Smooth Ambiguity, Wealth Dynamics and Asset Prices with Heterogeneous Beliefs

Bo Huang   Hening Liu*

*Alliance Manchester Business School, The University of Manchester

1. Abstract

We study a class of endowment economies with long-run risks in which agents have generalized recursive smooth ambiguity preferences and heterogeneous beliefs. The expected growth rate of aggregate consumption consists of a persistent component. Agents cannot observe the component but learn about it via Bayes' rule. Meanwhile, agents hold different beliefs about persistence of the long-run component. By examining a two-agent model, we find that:

- 1) the consumption share of the agent with the correct belief dominates in the long run, even when both agents have recursive preferences without smooth ambiguity.
- 2) smooth ambiguity, in conjunction with state uncertainty, generates uncertainty sharing motive that leads to long-run survival of both agents.
- 3) the time-varying weights of agents and posterior beliefs help explain the time variation of price-dividend ratios in the data.
- 4) in a model with an ambiguity-averse agent and an ambiguity-loving agent, both agents survive in the long run if they hold different beliefs.

2. Framework

- Long-run risk model and heterogeneous beliefs

$$\Delta c_{t+1} = \mu_c + x_t + \bar{\sigma}\epsilon_{c,t+1}$$
$$x_{t+1} = \rho_x x_t + \phi_x \bar{\sigma}\epsilon_{x,t+1}$$
$$\Delta d_{t+1} = \mu_d + \Phi x_t + \phi_{dc}\bar{\sigma}\epsilon_{c,t+1} + \phi_d \bar{\sigma}\epsilon_{d,t+1}$$
$$\epsilon_{c,t+1}, \epsilon_{d,t+1}, \epsilon_{x,t+1} \sim i.i.d.N(0,1)$$

  - x_t: long-run component, assumed to be unobservable, $\Delta c_{1:t}, \Delta d_{1:t}$: observable signals
  - Agents use Bayesian learning to update beliefs about x_t
  - Two agents have heterogeneous beliefs about ρ_x: ρ_x^h, $h = A, B$. Suppose agent A holds the correct belief $\rho_x^h = \rho_x$

- Generalized recursive smooth ambiguity preferences

$$U_t^h\left(C^h\right) = \left[\left(1-\beta^h\right)\left(C_t^h\right)^{1-\frac{1}{\psi^h}} + \beta^h\left(\mathcal{R}_t^h\left(U_{t+1}^h\left(C^h\right)\right)\right)^{1-\frac{1}{\psi^h}}\right]^{\frac{\psi^h}{\psi^h-1}}$$

$$\mathcal{R}_t^h\left(U_{t+1}^h\left(C^h\right)\right) = \left(\mathbb{E}_{\hat{x}_t}^h\left[\left(\mathbb{E}_{x_t}^h\left[\left(U_{t+1}^h\right)^{1-\gamma^h}\right]\right)^{\frac{1-\eta^h}{1-\gamma^h}}\right]\right)^{\frac{1}{1-\eta^h}}$$

  - γ^h: relative risk aversion (RRA)
  - $\psi^h > 0$: elasticity of intertemporal substitution (EIS)
  - η^h: ambiguity aversion. $\eta^h > \gamma^h \rightarrow$ ambiguity averse; $\eta^h = \gamma^h \rightarrow$ ambiguity neutral; $\eta^h < \gamma^h \rightarrow$ ambiguity loving
  - Belief distortions are embedded in the expectations $\mathbb{E}_{\hat{x}_t}^h$ and $\mathbb{E}_{x_t}$.

