal{B}$. Also, let p be a mean-preserving spread of q. We say c is *weakly risk averse* (resp. *risk loving*) *over* $\mathcal{B}$ if $\{p, q\} \in \mathcal{B}$ implies $q \in c(A)$ (resp. $p \in c(A)$). If $c(\{p, q\}) = \{p, q\}$ whenever $\{p, q\} \in \mathcal{B}$, we say c is *risk neutral* over $\mathcal{B}$. We say that indifference curves *fan out* (resp. *fan in*) if they become weakly steeper (resp. flatter) in the first order stochastic dominance direction.

Proposition 2. *Suppose c admits an AREU representation. If c satisfies Transitivity over a Marchak-Machina triangle $\mathcal{B}$, then:*

[20]AREU does not automatically imply Betweenness preferences. Consider the Allais common ratio behavior where \$3000 for sure is chosen over 80% of \$4000 (and otherwise \$0), but 80% of \$4000 is chosen over [50% of \$3000 and 40% of \$4000]. This behavior violates Betweenness, but it is admissible by AREU where the former choice problem uses a more concave utility function.

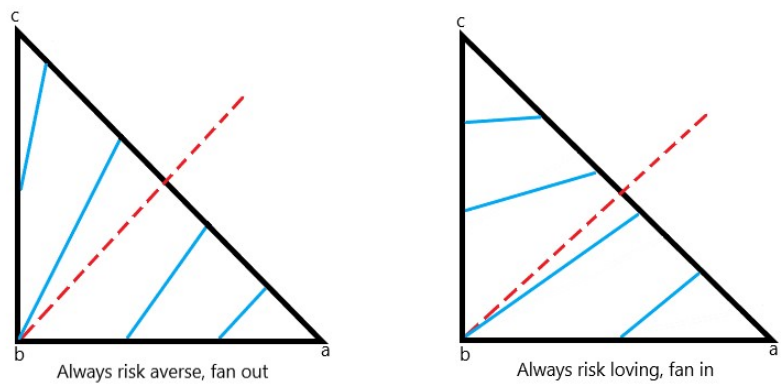

Figure 3.1: Let $a < b < c$. Dotted (red) lines are the mean-preserving spread lines. Solid (blue) lines are indifferent curves. Referring to Proposition 2, the picture on the left corresponds to point 3 and the picture on the right corresponds to point 4.

1. *c satisfies Betweenness over $\mathcal{B}$.*

2. *c is either weakly risk averse over $\mathcal{B}$, weakly risk loving over $\mathcal{B}$, or risk neutral over $\mathcal{B}$.*

3. *Indifference curves fan out if c is weakly risk averse over $\mathcal{B}$.*

4. *Indifference curves fan in if c is weakly risk loving over $\mathcal{B}$.*

Proposition 2 connects AREU to linear indifferent curves—a property of standard expected utility—and pins down the set of admissible indifferent curves.

While AREU allows a decision maker to have varying magnitudes of risk aversion, Transitivity puts a bound on this variation so that choices are either exclusively risk averse or exclusively risk loving (in this probability triangle). In each case, a particular direction of fanning is also prescribed (Figure 3.1), suggesting that risk averse (resp. risk loving) agents become even more risk averse (resp. risk loving) when lotteries become better.

These results provide testable predictions for AREU, and separates it from other models, which we discuss next.

3.8 Related literature

Various alternatives to expected utility were introduced by Quiggin (1982), Chew (1983); Fishburn (1983); Dekel (1986), Bell (1985); Loomes & Sugden (1986), Gul

(1991), Kőszegi & Rabin (2007), and Cerreia-Vioglio et al. (2015). We now use Proposition 1 and Proposition 2 to study their relationship to AREU.

The AREU model has a close relationship with *betweenness preferences* introduced by Chew (1983); Fishburn (1983); Dekel (1986). Although the two intersect only at expected utility, a direct implication of Proposition 1, the two make similar predictions for binary choices when Transitivity is added to AREU in a probability triangle.

Among models of betweenness preferences, Gul (1991)'s *disappointment aversion* is closest in spirit to AREU, but the two predict different behavior. In *disappointment aversion*, the set of possible outcomes of each lottery is decomposed into elevation prizes and disappointment prizes, and the utilities from disappointment prizes are discounted using a function of the probability of disappointment. An implication of disappointment aversion is the property of mixed fanning, in which indifference curves first fan in and then fan out, for example. AREU cannot accommodate mixed fanning, a direct application of Proposition 2, and so the two models differ in their coverage of non-expected utility behavior.

For the same reason, AREU and Cerreia-Vioglio et al. (2015)'s *cautious expected utility* put fourth different behavioral predictions. In their model, a decision maker evaluate each lottery as its worst certainty equivalence under a set of (Bernoulli) utility functions. The result is a behavior that resembles cautiousness. A property resembling mixed fanning is an implication of their model, where indifferent curves are steepest in the middle, a consequence of the axiom *Negative Certainty Independence*: $p \succsim \delta$ implies $p^\alpha q \succsim \delta^\alpha q$.

Like AREU, Kőszegi & Rabin (2007)'s *reference-dependent risk preferences* uses reference points to explain non-expected utility behavior. However, both the identification of reference points and the consequence of changing reference points differ. In AREU, reference alternatives are given by the safest alternatives in choice problems, and they serve as a proxy for changing risk preferences. In Kőszegi & Rabin (2007), a decision maker is subjected to gain-loss utility relative to a reference point, where the reference point is the lottery she expects to receive. We focus on *choice-acclimating personal equilibrium* (CPE), in which reference points are endogenously set as the eventually-chosen alternatives. Masatlioglu & Raymond (2016) shows that when a CPE specification satisfies first order stochastic dominance, the implied behavior can be explained by the *quadratic utility* functionals of Machina (1982); Chew et al. (1991). Yet, Chew et al. (1991) demonstrates that quadratic functionals intersect with betweenness preferences only at expected utility, and hence the CPE model of Kőszegi & Rabin (2007) intersects with AREU only at expected utility.[21]

The model closest to AREU, to my knowledge, is the *context-dependent gambling effect* by Bleichrodt & Schmidt (2002). In their model, a decision maker's preferences are explained by two (Bernoulli) utility functions, one for comparisons that involve a riskless option and another for the rest. Unlike AREU, their model only applies to binary decisions, which results in different axioms and applicability. Furthermore, when a degenerate lottery is slightly perturbed into a non-degenerate one, it produces a choice reversal, which seems implausible. Their model also does not accommodate violations of expected utility in choice problems without a riskless option, such as variations of the Allais paradox. Finally, while their axioms are separately imposed on binary decisions involving and not involving riskless options, our axioms are imposed on the choice correspondence without such discrimination.

4 Time Preference

Next, we provide an application of our unified framework for choices over delayed consumption. The canonical model for this setting is Discounted Utility, axiomatized by Fishburn & Rubinstein (1982), in which a decision maker evaluates each payment-time pair (x,t) by $\delta^t u(x)$. However, Discounted Utility has routinely faced empirical challenges as subjects tend to violate the Stationarity condition: The choice between two payments switches when the decision is made in advance, typically favoring the later option for the long-term decisions.[22] This effect is called the *present bias*.

In this section, we expand the scope of present bias to study reduced patience as a result of a WARP-violating *preference change*. We do so by weakening the axioms of Fishburn & Rubinstein (1982) using an approach analogous to Section 2's Reference Dependence. The outcome is a utility representation in which choices maximize exponentially discounted utilities using a discount factor that depends on the earliest available payday.

Although present bias has been explained by various models that reflect reduced patience for short-term decisions, this is often done under the assumption of WARP. This falls short of allowing us to study present bias as a matter of preference change,

[21] Similar conclusions of non-intersection with AREU (other than expected utility) can be made for Quiggin (1982)'s *rank dependent utility* (see Chew & Epstein (1989)) and Bell (1985); Loomes & Sugden (1986)'s *disappointment theory*. Some of these results, and a comprehensive summary, are provided by Masatlioglu & Raymond (2016).

[22] See for example Laibson (1997), Frederick et al. (2002), and Benhabib et al. (2010).

where the availability of certain options induces an overall decrease in patience that is manifested in a WARP violation.

To illustrate, suppose \$20 in 4 days is chosen over \$18 in 3 days, but \$18 today is chosen over \$20 tomorrow; this familiar choice pattern is present bias. Now suppose the decision maker encounters a new choice problem where she faces the choice of \$15 today, \$18 in 3 days, or \$20 in 4 days. In this situation, she might find it tempting to choose \$18 even though she would have chosen \$20 if \$15 were not available. This behavior reflects reduced patience in the form of a WARP violation. It is also consistent with intuition behind present bias, that the availability of something immediate induces impatience.

4.1 Preliminaries

Let $X = [a, b] \subset \mathbb{R}_{>0}$ where $b > a$ be an interval of positive payments and let $T = [0, \bar{t}] \subset \mathbb{R}_{+}$ where $\bar{t} > 0$ be an interval of non-negative time points. $Y = X \times T$ is the set of alternatives, in which an alternative $(x, t) \in X \times T$ is a (single) payment of x that arrives at time t. We endow $X \times T$ with the standard Euclidean metric. Let $\mathcal{A}$ be the set of all finite and nonempty subsets of $X \times T$. Finally, let $c : \mathcal{A} \to \mathcal{A}$, $c(A) \subseteq A$, be a choice correspondence. To avoid redundancy, we assume throughout that $(b, \bar{t}) \in c(\{(a, 0), (b, \bar{t})\})$, that is, it is possible to find payment at time $\bar{t}$ that the decision maker would choose when compared to a payment at time 0.

We maintain the following standard axioms for time preferences, that higher payments and sooner payments are always chosen from binary choice problems.

Axiom 6.

1. *Outcome Monotonicity: if* $x > y$, *then* $c(\{(x, t), (y, t)\}) = \{(x, t)\}$.
2. *Impatience: if* $t < s$, *then* $c(\{(x, t), (x, s)\}) = \{(x, t)\}$.

4.2 Time Reference Dependence

Time consistency in choice is captured by a well-known behavioral property called Stationarity. Under Stationarity, a decision maker's preference between two future payments is consistent regardless of when the decision is made. For this reason, Stationarity is often deemed a normative postulate in economic analysis.

Similar to what we did to weaken WARP and Independence in previous sections, we first define what it means for a choice correspondence c to satisfy Stationarity over a subset of all choice problems.

Definition 7. Let $c : \mathcal{A} \to \mathcal{A}$ be a choice correspondence and $\mathcal{S} \subseteq \mathcal{A}$. We say c satisfies *Stationarity* over $\mathcal{S}$ if for all $A, B \in \mathcal{S}$, $a > 0$,

$$(x,t) \in c(A)\,,\ (y,q) \in A,\ (y,q+a) \in c(B)\,, \text{ and } (x,t+a) \in B \Rightarrow (x,t+a) \in c(B)\,.$$

Supplied with Axiom 6, a direct adaption of Fishburn & Rubinstein (1982) into the framework of choice gives that c satisfies WARP and Stationarity over $\mathcal{A}$ if and only if it admits a (exponential) Discounted Utility representation.

A choice correspondence that exhibits time inconsistency fails to satisfy Stationarity over $\mathcal{A}$. However, the choice correspondence may still satisfy Stationary over some subsets of $\mathcal{A}$. Consider the following axiom, which states that Stationarity is satisfied between any two choice problems that share an earliest payment.

Definition 8. For each $A \in \mathcal{A}$, define by $\Psi(A) := \{(x,t) \in A : t \leq q \text{ for all } (y,q) \in A\}$ the set of *earliest payments* in A.

Axiom 7 (Time Reference Dependence)**.** *For any $A, B \in \mathcal{A}$, if $\Psi(A) \cap \Psi(B) \neq \emptyset$ (A and B share an earliest payment), then c satisfies WARP and Stationary over $\{A, B\}$.*

The axiom posits that a violation of WARP and Stationarity between two choice problems can only occur if they do not share an earliest payment. If we interpret compliance with Stationarity as having a stable level of patience, the axiom proposes that patience may depend on how soon any payment can be attained. This allows us to capture behavior in which compliance with WARP and Stationarity is not necessarily upheld between long-term and short-term choice problems, such as those exhibited in time consistency experiments.

Note that this postulate can be rewritten in the style of Reference Dependence (Axiom 1) and Risk Reference Dependence (Axiom 4) from previous sections, stated formally in the following lemma.

Lemma 1. *Fix a choice correspondence c, the following are equivalent.*

1. *c satisfies Axiom 7.*

2. *For every choice problem $A \in \mathcal{A}$ and every earliest payment (x, t) in it, c satisfies WARP and Stationarity over $\{B \subseteq A : (x, t) \in B\}$.*

Albeit straightforward, the lemma reassures us that the unified method of weakening standard postulates proposed in this paper is not dissimilar to demanding compliance between pairs of choice problems.

In fact, their equivalence in this setting is due to two details. First, unlike our general model (Section 2) and application in the risk domain (Section 3), in which the reference order is either fully or partly subjective, the reference points in the present setting is completely objective—the earliest payments in the choice sets. Because of this objectivity, the reference order is pinned down axiomatically, and the axiom does not involve an existential statement that allows for subjectivity in determining reference points. Second, WARP and Stationarity are properties between pairs of choices (and not more). This is not the case for all postulates. For example, Transitivity is an axiom that is trivially satisfied between any pair of choices, but a violation can be found when more choices are considered. Identifying this equivalence, and the reasons thereof, allows us to design more efficient tests of the axioms in our unified framework.

4.3 Present Bias

We postulate that patience (may) increase when options are postponed.

Consider prizes $x_1 < x_3$ arriving at time $t_1 < t_3$ respectively. We posit that by *postponing* the options by $d > 0$, the decision maker is (weakly) more patient and will choose $(x_3, t_3 + d)$ over $(x_1, t_1 + d)$ if she chose (x_3, t_3) over (x_1, t_1). The postulate differs from Stationarity as it allows for the choice of (x_1, t_1) over (x_3, t_3) but $(x_3, t_3 + d)$ over $(x_1, t_1 + d)$, or *present bias.* To summarize, it allows for violation of Stationarity in one direction but not the other.

However, this falls short of capturing changes in patience. Difference in delay aversion between individuals cannot be directly categorized into difference in discounting and difference in consumption utility, an issue discussed in Ok & Benoît (2007). Just because decision maker A chooses a sooner option, and decision maker B chooses a later one, it is not conclusive that the first decision maker discounts more. Instead, it could be due to a difference in consumption utility, where A's marginal utility for money is sufficiency lower than that of B, which induces the choice of a sooner but smaller payment.

To resolve this issue, we introduce an axiomatic observation that captures *fixed* consumption utilities under *varying* discounting/patience. This can be used to characterize a set of individuals whose consumption utility is the same but may differ in their patience levels.

Consider $c(\{(x,t),(y,q),(z,s)\}) = \{(x,t),(y,q),(z,s)\}$, where (x,t) gives the smallest payment but arrives earliest, (z,s) gives the largest payment but arrives latest, and (y,q) is intermediate in both. Now consider the new choice problem $\{(x,\lambda t),(y,\lambda q),(z,\lambda s)\}$, where $0 < \lambda < 1$; that is, all payments will now arrive at a (common) fraction of time. Under Stationarity, a decision maker only cares about the delay between the alternatives and would now strictly prefer the latest option since the time-difference between any two options is smaller. Yet it is ambiguous how a decision maker of the present model would behave. On one hand, an earlier choice problem causes the decision maker to choose more impatiently; on the other hand, delays between alternatives have decreased, which favor later options. The competing forces render the choice ambiguous. The same competing forces occur when $\lambda > 1$: the decision maker is more patient, but delays between options are larger. In these situations, we restrict the decision maker's behavior in the following way: if the decision maker chooses both the earliest and the latest alternatives after such a transformation (and recall that he was indifferent between all three before), then he also chooses the intermediate option in the new choice problem.

The same restriction is imposed when payments are uniformly delayed/advanced. This restriction is non-trivial only when the aforementioned competing forces are present: when "$\lambda < 1$ *and* $\lambda t + d < t$", where the decision maker becomes less patient but has to wait shorter for a better payment, and when "$\lambda > 1$ *and* $\lambda t + d > t$", where she is more patient but has to wait longer for a better payment.

This gives rise to the following axiom.

Axiom 8 (Present Bias)**.** *For any $t_1 < t_2 < t_3$, $A = \{(x_1,t_1),(x_2,t_2),(x_3,t_3)\}$, and $A' = \{(x_1,\lambda t_1 + d),(x_2,\lambda t_2 + d),(x_3,\lambda t_3 + d)\}$,*

1. $c(\{(x_1,t_1),(x_3,t_3)\}) = \{(x_3,t_3)\} \Rightarrow c(\{(x_1,t_1+d),(x_3,t_3+d)\}) = \{(x_3,t_3+d)\}$ *for all* $d > 0$,

2. $c(A) = A$ *and* $(x_1,\lambda t_1 + d),(x_3,\lambda t_3 + d) \in c(A') \Rightarrow (x_2,\lambda t_2 + d) \in c(A')$ *for all* $0 < \lambda < 1$ *and* $d \in \mathbb{R}$.

This postulate is trivially satisfied by a decision maker whose behavior fully complies with Stationarity, since she can neither be more patient nor less patient when options are symmetrically postponed.

Moreover, Part 2 is a necessary condition for behavior across individuals who share the same consumption utility but have varying discounting.[23]

4.4 Representation theorem

We are ready for the utility representation and representation theorem.

Definition 9. c admits a Present-Biased Discounted Utility representation (PBDU) if there exist a strictly increasing and continuous utility function $u : X \to \mathbb{R}$ and a set of time-indexed discount factors $\{\delta_t\}_{t\in T}$ such that

$$c(A) = \arg\max_{(x,t)\in A} \delta_{r(A)}^t u(x),$$

where

- $r(A) = \min\{t : (x,t) \in A\}$,
- $t < t'$ implies $\delta_t \leq \delta_{t'}$,
- $\arg\max_{(x,t)\in A} \delta_{r(A)}^t u(x)$ has a closed-graph.

Theorem 3. *Let $c : \mathcal{A} \to \mathcal{A}$ be a choice correspondence. The following are equivalent:*

1. *c satisfies Time Reference Dependence, Present Bias, Outcome Monotonicity, Impatience, and Continuity.*

2. *c admits a PBDU representation.*

Furthermore, in every PBDU representation, discount factors $\{\delta_t\}_{t\in T\setminus\{\bar{t}\}}$ are unique given u.

In this model, it is *as if* the decision maker maximizes exponentially discounted utility, but with discount factors that depend on the timing of the earliest available

[23]Take any two utility functions $\delta_A u(x)$ and $\delta_B u(x)$ such that $\delta_A, \delta_B \in (0,1)$. Suppose for some $(x_1,t_1),(x_2,t_2),(x_3,t_3)$ we have $\delta_A^{t_1} u(x_1) = \delta_A^{t_2} u(x_2) = \delta_A^{t_3} u(x_3)$. Moreover, suppose for some $0 < \lambda < 1$ and $d \in \mathbb{R}$, we have $\delta_B^{\lambda t_1+d} u(x_1) = \delta_B^{\lambda t_3+d} u(x_2)$. Then $\delta_B^{\lambda t_1+d}/\delta_B^{\lambda t_3+d} = u(x_3)/u(x_1) = \delta_A^{t_1}/\delta_A^{t_3}$, or $\delta_B = \delta_A^{1/\lambda}$. Then by $\delta_A^{t_1} u(x_1) = \delta_A^{t_2} u(x_2)$ we have $\delta_B^{\lambda t_1} u(x_1) = \delta_A^{t_1} u(x_1) = \delta_A^{t_2} u(x_2) = \delta_B^{\lambda t_2} u(x_2)$, or $\delta_B^{\lambda t_1+d} u(x_1) = \delta_B^{\lambda t_2+d} u(x_2)$.

payment. When the earliest available payment arrives sooner in one choice problem than another, then the decision maker uses a lower discount factor in the former. Since discount factors are often interpreted as a measure of *patience*, our model can be viewed as one in which the decision maker's patience changes systematically across choice problems, where she is less patient when an earlier payment is available.

This model conforms with present bias, the empirically prevalent failure of dynamic consistency in which decision makers exhibit less delay aversion for long-term decisions. Take for example the classic observation of present bias, where \$$x$ today is preferred to \$$y$ tomorrow (choice problem A) but the opposite decision is made when both payments are postponed by a year (choice problem B). The model we propose explains the behavior with the simple interpretation that, since the earliest alternative for A arrives sooner than that for B (i.e., $r(A) < r(B)$), the decision maker is less patient in the former (i.e., $\delta_{r(A)} < \delta_{r(B)}$).

Moreover, even though the choice between \$$x$ at time t and \$$y$ a day later is not consistent across the time horizon t, the model predicts that as we gradually postpone both options with s, the choice can only switch from $(x, t+s)$ to $(y, t+1+s)$. That is, if there is a point in time at which the decision maker becomes sufficiently patient to choose \$$y$ over \$$x$, she must continue to do so as we further postpone both options. This "single switching" property is the direct consequence of the fact that references are *ordered* and preference changes are unidirectional along this order.

When we move from binary choice problems to larger choice sets, PBDU predicts that present bias can be manifested in the form of a WARP violation. Consider the present bias example in which \$20 in 4 days is chosen over \$18 in 3 days, but \$18 today is chosen over \$20 tomorrow. In PBDU, this behavior is explained by the use of a lower discount factor for latter choice problem, since the earliest available payment arrives today. However, it would suggest that it is also possible to induce a choice of \$18 in 3 days over \$20 in 4 days if a lower discount factor is used. In PBDU, this is possible by introducing \$15 today as a third option, which changes the arrival of the earliest available payment from 3 days later to today. This in turn induces a lower discount factor, and \$18 in 3 may be chosen over \$20 in 4 days, even if the opposite is true be when \$15 today is not available. This behavior reflects reduced patience in the form of a WARP violation and provides a new perspective to study present bias.

