All-in Fighting*

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Abstract

How do reputational motives affect the intensity of repeated conflicts? We propose a model where players fight in a finite sequence of battles and privately know whether they are rational (and choose fighting efforts to maximize payoff) or automatons locked into fighting "all-in" in every battle. In the unique symmetric equilibrium, rational players fight all-in in early battles as doing so buys a beneficial "all-in look" that intimidates rivals in future battles. A rational player has a strictly positive payoff only if she monopolizes, among all players, the reputation for fighting all-in. In a period with reputational oligopoly, a war of attrition to become the reputational monopolist may yield overdissipation (expected fighting efforts exceeding the per-period prize). In a period with reputational monopoly, overdissipation never happens and the monopolist mixes between fighting all-in to boost her reputation tomorrow and a continuum of non-all-in fighting efforts to cash in on her reputation today. While a monopoly may last indefinitely, an oligopoly does not. Applications include turf wars, sea piracy, mafias, and litigation.

JEL classification codes: C72, D82, D83.

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1 Introduction

In legal markets, defense in case of conflicts often comes in the form of a lawyer in a courtroom. Black markets—such as the multi-billion dollar industry of smuggling drugs, firearms, liquor, or people—lack the typical legal channels to resolve conflicts. To fill this void, organized criminal groups look for other means of dispute resolution; typically, violence (e.g., MacCoun and Reuter, 2001). Lawyers in courtrooms are replaced by armed thugs on the streets, and legal expenditures by acts of violence. Acts of particularly heinous violence are often committed by organized criminal groups not only to resolve today's dispute, but also to build a reputation as tough "all-in" fighters in the eyes of their rivals, a strategy that may be beneficial to resolve *future* disputes. We provide a simple theoretical framework to analyze how reputational motives affect repeated conflicts; players repeatedly fight over time for a sequence of fixed prizes and, by fighting particularly hard (all-in) today, a player "buys an all-in look" for the future.

The importance of a reputation for fighting all-in is empirically and anecdotally well-documented by the literature on the social and economic impact of turf wars.[1] Similarly to turf wars, "pirates sought to develop a reputation as men who would unleash unspeakable savagery on those who crossed them" (Leeson, 2010, p. 504), "a Mafioso makes himself respected by winning a reputation for toughness and courage" (Shvarts, 2002; p. 70), and "one way that bullying may help to establish and maintain social dominance is by cultivating a reputation for toughness" (Shackelford and Weekes-Shackelford, 2012; p. 273). More broadly, the "[r]eputational logic has been used to explain the origins and escalation of conflicts from the Peloponnesian War to the Vietnam War" (Dafoe and Caughey, 2016; p. 341)."[2] Recently, when experts across the world analyze the current conflict in Ukraine, they

[1] For instance, driveby shootings against rival gangs are often motivated by the need to maintain and enhance the reputation for fighting all-in in the hope of muscling out the competition for valuable turf and drug markets (see Sanders, 2017). For organized criminal groups, "investing in building a reputation for violence is an effective way of lowering the future expenses of running and maintaining a criminal enterprise" (Livingston, 2011; p. 7), and "newcomers must 'invest' in developing their reputation"; such an investment typically comes in the form of costly acts of violence (Caulkins et al., 2006; p. 14). For a model of repeated conflict—without reputation—over the control of illegal markets (and a recent literature review) see Castillo and Kronick (2020).

[2] Dafoe and Caughey (2016; p. 371) provide the intriguing example of Americans raised in the American South who "tend to be more concerned with reputation for resolve than non-Southerners" and find that, over the past two centuries, the militarized disputes involving the US that "occurred under Southern presidents have been twice as likely to involve the use of force ... and have been three times as likely to be won by the United States." In her analysis of territorial disputes, Wiegand (2011; p. 101) finds that the challenger state often uses force in a dispute "to credibly demonstrate resolve to other adversaries in other disputes—an attempt to transfer reputation for resolve." In her analysis of separatist movements, Walter (2006; p. 313) finds that "[i]f a government believes it could face multiple additional challenges over numerous pieces of territory, it has greater incentives to invest in building a reputation for toughness." In fact, "the government's behavior in the first period could affect decisions by other separatists later on." Similar findings appear in

posit that "Putin the Rational may be pretending to be Vlad the Mad" (Rachman, 2022) in order to build a reputation for being committed to a full-on invasion (all-in fighting) regardless of obstacles, costs, sanctions, and losses that happen along the way. Finally, in litigation, parties choosing particularly high legal expenditures in pursuing today's case build a litigious look that is typically beneficial in future litigations (Allison et al., 2010; Hovenkamp, 2013).