- Complete markets and the social planner's problem

$$\max_{\{C^1,\ldots,C^H\}} U\left(\{\boldsymbol{C}\};\boldsymbol{\lambda}\right) = \sum_{h=1}^{H}\lambda_0^h U^h\left(C^h\right)$$

subject to the market-clearing condition

$$\sum_{h=1}^{H} C^h\left(y^t\right) = C\left(y^t\right)$$

where $C\left(y^t\right)$ is aggregate consumption following the long-run risk dynamics and initial welfare (Negishi) weights are $\lambda_0^h, h = 1, \cdots, H$

- The optimal condition for the individual consumption decision

$$\underline{\lambda}_t^h\left(1-\beta^h\right)\left(s_t^h\right)^{-\frac{1}{\psi^h}} = \underline{\lambda}_t^1\left(1-\beta^1\right)\left(s_t^1\right)^{-\frac{1}{\psi^1}}$$

$$\sum_{h=1}^{H} s_t^h = 1$$

- The dynamics of Negishi weights

$$\underline{\lambda}_{t+1}^h = \frac{\underline{\lambda}_t^h \underline{\Pi}_{t+1}^h}{\sum_{h=1}^{H}\underline{\lambda}_t^h \underline{\Pi}_{t+1}^h}, \quad h \in \{2,\ldots,H\}$$

$$\underline{\Pi}_{t+1}^h = \left(\beta^h e^{-\frac{1}{\psi^h}\Delta c_{t+1}}\frac{d\Lambda_{t,t+1}^h}{d\Lambda_{t,t+1}}\frac{d\zeta_t^h}{d\zeta_t}\right)\left(\frac{v_{t+1}^h e^{\Delta c_{t+1}}}{\mathcal{R}_t^h\left[v_{t+1}^h e^{\Delta c_{t+1}}\right]}\right)^{\frac{1}{\psi^h}-\gamma^h} \times \left(\frac{\left(\mathbb{E}_{x_t}^h\left[\left(v_{t+1}^h e^{\Delta c_{t+1}}\right)^{1-\gamma^h}\right]\right)^{\frac{1}{1-\gamma^h}}}{\mathcal{R}_t^h\left[v_{t+1}^h e^{\Delta c_{t+1}}\right]}\right)^{\gamma^h-\eta^h}$$

- Stochastic discount factor

$$M_{t+1}^h = \underbrace{\beta^h \cdot e^{-\rho^h\cdot\Delta c_{t+1}}}_{\text{CRRA}}\underbrace{\left(\frac{s_{t+1}^h}{s_t^h}\right)^{-\rho^h}}_{\text{Consumption Share}}\underbrace{\left(\frac{v_{t+1}^h e^{\Delta c_{t+1}}}{R_t^h\left(v_{t+1}^h e^{\Delta c_{t+1}}\right)}\right)^{\rho^h-\gamma^h}}_{\text{EZ}}\underbrace{\left(\frac{\left(E_{x_t}^h\left[\left(v_{t+1}^h e^{\Delta c_{t+1}}\right)^{1-\gamma^h}\right]\right)^{1/(1-\gamma^h)}}{\mathcal{R}_t^h\left(v_{t+1}^h e^{\Delta c_{t+1}}\right)}\right)^{-(\eta^h-\gamma^h)}}_{\text{Smooth Ambiguity}}$$

- The dynamics of the Consumption share ratio

$$\frac{s_{t+1}^B}{s_{t+1}^A} = \frac{s_t^B}{s_t^A}\underbrace{\left(\frac{d\Lambda_{t+1}^B d\zeta_t^B}{d\Lambda_{t+1}^A d\zeta_t^A}\right)^{\psi}}_{\text{estimation channel}}\underbrace{\left(\frac{\frac{v_{t+1}^B}{\mathcal{R}_t^B\left[v_{t+1}^B e^{\Delta c_{t+1}}\right]}}{\frac{v_{t+1}^A}{\mathcal{R}_t^A\left[v_{t+1}^A e^{\Delta c_{t+1}}\right]}}\right)^{1-\gamma\psi}}_{\uparrow\ \text{risk sharing channel}}\underbrace{\left(\frac{\frac{\left(\mathbb{E}_{x_t}^B\left[\left(v_{t+1}^B e^{\Delta c_{t+1}}\right)^{1-\gamma}\right]\right)^{\frac{1}{1-\gamma}}}{\mathcal{R}_t^B\left[v_{t+1}^B e^{\Delta c_{t+1}}\right]}}{\frac{\left(\mathbb{E}_{x_t}^A\left[\left(v_{t+1}^A e^{\Delta c_{t+1}}\right)^{1-\gamma}\right]\right)^{\frac{1}{1-\gamma}}}{\mathcal{R}_t^A\left[v_{t+1}^A e^{\Delta c_{t+1}}\right]}}\right)^{\psi(\gamma-\eta)}}_{\uparrow\ \text{smooth ambiguity; uncertainty sharing channel}}$$

