Optimal Time-Consistent Debt Policies*

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Abstract

We study time-consistent debt policies in a trade-off model of debt in which the firm can freely issue new debt and repurchase existing debt. A debt policy is time-consistent if in any state equityholders prefer to follow it rather than to deviate from it but lose credibility in sustaining debt discipline in the future. In a class of policies, the optimal time-consistent debt policy consists of an interest coverage ratio (ICR) target and two regions for the ICR: the stable and the distress regions. In the stable region, the firm actively manages liabilities to the ICR target by issuing/repurchasing debt. A sufficiently large negative shock to cash flows pushes the firm into the distress region, where it abandons the target and waits until either cash flows recover or further negative shocks trigger bankruptcy. Credit spreads are sensitive to cash flow shocks in the distress region but not in the stable region. The optimal policy captures realistic features of debt dynamics, such as active debt management in both directions, interior optimal debt maturity, and dynamics of "fallen angels."

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1 Introduction

Understanding corporate leverage and its dynamics is a fundamental question of corporate finance. One of the most influential theories in corporate finance, the static trade-off theory of capital structure, posits existence of a leverage target that trades off the present values of tax benefits of debt against the costs of financial distress in an optimal way. Firms are supposed to manage liabilities to stay at the target, issuing additional debt after positive shocks to firm value and reducing borrowing after negative shocks. However, as Admati et al. (2013) recently emphasized, this policy is not dynamically consistent: Reducing leverage is not in the ex-post interest of equity holders because of the debt overhang problem, a phenomenon termed the leverage ratchet effect. As Peter DeMarzo stresses in his 2019 AFA presidential address to the American Financial Association, understanding the firm's choice of debt policies over time requires thinking through the implications of this commitment problem.

In an insightful paper, DeMarzo and He (forthcoming) solve for the Markov perfect equilibria (henceforth, MPE) in the firm's problem of debt choice when it cannot commit to future debt decisions. The implied equilibrium debt policy has stark properties manifesting the leverage ratchet effect in a strong form: The firm never reduces its stock of debt unless it matures, it always borrows more over time, and in equilibrium leverage adds no value because all tax benefits of debt get offset by increased bankruptcy costs. Because of the latter, equity holders are indifferent between debt of different maturities.

In this paper, like DeMarzo and He (forthcoming), we are interested in exploring leverage dynamics without commitment. However, we ask a different research question: How well could the firm do by optimizing over its dynamic debt policies subject to the constraint that a debt policy must be time-consistent, that is, equity holders ex-post prefer to stick to the policy they chose ex-ante? And what are the properties of the optimal time-consistent debt policy? Answering these questions provides an alternative benchmark to DeMarzo and He (forthcoming) of leverage dynamics in a trade-off model when the firm cannot commit to future debt issuance and repurchase decisions.

The model setup captures standard ingredients of the classic trade-off between the tax benefits of debt and costs of default and follows DeMarzo and He (forthcoming). The firm's debt is associated with tax benefits, but it also makes (endogenous) default more likely, which destroys value. The equity holders control the firm's debt dynamics and can issue new debt and repurchase existing debt costlessly at any moment in time. They can also default at any point in time. Equity holders cannot commit to their future debt or default policy. New debt issues/repurchases are priced by the debt holders based on their expectations about the firm's future leverage choices and default likelihood.

We introduce two novel features into this setting. First, we depart from the assumption that the cash flows follow the diffusion process and introduce Poisson downward jumps. As we show, the potential for large negative cash flow shocks turns out to be an important determinant of optimal financial policies.

The second and more important novel feature is our focus on the optimal debt policy that is time-consistent. Specifically, a debt policy is a rule that for each level of the interest coverage ratio (henceforth, ICR), which is the state variable in the model, prescribes a certain fraction of debt to be issued or repurchased. We consider a rich class of policies, in which the equity holders issue new debt when the state reaches the issuance boundary and repurchase debt when the state falls into a certain repurchase region. Since the equity holders can freely adjust their future leverage, they face temptation of deviating from a particular debt policy.

We analyze *time-consistent* debt policies, which are policies immune to any such deviations. We specify that if such a deviation occurs, then the equity holders lose credibility in sustaining discipline in debt management in the future and the play switches to the MPE of the debt issuance game, characterized in DeMarzo and He (forthcoming). Formally, a time-consistent debt policy should satisfy *credibility constraints* requiring that in any state, the equity value under the policy exceeds the value of a deviation to alternative debt issuance/repurchase given that the price of newly issued/repurchased debt and the continuation equity value are as in the MPE.

We characterize the debt price and the equity value under different debt policies in closed form and derive the optimal time-consistent debt policy in our class of policies. The optimal policy consists of a particular ICR target and two regions for the ICR: the stable and the distress regions. The equity holders issue initial debt to reach the ICR target. As long as shocks to cash flows are small so that the ICR remains in the stable region, the equity holders issue or repurchase debt in order to compensate these shocks and stay at the chosen ICR target. In these times, the debt price is stable and does not react to shocks to cash flows, even though, debt is not risk-free and there is a positive credit spread. If a sufficiently large negative shock to cash flows arrives, the ICR drops and the firm enters the distress region. In this region, the firm waits until either the cash flows recover, at which point the firm repurchases a bulk of debt to reach the target again, or further negative shocks push the firm into bankruptcy. In the distress region, the debt price is sensitive to further shocks to cash flows.

As this description illustrates, this policy combines the features of the optimal financial policy in the static trade-off theory of capital structure with the implications of the leverage ratchet effect, stressed in Admati et al. (2013) and DeMarzo and He (forthcoming). The way the firm manages its finances in normal times is exactly as prescribed in the static trade-off theory: The firm sets a target and actively manages liabilities to get back to the target from either side. In contrast, the way the firm manages its finances in distress times reflects the leverage ratchet effect: The firm does not repurchase debt, even though it is over-levered, taking a wait-and-see approach.

We next study in what way the equity holders' lack of commitment constrains their debt policy choices, and how various parameters affect the severity of commitment issue, and through it, the firm's optimal debt policies. To study the former issue, we compare the optimal time-consistent policy to the optimal policy with "commitment," when the equity holders can commit to the debt policy (but not to default decisions), and hence, the credibility constraints are not relevant. Such an optimal policy also takes the form of targeted ICR, however, it compensates larger drops in cash flows with repurchases. This allows the firm to borrow more in the optimum with commitment. While these larger repurchases make debt safer and increase the overall firm value, they are too costly for the equity holders to execute, and hence, are not credible in the absence of commitment.

Interestingly, credibility constraints do not bind, when the equity holders merely promise to refrain from debt issuance close to default. Specifically, if we consider only policies that allow for debt issuances, but not repurchases, then the credibility constraints do not bind in the optimal time-consistent debt policy (which also coincides with the optimal policy with commitment). Thus, in order to explain how lack of commitment limits firm's leverage, it is important to have large repurchases in the optimum, which is attained in our model by allowing for both repurchases and large drops in cash flows.

Taking into consideration binding credibility constraints reveals the economic mechanisms behind the comparative statics, which are less apparent when one simply compares the MPE and the optimal policy with commitment. As an example, the comparative statics with respect to the volatility of Brownian shocks are quite nuanced and depend on whether the credibility constraints bind or not. When the credibility constraints do not bind, higher volatility leads to higher leverage. To see this, recall that in the classic Leland (1994) model, higher volatility has an ambiguous effect on firm value. On the one hand, when the firm is close to default, it increases the chances of escaping the default. On the other hand, when the firm is far from the default, it increases the chances of cash flows deteriorating and the firm sliding to default. While the former force is present under the targeted ICR policy, the latter is attenuated, because the equity holders compensate all negative Brownian shocks with debt repurchases when they are at the target ICR. Because of that, higher diffusion volatility tends to increase leverage in the case when the credibility constraints do not play a role. When the credibility constraints bind, the comparative statics with respect to the volatility of diffusion is reversed. Higher volatility of diffusion increases the equity value in the MPE, which makes the credibility constraints more restrictive. This, in turn, reduces the leverage and makes the debt more risk, because the maximal credible repurchase that the equity holders can make is reduced.

The comparatives statics is unambiguous with respect to intensity/frequency of Poisson jumps. Naturally, more severe or more frequent downward jumps in cash flows reduce the leverage ratio and increase the ICR target. Interestingly, lower leverage does not compensate completely for the increased riskiness of cash flows, which results in higher credit spreads at the ICR target.

The optimal time-consistent policy captures many realistic features of debt management by companies, namely, (i) switches in debt dynamics between normal and distress periods, (ii) realistic repurchases, and (iii) interior optimal maturity. Importantly, these features arise within a classical trade-off theory and do not require introduction of any additional frictions in the model apart from limited commitment.

First, the model captures the following evolution of company's leverage. Companies often announce at the initiation targets for financial policies and subsequently try to stay close to them. However, following significant negative shocks to cash flows, the company's debt can be downgraded and the company becomes a "fallen angel." Further, such companies sometimes return to the rank of investment grade by showing good performance. Interestingly, in the model fallen angels limit borrowing in the turbulent region not because of increased costs of borrowing, but rather because such a policy improves the pricing of issued bonds in the normal region when the firm's leverage is at the target level.

Another implication of the model is that debt price dynamics is qualitatively different in normal times from when the company becomes a fallen angel. While in normal times, the debt price is stable despite the presence of a positive credit spread, in distress times, the credit spreads increase significantly and the company's debt price becomes sensitive to news about underlying cash flows.

Second, our analysis provides a justification for debt repurchases. The key to this result is the addition of downward jumps in cash flows. Specifically, if cash flows follow the diffusion process (as is assumed in most models), a repurchase boundary completely removes default risk, which trivializes the problem of finding an optimal debt policy and leads to a counter-factual prediction that corporate bonds are risk-free. For this reason, the literature on optimal debt policies so far ruled out repurchases either by assumption or by arguing that they are too costly (e.g., because of high costs of raising equity, as in Benzoni et al. 2019). This is no longer the case in our model with Poisson negative jumps, because after a significant downward jump, the equity holders may find it optimal not to repurchase and even to default. Under the optimal time-consistent policy, the equity holders repurchase debt to compensate for relatively small drops in cash flows and stay at the target ICR.

Third, our theory predicts interior optimal debt maturity when the equity holders can also choose debt maturity at the firm's origination. Intuitively, unlike debt interest, debt principal is not tax-deductible, and so, as in Leland and Toft 1996, debt of longer maturities better captures tax benefits. At the same time, debt of shorter maturities commits the firm to reduce the debt burden, which is valuable as equity holders cannot credibly promise debt repurchases in the distress region. The optimal maturity solves this trade-off.

Literature Review Our paper contributes to the recent literature on leverage dynamics without commitment (Admati et al. 2013, DeMarzo and He forthcoming, DeMarzo 2019, DeMarzo et al. 2019). As described above, our contribution is primarily in using a different solution concept. Intuitively, our solution concept gives the maximum power to the self-sustained reputation in debt management, while the solution concept in DeMarzo and He (forthcoming) gives the maximum power to the leverage ratchet effect. As we described in detail above, this difference leads to different leverage dynamics, which has features of both static trade-off theory and the leverage ratchet effect.

It can be natural to conjecture that MPE derived in DeMarzo and He (forthcoming) can be used as a punishment to potentially sustain a variety of leverage policies, in particular, one would expect that the optimal policy with full commitment should be sustainable. Our analysis reveals that this is not always the case. The credibility constraints can be binding in the time-consistent optimum due to the fact that the promises of large repurchases are not credible. These binding constraints, in turn, affect the direction of the comparative statics.

Benzoni et al. (2019) is perhaps the most closely related paper, because they also study how promises to better capital structure policies can be sustained by creditors punishing deviations by pricing debt at the level given in the MPE. While Benzoni et al. (2019)

and our paper use the same equilibrium concept, they differ significantly in assumptions and thus lead to different results. Specifically, Benzoni et al. (2019) assume that there are fixed issuance costs and that all cash flow shocks are small (the cash flow process is a diffusion). The firm in their model does not repurchase debt, and the equilibrium financing policy has the firm issuing more debt in lumpy amounts when its cash flows hit higher thresholds. In contrast, we assume that there no issuance costs (as in DeMarzo and He forthcoming), but the cash flow process can experience both small (diffusion) and large (jump) negative shocks. As a result, the firm actively manages liabilities by issuing and repurchasing debt to the ICR target, but does not only in normal times: A big enough negative jump puts the firm into a distress region in which it abandons the target. Because the credibility constraint often binds in the optimum, the optima with and without commitment are generally different in our setting, while the two coincide in Benzoni et al. (2019).

The paper is related to earlier papers that study optimal dynamic capital structure decisions based on the trade-off between tax benefits of debt and costs of financial distress. An incomplete list of these papers include Fischer et al. (1989), Leland (1994, 1998), Leland and Toft (1996), Goldstein et al. (2001), Strebulaev (2007), He (2011). These papers typically assume that the firm must retire all existing debt before issuing new debt (e.g., Fischer et al. 1989, Goldstein et al. 2001) or that the firm issues debt at the initial date only (e.g., Leland (1994), He (2011)).

In Arellano and Ramanarayanan (2012), a sovereign optimally chooses a portfolio of short- and long-term bonds by trading off the dilution-resistance of the short-term debt and better hedging properties of long-term debt for the risk-averse sovereign. They show that the portfolio is biased towards shorter maturities during the crisis periods when spreads are higher, while the duration of bond portfolio lengthens during "good" times. The forces behind optimal interior maturity in our model are different than in Arellano and Ramanarayanan (2012). The equity holders are risk-neutral, hence, there are no hedging benefits of long-term debt, rather, long-term debt allows them to better capture tax benefits due to the tax deductibility of interest but not principal payments. More importantly, the benefits of short term debt in our model stem from the commitment problem: shorter maturity debt serves as a commitment device for the equity holders to lower debt burden in states when large repurchases are not credible. In contrast, the dilution-resistance of the short-term debt in Arellano and Ramanarayanan (2012) is analogous to that in Bizer and DeMarzo (1992).

The structure of the paper is as follows. Section 2 presents the model. Section 3 derives debt and equity values under debt policies. Section 4 derives the optimal time-consistent debt policy. Section 5 presents empirical implications. Section 6 concludes.

2 The Model

The model adopts the setup in DeMarzo and He (forthcoming) with lognormal cash flows, but introduces downward jump shocks to the cash flow process and uses a different solution concept. Time t is continuous. The firm's operating cash flow Y_t follows the geometric Brownian motion with downward Poisson jumps

$$\frac{dY_t}{Y_{t-}} = \underbrace{\hat{\mu}dt + \sigma dZ_t}_{\text{Brownian shocks}} + \underbrace{d\left(\sum_{i=1}^{N_t}(S_i - 1)\right)}_{\text{Poisson shocks}}.$$

Shocks to cash flows consist of the Brownian shocks and Poisson jumps. Here, $\hat{\mu}$ is the drift parameter, σ is the volatility of Brownian shocks, Z_t is the standard Brownian motion, dN_t is the Poisson process with constant intensity $\lambda > 0$. The size of the downward jumps $\tilde{S}_i \equiv -\ln S_i$ is exponentially distributed with parameter $\eta > 0$. The expected jump size is $\zeta \equiv \mathbb{E}[S_i - 1] = -1/(\eta+1)$. The expected cash flow growth is $\mu \equiv \hat{\mu} + \lambda\zeta$. We suppose that $\mu \in (0, r)$.

At any time t, the firm can issue/repurchase debt. The outstanding debt has face value F_{t-}. The debt pays coupon c and matures exponentially at rate ξ. Specifically, each instant $\xi F_{t-}dt$ is the principal repayment from maturing bonds and $cF_{t-}dt$ is the coupon payment on outstanding bonds. Let $y_t \equiv Y_t/F_{t-}$ and note that y_t/c is the interest coverage ratio (ICR), i.e., the ratio of pre-tax operating cash flows to the promised coupon payment at time t. The firm's taxes at time t equal $\pi(Y_t - cF_{t-})dt$, $\pi \in (0, 1)$.

All agents are risk neutral and discount time at rate $r > 0$. We suppose further that $r > \mu$.

The timing in the interval $[t, t + dt]$ is as follows. First, current cash flows Y_t are realized. Second, the equity holders observe realizations of cash flows up to and including time t, $(Y_s)_{s\leq t}$, and past debt dynamics up to time t, $(F_{s-})_{s\leq t}$. They decide whether to default or not. For simplicity, there is zero recovery after default, which allows us to abstract from the seniority of different debt issuances. Third, equity holders decide how much debt to issue or repurchase. Forth, the debt holders observe past cash flows and debt dynamics up to and including time t, $(Y_s, F_s)_{s\leq t}$. That is, they observe everything that the equity holders observe at time t, and in addition, how much debt was issued/repurchased

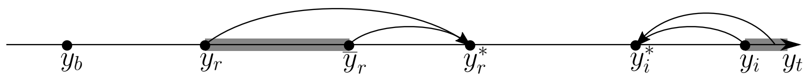

Figure 1: ***Policy thresholds***

The gray region is the action region where the firm issues or repurchases debt.

at time t. Given this information, they determine competitively the price of the newly issued debt, p_t (described below).

We study debt policies that are time-consistent in the sense that the equity holders prefer to stick to them rather than deviate and lose credibility in exerting discipline in debt issuance/repurchase in the future. We next formalize this definition.

Debt Policies A *debt policy* Σ is a Markov process that specifies (i) the initial debt issuance F_0 for a given Y_0, (ii) for any y_t and F_{t-}, the debt issuance/repurchase amount, $d\Gamma_t$. Then,

$$dF_t = d\Gamma_t - \xi F_{t-} dt.$$

Without loss of generality, the equity holders choose directly $dF_t = d\Sigma(y_t, F_{t-})$ at any t. We impose the no-Ponzi restriction that for some $M > 0$, $F_t \leq MY_t$ for all t.

We focus on the class of debt policies $\mathbb{S}$ that are characterized by the issuance boundary y_i and repurchase region $[y_r, \overline{y}_r]$, $\overline{y}_r \leq y_i$, as well as target ICRs y_i^* and y_r^* such that $y_i^* \leq y_i$ and $y_r^* \geq \overline{y}_r$ (see Figure 1). When the firm's ICR reaches y_i, the firm issues debt to lower the ICR to y_i^*; when the firm's ICR falls into the interval $[y_r, \overline{y}_r]$, the firm repurchases debt to increase the ICR to y_r^*.[1] Formally,

$$d\Sigma(y_t, F_{t-}) = \begin{cases} 0, & \text{if } y_r < y_t < y_i \text{ or } y_t < y_r, \\ F_{t-}(y_t - y_i^*)/y_i^*, & \text{if } y_t \geq y_i, \\ F_{t-}(y_t - y_r^*)/y_r^*, & \text{if } y_r \leq y_t \leq \overline{y}_r. \end{cases} \tag{1}$$

It is without loss of generality to suppose that $y_r^* \leq y_i$ (by (1)). We include in $\mathbb{S}$ limits of policies as $y_i^* \to y_i$ (reflecting issuance boundary) and/or $y_r^* \to \overline{y}_r$ (reflecting repurchase boundary) and/or $\overline{y}_r \to y_i$ (targeted ICR).[2] Figure 2 depicts an example of ICR and debt

[1] The class $\mathbb{F}$ excludes certain potentially interesting policies, for example, ones specifying debt issue/repurchase at a certain rate: $dF_t = g(y_t)F_{t-}dt$ for some function $g(\cdot)$. We conjecture that our results would not be affected by allowing for such more complex debt policies, yet, the verification of this conjecture requires a significant generalization of the techniques developed in the present paper and is left for future research.

[2] Note that issuance-only debt policies are a special case of (1) when $\overline{y}_r = 0$.

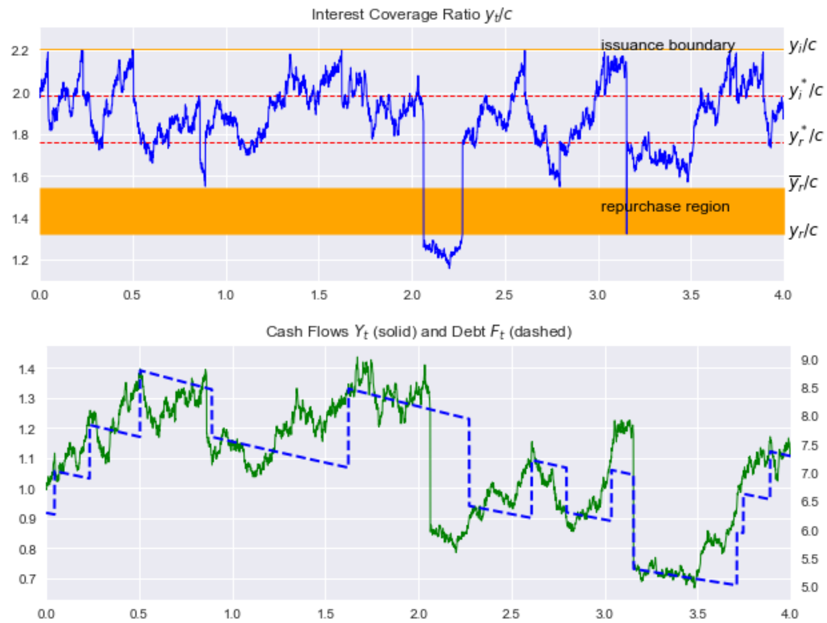

Figure 2: ***Dynamics under a generic policy in*** $\mathbb{S}$

Top panel depicts evolution of ICR y_t/c and bottom panel depicts evolution of cash flows Y_t and debt F_t.

dynamics under a generic debt policy in $\mathbb{S}$. The equity holders default at the first time τ_b when $y_t < y_b$ for some fixed y_b.

Remark 1. *Our restriction to class of policies $\mathbb{S}$ is not without loss of generality in the sense that there might be debt policies outside of class $\mathbb{S}$ that lead to a higher firm value. We focus on class $\mathbb{S}$, as it (1) incorporates all major classes of debt policies previously studied in the literature (e.g., policies with issuance and repurchase boundaries); (2) is sufficiently rich to provide new economic insights about the optimal debt policies without commitment.*

In Online Appendix C, we study two richer classes of debt policies and demonstrate numerically that the gain in the firm value tends to be very small (at most 0.45% across parameter values that we use in our comparative statics in Section 5). Although the comprehensive exploration of this issue is beyond the scope of the current paper, these exercises suggest that the class $\mathbb{S}$ is likely to provide a close approximation to the optimal policy. We also find that our comparative statics in Section 5 are not affected significantly by considering a richer class of debt policies.

The debt holders who expect the equity holders to follow policy Σ and default at time

τ_b price debt at

$$p\left(y_t|\Sigma\right) = \mathbb{E}\left[\int_t^{\tau_b \wedge \tau_m} ce^{-r(s-t)}dt + e^{-r(\tau_m - t)}1\{\tau_m \le \tau_b\}\middle| y_t, \Sigma\right], \tag{2}$$

where τ_m is the stopping time when the bond matures. Note that the debt price depends on Σ indirectly through its effect has on the endogenous default time τ_b. Given this debt pricing, the equity holders' value from following the debt policy Σ and defaulting at time τ_b is given by

$$E\left(Y_t, F_{t-}|\Sigma\right) = \mathbb{E}\left[\int_t^{\tau_b} e^{-r(s-t)}\left[(1-\pi)(Y_s - cF_{s-})ds - \xi F_{s-}ds + p\left(y_s|\Sigma\right)d\Gamma_s\right]\middle| Y_t, F_{t-}, \Sigma\right]. \tag{3}$$

The equity holders default strategically, which is captured by the smooth-pasting condition at the default boundary: $\partial E(Y,F)/\partial Y = 0$ for all $(Y,F) \in \mathcal{B}$ where $\mathcal{B}$ is the boundary of the default region.

Let

$$W(\Sigma) = \max_{F_0 \ge 0}\left\{p\left(Y_0/F_0|\Sigma\right)F_0 + E\left(Y_0, F_0|\Sigma\right)\right\}$$

is the equity holders' revenue from issuing F_0 at $t = 0$ and expected continuation value from following policy Σ in the future. Observe that it coincides with the maximal firm value.

Leverage Dynamics after Losing Credibility We suppose that the first time the equity holders deviate from the announced debt policy, the dynamics switches to the Markov Perfect Equilibrium (henceforth, MPE) of the game, which takes the following form. In the MPE, the equity holders default if and only if $y_t < y_{bm}$ for some y_{bm}, and the debt issuance process is given by some Σ^m. We derive y_{bm} and Σ^m in the next section. The debt holders expect the equity holders to follow this issuance and default strategy and price the newly issued debt accordingly at

$$p_m(y_t) = \mathbb{E}\left[\int_t^{\tau_{bm} \wedge \tau_m} ce^{-r(s-t)}ds + e^{-r(\tau_m - t)}1\{\tau_m \le \tau_b\}\middle| y_t, \Sigma^m\right],$$

where τ_{bm} is the stopping time when $y_t < y_{bm}$ for the first time. The distribution of τ_{bm} depends on the MPE issuance strategy Σ^m through its effect on the evolution of y_t. The equity value $E_m(Y,F)$ satisfies

$$E_m(Y_t, F_{t-}) = (1-\pi)(Y_t - cF_{t-})dt - \xi F_{t-}dt \\ + \max_{dF\in\mathbb{R}} \{p_m\left(Y_t/(F_{t-}+dF)\right)dF + (1-rdt)\mathbb{E}\left[E_m\left(Y_t+dY_t, F_{t-}+dF\right)\right]\}$$

The first two terms are the post-tax profit during dt and the payment to maturing debt. The third term is the optimal revenue from issuing or repurchasing new debt plus the continuation equity value. The debt price equals the value of debt after the equity holders adjust the debt level by dF. The last term is the continuation value of the equity holders given the new debt level.

The equity holders do not have credibility in exerting discipline in managing debt, hence, they choose dF in every state without taking into account that their debt management affects the pricing of debt in different states. Further, the equity holders default strategically, which is captured by the smooth-pasting condition at the default boundary: $\partial E_m(Y,F)/\partial Y = 0$ for all (Y,F) such that $Y/F = y_{bm}$. In Section 3, we derive the MPE and show that, as in DeMarzo and He (forthcoming), the equity holders' lack of credibility depresses the price of the current debt issue so that the equity holders do not capture any tax benefits of any issuances beyond $t=0$.

Time-Consistent Debt Policies Let $\mathcal{R}(\Sigma)$ be the set of all states (Y,F) that can be reached from the initial state (Y_0, F_0) under the debt policy Σ.

Definition 1. *A debt policy* Σ *is* time-consistent *if*

$$E\left(Y,F|\Sigma\right) \geq \sup_{\hat{F}\geq 0}\left\{\left(\hat{F}-F\right)p_m(Y/\hat{F}) + E_m(Y,\hat{F})\right\}, \text{ for all } (Y,F)\in\mathcal{R}(\Sigma), \tag{4}$$

We refer to conditions (4) *as the* credibility constraints*. We denote by* $\mathbb{S}_{TC}$ *the class of all time-consistent debt policies.*

The idea behind credibility contraints (4) is that a debt policy should be supported by a threat to revert to the MPE. Initially, the debt holders believe that the equity holders will stick to the debt policy Σ. As long as the equity holders continue following the policy Σ, the debt holders continue trusting the equity holders to do so in the future, and so, they price the debt accordingly at $p(y|\Sigma)$. If the equity holders deviate from the debt policy in some state y and issue amount $d\Gamma_t = \hat{F} - F$, then the state transitions from Y/F to $Y/\hat{F}$. Note that we allow the equity holders to deviate from Σ to any other debt policy, which in particularly, need not be in class $\mathbb{S}$. After this deviation, the equity holders lose credibility in exerting any debt discipline in the sense that the debt holders expect the debt issuance to be as in the MPE. They price the debt issuance $\hat{F} - F$ at $p_m(Y/\hat{F})$ and the continuation value of the equity holders is equal to $E_m(Y, \hat{F})$, which is the right-hand side of (4).

Definition 1 states that the debt policy is time-consistent if the equity holders never have incentives to deviate from it, if this entails that they lose credibility in exerting the debt discipline in the future. We are interested in characterizing the optimal time-consistent debt policies defined as follows:

Definition 2. *A debt policy* Σ^* *is the* optimal time-consistent debt policy *if*

$$\Sigma^* \in \underset{\Sigma \in \mathbb{S} \cap \mathbb{S}_{TC}}{\arg\max} W(\Sigma). \tag{5}$$

Time-consistent debt policies are intended to capture outcomes of non-Markov perfect equilibria of the continuous-time debt management game of DeMarzo and He (forthcoming). Abreu (1988) shows that in discrete-time games, to characterize all subgame perfect Nash equilibria (SPNE), it is sufficient to construct an optimal penal code for each player (in our case, only for the equity holders, as the debt holders do not take any actions), which is the SPNE that delivers the lowest continuation payoff to that player across all SPNEs, and then, consider all strategies on the equilibrium path that can be supported by this optimal penal code. In continuous time, there are well-known technical difficulties in defining subgame perfect Nash equilibria (see Simon and Stinchcombe (1989)), which led the literature to focus on the subclass of Markov perfect equilibria instead. For this reason, in this paper, we do not formulate time-consistent debt policies as on-path strategies of the non-Markov perfect equilibria in the continuous-time game of DeMarzo and He (forthcoming). Rather, in line with Abreu (1988), we consider all debt policies that can be supported by the treat of reversal to the MPE, which DeMarzo and He (forthcoming) show to be the optimal penal code for the equity holders' deviations. In this sense, our focus in this paper is on non-Markov equilibrium outcomes. In a recent paper, Panov (2019) proposes a method of defining subgame perfect equilibria in continuous time games that is consistent with our approach. In particular, he uses this method to formally define non-Markov perfect equilibria in the game of DeMarzo and He (forthcoming) that we study in the current paper. The interested reader should refer to his paper for formal details.

3 Debt and Equity Values

In this section, we characterize debt and equity values under different debt policies and in the MPE. This will allow us to write more explicitly the program (5).

For notational convenience, we will omit in the notations dependence on the debt policy Σ, and write, $p(y)$ instead of $p(y|\Sigma)$, $E(Y,F)$ instead of $E(Y,F|\Sigma)$, and so on. For $y \in (y_b, y_r) \cup (\overline{y}_r, y_i)$, $p(y)$ satisfies the HJB equation

$$(r+\lambda+\xi)p(y) = c+\xi+(\hat{\mu}+\xi)yp'(y)+\frac{1}{2}\sigma^2y^2p''(y)+\lambda\mathbb{E}\left[p(Sy)\right]. \tag{6}$$

We conjecture that

$$p(y) = \begin{cases} 0, & y < y_b \\ \frac{c+\xi}{r+\xi}\left(1+\sum_{k=1}^{3} b_k(y/y_b)^{-\gamma_k}\right), & y \in [y_b, y_r], \\ p_r^*, & y \in (y_r, \overline{y}_r), \\ \frac{c+\xi}{r+\xi}\left(1+\sum_{k=1}^{3} B_k(y/y_b)^{-\gamma_k}\right), & y \in [\overline{y}_r, y_i], \\ p_i^*, & y > y_i; \end{cases} \tag{7}$$

where γ_ks are roots of the characteristic equation

$$\tfrac{1}{2}\sigma^2\gamma^2 - \left(\hat{\mu}+\xi-\tfrac{1}{2}\sigma^2\right)\gamma + \frac{\lambda\eta}{\eta-\gamma} = r+\lambda+\xi. \tag{8}$$

At the default boundary,

$$p(y_b) = 0. \tag{9}$$

The debt holders anticipate that when the ICR reaches y_i/c, the equity holders issue debt to lower the ICR to y_i^*/c, thus,

$$p(y_i) = p_i^* \equiv p(y_i^*). \tag{10}$$

Similarly, the debt holders anticipate that if y_t falls into the repurchase region $[y_r, \overline{y}_r]$, then the equity holders repurchase debt to increase the ICR to y_r^*/c, thus,

$$p(y) = p_r^* \equiv p(y_r^*), \text{ for all } y \in [y_r, \overline{y}_r]. \tag{11}$$

If y_i or $\overline{y}_r$ are reflecting boundaries, then conditions (10) and (11) are replaced by their limits as $y_i^* \to y_i$ and $y_r^* \to \overline{y}_r$. These conditions are $p'(y_i) = 0$ and $p'(\overline{y}_r) = 0$, respectively (see Online Appendix B).

Coefficients b_ks and B_ks are determined by the boundary conditions (9) − (11) and the following two additional conditions:

$$\mathbb{E}\left[\frac{c+\xi}{r+\xi}\left(1+\sum_{k=1}^{3} b_k S^{-\gamma_k}\right)\right] = 0, \tag{12}$$

$$\mathbb{E}\left[\frac{c+\xi}{r+\xi}\left(1+\sum_{k=1}^{3} B_k (S\overline{y}_r/y_b)^{-\gamma_k}\right)\right] = \mathbb{E}\left[p\left(S\overline{y}_r\right)\right]. \tag{13}$$

Equations (12) and (13) arise because of the presence of downward jumps in cash flows. They require that even if the conjectures for p on $[y_b, y_r]$ and $[\overline{y}_r, y_i]$ were applied beyond these ranges, this would not change debt pricing on $[y_b, y_r]$ and $[\overline{y}_r, y_i]$. Indeed, by the memoryless property of the exponential distribution of downward jumps, the debt price for $y \in [y_b, y_r]$ would not change if we changed $p(y)$ on $y < y_b$ (from the specification in (7)) as long as $\mathbb{E}[p(Sy_b)] = 0$. Condition (12) requires that $\mathbb{E}[p(Sy_b)] = 0$ even if the conjecture for p on $[y_b, y_r]$ is extended to ys below y_b.

Similarly, the memoryless property implies that the debt price in the region $[\overline{y}_r, y_i]$ is not affected by a change in $p(y)$ below $\overline{y}_r$ (from the specification in (7)) as long as $\mathbb{E}[p(S\overline{y}_r)]$ stays the same. Condition (13) requires that this is indeed the case if the conjecture for p on $[\overline{y}_r, y_i]$ is extended to ys below $\overline{y}_r$.[3]

Due to the homogeneity of the setup, the equity value takes the form $E(Y, F) = e(y)F$. The value of equity satisfies the following HJB on $(y_b, y_r) \cup (\overline{y}_r, y_i)$,

$$(r+\lambda+\xi)e(y) = (1-\pi)(y-c) - \xi + (\hat{\mu}+\xi)ye'(y) + \tfrac{1}{2}\sigma^2 y^2 e''(y) + \lambda\mathbb{E}\left[e(Sy)\right]. \tag{14}$$

The boundary and smooth-pasting conditions at the default boundary y_b imply

$$e(y_b) = 0, \tag{15}$$

$$e'(y_b) = 0. \tag{16}$$

At the issuance boundary y_i, $e(y_i)F_{t-} = e(y_i^*)F_t + p(y_i)(F_t - F_{t-})$, or using $p(y_i) = p(y_i^*) = p_i^*$,

$$\frac{e(y_i)+p_i^*}{y_i} = \frac{e(y_i^*)+p_i^*}{y_i^*}. \tag{17}$$

[3]Online Appendix A provides explicit expressions for (12) and (13) and derives them.

Analogously, boundary conditions at repurchase boundaries y_r and $\overline{y}_r$ imply

$$\frac{e(y_r)+p_r^*}{y_r} = \frac{e(y_r^*)+p_r^*}{y_r^*}, \tag{18}$$

$$\frac{e(\overline{y}_r)+p_r^*}{\overline{y}_r} = \frac{e(y_r^*)+p_r^*}{y_r^*}. \tag{19}$$

Since the equity holders repurchase debt at any $y \in [y_r, \overline{y}_r]$, for any such y it holds $\frac{e(y)+p_r^*}{y} = \frac{e(y_r^*)+p_r^*}{y_r^*}$. Note that $(e(y)+p(y))/y$ is the EV/EBIT multiple (enterprise value divided by pre-tax earnings). Equations (17) − (19) state that the EV/EBIT multiple does not change when there is an expected adjustment of the leverage.