Finally, the underpinning of PBDU is the simultaneous weakening of WARP and Stationarity in a reference-dependent approach. Reminiscent of the observation made in Proposition 1, WARP and Stationarity are interconnected in our model in the sense that neither of them can be independently weakened:

Proposition 3. *Suppose c admits a PBDU representation. Then the following are equivalent:*

1. *c satisfies WARP (over $\mathcal{A}$).*
2. *c satisfies Stationarity (over $\mathcal{A}$).*
3. *c admits an exponential discounting utility representation.*
4. *c admits a utility representation.*

4.5 Related models of time preferences

The biggest difference between Present-Biased Discounted Utility (PBDU) and *hyperbolic discounting,* a class of models in which future options are discounted disproportionately less, is that PBDU (when non-trivial) necessitates WARP violations and hyperbolic discounting models satisfy WARP. Furthermore, unlike models of hyperbolic discounting, PBDU evaluates all alternatives in a choice problem using a single discount factor.[24] However, the empirically informed intuition that discount factors vary across time is shared between models of hyperbolic discounting and PBDU, albeit implemented differently. For our model, PBDU, discount rate changes at the choice problem level, whereas for hyperbolic discounting it changes at the alternative level. The difference is stark when we consider choice problems that contain more than two alternatives. In hyperbolic discounting, the preference between any two options stays the same regardless of what choice problems they appear in, hence WARP is never violated. This is not the case for PBDU, where a sooner option may become superior to a later one from the introduction of a third (but not necessarily chosen) alternative, and results in WARP violations in PBDU.

Exponential discounting has advantageous properties in economic applications, propelling Laibson (1997)'s well-known *quasi-hyperbolic discounting.* In their model, behavior complies with Stationarity as long as the choice is between two future payments, and present bias only arises when an immediate payment is involved. This is not the case in PBDU, as the switch from choosing the earlier payment to choosing the later one can occur at any time as we gradually shift both payments into the future. Another

[24]See for instance Loewenstein & Prelec (1992) and Laibson (1997).

implication of quasi-hyperbolic discounting in our setting is the failure of continuity, where an instantaneous change in choice occurs when the earlier payment arrives at time 1 ("today"). Our model complies with continuity of choice, and instead forgoes WARP to explain dynamic inconsistency.

We now turn to two other models that both explain dynamic inconsistency and can explain WARP violations.

Lipman et al. (2013) provides an explanation of dynamic inconsistency that builds on Gul & Pesendorfer (2001)'s introduction of *temptation*. In Gul & Pesendorfer (2001), a decision maker has commitment utilities and temptation utilities, and chooses a menu (a choice problem) taking into account both. The result is that a larger menu may be inferior, a departure from the conventional understanding that more options should never be worse. Lipman et al. (2013) extends this to the setting of time preference and proposes that a decision maker assess current consumption using temptation utility and future consumption using commitment utility. When making decisions in advance, it is as if the decision maker is choosing between singleton menus for her future self, and the absence of temptation utility allows her to make a more patient decision relative to choices over immediate consumption. Like *quasi-hyperbolic discounting*, present-bias is restricted to immediate consumption, whereas PBDU allows present-bias to kick in at time frame and as long as its effect is persistent when both options are further postponed—the "single switching" property discussed earlier.

More recently, Freeman (2016) introduced a framework in which WARP is weakened, and reversals are explained by time-inconsistent preferences. In their model, a decision maker chooses when to complete a task, and may exhibit choice reversal when additional opportunities for completions are introduced (a expansion of the choice set). In particular, in response to the addition of an opportunity for completion, a sophisticated decision maker may choose to complete the task earlier (and never later) in anticipation that allowing her future self to make that decision would result in an eventual completion time that is worse than completing the task now. A naive decision maker, however, could only end up completing later.

Although PBDU allows for choice reversal in the direction of choosing an earlier option when the choice set expands, it is incompatible with the behavior in Freeman (2016). In PBDU, a reversal can only occur when the discount factor changes, which only happens when an alternative earlier than any other already available is added. However in Freeman (2016), when this kind of alternatives is added, either that this added alternative is chosen (which is not a reversal) or the choice remains unchanged, and so WARP is complied with. This observation is echoed by the fact that the necessary conditions of their model, Irrelevant Alternatives Delay (for a naive agent) and Irrelevant Alternatives Expedite (for a sophisticated agent) only hold in PBDU if WARP were to hold.

5 Social Preference

We now turn to our last application.

Consider a decision maker who has a particular type of set-dependent social preference—her willingness to share is greater when greater equality is possible. Experiments in economics and psychology have shown that, instead of being fully selfish and maximize monetary payment to oneself, people are often willing to share their wealth. This leads to models of *other-regarding preferences* and *inequality aversion,* first introduced by Fehr & Schmidt (1999); Bolton & Ockenfels (2000); Charness & Rabin (2002). Furthermore, one's desire to share, or inequality aversion, may be affected by *available* options. When options that promote equality are possible, decision makers end up sharing more, even if this causes them to violate WARP. One explanation for this behavior is outcome-based, where a decision maker becomes more inequality averse in the presence of more balanced distributions. Another explanation is intention-based, where the decision maker seeks to be perceived as fair.[25] Our model does not distinguish between these two causes for increased altruism, we refer interested readers to surveys by Fehr & Schmidt (2006); Kagel & Roth (2016) for the vast evidence and suggested explanations.

To illustrate, suppose a decision maker is endowed with \$10 and is asked to share it with another individual. However, instead of choosing any split of this \$10, the designer is only giving her a few options to choose from. When she is asked to choose between giving \$2 and giving \$3, giving \$2 may seem like a fair decision. However, when the choice is between giving \$2, \$3 or \$5, the decision maker may opt for giving \$3 instead. The pair of choices (over income distributions) $c\left(\left\{\left(\$8,\$2\right),\left(\$7,\$3\right)\right\}\right)=\left\{\left(\$8,\$2\right)\right\}$ and $c\left(\left\{\left(\$8,\$2\right),\left(\$7,\$3\right),\left(\$5,\$5\right)\right\}\right)=\left\{\left(\$7,\$3\right)\right\}$ violates WARP. Hence the assumption of utility maximization, even if the utility function captures other-regarding preferences and inequality aversion, is incapable of explaining this behavior.

Using our unified framework, where both WARP and a standard postulate are

[25] See for example Ainslie (1992), Rabin (1993), Nelson (2002), and Sutter (2007).

weakened to capture reference-dependence, we provide a model in which a decision maker's willingness to share increases when balanced options are added to a choice problem.

5.1 Preliminaries

Let $Y = X = [w, +\infty) \times [w, +\infty)$, where $w > 0$, be a set of positive monetary payments. We call a pair $(x, y) \in X$ an income distribution, where x is the dollar amount the decision maker will receive for herself and y is the dollar amount for another individual. We endow X with the standard Euclidean metric. Let $\mathcal{A}$ be the set of all finite and nonempty subsets of X and $c : \mathcal{A} \to \mathcal{A}$, $c(A) \subset A$ a choice correspondence.

The first axiom is standard, an income distribution that gives everyone weakly more, and at least one person strictly more, is strictly preferred.

Axiom 9 (Monotonicity). $c(\{(x, y), (x', y')\}) = \{(x, y)\}$ *whenever* $x \geq x'$, $y \geq y'$, *and* $(x, y) \neq (x', y')$.

5.2 Equality Reference Dependence

Our first axiom for this section is a specialization of Axiom 1 from Section 2. It characterizes behavior in which choices from choice problems that have the same amount of *attainable equality* conform with quasi-linear preferences. The use of quasi-linear preferences for choices involving money is common in the economics. Since our model introduces reference-dependent utility functions, using quasi-linear utilities when preferences are stable provides meaningful restrictions to choices for our purpose.

Definition 10. Let $c : \mathcal{A} \to \mathcal{A}$ be a choice correspondence and $\mathcal{S} \subseteq \mathcal{A}$. We say c satisfies *Quasi-linearity* over $\mathcal{S}$ if for all $A, B \in \mathcal{S}$ and $a \in \mathbb{R} \setminus \{0\}$,

$$(x, y) \in c(A),\ (x', y') \in A,\ (x' + a, y') \in c(B), \text{ and } (x + a, y) \in B \Rightarrow (x + a, y) \in c(B).$$

In order to characterize attainable equality, we first need a measure of equality. A natural candidate is the Gini coefficient (as measured by the relative mean absolute difference in incomes),

$$G(x, y) = \frac{|x - y| + |y - x|}{4(x + y)}.$$

In a setting of two incomes, this coefficient ranges from 0 (most balanced) to 0.5 (least balanced). Moreover, since $x, y \geq w > 0$ in our setting, $G : X \to [0, 0.5)$.

Analogous to our approach in other domains, we demand that choices comply with WARP and Quasi-linearity when the most balanced income distribution of a choice problem is unchanged (and therefore allow for violation of WARP and Quasi-linearity when the most balanced income distribution has changed). Formally, we impose a weakening of WARP and Quasi-linearity in the following way:

Definition 11. For each $A \in \mathcal{A}$, define by $\Psi(A) := \{(x,y) \in A : G(x,y) \leq G(x',y') \text{ for all } (x',y') \in A\}$ the set of *most balanced income distributions* in A.

Axiom 10 (Equality Dependence)**.** *For every choice problem $A \in \mathcal{A}$ and any most balanced income distribution $(x,y) \in \Psi(A)$, c satisfies WARP and Quasi-linearity over $\{B \subseteq A : (x,y) \in B\}$.*

5.3 Fairness

We study choices that exhibit increased sharing when greater equality is attainable, that is, when the most balanced income distribution in one choice problem is more balanced than the most balanced income distribution of another.

Consider the following postulate. Suppose in a choice problem the decision maker chooses to share more (x,y) than to share less (x',y') (where $y > y'$). We postulate that by making more options available, since this will only weakly increase attainable equality, she does not switch from sharing more to sharing less. Effectively, this restriction imposes a direction on which willingness to share changes—the decision maker is weakly more altruistic when more options are available. Formally:

Axiom 11 (Fairness)**.** *For any $A, B \in \mathcal{A}$ such that $A \subset B$ and $(x,y), (x',y') \in A$ such that $y > y'$, if $(x,y) \in c(A)$ and $(x',y') \notin c(A)$, then $(x',y') \notin c(B)$.*

5.4 Representation theorem

Consider the following utility representation in which utility from receiving the amount $\$x$ is always evaluated consistently but utility from giving amount $\$y$ depends on how much equality is attainable from the choice problem.

Definition 12. c admits a Fairness-based Social Preference Utility representation (FSPU) if there exists a set of strictly increasing functions $\{v_r : [w, +\infty) \to \mathbb{R}\}_{r \in [0,0.5)}$

such that

$$c(A) = \arg\max_{(x,y)\in A} x + v_{r(A)}(y),$$

where

- $r(A) = \min_{(x,y)\in A} G(x,y)$,
- $r > r'$ implies $v_r(y) - v_r(y') \geq v_{r'}(y) - v_{r'}(y')$ for all $y > y'$,
- $\arg\max_{(x,y)\in A}\ x + v_{r(A)}(y)$ has a closed-graph.

Theorem 4. *Let $c : \mathcal{A} \to \mathcal{A}$ be a choice correspondence. The following are equivalent:*

1. *c satisfies Equality Reference Dependence , Fairness, Monotonicity, and Continuity.*
2. *c admits a FSPU representation.*

Furthermore, in every FSPU representation, v_r is unique for all r.

In this model, the decision maker's utility from giving y is $v_{r(A)}(y)$. It depends on $r(A)$, which measures how much equality is attainable in the underlying choice problem A. In particular, recall that a lower Gini coefficient $G(x,y)$ corresponds to greater equality, and hence attainable equality for choice problem A is simply the lowest $G(x,y)$ among available income distributions $(x,y) \in A$. When $r(A)$ is lower, the utility difference between sharing more (y) and sharing less (y') increases, which reflects increased willingness to share. Consequently, even though a decision maker chose to share less (x', y') over sharing more (x,y) in some choice problem, where $y > y'$, the introduction of a very balanced option could cause the a switch to sharing more: (x,y).

This is a model in which decisions makers agree on how to measure *equality*, but retain their own interpretations on what is *fair*. Each decision maker bases her choice on the lowest attainable Gini coefficient of a choice problem, $r(A)$, but their ultimate decisions depend on their subjective utility $v_{r(A)}(y)$.

The model explains increased willingness to share when distributing a fixed pie with different splitting options. To illustrate, suppose a decision maker must allocate a fixed amount of money, say \$100, between her and another individual, but she is not allowed to split the amount however she likes. Instead, there is a set of feasible distributions characterized by $D \subset [0,1]$; she can choose to allocate $\alpha \cdot \$100$ to herself if and only if $\alpha \in D$. By specifying two different sets of feasible distributions, D and D', we have effectively specified two choice problems in our setup. Say $D = \{0.5, 0.6, 0.7\}$ and $D' = \{0.6, 0.7\}$. If $\alpha = 0.7$ is chosen in D' (the decision maker keeps \$70 for herself and \$30 is given to the other individual), she might choose to keep less in D due to increased altruism from greater attainable equality. However, if she chose $\alpha = 0.6$ in D', then she must not choose $\alpha = 0.7$ in D; this is a testable prediction.

In FSPU, altruism is maximal when a perfectly balanced income distribution is available. In particular, the model captures increased altruism not as the result of the opportunity to *give more*; instead, it is due to the opportunity to *be equal*. To illustrate the difference, consider the same example but with $D = \{0.5, 0.3, 0.2\}$ and $D' = \{0.3, 0.2\}$. Even though D contains alternatives that achieve greater equality, the decision maker's ability to give is the same across the two choice problems. Yet, since the feasible allocations are always unfavorable to her (she can never keep more than half), higher attainable equality results from her ability to *take* more. In this setting, our decision maker can be interpreted as being less altruistic when the world is unfair to her, and she becomes more altruistic when more balanced options are added.

We consider one last application, where FSPU allows for willingness to forgo a greater surplus in favor of giving more. Suppose the decision maker must choose between $(\$30, \$20)$ and $(\$60, \$0)$. The second option is appealing in that the total amount of money extracted is greater, whereas the first option sacrifices both surplus and payment to oneself in favor of providing a share to the other individual. Suppose $(\$60, \$0)$ is chosen. The model allows for the behavior in which the addition of $(\$25, \$25)$ to the choice set causes the decision maker to switch from $(\$60, \$0)$ to $(\$30, \$20)$ due to increased generosity. While this behavior is reasonable, it cannot be explained by any model that complies with WARP.

Like in Section 3 and Section 4, the familiar linkage between WARP violation and the violation of a standard postulate, in this case Quasi-linearity, is summarized in the following statement. This irreducible connection suggests that, in our model, failure of quasi-linear utility for money is tied to changes in preferences. Instead, if the decision maker's choices can be represented by single utility function, her behavior must also comply with quasi-linear utility.

Proposition 4. *Suppose c admits a PBDU representation. Then the following are equivalent:*

1. *c satisfies WARP (over $\mathcal{A}$).*

2. *c satisfies Quasi-linearity (over $\mathcal{A}$).*

3. *c admits a quasi-linear utility representation.*

4. *c admits a utility representation.*

5.5 Related literature

Other-regarding preferences have been studied extensively, and well-known models are introduced by Fehr & Schmidt (1999); Bolton & Ockenfels (2000). However, the primary focus of these models is to capture *inequality aversion* using functional forms. In particular, a single and persistent preference ranking of income distributions is assumed throughout these models. Charness & Rabin (2002) introduced a departure that allows for *reciprocity* by introducing a utility function that lowers the utility from giving when the other player is deemed to have "misbehaved".

FSPU, departs from these models by introducing preferences over income distributions that may change from one choice problem to another. In particular, utility from giving depends on how much equality is attainable in the underlying choice set. The vast literature on distributional preferences provides suggestive evidence of this behavior. List (2007); Bardsley (2008); Korenok et al. (2014) showed that in a dictator game, adding (or increasing) the option to take from the receiver significantly reduces a dictator's willingness to give, and in some cases result in choice reversals (WARP violations). However, although the narratives are related, the design of their experiments does not provide a complete test for the predictions of FSPU, as additions of less balanced distributions do not affect preferences in FSPU.

The study of *audience effect* also provides empirical evidence that decision makers care about how others perceive their choices. In Dana et al. (2006), dictators were given the option exit (avoid) a \$10 dictator game and receive \$9, a option that leaves the receiver with nothing. Since a payoff of \$9 (and \$10) can be achieved by going through with the dictator game, exiting is interpreted as a costly effort to avoid the dictator game. 28% of the subjects chose to exit. When the game is conducted such that the decision to exit or not is completely veiled from the receivers, only 4% chose to exit.

In a separate study, Dana et al. (2007) provides dictators a costless opportunity to find out how much the receivers will receive from each of their two options, $(6, 1)$ and $(5, 5)$, before making a choice (payoffs to themselves, the first number in each pair, are always displayed). 44% of dictators chose not to find out, and among them 86% chose "$(6, ?)$" over "$(5, ?)$". Only 47% of dictators chose to reveal the payoffs and subsequently chose $(5,5)$ over $(6,6)$. In the baseline, in which all payoffs are displayed by default, 74% of subject chose $(5,5)$ over $(6,6)$. Based on subjects' apparent exploitation of this "moral wiggle room", the authors conclude that fair behavior is primarily motivated by the desire to appear fair, either to themselves or to others.

In game theoretic settings, Rabin (1993)'s pioneering work introduced intention based reciprocity through a notion of *kindness.* In their model, kindness is measured using the set of payoffs an opponent *could* induce. A player's kindness depends on how kind the opponent is, due to the desire to be *fair*, and vice versa, leading to the solution concept term *fairness equilibrium.* Since kindness is measured using the set of available actions, the Rabin (1993)'s model and FSPU share some conceptual similarity. However, since FSPU is built on a decision theoretic framework, it is unable to capture the type of reciprocity concerns depicted in Rabin (1993). The same argument separates FSPU from related models in game theory.

To my knowledge, Cox et al. (2016) is the only other paper, with a decision theoretic setup, that introduces a model to explain WARP violations of this kind. Unlike FSPU, they take *endowment* into account, which allows for the study of *giving* versus *taking.* This is different to the approach in FSPU, where only income distributions are relevant and endowments are not part of the primitive. Based on an intuition related to FSPU, Cox et al. (2016) uses *moral reference points* to explain changes in dictator's willingness to allocate, where a moral reference point more favorable to the dictator (and/or less favorable to the receiver) results in allocating more to the dictator herself. However, unlike FSPU, their reference points are not alternatives, but instead a vector of reference payoffs that depend on multiple allocations within the feasible set as well as the endowment. Consequently, there are many choice problems in which the addition of a more balanced alternative cannot result in choice reversal in their model, since it does not affect the moral reference point, yet preference reversals as a result of adding more balanced alternatives is precisely the behavioral tenet in FSPU.[26] Although the two models are different in many ways, they both seek to capture the increasingly evident intuition that social preference depends on the set of feasible allocations, which results in WARP-violating behavior.

[26]For example, if a choice problem contains income distributions $(0,1)$ and $(1,0)$, then adding (x,x) for any $x \in (0,1)$ will not change Cox et al. (2016)'s moral reference point, and their model demands compliance with WARP.

6 Conclusion

This paper presents a unified framework for ordered reference dependent choice. Built on a reference dependence approach of weakening behavioral postulates, we demonstrate it usefulness by providing applications in the context of risk, time, and social preferences. In each setting, predictable changes in preferences governed by reference dependency account for well-known behavioral anomalies.

Indispensable to our models is the joint weakening of WARP and structural postulates, which gives us a new way of weakening standard assumptions using behavior in a choice correspondence. This sets them apart from models that propose a single non-standard preference ranking. It also suggests new empirical directions, where behavioral anomalies that had primarily been studied with preference relations are now manifested in the form of a WARP violation in choice behavior.

A natural question is the generality of this exercise—does every choice theoretic model have an ordered reference dependence version by simply having their axioms weakened using an adapted version of Axiom 1? We provide some answers to this question in Appendix A.

In Appendix A, we provide a sufficient condition for an arbitrary behavioral postulate to be accommodated by our method. We call these standard postulates *finite properties*, they are axioms that are satisfied whenever a violation fails to be substantiated with just finitely many observations. For example, WARP is a finite property, since it is inherently a property between a pair (hence "finite") of choices. To put it differently, if a choice correspondence fails WARP, a violation can be substantiated with just two observations. A non-example is continuity, since a choice correspondence can fail continuity whilst a violation can never be substantiated with finitely many observations. However, at this level of generality, we are only able to achieve a result for ordered-reference dependent choice, and not for ordered-reference dependent utility representations. We formalize and discuss this limitation in Appendix A.

Appendix A: Unified Framework and Finite Properties

In this technical section, we state a companion result to Theorem 1 that allows for (i) either fully or partially prescribing the reference order R and (ii) expanding the accommodated property from WARP to a much larger class. For the latter, we call them *finite properties*, which we will now define. Theses two expansions are then used in our applications in the risk, time, and social domain in Section 3, Section 4, and Section 5.

Let Y be an arbitrary set of alternatives, $\mathcal{A}$ the set of all finite and nonempty subsets of Y. For any $\mathcal{B} \subseteq \mathcal{A}$, we call $c : \mathcal{B} \to \mathcal{A}$, where $c(A) \subseteq A$ for all $A \in \mathcal{B}$, a choice correspondence. Let $\mathcal{C}$ be the set of all choice correspondences one can possibly observe from Y and $\mathcal{A}$. Formally,

$$\mathcal{C} := \{c : \mathcal{B} \to \mathcal{A} \text{ s.t. } \mathcal{B} \subseteq \mathcal{A}\}.$$

A property imposed on a choice correspondence can be viewed as a subset of $\mathcal{C}$ that is itself closed under subset operations (where each choice correspondence, a member of $\mathcal{C}$, is viewed as a set of pairs). For instance, the set of all choice correspondences satisfying WARP form a collection of choice correspondences defined by the WARP property. We use this notation to characterize an arbitrary property, formally:

Definition 13. We call $\mathcal{T} \subseteq \mathcal{C}$ a *property* if for all $c, \hat{c} \in \mathcal{C}$ such that $\hat{c} \subset c$, $c \in \mathcal{T}$ implies $\hat{c} \in \mathcal{T}$.