3.1. The influence of the estimation channel on the survival of agents

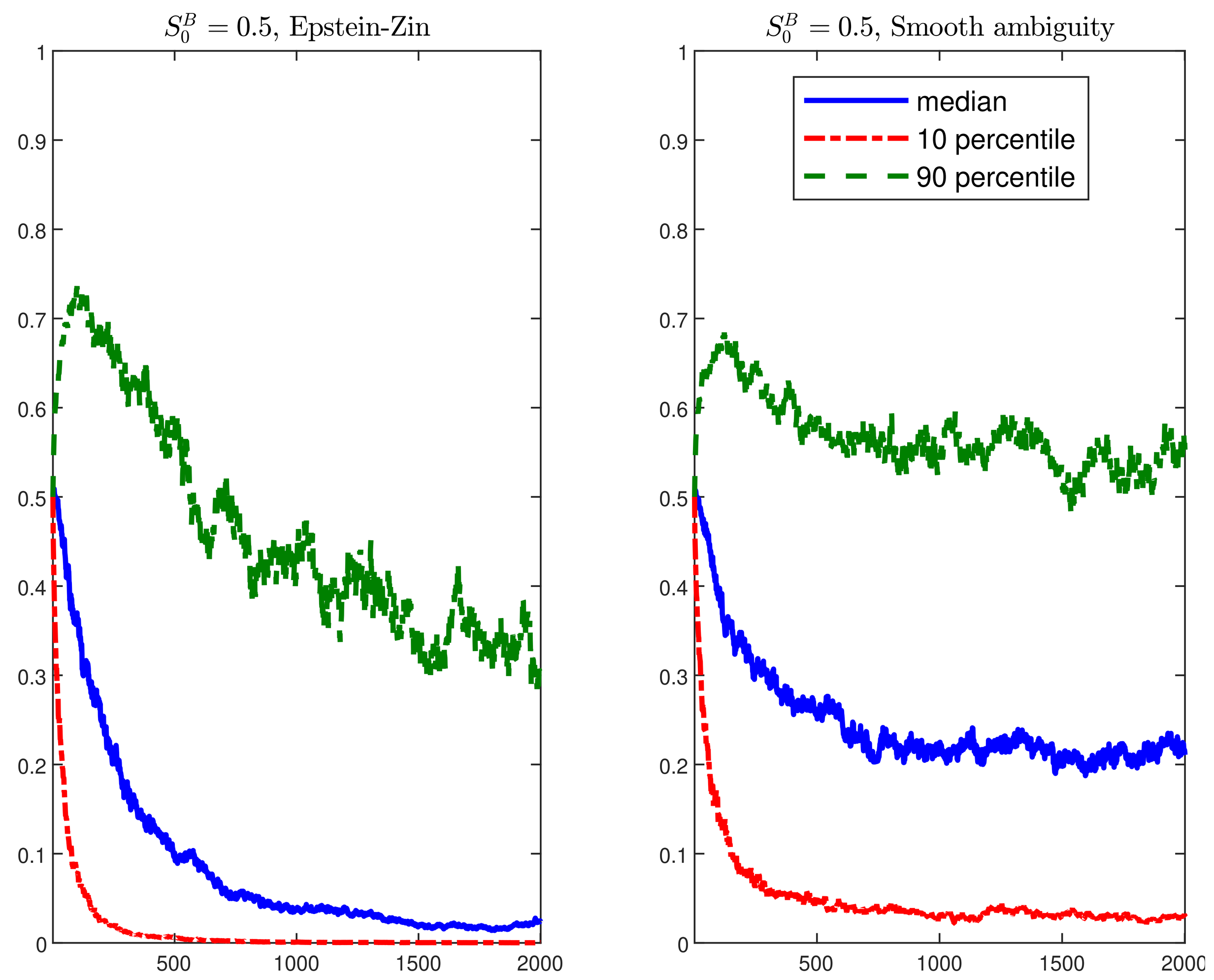
- Borovička (2019,JPE); Pohl, Schmedders and Wilms (2021,JFE): the risk-sharing channel is very important such that it can overturn the traditional market selection result.
- Our result: Not really if learning about long-run risk is taken into account
  1. The agent with correct beliefs tends to dominate in the long run via the estimation channel.
  2. Both short-run shocks and long-run shocks contribute to the estimation channel.
- Therefore, in our model, the share of the agent with incorrect beliefs diminishes toward zero even when both agents have Epstein-Zin recursive utility with EIS>1 (of course, also holds with time-separable CRRA utility).