If y_i or $\overline{y}_r$ are reflecting boundaries, then conditions (17) and (19) are replaced by their limits as $y_i^* \to y_i$ and $y_r^* \to \overline{y}_r$. These conditions are $e'(y_i) = (e(y_i)+p(y_i))/y_i$ and $e'(\overline{y}_r) = (e(\overline{y}_r)+p(\overline{y}_r))/\overline{y}_r$, respectively (see Online Appendix B).

We conjecture that the equity value per unit of debt takes the form:

$$e(y) = \begin{cases} 0, & y < y_b; \\ \phi y - \rho + \sum_{k=1}^{3} c_k (y/y_b)^{-\gamma_k}, & y \in [y_b, y_r], \\ y\frac{e(y_r^*)+p_r^*}{y_r^*} - p_r^*, & y \in (y_r, \overline{y}_r), \\ \phi y - \rho + \sum_{k=1}^{3} C_k (y/y_b)^{-\gamma_k}, & y \in [\overline{y}_r, y_i], \\ y\frac{e(y_i^*)+p_i^*}{y_i^*} - p_i^*, & y > y_i; \end{cases} \tag{20}$$

where $\phi \equiv \frac{1-\pi}{r-\mu}$ and $\rho \equiv \frac{c(1-\pi)+\xi}{r+\xi}$, and γ_is solve the characteristic equation (8). Coefficients c_ks and C_ks and the default boundary y_b are pinned down by (15) − (19), and in addition,

$$\mathbb{E}\left[\phi y - \rho + \sum_{k=1}^{3} S^{-\gamma_k}\right] = 0 \tag{21}$$

$$\mathbb{E}\left[\phi S\overline{y}_r - \rho + \sum_{k=1}^{3} C_k (S\overline{y}_r/y_b)^{-\gamma_k}\right] = \mathbb{E}\left[e(S\overline{y}_r)\right]. \tag{22}$$

Similarly to conditions (12) and (13) for debt pricing, conditions (21) and (22) require that even if the conjectures for e on $[y_b, y_r]$ and $[\overline{y}_r, y_i]$ were applied beyond these ranges, this would not change the equity value on $[y_b, y_r]$ and $[\overline{y}_r, y_i]$. By the memoryless property of the exponential distribution of downward jumps, this would be the case if $\mathbb{E}[e(Sy_b)]$ and $\mathbb{E}\left[e(S\overline{y}_r)\right]$ did not change, which are the conditions (21) and (22).[4]

[4]Online Appendix A provides explicit expressions for (21) and (22) and derives them.

3.1 Credibility Constraints

To make the credibility constraints (4) more explicit, we will derive the debt price and equity value in the MPE, which serves as a punishment to equity holders for deviating from the debt policy.

We can extend the analysis in DeMarzo and He (forthcoming) to characterize the MPE in the case of downward jumps in cash flows. The equity value $E_m(Y,F)$ satisfies the HJB equation:

$$\begin{aligned}(r+\lambda)E_m(Y,F) = \max_G \Big\{&(1-\pi)(Y-cF) - \xi F + Gp_m(Y/F) + (G-\xi F)\tfrac{\partial}{\partial F}E_m(Y,F) \\ &+ \hat{\mu}Y\tfrac{\partial}{\partial Y}E_m(Y,F) + \tfrac{1}{2}\sigma^2 Y^2 \tfrac{\partial^2}{\partial Y^2}E_m(Y,F) + \lambda\mathbb{E}\left[E_m\left(SY,F\right)\right]\Big\},\end{aligned}$$

where G denotes the debt issuance/repurchase amount. DeMarzo and He (forthcoming) argue in Proposition 1 that in the MPE, the equity holders are indifferent between issuing or repurchasing any amount of debt, which implies that for all $y = Y/F > y_{bm}$,

$$p_m(Y/F) + \tfrac{\partial}{\partial F}E_m(Y,F) = 0. \tag{23}$$

Thus,

$$\begin{aligned}(r+\lambda+\xi)E_m(Y,F) = &(1-\pi)(Y-cF) - \xi F \\ &+ (\hat{\mu}+\xi)\,Y\tfrac{\partial}{\partial Y}E_m(Y,F) + \tfrac{1}{2}\sigma^2 Y^2 \tfrac{\partial^2}{\partial Y^2}E_m(Y,F) + \lambda\mathbb{E}\left[E_m\left(SY,F\right)\right]. \end{aligned} \tag{24}$$

We conjecture that $E_m(Y,F) = e_m(y)F$ and $e_m(y) = \phi y - \rho + \sum_{k=1}^{3} c_{km}(y/y_{bm})^{-\gamma_k}$. By (24), e_m satisfies the HJB equation (14) with boundary conditions $e_m(y_{bm}) = e'_m(y_{bm}) = 0$, transversality condition $\lim_{y\to\infty} e_m(y) - (\phi y - \rho) = 0$, and condition (21), which pin down coefficients c_{km}s and the default boundary y_{bm}. The debt price in the MPE is determined from (23):

$$p_m(y) = ye'_m(y) - e_m(y).$$

In the MPE, the equity holders issue debt continuously with intensity $g(y)F \equiv \frac{\pi c}{yp'_m(y)}F$ so that the newly issued debt is priced exactly at $p_m(y)$.

An important property of the MPE showed by DeMarzo and He (forthcoming) is that whenever e_m is strictly convex, deviations to large debt issuances/repurchases (of order larger than dt) are not profitable. This property allows us to simplify the credibility constraints (4):

Proposition 1. *e_m is strictly convex on $[y_{bm}, \infty)$. Further, credibility constraints* (4) *are equivalent to*

$$e(y|\Sigma) \geq e_m(y), \text{ for all } y \in [y_b, y_i]. \tag{25}$$

To prove Proposition 1, we first show that $p_m(y)$ is strictly increasing in y on $[y_{bm}, \infty)$. Let us differentiate (24) with respect to F and use (23) to get

$$(r+\lambda+\xi)p_m(y) = c(1-\pi) + \xi + (\hat{\mu}+\xi)yp_m'(y) + \tfrac{1}{2}\sigma^2 y^2 p_m''(y) + \lambda \mathbb{E}\left[p_m(Sy)\right]. \tag{26}$$

The boundary conditions for the debt price are $p_m(y_{bm}) = 0$ and $\lim_{y\to\infty} p_m(y) = \rho$. From (26), we get that the debt price can be obtained from the auxiliary environment in which the equity holders do not issue any debt, the debt pays coupon $1-\pi$, and the default is triggered when the interest coverage ratio y_t reaches y_{bm}. Thus,

$$p_m(y) = \mathbb{E}\left[\int_t^{\tau_{bm}\wedge\tau_m} e^{-r(s-t)}c(1-\pi)ds + 1\{\tau_m \leq \tau_{bm}\}\middle| y_t = y, \Sigma^0\right], \tag{27}$$

where Σ^0 denotes the debt policy in which the equity holders do not issue/repurchase debt at any $t > 0$. In Appendix, we show that (27) implies that p_m is strictly increasing on $y \geq y_{bm}$. Intuitively, for larger y the default is less likely, and so, the debt holders expect to receive the coupon for longer.

By $p_m'(y) = ye_m''(y)$, e_m is strictly convex on $[y_{bm}, \infty)$. Proposition 3 in DeMarzo and He (forthcoming) show that when E_m is strictly convex in F, no global deviations from the strategy g are profitable in the MPE, i.e., $E_m(Y,F) \geq \max_{\hat{F}} \left\{\left(\hat{F} - F\right) p_m(Y/\hat{F}) + E_m(Y, \hat{F})\right\}$ for all Y, which implies that for time-consistent debt policies (4) are equivalent to $E(Y,F) \geq E_m(Y,F)$. Coupled with the homogeneity of E and E_m, this implies that the credibility constraints (4) are equivalent to (25).

4 Optimal Time-Consistent Policy

4.1 Targeted ICR

We first describe the debt policy, which as we show in the next subsection is the optimal time-consistent policy.

The *targeted ICR policy* is characterized by the ICR target $\hat{y}/c$ and the repurchase boundary y_r. Denote the class of such policies by $\hat{\mathbb{S}}$. The equity holders issue or repurchase the debt to compensate small shocks to the ICR for which $y_t \geq y_r$ to ensure that y_t stays at the target level $\hat{y}$. For larger shocks to y_t for which $y_t \in (y_b, y_r)$, the equity holders do not issue or repurchase any debt and wait until either y_t hits y_r, at which point they repurchase debt to restore the ICR to the target level $\hat{y}$, or y_t drops below y_b, at which point the equity holders default.[5]

Let us derive debt price and equity value under the targeted ICR policy. As before, the debt price satisfies the HJB equation (6) for $y \in [y_b, y_r]$. We conjecture that the debt price is given by

$$p(y) = \begin{cases} 0, & y \in (0, y_b], \\ \frac{c+\xi}{r+\xi}\left(1 + \sum_{k=1}^{3} b_k (y/y_b)^{-\gamma_k}\right), & y \in [y_b, y_r], \\ \hat{p}, & y \in [y_r, \infty); \end{cases}$$

where $\hat{p} \equiv p(\hat{y})$. The coefficients b_ks satisfy the boundary conditions $p(y_b) = 0$ and $p(y_r) = \hat{p} \equiv p(\hat{y})$, as well as condition (12). Further, the price of debt $\hat{p}$ at the target ICR $\hat{y}$ is given by

$$\hat{p} = (c+\xi)dt + (1-rdt-\xi dt)\left\{ \underbrace{(1-\lambda dt)\hat{p}}_{\text{Brownian shocks}} + \underbrace{\lambda dt \left(\int_0^{\ln(\hat{y}/y_r)} \eta e^{-\eta\tilde{s}} d\tilde{s}\right)\hat{p}}_{\text{small jumps}} + \underbrace{\lambda dt \int_{\ln(\hat{y}/y_r)}^{\ln(\hat{y}/y_b)} p(e^{-\tilde{s}}\hat{y})\eta e^{-\eta\tilde{s}} d\tilde{s}}_{\text{large jumps}} \right\}.$$

The bond pays a unit flow payoff. With probability $1 - \lambda dt$, only Brownian shocks occur and they are compensated by the equity holders' issuance or repurchase of debt so that the ICR still equals $\hat{y}$. With probability λdt, a negative jump shock to y arrives. Then, it is compensated by the equity holders only if it is sufficiently small. In this case, the price of debt continues to be equal to $\hat{p}$. Otherwise, when the shock is sufficiently large, the price of debt drops to $p(e^{-\tilde{s}}\hat{y})$. We can rewrite this equation as the HJB equation:

$$(r + \lambda + \xi)\hat{p} = c + \xi + \lambda\hat{p}\int_0^{\ln(\hat{y}/y_r)} \eta e^{-\eta\tilde{s}} d\tilde{s} + \lambda \int_{\ln(\hat{y}/y_r)}^{\ln(\hat{y}/y_b)} p(e^{-\tilde{s}}\hat{y})\eta e^{-\eta\tilde{s}} d\tilde{s}. \quad (28)$$

These four conditions pin down b_ks as well as price $\hat{p}$.

[5] Note that the targeted ICR debt policy can be obtained as the limit of the debt issuance/repurchase policy in $\mathbb{S}$ if we allow $\overline{y}_r$, y_r^*, y_i^*, and y_i all converge to $\hat{y}$.

Further, we conjecture that the equity value per unit of debt equals

$$e(y) = \begin{cases} 0, & y \in (0, y_b], \\ \phi y - \rho + \sum_{k=1}^{3} c_k (y/y_b)^{-\gamma_k}, & y \in [y_b, y_r], \\ \left(\frac{y}{\hat{y}} - 1\right)\hat{p} + \frac{y}{\hat{y}}\hat{e}, & y \in [y_r, \infty), \end{cases} \tag{29}$$

where $\hat{e} \equiv e(\hat{y})$. As before, the coefficients c_ks satisfy $e(y_b) = e'(y_b) = 0$ and equation (21). Further, the value of equity at the target $\hat{y}$ is equal to

$$\begin{aligned} E(\hat{y}F_{t-}, F_{t-}) = & \underbrace{(1-\pi)(\hat{y}F_{t-} - cF_{t-})dt - \xi F_{t-}dt}_{\text{flow payoff}} + \underbrace{\mathbb{E}\left[\hat{p}d\Gamma_t\right]}_{\text{issuance/repurchase revenue}} \\ & + \underbrace{(1 - rdt - \lambda dt)\mathbb{E}\left[E(\hat{y}F_t, F_t)|dN_t = 0\right]}_{\text{only Brownian shocks}} \\ & + (1 - rdt)\left\{ \underbrace{\lambda dt \int_0^{\ln(\hat{y}/y_r)} E(\hat{y}F_t, F_t)\eta e^{-\eta\tilde{s}} d\tilde{s}}_{\text{small jumps}} + \underbrace{\lambda dt \int_{\ln(\hat{y}/y_r)}^{\ln(\hat{y}/y_b)} E(e^{-\tilde{s}}\hat{y}F_{t-}, F_{t-})\eta e^{-\eta\tilde{s}} d\tilde{s}}_{\text{large jumps}} \right\}. \end{aligned} \tag{30}$$

In equation (30), the first term is the (flow) payoff from cash flows after coupon payments and taxes. The second term is the revenue from debt issuance or costs of debt repurchases. The third term is the continuation value when no downward jumps occur. The forth term is the continuation value when the jump is sufficiently small (so that $\exp(-\tilde{S}_t) > y_r/\hat{y}$). The fifth term is the continuation value when the jump is large, but not so large to trigger the default, $\exp(-\tilde{S}_t) \in (y_b/\hat{y}, y_r/\hat{y})$. In Appendix A.3, we show that equation (30) can be re-written as:

$$\begin{aligned} (r + \lambda - \hat{\mu})\hat{e} = & (1-\pi)(\hat{y} - c) - \xi + \hat{p}\left(\hat{\mu} + \xi\right) \\ & + \lambda \int_0^{\ln(\hat{y}/y_r)} \left(\left(e^{-\tilde{s}} - 1\right)\hat{p} + e^{-\tilde{s}}\hat{e}\right)\eta e^{-\eta\tilde{s}} d\tilde{s} + \lambda \int_{\ln(\hat{y}/y_r)}^{\ln(\hat{y}/y_b)} e(e^{-\tilde{s}}\hat{y})\eta e^{-\eta\tilde{s}} d\tilde{s}. \end{aligned} \tag{31}$$

To pin down $(c_1, c_2, c_3, y_b, \hat{e})$, we need an additional boundary conditions at y_r:

$$\frac{e(y_r) + \hat{p}}{y_r} = \frac{\hat{e} + \hat{p}}{\hat{y}}. \tag{32}$$

Targeted ICR policies have several useful properties. First, we show that there exists a credible targeted ICR policy that increases the firm value compared to the MPE.

Proposition 2. *There exists* $\Sigma \in \hat{\mathbb{S}}$ *such that*

$$e\left(y|\Sigma\right) > e_m(y) \text{ and } p\left(y|\Sigma\right) > p_m(y), \text{ for all } y \in \left(y_b, \hat{y}\right]. \tag{33}$$

To prove Proposition 2, we construct a simple targeted ICR policy with $\hat{y} = y_r$, call it Σ, in which the firm starts issuing/repurchasing debt only when y_t reaches the target level $\hat{y}$. It issues/repurchases debt to replace maturing debt and compensate for Brownian, but not Poisson shocks to cash flows.

For $\hat{y}$ sufficiently high, the debt is close to safe, and hence, is priced at close to $(c+\xi)/(r+\xi)$. At the same time, by (27), the debt is priced at most at $(1-\pi)(c+\xi)/(r+\xi)$ in the MPE, because of the ratched effect. By the argument in DeMarzo and He (forthcoming), the equity holders are indifferent between any rate of debt issuance in the MPE, in particular, the debt policy Σ. Debt pricing is more favorable under Σ compared to the MPE. Further, when in state $\hat{y}$, the equity holders expect to issue more debt than they repurchase, because cash flows in the absence of Poisson shocks have positive drift $\hat{\mu}$ and the equity holders replace maturing debt with new debt issues. Therefore, the equity holders are strictly better off under the debt policy Σ compared to the MPE.

The second important property of the targeted ICR policies is that for them, credibility constraints are particularly easy to check: We only need to check that the default boundary y_b is below that in the MPE, and the policy improves on the MPE at $\hat{y}$ and some $y \in (y_b, y_{bm}]$.

Proposition 3. *A targeted ICR policy satisfies the credibility constraints* (*25*) *if and only if* $y_b \leq y_{bm}$, $e(y) \geq 0$ *for some* $y \in (y_b, y_{bm}]$, *and* $e(\hat{y}) \geq e_m(\hat{y})$.

Therefore, the continuum of credibility constraints is reduced to only three constraints that should be verified in order to ensure that a targeted ICR policy is credible.

4.2 Optimality of Targeted ICR

Given the debt and equity values, we can compute the firm value per unit of debt:

$$v(y) \equiv e(y) + p(y) = \begin{cases} 0, & y \in (0, y_b], \\ \phi y + \frac{c\pi}{r+\xi} + \sum_{k=1}^{3} a_k (y/y_b)^{-\gamma_k}, & y \in [y_b, y_r], \\ \frac{e(y_r^*)+p_r^*}{y_r^*} y, & y \in [y_r, \overline{y}_r], \\ \phi y + \frac{c\pi}{r+\xi} + \sum_{k=1}^{3} A_k (y/y_b)^{-\gamma_k}, & y \in [\overline{y}_r, y_i], \\ \frac{e(y_i^*)+p_i^*}{y_i^*} y, & y \in [y_i, \infty). \end{cases} \tag{34}$$

Let $w(y) \equiv v(y)/y$. Then, the objective function in (5) can be written as $W(\Sigma) = Y_0 \max_{y \geq 0} w(y)$.

Auxiliary Program We first consider an auxiliary program in which y_b is fixed at the ex-post optimal level y_{b0} (i.e., for which the smooth-pasting condition is satisfied), and the credibility constraints are ignored:

$$\max_{\Sigma \in \mathbb{S}} \left\{ W(\Sigma) : y_b = y_{b0} \right\}. \tag{35}$$

To solve this program, we suppose that the smooth-pasting and super-contact principles hold. Say that the smooth-pasting principle holds if whenever the optimal issuance (repurchase) boundary is an impulse control, i.e., $y_i > y_i^*$ ($\overline{y}_r < y_r^*$, respectively), $w'(y_i) = w'(y_i^*) = 0$ ($w'(\overline{y}_r) = w'(y_r^*) = 0$, respectively). Say that the super-contact principle holds if whenever the optimal issuance (repurchase) boundary is an instantaneous control, i.e., $y_i = y_i^*$ ($\overline{y}_r = y_r^*$, respectively), $w''(y_i) = 0$ ($w''(\overline{y}_r) = 0$, respectively). These principles hold in the theory of optimal control of Brownian motion (see Dixit 1991, Dumas 1991 or for a more rigorous treatment Harrison et al. (1983), Harrison and Taksar (1983)). We can use the standard argument in Dixit 1991, Dumas 1991 to show that these principles hold in our setup without jumps. We verify numerically that these principles hold in all our parameter specifications.[6]

[6] Specifically, for each set of parameters, we find numerically the optimal time-consistent debt policy (see the algorithm described after Proposition 5 and footnote 7) and verify that it is indeed the targeted ICR policy with policy parameters $\hat{y}$ and y_r. Then, perturb the parameters of the optimal policy so that $y_i > y_i^*$ and $\overline{y}_r < y_r^*$, and verify that $w'(y_i)$, $w'(y_i^*)$, $w'(\overline{y}_r)$, and $w'(y_r^*)$ all converge to zero as y_i, y_i^*, $\overline{y}_r$, and y_r^* all converge to $\hat{y}$. Similarly, we perturb the parameters of the optimal policy so that reflecting issuance/repurchase boundaries y_i and $\overline{y}_r$ are distinct, and verify that $w''(y_i)$ and $w''(\overline{y}_r)$ converge to zero as y_i and $\overline{y}_r$ converge to $\hat{y}$.

Intuitively, the optimality principles state that the marginal value of control before and after it is applied should equal to marginal costs. In our environment, while the debt adjustment is costly/beneficial for the equity holders, as they repurchase or issue new debt at prices p_r^* or p_i^*, for the firm as a whole, debt adjustment involves neither lump sum nor proportional adjustment costs. At the same time, marginal value of controls are $w'(\overline{y}_r)$ and $w'(y_i)$ before the leverage adjustment, and $w'(y_r^*)$ and $w'(y_i^*)$ after the leverage adjustment. Thus, the smooth-pasting principle requires $w'(y_i) = w'(y_i^*) = 0$ and $w'(\overline{y}_r) = w'(y_r^*) = 0$. The intuition for the super-contact principle is analogous when applied to the limit case $y_i - y_i^* \to 0$ and $\overline{y}_r - y_r^* \to 0$.

Proposition 4. *The debt policy solving* (*35*) *is the targeted ICR policy.*

The proof of Proposition 4 is provided in Appendix and it boils down to determining appropriate boundary conditions for the function v. First, consider the case of impulse control at boundaries y_i and $\overline{y}_r$. By the smooth-pasting principle,

$$v'(y_i)y_i - v(y_i) = v'(y_i^*)y_i^* - v(y_i^*) = v'(y_r^*)y_r^* - v(y_r^*) = v'(\overline{y}_r)\overline{y}_r - v(\overline{y}_r) = 0.$$

As we argue in Appendix, this implies that $\partial a_k/\partial \tilde{y} = \partial A_k/\partial \tilde{y} = 0$ for $\tilde{y} \in \{y_i, y_i^*, \overline{y}_r, y_r^*\}$, which in turn, implies that $\partial b_k/\partial \tilde{y} = \partial B_k/\partial \tilde{y} = 0$. The latter implies that $p'(y_i) = p'(y_i^*) = 0$ and $p'(\overline{y}_r) = p'(y_r^*) = 0$. Thus, the issuance/repurchase boundaries can be replaced with reflecting boundaries. This implies that the appropriate boundary conditions at the issuance/repurchase boundaries are

$$\begin{aligned} v'(y_i)y_i &= v(y_i), \\ v'(\overline{y}_r)\overline{y}_r &= v(\overline{y}_r), \\ p'(y_i) &= p'(\overline{y}_r) = 0. \end{aligned}$$

By the super-contact principle:

$$v''(y_i) = v''(\overline{y}_r) = 0.$$

Again, we can show that

$$p''(y_i) = p''(\overline{y}_r) = 0.$$

Note that the targeted ICR policies satisfy these conditions. Indeed, in such policies, $\overline{y}_r \to y_r^*$ and $y_i \to y_i^*$, and the function v converges to a linear around $\hat{y}$. However, no other policies can satisfy these equations. Intuitively, if there were positive proportional adjustment costs, then the issuance and repurchase boundaries in general would be different. However, with zero proportional adjustment costs, there are no gains from keeping them apart.

Optimal Time-Consistent Policy Let

$$\hat{\Sigma} = \arg\max_{\Sigma \in \hat{\mathbb{S}}} \{W(\Sigma) : y_b \leq y_{bm}\} \tag{36}$$

be the optimal time-consistent targeted ICR debt policy. Program (5) ignores all credibility constraints but $y_b \leq y_{bm}$. Yet, Proposition 3 ensures that the rest of credibility constraints are nevertheless satisfied, as long as $e(y|\hat{\Sigma}) \geq 0$ for some $y \in (y_b, y_{bm}]$ and $e(\hat{y}) \geq e_m(\hat{y})$. Further, we show in the proof of Proposition 5in Appendix that condition $e(\hat{y}) \geq e_m(\hat{y})$ is automatically satisfied for $\hat{\Sigma}$.

We will now show that $\hat{\Sigma}$ is in fact an optimal time-consistent policy in a richer class $\mathbb{S}$. Consider Σ^* that solves (5) and denote by y_b^* the default boundary under Σ^*. By the credibility constraints, it is necessary that $y_b^* \leq y_{bm}$. By Proposition 4, there is a targeted ICR policy $\tilde{\Sigma}$ that weakly dominates Σ^* and has the same default boundary y_b^*. By (36), policy $\hat{\Sigma}$ weakly dominates $\tilde{\Sigma}$, and hence, also Σ^*. Thus,

Proposition 5. *Suppose that $e(y|\hat{\Sigma}) \geq 0$ for some $y \in (y_b, y_{bm}]$. Then, $\hat{\Sigma}$ is the optimal time-consistent policy in $\mathbb{S}$.*

Proposition 5 provides a simple way of solving a potentially complex problem of finding the optimal time-consistent debt policy. Specifically, one needs to simply find the optimal targeted ICR policy such that $y_b \leq y_{bm}$ and verify that under this policy the equity value is non-negative for some $y \in (y_b, y_{bm}]$. In fact, due to the closed-form solutions in Section 4.1, this problem is reduced to a single variable optimization program.[7]

4.3 Effect of Credibility on Leverage

We next explore how the credibility requirement affects the firm's leverage choice.

[7] Note that even without Proposition 4, there is a computationally simple algorithm for finding optimal time-consistent policy. Specifically, we first use closed-form solutions for w to find the optimal policy subject to only $y_b \leq y_{bm}$ (e.g., via grid search over $x_r, \overline{x}_r, x_r^*, x_i, x_i^*$). If this policy is a targeted ICR policy, then Proposition 3 implies that it also satisfies all the rest of credibility constraints, and hence, is the optimal time-consistent policy. We use this algorithm to verify the optimality of the targeted ICR policy in all our specifications.

Figure 3: ***Comparison to Commitment Solution***

Parameters: $\mu = 2\%$, $r = 5\%$, $\pi = 10\%$, $\sigma = 25\%$, $\lambda = 1/3$, $\eta = 5.6$. *The solid blue lines depicts values under reflecting issuance boundary at* $y_i = 1$, *the dotted yellow line depicts values in the MPE, the dashed green line depicts values under the optimal time-consistent debt policy. The parameters of the optimal time-consistent policy are* $\hat{y} = 2.89$, $y_r = 2.31$, $y_b = 0.289$. *The parameters of the optimal policy with commitment are* $\hat{y} = 1.44$, $y_r = 0.67$, $y_b = 0.396$.

To do so, we first compare the optimal time-consistent policy Σ^* to the optimal policy with commitment to future debt policies, which we define as follows. Suppose that the equityholders can commit to a particular issuance/repurchase policy Σ that need not satisfy credibility constraints. At the same time, they still cannot commit to the default policy, so the default boundary must satisfy the smooth-pasting condition. Then, the optimal policy with commitment is

$$\Sigma^c \equiv \arg\max_{\Sigma \in \mathbb{S}} W(\Sigma).$$

By Proposition 4, the optimal policy with commitment also takes a form of a targeted ICR policy, yet, it will generally differ from the optimal time-consistent policy. In other words, the credibility constraints sometimes bind in the optimal time-consistent policy.

To get an insight into why the optimal policy with commitment violates the credibility constraints, consider the illustration in Figure 3. The optimal policy with commitment Σ^c differs from the optimal time-consistent policy Σ^* in several respects. Naturally, the firm value is higher in Σ^c. Under Σ^c, the debt price at target ICR is close to the risk-free debt price due to the fact that the equity holders are committed to compensate with repurchase even very large shocks to cash flows (up to 53% drop). However, the repurchase of such a large amount is not credibility as illustrated in the first panel of Figure 3, where the equity value under Σ^c is below that in the MPE for low ICRs. The reason is that in these states, the equity holders need to repurchase a substantial amount of debt at a very high price (which is close to the price of risk-free debt). Thus, after significant cash flow drops, the equity holders would prefer to abandon Σ^c and switch to the MPE dynamics,

	leverage ratio	ICR target	credit spread	repurchase boundary	default boundary	MPE default boundary	maximal repurchase
Base case	21%	2.89	45 bps	2.32	0.289	0.289	20%
$\tau = 25\%$	48%	1.17	100 bps	0.90	0.289	0.289	23%
$\sigma = 35\%$	14%	4.08	72 bps	3.60	0.218	0.218	12%

Table 1: ***Effect of Credibility Constraints on Leverage***

Base case specification: $\tau = 10\%$, $\sigma = 25\%$, $\lambda = 1/3$, $\zeta = -15\%$, $\mu = 2\%$, $r = 5\%$. The credit spread is computed at $\hat{y}$ and is equal to $1/\hat{p} - r$. The leverage ratio is computed at $\hat{y}$ and is equal to $\hat{p}/(\hat{e} + \hat{p})$. Maximal repurchase is in percentage of outstanding debt and is equal to $(1 - y_r/\hat{y})$.

which makes Σ^c not time-consistent. In order to maintain time-consistency, in the optimal time-consistent policy Σ^*, the equity holders compensate only moderate shocks to cash flows with repurchase (up to 20%), however, have a wider range of ICRs for which they wait until the cash flows recover sufficiently so that they can get back to target.

Thus, it is repurchases that are particularly costly for the equity holders and the credibility constraints limit the maximal amount of repurchase that the equity holders can credibly promise to make. This is illustrated in Table 1. Since the equity holders do not capture any tax benefits in the MPE, while they get tax benefits from following certain policies in $\mathbb{F}$, higher tax benefits relax the credibility constraints. This leads to a higher maximal repurchase in the optimal time-consistent debt policy (23% versus 20% in the baseline) as well as higher leverage. On the contrary, a higher volatility of Brownian shocks σ makes the equity value in the MPE higher, hence, makes the credibility constraints stricter. Thus, the maximal repurchase in the optimal time-consistent policy is lower (12% versus 20% in the baseline), and the leverage is lowered as well (14% versus 21% in the baseline).

It is interesting to note that if we focus on issuance only policies, then the commitment solution coincides with the optimal time-consistent policy, and in particular, the credibility constraints would not affect the leverage dynamics. The policy depicted in Figure 4 is the optimal issuance only policy, and it results in the equity value that is above that in the MPE for all $y \geq y_{bm}$, hence, it is time-consistent. This observation extends the result in Benzoni et al. (2019) to the case of jump-diffusion process for cash flows. The important conceptual point is that it is large repurchases that are particularly costly for equity holders and can lead to the violation of credibility constraints. For this reason, the gap between the commitment and no-commitment solutions arises in our model, but not

Figure 4: ***Comparison to Issuance Only Policies***
Parameters: $\mu = 2\%$, $r = 5\%$, $\pi = 35\%$, $\sigma = 25\%$, $\lambda = 1/3$, $\eta = 5.6$. *The solid blue lines depicts values under optimal issuance only policy, the dotted yellow line depicts values in the MPE, the dashed green line depicts values under the optimal time-consistent debt policy. Depicted is the optimal issuance only policy characterized by* $y_i = y_i^* = 1.64$ *and* $y_b = 0.255$.

in the model with only issuance policies or with diffusion process for cash flows.

5 Empirical Implications

We describe implications of our model for leverage dynamics, the effect of volatility of different shocks, and optimal debt maturity.

5.1 Leverage Dynamics

We first describe leverage dynamics under the optimal time-consistent debt policy. In the baseline specification, the firm issues console. The risk-free rate is $r = 5\%$. The drift of cash-flows under the risk-neutral measure is $\mu = 2\%$ and the volatility is $\sigma = 25\%$. Following Graham (2000), we set tax benefits of debt to $\pi = 10\%$. Downward Poisson jumps occur on average every three years ($\lambda = 1/3$) and their average size is 15% ($\zeta = -15\%$).

Figure 5 illustrates the dynamics under the optimal time-consistent debt policy. There are two qualitatively very different regimes: the stable and the distress regimes. In the stable regime, the equityholders stick to the ICR target of $\hat{y} = 2.9$ and maintain a relatively low leverage ratio of 21% by compensating all positive or sufficiently small negative shocks to cash flows (up to 20% downward jumps) through the debt issuance or repurchase, respectively. Because of that, the price of debt is stable and close to the price of risk-free debt (the credit spread is 45 bps).

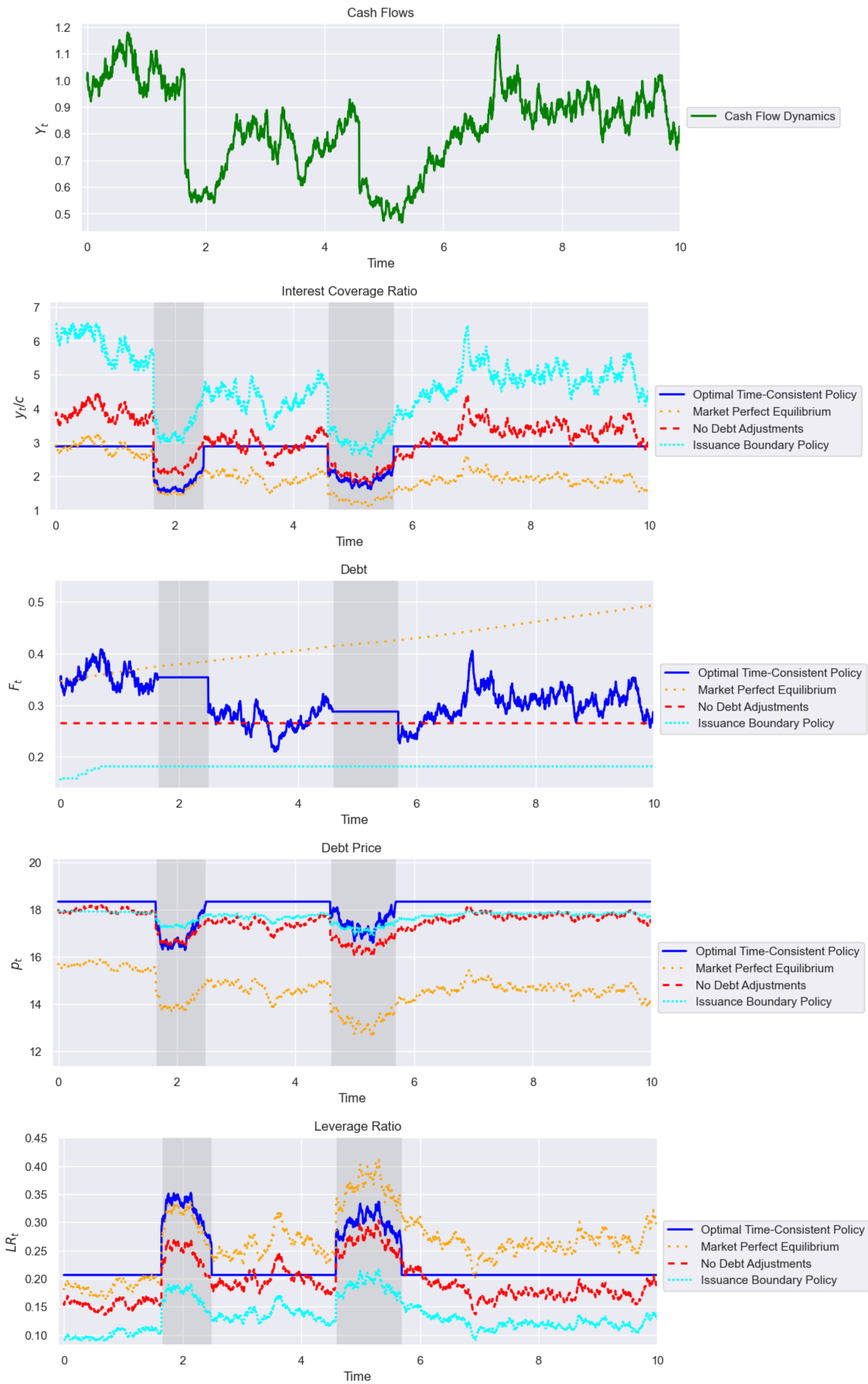

Figure 5: ***Leverage dynamics for the baseline specification***

Solid lines depict dynamics under the optimal time-consistent policy, dotted lines depict dynamics under DeMarzo and He (forthcoming) policy, dashed lines depict dynamics under Leland (1994) policy. The distress regime of the optimal time-consistent policy is highlighted in gray.