We use "c satisfies $\mathcal{T}$" and "$c \in \mathcal{T}$" interchangeably.

In decision theoretic terms, what we call properties here are features of a choice correspondence that are more likely satisfied when we have less observations (i.e. instead of observing c, we only observe $\hat{c}$). For example, WARP ($A \subset B$ and $c(A) \cap B \neq \emptyset \Rightarrow c(A) \cap B = c(B)$) is a property defined on a pair of a choice sets and their corresponding choices. If the statement of WARP is satisfied for some $c : \mathcal{B} \to \mathcal{A}$, that is, all pairs of choice sets and their corresponding choices satisfy WARP, and $\hat{c} : \mathcal{B}' \to \mathcal{A}$ is where $\mathcal{B}' \subset \mathcal{B}$ and $\hat{c}(B) = c(B)$, then the statement of WARP is also satisfied for $\hat{c}$.

Fact. *The intersection of properties is a property.*[27]

Now, we consider a subset of all properties:

Definition 14. Let $\mathcal{T}$ be a property. We call $\mathcal{T}$ a *finite property* if for all $c \in \mathcal{C}$, $c \notin \mathcal{T}$ if and only if there exists a finite set of choice sets $A_1, ..., A_n \in \text{dom}(c)$ such that $\hat{c} : \{A_1, ..., A_n\} \to \mathcal{A}$, where $\hat{c}(B) = c(B)$, is not in $\mathcal{T}$.

In words, a finite property is (defined as) a property in which non-compliance can be concluded with finitely many observations (i.e. choices from finitely many choice sets). The majority of decision theoretic axioms are finite properties.

[27]To see this: Consider any $c \in \mathcal{C}$ such that $c \in \mathcal{T}_1 \cap \mathcal{T}_2$. So $c \in \mathcal{T}_1, \mathcal{T}_2$. And since $\mathcal{T}_1, \mathcal{T}_2$ are properties, we have $\hat{c} \in \mathcal{T}_1, \mathcal{T}_2$, and hence $\hat{c} \in \mathcal{T}_1 \cap \mathcal{T}_2$, for all $\hat{c} \in \mathcal{C}$ and $\hat{c} \subset c$.

Fact. *When Y is finite, any property is a finite property.*[28]

When Y is infinite, examples of finite properties include Convexity (either $a^\alpha b \in c(\{a^\alpha b, a\})$ or $a^\alpha b \in c(\{a^\alpha b, b\})$), Monotonicity ($c(\{a,b\}) = \{a\}$ if $a > b$), Transitivity ($a \in c(\{a,b\})$ and $b \in c(\{b,d\})$ implies $a \in c(\{a,d\})$), von Neumann-Morgenstern (vNM) Independence ($p \in c(\{p,q\})$ if and only if $\alpha p + (1-\alpha) r \in c(\{\alpha p + (1-\alpha) r, \alpha q + (1-\alpha) r\})$), Betweenness, Stationarity, and Separability, to name a few.

Non-examples of finite properties (that are nonetheless properties) include various versions of continuity (e.g., $x_n \in c(A_n)$, $x_n \to x$, $A_n \to A$ implies $x \in c(A)$) and infinite acyclicity ($a_i \in c(\{a_i, a_{i+1}\})$ for $i = 1, 2, ..., \sigma$, where σ is an ordinal number, implies $a_1 \in c(\{a_1, a_\sigma\})$). Usually, the determination of whether a property is a finite property is immediate when a property is defined algorithmically (as in the axioms in this paragraph) as opposed to defined as an arbitrary subset of $\mathcal{C}$.[29]

Fact. *The intersection of finite properties is a finite property.*[30]

For instance, let $\mathcal{T}_1$ be the subset of all choice correspondences that satisfy WARP and $\mathcal{T}_2$ the subset of all choice correspondences that satisfy vNM Independence. These are both finite properties. We can define "WARP and vNM Independence" as a single finite property $\mathcal{T}_1 \cap \mathcal{T}_2$. It characterizes the set of all choice correspondences that satisfies *both* WARP and Independence.

Fact. *The intersection of finite properties and properties that are not finite properties may or may not be finite properties.*[31]

[28] To see this: Fix any c. Sufficiency is a direct result of the definition of a property. Necessity is also straightforward: Let $A_1, ..., A_n = \mathrm{dom}(c)$, then $\hat{c} = c$, so $c \notin \mathcal{T}$ completes the proof.

[29] The empirical falsifiability of a property (that with finitely many observations the property can be falsified) is not sufficient to establish that it is a finite property. Consider the combination of WARP and continuity, there is no reason why this cannot be defined as a single property. It is empirically falsifiable, since WARP needs only two observations to falsify. Yet in the absence of a violation of WARP, a choice correspondence can very well violate the continuity portion, rendering the property unsatisfied but not falsified with finitely many observations. Conversely, if a property is empirically non-falsifiable, then it is a finite property if and only if it is always trivially satisfied.

[30] To see this: Suppose $\mathcal{T}_1$ and $\mathcal{T}_2$ are both finite properties, $\mathcal{T}_1 \cap \mathcal{T}_2$ is a property. We check Definition 14 that $\mathcal{T}_1 \cap \mathcal{T}_2$ is a finite property. Fix any $c \in \mathcal{C}$. Suppose $c \notin \mathcal{T}_1 \cap \mathcal{T}_2$. Then without loss of generality say $c \notin \mathcal{T}_1$, take the choice sets $A_1, ..., A_n \in \mathrm{dom}(c)$ such that $\hat{c} : \{A_1, ..., A_n\} \to \mathcal{A}$, where $\hat{c}(B) = c(B)$. Since $\hat{c} \notin \mathcal{T}_1$, so $\hat{c} \notin \mathcal{T}_1 \cap \mathcal{T}_2$, and the rest is straightfoward. Now suppose there exist $A_1, ..., A_n \in \mathrm{dom}(c)$ such that $\hat{c} : \{A_1, ..., A_n\} \to \mathcal{A}$, where $\hat{c}(B) = c(B)$, is not in $\mathcal{T}_1 \cap \mathcal{T}_2$. Without loss of generality say $\hat{c} \notin \mathcal{T}_1$, so $c \notin \mathcal{T}_1$, so $c \notin \mathcal{T}_1 \cap \mathcal{T}_2$.

[31] We provide examples. Take $Y = [0, 1]$. The intersection of WARP and Continuity is clearly not

Let $\Psi : \mathcal{A} \to \mathcal{A}$ be a correspondence with $\Psi(A) \subseteq A$ such that $a \in B \subset A$ and $a \in \Psi(A)$ implies $a \in \Psi(B)$.

Definition. We say that a linear order (R, Y) is Ψ-*consistent* if $y \in A \backslash \Psi(A)$ implies xRy for some $x \in \Psi(A)$.

Going forward, given a linear order (R, Y), we use the notation $\arg\max_{y \in B} R$ to characterize the element $x \in B$ such that xRy for all $y \in B$.

Lemma 2. *Consider a choice correspondence $c : \mathcal{A} \to \mathcal{A}$, a finite property $\mathcal{T}$, and a correspondence Ψ. The following are equivalent:*

1. *For every finite $A \in \mathcal{A}$, there exists $x \in \Psi(A)$ such that the choice correspondence $\tilde{c} : \{B : B \subset A \text{ and } x \in B\} \to \mathcal{A}$, where $\tilde{c}(B) = c(B)$, is in $\mathcal{T}$.*

2. *There exists a complete, transitive, antisymmetric, and Ψ-consistent binary relation (R, Y) such that for all $x \in Y$, the choice correspondence $\tilde{c}$: $\left\{B : \arg\max_{y \in B} R = x\right\} \to \mathcal{A}$, where $\tilde{c}(B) = c(B)$, is in $\mathcal{T}$.*

First, consider the case in which $\Psi(A) := \mathrm{id}(A) = A$. The first condition in Lemma 2 is satisfied when, for each choice problem, an alternative serves as an anchor that guarantees compliance with finite property $\mathcal{T}$ among choices from subsets of the choice problem containing this anchor. Like before, this anchor is a potential reference alternative with which desirable properties of c hold. When Ψ is not the identity function, we are demanding that at least one alternative in a restricted set of each choice problem (restricted according to Ψ) is a potential reference alternative.

This lemma is the backbone of the models in Section 3, Section 4, and Section 5. For now, we present a simple demonstration. Consider again the wine example, but now the set of all alternatives Y contains multiple entries of the same wine at different prices. Each alternative is hence a wine-price pair (x, p). Like before, a decision maker was seen choosing a more expensive wine over a cheaper one, but sometimes the reverse (at the exact same prices). The economist postulates that for each choice problem, it is either the cheapest or the most expensive wine that the consumer's underlying

a finite property, since WARP can hold whereas Continuity will (trivially) hold for any set of choices from finitely many choice sets, but fails to hold in general. The intersection of Monotonicity (that $x > y \Leftrightarrow y \notin c(A)$ for all $A \ni x, y$) and Continuity, on the other hand, is a finite property; essentially, Monotonicity is so strong that Continuity holds whenever Monotonicity does, and since Monotonicity is a finite property (in fact, it is one where a violation can be detected with just the choice from one choice set), their intersection is a finite property.

preference depends on. Given this postulate, let $\Psi(A)$ be the set of cheapest and most expensive wine-price pairs in A. Furthermore, in addition to WARP, the economist would like to postulate that for a fixed reference, if the decision maker chooses wine x at price p over wine y at price q, then he would also choose wine x at price p over wine y at price $q' > q$; we will call this property "Money is Good". This is an example of a finite property on c.

Lemma 2 establishes that, for a choice correspondence that satisfies these postulates, a reference order (R, Y) can be built such that WARP and Money is Good are satisfied among choice sets that share the same $R-$maximal element. Furthermore, for any three wine-price pairs, the intermediate-in-price option is either reference dominated by the more expensive option, the cheaper option, or both. A prediction follows: If a wine-price pair (x, p) reference dominates another wine-price pair (y, q), then all wine-price pairs (z, s) such that $s \in [\min\{p, q\}, \max\{p, q\}]$ are reference dominated by (x, p). That is, even if the economist hasn't fully pinned down this partially subjective R, she can conclude that among choice sets that contain (x, p) and (z, s), where s is between p and q, choices satisfy WARP and Money is Good.

If instead the economist makes the weaker postulate that some reference alternative exists (i.e., $\Psi = id$), then no structure on R can be guaranteed (other than it is a linear order). Conversely, if the economist makes the stronger postulate that the cheapest wine is exactly the reference alternative, then for any two wine-price pairs, the cheaper option reference dominates the other. This demonstrates the flexibility Ψ provides in the trade-off between explanation and prediction. If $\Psi(A)$ is a very restrictive set, such as a singleton, then the model is easy to test and provides strong predictions. If $\Psi(A)$ is very nonrestrictive, such as $\Psi(A) = A$, then the model is harder to test but accommodates more behavior.

To summarize, we expanded the result of Theorem 1 to include (i) how properties of R can be axiomatically introduced and (ii) what kind of properties, beyond WARP, of a choice correspondence, can be accommodated in this framework. Theses two expansions are then used in our applications in the risk, time, and social domain.

Lemma 2 falls short of achieving a utility representation. The underlying difficulty is related to the literature on limited datasets, in which one observes choices from a strict subset of all choice problems. de Clippel & Rozen (2014) points out that, in this case, even if observed choices are consistent with behavioral postulates, it need not be sufficient for a corresponding utility representation. In our case, even though we started with an exhaustive dataset ($c : \mathcal{A} \to \mathcal{A}$), we have effectively created a partition such that each part contains only a subset of all choice problems. Nevertheless, as demonstrated in Section 3, ordered-reference dependent expected utility can be achieved with normative restrictions on Ψ.

(For Online Publication) Appendix B: Additional Materials

ORDU vs other non-WARP models

In this section, we study ORDU in comparison with other models that either involve reference formation or explains WARP violations through the addition/removal of certain alternatives. To simplify notation, we use "$\{\underline{a}, b, c\}$" for $c(\{a, b, c\}) = \{a\}$.

Under comparable setups, ORDU neither nests nor is nested by any of the following models: (i) Ok et al. (2015)'s *revealed (p)reference*, (ii) Kőszegi & Rabin (2006)'s *reference-dependent preferences* (*personal equilibrium*), (iii) Manzini & Mariotti (2007)'s *rational shortlist method*, and (iv) Masatlioglu et al. (2012)'s *choice with limited attention*.[32] The former two models involve reference formation. In the latter two models, the addition or removal of alternatives directly contribute to WARP violations.

A key observation separates ORDU from other models: In ORDU, because reference points are given by a reference order and choices maximize reference-dependent utilities, then either $c(\{a, b\})$ or $c(\{b, c\})$ must agree with $c(\{a, b, c\})$ in the maximization of a single utility function. This is because the reference point of $\{a, b, c\}$ is necessarily in $\{a, b\}$ or $\{b, c\}$ (or both). This defining feature of ORDU can be generalized:

Remark 1. Suppose c admits an ORDU representation and take any finite collection of choice problems $A_1, ..., A_n$. For some $x \in A = A_1 \cup A_2 \cup ... \cup A_n$, choices between A and A_i such that $x \in A_i$ must comply with standard utility maximization.

In Ok et al. (2015)'s *(endogeneous) reference dependent choice*, the decision maker maximizes a single utility function, but only chooses from alternatives that are better than the reference in all (endogenously determined) attributes. However, references do not necessarily come from an order. Consider the following choices accommodated by

[32] That is, for each of the four external models we consider, there are choice correspondences that admit ORDU but not the eternal model and vice versa. Complete specifications of the choice correspondences used are provided in Appendix B.

their model but not ORDU.[33]

$$\boxed{\boldsymbol{\{\underline{a}, b, c, d\}} \quad \boldsymbol{\{a, \underline{b}, d\}} \quad \boldsymbol{\{a, \underline{c}, d\}}}$$

In their model, a *decoy* d blocks the choice of a in $\{a, \underline{b}, d\}$ and $\{a, \underline{c}, d\}$ due to the *attraction effect,* where b and c are *elevated* because they are better than d in all attributes while a isn't. However, since reference formation is flexible as contrasted with the use of a reference order in ORDU, d need not be the decoy in $\{\underline{a}, b, c, d\}$, resulting in the choice of a. On the contrary, ORDU requires that either the reference point of $\{a, b, d\}$ or the reference point of $\{a, c, d\}$ is the reference point of $\{a, b, c, d\}$. As a result, Remark 1 excludes this behavior from ORDU. On the other hand, intransitive behavior in binary choice problems, such as $\{\underline{a}, b\}, \{\underline{b}, c\}, \{\underline{c}, a\}$, can be explained by ORDU but are ruled out by Ok et al. (2015), since the absence of a third alternative impedes their decoy-effect from taking place. Hence the two models are not nested.

Kőszegi & Rabin (2006)'s *reference-dependent preferences* is another related model. Gul et al. (2006) provided the axiomatic foundation for *personal equilibrium* (PE), in which a decision maker has a joint utility function $v : X \times X \to \mathbb{R}$ and chooses $PE\,(A) = \{x : v\,(x|x) \geq v\,(y|x)\ \forall y \in A\}$. That is, the choice maximizes a reference-dependent utility function, and the reference point is itself the eventually chosen alternative (therefore "equilibrium"). This leads to the following behavior.[34]

$$\boxed{\boldsymbol{\{\underline{a}, b, c, d\}} \quad \boldsymbol{\{\underline{a}, \underline{b}, c\}} \quad \boldsymbol{\{\underline{a}, \underline{d}\}}}$$

In this example, b is not chosen in $\{\underline{a}, b, c, d\}$. Yet it is chosen in the subset $\{\underline{a}, \underline{b}, c\}$, in which d—an alternative better than b under $v\,(\cdot|b)$—was removed. Furthermore, d is also chosen in the subset $\{\underline{a}, \underline{d}\}$ for the same reason—c, an alternative better than d

[33] The complete choice correspondence is

$\boldsymbol{\{\underline{a}, b, c, d\}}$	$\{\underline{a}, b, c\}$	$\boldsymbol{\{a, \underline{b}, d\}}$	$\boldsymbol{\{a, \underline{c}, d\}}$	$\{\underline{b}, c, d\}$	$\{\underline{a}, b\}$
$\{\underline{a}, c\}$	$\{\underline{a}, d\}$	$\{\underline{b}, c\}$	$\{\underline{b}, d\}$	$\{\underline{c}, d\}$	

Using an Ok et al. (2015) specification where $u\,(a) > u\,(b) > u\,(c) > u\,(d)$. $r\,(\{a, b, d\}) = r\,(\{a, c, d\}) = r\,(\{b, d\}) = r\,(\{c, d\}) = d$, $r\,(A) = \diamond$ otherwise, and $\mathcal{U} = \{U\}$ where $U\,(b) > U\,(c) > U\,(d) > U\,(a)$.

[34] The complete choice correspondence is

$\boldsymbol{\{\underline{a}, b, c, d\}}$	$\boldsymbol{\{\underline{a}, \underline{b}, c\}}$	$\{\underline{a}, b, \underline{d}\}$	$\{\underline{a}, c, d\}$	$\{b, \underline{c}, d\}$	$\{\underline{a}, \underline{b}\}$
$\{\underline{a}, c\}$	$\boldsymbol{\{\underline{a}, \underline{d}\}}$	$\{\underline{b}, \underline{c}\}$	$\{b, \underline{d}\}$	$\{\underline{c}, d\}$	

Gul et al. (2006) shows that PE is equivalent to choices maximizing a complete (but not necessarily transitive) preference relation. This choice correspondence is explained by $a \sim b$, $a \succ c$ $a \sim d$, $b \sim c$, $d \succ b$, $c \succ d$.

under $v(\cdot|d)$—was removed. While ORDU also allows for $x \in c(B) \backslash c(A)$ where $B \subset A$, it does so with two implications: (i) an alternative $y \in A \backslash B$ must had been the reference point of A and therefore (ii) choices are consistent between $c(A)$ and $c(T)$ for all $T \subset A$ that contains y. This is not satisfied in our example: the first WARP violation implies that d is the reference point, yet the second WARP violation occurs while d is remains present. As a consequence, Remark 1 excludes this behavior from ORDU.

Conversely, an immediate implication of PE is, if $x \in c(A)$ and $x \in B \subset A$, then $x \in c(B)$. A simple intransitive choice pattern $\{\underline{a}, b\}, \{\underline{b}, c\}, \{\underline{c}, a\}$ is therefore admissible by ORDU but not PE.[35] We conclude that the two models are not nested.

Manzini & Mariotti (2007) proposes a non-WARP model without a reference point interpretation. In *rational shortlist method* (RSM), decision makers are endowed with two asymmetric relations P_1 and P_2. Facing a choice problem A, she first creates a shortlist by eliminating inferior alternatives according to P_1 (eliminate x if yP_1x for some $y \in A$), and then choose from this shortlist according to P_2. WARP violation occurs when an alternative x is eliminated in a set S but not in its subset $T \subset S$, where x is subsequently chosen over the choice from S. An example of this behavior is the follow choice pattern.[36]

$$\boldsymbol{\{\underline{a}, b, c, d\} \quad \{a, b, \underline{d}\} \quad \{b, \underline{c}, d\} \quad \{\underline{b}, c\}}$$

For ORDU to reconcile with this behavior, c must be deemed the reference of $\{a, b, c, d\}$, but then choices from $\{b, c, d\}$ and $\{b, c\}$ must comply with standard utility maximization. This behavior is therefore excluded by Remark 1.

RSM, however, cannot accommodate an alternative that makes it to the shortlists for a small and large choice problems (A_1 and A_2 where $A_1 \subset A_2$) but not an intermediate one (B where $A_1 \subset B \subset A_2$), generating choices accommodated by ORDU but not RSM, such as the following.[37]

[35] Gul et al. (2006) shows that Kőszegi & Rabin (2006)'s *personal equilibrium* is equivalent to the maximization of a complete (but not necessarily transitive) preference relation.

[36] The complete choice correspondence is

$\boldsymbol{\{\underline{a}, b, c, d\}}$	$\{\underline{a}, b, c\}$	$\boldsymbol{\{b, \underline{c}, d\}}$	$\{\underline{a}, c, d\}$	$\boldsymbol{\{a, b, \underline{d}\}}$	$\{\underline{a}, b\}$
$\{\underline{a}, c\}$	$\{a, \underline{d}\}$	$\boldsymbol{\{\underline{b}, c\}}$	$\{b, \underline{d}\}$	$\{\underline{c}, d\}$	

,

induced by (aP_1b, aP_1c, cP_1d, dP_1b) and (aP_2b, aP_2c, dP_2a, bP_2c, dP_2b, cP_2d).

[37] The complete choice correspondence is

$\boldsymbol{\{\underline{a}, b, c, d\}}$ $\quad$ $\boldsymbol{\{a, \underline{b}, d\}}$ $\quad$ $\boldsymbol{\{\underline{a}, b\}}$

The intuition is that while ORDU is constrained when reference points are fixed, the model is more flexible than RSM when reference points change, since no restriction is put on the new utility functions.
Since this is not the case, ORDU does not nest RSM. While ORDU is constrained by fixed reference points, the model is more flexible than RSM when reference points do change, since no restriction is put on the new utility function. RSM, however, cannot accommodate a choice that makes the shortlists in a small and large set but not an intermediate one. We conclude that ORDU and RSM are not nested.

Last, we compare ORDU to Masatlioglu et al. (2012)'s *choice with limited attention* (CLA). A decision maker has a complete and transitive ranking $\succ_{CLA}$ of alternatives and an attention filter that limits choices to a subset of each choice problem, the "consideration set". When another choice problem is derived by removing choices not originally considered, the consideration set remains the same. Although a single ranking is used—as opposed to ORDU's many utility functions—flexibility in constructing consideration sets easily allows for behavior not accommodated by ORDU, for example the following, which is excluded from ORDU by Remark 1.