3.2. The effect of the uncertainty-sharing channel on the survival of agents

When can both agents survive in the long-run equilibrium?

- Our finding: when both agents have the generalized recursive smooth ambiguity preferences.
- Mechanism of uncertainty-sharing channel: long-run shocks realize $\longrightarrow$ create a wedge between the true hidden state and its estimate $\longrightarrow$ estimation uncertainty materializes $\xrightarrow[\text{and ambiguity aversion}]{\text{interaction of state uncertainty}}$ the agent believing in higher persistence (assumed to be the correct belief) is willing to compensate for the other agent believing in lower persistence (incorrect)

Figure 1. Long-run simulation: ρ_x^B (incorrect belief)



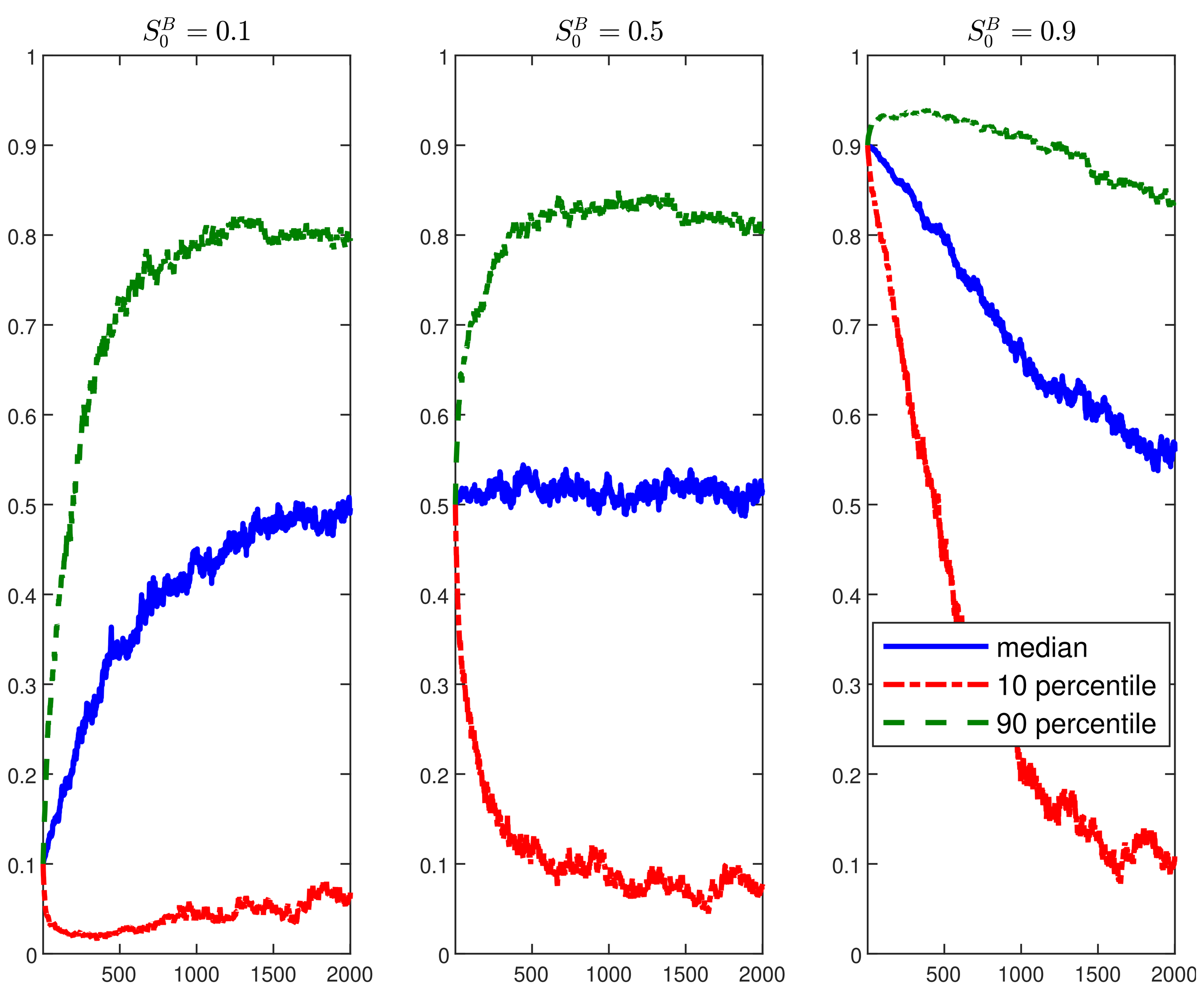
4. Preference heterogeneity and belief heterogeneity

What about preference heterogeneity?

We consider ambiguity aversion v.s. ambiguity loving

- Straightforward: if both agents hold identical beliefs, the ambiguity-loving agent will dominate in the long-run equilibrium.
- When can the ambiguity-averse agent exist in the long-run equilibrium?
- Our finding: when the ambiguity-averse agent believes in lower persistence (incorrect belief) of long-run consumption risk.

Figure 2. Long-run simulation: Consumption share of ambiguity aversion agent



5. The predictability of returns

When we calibrate the heterogeneous-agents model with smooth ambiguity to the post-war consumption growth and dividend growth data, we find

- the model produces simulated P/D highly correlated with actual P/D
- the simulated P/D can reproduce predictability of returns
- either the heterogeneous-agent model with Epstein-Zin utility or the representative-agent models (with Epstein-Zin utility or smooth ambiguity utility) cannot explain time variations in P/D data.