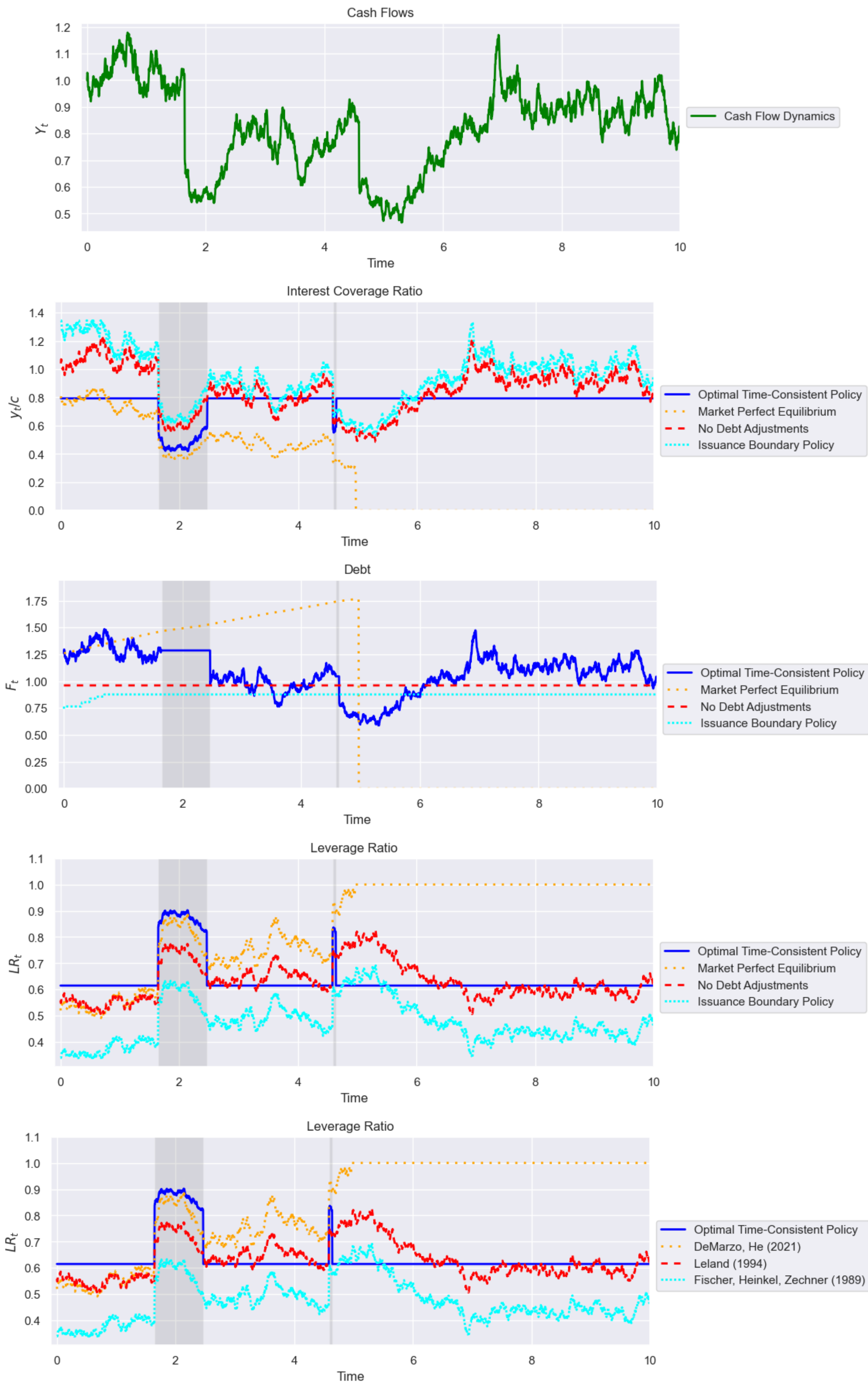

Figure 6: ***Leverage dynamics for the high tax benefits specification***

Solid lines depict dynamics under the optimal time-consistent policy, dotted lines depict dynamics under DeMarzo and He (forthcoming) policy, dashed lines depict dynamics under Leland (1994) policy. The distress regime of the optimal time-consistent policy is highlighted in gray.

Large negative shocks to cash flows transition the firm into the distress regime, where the equityholders temporarily abandon the ICR target and do not issue/repurchase debt until either the fundamentals improve or the firm goes bankrupt (see gray regions in Figure 5). As a result, the price of debt drops after the shock and becomes sensitive to further cash flow shocks. Despite a lower debt price and a constant debt level, the firm's leverage ratio jumps up and stays above the target leverage ratio of 21%, because of the drop in the equity value as the default becomes more likely. If the cash flows recover to the ICR level of 2.3, the firm repurchases a chunk of debt and returns to the stable regime. Otherwise, default occurs when the ICR drops below 0.3.

Interestingly, the two regimes and the ICR target arise endogenously as the equity holders' optimal credible response to the magnitude of shocks and the size of tax benefits within a rich class of policies that allow for repurchase and issuance regions with discrete as well as incremental debt adjustments. Changes in the underlying environment will affect the ICR target and the relative size of the stable and distress regions.

For example, consider the case of high tax benefits when $\pi = 40\%$. Naturally, the leverage is much higher compared to the low tax benefits case (initial leverage ratio becomes 61% compared to 21%), and correspondingly, ICR target is lower. The endogenous reaction to the shocks changes. The equity holders prefer to exit the distress regime faster by repurchasing earlier. In particular, when they are at the ICR target, they are willing to compensate larger drops in cash flows with repurchase (up to 24% compared to 20% in the baseline). This explains why in Figure 6 the second distress regime is much shorter compared to the baseline case depicted in Figure 5. Interestingly, the increase in leverage outweighs these larger repurchases by the equity holders, and as a result, debt is more risky and the credit spread at the target ICR is higher (141 bps versus 45 bps in the baseline).

Comparison to Benchmarks Properties of the optimal time-consistent debt policy are quite different from the alternative benchmark proposed by DeMarzo and He (forthcoming). Both papers consider the environment with costless leverage adjustments. Although somewhat extreme (as firms do face issuance/repurchase costs and often embed contractual commitments into debt contract), this environment provides a natural theoretical benchmark for assessing the value added of debt covenants, state-contingencies, exogenous commitment devices, etc., and hence, the scope of their use in practice.

DeMarzo and He (forthcoming) analyze the MPE in this environment when no exogenous commitment devices are available, such as collateral, covenants, state contingencies, etc. As described in Section 3.1, in the MPE debtholders do not believe that the equityholders can resist temptation to issue new debt in the future. This depresses the price of current debt issuance and leads to a particularly bad outcome in which the ratchet effect dissipates all surplus from the new debt issuances. When compared to this benchmark, various exogenous commitment devices have value, as they alleviate the ratchet effect. In contrast, in our paper debt policies can be sustained endogenously as long as the promise to follow them is credible. This presents a higher bar for justification of exogenous commitment devices.

The leverage dynamics are quite different under the optimal time-consistent policy and in the MPE. In Figure 5, dotted lines correspond to dynamics in the MPE and solid lines represent dynamics under the optimal time-consistent policy.[8] In the MPE, the firm constantly issues debt at a speed that varies with the level of interest coverage. Hence, the debt level grows steadily over time. Because the equity holders cannot sustain any debt discipline, the price of debt is significantly depressed compared to our optimal time-consistent debt policy. The credit spread is on average 186 bps in the no-credibility outcome compared to the average credit spread of 52 bps in the optimum. Further, the debt price is sensitive to cash flow shocks in the MPE.

Interestingly, lack of credibility need not imply that the debt level is always higher in the MPE compared to the optimal time-consistent policy. Rather, in the optimum, the equity holders tailor better the debt issuance/repurchase to the economic conditions. This can be clearly seen from the evolution of the debt levels and leverage ratios in Figure 5. During most of the first year, the firm accumulates debt faster under the optimal time-consistent policy than in the MPE, because on average the cash flows grow during this period. After the first significant negative shocks to cash flows (around year 2), the leverage ratio in both outcomes jumps up. Yet, after the cash flows improve, the firm repurchases debt and lowers leverage under the optimal time-consistent policy, while the leverage stays high in the MPE. With subsequent distress periods, this divergence in the firm's leverage continues to increase as seen in Figure 5. The ability to credibly get back to the stable regime under the optimal time-consistent debt policy is what allows the firm to gain from debt issuance when cash flows are high.

[8]In DeMarzo and He (forthcoming), the firm prefers to have zero leverage at $t = 0$. In the simulations, we suppose that in both optimal time-consistent outcome and no-credibility outcome, the firm starts with debt level F_0 as in the optimal time-consistent policy. Note that in Figure 5, the debt face value F_t is gradually rising over time. This is because in this section we consider consoles. Generally, if debt has a finite maturity, DeMarzo and He (forthcoming) show that debt face value can could both increase and decrease over time.

We also compare our dynamics to that under the policy of no debt adjustment after the initial issuance (Leland 1994) and the policy under which the firm can issue additional debt at a certain issuance boundary y_i (Fischer, Heinkel and Zechner 1989, Goldstein, Ju and Leland 2001). As can be seen from Figure 5, in both cases, the firm issues a lower debt level compared to the optimal time-consistent policy. Because the firm does not actively manage debt, both interest coverage and debt price are sensitive to cash flow fluctuations. Further, the leverage ratio is lower than in the optimal outcome. This extra precaution is explained by the fact that the firm cannot actively manage debt (in particular, repurchase it after negative shocks).

Finally, note that unless the credibility constraints binds (as is the case in Figure 5), the default threshold in the optimal time-consistent policy is below that in the MPE and other benchmarks. Figure 6 demonstrates that this makes debt riskier in the MPE. Specifically, the firm defaults around year 5 in the MPE, while it avoids default (for this sample path of the cash flows) under the optimal policy.

5.2 Comparative Statics

We next analyze how different types of shocks affect optimal time-consistent debt policy. We consider changes in the policy boundaries, leverage ratio, and credit spread at ICR target as we vary the size ζ and intensity λ of Poisson jumps and volatility of Brownian shocks σ.[9] Given that the leverage dynamics is very different in the normal and distress regions, we also analyze median leverage ratio and credit spread right after a sufficiently large shock that puts the firm in the distress region but does not bankrupt it.[10] These statistics capture conditions of "fallen angels," firms that are recently downgraded from investment grade to speculative grade.

We report comparative statics for three scenarios: (i) the *base case* of 10-year bonds and tax benefits $\pi = 10\%$; (ii) the *high tax benefits case* of 10-year bonds and $\pi = 40\%$; and (iii) the *case of console* with tax benefits $\pi = 10\%$. This comparison allows us to analyze the effect of binding credibility constraints and debt maturity on comparative statics.

First, consider the base case in Table 2. Higher intensity or higher expected losses

[9] In our comparative statics, we adjust $\hat{\mu}$ as we vary ζ or λ so that to hold the drift of the cash flow process $\mu = \hat{\mu} - \frac{\lambda}{\eta+1}$ constant.

[10] Leverage ratio at y equals $\frac{p(y)}{e(y)+p(y)}$ and the credit spread at y equals $\frac{c+\xi}{p(y)} - \xi - r$. We consider median leverage ratio and credit spread rather than their means, because the latter are not guaranteed to exist due to leverage ratio and credit spread going to infinity as y approaches y_b.

	at target		in distress		optimal policy			MPE
	leverage ratio	credit spread	median leverage ratio	median credit spread	target ICR	repurchase ICR	default ICR	default ICR
Base case	19%	34 bps	54%	145 bps	2.55	1.05	0.34	0.34
$\zeta = -20\%$	27%	23 bps	70%	165 bps	1.73	0.76	0.36	0.36
$\zeta = -25\%$	19%	34 bps	54%	145 bps	2.55	1.05	0.34	0.34
$\zeta = -30\%$	12%	48 bps	39%	147 bps	3.86	1.53	0.33	0.33
$\lambda = 1/4$	21%	31 bps	61%	156 bps	2.23	0.91	0.35	0.35
$\lambda = 1/3$	19%	34 bps	54%	145 bps	2.55	1.05	0.34	0.34
$\lambda = 1/2$	15%	39 bps	43%	133 bps	3.20	1.33	0.32	0.32
$\sigma = 10\%$	20%	30 bps	62%	135 bps	2.41	0.89	0.40	0.40
$\sigma = 25\%$	19%	34 bps	54%	145 bps	2.55	1.05	0.34	0.34
$\sigma = 40\%$	16%	43 bps	38%	146 bps	3.05	1.51	0.27	0.27

Table 2: ***Comparative statics in the base case***
Parameters: $\xi = 1/10$, $\pi = 10\%$, $c = 8\%$, $\sigma = 25\%$, $\lambda = 1/3$, $\zeta = -25\%$, $\mu = 2\%$, $r = 5\%$.

from downward jumps lead to a higher ICR target $\hat{y}/c$, a lower leverage ratio and a higher credit spread at $\hat{y}$. Observe that $\hat{y}/y_b$ also increases as ζ becomes smaller/λ becomes larger. Intuitively, when downward jumps become larger or more frequent, it is optimal for the firm to create a larger safety buffer by lowering the leverage and targeting a higher ICR relative to the default boundary. However, this safety buffer is not sufficient to compensate for the higher risk of default, and so, credit spreads at $\hat{y}$ increase. Quantitatively, the variation in credit spreads across various values of ζ and λ is small compared to the variation in credit spreads between the normal and distress regimes. This fact reflects the role of active debt management in the normal regime versus passive debt policy in the distress regime, which compensates for moderate negative shocks to cash flows.

Interestingly, when downward jumps are more severe/frequent, lower leverage chosen by the firm also translates into a smaller spike in the credit spread and leverage ratio when the firm becomes a "fallen angel." For example, credit spread goes up on average 5 times after a downgrade for a firm with infrequent jumps ($\lambda = 1/4$) as opposed to only 3.4 times for a firm with higher jump frequency ($\lambda = 1/2$). Thus, lower leverage prevents credit spreads from jumping too high in distress. The leverage ratio jumps as well after a downgrade (roughly 2.8 times), despite the fact that the firm stops issuing debt. This occurs, because of the proximity to default, which depreciates significantly equity value.

The effect of downward jumps is qualitatively similar in the high tax benefits case (see Table 3). Quantitatively, higher tax benefits incentivize the firm to increase leverage in the optimal time-consistent policy, which leads to higher credit spreads both at the target ICR and in distress.

The comparative statics are more nuanced with respect to the volatility of Brownian shocks. In the base case in Table 2, leverage ratio is decreasing and the target ICR is increasing with respect to σ, while directions are reversed in the high tax benefits case in Table 3. This qualitative difference arises because the credibility constraints in the optimal time-consistent policy are slack when π is high ($y_b < y_{bm}$ in Table 3), but some constraints are binding when π is low ($y_b = y_{bm}$ in Table 2).

To explain the intuition for these comparative statics, consider the Leland model as a benchmark (Leland 1994). Volatility σ has two effects there. On the one hand, the probability that the cash flow process declines to any given bankruptcy threshold and the firm will default increases (i.e., cash flows become riskier). This effect increases expected bankruptcy costs and therefore reduces optimal borrowing. On the other hand, higher volatility reduces the default boundary due to the option value effect. Intuitively, when deciding whether to inject cash into a money-losing firm or to announce default, equity holders trade-off saving interest payments (the benefit of default) against the possibility that things improve in the future and the firm recovers (the benefit of waiting). Higher volatility increases the latter and therefore reduces the default boundary. This effect reduces expected bankruptcy costs and increases optimal borrowing. In the Leland model, the first effect dominates when volatility is low, while the second effect dominates when volatility is high (see Table II and Figure 8 in Leland (1994)).

In our model, the second effect dominates when the credibility constraints do not bind, which occurs when tax benefits are high (see Table 3). Intuitively, when the firm is at its ICR target, a marginally higher volatility does not matter much for the firm riskiness, because the firm smoothes out these shocks by repurchasing and issuing debt. Volatility only matters when the firm gets hit by a big shock that gets it into the financial distress region, which is a relatively small probability event. At the same time, the second effect is as significant as in the Leland model, because the default boundary is determined by the same trade-off as in the Leland model, when the firm is already in the financial distress region. Thus, a marginal increase in σ increases the chances of recovering from large negative shocks, but only has a relatively small positive effect on the probability with which the cash flow process hits any given lower threshold. Consequently, the equity holders expand the distress region (ratio y_r/y_b increases with σ) to increase their chances

	at target		in distress		optimal policy			MPE
	leverage ratio	credit spread	median leverage ratio	median credit spread	target ICR	repurchase ICR	default ICR	default ICR
High benefits	37%	138 bps	85%	453 bps	1.41	0.66	0.40	0.44
$\zeta = -20\%$	44%	97 bps	90%	476 bps	1.11	0.57	0.39	0.46
$\zeta = -25\%$	37%	138 bps	85%	453 bps	1.41	0.66	0.40	0.44
$\zeta = -30\%$	31%	190 bps	79%	474 bps	1.72	0.77	0.39	0.42
$\lambda = 1/4$	40%	133 bps	87%	479 bps	1.27	0.63	0.39	0.45
$\lambda = 1/3$	37%	138 bps	85%	453 bps	1.41	0.66	0.40	0.44
$\lambda = 1/2$	33%	146 bps	84%	460 bps	1.61	0.69	0.39	0.42
$\sigma = 10\%$	36%	133 bps	81%	291 bps	1.46	0.72	0.45	0.51
$\sigma = 25\%$	37%	138 bps	85%	453 bps	1.41	0.66	0.40	0.44
$\sigma = 40\%$	39%	153 bps	87%	659 bps	1.28	0.62	0.34	0.35

Table 3: ***Comparative statics in the high tax benefits case***
Parameters: $\xi = 0.1$, $\pi = 40\%$, $c = 8\%$, $\sigma = 25\%$, $\lambda = 1/3$, $\zeta = -25\%$, $\mu = 2\%$, $r = 5\%$.

of escaping from it and borrow more. This explains why the leverage ratio increases in volatility σ whenever the credibility constraints do not bind.

The comparative statics are the opposite once the credibility constraints start to bind, which occurs in the base case (see Table 2). An increase in volatility increases the value of equity upon deviation from the debt policy and makes it harder to satisfy the credibility constraints. Thus, in this case the maximal repurchase amount must be sufficiently small so that the equity holders' promise of such a repurchase is credible. In Table 2, as σ increases from 10% to 40%, the maximal shock that the equity holders compensate with repurchase goes down from 63% to 50%. This makes debt riskier, and the firm borrows less, despite the fact that the equity holders expand the distress region (ratio y_r/y_b increases from 2.23 to 5.59 with σ) to increase their chances of escaping default while in the distress region.

Finally, we consider the effect of longer maturity. Table 4 reports comparative statics for the case of console debt. This case is qualitatively similar to the base case, because the credibility constraints bind at the optimal time-consistent debt policy. The quantitative difference is that the leverage ratios are lower, while the credit spreads are higher when the firm issues console. Intuitively, debt of shorter maturity serves as a commitment device to

	at target		in distress		optimal policy			MPE
	leverage ratio	credit spread	median leverage ratio	median credit spread	target ICR	repurchase ICR	default ICR	default ICR
Console	11%	77 bps	19%	106 bps	5.31	3.67	0.26	0.26
$\zeta = -15\%$	15%	60 bps	23%	83 bps	3.88	2.88	0.27	0.27
$\zeta = -20\%$	11%	77 bps	19%	106 bps	5.31	3.67	0.26	0.26
$\zeta = -25\%$	8%	99 bps	15%	133 bps	7.35	4.75	0.24	0.24
$\lambda = 1/4$	13%	68 bps	22%	98 bps	4.46	3.20	0.27	0.27
$\lambda = 1/3$	11%	77 bps	19%	106 bps	5.31	3.67	0.26	0.26
$\lambda = 1/2$	8%	93 bps	15%	121 bps	7.00	4.62	0.23	0.23
$\sigma = 10\%$	14%	56 bps	27%	82 bps	4.32	2.66	0.35	0.35
$\sigma = 25\%$	11%	77 bps	19%	106 bps	5.31	3.67	0.26	0.26
$\sigma = 40\%$	7%	119 bps	11%	151 bps	7.38	5.94	0.18	0.18

Table 4: ***Comparative statics in the case of console***
Parameters: $\xi = 0$, $\pi = 10\%$, $c = 8\%$, $\sigma = 25\%$, $\lambda = 1/3$, $\zeta = -25\%$, $\mu = 2\%$, $r = 5\%$.

reduce debt burden. While this commitment does not affect the firm much in normal times (because it issues new debt to stay at the target ICR), in distress times, this commitment device is valuable as it allows the firm to reduce the likelihood of bankruptcy in states where large repurchase promises are not credible. For this reason, the firm optimally borrows more in debt of shorter maturity.

5.3 Optimal Maturity

We next analyze the optimal debt maturity. Suppose that the equityholders choose debt maturity ξ and the optimal time-consistent targeted ICR policy $\hat{\Sigma}$ at $t = 0$. Following Leland and Toft (1996), coupon c is set so that the debt is priced at par (i.e., $\hat{p} = 1$). We suppose that the same coupon is applied in computing the MPE of the debt issuance game.

Figure 7 depicts the firm value as a function of maturity in the baseline specification. The optimal maturity is interior and equals (approximately) 5 years. The optimal maturity balances the following two forces. On the one hand, principal payments are not tax deductible, and hence, as in Leland and Toft (1996), debt with longer maturity allows the equity holders to better exploit tax benefits (this force makes the optimal maturity in

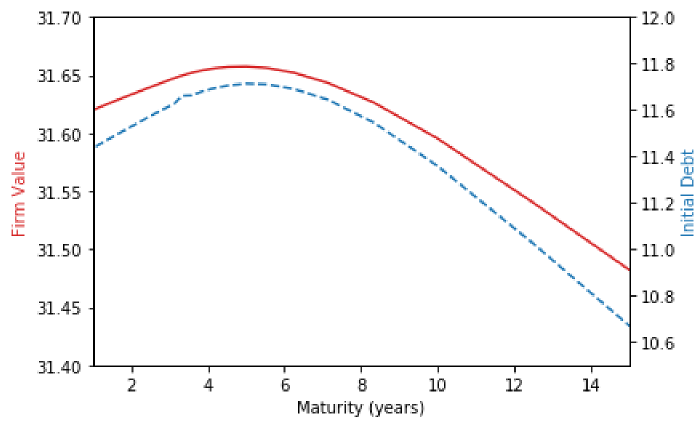

Figure 7: ***Firm Value (solid line) and Initial Debt Issued (dashed line) as a Function of Maturity***

Parameters: $\mu = 2\%$, $r = 5\%$, $\pi = 10\%$, $\sigma = 25\%$, $\lambda = 1/3$, $\zeta = -15\%$.

Leland and Toft (1996) infinite). On the other hand, shorter maturity allows the equityholders to reduce the likelihood of default while in the turbulent region. In the turbulent region, repurchase promises are not credible, and hence, the equityholders have to restrain themselves from active debt management. Shorter debt maturity allows the equityholders to commit to reduce the debt burden in the distress region when debt repurchases are not credible.

Figure 7 also shows that at the optimal maturity, the firm issues close to maximal amount of debt. Intuitively, at longer maturities the debt is too risky and the equity holders decide to reduce leverage. Debt of shorter maturities entails smaller tax benefits, and hence, it is optimal for the firm to issue less debt. Table 5 demonstrates this relationship between debt maturity and optimal firm leverage for each maturity level. Interestingly, despite the fact that the leverage is reduced for longer maturities, the debt is still more risky, which is reflected in a higher coupon.

Further, Table 5 shows default thresholds for the optimal time-consistent targeted ICR policy and the MPE. The credibility constraints bind for longer maturities, but not for shorter maturities. To see why, recall that the equity value in the MPE is the same as if the firm did not issue any new debt after the initial issuance. The faster the debt matures, the smaller tax benefits from the initial debt issuance captured by the equity holders. Thus, the equity value in the MPE is increasing with debt maturity, and so, the credibility constraints are more stringent for longer maturity debt.

We next consider the effect of volatility of different shocks on the optimal debt ma-

maturity	coupon	firm value	optimal leverage	target ICR	default ICR	MPE default ICR
console	5.45%	30.80	21%	2.88	0.289	0.289
30 years	5.21%	31.22	29%	2.13	0.386	0.386
10 years	5.12%	31.60	36%	1.72	0.486	0.486
5 years	**5.11%**	**31.66**	**37%**	**1.67**	**0.548**	**0.548**
1 year	5.10%	31.62	36%	1.71	0.607	0.627
6 months	5.10%	31.61	36%	1.72	0.614	0.640

Table 5: ***The Effect of Debt Maturity on Optimal Leverage and Debt Policy***
Parameters: $\mu = 2\%$, $r = 5\%$, $\pi = 10\%$, $\sigma = 25\%$, $\lambda = 1/3$, $\zeta = -15\%$

turity. Volatility of Brownian shocks affects significantly only the debt maturity but not other characteristics of the policy. When Brownian component of the cash flow process is more volatile, the firm shortens debt maturity, but does not change significantly the ICR target or the leverage ratio. As a result, the credit spread remains virtually unchanged. These effects hold for both high and low tax benefits (Table 6a and 6b).

At the same time, Poisson shocks affect both debt maturity and the leverage ratio. We first consider the baseline case of small tax benefits in Table 6a. When downward jumps are more frequent or more severe, maturity of debt shortens, the leverage is reduced, and the ICR target increases. As a result, credit spreads are not affected significantly by the volatility of shocks.

The effects of the frequency of Poisson shocks is amplified when the tax benefits are larger and the firm borrows more (see Table 6b). Interestingly, the effect of the size of downward shocks is non-monotonic. When the size of downward shocks is moderate, larger shock size shorten the maturity and reduce the leverage ratio (shift from $\zeta = -10\%$ to $\zeta = -15\%$). However, when downward shocks are sufficiently large, it is optimal for the firm to issue console and significantly reduce leverage and increase ICR target. Nevertheless, the credit spread increases significantly (approximately three times as ζ drops from -15% to -20%).

	maturity	leverage ratio	credit spread at target	target ICR	repurchase boundary ICR	default boundary ICR
Base case $\tau = 10\%$	4.9 years	37%	11 bps	1.67	0.70	0.55
$\sigma = 10\%$	12 years	37%	11 bps	1.67	0.74	0.57
$\sigma = 25\%$	4.9 years	37%	11 bps	1.67	0.70	0.55
$\sigma = 40\%$	2.4 years	37%	11 bps	1.67	0.69	0.54
$\zeta = -10\%$	6.5 years	51%	7 bps	1.18	0.65	0.54
$\zeta = -15\%$	4.9 years	37%	11 bps	1.67	0.70	0.55
$\zeta = -20\%$	3.7 years	26%	15 bps	2.38	0.76	0.56
$\lambda = 1/4$	5.4 years	39%	11 bps	1.58	0.69	0.56
$\lambda = 1/3$	4.9 years	37%	11 bps	1.67	0.70	0.55
$\lambda = 1/2$	4.2 years	32%	11 bps	1.81	0.71	0.56

(a) Low Tax Benefits $\tau = 10\%$

	maturity	leverage ratio	credit spread at target	target ICR	repurchase boundary ICR	default boundary ICR
High Benefits $\tau = 25\%$	21.3 years	48%	45 bps	1.27	0.75	0.44
$\sigma = 10\%$	console	48%	55 bps	1.24	0.82	0.43
$\sigma = 25\%$	21.3 years	48%	45 bps	1.27	0.75	0.44
$\sigma = 40\%$	8.6 years	48%	44 bps	1.26	0.73	0.44
$\zeta = -10\%$	23.6 years	61%	25 bps	0.97	0.65	0.445
$\zeta = -15\%$	21.3 years	48%	45 bps	1.27	0.75	0.44
$\zeta = -20\%$	console	40%	134 bps	1.38	0.98	0.28
$\lambda = 1/5$	31.4 years	53%	47 bps	1.12	0.74	0.42
$\lambda = 1/3$	21.3 years	48%	45 bps	1.27	0.75	0.44
$\lambda = 1$	9.5 years	39%	42 bps	1.62	0.77	0.50

(b) High Tax Benefits $\tau = 25\%$

Table 6: ***Comparative Statics of Optimal Maturity with respect to Shock Volatility***

In comparative statics with respect to ζ and λ, parameter μ is adjusted to keep the drift of cash flows $\hat{\mu}$ at the level in the base case. The credit spread is computed at $\hat{y}$ and is equal to $c - r$. The leverage ratio is computed at $\hat{y}$ and is equal to $\hat{p}/(\hat{e} + \hat{p})$.

6 Conclusion

In this paper, we contribute to understanding of limited commitment in the trade-off theory of dynamic capital structure by posing the following problem. Suppose that the firm designs and announces a debt policy, which specifies how much debt it plans to issue or repurchase at each interest coverage level. Call a debt policy time-consistent if ex-post the equity holders prefer not to deviate from it, provided that the debt holders trust that the firm will follow the debt policy before the deviation but the equity holders lose credibility in sustaining any debt discipline after a deviation. What will be the optimal time-consistent policy? Answering this question is important because it shows the extent to which commitment problems in debt management can be resolved via self-sustained reputation and because it can help us better understand the fit between the trade-off theory of dynamic capital structure and the data.

In a class of policies, the optimal time-consistent policy consists of an ICR target and two regions, the normal region and the distress region. In the normal region, the firm actively manages its liabilities to keep interest coverage (and leverage) at the target by issuing and repurchasing debt. Thus, the way the firm operates in the normal region is similar to the prescriptions of the static trade-off theory of capital structure. However, a sufficiently large negative shock to cash flows puts the firm into the distress region. In this case, the firm abandons active debt management, despite being over-levered, and waits until either subsequent good news get the firm out of the distress region or subsequent bad news lead to bankruptcy.

Credit spreads also behave qualitative differently in the two regions. In the stable region, the credit spread is small, and the debt price does not respond to small shocks to fundamentals. In contrast, in the distress region, credit spreads are high with the debt price being sensitive to cash flows shocks. The model also implies an optimal maturity as a solution to the trade-off between higher tax benefits of debt, which favor longer maturity, and commitment to reduce the debt burden in the distress region, which favor shorter maturity.

We show that credibility constraints often bind and taking them into account affects the comparative statics. While more severe/frequent jumps tend to reduce leverage, the volatility of Brownian shocks increases leverage when credibility constraints do not bind, but it decreases leverage when they are binding.

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A Appendix

A.1 Notations

Variable	Explanation	Variable	Explanation
Y_t	cash flows	Σ	a debt policy
Z_t	standard Brownian motion	$W(\Sigma)$	firm value given policy Σ
S_i	size of downward Poisson shocks	$\mathbb{S}$	class of debt policies
N_t	arrival process for Poisson shocks	$\mathbb{S}_{TC}$	class of time-consistent debt policies
F_t	debt face value	$\hat{\mathbb{S}}$	class of targeted ICR debt policies
y_t	$= Y_t/F_t$	τ_b	default time
c	coupon	y_b	default boundary
π	tax rate	y_r	lower bound of repurchase region
r	discount rate	$\overline{y}_r$	upper bound of repurchase region
μ	expected cash flow growth	y_r^*	target for y_t after repurchase
σ	volatility of Brownian shocks	y_i	issuance boundary
λ	intensity of Poisson shocks	y_i^*	target for y_t after issuance
ζ	expected jump size	$p(y)$	debt price
η	$= -(\zeta+1)/\zeta$	$e(y)$	equity value per unit of debt face value
$\hat{\mu}$	$= \mu - \lambda\zeta$	$v(y)$	firm value per unit of debt face value
ξ	rate of debt maturity	$w(y)$	$= v(y)/y$
ϕ	$= \frac{1-\pi}{r-\mu}$	$\hat{p}$	debt price at target ICR
ρ	$= \frac{c(1-\pi)+\xi}{r+\xi}$	$\hat{y}$	target ICR
		$\hat{e}$	equity value per unit of debt face value at target ICR

A.2 Omitted Proofs

Derivation of Debt Issuance Strategy in MPE The debt price $p_m(y)$ in the MPE is given by:

$$p_m(y) = cdt + \xi dt + (1 - rdt - \xi dt - \lambda dt)\mathbb{E}\left[p_m\left(\frac{Y+dY}{F+dF}\right)\middle| dN_t\right] + \lambda dt\mathbb{E}\left[p_m\left(e^{-\tilde{s}}y\right)\right],$$

or equivalently,

$$(r+\lambda+\xi)\,p_m(y)dt = (c+\xi)dt + \mathbb{E}\left[p_m'(y)\frac{dY}{F} - p_m'(y)\frac{Y}{F^2}dF\middle| dN_t\right] + \lambda dt\mathbb{E}\left[p_m\left(e^{-\tilde{s}}y\right)\right].$$

We suppose that in the MPE, $d\Gamma_t = g(y_t)F_{t-}dt$. This implies that p_m satisfies the HJB equation:

$$(r+\lambda+\xi)p_m(y) = c + \xi + p_m'(y)y\left(\hat{\mu} - g(y) + \xi\right) + \lambda\mathbb{E}\left[p\left(e^{-\tilde{s}}y\right)\right]. \tag{37}$$

On the other hand, differentiating (24) with respect to F and using (23),

$$(r+\lambda+\xi)p_m(y) = (1-\pi)c + \xi + (\hat{\mu}+\xi)\, Y \tfrac{\partial}{\partial Y} p_m(y) + \tfrac{1}{2}\sigma^2 Y^2 \tfrac{\partial^2}{\partial Y^2} p_m(y) + \lambda \mathbb{E}\left[p_m(e^{-\tilde{s}}y)\right]. \quad (38)$$

Comparing (37) and (38),

$$g(y) = \frac{\pi c}{y p_m'(y)},$$

which is the desired expression for $g(y)$.

Proof of Proposition 1. The argument for (25) is provided in the text, and we are left to prove that E_m is strictly convex in F. Consider $y_{bm} < y' < y''$. For any sequence of shocks dZ_t, dN_t, and dS_t, the process y_t that starts at y'' is always higher than the process that starts at y'. Thus, when the process y_t starts at y'', τ_{bm} occurs later than when it starts at y', which implies that

$$\mathbb{E}\left[\int_t^{\tau_{bm}\wedge\tau_m} ce^{-r(s-t)}ds + 1\{\tau_m \le \tau_{bm}\}\middle| y_t = y', \Sigma^0\right] < \mathbb{E}\left[\int_t^{\tau_{bm}\wedge\tau_m} ce^{-r(s-t)}ds + 1\{\tau_m \le \tau_{bm}\}\middle| y_t = y'', \Sigma^0\right].$$

Given (27), $p_m(y') < p_m(y'')$. Thus, $p_m'(y) > 0$ for $y > y_{bm}$. Finally,

$$\frac{\partial^2}{\partial F^2} E_m(Y,F) = \frac{\partial}{\partial F}\left\{e_m(y) - e_m'(y)y\right\} = e_m''(y)\frac{y^2}{F} = p_m'(y)\frac{y}{F} > 0,$$

where we used $p_m'(y) = y e_m''(y)$ in the last equality. This completes the proof of strict convexity of E_m in F. □

Proof of Proposition 2. We will construct a targeted ICR policy with $\hat{y} = y_r$, call it Σ, that satisfies (33). Under policy Σ, when y reaches $\hat{y}$, the equity holders compensate all Brownian shocks at $\hat{y}$ with debt issuances or repurchases, but not Poisson shocks. For simplicity, we omit in the notation dependence of e and p on Σ. The argument proceeds in three step.

Step 1: for any $\varepsilon > 0$, there is $\hat{y}$ sufficiently large such that $\hat{p} = p(\hat{y}) > (c+\xi)/(r+\xi) - \varepsilon$. Indeed, as $\hat{y} \to \infty$, the debt is close to safe near $\hat{y}$, thus, its price converges to the price of the safe debt, $(c+\xi)/(r+\xi)$. Let us choose $\varepsilon < \pi(c+\xi)/(r+\xi)$, which implies that $(c+\xi)/(r+\xi) - \varepsilon > (1-\pi)(c+\xi)/(r+\xi)$.