$\boldsymbol{\{a, \underline{b}, c\}}$ $\quad$ $\boldsymbol{\{\underline{a}, b\}}$ $\quad$ $\boldsymbol{\{b, \underline{c}\}}$ $\quad$ $\{\underline{a}, c\}$

Interestingly, CLA is provided under the framework of choice functions (no indifference), and with that restriction ORDU is nested by CLA.[38] However, the two models make different predictions when indifferences are allowed. For the analysis, we modify CLA by allowing for indifferences in the ranking of alternatives (replacing $\succ_{CLA}$ with $\succsim_{CLA}$), but preserve in entirety the attention filter/ consideration set component. The following behavior is accommodated by ORDU but not CLA.[39]

$\boldsymbol{\{\underline{a}, b, c, d\}}$	$\{a, \underline{b}, c\}$	$\boldsymbol{\{a, \underline{b}, d\}}$	$\{\underline{a}, c, d\}$	$\{\underline{b}, c, d\}$	$\boldsymbol{\{\underline{a}, b\}}$
$\{\underline{a}, c\}$	$\{\underline{a}, d\}$	$\{\underline{b}, c\}$	$\{\underline{b}, d\}$	$\{\underline{c}, d\}$	

This is explained by the ORDU specification: $bRaRcRd$, $u_i(a) > u_i(b) > u_i(c) > u_i(d)$ when $i \in \{a, b, d\}$, and $u_c(b) > u_c(a) > u_c(c) > u_c(d)$.

[38]Consider any choice function c that admits an ORDU representation, define CLA's parameters as follows: attention filter $\Gamma(A) := \left\{\min(A, R), \arg\max_{x \in A} u_{\min(A,R)}(x)\right\}$ (singleton if $\min(A, R) = \arg\max_{x \in A} u_{\min(A,R)}(x)$) and CLA's preference $x \succ y$ if xRy.

[39]The complete choice correspondence is

$\boldsymbol{\{\underline{a}, \underline{b}, c, d\}}$	$\boldsymbol{\{a, \underline{b}, \underline{c}\}}$	$\{\underline{a}, \underline{b}, d\}$	$\{\underline{a}, c, d\}$	$\{\underline{b}, c, d\}$	$\{a, \underline{b}\}$
$\{a, \underline{c}\}$	$\{\underline{a}, d\}$	$\boldsymbol{\{\underline{b}, c\}}$	$\{\underline{b}, d\}$	$\{\underline{c}, d\}$	

This is explained by the ORDU specification: $cRbRaRd$, $u_d(a) = u_d(b) > u_d(c) > u_d(d)$,

$\{\underline{a}, \underline{b}, c, d\}$ $\{a, \underline{b}, \underline{c}\}$ $\{\underline{b}, c\}$

When indifferences are allowed, the single ranking limitation of CLA becomes the bottleneck in explaining behavior. The two models are hence not nested under comparable setups.

(For Online Publication) Appendix C : Proofs

Proof of Lemma 2

Lemma 3. *Let Z be a set, and $\mathbb{Z}$ be the set of all finite and nonempty subsets of Z. Let$\mathcal{R}$ be a self map on $\mathbb{Z}$, $\mathcal{R}(S) \subseteq S$, such that*

(i) For all $S \in \mathbb{Z}$, $\mathcal{R}(S) \neq \{\emptyset\}$, and

(ii) α - for all $T, S \in \mathbb{Z}$, $x \in T \subseteq S$, if $x \in \mathcal{R}(S)$, then $x \in \mathcal{R}(T)$.

Then, there exist $\mathcal{R}^* \subseteq \mathcal{R}$ such that

(i) For all $S \in \mathbb{Z}$, $\mathcal{R}^*(S) \neq \{\emptyset\}$,

(ii) α - for all $T, S \in \mathbb{Z}$, $x \in Z$ such that $x \in T \subseteq S$, if $x \in \mathcal{R}^*(S)$, then $x \in \mathcal{R}^*(T)$, and

(iii) β - for all $T, S \in \mathbb{Z}$, $x, y \in Z$ such that $x, y \in T \subseteq S$, if $x \in \mathcal{R}^*(T)$ and $y \in \mathcal{R}^*(S)$, then $x \in \mathcal{R}^*(S)$.

Proof. We prove by construction.

1. We say $\mathcal{R}' \subseteq \mathcal{R}$ if $\mathcal{R}'(S) \subseteq \mathcal{R}(S)$ $\forall S \in \mathbb{Z}$. Assume and invoke Zermelo's theorem to well-order the set of all doubletons in the domain of $\mathcal{R}$ (there may be uncountable many of them, hence Zermelo's theorem). Now we start the transfinite recursion using this order.

2. In the zero case, we have $\mathcal{R}_0 = \mathcal{R}$. This correspondence satisfies α and is nonempty-valued. Suppose $\mathcal{R}_\sigma$ satisfies α and is nonempty-valued.

$u_a(b) = u_a(c) > u_a(a)$, $u_b(b) > u_b(c)$. Now we show non-compliance with CLA (with the indifference extension): Since $a \in c(\{a, b, c, d\}) \setminus c(\{a, b, c\})$, CLA reconciles this by setting the consideration sets $\Gamma(\{a, b, c, d\}) = \{a, b, d\}$ and $\Gamma(\{a, b, c\}) = \{b, c\}$, so a is not considered in the smaller set. However, the property of consideration sets then requires $\Gamma(\{b, c\}) = \{b, c\}$, and $\{a, c\}$

3. For the successor ordinal $\sigma+1$, we take the corresponding doubleton $B_{\sigma+1}$ and take $x \in B_{\sigma+1}$ such that $\forall S \supset B_{\sigma+1}$, $\mathcal{R}(S) \setminus \{x\} \neq \emptyset$. Suppose such an x does not exist, then for both $x, y \in B_{\sigma+1}$, there are $S_x \supset B_{\sigma+1}$ and $S_y \supset B_{\sigma+1}$ such that $\mathcal{R}_\sigma(S_x) = \{x\}$ and $\mathcal{R}_\sigma(S_y) = \{y\}$ since $\mathcal{R}_\sigma$ is nonempty-valued. Consider $S_x \cup S_y \in \mathbb{Z}$. Since $\mathcal{R}_\sigma$ is nonempty-valued, $\mathcal{R}_\sigma(S_x \cup S_y) \neq \emptyset$. But since $\mathcal{R}_\sigma$ satisfies α, it must be that $\mathcal{R}_\sigma(S_x \cup S_y) \subseteq \mathcal{R}_\sigma(S_x) \cup \mathcal{R}_\sigma(S_y)$, hence $\mathcal{R}_\sigma(S_x \cup S_y) \subseteq \{x, y\}$. Suppose without loss $x \in \mathcal{R}_\sigma(S_x \cup S_y)$, then due to α again and that $x \in B_{\sigma+1} \subset S_y$, it must be that $x \in \mathcal{R}_\sigma(S_y)$, which contradicts $\mathcal{R}_\sigma(S_y) = \{y\}$. (That is, we showed that with nonempty-valuedness and α, no two elements can each have a unique appearance in the $\mathcal{R}_{(\cdot)}$-image of a set containing those two elements.) Hence, $\exists x \in B_{\sigma+1}$ such that $\forall S \supset B_{\sigma+1}$, $\mathcal{R}(S) \setminus \{x\} \neq \emptyset$. Define $\mathcal{R}_{\sigma+1}$ from $\mathcal{R}_\sigma$ in the following way: $\forall S \supset B_{\sigma+1}$, $\mathcal{R}_{\sigma+1}(S) := \mathcal{R}_\sigma(S) \setminus \{x\}$. Note: (i) Since x is deleted from $\mathcal{R}_\sigma(T)$ only if it is also deleted (if it is in it at all) from $\mathcal{R}_\sigma(S)$ $\forall S \supset T$, we are preserving α, and (ii) since x is never the unique element in $\mathcal{R}_\sigma(S)$ $\forall S \supset B_{\sigma+1}$, we preserve nonempty-valuedness.

4. For a limit ordinal λ, define $\mathcal{R}_\lambda = \cap_{\sigma<\lambda} \mathcal{R}_\sigma$. Note that since $\mathcal{R}_{\sigma'} \subset \mathcal{R}_{\sigma''}$ $\forall \sigma' > \sigma''$, $\cap_{\sigma \leq \bar{\sigma}} = \mathcal{R}_{\bar{\sigma}}$. Furthermore, for any $\sigma < \lambda$, $\mathcal{R}_\sigma$ is constructed such that α and nonempty-valuedness are preserved. Hence $\mathcal{R}_\lambda$ satisfies α and is nonempty-valued.

5. Note that this process terminates when all the doubletons have been visited, for we would otherwise have constructed an injection from the class of all ordinals to the set of all doubletons in $\mathbb{Z}$, which is impossible.

6. Finally, we check that $|\mathcal{R}_\lambda(S)| = 1$ for all $S \in \mathbb{Z}$, hence β is satisfied trivially. Suppose it is not a function, hence $\exists S \in \mathbb{Z}$ such that $x, y \in \mathcal{R}_\lambda(S)$. Then by α we have that $x, y \in \mathcal{R}_\lambda(\{x, y\})$, which is not possible as the recursion process has visited it and deleted something from $\mathcal{R}(\{x, y\})$.

7. Set $\mathcal{R}_\lambda = \mathcal{R}^*$.

□

We state the following observation. Let $c : \mathcal{A} \to \mathcal{A}$ be a choice correspondence. Recall that $\mathcal{A}$ is the set of all finite and nonempty subsets of Y. For any $S \subseteq Y$ and $x \in S$, define

$$\mathbb{A}_S^x := \{A \subseteq S : A \in \mathcal{A} \text{ and } x \in A\}.$$

We use the notation $(c, \mathbb{A}_A^x)$ to denote the choice correspondence $\tilde{c} : \mathbb{A}_A^x \to \mathcal{A}$ where $\tilde{c}(B) = c(B)$. In other words, $(c, \mathbb{A}_A^x)$ is a subset of c where the domain is restricted to $\mathbb{A}_A^x$– the set of all subsets of A containing x.

Remark 2. Let $c : \mathcal{A} \to \mathcal{A}$ be a choice correspondence and $\mathcal{T}$ a finite property as defined in Definition 14. Define $\Gamma(S) := \{x \in S : (c, \mathbb{A}_A^x) \in \mathcal{T}\}$.

1. If $y \in \Gamma(A)$, then $y \in \Gamma(B)$ whenever $B \subset A$.

2. If $y \in \Gamma(A)$ for all finite $A \subseteq D$, then $y \in \Gamma(D)$.

We call x a reference alternative for S if $x \in \Gamma(S)$. Remark 2 states that if x is a reference alternative for some choice problem A, i.e., $(c, \mathbb{A}_A^x) \in \mathcal{T}$, then x is also a reference alternative for $B \subseteq A$. This is an immediate consequence of the definition of a property (and the fact that $\mathbb{A}_B^x \subseteq \mathbb{A}_A^x$ whenever $B \subseteq A$). In words, if a violation is undetected with more observations, then it cannot be detected with less. Furthermore, if x is a reference alternative for all finite subsets of an arbitrary set of alternatives D, then x is also a reference alternative for D; this, is due to $\mathcal{T}$ being a finite property. Otherwise, take a finite set of choice problems $\mathcal{S} = A_1, ..., A_n$, each of which a subset of D containing x, such that a finite property is violated, i.e., $\tilde{c} : \mathcal{S} \to \mathcal{A}$, where $\tilde{c}(B) = c(B)$, is not in $\mathcal{T}$. Since this is a finite tuple of finite choice problems, consider the finite set $A := \cup_i A_i$. Clearly, $x \notin \Gamma(A)$, but A is a finite subset of D, hence a contradiction. Intuitively, if x is not a reference alternative for some arbitrary set of alternatives D, then violation of a finite property would have been detected in a finite subset of D, rendering x not a reference alternative for some choice problem $A \subseteq D$.

Now, let $\mathcal{R}' : \mathcal{A} \to \mathcal{A} \cup \{\emptyset\}$ be a set valued function that picks out, for each choice problem $A \in \mathcal{A}$, the set of all reference alternatives $\mathcal{R}'(A) \subseteq A$; formally, $\mathcal{R}'(A) := \{x \in S : (c, \mathbb{A}_A^x) \in \mathcal{T}\}$. Since $\mathcal{T}$ is a finite property, by Remark 2, $\mathcal{R}'$ satisfies property α (as defined in Lemma 3). Furthermore, the hypothesis in Lemma 2 gives that $\mathcal{R}'(A) \cap \Psi(A)$ is nonempty for all $A \in \mathcal{A}$. Finally, define $\mathcal{R} : \mathcal{A} \to \mathcal{A}$ by $\mathcal{R}(A) := \mathcal{R}'(A) \cap \Psi(A)$. Since both $\mathcal{R}'(A)$ and $\Psi(A)$ satisfy property α, $\mathcal{R}(A)$ satisfies property α.

Putting our $\mathcal{R}$ through Lemma 3, we get a set-valued function $\mathcal{R}^* : \mathcal{A} \to \mathcal{A}$ that is now a singleton everywhere (i.e., $|\mathcal{R}^*(A)| = 1$ for all $A \in \mathcal{A}$). Furthermore, this function satisfies property α, and satisfies property β trivially. With this, we build the order (R, Y) by setting xRy if $\{x\} = \mathcal{R}^*(\{x, y\})$, and xRx. The result is a complete, transitive, and antisymmetric binary relation.

Lemma 4. *For an* (R, Y) *constructed according to the the aforementioned procedure,* $y \in A \setminus \Psi(A) \Rightarrow xRy$ *for some* $x \in \Psi(A)$ *(i.e.* R *is* Ψ*-consistent).*

Proof. Suppose not, say $y \in A \setminus \Psi(A)$ but yRx for all $x \in \Psi(A)$. Consider $\{\{x, y\} : x \in \Psi(A)\}$. Since this is a finite set of doubletons, suppose without loss of generality $\{x^*, y\}$ is the last one (in $\{\{x, y\} : x \in \Psi(A)\}$) visited by the procedure in Lemma 3, and denote the step corresponding to $\{x^*, y\}$ by the ordinal $\sigma_{\{x^*,y\}}$. Since yRx for all $x \in \Psi(A)$ such that $x \neq x^*$, $\mathcal{R}_{\sigma_{\{x^*,y\}}}(A) \cap \Psi(A) = \{x^*\}$. Since $\mathcal{R}_\sigma \subseteq \mathcal{R}_0 := \mathcal{R}' \cap \Psi$ for all σ, $\mathcal{R}_{\sigma_{\{x^*,y\}}}(A) = \{x^*\}$. Hence x^* uniquely appears in the image of $\mathcal{R}_{\sigma_{\{x^*,y\}}}$ evaluated at some superset of $\{x^*, y\}$, and the recursion procedure sets, ultimately, $\mathcal{R}^*(\{x^*, y\}) = \{x^*\}$. But this implies x^*Ry, a contradiction. □

Finally, consider the set

$$R^{\downarrow}(x) := \{y \in Y : xRy\}.$$

This is a set of alternatives that are, according to our binary relation R, reference dominated by x (including x itself). For any finite subset $A \subseteq R^{\downarrow}(x)$ such that $x \in A$, we have $x \in \mathcal{R}^*(A) \subseteq \mathcal{R}(A) \subseteq \mathcal{R}'(A)$, which by definition implies x is a reference alternative of A. Using point 2 in Remark 2, we conclude that x is reference alternative for $R^{\downarrow}(x)$, which need not be finite.

To summarize, we have effectively created a partition of $\mathcal{A}$ where the parts are characterized by $\left\{\mathbb{A}^x_{R^{\downarrow}(x)}\right\}_{x \in Y}$. To see this, take any $A \in \mathcal{A}$, since R is a linear order, there is a unique $z \in A$ such that zRy for all $y \in A$, and so $A \in \mathbb{A}^z_{R^{\downarrow}(z)}$ and $A \notin \mathbb{A}^y_{R^{\downarrow}(y)}$ for any $y \neq z$. Furthermore for each part $\mathbb{A}^x_{R^{\downarrow}(x)}$, $\left(c, \mathbb{A}^x_{R^{\downarrow}(x)}\right)$ is in $\mathcal{T}$. Since $\left\{B \in \mathcal{A} : \arg\max_{y \in B} R = z\right\}$ is simply $\mathbb{A}^z_{R^{\downarrow}(z)}$, the proof is complete.

Proof of Theorem 1, Part 1

Suppose Y is finite. We provide an independent proof (without the use of Lemma 2) that a choice correspondence c that satisfies Axiom 1 has an ORDU representation.

Let $\Gamma(A)$ be the set of reference alternatives for A. We create a list of alternatives in the following way; list $\Gamma(Y)$ with an arbitrary order. Since $Y \setminus \Gamma(Y)$ is again finite, list $\Gamma(Y \setminus \Gamma(Y))$ with an arbitrary order; and continue until all $x \in Y$ are listed. Finally, let i_x denote the position of x in the list. For any $x, y \in Y$, construct xRy if $i_x > i_y$ and xRx.

We now construct $\succsim_x$ for each $x \in Y$. Consider the set

$$R^{\downarrow}(x) := \{y : xRy\}.$$

Consider the choice correspondence c over the subset of choice problems

$$\mathbb{A}^x_{R^{\downarrow}(x)} := \left\{A \in \mathcal{A} : A \subseteq R^{\downarrow}(x) \text{ and } x \in A\right\},$$

which by construction satisfies WARP.

Construction of $\succsim_x$

1. First we set $x \succsim_x x$ for all $x \in Y$.

2. Using the doubletons in $\mathbb{A}^x_{R^{\downarrow}(x)}$, all of which would contain x, we set, for all $y \in R^{\downarrow}(x)$, either $y \succsim_x x$, or $x \succsim_x y$, or both, according to $c(\{x, y\})$.

3. Now for all $y_1, y_2 \succsim_x x$, we set either $y_1 \succsim_x y_2$, or $y_2 \succsim_x y_1$, or both, according to $c(\{x, y_1, y_2\})$, using tripletons in $\mathbb{A}^x_{R^{\downarrow}(x)}$. Due to WARP (of c on $\mathbb{A}^x_{R^{\downarrow}(x)}$), $\succsim_x$ is now complete on the set

$$\mathbb{P}^x := \{y : y \succsim_x x\} = \left\{y \in R^{\downarrow}(x) : y \in c(\{x, y\})\right\},$$

which we call the *prediction set* of x. It consists of all alternatives that are both reference dominated by x (i.e. xRy) and are weakly better than x in binary comparison (i.e. $y \in c(\{y, x\})$).

4. Now consider

$$Y \backslash \mathbb{P}^x = \{y : yRx \text{ or } x \succ_x y\}.$$

We set $y_1 \sim_x y_2$ for all $y_1, y_2 \in Y \backslash \mathbb{P}^x$, and $y_1 \succ_x y_2$ for all $y_1 \in \mathbb{P}^x$, $y_2 \in Y \backslash \mathbb{P}^x$. Our constructed $\succsim_x$ is now complete (on Y).[40]

Using quadrupletons in $\mathbb{A}^x_{R^{\downarrow}(x)}$, we show that $\succsim_x$ constructed above is transitive: Suppose $y_1 \succsim_x y_2$ and $y_2 \succsim_x y_3$, and that $y_1, y_2, y_3 \in \mathbb{P}^x$ (if any of them is in $Y \backslash \mathbb{P}^x$ then the argument is straightforward by $\sim_x$), hence $y_1 \in c(\{x, y_1, y_2\})$ and $y_2 \in c(\{x, y_2, y_3\})$.

[40] That is, for any $y_1, y_2 \in Y$, either $y_1 \succsim_x y_2$, or $y_2 \succsim_x y_1$, or both.

Furthermore, since $y_1, y_2, y_3 \in \mathbb{P}^x$, we have $\{x, y_1, y_2, y_3\} \in \mathbb{A}^x_{R^\downarrow(x)}$, and c on $\mathbb{A}^x_{R^\downarrow(x)}$ satisfies WARP implies $y_1 \in c(\{x, y_1, y_2, y_3\})$, and hence $y_1 \in c(\{x, y_1, y_3\})$, which implies $y_1 \succsim_x y_3$.

Finally, we show that (R, Y) and $\{(\succsim_x, Y)\}_{x \in Y}$ explain c. Take any $A \in \mathcal{A}$, since A is finite, and R is antisymmetric, there is a unique $R-$maximizer $x \in A$ (i.e., xRy for all $y \in A$), hence $A \subseteq R^\downarrow(x)$. Suppose for contradiction $c(A) \nsubseteq \{y \in A : y \succsim_x z \, \forall z \in A\}$; so for some $a \in c(A)$, $a' \succ_x a$ for some $a' \in A$. Then $a \notin c(\{x, a', a\})$. Since $\{x, a', a\}$ is a subset of A, and both choice problems are elements of $\mathbb{A}^x_{R^\downarrow(x)}$, this is a violation of the statement c satisfies WARP on $\mathbb{A}^x_{R^\downarrow(x)} := \left\{A \subseteq R^\downarrow(x) \cap \mathcal{A} : x \in A\right\}$, hence a contradiction. Suppose for contradiction $c(A) \nsupseteq \{y \in A : y \succsim_x z \, \forall z \in A\}$, so for some $a \in A$, $a \succsim_x z$ for all $z \in A$, but $a \notin c(A)$. Take $a' \in c(A)$; since $a \succsim_x a'$, $a \in c(\{x, a', a\})$. Since $\{x, a', a\}$ is a subset of A, and both choice problems are elements of $\mathbb{A}^x_{R^\downarrow(x)}$, a contradiction of the statement c satisfies WARP on $\mathbb{A}^x_{R^\downarrow(x)}$ is reached. Hence $c(A) = \{y \in A : y \succsim_x z \, \forall z \in A\}$.

It remains to show that for each alternative-indexed preference relation $\succsim_x$, we can construct a utility function that represents it. Since Y is finite, and each $\succsim_x$ is a complete and transitive preference relation, this is standard.