Figure 3. Time series $p-d$: the Epstein-Zin model

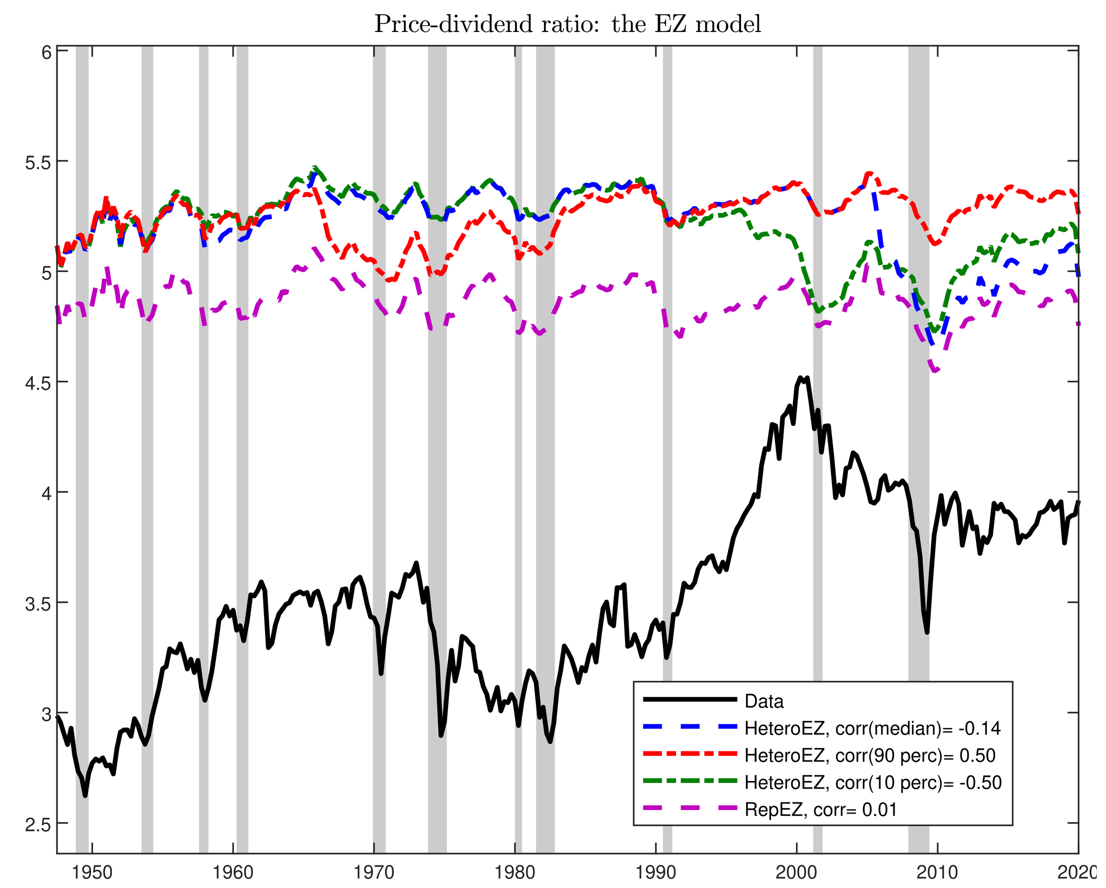
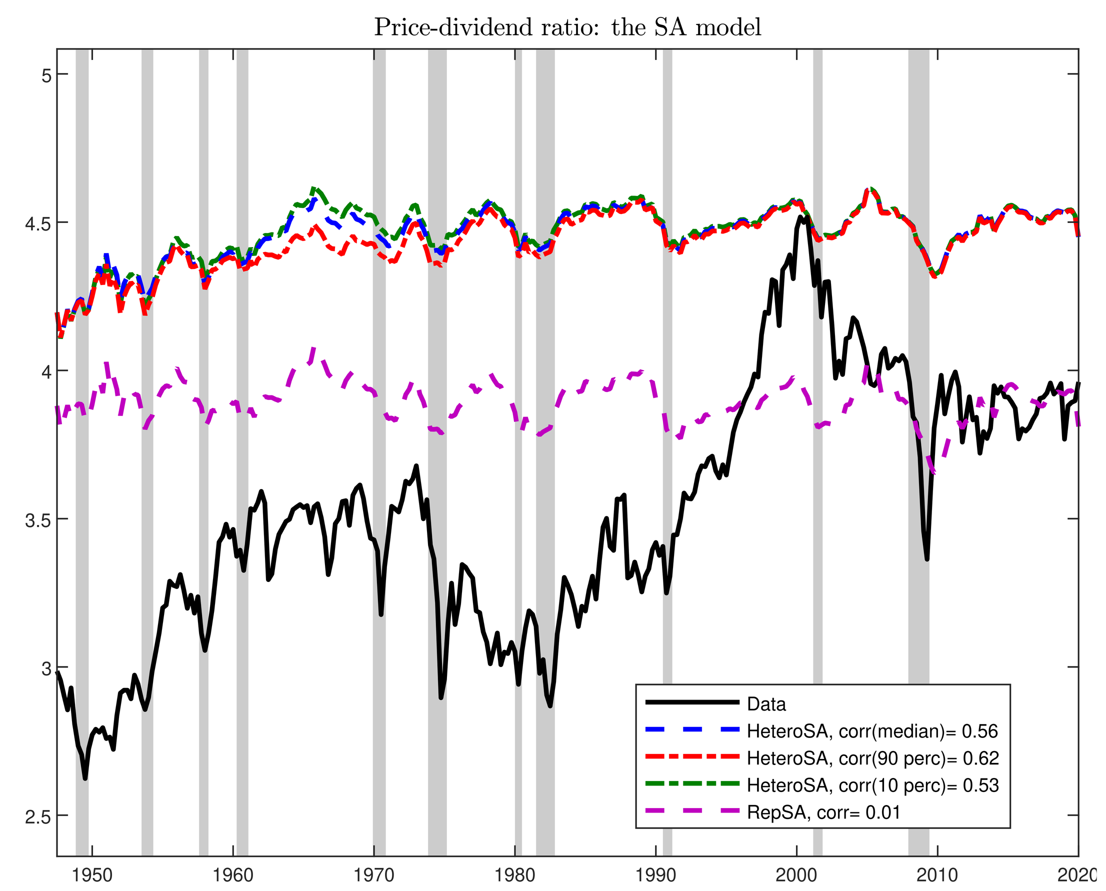