Step 2: for any $y \in (y_b, \hat{y}]$,

$$
\begin{aligned}
e(y) &= \mathbb{E}\left[\int_t^{\tau_b} e^{-r(s-t)}\left[(1-\pi)(y_s-c)ds - \xi ds + p(y_s)d\Gamma_s\right]\middle| y_t = y, \Sigma\right] \\
&= \mathbb{E}\left[\int_t^{\tau_b} e^{-r(s-t)}\left[(1-\pi)(y_s-c)ds - \xi ds + \hat{p}d\Gamma_s\right]\middle| y_t = y, \Sigma\right] \\
&\geq \mathbb{E}\left[\int_t^{\tau_{bm}} e^{-r(s-t)}\left[(1-\pi)(y_s-c)ds - \xi ds + \hat{p}d\Gamma_s\right]\middle| y_t = y, \Sigma\right] \\
&= \mathbb{E}\left[\int_t^{\tau_{bm}} e^{-r(s-t)}\left[(1-\pi)(y_s-c)ds - \xi ds\right]\middle| y_t = y, \Sigma\right] + \hat{p}\mathbb{E}\left[\int_t^{\tau_{bm}} e^{-r(s-t)}d\Gamma_s \middle| y_t = y, \Sigma\right] \\
&> \mathbb{E}\left[\int_t^{\tau_{bm}} e^{-r(s-t)}\left[(1-\pi)(y_s-c)ds - \xi ds\right]\middle| y_t = y, \Sigma\right] + \left(\frac{c+\xi}{r+\xi} - \varepsilon\right)\mathbb{E}\left[\int_t^{\tau_{bm}} e^{-r(s-t)}d\Gamma_s \middle| y_t = y, \Sigma\right] \\
&> \mathbb{E}\left[\int_t^{\tau_{bm}} e^{-r(s-t)}\left[(1-\pi)(y_s-c)ds - \xi ds\right]\middle| y_t = y, \Sigma\right] + (1-\pi)\frac{c+\xi}{r+\xi}\mathbb{E}\left[\int_t^{\tau_{bm}} e^{-r(s-t)}d\Gamma_s\right]\middle| y_t = y, \Sigma\right] \\
&\geq \mathbb{E}\left[\int_t^{\tau_{bm}} e^{-r(s-t)}\left[(1-\pi)(y_s-c)ds - \xi ds + p_m(y_i)d\Gamma_s\right]\middle| y_t = y, \Sigma\right] \\
&= e_m(y).
\end{aligned}
$$

The first equality is by $d\Gamma_s \neq 0$ if only if $y_s = \hat{y}$, in which case $p(y_s) = p(\hat{y}) = \hat{p}$; the first inequality is by the fact that the default policy τ_b is the optimal default policy, and so, it dominates the default policy in the MPE τ_{bm}. Note that whenever $y_s = \hat{y}$, the equity holders issue/repurchase debt to compensate for all Brownian shocks and replace maturing debt so that $dy_t = dY_t/Y_t - (d\Gamma_t - \xi F_{t-})/F_{t-} = 0$. Hence,

$$
\mathbb{E}\left[d\Gamma_s | y_s = \hat{y}, dN_s = 0, \Sigma\right] = \mathbb{E}\left[F_{s-}dY_s/Y_s | y_s = \hat{y}, dN_s = 0, \Sigma\right] + \xi F_{s-}ds = (\hat{\mu} + \xi)\, F_{s-}ds > 0,
$$

and so, $\mathbb{E}\left[\int_t^{\tau_{bm}} e^{-r(s-t)}d\Gamma_s \middle| y_t = y, \Sigma\right] > 0$. Thus, the second and third inequalities follow by $\hat{p} > (c+\xi)/(r+\xi) - \varepsilon$ and $\varepsilon < \pi(c+\xi)/(r+\xi)$; the last inequality is by $p_m(y) \leq (1-\pi)(c+\xi)/(r+\xi)$ for all y (by (27)); and the last equality is by the fact that in the MPE, the equity holders are indifferent between any debt issuance policy, and so, they weakly prefer to follow Σ. Therefore, $e(y) > e_m(y)$ for all $y \in (y_b, \hat{y}]$.

Step 3: let $\hat{\tau}$ be the first time when y_t reaches $\hat{y}$. For any $y \in (y_b, \hat{y}]$,

$$
\begin{aligned}
p(y) &= \mathbb{E}\left[\int_t^{\tau_b \wedge \hat{\tau}} e^{-r(s-t)}ds + 1\{\tau_b > \hat{\tau}\}e^{-r\hat{\tau}}\hat{p} \middle| y_t = y, \Sigma\right] \\
&> \mathbb{E}\left[\int_t^{\tau_b \wedge \hat{\tau}} (1-\pi)e^{-r(s-t)}ds + 1\{\tau_b > \hat{\tau}\}e^{-r\hat{\tau}}p_m(\hat{y}) \middle| y_t = y, \Sigma\right] \\
&= \mathbb{E}\left[\int_t^{\tau_b \wedge \hat{\tau}} (1-\pi)e^{-r(s-t)}ds + 1\{\tau_b > \hat{\tau}\}e^{-r\hat{\tau}}p_m(\hat{y}) \middle| y_t = y, \Sigma^0\right] \\
&= p_m(y)
\end{aligned}
$$

where the inequality is by $\hat{p} > (c+\xi)/(r+\xi) - \varepsilon > (1-\pi)(c+\xi)/(r+\xi) \geq p_m(\hat{y})$ and $\pi > 0$; the first equality is by writing explicitly the debt price under Σ; the second equality is by the fact that under policy Σ, the firm does not issue/repurchase debt until y_t reaches $\hat{y}$; the third equality is by the law of total expectation and (27). Thus, for all $y \in (y_b, \hat{y}]$, $p(y) > p_m(y)$. This completes the proof of Proposition 2. □

Proof of Proposition 3. Fix some targeted ICR policy. It follows immediately from (25) that $y_b \leq y_{bm}$ and $e(\hat{y}) \geq e_m(\hat{y})$ are necessary conditions for the incentive constraints to be satisfied. To show that they are also sufficient, we proceed in two steps. Suppose that $y_b \leq y_{bm}$.

Step 1: $e(y) \geq e_m(y)$ **for all** $y \in (y_{bm}, y_r]$

We will show the contrapositive that if $e(\tilde{y}) < e_m(\tilde{y})$ for some $\tilde{y} \in (y_{bm}, y_r]$, then $y_b > y_{bm}$. Suppose to contradiction that $e(\tilde{y}) < e_m(\tilde{y})$, but $y_b \leq y_{bm}$. Let the stopping time $\tilde{\tau}$ be the first time when the state y_t reaches $\tilde{y}$ when the equity holders do not issue or repurchase new debt for $y_t \in (y_b, \tilde{y})$. For all $y \in (y_b, \tilde{y})$,

$$\begin{aligned}
e(y) &= \mathbb{E}\left[\int_t^{\tau_b \wedge \tilde{\tau}} e^{-r(s-t)}(1-\pi)(y_s - c)ds - \xi ds + 1\{\tilde{\tau} < \tau_b\} e^{-r(\tilde{\tau}-t)} e(\tilde{y}) \middle| y_t = y\right] \\
&< \mathbb{E}\left[\int_t^{\tau_b \wedge \tilde{\tau}} e^{-r(s-t)}(1-\pi)(y_s - c)ds - \xi ds + 1\{\tilde{\tau} < \tau_b\} e^{-r(\tilde{\tau}-t)} e_m(\tilde{y}) \middle| y_t = y\right] \\
&\leq \mathbb{E}\left[\int_t^{\tau_{bm} \wedge \tilde{\tau}} e^{-r(s-t)}(1-\pi)(y_s - c)ds - \xi ds + 1\{\tilde{\tau} < \tau_{bm}\} e^{-r(\tilde{\tau}-t)} e_m(\tilde{y}) \middle| y_t = y\right] \\
&= e_m(y).
\end{aligned}$$

The first equality is by the fact that the equity holders do not issue any debt until the state reaches $\tilde{y}$ when they follow the targeted ICR policy. The first inequality is by the fact that $e(\tilde{y}) < e_m(\tilde{y})$ and the event $\tilde{\tau} < \tau_b$ has a positive probability starting from any state $y \in (y_b, \tilde{y})$. The second inequality is by the fact that if the equity holders do not issue any debt after the initial issuance, then the equity value at state $\tilde{y}$ equals $e_m(\tilde{y})$ and the stopping time τ_{bm} is the optimal default time for the equity holders, and in particular, is weakly preferred to the stopping time τ_b. The last equality is by the fact that e_m gives the equity value per unit of debt face value when the equity holders do not issue any debt for $t > 0$. Therefore, $e(y) < e_m(y)$ for all $y \in (y_b, \tilde{y})$. Since $e_m(y) = 0$ for all $y \in (y_b, y_{bm}]$, $e(y) < 0$ for all $y \in (y_b, y_{bm}]$, which contradicts the fact that $e(y) \geq 0$ for some $y \in (y_b, y_{bm}]$. Thus, it must be that $y_b > y_{bm}$, which is the desired conclusion.

Step 2: $e(y) \geq e_m(y)$ **for all** $y \in (y_r, \hat{y}]$

Since for all $y \in (y_r, \hat{y}]$, $e(y)$ is a convex combination of $e(y_r)$ and $e(\hat{y})$, which are above $e_m(y_r)$ and $e_m(\hat{y})$, respectively. Since function e_m is convex (by Proposition 1), $e(y) \geq e_m(y)$ for $y \in (y_r, \hat{y}]$, which concludes the proof. □

Proof of Proposition 4. It is convenient to define parameters of the debt policy relative to the default boundary: $x_i \equiv y_i/y_{b0}$, $x_i^* \equiv y_i^*/y_{b0}$, $\overline{x}_r \equiv \overline{y}_r/y_{b0}$, $x_r^* \equiv y_r^*/y_{b0}$, $x_r \equiv y_r/y_{b0}$. Let

$$\tilde{w}(x) = \frac{c\pi}{(r+\xi)x} + \frac{1}{x}\sum_{k=1}^{3} A_k x^{-\gamma_k}.$$

Note that $w(y) = \phi + \frac{1}{y_b}\tilde{w}(y/y_b)$ for $y \in [\overline{y}_r, y_i]$ and the solution to program (35) can be obtained by solving

$$\max_{\mathbf{x}\in X} \left\{\tilde{w}(\mathbf{x}) : y_b = y_{b0}\right\},$$

where $\mathbf{x} = (x_r, x_i, x_i^*, \overline{x}_r, x_r^*)$ and X is the set of all admissible $\mathbf{x}$s. Using the fact that $a_k = c_k - b_k/r$ and $A_k = C_k - B_k/r$ together with conditions (15), (17) – (19), (21), and (22) on function e and conditions (9) – (13) on function p (see explicit formulas for those conditions in Online Appendix B), we can determine coefficients a_ks and A_ks from the following system of equations:

$$\phi y_{b0} + \frac{c\pi}{r+\xi} + \sum_{k=1}^{3} a_k = 0, \tag{39}$$

$$\frac{\phi\eta}{\eta+1} y_{b0} + \frac{c\pi}{r+\xi} + \sum_{k=1}^{3} a_k \frac{\eta}{\eta-\gamma_k} = 0, \tag{40}$$

$$\frac{c\pi}{(r+\xi)x_r} + \frac{1}{x_r}\sum_{k=1}^{3} a_k x_r^{-\gamma_k} = \frac{c\pi}{(r+\xi)\overline{x}_r} + \frac{1}{\overline{x}_r}\sum_{k=1}^{3} A_k \overline{x}_r^{-\gamma_k}, \tag{41}$$

$$\frac{c\pi}{(r+\xi)x_r^*} + \frac{1}{x_r^*}\sum_{k=1}^{3} A_k x_r^{*-\gamma_k} = \frac{c\pi}{(r+\xi)\overline{x}_r} + \frac{1}{\overline{x}_r}\sum_{k=1}^{3} A_k \overline{x}_r^{-\gamma_k}, \tag{42}$$

$$\frac{c\pi}{(r+\xi)x_i^*} + \frac{1}{x_i^*}\sum_{k=1}^{3} A_k x_i^{*-\gamma_k} = \frac{c\pi}{(r+\xi)x_i} + \frac{1}{x_i}\sum_{k=1}^{3} A_k x_i^{-\gamma_k}, \tag{43}$$

$$\sum_{k=1}^{3} \frac{\eta(1+\gamma_k)}{\eta-\gamma_k}\left[a_k x_r^{\eta-\gamma_k} - A_k \overline{x}_r^{\eta-\gamma_k}\right] = \frac{c\pi}{r+\xi}(\overline{x}_r^{\eta} - x_r^{\eta}). \tag{44}$$

Coefficients b_ks and B_ks are determined from

$$\sum_{k=1}^{3} b_k = -1, \tag{45}$$

$$1 + \sum_{k=1}^{3} b_k \frac{\eta}{\eta - \gamma_k} = 0, \tag{46}$$

$$\sum_{k=1}^{3} b_k x_r^{-\gamma_k} = \sum_{k=1}^{3} B_k \overline{x}_r^{-\gamma_k}, \tag{47}$$

$$\sum_{k=1}^{3} B_k \left(\overline{x}_r^{-\gamma_k} - x_r^{*-\gamma_k}\right) = 0, \tag{48}$$

$$\sum_{k=1}^{3} B_k \left(x_i^{-\gamma_k} - x_i^{*-\gamma_k}\right) = 0, \tag{49}$$

$$\sum_{k=1}^{3} b_k \frac{\gamma_k}{\eta - \gamma_k} x_r^{\eta-\gamma_k} - \sum_{k=1}^{3} B_k \frac{\gamma_k}{\eta - \gamma_k} \overline{x}_r^{\eta-\gamma_k} = 0. \tag{50}$$

The remaining condition is the smooth-pasting condition at the default boundary $e'(y_{b0}) = 0$, which becomes

$$\phi y_{b0} - \sum_{k=1}^{3} \left(a_k - b_k \frac{c + \xi}{r + \xi}\right) \gamma_k = 0. \tag{51}$$

Suppose that boundaries y_i and $\overline{y}_r$ are non-reflecting. We will show that then, they can be replaced by reflecting boundaries without any changes in the firm value. By the smooth-pasting principle, in the optimal $\mathbf{x}$,

$$\tilde{w}'(x_i) = \tilde{w}'(x_i^*) = \tilde{w}'(\overline{x}_r) = \tilde{w}'(x_r^*) = 0. \tag{52}$$

Differentiating equations (39) – (44) with respect to x_i and taking into account that y_{b0} is held fixed:

$$\sum_{k=1}^{3} \frac{\partial a_k}{\partial x_i} = 0,$$
$$\sum_{k=1}^{3} \frac{\partial a_k}{\partial x_i} \frac{\eta}{\eta - \gamma_k} = 0,$$
$$\sum_{k=1}^{3} \frac{\partial a_k}{\partial x_i} x_r^{-\gamma_k - 1} - \sum_{k=1}^{3} \frac{\partial A_k}{\partial x_i} \overline{x}_r^{-\gamma_k - 1} = 0,$$
$$\sum_{k=1}^{3} \frac{\partial A_k}{\partial x_i} \left(x_r^{*-\gamma_k - 1} - \overline{x}_r^{-\gamma_k - 1} \right) = 0,$$
$$\sum_{k=1}^{3} \frac{\partial A_k}{\partial x_i} \left(x_i^{*-\gamma_k - 1} - x_i^{-\gamma_k - 1} \right) = -\frac{c\pi}{(r+\xi) x_i^2} - \sum_{k=1}^{3} A_k (\gamma_k + 1) x_i^{-\gamma_k - 2},$$
$$\sum_{k=1}^{3} \frac{\eta(1+\gamma_k)}{\eta - \gamma_k} \left[\frac{\partial a_k}{\partial x_i} x_r^{\eta - \gamma_k} - \frac{\partial A_k}{\partial x_i} \overline{x}_r^{\eta - \gamma_k} \right] = 0.$$

Note that $\tilde{w}'(x_i) = -\frac{c\pi}{(r+\xi)x_i^2} - \sum_{k=1}^{3} A_k(\gamma_k + 1) x_i^{-\gamma_k - 2}$. Thus, using (52), $\frac{\partial a_k}{\partial x_i} = \frac{\partial A_k}{\partial x_i} = 0$, $k = 1, 2, 3$. Differentiating equations (45) – (48), (50), and (51) with respect to x_i, we get that $\frac{\partial b_k}{\partial x_i} = \frac{\partial B_k}{\partial x_i} = 0$, $k = 1, 2, 3$. Differentiating equation (49) with respect to x_i, we get that

$$\sum_{k=1}^{3} \frac{\partial B_k}{\partial x_i} \left(x_i^{-\gamma_k} - x_i^{*-\gamma_k} \right) - \sum_{k=1}^{3} \gamma_k B_k x_i^{-\gamma_k - 1} = 0.$$

Thus,

$$p'(y_i) = -\frac{c+\xi}{(r+\xi) y_{b0}} \sum_{k=1}^{3} \gamma_k B_k (y_i / y_{b0})^{-\gamma_k - 1} = 0.$$

By the analogous argument, we get that $p'(y_i^*) = p'(y_r^*) = 0$.

The argument is slightly different for $\overline{x}_r$. Differentiating equations (39) – (43), (45) – (46), and (51) with respect to $\overline{x}_r$:

$$\sum_{k=1}^{3} \frac{\partial a_k}{\partial \overline{x}_r} = 0,$$
$$\sum_{k=1}^{3} \frac{\partial a_k}{\partial \overline{x}_r} \frac{\eta}{\eta - \gamma_k} = 0,$$
$$\sum_{k=1}^{3} \frac{\partial a_k}{\partial \overline{x}_r} x_r^{-\gamma_k - 1} - \sum_{k=1}^{3} \frac{\partial A_k}{\partial \overline{x}_r} \overline{x}_r^{-\gamma_k - 1} = \tilde{w}'(\overline{x}_r),$$
$$\sum_{k=1}^{3} \frac{\partial A_k}{\partial \overline{x}_r} \left(x_r^{*-\gamma_k - 1} - \overline{x}_r^{-\gamma_k - 1} \right) = \tilde{w}'(\overline{x}_r),$$
$$\sum_{k=1}^{3} \frac{\partial A_k}{\partial \overline{x}_r} \left(x_i^{*-\gamma_k - 1} - x_i^{-\gamma_k - 1} \right) = 0,$$
$$\sum_{k=1}^{3} \frac{\partial b_k}{\partial \overline{x}_r} = -1,$$
$$1 + \sum_{k=1}^{3} \frac{\partial b_k}{\partial \overline{x}_r} \frac{\eta}{\eta - \gamma_k} = 0,$$
$$\sum_{k=1}^{3} \left(\frac{\partial a_k}{\partial \overline{x}_r} - \frac{c+\xi}{r+\xi} \frac{\partial b_k}{\partial \overline{x}_r} \right) \gamma_k = 0.$$

Using (52), $\frac{\partial a_k}{\partial \overline{x}_r} = \frac{\partial b_k}{\partial \overline{x}_r} = \frac{\partial A_k}{\partial \overline{x}_r} = 0$, $k = 1, 2, 3$. Differentiating (47) – (50) with respect to $\overline{x}_r$ and using $\frac{\partial b_k}{\partial \overline{x}_r} = 0$, $k = 1, 2, 3$, we get

$$\sum_{k=1}^{3} \frac{\partial B_k}{\partial \overline{x}_r} x_r^{*-\gamma_k} = \sum_{k=1}^{3} \frac{\partial b_k}{\partial \overline{x}_r} x_r^{-\gamma_k} = 0,$$
$$\sum_{k=1}^{3} \frac{\partial B_k}{\partial \overline{x}_r} \overline{x}_r^{-\gamma_k} + p'(\overline{y}_r) \bar{y}_b \frac{r+\xi}{c+\xi} = \sum_{k=1}^{3} \frac{\partial b_k}{\partial \overline{x}_r} x_r^{-\gamma_k} = 0,$$
$$\sum_{k=1}^{3} \frac{\partial B_k}{\partial \overline{x}_r} \left(x_i^{-\gamma_k} - x_i^{*-\gamma_k} \right) = 0,$$
$$\sum_{k=1}^{3} \frac{\partial B_k}{\partial \overline{x}_r} \frac{\gamma_k}{\eta - \gamma_k} \overline{x}_r^{\eta - \gamma_k} - \sum_{k=1}^{3} B_k \gamma_k \overline{x}_r^{\eta - \gamma_k - 1} = \sum_{k=1}^{3} \frac{\partial b_k}{\partial \overline{x}_r} \frac{\gamma_k}{\eta - \gamma_k} x_r^{\eta - \gamma_k} = 0,$$

Note that $p'(\overline{y}_r) = -\frac{c+\xi}{(r+\xi) y_{b0}} \sum_{k=1}^{3} B_k \gamma_k \overline{x}_r^{-\gamma_k - 1}$. Thus, inverting this system, we get that $\frac{\partial B_k}{\partial \overline{x}_r} = 0$, $k = 1, 2, 3$, and $p'(\overline{y}_r) = 0$. Therefore,

$$p'(y_i) = p'(y_i^*) = p'(y_r^*) = p'(\overline{y}_r) = 0.$$

By $w'(y_i) = p'(y_i) = 0$, function w would not change if we set reflecting issuance boundary at y_i. Thus, we can focus on reflecting issuance boundaries with $y_i = y_i^*$. Similarly, by $w'(\overline{y}_r) = p'(\overline{y}_r) = 0$, function w would not change if we set reflecting repurchase boundary at $\overline{y}_r$.

For reflecting repurchase/issuance boundaries at $\overline{y}_r$ and y_i, conditions (42) and (43) on A_ks are replaced by

$$\frac{c\pi}{r+\xi} + \sum_{k=1}^{3} A_k(\gamma_k+1)\overline{x}_r^{-\gamma_k} = 0, \tag{53}$$

$$\frac{c\pi}{r+\xi} + \sum_{k=1}^{3} A_k(\gamma_k+1)x_i^{-\gamma_k} = 0, \tag{54}$$

and conditions (48) and (49) on B_ks are replaced by

$$\sum_{k=1}^{3} \gamma_k B_k \overline{x}_r^{-\gamma_k} = 0, \tag{55}$$

$$\sum_{k=1}^{3} \gamma_k B_k x_i^{-\gamma_k} = 0. \tag{56}$$

By the super-contact principle, in the optimal $\mathbf{x}$,

$$\tilde{w}''(x_i) = \tilde{w}''(\overline{x}_r) = 0. \tag{57}$$

Note that

$$\begin{aligned}
\tilde{w}''(x_i) &= \frac{2c\pi}{(r+\xi)x_i^3} + \sum_{k=1}^{3}(\gamma_k+1)(\gamma_k+2)A_k x_i^{-\gamma_k-3} \\
&= \frac{2}{x_i^3}\left(\frac{c\pi}{r+\xi} + \sum_{k=1}^{3}(\gamma_k+1)A_k x_i^{-\gamma_k}\right) + \sum_{k=1}^{3}(\gamma_k+1)\gamma_k A_k x_i^{-\gamma_k-3} \\
&= \sum_{k=1}^{3}(\gamma_k+1)\gamma_k A_k x_i^{-\gamma_k-3},
\end{aligned}$$

where we used (54) in the last line. Differentiating equations (39) – (41), (44), and (53) – (54)

with respect to x_i:

$$\sum_{k=1}^{3} \frac{\partial a_k}{\partial x_i} = 0,$$
$$\sum_{k=1}^{3} \frac{\partial a_k}{\partial x_i} \frac{\eta}{\eta - \gamma_k} = 0,$$
$$\sum_{k=1}^{3} \frac{\partial a_k}{\partial x_i} x_r^{-\gamma_k - 1} - \sum_{k=1}^{3} \frac{\partial A_k}{\partial x_i} \overline{x}_r^{-\gamma_k - 1} = 0,$$
$$\sum_{k=1}^{3} \frac{\partial A_k}{\partial x_i} (\gamma_k + 1) \overline{x}_r^{-\gamma_k} = 0,$$
$$\sum_{k=1}^{3} \frac{\partial A_k}{\partial x_i} (\gamma_k + 1) x_i^{-\gamma_k} = \sum_{k=1}^{3} A_k (\gamma_k + 1) \gamma_k x_i^{-\gamma_k - 1} = \tilde{w}''(x_i) x_i^2,$$
$$\sum_{k=1}^{3} \frac{\eta(1+\gamma_k)}{\eta - \gamma_k} \left[\frac{\partial a_k}{\partial x_i} x_r^{\eta - \gamma_k} - \frac{\partial A_k}{\partial x_i} \overline{x}_r^{\eta - \gamma_k} \right] = 0.$$

Thus, by the super-contact principle, $\frac{\partial a_k}{\partial x_i} = \frac{\partial A_k}{\partial x_i} = 0$, $k = 1, 2, 3$. Differentiating equations (45) – (47), (55), (50), and (51) with respect to x_i, we get that $\frac{\partial b_k}{\partial x_i} = \frac{\partial B_k}{\partial x_i} = 0$, $k = 1, 2, 3$. Note that

$$p''(y_i) = \frac{c+\xi}{(r+\xi) y_{b0}^2} \sum_{k=1}^{3} \gamma_k (\gamma_k + 1) B_k x_i^{-\gamma_k - 2} = \frac{c+\xi}{(r+\xi) y_{b0}^2} \sum_{k=1}^{3} \gamma_k^2 B_k x_i^{-\gamma_k - 2},$$

where we used (56) to get the last equality. Differentiating equation (56), we get

$$\sum_{k=1}^{3} \gamma_k \frac{\partial B_k}{\partial x_i} x_i^{-\gamma_k} - \sum_{k=1}^{3} \gamma_k^2 B_k x_i^{-\gamma_k - 1} = 0.$$

Therefore, $p''(y_i) = 0$.

Next, differentiating equations (39) − (41), (44), and (53) − (54) with respect to $\overline{x}_r$:

$$\sum_{k=1}^{3} \frac{\partial a_k}{\partial \overline{x}_r} = 0,$$
$$\sum_{k=1}^{3} \frac{\partial a_k}{\partial \overline{x}_r} \frac{\eta}{\eta - \gamma_k} = 0,$$
$$\sum_{k=1}^{3} \frac{\partial a_k}{\partial \overline{x}_r} x_r^{-\gamma_k - 1} - \sum_{k=1}^{3} \frac{\partial A_k}{\partial \overline{x}_r} \overline{x}_r^{-\gamma_k - 1} = - \sum_{k=1}^{3} A_k(\gamma_k + 1)\overline{x}_r^{-\gamma_k - 2},$$
$$\sum_{k=1}^{3} \frac{\eta(1+\gamma_k)}{\eta - \gamma_k} \left[\frac{\partial a_k}{\partial \overline{x}_r} x_r^{\eta - \gamma_k} - \frac{\partial A_k}{\partial \overline{x}_r} \overline{x}_r^{\eta - \gamma_k} \right] = \frac{c\pi}{r+\xi} \eta \overline{x}_r^{\eta - 1} + \sum_{k=1}^{3} \eta(1+\gamma_k) A_k \overline{x}_r^{\eta - \gamma_k - 1},$$
$$\sum_{k=1}^{3} \frac{\partial A_k}{\partial \overline{x}_r} (\gamma_k + 1) x_i^{-\gamma_k} = 0,$$
$$\sum_{k=1}^{3} \frac{\partial A_k}{\partial \overline{x}_r} (\gamma_k + 1) \overline{x}_r^{-\gamma_k} - \sum_{k=1}^{3} A_k \gamma_k (\gamma_k + 1) \overline{x}_r^{-\gamma_k - 1} = 0.$$

Thus, using the super-contact principle and (53), we get that the right-hand side of these equations are zero. Hence, $\frac{\partial a_k}{\partial \overline{x}_r} = \frac{\partial A_k}{\partial \overline{x}_r} = 0$, $k = 1, 2, 3$. Differentiating equations (45) − (47), (56), (50), and (51) with respect to $\overline{x}_r$, we get that $\frac{\partial b_k}{\partial x_i} = \frac{\partial B_k}{\partial x_i} = 0$, $k = 1, 2, 3$. Note that

$$p''(\overline{y}_r) = \frac{c+\xi}{(r+\xi)y_{b0}^2} \sum_{k=1}^{3} \gamma_k(\gamma_k + 1) B_k \overline{x}_r^{-\gamma_k - 2} = \frac{c+\xi}{(r+\xi)y_{b0}^2} \sum_{k=1}^{3} \gamma_k^2 B_k \overline{x}_r^{-\gamma_k - 2},$$

where we used (55) to get the last equality. Differentiating equation (55), we get

$$\sum_{k=1}^{3} \gamma_k \frac{\partial B_k}{\partial \overline{x}_r} \overline{x}_r^{-\gamma_k} - \sum_{k=1}^{3} \gamma_k^2 B_k \overline{x}_r^{-\gamma_k - 1} = 0.$$

Therefore, $p''(\overline{y}_r) = 0$.

To sum up, both x_i and $\overline{x}_r$ satisfy the following equations in X:

$$w'(X) = 0 : \sum_{k=1}^{3} A_k(\gamma_k + 1)X^{-\gamma_k} = -\frac{c\pi}{r+\xi},$$
$$w''(X) = 0 : \sum_{k=1}^{3} \gamma_k(\gamma_k + 1)A_k X^{-\gamma_k} = 0,$$
$$p'(X\bar{y}_b) = 0 : \sum_{k=1}^{3} \gamma_k B_k X^{-\gamma_k} = 0,$$
$$p''(X\bar{y}_b) = 0 : \sum_{k=1}^{3} \gamma_k(\gamma_k + 1)B_k X^{-\gamma_k} = 0.$$

Note that $(X^{-\gamma_1}, X^{-\gamma_2}, X^{-\gamma_3})$ can be determined from either first three equations or first two equations and the last. Since both x_i and $\overline{x}_r$ satisfy these equations, it must be that both systems do not have full rank. Thus, $\frac{(\gamma_k+1)A_k}{B_k}$ is a constant across k, and $\frac{A_k}{B_k}$ is a constant across k. But this means that γ_k should be constant across k, which is a contradiction to the fact that γ_ks are distinct solutions to the characteristic equation. Thus, it is necessary that $\overline{y}_r = y_i$, which is the targeted ICR policy. □

Proof of Proposition 5. The proof is provided in the main text before Proposition 5, and it only remains to show that Proposition 3 implies that $\hat{\Sigma}$ is credible, whenever $e(y|\hat{\Sigma}) \geq 0$ for some $y \in (y_b, y_{bm}]$. By (36), $y_b \leq y_{bm}$ under $\hat{\Sigma}$, hence, we need to show that the condition $e(\hat{y}|\hat{\Sigma}) \geq e_m(\hat{y})$ is satisfied in this case.

To see this, note that $e(\hat{y}|\hat{\Sigma}) + \hat{p} > e_m(\hat{y}) + p_m(\hat{y})$. Indeed, by Proposition 2, under the optimal time-consistent policy Σ^*, the firm value is higher compared to the MPE:

$$W_m \equiv \max_{y \geq 0} \frac{p_m(y) + e_m(y)}{y} Y_0 < \max_{y \geq 0} \frac{p(y|\Sigma^*) + e(y|\Sigma^*)}{y} Y_0 = W(\Sigma^*).$$

Further, as we argued in the main text, $W(\Sigma^*) \leq W(\hat{\Sigma}) = \frac{p(\hat{y}|\hat{\Sigma}) + e(\hat{y}|\hat{\Sigma})}{\hat{y}} Y_0$. Hence, $p_m(\hat{y}) + e_m(\hat{y}) < p(\hat{y}|\hat{\Sigma}) + e(\hat{y}|\hat{\Sigma})$. Thus, either $e(\hat{y}|\hat{\Sigma}) > e_m(\hat{y})$ or $\hat{p} > p_m(\hat{y})$. We will next show that $\hat{p} > p_m(\hat{y})$ implies that $e(\hat{y}|\hat{\Sigma}) > e_m(\hat{y})$.

(DeMarzo and He forthcoming) show that convexity of e_m (by Proposition 1) implies that no global deviations are profitable in the MPE. In particular, the equity holders do not have incentives in state y_r to repurchase $(Y_t/\hat{y} - Y_t/y_r)$ of debt at price $p_m(\hat{y})$ to transition to state $\hat{y}$:

$$\frac{e_m(y_r)}{y_r} \geq \frac{e_m(\hat{y})}{\hat{y}} + p_m(\hat{y}) \left(\frac{1}{\hat{y}} - \frac{1}{y_r} \right) \tag{58}$$

Further, by the argument in Step 1 in the proof of Proposition 3, $e(y_r) \geq e_m(y_r)$. Combining

(32) and (58):

$$e(\hat{y}|\hat{\Sigma}) = e(y_r|\hat{\Sigma})\frac{\hat{y}}{y_r}+\hat{p}\left(\frac{\hat{y}}{y_r}-1\right) > e(y_r|\hat{\Sigma})\frac{\hat{y}}{y_r}+p_m(\hat{y})\left(\frac{\hat{y}}{y_r}-1\right) > e_m(y_r)\frac{\hat{y}}{y_r}+p_m(\hat{y})\left(\frac{\hat{y}}{y_r}-1\right) \geq e_m(\hat{y}),$$

where the first inequality uses $\hat{p} > p_m(\hat{y})$ and $\hat{y} > y_r$, and the second inequality uses $e(y_r|\hat{\Sigma}) \geq e_m(y_r)$. Thus, $e(\hat{y}|\hat{\Sigma}) > e_m(\hat{y})$, which is the desired conclusion. □

A.3 Derivation of Equation (31)

To determine $\mathbb{E}\left[\hat{p}d\Gamma_t\right]$ in equation (30), we consider separately the cases with and without Poisson jumps. If no jumps occur in the interval $[t, t+dt]$ (i.e., $dN_t = 0$), then the equity holders issue/repurchase debt to compensate for all Brownian shocks and reissue maturing debt so that

$$dy_t = dY_t/Y_t - dF_t/F_{t-} = dY_t/Y_t - (d\Gamma_t - \xi F_{t-})/F_{t-} = 0.$$

Hence, in this case

$$\mathbb{E}\left[d\Gamma_t|dN_t = 0\right] = \mathbb{E}\left[F_{t-}dY_t/Y_t|dN_t = 0\right] + \xi F_{t-}dt = \left(\hat{\mu}+\xi\right)F_{t-}dt,$$

and

$$\mathbb{E}\left[dF_t|dN_t = 0\right] = \mathbb{E}\left[F_{t-}dY_t/Y_{t-}|dN_t = 0\right] = \hat{\mu}F_{t-}dt.$$

The continuation value in this case is equal to

$$\begin{aligned}\mathbb{E}\left[E(\hat{y}F_t, F_t)|dN_t = 0\right] &= \mathbb{E}\left[e(\hat{y})(F_{t-}+dF_t)|dN_t = 0\right] \\ &= E(\hat{y}F_{t-}, F_{t-}) + \mathbb{E}\left[e(\hat{y})dF_t|dN_t = 0\right] \\ &= E(\hat{y}F_{t-}, F_{t-}) + \hat{e}\hat{\mu}F_{t-}dt.\end{aligned}$$

If there is a Poisson jump $dY_t/Y_{t-} = e^{-\tilde{S}_t} - 1$ so that $\hat{y}e^{-\tilde{S}_t} > y_r$, then the equity holders compensate this jump so that the state returns to $\hat{y}$. In order to do so, they repurchase $F_{t-}(1 - e^{-\tilde{s}})$ units of debt (i.e., $dF_t = F_{t-}(e^{-\tilde{s}} - 1) < 0$). Thus, if $\tilde{S}_t = \tilde{s} < \ln(\hat{y}/y_r)$, then

$$\mathbb{E}\left[d\Gamma_t\middle|dN_t = 1, \tilde{S}_t = \tilde{s}\right] = F_{t-}\left(e^{-\tilde{s}}-1\right),$$

and the continuation value is

$$\begin{aligned}\mathbb{E}\left[E(\hat{y}F_t, F_t)\middle|dN_t = 1, \tilde{S}_t = \tilde{s}\right] &= \mathbb{E}\left[e(\hat{y})\left(F_{t-}+dF_t\right)\middle|dN_t = 0, \tilde{S}_t = \tilde{s}\right] \\ &= E(\hat{y}F_{t-}, F_{t-})e^{-\tilde{s}}.\end{aligned}$$

Therefore, we can rewrite equation (30) as

$$
\begin{aligned}
E(\hat{y}F_{t-}, F_{t-}) = {} & (1-\pi)(\hat{y}F_{t-} - cF_{t-})dt - \xi F_{t-}dt + \hat{p}\left(\hat{\mu}+\xi\right)F_{t-}dt + \lambda dt \int_0^{\ln(\hat{y}/y_r)} \hat{p}F_{t-}(e^{-\tilde{s}}-1)\eta e^{-\eta\tilde{s}}d\tilde{s} \\
& + (1 - rdt - \lambda dt)\left(E(\hat{y}F_{t-}, F_{t-}) + \hat{e}\hat{\mu}F_{t-}dt\right) \\
& + (1 - rdt)\left\{\lambda dt \int_0^{\ln(\hat{y}/y_r)} E(\hat{y}F_{t-}, F_{t-})e^{-\tilde{s}}\eta e^{-\eta\tilde{s}}d\tilde{s} + \lambda dt \int_{\ln(\hat{y}/y_r)}^{\ln(\hat{y}/y_b)} E(e^{-\tilde{s}}\hat{y}F_{t-}, F_{t-})\eta e^{-\eta\tilde{s}}d\tilde{s}\right\}.
\end{aligned}
$$

Normalizing by F_{t-}, we get

$$
\begin{aligned}
\hat{e} = {} & (1-\pi)(\hat{y}-c)dt - \xi dt + \hat{p}\left(\hat{\mu}+\xi\right)dt + \lambda dt \int_0^{\ln(\hat{y}/y_r)} \hat{p}(e^{-\tilde{s}}-1)\eta e^{-\eta\tilde{s}}d\tilde{s} \\
& + (1 - rdt - \lambda dt)\left(\hat{e} + \hat{e}\hat{\mu}dt\right) \\
& + (1 - rdt)\left\{\lambda dt \int_0^{\ln(\hat{y}/y_r)} \hat{e}e^{-\tilde{s}}\eta e^{-\eta\tilde{s}}d\tilde{s} + \lambda dt \int_{\ln(\hat{y}/y_r)}^{\ln(\hat{y}/y_b)} e(e^{-\tilde{s}}\hat{y})\eta e^{-\eta\tilde{s}}d\tilde{s}\right\}.
\end{aligned}
$$

Hence, we get the HJB equation (31).