Proof of Theorem 1, Part 2

We invoke Lemma 2 to prove the intermediary result that, if c satisfies Reference Dependence (Axiom 1) and Continuity (Axiom 2), then there exists a linear order (R, Y) and a set of complete and transitive preference relations $\{(\succsim_x, Y)\}_{x \in Y}$ such that for all $A \in \mathcal{A}$ and $c(A) = \left\{y \in A : y \succsim_{r(A)} z \, \forall z \in A\right\}$ where

$$r(A) := \{x \in A : xRy \, \forall y \in A\}.$$

Using the terminology developed in Appendix A, define $\mathcal{T}$ to be the property WARP. By Lemma 2, there exists a Ψ-consistent linear order (R, Y) such that c on $\{A \in \mathcal{A} : r(A) = x\}$ satisfies $\mathcal{T}$ for all $x \in Y$. Notice that $\{A \in \mathcal{A} : r(A) = x\} = \mathbb{A}^x_{R^\downarrow(x)}$, and so we conclude that for all $T, S \in \mathbb{A}^x_{R^\downarrow(x)}$, $c(S) \cap T = c(T)$ whenever $T \subset S \subseteq A$ and $c(S) \cap T \neq \emptyset$.

We proceed to build $\{(\succsim_x, Y)\}_{x \in Y}$ using the method outlines in the proof for Theorem 1 Part 1, which gives us a complete and transitive $\succsim_x$ for all x such that $c(A) = \left\{y \in A : y \succsim_{r(A)} z \, \forall z \in A\right\}$.

It remains to show that for each alternative-indexed preference relation $\succsim_x$, we can construct a utility function that represents it.

Based on our construction, $\succsim_x$ is complete and transitive on Y. Moreover, it is continuous when restricted to the prediction set $\mathbb{P}^x$, otherwise a contradiction of Continuity (Axiom 2) would be rendered in the standard way using choices from a sequence of choice problems of the form $\{x, y_n, z_n\}$ that converges to the discontinuous point $\{x, y, z\}$. Therefore, along with the fact $\mathbb{P}^x$ is a subset of the separable metric space Y, $\succsim_x$ admits a (continuous) utility function $u : \mathbb{P}^x \to [0, 1]$ that represents $\succsim_x$ when restricted to the alternatives in $\mathbb{P}^x$. Now define $u(z) = -1$ for all $z \in Y \backslash \mathbb{P}^x$. Now u also represents $y \succ_x z$ for all $y \in \mathbb{P}^x$ and $z \in Y \backslash \mathbb{P}^x$ and $z \sim_x z'$ for all $z, z' \in Y \backslash \mathbb{P}^x$. And we are done. (This proof does not give us a utility function $u : Y \to \mathbb{R}$ that is continuous everywhere, which we do not need.)

Finally, since our system of (R, Y) and $\{(\succsim_x, Y)\}_{x \in Y}$ explains c, which satisfies Continuity by Axiom 2, $c(A) = \arg\max_{y \in A} u_{r(A)}(y)$ has a closed-graph.

Proof of Theorem 2

We interpret $\Delta(X)$ as a $|X| - 1$ dimensional simplex, as is conventional, and hence *full-dimensional* means $|X| - 1$ dimensional.

First, we split $\Delta(X)$ into two groups, those that are simply the mixture of δ_b and δ_w, and everything else:

$$E := \Delta(\{b, w\})$$
$$I := \Delta(X) \backslash E.$$

Recall that $\Psi(A) = A \backslash (MPS(A) \cup ES(A))$, and therefore $a \in \Psi(B)$ if $a \in B \subseteq A$ and $a \in \Psi(A)$. Applying Lemma 2, we get a linear order $(R, \Delta(X))$ that gives a partition of $\mathcal{A}$, with parts $\left\{\mathbb{A}^r_{R^\downarrow(r)}\right\}_{r \in \Delta(X)}$, such that c satisfies WARP and Independence over $\mathbb{A}^r_{R^\downarrow(r)}$ for any $r \in \Delta(X)$. Furthermore, since R is Ψ-consistent, or $\max(A, R) \in \Psi(A)$, we have $\max(A, R) \, Rp$ for all $p \in A \backslash \Psi(A)$, and therefore rRp if pMPSr or pESr, which means R is risk-consistent.

Lemma 5. *For any $r \in I$ and any open ball B_r that contains r, $B_r \cap R^\downarrow(r)$ contains a full-dimensional convex subset of $\Delta(X)$.*

Proof. Take any $r \in I$. By definition, $r(x) \neq 0$ for some $x \neq b, w$ ($r(x)$ is the probability that lottery r gives prize x). Consider the following set

$$\mathbb{C}(r) := ES(\{r\}) \cup \{r\} \cup (MPS(ES(\{r\}) \cup \{r\})),$$

which consists of r, all extreme spreads of r, and all of their mean-preserving spreads.

First we show that $\mathbb{C}(r)$ is convex: First note that since $ES(\{r\})$ is a convex set and r is on the boundary of $ES(\{r\})$, $ES(\{r\}) \cup \{r\}$ is convex. Take any two lotteries $p_1, p_2 \in \mathbb{C}(r)$ and consider their convex combination $(p_1)^{\alpha}(p_2)$ for some $\alpha \in (0,1)$. Since $p_1, p_2 \in \mathbb{C}(r)$, there exist $e_1, e_2 \in ES(\{r\}) \cup \{r\}$ such that either $p_1 = e_1$ or $p_1 \text{MPS} e_1$ and either $p_2 = e_2$ or $p_2 \text{MPS} e_2$. If $p_i = e_i$ for both $i = 1, 2$, then $(p_1)^{\alpha}(p_2) = (e_1)^{\alpha}(e_2)$ and by convexity of $ES(\{r\}) \cup \{r\}$ we are done. Suppose $p_i \neq e_i$ for some $i = 1, 2$, then since the mean-preserving spread relation is preserved under convex combinations, we have $(p_1)^{\alpha}(p_2) \, \text{MPS} \, (e_1)^{\alpha}(e_2)$. Finally, since $(e_1)^{\alpha}(e_2) \in ES(\{r\}) \cup \{r\}$ by the convexity of $ES(\{r\}) \cup \{r\}$, we have $(p_1)^{\alpha}(p_2) \in MPS(ES(\{r\}) \cup \{r\}) \subseteq \mathbb{C}(r)$.

Moreover, we show that $\mathbb{C}(r)$ is full dimensional: For any $p \in I$, $ES(\{p\})$ is not nested in $MPS(\{p\})$, which is itself a $|X-2|$ dimensional object, and therefore $ES(\{p\}) \cup MPS(\{p\})$ is full-dimensional. This means $\mathbb{C}(r)$ is full dimensional as well since it contains $ES(\{p\}) \cup MPS(\{p\})$ for some $p \in I$.

Finally, we show that $\mathbb{C}(r) \subseteq R^{\downarrow}(r)$: If $p \in ES(\{r\})$, rRp since R is risk-consistent. If $q \in MPS(ES(\{r\}) \cup \{r\})$, qRp for some $p \in ES(\{r\}) \cup \{r\}$ since R is risk-consistent, and by transitivity of R we have qRr. Since B_r is also a full-dimensional and convex set, $B_r \cap \mathbb{C}(r)$ is a full-dimensional convex subset of $B_r \cap R^{\downarrow}(r)$. □

Define for each $r \in \Delta(X)$ the *strict prediction set*

$$\mathbb{P}_{+}^{r} := \left\{p \in R^{\downarrow}(r) : r \notin c(\{p, r\})\right\},$$

this set consists of all lotteries that are both reference dominated by r and is *chosen over* r in a binary choice problems.

Lemma 6. *For any $r \in I$, $\mathbb{P}_{+}^{r}$ contains a full-dimensional convex subset of $\Delta(X)$.*

Proof. Take $r \in I$. Suppose for contradiction $r \in c(\{e, r\})$ for all e an extreme spread of r; then since the lottery $(\delta_w)^{r(w)}(\delta_b)$ is in the closure of the extreme spread of r, continuity of c implies $r \in c\left(\left\{r, (\delta_w)^{r(w)}(\delta_b)\right\}\right)$, which is a violation of FOSD (Axiom 3). Hence there is an extreme spread of r, which we denote by e, such that $r \notin c(\{r, e\})$. Since $r^{\alpha}e \in R^{\downarrow}(r)$ and c satisfies Independence over $\mathbb{A}_{R^{\downarrow}(r)}^{r}$, we can find $p := r^{\alpha}e \in \mathbb{P}_{+}^{r}$

where $\alpha \in (0,1)$, hence $p \in I$. By continuity of c, there exists an open ball B_p around p such that $r \notin c(\{r,q\})$ for all $q \in B_p$. By Lemma 5, $B_p \cap R^{\downarrow}(p)$ contains a full-dimensional convex subset of $\Delta(X)$. Since rRp, $B_p \cap R^{\downarrow}(p) \subseteq B_p \cap R^{\downarrow}(r)$, hence $\mathbb{P}^r_+$ contains a full-dimensional convex subset of $\Delta(X)$. $\square$

Immediately, this implies that for $r \in I$, we can build an increasing $u_r : X \to \mathbb{R}$, unique up to an affine transformation, such that $c(A) = \arg\max_{p \in A} \mathbb{E}_p u_r(x)$ if $A \in \mathbb{A}^r_{R^{\downarrow}(r)}$. The technique is standard: Let $\mathbb{P}$ be a full-dimensional convex subset of $\mathbb{P}^r_+$. First, notice that for all $p, q \in \mathbb{P}$, we have $\{r,p,q\} \in \mathbb{A}^r_{R^{\downarrow}(r)}$ and $r \notin c(\{r,p,q\})$. Recall that c satisfies WARP and Independence over $\mathbb{A}^r_{R^{\downarrow}(r)}$. By defining $p \succsim_r q$ if $p \in c(\{r,p,q\})$, we get a binary relation $(\succsim_r, \mathbb{P})$ that is complete, transitive, continuous, and satisfies the standard von Neumann-Morgenstern Independence. Since $\mathbb{P}$ is full-dimensional and convex, it contains a subset that is essentially a linear transformation of a $|X|-1$ dimensional simplex. Since $(\succsim_r, \mathbb{P})$ satisfies FOSD, an increasing utility function $u_r : X \to \mathbb{R}$, unique up to an affine transformation, such that $c(A) = \arg\max_{p \in A} \mathbb{E}_p u_r(x)$ for all $A \in \mathbb{A}^r_{\mathbb{P}}$. We normalize this function to $u_r : X \to [0,1]$, where $u_r(w) = 0$ and $u_b(b) = 1$.

We now show that this utility function works for c over $\mathbb{A}^r_{R^{\downarrow}(r)}$. First, for any two lotteries $p, q \in \Delta(X)$, there exist $p', q' \in \mathbb{P}$ such that $p' = p^\alpha s$ and $q' = q^\alpha s$ for some $s \in \Delta(X)$ and $\alpha \in [0,1]$; we call p', q' $\mathbb{P}$-*common mixtures* of p, q. This can be done by using an arbitrary point in the interior of $\mathbb{P}$, $s \in \text{Int}\,\mathbb{P}$, and take α small enough until both p' and q' enter $\mathbb{P}$. Take any $p \in R^{\downarrow}(r)$ and let r', p' be $\mathbb{P}$-common mixtures of r, p. Since c satisfies Independence over $\mathbb{A}^r_{R^{\downarrow}(r)}$, $i' \in c(\{r, r', p'\})$ if and only if $i \in c(\{r,p\})$, for $i = r, p$. Now take any $p, q \in R^{\downarrow}(r)$ such that $p \in c(\{r,p\})$ and $q \in c(\{r,q\})$, then again by Independence over $\mathbb{A}^r_{R^{\downarrow}(r)}$, $p' \in c(\{r, p', q'\})$ if and only if $p \in c(\{r,p,q\})$, where p', q' are $\mathbb{P}$-common mixtures of p, q.

We have thus shown that $c(\{r,p\}) = \arg\max_{s \in \{r,p\}} \mathbb{E}_s u_r(x)$ for all $\{r,p\} \in \mathbb{A}^r_{R^{\downarrow}(r)}$ and $c(\{r,p,q\}) = \arg\max_{s \in \{r,p,q\}} \mathbb{E}_s u_r(x)$ for all $\{r,p,q\} \in \mathbb{A}^r_{R^{\downarrow}(r)}$ where $p \in c(\{r,p\})$ and $q \in c(\{r,q\})$. Since c satisfies WARP over $\mathbb{A}^r_{R^{\downarrow}(r)}$, showing $c(A) = \arg\max_{p \in A} \mathbb{E}_p u_r(x)$ for all $A \in \mathbb{A}^r_{R^{\downarrow}(r)}$ is straightforward from here.

Corollary 1. *For any $r \in \Delta(X)$ and $p \in R^{\downarrow}(r) \cap I$, if $r \notin c(\{r,p\})$, then there exists $q \in R^{\downarrow}(r) \cap I$ such that $\{q\} = c(\{r,p,q\})$. Furthermore, $\mathbb{P}^r_{+p} := \left\{q \in R^{\downarrow}(r) : \{q\} = c(\{r,p,q\})\right\}$ contains a full-dimensional convex subset of $\Delta(X)$.*

Proof. The proof utilizes techniques in the proofs of Lemma 5 and Lemma 6. First, we show the existence of $q \in R^{\downarrow}(r) \cap I$ such that $\{q\} = c(\{r,p,q\})$. Consider the set of extreme spread of of p, we know that this set is a subset of $R^{\downarrow}(p)$, and is hence a subset of $R^{\downarrow}(r)$. Notice that $r \notin c(\{r, p, e\})$ for any extreme spread e of p since c satisfies WARP over $\mathbb{A}^r_{R^{\downarrow}(r)}$ and $r \notin c(\{r, p\})$. Using the same technique in the proof of Lemma 6, it must be that for some extreme spread e^* of p we have $p \notin c(\{r, p, e^*\})$, otherwise by continuity of c we have $p \in c\left(\left\{r, p, (\delta_w)^{p(w)}(\delta_b)\right\}\right)$, a violation of FOSD. Take any non-trivial convex combination $p^{\alpha}e^*$, this is in $R^{\downarrow}(p) \subseteq R^{\downarrow}(r)$, in I, and $\{p^{\alpha}e^*\} = c(\{r, p, p^{\alpha}e^*\})$, so let $q = p^{\alpha}e^*$. Finally, by continuity of c, take an open ball B_q such that $q' \in B_q$ implies $\{q'\} = c(\{r, p, q'\})$. By Lemma 5, $B_q \cap R^{\downarrow}(q)$ contains a full-dimensional convex subset of $\Delta(X)$. Moreover, $B_q \cap R^{\downarrow}(q) \subseteq B_q \cap R^{\downarrow}(r) \subseteq \mathbb{P}^r_{+p}$. So $\mathbb{P}^r_{+p}$ contains a full-dimensional convex subset of $\Delta(X)$. □

Lemma 7. *For any $r_1, r_2 \in I$, if $r_1 R r_2$, then $u_{r_1} = f \circ u_{r_2}$ for some concave and increasing function $f : [0, 1] \to [0, 1]$.*

Proof. This proof uses Axiom 5. Take any $r_1, r_2 \in I$ such that $r_1 R r_2$. u_{r_1} and u_{r_2} are defined above, let $\bar{f}$ be defined on the utility numbers $u_{r_2}(x)$, $x \in X$, such that $u_{r_1}(x) = \bar{f} u_{r_2}(x)$. Since u_{r_1} and u_{r_2} are strictly increasing, $\bar{f}$ is strictly increasing in its domain. We show that for any $x_1, x_2, x_3 \in X$ such that $x_1 < x_2 < x_3$, we have

$$\bar{f}(\alpha u_{r_2}(x_1) + (1-\alpha) u_{r_2}(x_3)) \geq \alpha \bar{f}(u_{r_2}(x_1)) + (1-\alpha) \bar{f}(u_{r_2}(x_3)),$$

where α solves

$$\alpha u_{r_2}(x_1) + (1-\alpha) u_{r_2}(x_3) = u_{r_2}(x_2). \tag{6.1}$$

Suppose not, then for some $\beta > \alpha$, we have

$$\begin{aligned}\bar{f}(\alpha u_{r_2}(x_1) + (1-\alpha) u_{r_2}(x_3)) &< \beta \bar{f}(u_{r_2}(x_1)) + (1-\beta) \bar{f}(u_{r_2}(x_3)) \\ &< \alpha \bar{f}(u_{r_2}(x_1)) + (1-\alpha) \bar{f}(u_{r_2}(x_3)).\end{aligned} \tag{6.2}$$

Consider the lotteries $\delta = \delta_{x_2}$ and $p = (\delta_{x_1})^{\beta}(\delta_{x_3})$. Equation 6.2 gives

$$\mathbb{E}_{\delta} u_{r_1}(x) < \mathbb{E}_p u_{r_1}(x)$$

and Equation 6.1 with $\beta > \alpha$ gives

$$\mathbb{E}_{\delta} u_{r_2}(x) > \mathbb{E}_p u_{r_2}(x).$$

Let δ_1, p_1 be $\mathbb{P}$-common mixtures of δ, p, where $\mathbb{P}$ here is a full-dimensional convex subset of $\mathbb{P}^{r_1}_{+r_2}$ if $r_1 \notin c(\{r_1, r_2\})$, and of $\mathbb{P}^{r}_{+}$ otherwise. Let δ_2, p_2 be $\mathbb{P}$-common mixtures of δ, p, where $\mathbb{P}$ here is a full-dimensional convex subset of $\mathbb{P}^{r_2}_{+}$. Since u_{r_1} explain c over $\mathbb{A}^{r_1}_{R^{\downarrow}(r)}$ and u_{r_2} explain c over $\mathbb{A}^{r_2}_{R^{\downarrow}(r)}$, we have $\{p_1\} = c(\{r_1, \delta_1, p_1\})$ and $\{\delta_2\} = c(\{r_2, \delta_2, p_2\})$. Notice that $A := \{r_1, r_2, \delta_1, \delta_2, p_1, p_2\} \in \mathbb{A}^{r_1}_{R^{\downarrow}(r_1)}$, so $c(A) = \arg\max_{q \in A} \mathbb{E}_q u_{r_1}(x)$. We established that $\mathbb{E}_{r_1} u_{r_1}(x) < \mathbb{E}_{p_1} u_{r_1}(x)$, $\mathbb{E}_{r_2} u_{r_1}(x) < \mathbb{E}_{p_1} u_{r_1}(x)$, and $\mathbb{E}_{\delta_i} u_{r_1}(x) < \mathbb{E}_{p_i} u_{r_1}(x)$ for $i = 1, 2$, so $c(\{r_1, r_2, \delta_1, \delta_2, p_1, p_2\}) \subseteq \{p_1, p_2\}$. But this along with $\{\delta_2\} = c(\{r_2, \delta_2, p_2\})$ violate Axiom 5. Finally, it is straightforward that one can extend $\bar{f}$ to a concave function $f : [0,1] \to [0,1]$ (for example, by connecting the points with straight lines). □

At this point we left with $r \in E = \Delta(\{b, w\})$.

Lemma 8. *For any $r \in E$ and $p \in R^{\downarrow}(r)$ such that $p \neq r$, either p first order stochastically dominates r or r first order stochastically dominates p.*

Proof. Take $r \in E$ and $p \in R^{\downarrow}(r)$, $p \neq r$. Let $\alpha = r(b)$, then $r(w) = 1 - \alpha$. If $p(b) < \alpha$ and $p(w) < (1-\alpha)$, then r is an extreme spread of p and pRr, so $p \notin R^{\downarrow}(r)$. Furthermore, it is not possible that $p(b) \geq \alpha$ and $p(w) \geq (1-\alpha)$ if $p \neq r$. Hence either $p(b) \geq \alpha$ and $p(w) \leq (1-\alpha)$ with at least one strict inequality, or $p(b) \leq \alpha$ and $p(w) \geq (1-\alpha)$ with at least one strict inequality. If the former, p FOSD r; if the latter, r FOSD p. □

With this observation in mind, we now construct u_r for $r \in E$. Define

$$E_1 := \left\{ r \in E : r \notin c(\{r, p\}) \text{ for some } p \in R^{\downarrow}(r) \cap I \right\}$$
$$E_2 := E \backslash E_1.$$

For any $r \in E_1$, $\mathbb{P}^{r}_{+}$ contains a full-dimensional convex subset of $\Delta(X)$ by Corollary 1, and so we can build u_r using the same method we used to build u_r for $r \in I$. We will construct u_r for $r \in E_2$ after the following result.

Corollary 2. *For any $r_1, r_2 \in I \cup E_1$, if $r_1 R r_2$, then $u_{r_1} = f \circ u_{r_2}$ for some concave and increasing function $f : [0,1] \to [0,1]$.*

Proof. Consider the proof in Lemma 7, but that when $r_2 \in E_1$, we simply let δ_1, p_1 be $\mathbb{P}$-common mixtures of δ, p, where $\mathbb{P}$ here is a full-dimensional convex subset of $\mathbb{P}^{+}_{r_1}$. Before, we let $\mathbb{P}$ be a full-dimensional convex subset of $\mathbb{P}^{+r_2}_{r_1}$ when $r_1 \notin c(\{r_1, r_2\})$, but now such subset need not exist since $r_2 \notin I$. To compensate for this, since $\delta_2, p_2 \in \mathbb{P}^{+}_{r_2}$ implies that

δ_2, p_2 FOSD r_2 due to Lemma 8, we replace the argument "$\mathbb{E}_{r_2} u_{r_1}(x) < \mathbb{E}_{p_1} u_{r_1}(x)$" with "$\mathbb{E}_{r_2} u_{r_1}(x) < \mathbb{E}_{p_2} u_{r_1}(x)$". Everything else goes through as in the proof in Lemma 7, giving us the desired result. □

For any $r \in E_2$, given Lemma 8, any increasing utility function $u_r : X \to [0,1]$ will accomplish $c(A) = \arg\max_{p \in A} \mathbb{E}_p u_r(x)$ for all $A \in \mathbb{A}^r_{R^{\downarrow}(r)}$. With this freedom, we construct u_r to our desired properties.