Figure 4. Time series $p-d$: the smooth ambiguity model



6. Asset pricing moments and predictability

Table 1. Asset Pricing Moments

	$\mathbb{E}[R_f]-1$	$\sigma(R_f)$	$\mathbb{E}(R-R_f)$	$\sigma(R)$	$\mathbb{E}(p-d)$	$\sigma(p-d)$
Data	0.60	1.57	7.85	16.00	3.54	0.42
Panel A: Smooth ambiguity						
Rep. $\rho_x = 0.96$	1.61	1.06	11.54	20.21	2.53	0.26
Two-agent	1.92	0.98	8.79	19.97	2.83	0.26
Rep. $\rho_x = 0.94$	2.67	0.94	5.70	17.68	3.14	0.20
Panel B: Epstein-Zin						
Rep. $\rho_x = 0.96$	2.67	1.07	5.88	24.05	3.50	0.33
Two-agent	2.67	1.03	5.33	23.49	3.62	0.32
Rep. $\rho_x = 0.94$	3.15	0.94	2.79	18.69	3.96	0.22

Note: This table reports unconditional moments for models with the generalized recusrive smooth ambiguity utility and models with Epstein-Zin's recursive utility. For each utility function, results are simulated from three models, the heterogeneous-agent model and two representative-agent models with $\rho_x^A = 0.96$ and $\rho_x^B = 0.94$ respectively. For each model, we run $N = 500$ simulations where each simulation contains 400 periods of states. The length of the burn-in stage is 2000.

Table 2. Predictive Regressions

		1Q	1Y	2Y	3Y	4Y	5Y
Excess return: $r_{t+H} - r_{f,t+H}$							
Data	slope	-0.025	-0.102	-0.185	-0.243	-0.290	-0.366
	R^2	0.017	0.066	0.116	0.147	0.169	0.204
SA Model	slope	-0.074	-0.479	-1.040	-1.531	-2.009	-2.524
	R^2	0.005	0.054	0.136	0.220	0.304	0.362
EZ Model	slope	-0.005	-0.077	-0.172	-0.256	-0.353	-0.468
	R^2	0.002	0.007	0.019	0.030	0.043	0.055

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