Online Appendix (Not for Publication)
"Optimal Time-Consistent Debt Policies"

A Auxiliary Derivations

This appendix presents auxiliary derivations for the results provided in the main text.

Derivation of Condition (12) Condition (12) can be written more explicitly by computing the expectation:

$$1 + b_1 \frac{\eta}{\eta - \gamma_1} + b_2 \frac{\eta}{\eta - \gamma_2} + b_3 \frac{\eta}{\eta - \gamma_3} = 0. \tag{59}$$

By $p(e^{-\tilde{s}}y) = 0$ for all $\tilde{s} > \ln(y/y_b)$, the HJB equation (6) for $y \in (y_b, y_r)$ can be written as

$$(r + \lambda + \xi)p(y) = c + \xi + (\hat{\mu} + \xi)yp'(y) + \tfrac{1}{2}\sigma^2 y^2 p''(y) + \lambda \int_0^{\ln(y/y_b)} \left\{ p\left(e^{-\tilde{s}}y\right) \eta e^{-\eta\tilde{s}} \right\} d\tilde{s}.$$

Let us compute the integral using the conjectured p in (7):

$$\begin{aligned}
&\int_0^{\ln(y/y_b)} \left\{ p\left(e^{-\tilde{s}}y\right) \eta e^{-\eta\tilde{s}} \right\} d\tilde{s} \\
=&\int_0^{\ln(y/y_b)} \left\{ \frac{c+\xi}{r+\xi} \left(1 + \sum_{k=1}^{3} b_k e^{\gamma_k \tilde{s}} (y/y_b)^{-\gamma_k} \right) \eta e^{-\eta\tilde{s}} \right\} d\tilde{s} \\
=&\frac{c+\xi}{r+\xi}\left(1 - (y/y_b)^{-\eta}\right) + \frac{c+\xi}{r+\xi} \sum_{k=1}^{3} b_k (y/y_b)^{-\gamma_k} \int_0^{\ln(y/y_b)} \eta e^{-(\eta-\gamma_k)\tilde{s}} d\tilde{s} \\
=&\frac{c+\xi}{r+\xi}\left(1 - (y/y_b)^{-\eta}\right) + \frac{c+\xi}{r+\xi} \sum_{k=1}^{3} b_k (y/y_b)^{-\gamma_k} \frac{\eta}{\eta-\gamma_k} \left(1 - (y/y_b)^{-(\eta-\gamma_k)}\right) \\
=&\frac{c+\xi}{r+\xi} + \frac{c+\xi}{r+\xi} \sum_{k=1}^{3} b_k \frac{\eta}{\eta-\gamma_k} (y/y_b)^{-\gamma_k} - (y/y_b)^{-\eta} \frac{c+\xi}{r+\xi} \left(1 + \sum_{k=1}^{3} b_k \frac{\eta}{\eta-\gamma_k} \right).
\end{aligned}$$

Thus, we can write the HJB equation (6) as

$$
\begin{aligned}
(r+\lambda+\xi)\frac{c+\xi}{r+\xi}\left(1+\sum_{k=1}^{3} b_k (y/y_b)^{-\gamma_k}\right) = {} & c+\xi-(\hat{\mu}+\xi)\frac{c+\xi}{r+\xi}\sum_{k=1}^{3}\gamma_k b_k (y/y_b)^{-\gamma_k} \\
& + \tfrac{1}{2}\sigma^2 \frac{c+\xi}{r+\xi}\sum_{k=1}^{3}\gamma_k(1+\gamma_k) b_k (y/y_b)^{-\gamma_k} \\
& + \lambda\frac{c+\xi}{r+\xi} + \lambda\frac{c+\xi}{r+\xi}\sum_{k=1}^{3} b_k \frac{\eta}{\eta-\gamma_k}(y/y_b)^{-\gamma_k} \\
& - \lambda (y/y_b)^{-\eta}\frac{c+\xi}{r+\xi}\left(1+\sum_{k=1}^{3} b_k\frac{\eta}{\eta-\gamma_k}\right).
\end{aligned}
$$

Cancelling the terms at the constant,

$$
\lambda(y/y_b)^{-\eta}\left(1+\sum_{k=1}^{3} b_k\frac{\eta}{\eta-\gamma_k}\right) = \sum_{k=1}^{3} b_k (y/y_b)^{-\gamma_k}\left\{-(\hat{\mu}+\xi)\gamma_k + \frac{\sigma^2}{2}\gamma_k(1+\gamma_k) + \lambda\frac{\eta}{\eta-\gamma_k} - (r+\lambda+\xi)\right\}.
$$

Thus, we get that γ_ks must solve the characteristic equation (8). Further, matching terms at $(y/y_b)^{-\eta}$, we get that coefficients b_ks satisfy condition (12), which is the desired result.

Condition (12) can be interpreted as follows. It requires that even if conjecture $p(y) = \frac{c+\xi}{r+\xi}\left(1+\sum_{k=1}^{3} b_k (y/y_b)^{-\gamma_k}\right)$ was applied to ys below y_b, it would not change debt pricing in the range $[y_b, y_r]$. Indeed, this conjecture only describes debt pricing on $[y_b, y_r]$, and $p(y) = 0$ for $y < y_b$. Because of the memoryless property of the exponential distribution of downward jumps, in order to derive the debt price in the region $[y_b, y_r]$, it is sufficient to ensure that $\mathbb{E}[p(Sy_b)] = 0$ rather than that $p(y) = 0$ for all $y < y_b$. Condition (12) is equivalent to the requirement that $\mathbb{E}[p(Sy_b)] = 0$ even if conjecture $p(y) = \frac{c+\xi}{r+\xi}\left(1+\sum_{k=1}^{3} b_k (y/y_b)^{-\gamma_k}\right)$ is extended to ys below y_b.

Derivation of Condition (13) Condition (13) can be written more explicitly by computing the expectation:

$$
\sum_{k=1}^{3} b_k \frac{\gamma_k}{\eta-\gamma_k}\left(\frac{y_r}{y_b}\right)^{\eta-\gamma_k} = \sum_{k=1}^{3} B_k \frac{\gamma_k}{\eta-\gamma_k}\left(\frac{\overline{y}_r}{y_b}\right)^{\eta-\gamma_k}. \tag{60}
$$

Let us derive it. For $y \in (y_b, y_r)$, the HJB equation (6) becomes

$$
(r+\lambda+\xi)p(y) = c+\xi+(\hat{\mu}+\xi)yp'(y) + \tfrac{1}{2}\sigma^2 y^2 p''(y) + \lambda\int_0^{\ln(y/y_b)}\left\{p\left(e^{-\tilde{s}}y\right)\eta e^{-\eta\tilde{s}}\right\}d\tilde{s},
$$

because $p(e^{-\tilde{s}}y) = 0$ for all $\tilde{s} > \ln(y/y_b)$. Using $p(y) = p_r^*$ for $y \in [y_r, \overline{y}_r]$, we get that $p(y)$ in the region $y \in (\overline{y}_r, y_i)$ satisfies

$$(r+\lambda+\xi)p(y) = c+\xi+(\hat{\mu}+\xi)yp'(y)+\tfrac{1}{2}\sigma^2y^2p''(y)+\lambda\int_{\ln(y/y_r)}^{\ln(y/y_b)}\left\{p\left(e^{-\tilde{s}}y\right)\eta e^{-\eta\tilde{s}}\right\}d\tilde{s}$$
$$+\lambda\int_{\ln(y/\overline{y}_r)}^{\ln(y/y_r)}\left\{p_r^*\eta e^{-\eta\tilde{s}}\right\}d\tilde{s}+\lambda\int_0^{\ln(y/\overline{y})}\left\{p\left(e^{-\tilde{s}}y\right)\eta e^{-\eta\tilde{s}}\right\}d\tilde{s}. \quad (61)$$

Using the conjecture (7), we can compute each integrals more explicitly as follow. To simplify the expressions, we use notation $x = y/y_b$, $x_r = y_r/y_b$, $\overline{x} = \overline{y}/y_b$, $\tilde{x} = \tilde{y}/y_b$, $x^* = y^*/y_b$, $\hat{x} = \hat{y}/y_b$. The first integral in (61) equals

$$\begin{aligned}
&\int_{\ln(y/y_r)}^{\ln(y/y_b)}\frac{c+\xi}{r+\xi}\left(1+\sum_{k=1}^{3}b_ke^{\gamma_k\tilde{s}}(y/y_b)^{-\gamma_k}\right)\eta e^{-\eta\tilde{s}}d\tilde{s}\\
=&\frac{c+\xi}{r+\xi}\left(\left(\frac{y}{y_r}\right)^{-\eta}-\left(\frac{y}{y_b}\right)^{-\eta}\right)+\frac{c+\xi}{r+\xi}\sum_{k=1}^{3}b_k(y/y_b)^{-\gamma_k}\frac{\eta}{\eta-\gamma_k}\left(\left(\frac{y}{y_r}\right)^{-(\eta-\gamma_k)}-\left(\frac{y}{y_b}\right)^{-(\eta-\gamma_k)}\right)\\
=&\frac{c+\xi}{r+\xi}x^{-\eta}\left(x_r^{\eta}-1\right)+\frac{c+\xi}{r+\xi}\sum_{k=1}^{3}b_kx^{-\eta}\frac{\eta}{\eta-\gamma_k}\left(x_r^{\eta-\gamma_k}-1\right).
\end{aligned}$$

Analogously, the last integral equals

$$\begin{aligned}
&\int_0^{\ln(y/\overline{y}_r)}\frac{c+\xi}{r+\xi}\left(1+\sum_{k=1}^{3}B_ke^{\gamma_k\tilde{s}}(y/y_b)^{-\gamma_k}\right)\eta e^{-\eta\tilde{s}}d\tilde{s}\\
=&\frac{c+\xi}{r+\xi}\left(1-\left(\frac{y}{\overline{y}_r}\right)^{-\eta}\right)+\frac{c+\xi}{r+\xi}\sum_{k=1}^{3}B_k(y/y_b)^{-\gamma_k}\frac{\eta}{\eta-\gamma_k}\left(1-\left(\frac{y}{\overline{y}_r}\right)^{-(\eta-\gamma_k)}\right)\\
=&\frac{c+\xi}{r+\xi}\left(1-x^{-\eta}\overline{x}_r^{\eta}\right)+\frac{c+\xi}{r+\xi}\sum_{k=1}^{3}B_k\frac{\eta}{\eta-\gamma_k}\left(x^{-\gamma_k}-x^{-\eta}\overline{x}_r^{\eta-\gamma_k}\right).
\end{aligned}$$

The second integral equals

$$\begin{aligned}
\int_{\ln(y/\overline{y}_r)}^{\ln(y/y_r)}p_r^*\eta e^{-\eta\tilde{s}}d\tilde{s} =&p_r^*\left(\left(\frac{y}{\overline{y}_r}\right)^{-\eta}-\left(\frac{y}{y_r}\right)^{-\eta}\right)\\
=&p_r^*x^{-\eta}\left(\overline{x}_r^{\eta}-x_r^{\eta}\right).
\end{aligned}$$

Therefore, the HJB equation becomes

$$(r+\lambda+\xi)\frac{c+\xi}{r+\xi}\left(1+\sum_{k=1}^{3}B_k x^{-\gamma_k}\right)=c+\xi-(\hat{\mu}+\xi)\frac{c+\xi}{r+\xi}\sum_{k=1}^{3}\gamma_k B_k x^{-\gamma_k}+\frac{\sigma^2}{2}\frac{c+\xi}{r+\xi}\sum_{k=1}^{3}\gamma_k(\gamma_k+1)B_k x^{-\gamma_k}$$
$$+\lambda\frac{c+\xi}{r+\xi}x^{-\eta}\left(x_r^{\eta}-1\right)+\lambda\frac{c+\xi}{r+\xi}\sum_{k=1}^{3}b_k x^{-\eta}\frac{\eta}{\eta-\gamma_k}\left(x_r^{\eta-\gamma_k}-1\right)+\lambda p_r^* x^{-\eta}\left(\overline{x}_r^{\eta}-x_r^{\eta}\right)$$
$$+\lambda\frac{c+\xi}{r+\xi}\left(1-x^{-\eta}\overline{x}_r^{\eta}\right)+\lambda\frac{c+\xi}{r+\xi}\sum_{k=1}^{3}B_k\frac{\eta}{\eta-\gamma_k}\left(x^{-\gamma_k}-x^{-\eta}\overline{x}_r^{\eta-\gamma_k}\right).$$

Given that γ_ks solve the characteristic equation (8),

$$x_r^{\eta}-1+\sum_{k=1}^{3}b_k\frac{\eta}{\eta-\gamma_k}\left(x_r^{\eta-\gamma_k}-1\right)+\frac{r+\xi}{c+\xi}p_r^*\left(\overline{x}_r^{\eta}-x_r^{\eta}\right)-\overline{x}_r^{\eta}-\sum_{k=1}^{3}B_k\frac{\eta}{\eta-\gamma_k}\overline{x}_r^{\eta-\gamma_k}=0.$$

Using (12),

$$x_r^{\eta}+\sum_{k=1}^{3}b_k\frac{\eta}{\eta-\gamma_k}x_r^{\eta-\gamma_k}+\frac{r+\xi}{c+\xi}p_r^*\left(\overline{x}_r^{\eta}-x_r^{\eta}\right)-\overline{x}_r^{\eta}-\sum_{k=1}^{3}B_k\frac{\eta}{\eta-\gamma_k}\overline{x}_r^{\eta-\gamma_k}=0,$$

or equivalently,

$$\sum_{k=1}^{3}b_k\frac{\eta}{\eta-\gamma_k}x_r^{\eta-\gamma_k}+\left(\frac{r+\xi}{c+\xi}p_r^*-1\right)\left(\overline{x}_r^{\eta}-x_r^{\eta}\right)-\sum_{k=1}^{3}B_k\frac{\eta}{\eta-\gamma_k}\overline{x}_r^{\eta-\gamma_k}=0.$$

Using $p_r^*=\frac{c+\xi}{r+\xi}\left(1+\sum_{k=1}^{3}b_k x_r^{-\gamma_k}\right)=\frac{c+\xi}{r+\xi}\left(1+\sum_{k=1}^{3}B_k\overline{x}_r^{-\gamma_k}\right)$,

$$\sum_{k=1}^{3}b_k\frac{\eta}{\eta-\gamma_k}x_r^{\eta-\gamma_k}+\sum_{k=1}^{3}B_k\overline{x}_r^{\eta-\gamma_k}-\sum_{k=1}^{3}b_k x_r^{\eta-\gamma_k}-\sum_{k=1}^{3}B_k\frac{\eta}{\eta-\gamma_k}\overline{x}_r^{\eta-\gamma_k}=0,$$

or

$$\sum_{k=1}^{3}b_k\frac{\gamma_k}{\eta-\gamma_k}x_r^{\eta-\gamma_k}=\sum_{k=1}^{3}B_k\frac{\gamma_k}{\eta-\gamma_k}\overline{x}_r^{\eta-\gamma_k},$$

which is the desired condition (13).

The interpretation of condition (13) is as follows. It requires that even if conjecture $p(y)=\frac{c+\xi}{r+\xi}\left(1+\sum_{k=1}^{3}B_k(y/y_b)^{-\gamma_k}\right)$ was applied to ys below $\overline{y}_r$, it would not change debt pricing in the range $[\overline{y}_r, y_i]$. Indeed, because of the memoryless property of the exponential distribution of downward jumps, the debt price in the region $[\overline{y}_r, y_i]$ depends on the price of debt below $\overline{y}_r$

only through $\mathbb{E}[p(S\overline{y}_r)]$. Condition (13) is equivalent to the requirement that

$$\mathbb{E}[p(S\overline{y}_r)] = \left(1 - \left(\frac{\overline{y}_r}{y_r}\right)^{-\eta}\right) p_r^* + \left(\frac{\overline{y}_r}{y_r}\right)^{-\eta} \mathbb{E}[p(Sy_r)]$$

still holds even if we use conjecture $p(y) = \frac{c+\xi}{r+\xi}\left(1 + \sum_{k=1}^{3} B_k (y/y_b)^{-\gamma_k}\right)$ in the left-hand side to compute $\mathbb{E}[p(S\overline{y}_r)]$.

Derivation of Condition (21) Condition (21) can be written more explicitly by computing the expectation:

$$\frac{\phi\eta}{\eta+1} y_b - \rho + \sum_{k=1}^{3} c_k \frac{\eta}{\eta - \gamma_k} = 0.$$

Let us derive it. In region $y \in [y_b, y_r]$, the evolution of e is given by

$$(r+\lambda+\xi)e(y) = (1-\pi)(y-c) - \xi + (\hat{\mu}+\xi)ye'(y) + \tfrac{1}{2}\sigma^2 y^2 e''(y) + \lambda \int_0^{\ln(y/y_b)} e\left(e^{-\tilde{s}}y\right) \eta e^{-\eta\tilde{s}} d\tilde{s}.$$

Using the conjectured form of e, the integral becomes

$$\begin{aligned}
&\int_0^{\ln(y/y_b)} \left(\phi e^{-\tilde{s}} y - \rho + \sum_{k=1}^{3} c_k \left(e^{-\tilde{s}} y/y_b\right)^{-\gamma_k}\right) \eta e^{-\eta\tilde{s}} d\tilde{s} \\
=&\frac{\phi y \eta}{\eta+1}\left(1 - \left(\frac{y}{y_b}\right)^{-(\eta+1)}\right) - \rho\left(1 - \left(\frac{y}{y_b}\right)^{-\eta}\right) + \sum_{k=1}^{3} c_k \frac{(y/y_b)^{-\gamma_k}\eta}{\eta-\gamma_k}\left(1 - \left(\frac{y}{y_b}\right)^{-(\eta-\gamma_k)}\right) \\
=&\frac{\phi\eta x y_b}{\eta+1}\left(1 - x^{-(\eta+1)}\right) - \rho\left(1 - x^{-\eta}\right) + \sum_{k=1}^{3} c_k \frac{x^{-\gamma_k}\eta}{\eta-\gamma_k}\left(1 - x^{-(\eta-\gamma_k)}\right).
\end{aligned}$$

Thus, we can re-write the HJB equation as

$$\begin{aligned}
(r+\lambda+\xi)\left(\phi y - \rho + \sum_{k=1}^{3} c_k x^{-\gamma_k}\right) &= (1-\pi)(y-c) - \xi + (\hat{\mu}+\xi)\left(\phi y - \sum_{k=1}^{3} \gamma_k c_k x^{-\gamma_k}\right) \\
&+ \tfrac{1}{2}\sigma^2 \sum_{k=1}^{3} \gamma_k(\gamma_k+1) c_k x^{-\gamma_k} \\
&+ \frac{\lambda\phi\eta x y_b}{\eta+1}\left(1 - x^{-(\eta+1)}\right) - \lambda\rho\left(1 - x^{-\eta}\right) + \sum_{k=1}^{3} c_k \frac{x^{-\gamma_k}\lambda\eta}{\eta-\gamma_k}\left(1 - x^{-(\eta-\gamma_k)}\right).
\end{aligned}$$

Since γ_ks satisfy the characteristic equation (8), terms at $c_k x^{-\gamma_k}$ disappear. Further, given the definition of ϕ and ρ, the terms at constant and at y disappear as well. Thus, we get that equation (21) must hold.

The interpretation of equation (21) is as follows. It requires that even if the conjecture $e(y) = \phi y - \rho + \sum_{k=1}^{3} c_k (y/y_b)^{-\gamma_k}$ were applied beyond the range $[y_b, y_r]$, this would not change the equity value on $[y_b, y_r]$. By the memoryless property of the exponential distribution of downward jumps, this would be the case if $\mathbb{E}\left[e(Sy_b)\right]$ did not change. Condition (22) requires that $\mathbb{E}\left[e(S\overline{y}_r)\right] = 0$ holds even if we use conjecture $e(y) = \phi y - \rho + \sum_{k=1}^{3} c_k (y/y_b)^{-\gamma_k}$ to compute $\mathbb{E}[e(S\overline{y}_r)]$ (rather than $e(y) = 0$ for $y < y_b$).

Derivation of Condition (22) Condition (22) can be written more explicitly by computing the expectation:

$$\sum_{k=1}^{3} \frac{\eta(1+\gamma_k)}{\eta-\gamma_k} \left[\left(c_k + b_k \frac{c+\xi}{r+\xi} \right) \left(\frac{y_r}{y_b} \right)^{\eta-\gamma_k} - \left(C_k + B_k \frac{c+\xi}{r+\xi} \right) \left(\frac{\overline{y}_r}{y_b} \right)^{\eta-\gamma_k} \right] = \frac{c\pi}{r+\xi} \left[\left(\frac{\overline{y}_r}{y_b} \right)_r^{\eta} - \left(\frac{y_r}{y_b} \right)^{\eta} \right].$$

Let us derive it. In region $y \in [\overline{y}_r, y_i]$, the evolution of e is given by

$$\begin{aligned}(r+\lambda+\xi)e(y) = (1-\pi)(y-c) - \xi + (\hat{\mu}+\xi)ye'(y) + \tfrac{1}{2}\sigma^2 y^2 e''(y) + \lambda \int_{\ln(y/y_r)}^{\ln(y/y_b)} e\left(e^{-\tilde{s}}y\right) \eta e^{-\eta\tilde{s}} d\tilde{s} \\ + \lambda \int_{\ln(y/\overline{y}_r)}^{\ln(y/y_r)} e\left(e^{-\tilde{s}}y\right) \eta e^{-\eta\tilde{s}} d\tilde{s} + \lambda \int_{0}^{\ln(y/\overline{y}_r)} e\left(e^{-\tilde{s}}y\right) \eta e^{-\eta\tilde{s}} d\tilde{s}.\end{aligned}$$

We use the conjectures in (20) to compute each integral. The first integral becomes

$$\begin{aligned}
&\int_{\ln(y/y_r)}^{\ln(y/y_b)} \left(\phi e^{-\tilde{s}} y - \rho + \sum_{k=1}^{3} c_k \left(e^{-\tilde{s}} y/y_b \right)^{-\gamma_k} \right) \eta e^{-\eta\tilde{s}} d\tilde{s} \\
=& \frac{\phi y \eta}{\eta+1} \left(\left(\frac{y}{y_r} \right)^{-(\eta+1)} - \left(\frac{y}{y_b} \right)^{-(\eta+1)} \right) - \rho \left(\left(\frac{y}{y_r} \right)^{-\eta} - \left(\frac{y}{y_b} \right)^{-\eta} \right) \\
&+ \sum_{k=1}^{3} c_k \frac{(y/y_b)^{-\gamma_k} \eta}{\eta-\gamma_k} \left(\left(\frac{y}{y_r} \right)^{-(\eta-\gamma_k)} - \left(\frac{y}{y_b} \right)^{-(\eta-\gamma_k)} \right) \\
=& \frac{\phi x^{-\eta} y_b \eta}{\eta+1} \left(x_r^{\eta+1} - 1 \right) - \rho x^{-\eta} \left(x_r^{\eta} - 1 \right) + \sum_{k=1}^{3} c_k \frac{x^{-\eta}\eta}{\eta-\gamma_k} \left(x_r^{\eta-\gamma_k} - 1 \right) \\
=& x^{-\eta} \left[\frac{\phi y_b \eta}{\eta+1} \left(x_r^{\eta+1} - 1 \right) - \rho \left(x_r^{\eta} - 1 \right) + \sum_{k=1}^{3} c_k \frac{\eta}{\eta-\gamma_k} \left(x_r^{\eta-\gamma_k} - 1 \right) \right].
\end{aligned}$$

Denote $e_r^* \equiv e(y_r^*)$. The second integral becomes

$$\begin{aligned}
&\int_{\ln(y/\overline{y}_r)}^{\ln(y/y_r)} \left(\frac{p_r^*+e_r^*}{y_r^*} y e^{-\tilde{s}} - p_r^*\right) \eta e^{-\eta\tilde{s}} d\tilde{s} \\
=&\frac{p_r^*+e_r^*}{y_r^*} y \int_{\ln(y/\overline{y}_r)}^{\ln(y/y_r)} \eta e^{-(\eta+1)\tilde{s}} d\tilde{s} - p_r^* \int_{\ln(y/\overline{y}_r)}^{\ln(y/y_r)} \eta e^{-\eta\tilde{s}} d\tilde{s} \\
=&\frac{p_r^*+e_r^*}{y_r^*} y \frac{\eta}{\eta+1} \left(\left(\frac{y}{\overline{y}_r}\right)^{-(\eta+1)} - \left(\frac{y}{y_r}\right)^{-(\eta+1)}\right) - p_r^* \left(\left(\frac{y}{\overline{y}_r}\right)^{-\eta} - \left(\frac{y}{y_r}\right)^{-\eta}\right) \\
=&y^{-\eta}\left[\frac{p_r^*+e_r^*}{y_r^*}\frac{\eta}{\eta+1}\left(\overline{y}_r^{\eta+1} - y_r^{\eta+1}\right) - p_r^*\left(\overline{y}_r^{\eta} - y_r^{\eta}\right)\right] \\
=&x^{-\eta}\left[\frac{p_r^*+e_r^*}{x_r^*}\frac{\eta}{\eta+1}\left(\overline{x}_r^{\eta+1} - x_r^{\eta+1}\right) - p_r^*\left(\overline{x}_r^{\eta} - x_r^{\eta}\right)\right].
\end{aligned}$$

The third integral becomes

$$\begin{aligned}
&\int_0^{\ln(y/\overline{y}_r)} \left(\phi e^{-\tilde{s}} y - \rho + \sum_{k=1}^3 C_k \left(e^{-\tilde{s}} y/y_b\right)^{-\gamma_k}\right) \eta e^{-\eta\tilde{s}} d\tilde{s} \\
=&\frac{\phi y \eta}{\eta+1}\left(1 - \left(\frac{y}{\overline{y}_r}\right)^{-(\eta+1)}\right) - \rho\left(1 - \left(\frac{y}{\overline{y}_r}\right)^{-\eta}\right) + \sum_{k=1}^3 C_k \frac{(y/y_b)^{-\gamma_k}\eta}{\eta-\gamma_k}\left(1 - \left(\frac{y}{\overline{y}_r}\right)^{-(\eta-\gamma_k)}\right) \\
=&\frac{\phi y_b x \eta}{\eta+1} - \rho + \sum_{k=1}^3 C_k \frac{\eta}{\eta-\gamma_k} x^{-\gamma_k} + x^{-\eta}\left[\rho \overline{x}_r^{\eta} - \frac{\phi\eta y_b}{\eta+1}\overline{x}_r^{\eta+1} - \sum_{k=1}^3 C_k \frac{\eta}{\eta-\gamma_k} x^{-\eta}\overline{x}_r^{\eta-\gamma_k}\right]
\end{aligned}$$

Thus,

$$\begin{aligned}
(r+\lambda+\xi)\left(\phi y_b x - \rho + \sum_{k=1}^3 C_k x^{-\gamma_k}\right) =& (1-\pi)(y_b x - c) - \xi + (\hat{\mu}+\xi)\left(\phi y_b x - \sum_{k=1}^3 \gamma_k C_k x^{-\gamma_k}\right) \\
&+ \tfrac{1}{2}\sigma^2 \sum_{k=1}^3 \gamma_k(\gamma_k+1) C_k x^{-\gamma_k} \\
&+ \lambda x^{-\eta}\left[\frac{\phi y_b \eta}{\eta+1}\left(x_r^{\eta+1} - 1\right) - \rho\left(x_r^{\eta} - 1\right) + \sum_{k=1}^3 \frac{c_k\eta}{\eta-\gamma_k}\left(x_r^{\eta-\gamma_k} - 1\right)\right] \\
&+ \lambda x^{-\eta}\left[\frac{p_r^*+e_r^*}{x_r^*}\frac{\eta}{\eta+1}\left(\overline{x}_r^{\eta+1} - x_r^{\eta+1}\right) - p_r^*\left(\overline{x}^{\eta} - x_r^{\eta}\right)\right] \\
&+ \lambda\left[\frac{\phi y_b x \eta}{\eta+1} - \rho + \sum_{k=1}^3 \frac{C_k\eta}{\eta-\gamma_k} x^{-\gamma_k}\right] \\
&+ \lambda x^{-\eta}\left[\rho\overline{x}_r^{\eta} - \frac{\phi\eta y_b}{\eta+1}\overline{x}_r^{\eta+1} - \sum_{k=1}^3 \frac{C_k\eta}{\eta-\gamma_k} x^{-\eta}\overline{x}_r^{\eta-\gamma_k}\right].
\end{aligned}$$

Matching the coefficients at $\lambda x^{-\eta}$,

$$\frac{\phi y_b \eta}{\eta+1}\left(x_r^{\eta+1}-1\right)-\rho\left(x_r^{\eta}-1\right)+\sum_{k=1}^{3}\frac{c_k\eta}{\eta-\gamma_k}\left(x_r^{\eta-\gamma_k}-1\right)$$
$$+\frac{p_r^*+e_r^*}{x_r^*}\frac{\eta}{\eta+1}\left(\overline{x}_r^{\eta+1}-x_r^{\eta+1}\right)-p_r^*\left(\overline{x}_r^{\eta}-x_r^{\eta}\right)$$
$$+\rho\overline{x}_r^{\eta}-\frac{\phi\eta y_b}{\eta+1}\overline{x}_r^{\eta+1}-\sum_{k=1}^{3}\frac{C_k\eta}{\eta-\gamma_k}\overline{x}_r^{\eta-\gamma_k}=0.$$

Given equation (21),

$$\frac{\phi y_b \eta}{\eta+1}x_r^{\eta+1}-\rho x_r^{\eta}+\sum_{k=1}^{3}\frac{c_k\eta}{\eta-\gamma_k}x_r^{\eta-\gamma_k}$$
$$+\frac{p_r^*+e_r^*}{x_r^*}\frac{\eta}{\eta+1}\left(\overline{x}_r^{\eta+1}-x_r^{\eta+1}\right)-p_r^*\left(\overline{x}_r^{\eta}-x_r^{\eta}\right)$$
$$+\rho\overline{x}_r^{\eta}-\frac{\phi\eta y_b}{\eta+1}\overline{x}_r^{\eta+1}-\sum_{k=1}^{3}\frac{C_k\eta}{\eta-\gamma_k}\overline{x}_r^{\eta-\gamma_k}=0.$$

Denote $\overline{e}_r \equiv e(\overline{y}_r)$ and $e_r \equiv e(y_r)$. Using $\frac{p_r^*+e_r^*}{x_r^*}=\frac{p_r^*+\overline{e}_r}{\overline{x}_r}=\frac{p_r^*+e_r}{x_r}$,

$$\frac{\phi y_b \eta}{\eta+1}x_r^{\eta+1}-\rho x_r^{\eta}+\sum_{k=1}^{3}\frac{c_k\eta}{\eta-\gamma_k}x_r^{\eta-\gamma_k}$$
$$+\left(p_r^*+\overline{e}_r\right)\frac{\eta}{\eta+1}\overline{x}_r^{\eta}-\left(p_r^*+e_r\right)\frac{\eta}{\eta+1}x_r^{\eta}-p_r^*\left(\overline{x}_r^{\eta}-x_r^{\eta}\right)$$
$$+\rho\overline{x}_r^{\eta}-\frac{\phi\eta y_b}{\eta+1}\overline{x}_r^{\eta+1}-\sum_{k=1}^{3}\frac{C_k\eta}{\eta-\gamma_k}\overline{x}_r^{\eta-\gamma_k}=0,$$

or

$$\frac{\phi y_b \eta}{\eta+1}x_r^{\eta+1}-\rho x_r^{\eta}+\sum_{k=1}^{3}\frac{c_k\eta}{\eta-\gamma_k}x_r^{\eta-\gamma_k}$$
$$-p_r^*\frac{1}{\eta+1}\overline{x}_r^{\eta}+\overline{e}_r\frac{\eta}{\eta+1}\overline{x}_r^{\eta}+p_r^*\frac{1}{\eta+1}x_r^{\eta}-e_r\frac{\eta}{\eta+1}x_r^{\eta}$$
$$+\rho\overline{x}_r^{\eta}-\frac{\phi\eta y_b}{\eta+1}\overline{x}_r^{\eta+1}-\sum_{k=1}^{3}\frac{C_k\eta}{\eta-\gamma_k}\overline{x}_r^{\eta-\gamma_k}=0.$$

Regrouping the terms,

$$\begin{aligned} &x_r^\eta\left(\frac{\phi y_b x_r\eta}{\eta+1}-\rho+\sum_{k=1}^{3}\frac{c_k\eta}{\eta-\gamma_k}x_r^{-\gamma_k}\right)\\ &-\overline{x}_r^\eta\left(\frac{\phi y_b \overline{x}_r\eta}{\eta+1}-\rho+\sum_{k=1}^{3}\frac{C_k\eta}{\eta-\gamma_k}\overline{x}_r^{-\gamma_k}\right)\\ &+p_r^*\frac{1}{\eta+1}x_r^\eta-p_r^*\frac{1}{\eta+1}\overline{x}_r^\eta+\overline{e}_r\frac{\eta}{\eta+1}\overline{x}_r^\eta-e_r\frac{\eta}{\eta+1}x_r^\eta=0, \end{aligned}$$

or

$$\begin{aligned} &x_r^\eta\left(\frac{\phi y_b x_r\eta}{\eta+1}-\rho+\sum_{k=1}^{3}\frac{c_k\eta}{\eta-\gamma_k}x_r^{-\gamma_k}-e_r\frac{\eta}{\eta+1}+p_r^*\frac{1}{\eta+1}\right)\\ &=\overline{x}_r^\eta\left(\frac{\phi y_b \overline{x}_r\eta}{\eta+1}-\rho+\sum_{k=1}^{3}\frac{C_k\eta}{\eta-\gamma_k}\overline{x}_r^{-\gamma_k}-\overline{e}_r\frac{\eta}{\eta+1}+p_r^*\frac{1}{\eta+1}\right), \end{aligned}$$

or

$$\begin{aligned} &x_r^\eta\left(\frac{\phi y_b x_r\eta}{\eta+1}-\rho+\sum_{k=1}^{3}\frac{c_k\eta}{\eta-\gamma_k}x_r^{-\gamma_k}-\left[\phi y_b x_r-\rho+\sum_{k=1}^{3}c_k x_r^{-\gamma_k}\right]\frac{\eta}{\eta+1}+p_r^*\frac{1}{\eta+1}\right)\\ &=\overline{x}_r^\eta\left(\frac{\phi y_b \overline{x}_r\eta}{\eta+1}-\rho+\sum_{k=1}^{3}\frac{C_k\eta}{\eta-\gamma_k}\overline{x}_r^{-\gamma_k}-\left[\phi y_b\overline{x}_r-\rho+\sum_{k=1}^{3}C_k\overline{x}_r^{-\gamma_k}\right]\frac{\eta}{\eta+1}+p_r^*\frac{1}{\eta+1}\right), \end{aligned}$$

or

$$\begin{aligned} &x_r^\eta\left(\sum_{k=1}^{3}\frac{c_k\eta}{\eta-\gamma_k}x_r^{-\gamma_k}-\sum_{k=1}^{3}c_k x_r^{-\gamma_k}\frac{\eta}{\eta+1}+(p_r^*-\rho)\frac{1}{\eta+1}\right)\\ &=\overline{x}_r^\eta\left(\sum_{k=1}^{3}\frac{C_k\eta}{\eta-\gamma_k}\overline{x}_r^{-\gamma_k}-\sum_{k=1}^{3}C_k\frac{\eta}{\eta+1}\overline{x}_r^{-\gamma_k}+(p_r^*-\rho)\frac{1}{\eta+1}\right), \end{aligned}$$

or

$$\sum_{k=1}^{3}\frac{\eta(1+\gamma_k)}{\eta-\gamma_k}c_k x_r^{\eta-\gamma_k}+(p_r^*-\rho)x_r^\eta=\sum_{k=1}^{3}\frac{\eta(1+\gamma_k)}{\eta-\gamma_k}C_k\overline{x}_r^{\eta-\gamma_k}+(p_r^*-\rho)\overline{x}_r^\eta.$$

Using $p_r^*=\frac{c+\xi}{r+\xi}(1+\sum_{k=1}^{3}b_k x_r^{-\gamma_k})=\frac{c+\xi}{r+\xi}(1+\sum_{k=1}^{3}B_k\overline{x}_r^{-\gamma_k})$, we re-write this equation as

$$\sum_{k=1}^{3}\left\{\frac{\eta(1+\gamma_k)}{\eta-\gamma_k}\left(c_k x_r^{\eta-\gamma_k}-C_k\overline{x}_r^{\eta-\gamma_k}\right)+\frac{c+\xi}{r+\xi}b_k x_r^{\eta-\gamma_k}-\frac{c+\xi}{r+\xi}B_k\overline{x}_r^{\eta-\gamma_k}\right\}=\frac{c\pi}{r+\xi}(\overline{x}_r^\eta-x_r^\eta),$$

or given equation (13),

$$\sum_{k=1}^{3}\left\{\frac{\eta(1+\gamma_k)}{\eta-\gamma_k}\left(c_k x_r^{\eta-\gamma_k}-C_k\overline{x}_r^{\eta-\gamma_k}\right)+\frac{c+\xi}{r+\xi}b_k x_r^{\eta-\gamma_k}\left(1+\frac{\gamma_k(\eta+1)}{\eta-\gamma_k}\right)-\frac{c+\xi}{r+\xi}B_k\overline{x}_r^{\eta-\gamma_k}\left(1+\frac{\gamma_k(\eta+1)}{\eta-\gamma_k}\right)\right\}=\frac{c\pi}{r+\xi}$$

or

$$\sum_{k=1}^{3}\left\{\frac{\eta(1+\gamma_k)}{\eta-\gamma_k}\left[\left(c_k+b_k\frac{c+\xi}{r+\xi}\right)x_r^{\eta-\gamma_k}-\left(C_k+B_k\frac{c+\xi}{r+\xi}\right)\overline{x}_r^{\eta-\gamma_k}\right]\right\}=\frac{c\pi}{r+\xi}(\overline{x}_r^{\eta}-x_r^{\eta}),$$

which is the desired condition (22).