Formally, consider, for an increasing utility function u_p, the object $\rho^p = \left(\rho^p_2, ..., \rho^p_{|X|-1}\right) \in (0,1)^{|X|-2}$ where

$$\rho^p_i := \frac{u_p(x_i) - u_p(x_{i-1})}{u_p(x_{i+1}) - u_p(x_{i-1})}$$

(that is, ρ^p_i satisfies $u_p(x_i) = \rho^p_i u_p(x_{i+1}) + (1-\rho^p_i) u_p(x_{i-1})$). There is a one-to-one relationship between u_p and ρ^ρ. It is an algebraic exercise to show that $u_p = f \circ u_q$ for some concave and increasing $f : [0,1] \to [0,1]$ if and only if $\rho^p_i \geq \rho^q_i$ for all $i \in \{2, ..., |X|-1\}$. Take $r \in E_2$ and define $\rho^r := \left(\inf_{p \in K}(\rho^p_2), ..., \inf_{p \in K}\left(\rho^p_{|X|-1}\right)\right)$, where $K_r := (I \cup E_2) \cap \{p : rRp\} \subseteq \Delta(X)$, and subsequently construct u_r using ρ^r. It is straightforward to show that R being risk-consistent implies K_r is nonempty for all $r \in E_2 \backslash \{\delta_b, \delta_w\}$, and so u_r is defined other than when $r \in \{\delta_b, \delta_w\}$.

For the non-generic case where, for some $j \in \{b, w\}$, we have $\delta_j \in E_2$ such that K_{δ_j} is not defined, this implies $\delta_j R p$ for all $p \in \Delta(X) \setminus \{\delta_b, \delta_w\}$. Then, we define

$$\rho^{\delta_j}_i := \frac{1}{2}(1) + \frac{1}{2} \sup_{p \in \Delta(X) \setminus \{\delta_b, \delta_w\}} \rho^p_i$$

for all i and construct u_{δ_j} correspondingly. Utility functions indexed by such a δ_j and that by any $p \in \Delta(X) \setminus \{\delta_j\}$ now satisfy $\rho^{\delta_j}_i \geq \rho^p_i$, with equality when p also is a δ_j falling into this special case (there are at most two of them, δ_b and δ_w).

We now show that for $r_1, r_2 \in \Delta(X)$ where $r_1 R r_2$, we have $\rho^{r_1} \geq \rho^{r_2}$. This is already shown for any $r_1, r_2 \in I \cup E_1$ by Corollary 2. It is also shown for the special cases in the preceding paragraph. Hence, we restrict attention to the remaining cases. Say $r_1 \in E_2$, $r_2 \in I \cup E_1$, but $\rho^{r_1}_i < \rho^{r_2}_i$ for some i. Then $\inf_{p \in K_{r_1}}(\rho^p_i) < \rho^{r_2}_i$, so $\rho^p_i < \rho^{r_2}_i$ for some $p \in K_{r_1}$. However, $p \in K_{r_1}$ implies pRr_2 since R is transitive; since $p \in I \cup E_2$, this contradicts Corollary 2. Say $r_1 \in I \cup E_1$, $r_2 \in E_2$, but $\rho^{r_1}_i < \rho^{r_2}_i$ for some i. Then $\rho^{r_1}_i < \inf_{p \in K_{r_2}}(\rho^p_i)$, so $\rho^{r_1}_i < \rho^p_i$ for all $p \in K_{r_2}$. But $r_1 \in K_{r_2}$, a contradiction. Finally, for $r_1, r_2 \in E_2$ and $r_2 R r_1$, either $K_{r_1} = K_{r_2}$ or $K_{r_1} \subsetneq K_{r_2}$. If it is the former, it is immediate that $\rho^{r_1} = \rho^{r_2}$. If it is the later, then $\rho_i^{r_1} = \inf_{p \in K_{r_1}} (\rho_i^p) \leq \inf_{p \in K_{r_2}} (\rho_i^p) = \rho_i^{r_2}$ for all i, as desired.

Thus, we have now shown that for any $r_1, r_2 \in \Delta(X)$, $\rho^{r_1} \geq \rho^{r_2}$ whenever $r_1 R r_2$, or equivalently $u_{r_1} = f \circ u_{r_2}$ for some concave and increasing $f : [0,1] \to [0,1]$.

Proof of Proposition 2

In this proof, we use the notation $\max(A, R)$ to characterize the element $x \in A$ such that xRy for all $y \in A$.

(1)

First, we prove the second property of Betweenness in Definition 5, that

$$c(\{p,q\}) = \{p,q\} \Rightarrow c(\{p, p^\alpha q\}) = \{p, p^\alpha q\} \text{ and } c(\{p^\alpha q, q\}) = \{p^\alpha q, q\}.$$

Then, we use it to prove the first property, that

$$c(\{p,q\}) = \{p\} \Rightarrow c(\{p, p^\alpha q\}) = \{p\} \text{ and } c(\{p^\alpha q, q\}) = \{p^\alpha q\}.$$

Suppose $c(\{p,q\}) = \{p,q\}$. Without loss of generality, either (i) $\max(\{p,q,p^\alpha q\}, R) = p$ or (ii) $\max(\{p,q,p^\alpha q\}, R) = p^\alpha q$. If (i), then $\max(\{p,q\}, R) = \max(\{p, p^\alpha q\}, R) = p$, hence the utility functions used in the two choice problems $c(\{p,q\})$ and $c(\{p, p^\alpha q\})$ are both u_p. It is immediate that, since $p^\alpha q$ is a mixture of p and q, we have $c(\{p, p^\alpha q\}) = \{p, p^\alpha q\}$, and by Transitivity we have $c(\{q, p^\alpha q\}) = \{q, p^\alpha q\}$.[41] If (ii), then $\max(\{p^\alpha q, q\}, R) = \max(\{p, p^\alpha q\}, R) = p^\alpha q$. Suppose for contradiction $p \notin c(\{p, p^\alpha q\})$, then $p^\alpha q \notin c(\{p^\alpha q, q\})$ since $p^\alpha q$ is a mixture of p and q.[42] But by Transitivity we would have $c(\{p,q\}) = \{q\}$, a contradiction. We have hence proved the second property of Betweenness in Definition 5.

The first property is given by this second property, Continuity and FOSD (the latter two are implied by AREU), which we now show.

Suppose $c(\{p,q\}) = \{p\}$. We start by ruling out $c(\{p, p^\alpha q\}) = \{p^\alpha q\}$ (and similarly $c(\{p^\alpha q, q\}) = \{q\}$): Suppose $c(\{p, p^\alpha q\}) = \{p^\alpha q\}$, then by Continuity, it is well-know that there exists a a (strict) mixture of q and $p^\alpha q$ such that $c(\{p, a\}) = \{p, a\}$. But

[41] Since $\sum_{x \in X} p(x) u_p(x) = \sum_{x \in X} q(x) u_p(x) = \sum_{x \in X} [\alpha p(x) + (1-\alpha) q(x)] u_p(x)$.

[42] Since $\sum_{x \in X} p(x) u_{p^\alpha q}(x) < \sum_{x \in X} [\alpha p(x) + (1-\alpha) q(x)] u_{p^\alpha q}(x)$ implies $\sum_{x \in X} [\alpha p(x) + (1-\alpha) q(x)] u_{p^\alpha q}(x) < \sum_{x \in X} q(x) u_{p^\alpha q}(x)$.

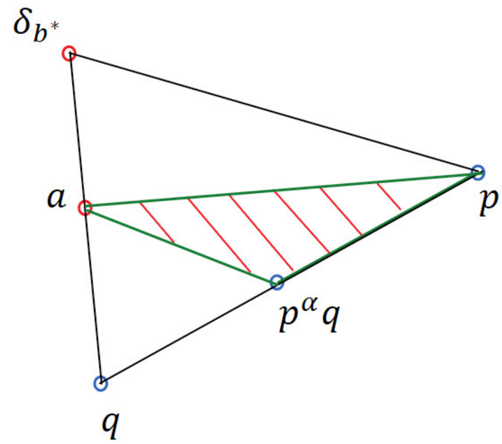

Figure 6.1: Highlighted region is a full-dimensional subset of the larger triangle, which necessitates a FOSD violation.

then by the second property in Definition 5, since $p^\alpha q$ is a mixture of a and p, we have $c(\{p, p^\alpha q\}) = \{p, p^\alpha q\}$, which is a contradiction. A analogous argument can be made to rule out $c(\{p^\alpha q, q\}) = \{q\}$.

Next we rule out $c(\{p, p^\alpha q\}) = \{p, p^\alpha q\}$. Suppose $c(\{p, p^\alpha q\}) = \{p, p^\alpha q\}$. Let δ_{b^*} be the degenerate lottery on $b^* := \max(\text{supp}(p^\alpha q)) \in X$. If $p = \delta_{b^*}$, then by FOSD we are done. Suppose $p \neq \delta_{b^*}$, then by FOSD we have $c(\{b, p\}) = \{b\}$. Since $c(\{p, q\}) = \{p\}$, by Continuity there exists a a (strict) mixture of δ_{b^*} and q such that $c(\{p, a\}) = \{p, a\}$. Now notice that the convex set formed by $p, p^\alpha q, a$, denoted Q, is a full-dimensional subset of the convex set formed by p, q, δ_{b^*}, within which by Transitivity every pairs of lotteries are indifferent to one another (in the sense that for all $p', q' \in Q$, $c(\{p', q'\}) = \{p', q'\}$), which necessitates a FOSD violation. Figure 6.1 provides an illustration. An analogous argument, using δ_{b^*} as the degenerate lottery on $\min(\text{supp}(p^\alpha q)) \in X$, rules out $c(\{p^\alpha q, q\}) = \{p^\alpha q, q\}$.

Now that we have ruled out $c(\{p, p^\alpha q\}) = \{p^\alpha q\}$, $c(\{p, p^\alpha q\}) = \{p, p^\alpha q\}$, $c(\{p^\alpha q, q\}) = \{q\}$ and $c(\{p^\alpha q, q\}) = \{p^\alpha q, q\}$, we conclude that $c(\{p, p^\alpha q\}) = \{p\}$ and $c(\{p^\alpha q, q\}) = \{p^\alpha q\}$.

(2) - (4) We first prove point 3. Using Proposition 2 Part 1, we know that indifference curves are linear and do not intersect. Take an arbitrary indifference curve and consider two points p, q on it that lie in the interior of the triangle. Let p' and q' be mean-preserving contractions of p and q, respectively, such that the line connecting p'

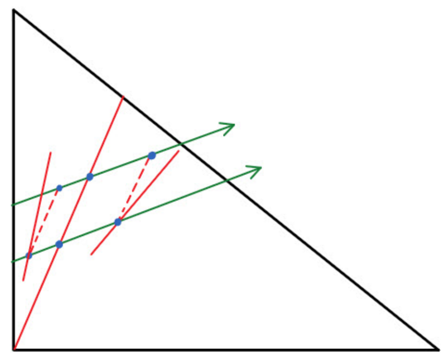

Figure 6.2: Indifference curves fan out when AREU is combined with Transitivity and risk aversion (Proposition 2). Arrows correspond to mean-preserving spreads.

and q' is parallel to the line connecting p and q. Since p', q' are mean-preserving contractions, $\max(\{p,q\}, R)\ R\ \max(\{p',q'\}, R)$, and therefore AREU posits that $c(\{p',q'\})$ is explained by a more concave utility function than the one used for $c(\{p,q\})$, which corresponds to a weakly steeper indifference curve. Figure **??** provides an illustration. Point 4 is proven analogously. Finally, the immediate consequence of these unidirectional fanning, along with Continuity, rules out the possibility of c being both strictly risk averse and strictly risk loving in this triangle, i.e., point 2 of the proposition.

Proof of Lemma 1

This result is a direct consequence from the fact that WARP and Stationarity are both properties between pairs of choice problems.

To see (1) $\Rightarrow$ (2): Take any set $A \in \mathcal{A}$ and any earliest payment $(x,t) \in \Psi(A)$. Consider the collection of choice problems $\mathcal{S} := \{B \subseteq A : (x,t) \in B\}$. For any pair of choice problems $B_1, B_2 \in \mathcal{S}$, since $(x,t) \in \Psi(B_1) \cap \Psi(B_2)$, c satisfies WARP and Stationarity over $\{B_1, B_2\}$. Therefore, c satisfies WARP and Stationarity over $\mathcal{S}$.

To see (2) $\Rightarrow$ (1): Take any B_1, B_2 such that $\Psi(B_1) \cap \Psi(B_2) \neq \emptyset$. Take $(x,t) \in \Psi(B_1) \cap \Psi(B_2)$. Consider $A := B_1 \cup B_2$. Since B_1 and B_2 are both finite, A is finite, and therefore $A \in \mathcal{A}$. Since $(x,t) \in \Psi(B_1) \cap \Psi(B_2)$, $(x,t) \in \Psi(A)$. By (2) this means c satisfies WARP and Stationarity over $\mathcal{S} := \{B \subseteq A : (x,t) \in B\}$, which contains B_1 and B_2. This means c must satisfy WARP and Stationarity over $\{B_1, B_2\}$ otherwise this is a direct contradiction of the preceding statement.

Proof of Theorem 3

The proof for necessity (utility representation implies axioms) is straightforward, the necessity of Axiom 8 is shown in the main body as a footnote. The proof for sufficiency (axioms imply utility representation) is three-fold.

In Stage 1, we show that with Axiom 6 and Axiom 7, for any time $r \in T$, the set of all choice problems such that the earliest payment arrives at time r can be explained by a nonempty set of Discounted Utility specifications, where a generic element of this set is $(\tilde{u}, \delta)$, a utility function and a discount factor. In Stage 2, we show that at least one utility function u can be supported for all $r \in T$, and for each $r \in T$ we set as δ_r the corresponding discount factor associated with u for r; this is the more involved portion of the proof and it uses Axiom 8. Lastly in Stage 3, with Axiom 8 again, we show the desired relationship between δ_r and $\delta_{r'}$ for any two r, r'.

Denote by $\Psi(A)$ the set of payments in A that arrive soonest, $\Psi(A) := \{(x,t) \in A : t \leq q \, \forall \, (y,q) \in A\}$.

Stage 1: DU representation for each $r \in T$

By Lemma 1 and Lemma 2, for any $x \in X$ and $r \in T$, c satisfies WARP and Stationarity over $S_{(x,r)} := \{A \in \mathcal{A} : (x,r) \in \Psi(A)\}$ ($S_{(x,r)}$ is the collection of choice sets such that the earliest payment is (x,r)). In fact, WARP and Stationarity hold even when we consolidate the collection of choice problems in which the earliest payment arrives at the same time (although the payments themselves may be different), which we now show.

Lemma 9. *For any $r \in T$, c satisfies WARP and Stationarity over $S_{(\cdot,r)} := \cup_{x \in X} S_{(x,r)}$.*

Proof. Take any two choice sets $A, B \in S_{(\cdot,r)}$ such that either WARP or Stationarity fails. Therefore, it must be that $\Psi(A) \cap \Psi(B) = \emptyset$. Now let's take the worse payment at r for each set: $(x^*, r) \in A$ such that $x^* \leq x$ for all $(x,r) \in A$ and $(y^*, r) \in B$ such that $y^* \leq$ for all $(y,r) \in B$. Suppose without loss of generality $x^* < y^*$ (strict inequality due to $\Psi(A) \cap \Psi(B) = \emptyset$). By Outcome Monotonicity (Axiom 6), adding (x^*, r) to B would not change the choice from B: $c(B \cup \{(x^*, r)\}) = c(B)$. Moreover, $A, B \cup \{(x^*, r)\} \in S_{(x^*,r)}$, and therefore neither WARP nor Stationarity fails between them. If it is Stationarity that is violated between A and B, a contradiction is established. If it is WARP that is violated, it remains to show that if $A \subseteq B$ then

$A \subseteq B \cup \{(x^*, r)\}$ and if $A \supseteq B$ then $A \supseteq B \cup \{(x^*, r)\}$, both of which are immediate since $(x^*, r) \in A$. $\square$

Now that we have established c satisfies WARP and Stationarity over $S_{(\cdot,r)} := \cup_{x \in X} S_{(x,r)}$, it is well-known (Fishburn & Rubinstein (1982)) that along with Axiom 6 we achieve (many) Discounted Utility (DU) representations, for instance by translating the time-index by $-r$ so that time r is time 0, etc.

Stage 2: $u_r = u_0$ for each $r \in T$

Take $r = 0$ and arbitrarily pick a DU representation (δ_0, u_0) for c on $S_{(x,r)}$, and define the ultimate utility function $U_0(x,t) : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ by $U_0(x,t) := \delta_0^t u_0(x)$.

We proceed to show that for every other $r \in T$, there exists a DU representation (δ_r, u_r) such that $u_r = u_0$ (and later on, that $\delta^* > \delta$). Fix any $r \in (a,b)$. Define $U_r(x,t) : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ by $U_r(x,t) := \delta_r^t u_t(x)$, a utility function for c on $S_{(x,r)}$.

Lemma 10. *For each $x \in (a,b)$, there exists an open set $(x^-, x^+) \subseteq (a,b)$ such that $x^- < x < x^+$ and for all x_1, x_2 where $x^- < x_1 < x < x_2 < x^+$, we have*

$$U_0\left(x, \alpha\tilde{t}_1 + (1-\alpha)\tilde{t}_2\right) = U_0\left(x_1, \tilde{t}_1\right) = U_0\left(x_2, \tilde{t}_2\right)$$

if and only if

$$U_r(x, \alpha t_1 + (1-\alpha) t_2) = U_r(x_1, t_1) = U_r(x_2, t_2).$$

Moreover, there exists $\lambda \in \mathbb{R}$ such that for all $z_1, z_2 \in (x^-, x^+)$,

$$U_0\left(z_1, \tilde{t}_1\right) = U_0\left(z_2, \tilde{t}_2\right) \text{ if and only if } U_r(z_1, t_1) = U_r\left(z_2, t_1 + \lambda\left(\tilde{t}_2 - \tilde{t}_1\right)\right).$$

Proof. This proof relies primarily on Part 2 of Axiom 8. Fix any $x \in (a,b)$. Since $U_r(\cdot,\cdot)$ is continuous and decreasing in it's second argument, there exists $q \in (r, \bar{t})$ such that $c(\{(a,r),(x,q)\}) = \{(x,q)\}$. Likewise, there exists $\tilde{q} \in (0,\bar{t})$ such that $c(\{(a,0),(x,\tilde{q})\}) = \{(x,\tilde{q})\}$. Since there is an open set in $(r,\bar{t})$ that contains q, by continuity of $U_r(\cdot,\cdot)$, there exists an open set O in X that contains x such that $x' \in O$ implies $c(\{(a,r),(x,q),(x',q')\}) = \{(x,q),(x',q')\}$ for some $q' \in (r,\bar{t})$. Likewise, there exists an open set $\tilde{O}$ in X that contains x such that $x' \in \tilde{O}$ implies $c(\{(a,0),(x,q),(x',q')\}) = \{(x,q),(x',q')\}$ for some $q' \in (0,\bar{t})$. Now take any open subset $(x^-, x^+) \subseteq O \cap \tilde{O}$ such that $x \in (x^-, x^+)$.

What we have done so far is to focus on an outcome space (x^-, x^+) small enough so that for any two outcomes $x_1, x_2 \in (x^-, x^+)$ such that $x_1 < x_2$, it is possible to find $t_1, t_2 \in (r, \bar{t})$ and $\tilde{t}_1, \tilde{t}_2 \in (0, \bar{t})$ such that (x_1, t_1) is (revealed) indifferent to (x_2, t_2) under U_r and $\left(x_1, \tilde{t}_1\right)$ is (revealed) indifferent to $\left(x_2, \tilde{t}_2\right)$ under U_0. Since U_r and U_0 are DU representations, subtracting a fixed amount of time from each pairs of options does not alter the indifferences, and therefore what we are trying to show is

$$U_0\left(x, \alpha 0 + (1-\alpha)\left(\tilde{t}_2 - \tilde{t}_1\right)\right) = U_0\left(x_1, 0\right) = U_0\left(x_2, \tilde{t}_2 - \tilde{t}_1\right) \tag{6.3}$$

if and only if

$$U_r\left(x, \alpha r + (1-\alpha)\left(t_2 - (t_1 - r)\right)\right) = U_r\left(x_1, r\right) = U_r\left(x_2, t_2 - (t_1 - r)\right). \tag{6.4}$$

Say Equation 6.3 holds, since $(x^-, x^+) \subseteq \tilde{O}$, $c(A) = A$ where $A = \left\{\left(x, \alpha 0 + (1-\alpha)\left(\tilde{t}_2 - \tilde{t}_1\right)\right), (x_1, 0), \left(x_2, \tilde{t}_2 - \tilde{t}_1\right)\right\}$ for some $\alpha \in (0,1)$. By the fact that $(x^-, x^+) \subseteq O$, there exists $t_2 - (t_1 - r) \in T$ such that $c\left(\{(x_1, r), (x_2, t_2 - (t_1 - r))\}\right) = \{(x_1, r), (x_2, t_2 - (t_1 - r))\}$. Consequently, a direct application of Part 2 of Axiom 8 gives $c(B) = B$ where $B = \left\{\left(x, \lambda\left(\alpha 0 + (1-\alpha)\left(\tilde{t}_2 - \tilde{t}_1\right)\right) + d\right), (x_1, r), (x_2, t_2 - (t_1 - r))\right\}$ and $\lambda, d \in \mathbb{R}$ solves $r = \lambda 0 + d$ and $t_2 - (t_1 - r) = \lambda\left(\tilde{t}_2 - \tilde{t}_1\right) + d$, the solution for which is

$$d = r$$
$$\lambda = \frac{t_2 - t_1}{\tilde{t}_2 - \tilde{t}_1}.$$

Then, the term $\lambda\left(\alpha 0 + (1-\alpha)\left(\tilde{t}_2 - \tilde{t}_1\right)\right) + d$ simplifies to $\alpha r + (1-\alpha)\left(t_2 - (t_1 - r)\right)$, which gives Equation 6.4 as desired. The proof that Equation 6.4 implies Equation 6.3 is analogous.

The second portion of the lemma is given by applying the first portion multiple times. □

Likewise, we can come up an analogous result for $x = a$ and $x = b$ (recall that $X = [a, b] \subseteq \mathbb{R}_{>0}$).