Condition (22) is interpreted as follows. It requires that even if the conjecture $e(y)=\phi y-\rho+\sum_{k=1}^{3}C_k(y/y_b)^{-\gamma_k}$ were applied beyond the range $[\overline{y}_r, y_i]$, this would not change the equity value on $[\overline{y}_r, y_i]$. By the memoryless property of the exponential distribution of downward jumps, this would be the case if $\mathbb{E}\left[e(S\overline{y}_r)\right]$ did not change. Condition (22) requires that

$$\mathbb{E}\left[e(S\overline{y}_r)\right]=\left(1-\left(\frac{\overline{y}_r}{y_r}\right)^{-\eta}\right)\mathbb{E}\left[\frac{e_r^*+p_r^*}{y_r^*}S\overline{y}_r-p_r^*\middle|S\in[y_r/\overline{y}_r,1]\right]+\left(\frac{\overline{y}_r}{y_r}\right)^{-\eta}\mathbb{E}\left[e(Sy_r)\right]$$

holds even if we use conjecture $e(y)=\phi y-\rho+\sum_{k=1}^{3}C_k(y/y_b)^{-\gamma_k}$ in the left-hand side to compute $\mathbb{E}[e(S\overline{y}_r)]$.

B Closed-Form Expressions

B.0.1 Reflecting Boundaries

If the issuance boundary y_i is a reflecting boundary so that $y_i=y_i^*$, then conditions (70) and (17) are replaced by appropriate limits of them. The former is replaced by

$$p'(y_i)=0, \tag{62}$$

which is obtained as the limit of the condition $\left(p(y_i)-p(y_i^*)\right)/\left(y_i-y_i^*\right)=0$ as $y_i^*\to y_i$:

$$\begin{aligned}\lim_{y_i^*\to y_i}\frac{\sum_{k=1}^{3}B_k\left[(y_i/y_b)^{-\gamma_k}-(y_i^*/y_b)^{-\gamma_k}\right]}{y_i-y_i^*}&=\lim_{y_i^*\to y_i}\frac{\sum_{k=1}^{3}B_k y_b^{\gamma_k}y_i^{-\gamma_k-1}\left[1-(y_i^*/y_i)^{-\gamma_k}\right]}{1-y_i^*/y_i}\\&=\lim_{y_i^*\to y_i}\sum_{k=1}^{3}-\gamma_k y_i^{-\gamma_k-1}B_k y_b^{\gamma_k}\\&=\lim_{y_i^*\to y_i}p'(y_i).\end{aligned}$$

Condition (17) is replaced by

$$e'(y_i)y_i = p(y_i) + e(y_i), \tag{63}$$

which is obtained by taking the limit of the condition (17).

If the repurchase boundary $\overline{y}_r$ is a reflecting boundary so that $\overline{y}_r = y_r^*$, then by the analogy with the reflecting issuance boundary, conditions (69) and (79) are replaced by appropriate limits of them:

$$p'(\overline{y}_r) = 0, \tag{64}$$

$$e'(\overline{y}_r)\overline{y}_r = p(\overline{y}_r) + e(\overline{y}_r). \tag{65}$$

B.0.2 Conditions Determining the Leverage Dynamics

Debt Price The debt price is determined by conditions:

$$p(y_b) = 0: \qquad \sum_{k=1}^{3} b_k = -1, \tag{66}$$

$$\text{eq. (12)}: \qquad 1 + \sum_{k=1}^{3} b_k \frac{\eta}{\eta - \gamma_k} = 0, \tag{67}$$

$$p(y_r) = p(\overline{y}_r): \qquad \sum_{k=1}^{3} b_k x_r^{-\gamma_k} = \sum_{k=1}^{3} B_k \overline{x}_r^{-\gamma_k}, \tag{68}$$

$$p(\overline{y}_r) = p(y_r^*): \qquad \sum_{k=1}^{3} B_k \left(\overline{x}_r^{-\gamma_k} - x_r^{*-\gamma_k}\right) = 0, \tag{69}$$

$$p(y_i) = p(y_i^*): \qquad \sum_{k=1}^{3} B_k \left(x_i^{-\gamma_k} - x_i^{*-\gamma_k}\right) = 0, \tag{70}$$

$$\text{eq. (13)}: \qquad x_r^{\eta} \sum_{k=1}^{3} b_k \frac{\gamma_k}{\eta - \gamma_k} x_r^{-\gamma_k} - \overline{x}_r^{\eta} \sum_{k=1}^{3} B_k \frac{\gamma_k}{\eta - \gamma_k} \overline{x}_r^{-\gamma_k} = 0. \tag{71}$$

Conditions (12) and (13) are obtained by plugging the conjecture (7) into the HJB equation (6) and matching the terms at $\lambda(y/y_b)^{-\eta}$. Conditions (66) – (71) give six equations on six parameters $(b_k, B_k)_{k=1,2,3}$, and allow us to derive the debt price without computing the equity value.

If y_i is a reflecting issuance boundary, then as we showed above, condition (70) is replaced by (62), or more explicitly:

$$p'(y_i) = 0: \sum_{k=1}^{3} \gamma_k B_k x_i^{-\gamma_k} = 0. \tag{72}$$

If $\overline{y}_r$ is a reflecting repurchase boundary, then as we showed above, condition (69) is replaced by (64), or more explicitly:

$$p'(\overline{y}_r) = 0: \sum_{k=1}^{3} \gamma_k B_k \overline{x}_r^{-\gamma_k} = 0. \tag{73}$$

Equity Value The equity value is determined by conditions:

$$e(y_b) = 0: \phi y_b - \rho + \sum_{k=1}^{3} c_k = 0, \tag{74}$$

$$e'(y_b) = 0: \phi y_b - \sum_{k=1}^{3} c_k \gamma_k = 0, \tag{75}$$

$$\text{eq. (21)}: \frac{\phi\eta}{\eta+1} y_b - \rho + \sum_{k=1}^{3} c_k \frac{\eta}{\eta-\gamma_k} = 0, \tag{76}$$

$$\frac{e(y_i)+p_i^*}{y_i} = \frac{e(y_i^*)+p_i^*}{y_i^*}: \frac{c\pi}{(r+\xi)x_i} + \frac{1}{x_i}\sum_{k=1}^{3}\left(C_k + B_k\frac{c+\xi}{r+\xi}\right)x_i^{-\gamma_k} = \frac{c\pi}{(r+\xi)x_i^*} + \frac{1}{x_i^*}\sum_{k=1}^{3}\left(C_k + B_k\frac{c+\xi}{r+\xi}\right)x_i^{*-\gamma_k}, \tag{77}$$

$$\frac{e(y_r)+p_r^*}{y_r} = \frac{e(y_r^*)+p_r^*}{y_r^*}: \frac{c\pi}{(r+\xi)x_r} + \frac{1}{x_r}\sum_{k=1}^{3}\left(c_k + b_k\frac{c+\xi}{r+\xi}\right)x_r^{-\gamma_k} = \frac{c\pi}{(r+\xi)x_r^*} + \frac{1}{x_r^*}\sum_{k=1}^{3}\left(C_k + B_k\frac{c+\xi}{r+\xi}\right)x_r^{*-\gamma_k}, \tag{78}$$

$$\frac{e(\overline{y}_r)+p_r^*}{\overline{y}_r} = \frac{e(y_r^*)+p_r^*}{y_r^*}: \frac{c\pi}{(r+\xi)\overline{x}_r} + \frac{1}{\overline{x}_r}\sum_{k=1}^{3}\left(C_k + B_k\frac{c+\xi}{r+\xi}\right)\overline{x}_r^{-\gamma_k} = \frac{c\pi}{(r+\xi)x_r^*} + \frac{1}{x_r^*}\sum_{k=1}^{3}\left(C_k + B_k\frac{c+\xi}{r+\xi}\right)x_r^{*-\gamma_k}, \tag{79}$$

$$\text{eq. (22)}: \sum_{k=1}^{3}\left\{\frac{\eta(1+\gamma_k)}{\eta-\gamma_k}\left[\left(c_k + b_k\frac{c+\xi}{r+\xi}\right)x_r^{\eta-\gamma_k} - \left(C_k + B_k\frac{c+\xi}{r+\xi}\right)\overline{x}_r^{\eta-\gamma_k}\right]\right\} = \frac{c\pi}{r+\xi}(\overline{x}_r^{\eta} - x_r^{\eta}). \tag{80}$$

Conditions (21) and (22) are obtained by plugging the conjecture (20) into the HJB equation (14) and matching the terms at $\lambda(y/y_b)^{-\eta}$. Conditions (74) $-$ (80) give seven equations on six parameters $(c_k, C_k)_{k=1,2,3}$ and default threshold y_b, and allow us to derive the equity value.

If y_i is a reflecting issuance boundary, then as we showed above, condition (77) is replaced by (63), or more explicitly:

$$\frac{c\pi}{r+\xi} + \sum_{k=1}^{3}\left(C_k(\gamma_k+1) + B_k\frac{c+\xi}{r+\xi}\right)x_i^{-\gamma_k} = 0. \tag{81}$$

If $\overline{y}_r$ is a reflecting repurchase boundary, then as we showed above, condition (79) is replaced by (65), or more explicitly:

$$\frac{c\pi}{r+\xi} + \sum_{k=1}^{3} \left(C_k(\gamma_k + 1) + B_k \frac{c+\xi}{r+\xi} \right) \overline{x}_r^{-\gamma_k} = 0. \tag{82}$$

Enterprise Value per Unit of Debt Using the conjectured form of function v in (34), the conditions on v can be written explicitly as

$$v(y_b) = 0 : \phi y_b + \frac{c\pi}{r+\xi} + \sum_{k=1}^{3} a_k = 0, \tag{83}$$

$$\text{eq. (12) and eq. (21)} : \frac{\phi\eta}{\eta+1} y_b + \frac{c\pi}{r+\xi} + \sum_{k=1}^{3} a_k \frac{\eta}{\eta-\gamma_k} = 0, \tag{84}$$

$$\frac{v(y_r)}{y_r} = \frac{v(y_r^*)}{y_r^*} : \frac{c\pi}{(r+\xi)x_r} + \sum_{k=1}^{3} a_k x_r^{-\gamma_k - 1} = \frac{c\pi}{(r+\xi)x_r^*} + \frac{1}{x_r^*} \sum_{k=1}^{3} A_k x_r^{*-\gamma_k}, \tag{85}$$

$$\frac{v(\overline{y}_r)}{\overline{y}_r} = \frac{v(y_r^*)}{y_r^*} : \frac{c\pi}{(r+\xi)\overline{x}_r} + \frac{1}{\overline{x}_r} \sum_{k=1}^{3} A_k \overline{x}_r^{-\gamma_k} = \frac{c\pi}{(r+\xi)x_r^*} + \frac{1}{x_r^*} \sum_{k=1}^{3} A_k x_r^{*-\gamma_k}, \tag{86}$$

$$\frac{v(y_i)}{y_i} = \frac{v(y_i^*)}{y_i^*} : \frac{c\pi}{(r+\xi)x_i} + \frac{1}{x_i} \sum_{k=1}^{3} A_k x_i^{-\gamma_k} = \frac{c\pi}{(r+\xi)x_i^*} + \frac{1}{x_i^*} \sum_{k=1}^{3} A_k x_i^{*-\gamma_k}, \tag{87}$$

$$\text{eq. (22)} : \sum_{k=1}^{3} \left\{ \frac{\eta(1+\gamma_k)}{\eta-\gamma_k} \left[a_k x_r^{\eta-\gamma_k} - A_k \overline{x}_r^{\eta-\gamma_k} \right] \right\} = \frac{c\pi}{r+\xi} (\overline{x}_r^{\eta} - x_r^{\eta}). \tag{88}$$

If y_i is the reflecting issuance boundary, then the condition $v(y_i)/y_i = v(y_i^*)/y_i^*$, is replaced by

$$v'(y_i) y_i = v(y_i), \tag{89}$$

or more explicitly,

$$\frac{c\pi}{r+\xi} + \sum_{k=1}^{3} A_k(\gamma_k + 1) x_i^{-\gamma_k} = 0. \tag{90}$$

If $\overline{y}_r$ is the reflecting repurchase boundary, then the condition $v(\overline{y}_r)/\overline{y}_r = v(y_r^*)/y_r^*$ is replaced by

$$v'(\overline{y}_r)\overline{y}_r = v(\overline{y}_r), \tag{91}$$

or more explicitly,

$$\frac{c\pi}{r+\xi} + \sum_{k=1}^{3} A_k(\gamma_k + 1) \overline{x}_r^{-\gamma_k} = 0. \tag{92}$$

B.0.3 Conditions Determining Leverage Dynamics under Targeted ICR Policies

Debt Price The coefficients b_ks satisfy the following conditions:

$$p(y_b) = 0 : b_1 + b_2 + b_3 = -1, \tag{93}$$

$$p(y_r) = \hat{p} : \frac{c+\xi}{r+\xi}\left(1 + b_1 x_r^{-\gamma_1} + b_2 x_r^{-\gamma_2} + b_3 x_r^{-\gamma_3}\right) = \hat{p}, \tag{94}$$

$$\text{eq. (12)} : b_1 \frac{\eta}{\eta-\gamma_1} + b_2 \frac{\eta}{\eta-\gamma_2} + b_3 \frac{\eta}{\eta-\gamma_3} = -1. \tag{95}$$

Further, we show that this HJB equation together with (94) imply

$$\sum_{k=1}^{3} b_k x_r^{-\gamma_k}\left(r + \xi - \frac{\lambda\gamma_k}{\eta-\gamma_k}\left(\frac{\hat{x}}{x_r}\right)^{-\eta}\right) = 0. \tag{96}$$

Using the conjectured solution, we can re-write the HJB equation (28) as follows

$$(r+\xi+\lambda)\hat{p} = c + \xi + \lambda\hat{p}\left(1 - \left(\frac{\hat{y}}{y_r}\right)^{-\eta}\right) + \lambda\frac{c+\xi}{r+\xi}\int_{\ln(\hat{y}/y_r)}^{\ln(\hat{y}/y_b)}\left(1 + \sum_{k=1}^{3} b_k (e^{-\tilde{s}}\hat{y}/y_b)^{-\gamma_k}\right)\eta e^{-\eta\tilde{s}} d\tilde{s},$$

or equivalently,

$$\left(r+\xi+\lambda\left(\hat{y}/y_r\right)^{-\eta}\right)\hat{p} = c+\xi+\lambda\frac{c+\xi}{r+\xi}\left(\left(\frac{\hat{y}}{y_r}\right)^{-\eta} - \left(\frac{\hat{y}}{y_b}\right)^{-\eta}\right)+\lambda\frac{c+\xi}{r+\xi}\sum_{k=1}^{3}\frac{b_k(\hat{y}/y_b)^{-\gamma_k}\eta}{\eta-\gamma_k}\left(\left(\frac{\hat{y}}{y_r}\right)^{-(\eta-\gamma_k)} - \left(\frac{\hat{y}}{y_b}\right)^{-(\eta-}\right.$$

Using the notation $\hat{x} = \hat{y}/y_b$ and $x_r = y_r/y_b$,

$$\left(r+\xi+\lambda\left(\frac{\hat{x}}{x_r}\right)^{-\eta}\right)\hat{p} = c+\xi+\lambda\frac{c+\xi}{r+\xi}\left(\left(\frac{\hat{x}}{x_r}\right)^{-\eta} - \hat{x}^{-\eta}\right)+\lambda\frac{c+\xi}{r+\xi}\sum_{k=1}^{3}\frac{b_k\hat{x}^{-\gamma_k}\eta}{\eta-\gamma_k}\left(\left(\frac{\hat{x}}{x_r}\right)^{-(\eta-\gamma_k)} - \hat{x}^{-(\eta-\gamma_k)}\right).$$

Plugging in $\hat{p}$, we get

$$\frac{c+\xi}{r+\xi}\left(r+\xi+\lambda\left(\frac{\hat{x}}{x_r}\right)^{-\eta}\right)\left(1+\sum_{k=1}^{3} b_k x_r^{-\gamma_k}\right) = c+\xi+\lambda\frac{c+\xi}{r+\xi}\left(\left(\frac{\hat{x}}{x_r}\right)^{-\eta} - \hat{x}^{-\eta}\right)+\lambda\frac{c+\xi}{r+\xi}\sum_{k=1}^{3}\frac{b_k\hat{x}^{-\gamma_k}\eta}{\eta-\gamma_k}\left(\left(\frac{\hat{x}}{x_r}\right)^{-(\eta-\gamma}\right.$$

Simplifying,

$$(r+\xi)\left(1+\sum_{k=1}^{3} b_k x_r^{-\gamma_k}\right)+\lambda\left(\frac{\hat{x}}{x_r}\right)^{-\eta}\left(1+\sum_{k=1}^{3} b_k x_r^{-\gamma_k}\right) = r+\xi+\lambda\left(\left(\frac{\hat{x}}{x_r}\right)^{-\eta} - \hat{x}^{-\eta}\right)+\lambda\sum_{k=1}^{3}\frac{b_k\hat{x}^{-\gamma_k}\eta}{\eta-\gamma_k}\left(\left(\frac{\hat{x}}{x_r}\right)^{-(\eta-\gamma_k)}\right.$$

or

$$(r+\xi)\sum_{k=1}^{3} b_k x_r^{-\gamma_k} + \lambda\left(\frac{\hat{x}}{x_r}\right)^{-\eta}\sum_{k=1}^{3} b_k x_r^{-\gamma_k} = -\lambda\hat{x}^{-\eta} + \lambda\sum_{k=1}^{3}\frac{b_k\hat{x}^{-\gamma_k}\eta}{\eta-\gamma_k}\left(\left(\frac{\hat{x}}{x_r}\right)^{-(\eta-\gamma_k)} - \hat{x}^{-(\eta-\gamma_k)}\right),$$

or

$$(r+\xi)\sum_{k=1}^{3} b_k x_r^{-\gamma_k} + \lambda\sum_{k=1}^{3} b_k\hat{x}^{-\eta}x_r^{\eta-\gamma_k} = -\lambda\hat{x}^{-\eta} + \lambda\sum_{k=1}^{3}\frac{b_k\eta}{\eta-\gamma_k}\left(\hat{x}^{-\eta}x_r^{\eta-\gamma_k} - \hat{x}^{-\eta}\right).$$

Using (95),

$$(r+\xi)\sum_{k=1}^{3} b_k x_r^{-\gamma_k} = \lambda\sum_{k=1}^{3}\frac{b_k\gamma_k}{\eta-\gamma_k}\hat{x}^{-\eta}x_r^{\eta-\gamma_k},$$

or

$$\sum_{k=1}^{3} b_k x_r^{-\gamma_k}\left(r+\xi-\frac{\lambda\gamma_k}{\eta-\gamma_k}\left(\frac{\hat{x}}{x_r}\right)^{-\eta}\right) = 0,$$

which is the desired condition (96).

Equity Value The equity value satisfies the following conditions:

$$e(y_b) = 0: \qquad -\rho + \phi y_b + c_1 + c_2 + c_3 = 0, \tag{97}$$

$$e'(y_b) = 0: \qquad \phi y_b - c_1\gamma_1 - c_2\gamma_2 - c_3\gamma_3 = 0, \tag{98}$$

$$\text{eq. (21)}: \qquad \frac{\phi\eta}{\eta+1}y_b - \rho + \sum_{k=1}^{3} c_k\frac{\eta}{\eta-\gamma_k} = 0. \tag{99}$$

We can re-write condition $\frac{e(y_r)+\hat{p}}{y_r} = \frac{\hat{e}+\hat{p}}{\hat{y}}$ more explicitly as

$$\frac{\phi y_b x_r - \rho + \sum_{k=1}^{3} c_k x_r^{-\gamma_k} + \hat{p}}{x_r} = \frac{\hat{e}+\hat{p}}{\hat{x}}. \tag{100}$$

Finally, we show that using our conjecture and (99), we can rewrite the HJB equation (31) as

$$\begin{aligned}\left(r-\mu+\frac{\lambda\eta}{\eta+1}\left(\hat{x}/x_r\right)^{-(\eta+1)}\right)\hat{e} &= (1-\pi)(\hat{x}y_b - c) - \xi \\ &\quad + \hat{p}\left[\mu+\xi-\frac{\lambda\eta}{\eta+1}(\hat{x}/x_r)^{-(\eta+1)} + \lambda(\hat{x}/x_r)^{-\eta}\right] \\ &\quad + \lambda(\hat{x}/x_r)^{-\eta}\left(\frac{\phi\eta x_r}{\eta+1}y_b - \rho + \sum_{k=1}^{3}\frac{c_k\eta}{\eta-\gamma_k}x_r^{-\gamma_k}\right).\end{aligned} \tag{101}$$

We can re-write the HJB equation (31) as

$$\left(r - \hat{\mu} + \frac{\lambda}{\eta+1} + \frac{\lambda\eta}{\eta+1}\left(\hat{y}/y_r\right)^{-(\eta+1)}\right)\hat{e} = (1-\pi)(\hat{y}-c) - \xi + \hat{p}\left(\hat{\mu}+\xi\right) + \lambda\frac{\hat{p}\eta}{\eta+1}\left(1-\left(\hat{y}/y_r\right)^{-(\eta+1)}\right) - \lambda\hat{p}\left(1-\left(\hat{y}/y_r\right)^{-\eta}\right) + \lambda\int_{\ln(\hat{y}/y_r)}^{\ln(\hat{y}/y_b)} e(e^{-\tilde{s}}\hat{y})\eta e^{-\eta\tilde{s}}d\tilde{s}.$$

Using the conjectured form of e:

$$\left(r - \hat{\mu} + \frac{\lambda}{\eta+1} + \frac{\lambda\eta}{\eta+1}\left(\hat{y}/y_r\right)^{-(\eta+1)}\right)\hat{e} = (1-\pi)(\hat{y}-c) - \xi + \hat{p}\left(\hat{\mu}+\xi\right) + \lambda\frac{\hat{p}\eta}{\eta+1}\left(1-\left(\hat{y}/y_r\right)^{-(\eta+1)}\right) - \lambda\hat{p}\left(1-\left(\hat{y}/y_r\right)^{-\eta}\right) + \lambda\int_{\ln(\hat{y}/y_r)}^{\ln(\hat{y}/y_b)}\left(\phi\hat{y}e^{-\tilde{s}} - \rho + \sum_{k=1}^{3} c_k(e^{-\tilde{s}}\hat{y}/y_b)^{-\gamma_k}\right)\eta e^{-\eta\tilde{s}}d\tilde{s}.$$

or

$$\begin{aligned}\left(r - \hat{\mu} + \frac{\lambda}{\eta+1} + \frac{\lambda\eta}{\eta+1}\left(\hat{y}/y_r\right)^{-(\eta+1)}\right)\hat{e} = {}& (1-\pi)(\hat{y}-c) - \xi + \hat{p}\left(\hat{\mu}+\xi\right) \\ & + \lambda\frac{\hat{p}\eta}{\eta+1}\left(1-\left(\hat{y}/y_r\right)^{-(\eta+1)}\right) - \lambda\hat{p}\left(1-\left(\hat{y}/y_r\right)^{-\eta}\right) \\ & + \frac{\lambda\phi\hat{y}\eta}{\eta+1}\left(\left(\hat{y}/y_r\right)^{-(\eta+1)} - \left(\hat{y}/y_b\right)^{-(\eta+1)}\right) \\ & - \lambda\rho\left(\left(\hat{y}/y_r\right)^{-\eta} - \left(\hat{y}/y_b\right)^{-\eta}\right) \\ & + \lambda\sum_{k=1}^{3}\frac{c_k(\hat{y}/y_b)^{-\gamma_k}\eta}{\eta-\gamma_k}\left(\left(\hat{y}/y_r\right)^{-(\eta-\gamma_k)} - \left(\hat{y}/y_b\right)^{-(\eta-\gamma_k)}\right),\end{aligned}$$

or using $\hat{x} = \hat{y}/y_b$ and $x_r = y_r/y_b$,

$$\begin{aligned}\left(r - \hat{\mu} + \frac{\lambda}{\eta+1} + \frac{\lambda\eta}{\eta+1}\left(\hat{x}/x_r\right)^{-(\eta+1)}\right)\hat{e} = {}& (1-\pi)(\hat{x}y_b - c) - \xi \\ & + \hat{p}\left[\hat{\mu} + \xi - \frac{\lambda}{\eta+1} - \frac{\lambda\eta}{\eta+1}(\hat{x}/x_r)^{-(\eta+1)} + \lambda(\hat{x}/x_r)^{-\eta}\right] \\ & + \frac{\lambda\phi\hat{x}y_b\eta}{\eta+1}\left(\left(\hat{x}/x_r\right)^{-(\eta+1)} - \hat{x}^{-(\eta+1)}\right) \\ & - \lambda\rho\left(\left(\hat{x}/x_r\right)^{-\eta} - \hat{x}^{-\eta}\right) \\ & + \lambda\sum_{k=1}^{3}\frac{c_k\hat{x}^{-\gamma_k}\eta}{\eta-\gamma_k}\left(\left(\hat{x}/x_r\right)^{-(\eta-\gamma_k)} - \hat{x}^{-(\eta-\gamma_k)}\right),\end{aligned}$$

or

$$\left(r-\hat{\mu}+\frac{\lambda}{\eta+1}+\frac{\lambda\eta}{\eta+1}\left(\hat{x}/x_r\right)^{-(\eta+1)}\right)\hat{e}=(1-\pi)(\hat{x}y_b-c)-\xi$$

$$+\hat{p}\left[\hat{\mu}+\xi-\frac{\lambda}{\eta+1}-\frac{\lambda\eta}{\eta+1}(\hat{x}/x_r)^{-(\eta+1)}+\lambda(\hat{x}/x_r)^{-\eta}\right]$$

$$+\frac{\lambda\phi\hat{x}y_b\eta}{\eta+1}(\hat{x}/x_r)^{-(\eta+1)}-\lambda\rho(\hat{x}/x_r)^{-\eta}$$

$$+\lambda\sum_{k=1}^{3}\frac{c_k\hat{x}^{-\gamma_k}\eta}{\eta-\gamma_k}(\hat{x}/x_r)^{-(\eta-\gamma_k)}$$

$$+\lambda\hat{x}^{-\eta}\left(\rho-\frac{\phi y_b\eta}{\eta+1}-\sum_{k=1}^{3}\frac{c_k\eta}{\eta-\gamma_k}\right).$$

Using (99),

$$\left(r-\hat{\mu}+\frac{\lambda}{\eta+1}+\frac{\lambda\eta}{\eta+1}\left(\hat{x}/x_r\right)^{-(\eta+1)}\right)\hat{e}=(1-\pi)(\hat{x}y_b-c)-\xi$$

$$+\hat{p}\left[\hat{\mu}+\xi-\frac{\lambda}{\eta+1}-\frac{\lambda\eta}{\eta+1}(\hat{x}/x_r)^{-(\eta+1)}+\lambda(\hat{x}/x_r)^{-\eta}\right]$$

$$+\lambda(\hat{x}/x_r)^{-\eta}\left(\frac{\phi\eta x_r}{\eta+1}y_b-\rho+\sum_{k=1}^{3}\frac{c_k\eta}{\eta-\gamma_k}x_r^{-\gamma_k}\right),$$

which after noting that $\mu=\hat{\mu}-\frac{\lambda}{\eta+1}$, gives the desired equation (101).

C Richer Classes of Policies

In this Online Appendix, we consider two richer classes of policies. We demonstrate numerically that certain natural more complex debt policies can lead to a negligible improvement in the firm value, do not our qualitative implications, and have very small quantitative effect on optimal leverage ratios and parameters of the policy.

The motivation for the class of policies that we consider is as follows. As we showed in the paper, requiring equity holders to make large repurchases is particularly costly in terms of equity holders' incentives and can cause the credibility constraints to bind. One may conjecture that policies that allow for more flexible repurchases might dominate the targeted ICR policy. In this Online Appendix, we consider two such policies.

1. In Online Appendix C.1, we consider policies in which after a sufficiently large negative cash flow shock, the firm repurchases a smaller amount of debt than is necessary to get back to the ICR target. In the continuation, the firm waits for a sequence of positive shocks to increase the ICR to the level at which it makes another repurchase of a chunk

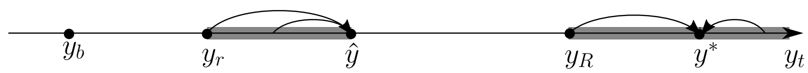

Figure 8: ***Complex-repurchase targeted ICR policy thresholds***

The gray region is the region where the firm issues or repurchases debt. Arrows indicate where the state y_t transitions when it falls into the gray region.

of debt and gets back to the ICR target.

2. In Online Appendix C.2, we consider policies in which after a sufficiently large negative cash flow shock, the firm repurchases a smaller amount of debt than is necessary to get back to the initial ICR target and instead switches to a lower ICR target in the continuation.

C.1 Complex-Repurchase Targeted ICR Policies

We first consider policies with two repurchase regions, which we call "complex-repurchase TICR policies." Formally, the repurchase region consists of two intervals: $[y_r, \hat{y}]$ and $[y_R, y^*]$ with $\hat{y} \leq y_R$ (see Figure 8). When at the ICR target y^*, the firm manages its liabilities to stay at the target y^* by compensating all positive shocks to y_t with debt issuances. After negative shocks that bring y_t into the higher repurchase region $[y_R, y^*)$, the firm repuchases debt to get back to the target y^*. After negative shocks that bring y_t into the lower repurchase region $[y_r, \hat{y}]$, the firm repuchases debt to get to $\hat{y}$. In regions $(\hat{y}, y_R)$ and (y_b, y_r), the firm does not manage liabilities.

C.1.1 Derivation of Value Functions

We next characterize the debt price, equity value, and enterprise value functions.