Lemma 11. *For $x = a$ (resp. $x = b$), there exists $x^+ \in (a, b)$ (resp. $x^- \in (a, b)$) such that for all x_1, x_2 where $a \leq x_1 < x < x_2 < x^+$ (resp. $x^- < x_1 < x < x_2 \leq b$), we have*

$$U_0\left(x, \alpha \tilde{t}_1 + (1-\alpha)\tilde{t}_2\right) = U_0\left(x_1, \tilde{t}_1\right) = U_0\left(x_2, \tilde{t}_2\right)$$

if and only if

$$U_r\left(x, \alpha t_1 + (1-\alpha) t_2\right) = U_r\left(x_1, t_1\right) = U_r\left(x_2, t_2\right).$$

Moreover, there exists $\lambda \in \mathbb{R}$ such that for all $a \leq z_1 < z_2 < x^+$ (resp. $x^- < z_1 < z_2 \leq b$),

$$U_0\left(z_1, \tilde{t}_1\right) = U_0\left(z_2, \tilde{t}_2\right) \text{ if and only if } U_r\left(z_1, t_1\right) = U_r\left(z_2, t_1 + \lambda\left(\tilde{t}_2 - \tilde{t}_1\right)\right).$$

Proof. The proof is essentially identical to the proof for Lemma 10, with the exception that it is impossible to build an open set around $x = a$ and $x = b$. □

Using Lemma 10, we now show that if indifference between some two outcomes are preserved when the time gap between them are expanded by a certain factor λ, then indifferences between any two outcomes are also preserved when the time gap between them are expanded by the factor λ.

Lemma 12. *If $c\left(\left\{(a,0), \left(y, \tilde{t}\right)\right\}\right) = \left\{(a,0), \left(y, \tilde{t}\right)\right\}$ and $c\left(\left\{(a,r), (y,t)\right\}\right) = \left\{(a,r), (y,t)\right\}$, then for all $x^* \in X$,*

$$U_0\left(a, \tilde{t}\right) = U_0\left(x^*, t^*\right) \text{ if and only if } U_r\left(a, t\right) = U_r\left(x^*, t + \lambda\left(t^* - \tilde{t}\right)\right)$$

where $\lambda = \frac{t-r}{\tilde{t}-0}$.

Proof. Note that it is possible to have $t^* > \bar{t}$, which is the case that x^* is so good it is impossible to make x^* and x indifferent within the time span allowed in T; nonetheless, these utility functions are well defined (e.g., $U_0\left(x^*, t^*\right) = \delta_0^{t^*} u\left(x^*\right)$ is a number).

Before we provide the complete proof, we illustrate the key component to this proof: Suppose $a < x^* < y < x^+$, where x^+ is defined in Lemma 11. Then, a direct application of Lemma 11 where $x_1 = a$, $x_2 = y$ and $x = x^*$ would achieve the desired result. Whereas if $a < y < x^* < x^+$, then a direct application of Lemma 11 where $x_1 = a$, $x_2 = x^*$ and $x = y$ would achieve the desired result. This illustration becomes insufficient when $x^* > x^+$. Therefore, the complete proof uses the fact that the collection of open intervals $\left(x^+, x^-\right)$, one for each x where $x \in (a, x^*]$, would create a chain of pseudo-indifferences from a to x^*.

If $x^* = a$, the proof is trivial.

Consider $x^* \in (a,b)$. Let $\hat{\mathbb{C}} := \left\{\left(x^+, x^-\right)_x : x \in (a, x^*]\right\}$ be the collection of open sets $\left(x^+, x^-\right)_x$ defined for each x in Lemma 10. Now add $[a, x^+)$, defined for $x = a$ in Lemma 11, to this collection and obtain $\mathbb{C} := \hat{\mathbb{C}} \cup \left\{[a, x^+)\right\}$. Since this is a collection of open neighborhoods (other than at the boundary a) around each element of an interval, there exists a sequence of intervals, each of which an element of $\mathbb{C}$, that starts with $[a, x^+)$ and ends with $(x^+, x^-)_{x^*}$ such that every two consequence intervals have non-empty intersection. (If a sequence, indexed by x, converges to a point $\ell < x^*$, then we can build another sequence by removing an infinite tail of original sequence and continuing with $(x^+, x^-)_{\ell}$.) Therefore, for every two consecutive intervals $(x^+, x^-)_{y_1}$, $(x^+, x^-)_{y_2}$, the second portion of Lemma 10 establishes that if

$$U_0\left(z_1, \tilde{t}_1\right) = U_0\left(z_2, \tilde{t}_2\right)$$
$$U_r\left(z_1, t_1\right) = U_r\left(z_2, t_1 + \hat{\lambda}\left(\tilde{t}_2 - \tilde{t}_1\right)\right)$$

for some $z_1, z_2 \in (x^+, x^-)_{y_1}$ and $\hat{\lambda} \in \mathbb{R}$, then

$$U_0\left(z_3, \tilde{t}_3\right) = U_0\left(z_4, \tilde{t}_4\right) \text{ if and only if } U_r\left(z_3, t_3\right) = U_r\left(z_4, t_3 + \hat{\lambda}\left(\tilde{t}_4 - \tilde{t}_3\right)\right)$$

for all $z_3, z_4 \in (x^+, x^-)_{y_2}$. (Loosely speaking, the intersection helped transfer $\hat{\lambda}$ from the first interval to the second.) Finally, iterating through this sequence of intersections gives $\hat{\lambda} = \lambda = \frac{t-r}{\tilde{t}-0}$, and completes the proof.

If instead $x^* = b$, then we do the same but with $\mathbb{C} := \hat{\mathbb{C}} \cup \{[a, x^+), (x^-, b]\}$, where $(x^-, b]$ is defined for $x = b$ in Lemma 11. □

Finally, since Continuity (or refer to the proof of Lemma 11) guarantees the existence of with $\left(y, \tilde{t}\right)$ such that $c\left(\left\{(a, 0), \left(y, \tilde{t}\right)\right\}\right) = \left\{(a, 0), \left(y, \tilde{t}\right)\right\}$ and $c\left(\left\{(a, r), (y, t)\right\}\right) = \left\{(a, r), (y, t)\right\}$, with Lemma 12, we conclude that (δ_r, u_r) such that $u_r = u_0$ and $\delta_r = \delta_0^{-\lambda}$ where $\lambda = \frac{t-r}{\tilde{t}-0}$ is a DU representation for c on $S_{(\cdot, r)}$.

The analysis thus far was for $r \in [0, \bar{t})$. When $r = \bar{t}$, since every choice problems in $S_{(\cdot, \bar{t})}$ contains only payments that arrive at time $\bar{t}$, a DU representation is trivially available with any positive $\delta_{\bar{t}}$ and any strictly increasing $u_{\bar{t}}$. Therefore, we set $u_{\bar{t}} = u_0$ and $\delta_{\bar{t}} = \sup_{r \in [0, \bar{t})} \delta_r$. Due to Impatience (Axiom 6), $\delta_{\bar{t}} \in (0, 1)$.

Stage 3: $\delta_r \geq \delta_{r'}$ for all $r > r'$

If $r = \bar{t}$, this is trivial from the construction of $\delta_{\bar{t}}$. Consider any $r, r' \in [0, \bar{t})$. By continuity of c (or refer to the proof of Lemma 10 and Lemma 11), there exists $y > a$ such that $c\left(\left\{(a, r), (y, t)\right\}\right) = \left\{(a, r), (y, t)\right\}$ and $c\left(\left\{(a, r'), (y, t')\right\}\right) = \left\{(a, r'), (y, t')\right\}$ for some $t, t' \in T$, and therefore $\delta_r^r u(a) = \delta_r^t u(y)$ and $\delta_{r'}^{r'} u(a) = \delta_{r'}^{t'} u(y)$.

Suppose for contradiction $\delta_{r'} > \delta_r$, then $t' - r' > t - r$, or $t' > t - r + r'$. Consider $(y, t - r + r')$, since $t \in T$ and $r > r'$, we have $t - r + r' \in T$, and therefore

$$c\left(\left\{\left(a, r'\right), \left(y, t - r + r'\right)\right\}\right) = \left\{\left(y, t - r + r'\right)\right\} \tag{6.5}$$

from $\delta_{r'}^{r'} u(a) = \delta_{r'}^{t'} u(y) < \delta_{r'}^{t-r+r'} u(y)$. However, by construction,

$$c\left(\left\{\left(a, r' + d\right), \left(y, \left(t - r + r'\right) + d\right)\right\}\right) = \left\{\left(a, r' + d\right), \left(y, \left(t - r + r'\right) + d\right)\right\} \tag{6.6}$$

where $d = r - r' > 0$. Equation 6.5 and Equation 6.6 jointly contradict Part 1 of Axiom 8.

Proof of Theorem 4

The proof for necessity (utility representation implies axioms) is straightforward. The proof for sufficiency (axioms imply utility representation) is three-fold.

The first two stages show that with Axiom 10 and Axiom 9, for each r, the set of all choice problems where the most balanced alternative has a Gini coefficient of r can be explained by the maximization of $x + v_r(y)$ for some unique $v_r : [w, +\infty) \to \mathbb{R}$. Then, the third stage shows that $r > r'$ and $y > y'$ implies $v_r(y) - v_r(y') \geq v_{r'}(y) - v_{r'}(y')$.

Stage 1: $x + v_{(x,y)}(y)$ **for each** $(x, y) \in X$

For each alternative $(x, y) \in X$, define the *reference-dominated* set of (x, y) by

$$R^{\downarrow}((x, y)) := \left\{(x', y') \in X : G(x, y) \leq G(x', y')\right\}$$

and the *prediction set* of (x, y) by

$$\mathbb{P}^{(x,y)} := \left\{(x', y') \in X : G(x, y) \leq G(x', y') \text{ and } (x', y') \in c\left(\left\{(x', y'), (x, y)\right\}\right)\right\}.$$

By Axiom 10, c satisfies WARP over the collection of choice problems $\mathcal{S} = \{A \in \mathcal{A} : r(A) = G(x, y) \text{ and } (x, y) \in A\}$. By Theorem 1, each alternative (x, y) admits a real-valued utility function $u^* : X \to \mathbb{R}$ that explains the choices from choice problems in $\mathcal{S}$, which for convenience we notate as $c : \mathcal{S} \to \mathcal{A}$.

Note that for all $(x', y') \in R^{\downarrow}((x, y))$, $u^*(x', y') \geq u^*(x, y) := \bar{u}$ if and only if $(x', y') \in \mathbb{P}^{(x,y)}$. By Continuity and Axiom 9 (Monotonicity), this is region is character-

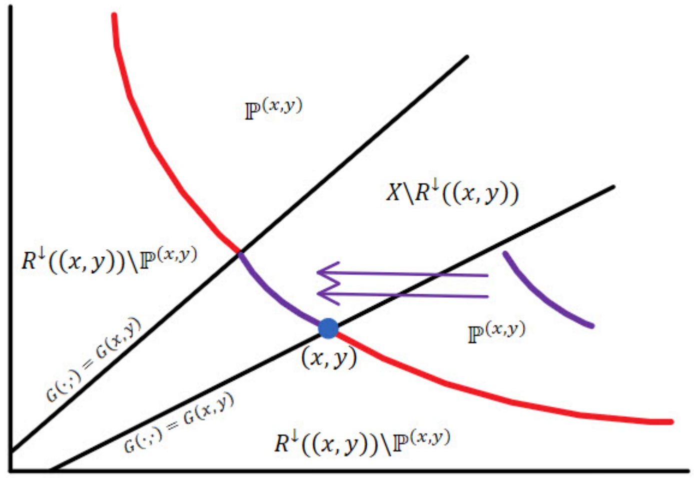

Figure 6.3: This figure illustrates the construction of $v_{(x,y)}$ for a fixed $(x,y) \in X$. The space X is divided into three regions: (1) $X\backslash R^{\downarrow}((x,y))$ is the set of alternatives that have a lower Gini coefficient than (x,y), and therefore they are never in a choice problem where (x,y) is the reference. The alternatives in $R^{\downarrow}((x,y))$ are then split into two groups: (2) those that are chosen when (x,y) is the reference, $\mathbb{P}^{(x,y)}$, and (3) those that are not, $R^{\downarrow}((x,y))\backslash\mathbb{P}^{(x,y)}$. These two groups are separated by the indifference curve passing through (x,y), the red curve, which partially defines $v_{(x,y)}$ (it is partial because $X\backslash R^{\downarrow}((x,y))$ separates the space). The rest of $v_{(x,y)}$ can be defined by using the different curve that connects $(x+a,y)$ and (x^*+a,y^*), the purple curve, where $G(x^*,y^*) = G(x,y)$, $c(\{(x,y),(x^*,y^*)\}) = (x,y),(x^*,y^*)$, and $\{(x+a,y),(x^*+a,y^*)\} \subset \mathbb{P}^{(x,y)}$.

ized by the upper graph, Moreover, since c satisfies Quasi-linearity over $\mathcal{S}$ (Axiom 10), the portion of $u^*(x',y')$ with domain $\mathbb{P}^{(x,y)}$ (which contains (x,y) itself) must be a strictly increasing transformation of $x'+v_{(x,y)}(y')$ for some unique $v_{(x,y)} : [w,+\infty) \to \mathbb{R}$. Also, the portion of $u^*(x',y')$ with domain $X\backslash R^{\downarrow}((x,y))$ is irrelevant, since no choice problem in $\mathcal{S}$ contains any alternative from this set. Finally, for the portion of $u^*(x',y')$ with domain $R^{\downarrow}((x,y))\backslash\mathbb{P}^{(x,y)}$, it must be that $x'+v_{(x,y)}(y') < u^*(x,y) = x+v_{(x,y)}(y)$: Otherwise, since for some a we have $\{(x+a,y),(x'+a,y')\} \subseteq \mathbb{P}^{(x,y)}$, and therefore $(x'+a,y') \in c\{(x,y),(x+a,y),(x'+a,y')\}$, the fact that $(x',y') \in R^{\downarrow}((x,y))\backslash\mathbb{P}^{(x,y)}$ and $\{\{(x,y),(x',y')\},\{(x+a,y),(x'+a,y')\}\} \subset \mathcal{S}$ contradicts the fact that c satisfies Quasi-linearity over $\mathcal{S}$ (Axiom 10).

Figure 6.3 provides an illustration of how $v_{(x,y)}$ is constructed.

Stage 2: $x + v_r(y)$ **for each** r

Fix an r, we now show that $v_{(x,y)}$ must coincide for all (x, y) where $G(x, y) = r$. Consider the collection of choice problems $\mathcal{S} := \{A \in \mathcal{A} : r(A) = r\}$. Note that c satisfies WARP and Quasi-linearity over $\mathcal{S}$. To see this, take any two choice problems A_1, A_2 in $\mathcal{S}$. For each $i = 1, 2$, there must be an alternative $(x_i, y_i) \in A_i$ such that $G(x_i, y_i) = r$ and $G(x', y') \geq r$ for all other (x', y') in A_i. Consider an income distribution (x^*, y^*) such that $x^* \leq \min\{x_1, x_2\}$ and $y^* \leq \min\{y_1, y_2\}$ and $G(x^*, y^*) = r$. Due to $(x_i, y_i) \in \Psi(A_i \cup \{(x^*, y^*)\})$, Axiom 10 (Equality Dependence), and Axiom 9 (Monotonicity), we have $c(A_i) = c(A_i \cup \{(x^*, y^*)\})$. But $(x^*, y^*) \in \Psi(A_1 \cup A_2 \cup \{(x^*, y^*)\})$, so by Axiom 10 again, $c(A_1 \cup \{(x^*, y^*)\})$ and $c(A_2 \cup \{(x^*, y^*)\})$, which as established are just $c(A_1)$ and $c(A_2)$, cannot result in a violation of WARP or Quasi-linearity. Consequently, $v_{(x,y)}$ must coincide for all (x, y) such that $G(x, y) = r$.

Stage 3: $r < r'$ **and** $y > y'$ **implies** $v_r(y) - v_r(y') \geq v_{r'}(y) - v_{r'}(y')$

Finally we show that for all $r < r'$, $v_r(y) - v_r(y') \geq v_{r'}(y) - v_{r'}(y')$ for all $y > y'$ (reminder: higher r implies greater attainable equality). Suppose not, our goal is to substantiate a contradiction of Axiom 11 in the choice correspondence. Fix any $y, y' \in \mathbb{R}_+$ such that $y > y'$. Define $\tilde{v}_r = v_r(y) - v_r(y')$ and $\tilde{v}_{r'} = v_{r'}(y) - v_{r'}(y')$. We want to show that $\tilde{v}_r \geq \tilde{v}_{r'}$.

Suppose for contradiction that this is not true, let z be any value such that $\tilde{v}_r < z < \tilde{v}_{r'}$.

Consider $(x_0, w), (x_1, w) \in X$ such that

$$G(x_0, w) = r' \tag{6.7}$$

$$G(x_1, w) = r, \tag{6.8}$$

which exist because $G(\cdot, w)$ is continuous and increasing in it's first argument from $G(w, w) = 0$ to $\lim_{x \to +\infty} G(x, w) = 0.5$ and $r, r' \in [0, 0.5)$. Consider $x := z + \Delta$, $x' := 2z + \Delta$ for some $\Delta > 0$ such that

$$\min(\{G(x, y), G(x', y')\}) \geq r' \tag{6.9}$$

and $x' > x > \max(\{x_0, x_1\})$, which is possible because for any fixed $\bar{y}$, $G(x, \bar{y})$ is asymptotically increasing in it's first argument and $\lim_{x \to +\infty} G(x, \bar{y}) = 0.5$, and $r' \in$

$[0, 0.5)$. Essentially, we have introduced reference points $(x_0, w), (x_1, w)$ that will not be chosen, forcing the choice to be between (x, y) and (x', y')

We now use the constructed alternatives $(x, y), (x', y'), (x_0, w), (x_1, w)$ to demonstrate a violation of Axiom 11. Consider the choice problem $A := \{(x, y), (x', y'), (x_0, w)\}$. Due to Equation 6.7 and Equation 6.9, $c(A)$ consists of all maximizers of the utility function $\hat{x} + v_{r'}(\hat{y})$, and (x_0, w) will not be chosen since $x_0 < x$ and $w \leq y$. Since $\tilde{v}_{r'} > z$, or equivalently $z + v_{r'}(y) > 2z + v_{r'}(y')$, we have $x + v_{r'}(y) > x' + v_{r'}(y')$ and therefore

$$c(\{(x, y), (x', y'), (x_0, w)\}) = \{(x, y)\}. \tag{6.10}$$

Likewise, due to Equation 6.8 and Equation 6.10, $c(A \cup \{(x_1, w)\})$ consists of all maximizers of the utility function $\hat{x} + v_r(\hat{y})$ and both (x_0, w), (x_1, w) will not be chosen. Since $z > \tilde{v}_r$, or equivalently $2z + v_r(y') > z + v_r(y)$, we have $x' + v_r(y') > x + v_r(y)$ and therefore

$$c(\{(x, y), (x', y'), (x_0, w), (x_1, w)\}) = \{(x', y')\}. \tag{6.11}$$

Since $y > y'$, Equation 6.10 and Equation 6.11 jointly contradict Axiom 11. This establishes $v_r(y) - v_r(y') \geq v_{r'}(y) - v_{r'}(y')$ for all $y > y'$ and $r < r'$.

Proof of Proposition 1, Proposition 3, and Proposition 4

In each of the three models, it is straightforward that we arrive at the standard models (expected utility, exponential discounting, quasi-linear utility) when the corresponding preference parameters converge. That is, for all r, r', $u_r = u_{r'}$ in the risk model, $\delta_r = \delta_{r'}$ in the time model, and $v_r = v_{r'}$ in the social preference model. Therefore, the key to showing WARP and structural postulates (Independence, Stationarity, Quasi-linearity) are independently sufficient for the standard models is by arguing that non-convergent preference parameters necessitates both WARP violations and violations of structural postulates.

The remaining statements, that WARP and structural postulates (Independence, Stationarity, Quasi-linearity) are necessary for standard models ((1) if (3), (2) if (3)), and that WARP is sufficient and necessary for a standard utility representation ((1) if and only if (4)), are straightforward and omitted.

Proof of Proposition 1

(1) WARP implies (3) Expected Utility We first show that if c satisfies Transitivity over $\mathcal{A}$ (defined in Subsection 3.7), a necessary condition for WARP, then indifferent curves are linear and parallel on the convex hull of p, δ_b, δ_w for every p. For any $p \in \Delta(X) \backslash \Delta(\{b, w\})$, define $p^* \in \Delta(X)$ by the probabilities

$$p^*(x) := \begin{cases} p(x) / (1 - p(b) - p(w)) & \text{if } x \in X \backslash \{b, w\} \\ 0 & \text{otherwise} \end{cases}.$$

Furthermore, let $\tau_p \subseteq \Delta(X)$ ("tau" for triangle) be the convex hull

$$\tau_p := \operatorname{conv}(\{p^*, \delta_b, \delta_w\}).$$

Note that $p \in \tau_p$, $\tau_{p^*} = \tau_p$, and $\cup_{p \in \Delta(X)} \tau_p = \Delta(X)$. Moreover, for every $q \in \tau_p$ such that $q(b), q(w) > 0$, q is an extreme spread of p^* and therefore $p^* R q$.

Also, denote by ∂_p the (partial) boundary

$$\partial_p := [\operatorname{conv}(\{p^*, \delta_b\}) \cup \operatorname{conv}(\{p^*, \delta_w\})] \backslash \{\delta_b, \delta_w\},$$

which is a subset of the boundary of τ_p.