Debt price We consider the debt price function of the following form:

$$p(y) = \begin{cases} 0, & y \in (0, y_b], \\ \frac{c+\xi}{r+\xi}\left(1 + \sum_{k=1}^{3} b_k (y/y_b)^{-\gamma_k}\right), & y \in [y_b, y_r], \\ \hat{p}, & [y_r, \hat{y}], \\ \frac{c+\xi}{r+\xi}\left(1 + \sum_{k=1}^{3} B_k (y/y_b)^{-\gamma_k}\right), & y \in [\hat{y}, y_R], \\ p^*, & y \in [y_R, \infty). \end{cases}$$

Coefficients b_ks and B_ks are pinned down by the following conditions:

$$p(y_b) = 0 : \sum_{k=1}^{3} b_k = -1, \tag{102}$$

$$p(y_r) = p(\hat{y}) \equiv \hat{p} : \frac{c+\xi}{r+\xi}\left(1 + \sum_{k=1}^{3} b_k x_r^{-\gamma_k}\right) = \frac{c+\xi}{r+\xi}\left(1 + \sum_{k=1}^{3} B_k \hat{x}^{-\gamma_k}\right) = \hat{p}, \tag{103}$$

$$\text{analogue of eq. (67)} : 1 + \sum_{k=1}^{3} b_k \frac{\eta}{\eta - \gamma_k} = 0, \tag{104}$$

$$\text{analogue of eq. (71)} : x_r^{\eta} \sum_{k=1}^{3} b_k \frac{\gamma_k}{\eta - \gamma_k} x_r^{-\gamma_k} - \hat{x}^{\eta} \sum_{k=1}^{3} B_k \frac{\gamma_k}{\eta - \gamma_k} \hat{x}^{-\gamma_k} = 0, \tag{105}$$

$$\text{analgoue of eq. (73)} : \sum_{k=1}^{3} \gamma_k B_k \hat{x}^{-\gamma_k} = 0, \tag{106}$$

$$p(y_R) = p^* : \frac{c+\xi}{r+\xi}\left(1 + \sum_{k=1}^{3} B_k x_R^{-\gamma_k}\right) = p^*, \tag{107}$$

where $x^* \equiv y^*/y_b$, $x_R \equiv y_R/y_b$, $\hat{x} \equiv \hat{y}/y_b$, $x_r \equiv y_r/y_b$. Finally, the price of debt p^* at the target ICR y^* is given by

$$p^* = (c+\xi)dt + (1 - rdt - \xi dt)\Bigg\{(1-\lambda dt)p^* + \lambda dt \int_0^{\ln(\hat{y}/y_R)} p^* \eta e^{-\eta\tilde{s}} d\tilde{s}$$
$$+\lambda dt \int_{\ln(\hat{y}/y_R)}^{\ln(\hat{y}/\overline{y}_r)} p(e^{-\tilde{s}}\hat{y})\eta e^{-\eta\tilde{s}} d\tilde{s} + \lambda dt \int_{\ln(\hat{y}/\overline{y}_r)}^{\ln(\hat{y}/y_r)} \hat{p}\eta e^{-\eta\tilde{s}} d\tilde{s} + \lambda dt \int_{\ln(\hat{y}/y_r)}^{\ln(\hat{y}/y_b)} p(e^{-\tilde{s}}\hat{y})\eta e^{-\eta\tilde{s}} d\tilde{s}\Bigg\}.$$

Simplifying:

$$(r+\xi+\lambda)p^* = c + \xi + \lambda\Bigg\{\int_0^{\ln(\hat{y}/y_R)} p^* \eta e^{-\eta\tilde{s}} d\tilde{s}$$
$$+ \int_{\ln(\hat{y}/y_R)}^{\ln(\hat{y}/\overline{y}_r)} p(e^{-\tilde{s}}\hat{y})\eta e^{-\eta\tilde{s}} d\tilde{s} + \int_{\ln(\hat{y}/\overline{y}_r)}^{\ln(\hat{y}/y_r)} \hat{p}\eta e^{-\eta\tilde{s}} d\tilde{s} + \int_{\ln(\hat{y}/y_r)}^{\ln(\hat{y}/y_b)} p(e^{-\tilde{s}}\hat{y})\eta e^{-\eta\tilde{s}} d\tilde{s}\Bigg\}. \tag{108}$$

Plugging in the functional forms for p,

$$(r+\xi+\lambda)p^* = c + \xi + \lambda p^*\left(1 - \left(\frac{\hat{y}}{y_R}\right)^{-\eta}\right) + \lambda\frac{c+\xi}{r+\xi}\int_{\ln(y^*/y_R)}^{\ln(y^*/\hat{y})}\left(1 + \sum_{k=1}^{3} B_k (e^{-\tilde{s}}y^*/y_b)^{-\gamma_k}\right)\eta e^{-\eta\tilde{s}} d\tilde{s}$$
$$+ \lambda\hat{p}\left(\left(\frac{y^*}{\hat{y}}\right)^{-\eta} - \left(\frac{y^*}{y_r}\right)^{-\eta}\right) + \lambda\frac{c+\xi}{r+\xi}\int_{\ln(y^*/y_r)}^{\ln(y^*/y_b)}\left(1 + \sum_{k=1}^{3} b_k (e^{-\tilde{s}}y^*/y_b)^{-\gamma_k}\right)\eta e^{-\eta\tilde{s}} d\tilde{s},$$

or

$$
\begin{aligned}
(r+\xi+\lambda)p^* =& c+\xi+\lambda p^*\left(1-\left(\frac{y^*}{y_R}\right)^{-\eta}\right)+\lambda\frac{c+\xi}{r+\xi}\left(\left(\frac{y^*}{y_R}\right)^{-\eta}-\left(\frac{y^*}{\hat{y}}\right)^{-\eta}\right) \\
&+\lambda\frac{c+\xi}{r+\xi}\sum_{k=1}^{3}\frac{B_k(y^*/y_b)^{-\gamma_k}\eta}{\eta-\gamma_k}\left(\left(\frac{y^*}{y_R}\right)^{-(\eta-\gamma_k)}-\left(\frac{y^*}{\hat{y}}\right)^{-(\eta-\gamma_k)}\right) \\
&+\lambda\hat{p}\left(\left(\frac{y^*}{\hat{y}}\right)^{-\eta}-\left(\frac{y^*}{y_r}\right)^{-\eta}\right)+\lambda\frac{c+\xi}{r+\xi}\left(\left(\frac{y^*}{y_r}\right)^{-\eta}-\left(\frac{y^*}{y_b}\right)^{-\eta}\right) \\
&+\lambda\frac{c+\xi}{r+\xi}\sum_{k=1}^{3}\frac{b_k(y^*/y_b)^{-\gamma_k}\eta}{\eta-\gamma_k}\left(\left(\frac{y^*}{y_r}\right)^{-(\eta-\gamma_k)}-\left(\frac{y^*}{y_b}\right)^{-(\eta-\gamma_k)}\right),
\end{aligned}
$$

Using the notation $x^* = y^*/y_b$, $x_R = y_R/y_b$, $\hat{x} = \hat{y}/y_b$, $x_r = y_r/y_b$,

$$
\begin{aligned}
(r+\xi+\lambda)p^* =& c+\xi+\lambda p^*\left(1-\left(\frac{x^*}{x_R}\right)^{-\eta}\right)+\lambda\frac{c+\xi}{r+\xi}\left(\left(\frac{x^*}{x_R}\right)^{-\eta}-\left(\frac{x^*}{\hat{x}}\right)^{-\eta}\right) \\
&+\lambda\frac{c+\xi}{r+\xi}\sum_{k=1}^{3}\frac{B_k(x^*)^{-\gamma_k}\eta}{\eta-\gamma_k}\left(\left(\frac{x^*}{x_R}\right)^{-(\eta-\gamma_k)}-\left(\frac{x^*}{\hat{x}}\right)^{-(\eta-\gamma_k)}\right) \\
&+\lambda\hat{p}\left(\left(\frac{x^*}{\hat{x}}\right)^{-\eta}-\left(\frac{x^*}{x_r}\right)^{-\eta}\right)+\lambda\frac{c+\xi}{r+\xi}\left(\left(\frac{x^*}{x_r}\right)^{-\eta}-(x^*)^{-\eta}\right) \\
&+\lambda\frac{c+\xi}{r+\xi}\sum_{k=1}^{3}\frac{b_k(x^*)^{-\gamma_k}\eta}{\eta-\gamma_k}\left(\left(\frac{x^*}{x_r}\right)^{-(\eta-\gamma_k)}-(x^*)^{-(\eta-\gamma_k)}\right).
\end{aligned}
$$

Using the expressions in (103) for $\hat{p}$ and the expression (107) for p^*,

$$
\begin{aligned}
\frac{c+\xi}{r+\xi}\left(1+\sum_{k=1}^{3}B_k x_R^{-\gamma_k}\right)(r+\xi+\lambda) =& c+\xi+\lambda\frac{c+\xi}{r+\xi}\left(1+\sum_{k=1}^{3}B_k x_R^{-\gamma_k}\right)\left(1-\left(\frac{x^*}{x_R}\right)^{-\eta}\right) \\
&+\lambda\frac{c+\xi}{r+\xi}\left(\left(\frac{x^*}{x_R}\right)^{-\eta}-\left(\frac{x^*}{\hat{x}}\right)^{-\eta}\right) \\
&+\lambda\frac{c+\xi}{r+\xi}\sum_{k=1}^{3}\frac{B_k(x^*)^{-\gamma_k}\eta}{\eta-\gamma_k}\left(\left(\frac{x^*}{x_R}\right)^{-(\eta-\gamma_k)}-\left(\frac{x^*}{\hat{x}}\right)^{-(\eta-\gamma_k)}\right) \\
&+\lambda\frac{c+\xi}{r+\xi}\left(1+\sum_{k=1}^{3}B_k\hat{x}^{-\gamma_k}\right)\left(\frac{x^*}{\hat{x}}\right)^{-\eta} \\
&-\lambda\frac{c+\xi}{r+\xi}\left(1+\sum_{k=1}^{3}b_k x_r^{-\gamma_k}\right)\left(\frac{x^*}{x_r}\right)^{-\eta} \\
&+\lambda\frac{c+\xi}{r+\xi}\left(\left(\frac{x^*}{x_r}\right)^{-\eta}-(x^*)^{-\eta}\right) \\
&+\lambda\frac{c+\xi}{r+\xi}\sum_{k=1}^{3}\frac{b_k(x^*)^{-\gamma_k}\eta}{\eta-\gamma_k}\left(\left(\frac{x^*}{x_r}\right)^{-(\eta-\gamma_k)}-(x^*)^{-(\eta-\gamma_k)}\right).
\end{aligned}
$$

Simplifying,

$$
\begin{aligned}
\frac{c+\xi}{r+\xi}\left(1+\sum_{k=1}^{3}B_k x_R^{-\gamma_k}\right)(r+\xi+\lambda) =& c+\xi+\lambda\frac{c+\xi}{r+\xi}\left(1-\left(\frac{x^*}{x_R}\right)^{-\eta}\right)+\lambda\frac{c+\xi}{r+\xi}\sum_{k=1}^{3}B_k x_R^{-\gamma_k}\left(1-\left(\frac{x^*}{x_R}\right)^{-\eta}\right)\\
&+\lambda\frac{c+\xi}{r+\xi}\left(\left(\frac{x^*}{x_R}\right)^{-\eta}-\left(\frac{x^*}{\hat{x}}\right)^{-\eta}\right)\\
&+\lambda\frac{c+\xi}{r+\xi}\sum_{k=1}^{3}\frac{B_k(x^*)^{-\gamma_k}\eta}{\eta-\gamma_k}\left(\left(\frac{x^*}{x_R}\right)^{-(\eta-\gamma_k)}-\left(\frac{x^*}{\hat{x}}\right)^{-(\eta-\gamma_k)}\right)\\
&+\lambda\frac{c+\xi}{r+\xi}\left(\left(\frac{x^*}{\hat{x}}\right)^{-\eta}-\left(\frac{x^*}{x_r}\right)^{-\eta}\right)\\
&+\lambda\frac{c+\xi}{r+\xi}\left(\sum_{k=1}^{3}B_k\hat{x}^{-\gamma_k}\left(\frac{x^*}{\hat{x}}\right)^{-\eta}-\sum_{k=1}^{3}b_k x_r^{-\gamma_k}\left(\frac{x^*}{x_r}\right)^{-\eta}\right)\\
&+\lambda\frac{c+\xi}{r+\xi}\left(\left(\frac{x^*}{x_r}\right)^{-\eta}-(x^*)^{-\eta}\right)\\
&+\lambda\frac{c+\xi}{r+\xi}\sum_{k=1}^{3}\frac{b_k(x^*)^{-\gamma_k}\eta}{\eta-\gamma_k}\left(\left(\frac{x^*}{x_r}\right)^{-(\eta-\gamma_k)}-(x^*)^{-(\eta-\gamma_k)}\right).
\end{aligned}
$$

Cancelling terms (in blue) and rearranging,

$$
\begin{aligned}
(c+\xi)\left(1+\sum_{k=1}^{3}B_k x_R^{-\gamma_k}\right) =& -\lambda\frac{c+\xi}{r+\xi}\left(1+\sum_{k=1}^{3}B_k x_R^{-\gamma_k}\right)\\
&+c+\xi+\lambda\frac{c+\xi}{r+\xi}\left(1-(x^*)^{-\eta}\right)\\
&+\lambda\frac{c+\xi}{r+\xi}\sum_{k=1}^{3}B_k x_R^{-\gamma_k}\left(1-\left(\frac{x^*}{x_R}\right)^{-\eta}\right)\\
&+\lambda\frac{c+\xi}{r+\xi}\sum_{k=1}^{3}\frac{B_k(x^*)^{-\gamma_k}\eta}{\eta-\gamma_k}\left(\left(\frac{x^*}{x_R}\right)^{-(\eta-\gamma_k)}-\left(\frac{x^*}{\hat{x}}\right)^{-(\eta-\gamma_k)}\right)\\
&+\lambda\frac{c+\xi}{r+\xi}\left(\sum_{k=1}^{3}B_k\hat{x}^{-\gamma_k}\left(\frac{x^*}{\hat{x}}\right)^{-\eta}-\sum_{k=1}^{3}b_k x_r^{-\gamma_k}\left(\frac{x^*}{x_r}\right)^{-\eta}\right)\\
&+\lambda\frac{c+\xi}{r+\xi}\sum_{k=1}^{3}\frac{b_k(x^*)^{-\gamma_k}\eta}{\eta-\gamma_k}\left(\left(\frac{x^*}{x_r}\right)^{-(\eta-\gamma_k)}-(x^*)^{-(\eta-\gamma_k)}\right),
\end{aligned}
$$

and further,

$$
\begin{aligned}
(c+\xi)\sum_{k=1}^{3} B_k x_R^{-\gamma_k} =& -\lambda\frac{c+\xi}{r+\xi}\hat{x}^{-\eta} - \lambda\frac{c+\xi}{r+\xi}\sum_{k=1}^{3} B_k x_R^{-\gamma_k}\left(\frac{x^*}{x_R}\right)^{-\eta} \\
&+\lambda\frac{c+\xi}{r+\xi}\sum_{k=1}^{3}\frac{B_k (x^*)^{-\gamma_k}\eta}{\eta-\gamma_k}\left(\left(\frac{x^*}{x_R}\right)^{-(\eta-\gamma_k)} - \left(\frac{x^*}{\hat{x}}\right)^{-(\eta-\gamma_k)}\right) \\
&+\lambda\frac{c+\xi}{r+\xi}\left(\sum_{k=1}^{3} B_k \hat{x}^{-\gamma_k}\left(\frac{x^*}{\hat{x}}\right)^{-\eta} - \sum_{k=1}^{3} b_k x_r^{-\gamma_k}\left(\frac{x^*}{x_r}\right)^{-\eta}\right) \\
&+\lambda\frac{c+\xi}{r+\xi}\sum_{k=1}^{3}\frac{b_k (x^*)^{-\gamma_k}\eta}{\eta-\gamma_k}\left(\left(\frac{x^*}{x_r}\right)^{-(\eta-\gamma_k)} - (x^*)^{-(\eta-\gamma_k)}\right),
\end{aligned}
$$

or

$$
\begin{aligned}
\frac{r+\xi}{\lambda}\sum_{k=1}^{3} B_k x_R^{-\gamma_k} =& -\hat{x}^{-\eta} - \sum_{k=1}^{3} B_k x_R^{-\gamma_k}\left(\frac{x^*}{x_R}\right)^{-\eta} \\
&+\sum_{k=1}^{3}\frac{B_k (x^*)^{-\gamma_k}\eta}{\eta-\gamma_k}\left(\left(\frac{x^*}{x_R}\right)^{-(\eta-\gamma_k)} - \left(\frac{x^*}{\hat{x}}\right)^{-(\eta-\gamma_k)}\right) \\
&+\sum_{k=1}^{3} B_k \hat{x}^{-\gamma_k}\left(\frac{x^*}{\hat{x}}\right)^{-\eta} - \sum_{k=1}^{3} b_k x_r^{-\gamma_k}\left(\frac{x^*}{x_r}\right)^{-\eta} \\
&+\sum_{k=1}^{3}\frac{b_k (x^*)^{-\gamma_k}\eta}{\eta-\gamma_k}\left(\left(\frac{x^*}{x_r}\right)^{-(\eta-\gamma_k)} - (x^*)^{-(\eta-\gamma_k)}\right).
\end{aligned}
$$

Using (104) to cancel terms in blue,

$$
\begin{aligned}
\frac{r+\xi}{\lambda}\sum_{k=1}^{3} B_k x_R^{-\gamma_k} =& - \sum_{k=1}^{3} B_k x_R^{-\gamma_k}\left(\frac{x^*}{x_R}\right)^{-\eta} \\
&+\sum_{k=1}^{3}\frac{B_k (x^*)^{-\gamma_k}\eta}{\eta-\gamma_k}\left(\left(\frac{x^*}{x_R}\right)^{-(\eta-\gamma_k)} - \left(\frac{x^*}{\hat{x}}\right)^{-(\eta-\gamma_k)}\right) \\
&+\sum_{k=1}^{3} B_k \hat{x}^{-\gamma_k}\left(\frac{x^*}{\hat{x}}\right)^{-\eta} - \sum_{k=1}^{3} b_k x_r^{-\gamma_k}\left(\frac{x^*}{x_r}\right)^{-\eta} \\
&+\sum_{k=1}^{3}\frac{b_k x_r^{-\gamma_k}\eta}{\eta-\gamma_k}\left(\frac{x^*}{x_r}\right)^{-\eta}.
\end{aligned}
$$

Rearranging,

$$
\begin{aligned}
\frac{r+\xi}{\lambda}\sum_{k=1}^{3} B_k x_R^{-\gamma_k} = &-\sum_{k=1}^{3} B_k x_R^{-\gamma_k}\left(\frac{x^*}{x_R}\right)^{-\eta} \\
&+\sum_{k=1}^{3}\frac{B_k(x^*)^{-\gamma_k}\eta}{\eta-\gamma_k}\left(\frac{x^*}{x_R}\right)^{-(\eta-\gamma_k)} \\
&-(x^*)^{-\eta}\sum_{k=1}^{3}\frac{B_k\gamma_k}{\eta-\gamma_k}\hat{x}^{\eta-\gamma_k} \\
&+(x^*)^{-\eta}\sum_{k=1}^{3}\frac{b_k\gamma_k}{\eta-\gamma_k}x_r^{\eta-\gamma_k}.
\end{aligned}
$$

Using (105),

$$
\frac{r+\xi}{\lambda}\sum_{k=1}^{3} B_k x_R^{-\gamma_k} = -\sum_{k=1}^{3} B_k x_R^{-\gamma_k}\left(\frac{x^*}{x_R}\right)^{-\eta} + \sum_{k=1}^{3}\frac{B_k(x^*)^{-\gamma_k}\eta}{\eta-\gamma_k}\left(\frac{x^*}{x_R}\right)^{-(\eta-\gamma_k)},
$$

or

$$
\frac{r+\xi}{\lambda}\sum_{k=1}^{3} B_k x_R^{-\gamma_k} = \sum_{k=1}^{3}\frac{B_k\gamma_k}{\eta-\gamma_k}(x^*)^{-\eta}x_R^{\eta-\gamma_k},
$$

Therefore,

$$
\sum_{k=1}^{3} B_k x_R^{-\gamma_k}\left(r+\xi-\frac{\lambda\gamma_k}{\eta-\gamma_k}\left(\frac{x^*}{x_R}\right)^{-\eta}\right) = 0, \tag{109}
$$

which is the last equation that pins down b_ks and B_ks.

Enterprise Value and Equity Value Functions The normalized equity value functions takes the form:

$$
e(y) = \begin{cases}
0, & y \in (0, y_b], \\
\phi y - \rho + \sum_{k=1}^{3} c_k (y/y_b)^{-\gamma_k}, & y \in [y_b, y_r], \\
\left(\frac{y}{\hat{y}} - 1\right) p(\hat{y}) + \frac{y}{\hat{y}} e(\hat{y}), & y \in [y_r, \hat{y}], \\
\phi y - \rho + \sum_{k=1}^{3} C_k (y/y_b)^{-\gamma_k}, & y \in [\hat{y}, y_R], \\
\left(\frac{y}{y^*} - 1\right) p^* + \frac{y}{y^*} e^*, & y \in [y_R, \infty).
\end{cases}
$$

The normalized enterprise value function $v(y) = e(y) + p(y)$ takes the form:

$$v(y) = \begin{cases} 0, & y \in (0, y_b], \\ \phi y + \frac{c\pi}{r+\xi} + \sum_{k=1}^{3} a_k (y/y_b)^{-\gamma_k}, & y \in [y_b, y_r], \\ \frac{v(\hat{y})}{\hat{y}} y, & y \in [y_r, \hat{y}], \\ \phi y + \frac{c\pi}{r+\xi} + \sum_{k=1}^{3} A_k (y/y_b)^{-\gamma_k}, & y \in [\hat{y}, y_R], \\ \frac{v^*}{y^*} y, & y \in [y_R, \infty), \end{cases}$$

where $v^* \equiv v(y^*)$. The coefficients of functions p, e, and v are related by $a_k = c_k + \frac{c+\xi}{r+\xi} b_k$ and $A_k = C_k + \frac{c+\xi}{r+\xi} B_k$, and $v^* = p^* + e^*$. Thus, the function e will be determined once we determine functions v and p. Coefficients a_ks, $A_k s$, v^*, and y_b satisfy:

$$v(y_b) = 0 : \phi y_b + \frac{c\pi}{r+\xi} + \sum_{k=1}^{3} a_k = 0, \tag{110}$$

$$\text{analogue of eq. (84)} : \frac{\phi\eta}{\eta+1} y_b + \frac{c\pi}{r+\xi} + \sum_{k=1}^{3} a_k \frac{\eta}{\eta - \gamma_k} = 0, \tag{111}$$

$$\frac{v(y_r)}{y_r} = \frac{v(\hat{y})}{\hat{y}} : \frac{c\pi}{(r+\xi)x_r} + \sum_{k=1}^{3} a_k x_r^{-\gamma_k - 1} = \frac{c\pi}{(r+\xi)\hat{x}} + \sum_{k=1}^{3} A_k \hat{x}^{-\gamma_k - 1}, \tag{112}$$

$$\text{analogue of eq. (88)} : \sum_{k=1}^{3} \left\{ \frac{\eta(1+\gamma_k)}{\eta - \gamma_k} \left[a_k x_r^{\eta - \gamma_k} - A_k \hat{x}^{\eta - \gamma_k} \right] \right\} = \frac{c\pi}{r+\xi} (\hat{x}^\eta - x_r^\eta), \tag{113}$$

$$\frac{v^*}{y^*} = \frac{v(y_R)}{y_R} : \frac{x_R}{x^*} v^* = \phi x_R y_b + \frac{c\pi}{r+\xi} + \sum_{k=1}^{3} A_k x_R^{-\gamma_k}, \tag{114}$$

$$e'(y_b) = 0 : \phi y_b - \sum_{k=1}^{3} \left(a_k - \frac{c+\xi}{r+\xi} b_k \right) \gamma_k = 0, \tag{115}$$

$$\text{analogue of eq. (92)} : \frac{c\pi}{r+\xi} + \sum_{k=1}^{3} A_k (1+\gamma_k) \hat{x}^{-\gamma_k} = 0. \tag{116}$$

To get the last condition, by the same argument as for the TICR, the equity value at target y^* is given by

$$
\begin{aligned}
(r+\lambda-\hat{\mu})e^* =& (1-\pi)(y^*-c) - \xi + p^*\left(\hat{\mu}+\xi\right) \\
&+ \lambda \int_0^{\ln(y^*/y_R)} \left(\left(e^{-\tilde{s}}-1\right)p^* + e^{-\tilde{s}}e^*\right)\eta e^{-\eta\tilde{s}}d\tilde{s} \\
&+ \lambda \int_{\ln(y^*/y_R)}^{\ln(y^*/\hat{y})} e(e^{-\tilde{s}}y^*)\eta e^{-\eta\tilde{s}}d\tilde{s} \\
&+ \lambda \int_{\ln(y^*/\hat{y})}^{\ln(y^*/y_r)} \left(\left(\frac{y^*e^{-\tilde{s}}}{\hat{y}}-1\right)p(\hat{y}) + \frac{y^*e^{-\tilde{s}}}{\hat{y}}e(\hat{y})\right)\eta e^{-\eta\tilde{s}}d\tilde{s} \\
&+ \lambda \int_{\ln(y^*/y_r)}^{\ln(y^*/y_b)} e(e^{-\tilde{s}}y^*)\eta e^{-\eta\tilde{s}}d\tilde{s}.
\end{aligned}
$$

Combining this with the expression for p^* in (108), we get

$$
\begin{aligned}
(r+\lambda-\hat{\mu})v^* =& (1-\pi)y^* + \pi c \\
&+ \lambda \int_0^{\ln(y^*/y_R)} e^{-\tilde{s}}v^*\eta e^{-\eta\tilde{s}}d\tilde{s} \\
&+ \lambda \int_{\ln(y^*/y_R)}^{\ln(y^*/\hat{y})} v(e^{-\tilde{s}}y^*)\eta e^{-\eta\tilde{s}}d\tilde{s} \\
&+ \lambda \int_{\ln(y^*/\hat{y})}^{\ln(y^*/y_r)} \frac{y^*e^{-\tilde{s}}}{\overline{y}_r}v(\hat{y})\eta e^{-\eta\tilde{s}}d\tilde{s} \\
&+ \lambda \int_{\ln(y^*/y_r)}^{\ln(y^*/y_b)} v(e^{-\tilde{s}}y^*)\eta e^{-\eta\tilde{s}}d\tilde{s}.
\end{aligned}
$$

Using the functional form for v,

$$
\begin{aligned}
(r+\lambda-\hat{\mu})v^* =& (1-\pi)y^* + \pi c \\
&+ \lambda v^* \frac{\eta}{\eta+1}\left(1-\left(\frac{y^*}{y_R}\right)^{-(\eta+1)}\right) \\
&+ \lambda \int_{\ln(y^*/y_R)}^{\ln(y^*/\hat{y})} \left(\phi e^{-\tilde{s}}y^* + \frac{c\pi}{r+\xi} + \sum_{k=1}^{3} A_k(e^{-\tilde{s}}y^*/y_b)^{-\gamma_k}\right)\eta e^{-\eta\tilde{s}}d\tilde{s} \\
&+ \lambda v(\hat{y})\frac{y^*}{\hat{y}}\frac{\eta}{\eta+1}\left(\left(\frac{y^*}{\hat{y}}\right)^{-(\eta+1)} - \left(\frac{y^*}{y_r}\right)^{-(\eta+1)}\right) \\
&+ \lambda \int_{\ln(y^*/y_r)}^{\ln(y^*/y_b)} \left(\phi e^{-\tilde{s}}y^* + \frac{c\pi}{r+\xi} + \sum_{k=1}^{3} a_k(e^{-\tilde{s}}y^*/y_b)^{-\gamma_k}\right)\eta e^{-\eta\tilde{s}}d\tilde{s}.
\end{aligned}
$$

Taking the integrals,

$$
\begin{aligned}
(r+\lambda-\hat{\mu})v^* =& (1-\pi)y^* + \pi c \\
&+ \lambda v^* \frac{\eta}{\eta+1}\left(1-\left(\frac{y^*}{y_R}\right)^{-(\eta+1)}\right) \\
&+ \lambda \phi y^* \frac{\eta}{\eta+1}\left(\left(\frac{y^*}{y_R}\right)^{-(\eta+1)} - \left(\frac{y^*}{\hat{y}}\right)^{-(\eta+1)}\right) + \lambda \frac{c\pi}{r+\xi}\left(\left(\frac{y^*}{y_R}\right)^{-\eta} - \left(\frac{y^*}{\hat{y}}\right)^{-\eta}\right) \\
&+ \lambda \sum_{k=1}^{3} A_k (y^*/y_b)^{-\gamma_k} \frac{\eta}{\eta-\gamma_k}\left(\left(\frac{y^*}{y_R}\right)^{-(\eta-\gamma_k)} - \left(\frac{y^*}{\hat{y}}\right)^{-(\eta-\gamma_k)}\right) \\
&+ \lambda \frac{v(\hat{y})}{\hat{y}} y^* \frac{\eta}{\eta+1}\left(\left(\frac{y^*}{\hat{y}}\right)^{-(\eta+1)} - \left(\frac{y^*}{y_r}\right)^{-(\eta+1)}\right) \\
&+ \lambda \phi y^* \frac{\eta}{\eta+1}\left(\left(\frac{y^*}{y_r}\right)^{-(\eta+1)} - \left(\frac{y^*}{y_b}\right)^{-(\eta+1)}\right) + \lambda \frac{c\pi}{r+\xi}\left(\left(\frac{y^*}{y_r}\right)^{-\eta} - \left(\frac{y^*}{y_b}\right)^{-\eta}\right) \\
&+ \lambda \sum_{k=1}^{3} a_k (y^*/y_b)^{-\gamma_k} \frac{\eta}{\eta-\gamma_k}\left(\left(\frac{y^*}{y_r}\right)^{-(\eta-\gamma_k)} - \left(\frac{y^*}{y_b}\right)^{-(\eta-\gamma_k)}\right).
\end{aligned}
$$

Using (111),

$$
\begin{aligned}
(r+\lambda-\hat{\mu})v^* =& (1-\pi)y^* + \pi c \\
&+ \lambda v^* \frac{\eta}{\eta+1}\left(1-\left(\frac{y^*}{y_R}\right)^{-(\eta+1)}\right) \\
&+ \lambda \phi y^* \frac{\eta}{\eta+1}\left(\left(\frac{y^*}{y_R}\right)^{-(\eta+1)} - \left(\frac{y^*}{\hat{y}}\right)^{-(\eta+1)}\right) + \lambda \frac{c\pi}{r+\xi}\left(\left(\frac{y^*}{y_R}\right)^{-\eta} - \left(\frac{y^*}{\hat{y}}\right)^{-\eta}\right) \\
&+ \lambda \sum_{k=1}^{3} A_k (y^*/y_b)^{-\gamma_k} \frac{\eta}{\eta-\gamma_k}\left(\left(\frac{y^*}{y_R}\right)^{-(\eta-\gamma_k)} - \left(\frac{y^*}{\hat{y}}\right)^{-(\eta-\gamma_k)}\right) \\
&+ \lambda \frac{v(\hat{y})}{\hat{y}} y^* \frac{\eta}{\eta+1}\left(\left(\frac{y^*}{\hat{y}}\right)^{-(\eta+1)} - \left(\frac{y^*}{y_r}\right)^{-(\eta+1)}\right) \\
&+ \lambda \phi y^* \frac{\eta}{\eta+1}\left(\frac{y^*}{y_r}\right)^{-(\eta+1)} + \lambda \frac{c\pi}{r+\xi}\left(\frac{y^*}{y_r}\right)^{-\eta} \\
&+ \lambda \sum_{k=1}^{3} a_k (y^*/y_b)^{-\gamma_k} \frac{\eta}{\eta-\gamma_k}\left(\frac{y^*}{y_r}\right)^{-(\eta-\gamma_k)}.
\end{aligned}
$$

Using the expressions in (112) for $v(\hat{y})$,

$$
\begin{aligned}
(r+\lambda-\hat{\mu})v^* =&(1-\pi)y^* + \pi c \\
&+ \lambda v^* \frac{\eta}{\eta+1}\left(1-\left(\frac{y^*}{y_R}\right)^{-(\eta+1)}\right) \\
&+ \lambda\phi y^* \frac{\eta}{\eta+1}\left(\left(\frac{y^*}{y_R}\right)^{-(\eta+1)} - \left(\frac{y^*}{\hat{y}}\right)^{-(\eta+1)}\right) + \lambda\frac{c\pi}{r+\xi}\left(\left(\frac{y^*}{y_R}\right)^{-\eta} - \left(\frac{y^*}{\hat{y}}\right)^{-\eta}\right) \\
&+ \lambda\sum_{k=1}^{3} A_k (y^*/y_b)^{-\gamma_k}\frac{\eta}{\eta-\gamma_k}\left(\left(\frac{y^*}{y_R}\right)^{-(\eta-\gamma_k)} - \left(\frac{y^*}{\hat{y}}\right)^{-(\eta-\gamma_k)}\right) \\
&+ \lambda\frac{\eta}{\eta+1}\frac{y^*}{\hat{y}}\left(\phi\hat{y} + \frac{c\pi}{r+\xi} + \sum_{k=1}^{3} A_k(\hat{y}/y_b)^{-\gamma_k}\right)\left(\frac{y^*}{\hat{y}}\right)^{-(\eta+1)} \\
&- \lambda\frac{\eta}{\eta+1}\frac{y^*}{y_r}\left(\phi y_r + \frac{c\pi}{r+\xi} + \sum_{k=1}^{3} a_k(y_r/y_b)^{-\gamma_k}\right)\left(\frac{y^*}{y_r}\right)^{-(\eta+1)} \\
&+ \lambda\phi y^* \frac{\eta}{\eta+1}\left(\frac{y^*}{y_r}\right)^{-(\eta+1)} + \lambda\frac{c\pi}{r+\xi}\left(\frac{y^*}{y_r}\right)^{-\eta} \\
&+ \lambda\sum_{k=1}^{3} a_k (y^*/y_b)^{-\gamma_k}\frac{\eta}{\eta-\gamma_k}\left(\frac{y^*}{y_r}\right)^{-(\eta-\gamma_k)}.
\end{aligned}
$$

Cancelling terms (in blue),

$$
\begin{aligned}
(r+\lambda-\hat{\mu})v^* =&(1-\pi)y^* + \pi c \\
&+ \lambda v^* \frac{\eta}{\eta+1}\left(1-\left(\frac{y^*}{y_R}\right)^{-(\eta+1)}\right) \\
&+ \lambda\phi y^* \frac{\eta}{\eta+1}\left(\frac{y^*}{y_R}\right)^{-(\eta+1)} + \lambda\frac{c\pi}{r+\xi}\left(\left(\frac{y^*}{y_R}\right)^{-\eta} - \left(\frac{y^*}{\hat{y}}\right)^{-\eta}\right) \\
&+ \lambda\sum_{k=1}^{3} A_k (y^*/y_b)^{-\gamma_k}\frac{\eta}{\eta-\gamma_k}\left(\left(\frac{y^*}{y_R}\right)^{-(\eta-\gamma_k)} - \left(\frac{y^*}{\hat{y}}\right)^{-(\eta-\gamma_k)}\right) \\
&+ \lambda\frac{\eta}{\eta+1}\frac{y^*}{\hat{y}}\left(\frac{c\pi}{r+\xi} + \sum_{k=1}^{3} A_k(\hat{y}/y_b)^{-\gamma_k}\right)\left(\frac{y^*}{\hat{y}}\right)^{-(\eta+1)} \\
&- \lambda\frac{\eta}{\eta+1}\frac{y^*}{y_r}\left(\frac{c\pi}{r+\xi} + \sum_{k=1}^{3} a_k(y_r/y_b)^{-\gamma_k}\right)\left(\frac{y^*}{y_r}\right)^{-(\eta+1)} \\
&+ \lambda\frac{c\pi}{r+\xi}\left(\frac{y^*}{y_r}\right)^{-\eta} + \lambda\sum_{k=1}^{3} a_k (y^*/y_b)^{-\gamma_k}\frac{\eta}{\eta-\gamma_k}\left(\frac{y^*}{y_r}\right)^{-(\eta-\gamma_k)}.
\end{aligned}
$$

Using the notation $x^* = y^*/y_b$, $x_R = y_R/y_b$, $\hat{x} = \hat{y}/y_b$, $x_r = y_r/y_b$,

$$
\begin{aligned}
(r+\lambda-\hat{\mu})v^* =& (1-\pi)y^* + \pi c \\
&+ \lambda v^* \frac{\eta}{\eta+1}\left(1-\left(\frac{x^*}{x_R}\right)^{-(\eta+1)}\right) \\
&+ \lambda \phi y^* \frac{\eta}{\eta+1}\left(\frac{x^*}{x_R}\right)^{-(\eta+1)} + \lambda \frac{c\pi}{r+\xi}\left(\left(\frac{x^*}{x_R}\right)^{-\eta} - \left(\frac{x^*}{\hat{x}}\right)^{-\eta}\right) \\
&+ \lambda \sum_{k=1}^{3} A_k (x^*)^{-\gamma_k} \frac{\eta}{\eta-\gamma_k}\left(\left(\frac{x^*}{x_R}\right)^{-(\eta-\gamma_k)} - \left(\frac{x^*}{\hat{x}}\right)^{-(\eta-\gamma_k)}\right) \\
&+ \lambda \frac{\eta}{\eta+1}\frac{c\pi}{r+\xi}\left(\frac{x^*}{\hat{x}}\right)^{-\eta} + \lambda \frac{\eta}{\eta+1}\sum_{k=1}^{3} A_k \hat{x}^{-\gamma_k}\left(\frac{x^*}{\hat{x}}\right)^{-\eta} \\
&- \lambda \frac{\eta}{\eta+1}\frac{c\pi}{r+\xi}\left(\frac{x^*}{x_r}\right)^{-\eta} - \lambda \frac{\eta}{\eta+1}\sum_{k=1}^{3} a_k x_r^{-\gamma_k}\left(\frac{x^*}{x_r}\right)^{-\eta} \\
&+ \lambda \frac{c\pi}{r+\xi}\left(\frac{x^*}{x_r}\right)^{-\eta} + \lambda \sum_{k=1}^{3} a_k (x^*)^{-\gamma_k} \frac{\eta}{\eta-\gamma_k}\left(\frac{x^*}{x_r}\right)^{-(\eta-\gamma_k)}.
\end{aligned}
$$

Simplifying,

$$
\begin{aligned}
(r+\lambda-\hat{\mu})v^* =& (1-\pi)y^* + \pi c \\
&+ \lambda v^* \frac{\eta}{\eta+1}\left(1-\left(\frac{x^*}{x_R}\right)^{-(\eta+1)}\right) \\
&+ \lambda \phi y^* \frac{\eta}{\eta+1}\left(\frac{x^*}{x_R}\right)^{-(\eta+1)} + \lambda \frac{c\pi}{r+\xi}\left(\left(\frac{x^*}{x_R}\right)^{-\eta} - \left(\frac{x^*}{\hat{x}}\right)^{-\eta}\right) \\
&+ \lambda \sum_{k=1}^{3} A_k (x^*)^{-\gamma_k} \frac{\eta}{\eta-\gamma_k}\left(\frac{x^*}{x_R}\right)^{-(\eta-\gamma_k)} \\
&+ \lambda \frac{\eta}{\eta+1}\frac{c\pi}{r+\xi}\left(\frac{x^*}{\hat{x}}\right)^{-\eta} - \lambda \sum_{k=1}^{3} A_k \frac{\eta(\gamma_k+1)}{(\eta-\gamma_k)(\eta+1)}(x^*)^{-\eta}\hat{x}^{\eta-\gamma_k} \\
&- \lambda \frac{\eta}{\eta+1}\frac{c\pi}{r+\xi}\left(\frac{x^*}{x_r}\right)^{-\eta} \\
&+ \lambda \frac{c\pi}{r+\xi}\left(\frac{x^*}{x_r}\right)^{-\eta} + \lambda \sum_{k=1}^{3} a_k \frac{\eta(\gamma_k+1)}{(\eta-\gamma_k)(\eta+1)}(x^*)^{-\eta} x_r^{\eta-\gamma_k},
\end{aligned}
$$

and further,

$$
\begin{aligned}
(r+\lambda-\hat{\mu})v^* =&(1-\pi)y^* + \pi c \\
&+ \lambda v^* \frac{\eta}{\eta+1}\left(1-\left(\frac{x^*}{x_R}\right)^{-(\eta+1)}\right) \\
&+ \lambda\left(\frac{x^*}{x_R}\right)^{-\eta}\left(\phi y_R \frac{\eta}{\eta+1} + \frac{c\pi}{r+\xi} + \sum_{k=1}^{3} A_k \frac{\eta}{\eta-\gamma_k} x_R^{-\gamma_k}\right) \\
&- \frac{\lambda}{\eta+1}\frac{c\pi}{r+\xi}\left(\frac{x^*}{\hat{x}}\right)^{-\eta} - \lambda \sum_{k=1}^{3} A_k \frac{\eta(\gamma_k+1)}{(\eta-\gamma_k)(\eta+1)}(x^*)^{-\eta}\hat{x}^{\eta-\gamma_k} \\
&+ \frac{\lambda}{\eta+1}\frac{c\pi}{r+\xi}\left(\frac{x^*}{x_r}\right)^{-\eta} + \lambda \sum_{k=1}^{3} a_k \frac{\eta(\gamma_k+1)}{(\eta-\gamma_k)(\eta+1)}(x^*)^{-\eta} x_r^{\eta-\gamma_k}.
\end{aligned}
$$

Using (113)

$$
\begin{aligned}
(r+\lambda-\hat{\mu})v^* =&(1-\pi)y^* + \pi c \\
&+ \lambda v^* \frac{\eta}{\eta+1}\left(1-\left(\frac{x^*}{x_R}\right)^{-(\eta+1)}\right) \\
&+ \lambda\left(\frac{x^*}{x_R}\right)^{-\eta}\left(\phi y_R \frac{\eta}{\eta+1} + \frac{c\pi}{r+\xi} + \sum_{k=1}^{3} A_k \frac{\eta}{\eta-\gamma_k} x_R^{-\gamma_k}\right)
\end{aligned}
$$

Using $\mu = \hat{\mu} - \lambda/(\eta+1)$,

$$
\left(r-\mu+\frac{\lambda\eta}{\eta+1}\left(\frac{x^*}{x_R}\right)^{-(\eta+1)}\right)v^* = (1-\pi)y^* + \pi c + \lambda\left(\frac{x^*}{x_R}\right)^{-\eta}\left(\phi y_R \frac{\eta}{\eta+1} + \frac{c\pi}{r+\xi} + \sum_{k=1}^{3} A_k \frac{\eta}{\eta-\gamma_k} x_R^{-\gamma_k}\right), \tag{117}
$$

which is the last condition on a_ks, A_ks, v^*, and y_b.