Lemma 13. *Suppose c admits an AREU representation and satisfies Transitivity over $\mathcal{A}$. For any $p \in \Delta(X) \backslash \Delta(\{b, w\})$, if $p_1, p_2 \in \tau_p \backslash \Delta(\{b, w\})$, then $u_{p_1}(x) = u_{p_2}(x)$ for all x.*

Proof. Fix $p \in \Delta(X) \backslash \Delta(\{b, w\})$. For each $q \in \partial_p$, due to FOSD and continuity of c, there exists a lottery that involves only the best and the worst prizes $e_q \in \Delta(\{b, w\})$ such that $c(\{q, e_q\}) = \{q, e_q\}$. Moreover due to FOSD, $e_q(b) > q(b)$ and $e_q(w) > q(w)$, and therefore e_q is an extreme spread of q, which implies $q R e_q$ and therefore $\mathbb{E}_q(u_q(x)) = e_q(b)$ since u_q explains $c(\{q, e_q\})$ and $\mathbb{E}_{q_e}(u_s(x)) = e_q(b)$ for all s (due to our normalization that $u_s(w) = 0$ and $u_s(b) = 1$ for all s).

Consider the line $L_q := \overline{q e_q} = \operatorname{conv}(\{q, e_q\})$. Every element $q' \in L_q$ is an extreme spread of q and therefore $q R q'$, which implies $c(\{q, q'\}) = \arg\max_{s \in \{q, q'\}} \mathbb{E}_s(u_q(x)) = \{q, q'\}$ (since q' is a convex combination of q and q_e, which are indifferent under u_q). By Transitivity, this means for all $q_1, q_2 \in L_q$, we have $c(\{q_1, q_2\}) = \{q_1, q_2\}$, and therefore for all $s \in L_q \backslash \{e_q\}$, we have $\mathbb{E}_q(u_s(x)) = q_e(b)$, a constant. However, any two utility functions u_1, u_2 in AREU are related by a concave transformation, say WLOG $u_1 = f \circ u_2$ for some concave f, and therefore either $u_1 = u_2$ or $u_1(x) > u_2(x)$ for all $x \in X \setminus \{b, w\}$ (recall that $u_i(w) = 0$ and $u_i(b) = 1$). Now, since $\mathbb{E}_q(u_{s_1}(x)) = \mathbb{E}_q(u_{s_2}(x))$ for all $s_1, s_2 \in L_q \setminus \{e_q\}$ and $q(x) > 0$ for some $x \in X \setminus \{b, w\}$, we conclude that for all $s_1, s_2 \in L_q \setminus \{e_q\}$, we have $u_{s_1} = u_{s_2}$.

We have established that for every line L_q where $q \in \partial_p$, the utility function indexed by every alternative on that line (except e_q) is identical to u_q. Also, since for any $s \in \tau_p$ there exists some $q \in \partial_p$ such that $c(\{s, e_q\}) = \{s, e_q\}$ (due to FOSD and continuity of c), and since for every $e \in \partial_p$ there exists $q \in \partial_p$ such that $e = e_q$ (due to continuity of c), we have $\cup_{q \in \partial_p} L_q = \tau_p$.

We now show that for any two $q_1, q_2 \in \partial_p$, we have $u_{q_1} = u_{q_2}$.

Consider the set $Q^+ := \{q_1^*, q_2^*, ...\}$ such that $q_i^* \in \partial_p$ is defined by

$$q_i^* := \alpha_i \delta_b + (1 - \alpha_i) p^*$$

where $\alpha_1 = 0$ and $\alpha_i = \frac{1}{2} e_{q_{i-1}^*}(b) + \frac{1}{2} q_{i-1}^*(b)$ for all $i > 1$. This defines an infinite sequence of lotteries that are convex combinations of δ_b and p^*, and converges to δ_b. Consider any consecutive pair $q_i^*, q_{i+1}^* \in Q^+$. By construction (and this is the reason we constructed them this way), there exist $s_1 \in L_{q_i^*} \setminus \{e_{q_i^*}\}$ and $s_2 \in L_{q_{i+1}^*} \setminus \left\{e_{q_{i+1}^*}\right\}$ such that s_1 is an extreme spread of s_2, and therefore $s_2 R s_1$, which implies $u_{q_{i+1}^*}$ is more concave than $u_{q_i^*}$. Also by construction, there exists $s_1' \in L_{q_i^*} \setminus \{e_{q_i^*}\}$ and $s_2' \in L_{q_{i+1}^*} \setminus \left\{e_{q_{i+1}^*}\right\}$ such that s_2' is an extreme spread of s_1', and therefore $s_1' R s_2'$, which implies $u_{q_i^*}$ is more concave than $u_{q_{i+1}^*}$. Together, this gives $u_{q_i^*} = u_{q_{i+1}^*}$. By iteration, this means $u_{q_i^*} = u_{p^*}$ for all $q_i^* \in Q^+$. Figure 6.4 provides an illustration.

Analogously, consider the set $Q^- := \{q_1^*, q_2^*, ...\}$ such that $q_i^* \in \partial_p$ is defined by

$$q_i^* := \alpha_i \delta_w + (1 - \alpha_i) p^*$$

where $\alpha_1 = 0$ and $\alpha_i = \frac{1}{2} e_{q_{i-1}^*}(w) + \frac{1}{2} q_{i-1}^*(w)$ for all $i > 1$. Using the same line of argument as above, we conclude that $u_{q_i^*} = u_{p^*}$ for all $q_i^* \in Q^-$.

Now consider any $q \in \partial_p$. If $q \in \operatorname{conv}(\{p^*, \delta_b\})$, consider $q_i^* \in Q^+$ such that $q \in \operatorname{conv}\left(\left\{q_{i-1}^*, q_i^*\right\}\right)$. If instead $q \in \operatorname{conv}(\{p^*, \delta_w\})$, consider $q_i^* \in Q^-$ such that $q \in \operatorname{conv}\left(\left\{q_{i-1}^*, q_i^*\right\}\right)$. By construction, there exist $s_1 \in L_q \setminus \{e_q\}$ and $s_2 \in L_{q_i^*} \setminus \{e_{q_i^*}\}$ such that s_1 is an extreme spread of s_2. Also by construction, there exist $s_1' \in L_q \setminus \{e_q\}$ and $s_2' \in L_{q_i^*} \setminus \{e_{q_i^*}\}$ such that s_2' is an extreme spread of s_1'. Then, by using the same line of argument as above, we conclude that $u_q = u_{q_i^*} = u_{p^*}$ for all $q \in \partial_p$.

Therefore, since $\cup_{q\in\partial_p} L_q = \tau_p$, we have $u_s = u_q = u_{p^*}$ for all $s \in \tau_p \backslash \Delta(\{b, w\})$ where $s \in L_q$ and $q \in \partial_p$, and we are done. □

Lemma 14. *Suppose c admits an AREU representation and satisfies Transitivity over $\mathcal{A}$. For any $p_1, p_2 \in \Delta(X) \backslash \Delta(\{b, w\})$, we have $u_{p_1}(x) = u_{p_2}(x)$ for all x.*

Proof. Fix any $p_1, p_2 \in \Delta(X) \backslash \Delta(\{b, w\})$, suppose for contradiction $u_{p_1}(x) \neq u_{p_2}(x)$ for some x. Note that by Lemma 13, $p_1^* \neq p_2^*$. Then, since u_{p_1} and u_{p_2} are related by a concave transformation (and that $u_i(w) = 0$ and $u_i(b) = 1$ for all i), either $u_{p_1}(x) > u_{p_2}(x)$ for all $x \in X \backslash \{b, w\}$ or $u_{p_1}(x) < u_{p_2}(x)$ for all $x \in X \backslash \{b, w\}$. By Lemma 13, either $u_{p_1^*}(x) > u_{p_2^*}(x)$ for all $x \in X \backslash \{b, w\}$ or $u_{p_1^*}(x) < u_{p_2^*}(x)$ for all $x \in X \backslash \{b, w\}$.

Suppose without loss of generality $p_1^* R p_2^*$. It is established in the proof of Lemma 13 that there exists $e_{p_1^*} \in \Delta(\{b, w\})$ such that

$$c\left(\left\{p_1^*, e_{p_1^*}\right\}\right) = \left\{p_1^*, e_{p_1^*}\right\}. \tag{6.12}$$

Furthermore, since $e_{p_1}^* \in \tau_{p_2^*}$, there exists a line $\overline{qe_{p_1^*}} \subseteq \tau_{p_2}$ where $q \in \partial_{p_2^*}$ such that for all $s \in \overline{qe_{p_1^*}}$, we have $\mathbb{E}_s\left(u_{p_1^*}(x)\right) = \mathbb{E}_{p_1^*}\left(u_{p_1^*}(x)\right) = e_{p_1^*}(b)$ (the line $\overline{qe_{p_1^*}}$ is the intersection between $\tau_{p_2^*}$ and the indifference hyperplane consists of all $s \in \Delta(X)$ such that $\mathbb{E}_s\left(u_{p_1^*}(x)\right) = \mathbb{E}_{p_1^*}\left(u_{p_1^*}(x)\right)$). Denote the interior of a set S by Int S. Since Int $\tau_{p_2^*} \subseteq R^{\downarrow}(p_2^*) \subseteq R^{\downarrow}(p_1^*)$ (the first inclusion is due to $s \in$ Int $\tau_{p_2^*}$ implies s is an extreme spread of p_2^*, the second inclusion is because R is transitive), we have $c(\{p_1^*, s\}) = \{p_1^*, s\}$ for all $s \in$ Int $\overline{qe_{p_1^*}}$. Therefore, by continuity c we have

$$c(\{p_1^*, q\}) = \{p_1^*, q\}. \tag{6.13}$$

Recall that either $u_{p_1^*}(x) > u_{p_2^*}(x)$ for all $x \in X \backslash \{b, w\}$ or $u_{p_1^*}(x) < u_{p_2^*}(x)$ for all $x \in X \backslash \{b, w\}$. Therefore, by Lemma 13, we have

$$e_{p_1^*}(b) = \mathbb{E}_q\left(u_{p_1^*}(x)\right) \neq \mathbb{E}_q(u_q(x)) = e_q(b),$$

and therefore

$$c(\{q, e_q\}) = \{q, e_q\} \tag{6.14}$$

for some $e_q \in \Delta(\{b, w\})$ where $e_q \neq e_{p_1^*}$. But by Transitivity of c, Equation 6.12,

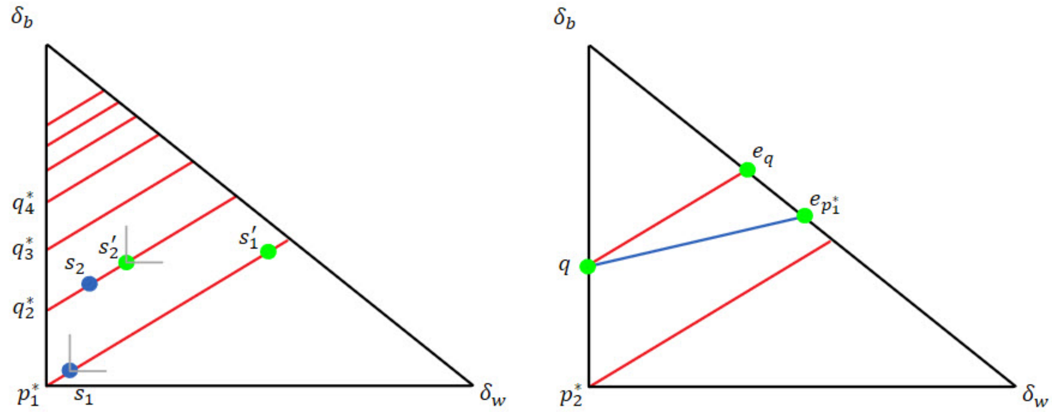

Figure 6.4: The figure on the left provides an illustration for the proof of Lemma 13, the figure on the right provides an illustration for the proof of Lemma 14.

Equation 6.13, and Equation 6.14 would imply $c\left(\left\{e_q, e_{p_1^*}\right\}\right) = \left\{e_q, e_{p_1^*}\right\}$, a violation of FOSD. Figure 6.4 provides an illustration. □

Corollary 3. *Suppose c admits an AREU representation and satisfies Transitivity over $\mathcal{A}$. Then c admits an AREU representation such that for any $p_1, p_2 \in \Delta(X)$, we have $u_{p_1}(x) = u_{p_2}(x)$ for all x.*

Proof. We extend our result from Lemma 14 to all p_1, p_2 by means of constructing an AREU representation. Consider the original AREU representation $\left(\tilde{R}, \{\tilde{u}_r\}_r\right)$. Let $R := \tilde{R}$ and $u_r := \tilde{u}_r$ for all $r \in \Delta(X) \setminus \Delta(\{b, w\})$. Also, for any $r \in \Delta(\{b, w\})$ such that $R^{\downarrow}(r) \nsubseteq \Delta(\{b, w\})$, let $u_r := \tilde{u}_r$. Consider any $s_1 \in R^{\downarrow}(r) \setminus \Delta(\{b, w\})$ and $s_2 \in \Delta(X) \setminus \Delta(\{b, w\})$ such that r is an extreme spread of s_2. Then, $s_2 R r R s_1$ and $s_1, s_2 \in \Delta(X) \setminus \Delta(\{b, w\})$, which implies $u_r = u_s$ for all $s \in \Delta(X) \setminus \Delta(\{b, w\})$. Finally, consider any $r \in \Delta(\{b, w\})$ such that $R^{\downarrow}(r) \subseteq \Delta(\{b, w\})$. Note that any u_r where $u_r(b) = 1$ and $u_r(w) = 0$ would explain c over $\mathbb{A}^r_{R^{\downarrow}(r)} = \left\{A \in \mathcal{A} : A \subseteq R^{\downarrow}(r) \text{ and } r \in A\right\}$. Therefore, we set $u_r := u_s$ for some $s \in \Delta(X) \setminus \Delta(\{b, w\})$. □

We are ready to complete the proof. Suppose c satisfies WARP over $\mathcal{A}$, then c satisfies Transitivity over $\mathcal{A}$. Therefore, by Corollary 3, if c admits an AREU representation, it also admits an AREU representation such that for any $p_1, p_2 \in \Delta(X)$, we have $u_{p_1}(x) = u_{p_2}(x)$ for all x, which means c admits an expected utility representation.

(2) Independence implies (3) Expected Utility Suppose c satisfies Independence over $\mathcal{A}$. Since c admits an AREU representation $\left(\tilde{R}, \{\tilde{u}_r\}_r\right)$, we argue that c also admits an AREU representation with specifications $(R, \{u_r\}_r)$ where $R = \tilde{R}$ and $u_r = \tilde{u}_{r^*}$ for all r, where r^* is the lottery that assigns equal probability to every prize (the choice of r^* is not important for the proof as long as $r^* \notin \Delta(\{b, w\})$.

Since $r^* \notin \Delta(\{b, w\})$, by Lemma 6, $\mathbb{P}_+^{r^*} := \left\{p \in R^{\downarrow}(r^*) : r \notin c(\{p, r^*\})\right\}$ contains a full-dimensional convex subset of $\Delta(X)$. Now suppose for contradiction that for some $r \in \Delta(X)$, u_r cannot explain c over $\mathbb{A}^r_{R^{\downarrow}(r)} := \left\{A \in \mathcal{A} : A \subseteq R^{\downarrow}(r) \text{ and } r \in A\right\}$, then by Lemma 6 and Corollary 1, $\mathbb{P}^r_+$ also contains a full-dimensional convex subset of $\Delta(X)$. Consider lotteries $p_1, q_1 \in \mathbb{P}^r_+$such that $\mathbb{E}_{p_1}(\tilde{u}_r(x)) = \mathbb{E}_{q_1}(\tilde{u}_r(x))$ but $\mathbb{E}_{p_1}(u_{r^*}(x)) \neq \mathbb{E}_{q_1}(u_{r^*}(x))$, which exists since $\mathbb{P}^r_+$ contains a full dimensional convex subset of $\Delta(X)$. By construction, we have

$$c(\{r, p_1, q_1\}) = \{p_1, q_1\}. \tag{6.15}$$

Consider also lotteries $p_2, q_2 \in \mathbb{P}_+^{r^*}$ related to p_1, q_1 by a common mixture, which exists since $\mathbb{P}_+^{r^*}$ contains a full dimensional convex subset of $\Delta(X)$. By construction, we have

$$r \notin c(\{r^*, p_2, q_2\}) \neq \{p_2, q_2\} \tag{6.16}$$

since $\mathbb{E}_{p_2}(u_{r^*}(x)) - \mathbb{E}_{q_2}(u_{r^*}(x)) = \mathbb{E}_{p_1}(u_{r^*}(x)) - \mathbb{E}_{q_1}(u_{r^*}(x)) \neq 0$, but then Equation 6.15 and Equation 6.16 jointly violate Independence.

Proof of Proposition Proposition 3

(1) WARP / (2) Stationarity implies (3) Exponential Discounting Utility Suppose c admits an PBDU representation with specification $(\{\delta_r\}_r, u)$. We show that if $\delta_r \neq \delta_{r'}$ for some $r, r' \in [0, \bar{t})$, then c violates both WARP and Stationarity. ($\delta_{\bar{t}}$ only plays a role for choice problems $A \in \mathcal{A}$ where $(x, t) \in A$ only if $t = \bar{t}$, and therefore we may set it as $\delta_{\bar{t}} = \delta_0$.)

Suppose for contradiction $\delta_r \neq \delta_{r'}$ for some $r, r' \in [0, \bar{t})$. Say without loss of generality $r > r' \geq 0$, then $\delta_r > \delta_{r'} \geq \delta_0$. Recall that $X = [a, b]$. Consider alternatives

$(b-\Delta_x, 0), (b, 0+\Delta_t) \in X \times T$ such that

$$\Delta_x \in (0, b-a) \tag{6.17}$$

$$\Delta_t \in (0, \bar{t}-r) \tag{6.18}$$

$$\delta_r^0 u(b-\Delta_x) < \delta_r^{0+\Delta_t} u(b). \tag{6.19}$$

$$\delta_0^0 u(b-\Delta_x) > \delta_0^{0+\Delta_t} u(b) \tag{6.20}$$

This is possible due to the assumption that $(b, \bar{t}) \in c(\{(a,0), (b,\bar{t})\})$. Equation 6.17 and Equation 6.18 guarantee that $(b-\Delta_x, 0), (b, 0+\Delta_t), (b-\Delta_x, r), (b, r+\Delta_t) \in X \times T$. Then, Equation 6.19 gives

$$c(\{(b-\Delta_x, r), (b, r+\Delta_t)\}) = \{(b, r+\Delta_t)\}, \tag{6.21}$$

and Equation 6.20 gives

$$c(\{(b-\Delta_x, 0), (b, 0+\Delta_t)\}) = \{(b-\Delta_x, 0)\} \tag{6.22}$$

$$c(\{(a,0), (b-\Delta_x, r), (b, r+\Delta_t)\}) = \{(b-\Delta_x, r)\}, \tag{6.23}$$

where Equation 6.23 is due in part to the assumption that $(b, \bar{t}) \in c(\{(a,0), (b,\bar{t})\})$.

Note that Equation 6.21 and Equation 6.22 jointly violate Stationarity, whereas Equation 6.21 and Equation 6.23 jointly violate WARP.

We conclude that if either WARP or Stationarity (or both) holds, then c admits an exponential discounting utility representation.

Proof of Proposition Proposition 4

(1) WARP / (2) Quasi-linearity implies (3) Quasi-linear Utility Suppose c admits an FSPU representation with specification $\{v_r\}_r$. We show that if $v_r(y) - v_r(y') \neq v_{r'}(y) - v_{r'}(y')$ for some r, r' and $y > y'$, then c violates both WARP and Quasi-linearity.

Suppose for contradiction $v_r(y) - v_r(y') \neq v_{r'}(y) - v_{r'}(y')$ for some $r, r' \in [0, 0.5)$ and $y > y'$. Without loss of generality, say $r > r' \geq 0$. Then $v_r(y) - v_r(y') < v_{r'}(y) - v_{r'}(y') \leq v_0(y) - v_0(y')$, and therefore there exist $\tilde{x}, \tilde{x}' \in [w, +\infty)$ such that $\tilde{x}' + v_r(y') > \tilde{x} + v_r(y)$ and $\tilde{x}' + v_0(y') < \tilde{x} + v_0(y)$.

Consider $(x^*, y^*) \in X$ such that $y^* = w$ and $G(x^*, y^*) = r$, which is possible since $G(\cdot, w)$ is continuous increasing in it's first argument from $G(w,w) = 0$ to

$\lim_{x\to+\infty} G(x,w) = 0.5$. Since for any fixed $\bar{y}$, $G(\cdot,\bar{y})$ is asymptotically increasing in it's first argument, there exists $\Delta > 0$ such that $\min(\{G(\tilde{x}+\Delta, y), G(\tilde{x}'+\Delta, y')\}) \geq r$ and $\tilde{x}+\Delta, \tilde{x}'+\Delta > x^*$. Let $x := \tilde{x}+\Delta$ and $x' := \tilde{x}'+\Delta$. We have now established that

$$\min(\{x,x'\}) > x^* \geq w, \min(\{y,y'\}) \geq y^* = w \tag{6.24}$$

$$\min(\{G(x,y), G(x',y')\}) \geq G(x^*,y^*) = r \tag{6.25}$$

$$x' + v_r(y') > x + v_r(y) \tag{6.26}$$

$$x' + v_0(y') < x + v_0(y). \tag{6.27}$$

Then, Equation 6.24, Equation 6.25, and Equation 6.26 give

$$c(\{(x^*,y^*),(x,y),(x',y')\}) = \{(x',y')\}. \tag{6.28}$$

Separately, Equation 6.24 and Equation 6.27 give

$$c(\{(w,w),(x,y),(x',y')\}) = \{(x,y)\} \tag{6.29}$$

$$c(\{(w,w),(x+\epsilon,y),(x'+\epsilon,y')\}) = \{(x+\epsilon,y)\} \tag{6.30}$$

for any $\epsilon > 0$.

Note that Equation 6.28 and Equation 6.30 jointly violate Quasi-linearity. Separately, by WARP, Equation 6.28 and Equation 6.29 imply $c(\{(x,y),(x',y')\}) = \{(x',y')\}$ and $c(\{(x,y),(x',y')\}) = \{(x,y)\}$ respectively, which is also contradiction.

We conclude that if either WARP or Quasi-linearity (or both) holds, then c admits a quasi-linear utility representation.

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