C.1.2 Results

We next compare numerically the optimal time-consistent complex-repurchase targeted ICR policy to the optimal time-consistent targeted ICR policy. We do so for all parameters analyzed in Section 5.2. Table 7 reports the results. Complex-repurchase targeted ICR policies produce only a small improvement in the firm value over targeted ICR policies. Across parameter specifications that we consider, the gain in the firm value is at most 0.45% and in many cases (e.g., for all parameters of the baseline specification), the gain is less than 0.01%. This observation suggest that even though the targeted ICR policy might not be optimal, it might be very close to the optimal policy. Table 7 indicates that the targeted ICR policy is closer to the optimal complex-repurchase targeted ICR policy, whenever tax benefits are not too high, debt maturity is shorter, and Brownian volatility is smaller.

Further, in the optimal compex-repurchase policy, $y_r = \hat{y}$. This means that when at the ICR target y^*, the equity holders compensate moderate jumps (up to y_R) with repurchases. For sufficiently large downmward jumps (that bring y_t below y_R), the firm does not do large repurchases (of order greater than dt). However, it might do small repurchases of order dt at boundary $\hat{y}$ (which coincides with y_r) to compensate for Brownian shocks. Thus, optimal complex-repurchase targeted ICR policies can be interpreted as having two ICR targets: the higher ICR target (y^*) for which the equity holders compensate moderate jumps and the lower ICR target ($\hat{y}$) for which the equity holders compensate only Brownian downward shocks.

We next report the parameters of the optimal complex-repurchase targeted ICR policy. Tables 8, 9, and 10 present the results for the base case, high-tax benefits case, and the case of console. We find that the targeted ICR y^*, the leverage ratio at the target, the spread at the target, ICR targets ($\hat{y}$for TICR and y^* for complex-repurchase TICR policies), and default boundaries are very close to those for the optimal targeted ICR policy in Tables 2, 3, and 4, respectively. Further, differences in the ICR targets. The difference in the leverage ratios and credit spreads in distress regions across two classes of policies is somewhat larger (although still quite small). Further, the lower boundary of the repurchase region y_r in the TICR policy lies in between the lower boundary of the repurchase region y_R and the reflecting repurchase boundary $\hat{y}$ in the complex-repurchase TICR policy.

C.2 Dual Targeted ICR Policies

Next, we consider policies with two ICR targets, which we call "dual targeted ICR policies." Formally, there is a lower ICR target $\hat{y}$ and an upper ICR target $y^* > \hat{y}$. When at target $\hat{y}$, the firm manages its liabilities to stay at the target $\hat{y}$ by compensating all positive shocks to y_t with debt issuances and all negative shocks that fall into the repurchase region $[y_r, \hat{y})$ with debt repuchases. When at target y^*, the firm manages its liabilities to stay at the target y^* by compensating all positive shocks to y_t with debt issuances and all negative shocks that fall into the repurchase region $[y_R, y^*)$ with debt repuchases (see Figure 9). We suppose that $\hat{y} \leq y_R$. In other regions of y, the firm does not issue/repurchase debt. As before, y_b is the default threshold.

Notice that the only difference of the dual targeted ICR policy from the complex-repurchase targeted ICR policy is that at the upper boundary of the lower repurchase region $\hat{y}$, the firm issues debt, hence, preventing y_t going above $\hat{y}$. This way, after the first negative drop in y_t that brings it below $\hat{y}$, the firm never returns to its original ICR target y^* in the future. In contrast, under the complex-repurchase targeted ICR policy, the state can drop below $\hat{y}$, but then recover and reach y^* again in the future.

Parameters	Difference in % between the optimal complex-repurchase TICR (super-indexed CR) and the optimal TICR policies (super-indexed $TICR$)				$\frac{\hat{y}^{CR}}{y_r^{CR}}-1$	$\frac{y^{*CR}-y_R^{CR}}{y^{*CR}-y_r^{CR}}$
	Firm value	$y^{*CR}/\hat{y}^{TICR}-1$	$y_r^{CR}/y_r^{TICR}-1$	$y_b^{CR}/y_b^{TICR}-1$		
$\pi=10\%$, $\xi=1/10$						
$\zeta=-25\%$, $\lambda=1/3$, $\sigma=25\%$	0.0015%	-0.02%	-2.85%	0%	0%	91.87%
$\zeta=-20\%$, $\lambda=1/3$, $\sigma=25\%$	0.0017%	-0.02%	-1.53%	0%	0%	92.86%
$\zeta=-30\%$, $\lambda=1/3$, $\sigma=25\%$	0.0017%	0.04%	-6.08$	0%	0%	89.05%
$\zeta=-25\%$,$\lambda=1/4$,$\sigma=25\%$	0.0028%	0.28%	-3.49%	0%	0%	90.02%
$\zeta=-25\%$, $\lambda=1/2$, $\sigma=25\%$	0.0008%	0.13%	-2.68%	0%	0%	92.82%
$\zeta=-25\%$,$\lambda=1/3$, $\sigma=10\%$	0%	0%	-0.05%	0%	0%	99.68%
$\zeta=-25\%$, $\lambda=1/3$, $\sigma=40\%$	0.006%	-0.05%	-3.69%	0%	0%	76.2%
$\pi=40\%$, $\xi=1/10$						
$\zeta=-25\%$, $\lambda=1/3$, $\sigma=25\%$	0.20%	-0.95%	-20.02%	1.67%	0%	73.4%
$\zeta=-20\%$, $\lambda=1/3$, $\sigma=25\%$	0.22%	-0.69%	-9.18%	0.47%	0%	76.19%
$\zeta=-30\%$, $\lambda=1/3$, $\sigma=25\%$	0.13%%	-0.97%	-34.43%	3.16%	0%	69.84%
$\zeta=-25\%$,$\lambda=1/4$,$\sigma=25\%$	0.45%	-1.53%	-18.85%	1.06%	0%	65.85%
$\zeta=-25\%$, $\lambda=1/2$, $\sigma=25\%$	0.02%	-0.10%	-4.26%	0.05%	0%	90.67%
$\zeta=-25\%$,$\lambda=1/3$, $\sigma=10\%$	0.0008%	0%	-1.03%	-0.01%	0%	97.37%
$\zeta=-25\%$, $\lambda=1/3$, $\sigma=40\%$	0.21%	-0.67%	-7.16%	-0.27%	0%	67.35%
$\pi=10\%$, $\xi=0$						
$\zeta=-25\%$, $\lambda=1/3$, $\sigma=25\%$	0.0205%	1.87%	-15.89%	0%	0%	14.84%
$\zeta=-20\%$, $\lambda=1/3$, $\sigma=25\%$	0.0149%	-2.31%	-5.81%	0%	0%	39.91%
$\zeta=-30\%$, $\lambda=1/3$, $\sigma=25\%$	0.0163%	2.24%	-21.63%	0%	0%	23.24%
$\zeta=-25\%$,$\lambda=1/4$,$\sigma=25\%$	0.0276%	3.12%	-8.35%	0%	0%	11.49%
$\zeta=-25\%$, $\lambda=1/2$, $\sigma=25\%$	0.0143%	1.26%	-18.4%	0%	0%	37.65%
$\zeta=-25\%$,$\lambda=1/3$, $\sigma=10\%$	0.0010%	0.15%	-6.62%	0%	0%	83.47%
$\zeta=-25\%$, $\lambda=1/3$, $\sigma=40\%$	0.0058%	0.86%	-0.83%	0%	0%	5.46%

Table 7: ***Comparison of the optimal complex-repurchase targeted ICR policy to the optimal targeted ICR policy***

Notes: Super-script CR refers to the complex-repurchase targeted ICR policy and superscript $TICR$ refers to the targeted ICR policy.

Parameters: $c=8\%$, $\mu=2\%$, $r=5\%$.

	at target		in distress		optimal policy					MPE
	leverage ratio	credit spread	median leverage ratio	median credit spread	y^*/c	y_R/c	$\hat{y}/c$	y_r/c	y_b/c	default ICR
Base case	19%	34 bps	55%	160 bps	2.55	1.14	1.02	1.02	0.34	0.34
$\zeta = -20\%$	27%	23 bps	70%	179 bps	1.73	0.82	0.75	0.75	0.36	0.36
$\zeta = -25\%$	19%	34 bps	55%	160 bps	2.55	1.14	1.02	1.02	0.34	0.34
$\zeta = -30\%$	12%	48 bps	41%	169 bps	3.86	1.70	1.44	1.44	0.33	0.33
$\lambda = 1/4$	21%	31 bps	63%	178 bps	2.23	1.01	0.88	0.88	0.35	0.35
$\lambda = 1/3$	19%	34 bps	55%	160 bps	2.55	1.14	1.02	1.02	0.34	0.34
$\lambda = 1/2$	15%	39 bps	44%	145 bps	3.20	1.43	1.29	1.29	0.32	0.32
$\sigma = 10\%$	20%	30 bps	62%	135 bps	2.41	0.90	0.90	0.90	0.40	0.40
$\sigma = 25\%$	19%	34 bps	55%	160 bps	2.55	1.14	1.02	1.02	0.34	0.34
$\sigma = 40\%$	16%	43 bps	39%	161 bps	3.05	1.83	1.45	1.45	0.27	0.27

Table 8: ***Comparative statics for optimal complex-repurchase policy in the base case***
Baseline parameters: $\xi = 1/10$, $\pi = 10\%$, $c = 8\%$, $\sigma = 25\%$, $\lambda = 1/3$, $\zeta = -25\%$, $\mu = 2\%$, $r = 5\%$.

	at target		in distress		optimal policy					MPE
	leverage ratio	credit spread	median leverage ratio	median credit spread	y^*/c	y_R/c	$\hat{y}/c$	y_r/c	y_b/c	default ICR
High benefits	37%	143 bps	94%	1033 bps	1.40	0.76	0.53	0.53	0.40	0.46
$\zeta = -20\%$	44%	100 bps	94%	787 bps	1.10	0.65	0.51	0.51	0.39	0.46
$\zeta = -25\%$	37%	143 bps	94%	1033 bps	1.40	0.76	0.53	0.53	0.40	0.46
$\zeta = -30\%$	31%	198 bps	96%	1521 bps	1.71	0.87	0.50	0.50	0.40	0.42
$\lambda = 1/4$	40%	140 bps	95%	1077 bps	1.25	0.76	0.51	0.51	0.40	0.45
$\lambda = 1/3$	37%	143 bps	94%	1033 bps	1.40	0.76	0.53	0.53	0.40	0.46
$\lambda = 1/2$	33%	147 bps	85%	534 bps	1.61	0.75	0.66	0.66	0.39	0.42
$\sigma = 10\%$	36%	133 bps	81%	301 bps	1.46	0.73	0.71	0.71	0.45	0.51
$\sigma = 25\%$	37%	143 bps	94%	1033 bps	1.40	0.76	0.53	0.53	0.40	0.46
$\sigma = 40\%$	39%	160 bps	89%	840 bps	1.27	0.80	0.57	0.57	0.34	0.35

Table 9: ***Comparative statics for optimal complex-repurchase policy in the high tax benefits case***
Parameters: $\xi = 0.1$, $\pi = 40\%$, $c = 8\%$, $\sigma = 25\%$, $\lambda = 1/3$, $\zeta = -25\%$, $\mu = 2\%$, $r = 5\%$.

	at target		in distress		optimal policy					MPE
	leverage ratio	credit spread	median leverage ratio	median credit spread	y^*/c	y_R/c	$\hat{y}/c$	y_r/c	y_b/c	default ICR
Console	10%	81 bps	22%	125 bps	5.42	5.07	3.09	3.09	0.26	0.26
$\zeta = -20\%$	15%	62 bps	24%	89 bps	3.79	3.36	2.71	2.71	0.27	0.27
$\zeta = -25\%$	10%	81 bps	22%	125 bps	5.42	5.07	3.09	3.09	0.26	0.26
$\zeta = -30\%$	7%	103 bps	18%	162 bps	7.51	6.63	3.72	3.72	0.24	0.24
$\lambda = 1/4$	13%	69 bps	23%	108 bps	4.60	4.41	2.93	2.93	0.27	0.27
$\lambda = 1/3$	10%	81 bps	22%	125 bps	5.42	5.07	3.09	3.09	0.26	0.26
$\lambda = 1/2$	8%	97 bps	17%	144 bps	7.09	5.84	3.77	3.77	0.23	0.23
$\sigma = 10\%$	14%	56 bps	28%	90 bps	4.33	2.79	2.48	2.48	0.35	0.35
$\sigma = 25\%$	10%	81 bps	22%	125 bps	5.42	5.07	3.09	3.09	0.26	0.26
$\sigma = 40\%$	7%	119 bps	11%	152 bps	7.45	7.36	5.89	5.89	0.18	0.18

Table 10: ***Comparative statics for optimal complex-repurchase policy in the case of console***

Parameters: $\xi = 0$, $\pi = 10\%$, $c = 8\%$, $\sigma = 25\%$, $\lambda = 1/3$, $\zeta = -20\%$, $\mu = 2\%$, $r = 5\%$.

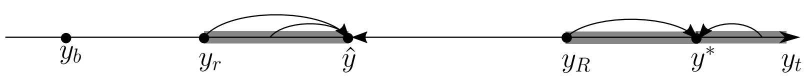

Figure 9: ***Dual targeted ICR policy thresholds***

The gray region is the action region where the firm issues or repurchases debt. Arrows indicate where the state y_t transitions when it falls into the action region.

C.2.1 Derivation of Value Functions

We next characterize the debt price, equity value, and enterprise value functions.

Debt price We consider the normalized debt price function of the following form:

$$p(y) = \begin{cases} 0, & y \in (0, y_b], \\ \frac{c+\xi}{r+\xi}\left(1 + \sum_{k=1}^{3} b_k (y/y_b)^{-\gamma_k}\right), & y \in [y_b, y_r], \\ \hat{p}, & y \in [y_r, \hat{y}], \\ \frac{c+\xi}{r+\xi}\left(1 + \sum_{k=1}^{3} B_k (y/y_b)^{-\gamma_k}\right), & y \in [\hat{y}, y_R], \\ p^*, & y \in [y_R, \infty). \end{cases}$$

Normalized debt price function for the dual targeted ICR policy satisfies the same conditions below $\hat{y}$ as for the targeted ICR policy in Online Appendix B.0.3. Specifically, the coefficients b_ks and $\hat{p}$ satisfy the following conditions:

$$p(y_b) = 0 : \sum_{k=1}^{3} b_k = -1, \tag{118}$$

$$p(y_r) = p(\hat{y}) \equiv \hat{p} : \frac{c+\xi}{r+\xi}\left(1 + \sum_{k=1}^{3} b_k x_r^{-\gamma_k}\right) = \frac{c+\xi}{r+\xi}\left(1 + \sum_{k=1}^{3} B_k \hat{x}^{-\gamma_k}\right) = \hat{p}, \tag{119}$$

$$\text{analogue of eq. (67)} : 1 + \sum_{k=1}^{3} b_k \frac{\eta}{\eta - \gamma_k} = 0, \tag{120}$$

$$\text{analgoue of eq. (71)} : x_r^{\eta} \sum_{k=1}^{3} b_k \frac{\gamma_k}{\eta - \gamma_k} x_r^{-\gamma_k} - \hat{x}^{\eta} \sum_{k=1}^{3} B_k \frac{\gamma_k}{\eta - \gamma_k} \hat{x}^{-\gamma_k} = 0, \tag{121}$$

$$\text{analogue of eq. (96)} : \sum_{k=1}^{3} b_k x_r^{-\gamma_k} \left(r + \xi - \frac{\lambda \gamma_k}{\eta - \gamma_k} \left(\frac{\hat{x}}{x_r}\right)^{-\eta}\right) = 0, \tag{122}$$

$$p(y_R) = p^* : \frac{c+\xi}{r+\xi}\left(1 + \sum_{k=1}^{3} B_k x_R^{-\gamma_k}\right) = p^*, \tag{123}$$

Note that the only difference from (102)-(107) is in the equation (106) due to the fact that $\hat{y}$ is the new ICR target under the dual targeted ICR policy. Finally, by the same argument as in (109), we have the following condition

$$\sum_{k=1}^{3} B_k x_R^{-\gamma_k} \left(r + \xi - \frac{\lambda \gamma_k}{\eta - \gamma_k} \left(\frac{x^*}{x_R}\right)^{-\eta}\right) = 0, \tag{124}$$

which is the last equation that pins down B_ks and p^*.

Equity Value and Enterprise Value Functions The normalized equity value functions takes the form:

$$
e(y) = \begin{cases} 0, & y \in (0, y_b], \\ \phi y - \rho + \sum_{k=1}^{3} c_k (y/y_b)^{-\gamma_k}, & y \in [y_b, y_r], \\ \left(\frac{y}{\hat{y}} - 1\right) \hat{p} + \frac{y}{\hat{y}} \hat{e}, & y \in [y_r, \hat{y}], \\ \phi y - \rho + \sum_{k=1}^{3} C_k (y/y_b)^{-\gamma_k}, & y \in [\hat{y}, y_R], \\ \left(\frac{y}{y^*} - 1\right) p^* + \frac{y}{y^*} e^*, & y \in [y_R, \infty). \end{cases}
$$

The normalized enterprise value function $v(y) = e(y) + p(y)$ takes the form

$$
v(y) = \begin{cases} 0, & y \in (0, y_b], \\ \phi y + \frac{c\pi}{r+\xi} + \sum_{k=1}^{3} a_k (y/y_b)^{-\gamma_k}, & y \in [y_b, y_r], \\ \frac{\hat{v}}{\hat{y}} y, & y \in [y_r, \hat{y}], \\ \phi y + \frac{c\pi}{r+\xi} + \sum_{k=1}^{3} A_k (y/y_b)^{-\gamma_k}, & y \in [\hat{y}, y_R], \\ \frac{v^*}{y^*} y, & y \in [y_R, \infty); \end{cases}
$$

where $v^* \equiv v(y^*)$ and $\hat{v} \equiv v(\hat{y})$. The coefficients of functions p, e, and v are related by $a_k = c_k + \frac{c+\xi}{r+\xi} b_k$ and $A_k = C_k + \frac{c+\xi}{r+\xi} B_k$, and $v^* = p^* + e^*$. Thus, the function e will be determined once we determine functions v and p. Coefficients a_ks, $A_k s$, v^*, and y_b satisfy:

$$
v(y_b) = 0 : \phi y_b + \frac{c\pi}{r+\xi} + \sum_{k=1}^{3} a_k = 0, \tag{125}
$$

$$
\text{analogue of eq. (84)} : \frac{\phi\eta}{\eta+1} y_b + \frac{c\pi}{r+\xi} + \sum_{k=1}^{3} a_k \frac{\eta}{\eta - \gamma_k} = 0, \tag{126}
$$

$$
\frac{v(y_r)}{y_r} = \frac{v(\hat{y})}{\hat{y}} : \frac{c\pi}{(r+\xi)x_r} + \sum_{k=1}^{3} a_k x_r^{-\gamma_k - 1} = \frac{c\pi}{(r+\xi)\hat{x}} + \sum_{k=1}^{3} A_k \hat{x}^{-\gamma_k - 1} = \frac{\hat{v}}{\hat{y}}, \tag{127}
$$

$$
\text{analogue of eq. (88)} : \sum_{k=1}^{3} \left\{ \frac{\eta(1+\gamma_k)}{\eta - \gamma_k} \left[a_k x_r^{\eta - \gamma_k} - A_k \hat{x}^{\eta - \gamma_k} \right] \right\} = \frac{c\pi}{r+\xi} (\hat{x}^\eta - x_r^\eta), \tag{128}
$$

$$
\frac{v^*}{y^*} = \frac{v(y_R)}{y_R} : \frac{x_R}{x^*} v^* = \phi x_R y_b + \frac{c\pi}{r+\xi} + \sum_{k=1}^{3} A_k x_R^{-\gamma_k}, \tag{129}
$$

$$
e'(y_b) = 0 : \phi y_b - \sum_{k=1}^{3} \left(a_k - \frac{c+\xi}{r+\xi} b_k \right) \gamma_k = 0, \tag{130}
$$

$$
\text{eq. (96) and eq. (101)} : \frac{c\pi}{r+\xi} + \sum_{k=1}^{3} A_k (1+\gamma_k) \hat{x}^{-\gamma_k} = 0. \tag{131}
$$

These are the same equations as equations (110)-(115). Given that $\hat{y}$ is the new ICR target it satisfies the equation (101). Rearranging,

$$\left(r-\mu+\frac{\lambda\eta}{\eta+1}\left(\hat{x}/x_r\right)^{-(\eta+1)}\right)\hat{v}=(1-\pi)(\hat{x}y_b-c)-\xi+\hat{p}(r+\xi)$$
$$+\lambda(\hat{x}/x_r)^{-\eta}\left(\frac{\phi\eta x_r}{\eta+1}y_b+\hat{p}-\rho+\sum_{k=1}^{3}\frac{c_k\eta}{\eta-\gamma_k}x_r^{-\gamma_k}\right).$$

Using (119) to substitute for $\hat{p}$,

$$\left(r-\mu+\frac{\lambda\eta}{\eta+1}\left(\hat{x}/x_r\right)^{-(\eta+1)}\right)\hat{v}=(1-\pi)(\hat{x}y_b-c)-\xi+(c+\xi)\left(1+\sum_{k=1}^{3}b_kx_r^{-\gamma_k}\right)$$
$$+\lambda(\hat{x}/x_r)^{-\eta}\left(\frac{\phi\eta x_r}{\eta+1}y_b+\frac{c+\xi}{r+\xi}\left[1+\sum_{k=1}^{3}b_kx_r^{-\gamma_k}\right]-\rho+\sum_{k=1}^{3}\frac{c_k\eta}{\eta-\gamma_k}x_r^{-\gamma_k}\right),$$

or simplifying,

$$\left(r-\mu+\frac{\lambda\eta}{\eta+1}\left(\hat{x}/x_r\right)^{-(\eta+1)}\right)\hat{v}=(1-\pi)\hat{y}+\pi c+(c+\xi)\sum_{k=1}^{3}b_kx_r^{-\gamma_k}$$
$$+\lambda(\hat{x}/x_r)^{-\eta}\left(\frac{\phi\eta x_r}{\eta+1}y_b+\frac{c\pi}{r+\xi}+\frac{c+\xi}{r+\xi}\sum_{k=1}^{3}b_kx_r^{-\gamma_k}+\sum_{k=1}^{3}\frac{c_k\eta}{\eta-\gamma_k}x_r^{-\gamma_k}\right).$$

Using (122),

$$\left(r-\mu+\frac{\lambda\eta}{\eta+1}\left(\hat{x}/x_r\right)^{-(\eta+1)}\right)\hat{v}=(1-\pi)\hat{y}+\pi c+\lambda(\hat{x}/x_r)^{-\eta}\frac{c+\xi}{r+\xi}\sum_{k=1}^{3}\frac{b_k\gamma_k}{\eta-\gamma_k}x_r^{-\gamma_k}$$
$$+\lambda(\hat{x}/x_r)^{-\eta}\left(\frac{\phi\eta x_r}{\eta+1}y_b+\frac{c\pi}{r+\xi}+\frac{c+\xi}{r+\xi}\sum_{k=1}^{3}b_kx_r^{-\gamma_k}+\sum_{k=1}^{3}\frac{c_k\eta}{\eta-\gamma_k}x_r^{-\gamma_k}\right),$$

or simplifying,

$$\left(r-\mu+\frac{\lambda\eta}{\eta+1}\left(\hat{x}/x_r\right)^{-(\eta+1)}\right)\hat{v}=(1-\pi)\hat{y}+\pi c+\lambda(\hat{x}/x_r)^{-\eta}\left(\frac{\phi\eta x_r}{\eta+1}y_b+\frac{c\pi}{r+\xi}+\sum_{k=1}^{3}a_k\frac{\eta}{\eta-\gamma_k}x_r^{-\gamma_k}\right).\tag{132}$$

Finally, by the same argument as in (117), we have the following condition

$$\left(r-\mu+\frac{\lambda\eta}{\eta+1}\left(\frac{x^*}{x_R}\right)^{-(\eta+1)}\right)v^*=(1-\pi)y^*+\pi c+\lambda\left(\frac{x^*}{x_R}\right)^{-\eta}\left(\phi y_R\frac{\eta}{\eta+1}+\frac{c\pi}{r+\xi}+\sum_{k=1}^{3}A_k\frac{\eta}{\eta-\gamma_k}x_R^{-\gamma_k}\right),\tag{133}$$

which is the last condition on a_ks, A_ks, $\hat{v}$, v^*, and y_b.

C.2.2 Results

We compare numerically the optimal time-consistent dual targeted ICR policy to the optimal time-consistent targeted ICR policy. We do so for all parameters analyzed in Section 5.2. Table 11 reports the results. Dual targeted ICR policies produce only a small improvement in the firm value over targeted ICR policies. Across parameter specifications that we consider, the gain in the firm value is at most 0.042% and in many cases (e.g., for all parameters of the baseline specification), the gain is less than 0.005%. This again suggests that the targeted ICR policy might be very close to the optimal policy. Table 11 indicates that the targeted ICR policy is closer to the optimal dual targeted ICR policy, whenever tax benefits are not too high, debt maturity is shorter, and Brownian volatility is smaller.

Further, we find that for all parameter specifications, $\hat{y} = y_R$. This means that the optimal dual targeted ICR policy takes the following form. The firm initially chooses a higher ICR target y^* and compensates all negative downward jump with repurtchases as long as y_t is above $\hat{y}$. If the ICR drops below $\hat{y}$, then from this moment on, the firm follows the targeted ICR policy with a new ICR target of $\hat{y}$. Table 11 shows that in the base and high tax benefit case, the lower ICR target and the lower repurchase boundaries are withing 2% of those in the optimal TICR policy. Because these optimal policy thresholds are very close in two classes of policies, the comparative statics in Section 5.2 do not change. For the case of console, we verify in Table 12 that the comparative statics with dual targeted ICR policies are qualitatively and quantitatively to those with targeted ICR policies in Section 5.2.

Parameters	Difference in % between the optimal dual TICR (super-indexed $2TICR$) and the optimal TICR policies (super-indexed $TICR$)				$y^{*2TICR}/\hat{y}^{2TICR}-1$
	Firm value	$\hat{y}^{2TICR}/\hat{y}^{TICR}-1$	$y_r^{2TICR}/y_r^{TICR}-1$	$y_b^{2TICR}/y_b^{TICR}-1$	
$\pi=10\%$, $\xi=1/10$					
$\zeta=-25\%$, $\lambda=1/3$, $\sigma=25\%$	0.002%	-0.69%	-0.27%	0%	5.73%
$\zeta=-20\%$, $\lambda=1/3$, $\sigma=25\%$	0.001%	-0.37%	-0.11%	0%	3.47%
$\zeta=-30\%$, $\lambda=1/3$, $\sigma=25\%$	0.002%	-1.05%	-0.50%	0%	8.19%
$\zeta=-25\%$,$\lambda=1/4$,$\sigma=25\%$	0.002%	-0.35%	-0.13%	0%	4.58%
$\zeta=-25\%$, $\lambda=1/2$, $\sigma=25\%$	0.002%	-0.72%	-0.31%	0%	8.45%
$\zeta=-25\%$,$\lambda=1/3$, $\sigma=10\%$	0.001%	-0.49%	-0.20%	0%	4.26%
$\zeta=-25\%$, $\lambda=1/3$, $\sigma=40\%$	0.004%	-1.22%	-0.50%	0%	9.13%
$\pi=40\%$, $\xi=1/10$					
$\zeta=-25\%$, $\lambda=1/3$, $\sigma=25\%$	0.028%	-1%	-0.29%	-0.10%	7.67%
$\zeta=-20\%$, $\lambda=1/3$, $\sigma=25\%$	0.027%	-0.69%	-0.17%	-0.09%	6.01%
$\zeta=-30\%$, $\lambda=1/3$, $\sigma=25\%$	0.028%	-1.36%	-0.30%	-0.16%	9.21%
$\zeta=-25\%$,$\lambda=1/4$,$\sigma=25\%$	0.041%	-1.29%	-0.41%	-0.12%	7.95%
$\zeta=-25\%$, $\lambda=1/2$, $\sigma=25\%$	0.018%	-0.72%	-0.19%	-0.09%	7.65%
$\zeta=-25\%$,$\lambda=1/3$, $\sigma=10\%$	0.021%	-0.85%	-0.28%	-0.08%	6.72%
$\zeta=-25\%$, $\lambda=1/3$, $\sigma=40\%$	0.042%	-1.22%	-0.30%	-0.16%	9.05%
$\pi=10\%$, $\xi=0$					
$\zeta=-25\%$, $\lambda=1/3$, $\sigma=25\%$	0.022%	-9.22%	-7.52%	0%	32.92%
$\zeta=-20\%$, $\lambda=1/3$, $\sigma=25\%$	0.043%	-11.33%	-9.10%	0%	35.45%
$\zeta=-30\%$, $\lambda=1/3$, $\sigma=25\%$	0.012%	-8.08%	-6.66%	0%	31.08%
$\zeta=-25\%$,$\lambda=1/4$,$\sigma=25\%$	0.031%	-11.08%	-8.84%	0%	33.56&
$\zeta=-25\%$, $\lambda=1/2$, $\sigma=25\%$	0.013%	-7.20%	-6.02%	0%	33.19%
$\zeta=-25\%$,$\lambda=1/3$, $\sigma=10\%$	0.013%	-5.06%	-4.24%	0%	21.09%
$\zeta=-25\%$, $\lambda=1/3$, $\sigma=40\%$	0.033%	-16.62%	-13.31%	0%	52.33%

Table 11: ***Comparison of the optimal dual targeted ICR policy to the optimal targeted ICR policy***
Parameters: $c=8\%$, $\mu=2\%$, $r=5\%$.

	at target		in distress		optimal policy				MPE
	leverage ratio	credit spread	median leverage ratio	median credit spread	y^*/c	$\hat{y}/c$	y_r/c	y_b/c	default ICR
Console	12%	85 bps	20%	114 bps	6.41	4.83	3.40	0.26	0.26
$\zeta = -20\%$	17%	69 bps	25%	94 bps	4.66	3.44	2.62	0.27	0.27
$\zeta = -25\%$	12%	85 bps	20%	114 bps	6.41	4.83	3.40	0.26	0.26
$\zeta = -30\%$	8%	106 bps	16%	141 bps	8.85	6.75	4.43	0.24	0.24
$\lambda = 1/4$	14%	78 bps	23%	109 bps	5.30	3.97	2.91	0.27	0.27
$\lambda = 1/3$	12%	85 bps	20%	114 bps	6.41	4.83	3.40	0.26	0.26
$\lambda = 1/2$	8%	99 bps	15%	127 bps	8.65	6.50	4.34	0.23	0.23
$\sigma = 10\%$	14%	60 bps	28%	86 bps	4.97	4.10	2.54	0.35	0.35
$\sigma = 25\%$	12%	85 bps	20%	114 bps	6.41	4.83	3.40	0.26	0.26
$\sigma = 40\%$	8%	137 bps	12%	170 bps	9.37	6.15	5.15	0.18	0.18

Table 12: ***Comparative statics for optimal dual targeted ICR policy in the case of console***

Parameters: $\xi = 0$, $\pi = 10\%$, $c = 8\%$, $\sigma = 25\%$, $\lambda = 1/3$, $\zeta = -20\%$, $\mu = 2\%$, $r = 5\%$.