In our simple model, n players fight repeatedly in a sequence of T battles, one in each of T periods. Each battle is modeled as a standard all-pay auction with a fixed prize of value 1; that is, players simultaneously exert efforts and the player who exerts the highest effort wins 1. The cost of effort equals the effort level itself, and it is paid regardless of victory or defeat. Each player is privately informed about her persistent type, which is either a standard utility-maximizing rational type or, with probability ε, an "all-in" automaton locked into exerting "all-in fighting effort" equal to 1 in all periods. As the per-period prize is 1, setting automatons' efforts to 1 is a simple way to capture all-in fighting (an act of particularly heinous violence in a turf war, for instance).[3] Efforts are observable. Thus, exerting a "non-all-in" effort (i.e., smaller than 1) immediately and forever unmasks a player as rational.[4] A player who fights all-in in today's battle builds a reputation in the eyes of her rivals who are uncertain whether she is an all-in automaton, or a rational type trying to buy an all-in look. We fully characterize the unique type-symmetric equilibrium of the (T, n, ε)-game (with $2 \leq T < \infty$, $n \geq 2$, and $\varepsilon \in [0, 1)$). The general implication of our analysis is that embedding a game of reputation for an all-in fighting posture in an otherwise standard setup of repeated contests can yield novel insights into understanding repeated conflicts. In particular, we find a stark difference in intensity and form of conflict in a period with "reputational monopoly" (when only one player "has reputation", as only she has always fought all-in thus far) and a period with "reputational oligopoly" (when multiple players have reputation). When confusion does not arise, in what follows we focus on rational players and omit the qualifier "rational". Furthermore, we refer to the cumulative payoff in a period as the sum of present and all future per-period payoffs.

In a period of reputational monopoly, the monopolist is not afraid of being up against all-in automatons as all others have been unmasked as rational at some prior point in time. Therefore, in equilibrium, the monopolist enjoys a strictly positive cumulative payoff because

Keels and Greig (2019). Finally, see Tingley and Walter (2011) for experimental evidence that subjects do invest in building reputation for fighting all-in in repeated games.

[3]Section 6 discusses the robustness of our results to alternative choices of automatons' efforts. Committed types locked into playing a fixed action are pioneered by Kreps and Wilson (1982), Milgrom and Roberts (1982), and Kreps, Milgrom, Roberts, and Wilson (1982). For an extensive literature analysis, see Mailath and Samuelson (2006).

[4]See Section 6 for the possibility of regaining reputation.

others fear that she is an all-in automaton and their efforts are thus discouraged. Importantly, in any period, a player has a strictly positive cumulative equilibrium payoff if and only if she has a reputational monopoly; in all other cases, cumulative equilibrium payoffs are zero. The value of such a strictly positive payoff is endogenously determined. Furthermore, as the fighting intensity (expected aggregate effort) in a reputational monopoly is lower than in the natural benchmark without reputation at all ($\varepsilon = 0$), one can say that reputation, when monopolized, *discourages* violence. This finding is in line with evidence from the literature on criminal organizations, which shows that the threat arising from having a reputation for particularly heinous violence may mitigate the need for actual violence (Reuter, 1985; Livingston, 2011).

In a period of reputational oligopoly, in equilibrium, players with reputation engage in a fierce fight because the stakes include the per-period prize *and* the prospect of becoming a reputational monopolist in the future and obtaining the monopoly's strictly positive cumulative equilibrium payoff. For this reason, the expected aggregate effort may be higher than in the benchmark without reputation; thus, reputation may *encourage* violence, when a reputational oligopoly occurs. Because of the fierce fight among reputational oligopolists, per-period expected equilibrium payoffs of players who fight all-in are negative; this is in line with the literature on acts of violence during turf wars, which shows that warring criminal organizations' profits are often negative (Levitt and Venkatesh, 2000). Moreover, in the parlance of contest theory, reputation provides a novel rationale for the commonly observed "overdissipation," as the expected aggregate effort in a period with reputational oligopoly may be larger than the per-period prize (Dechenaux, Kovenock, and Sheremeta, 2015).

A crucial role is played by equilibrium dynamics. Starting from a period of reputational oligopoly, the equilibrium may follow a war-of-attrition path over time; players fighting all-in incur per-period losses in the hope that their reputation will outlast, at some point in time, that of all their rivals. Along this path, beliefs that others are all-in automatons increase over time until the net cost of fighting all-in becomes too large because the risk of facing all-in automatons is too high: the fear that others are all-in automatons dominates the benefit of possibly outlasting the rivals' reputations. When this happens, players with reputation stop fighting all-in and are all unmasked as rational. In other words, the fierce fight in the reputational oligopoly may end suddenly without passing through a reputational monopoly—the war of attrition ends without winners. While the war of attrition during a reputational oligopoly has a maximum duration, the reputational monopolist may fight all-in up until the very last period is reached, because she is never afraid of being up against all-in automatons.

Structure of the Paper. Section 2 discusses the related literature. Section 3 describes the model. Section 4 provides an illustrative example with three periods and two players that helps gradually build intuition. Section 5 formally characterizes and analyzes the unique type-symmetric equilibrium with any number of periods and players (i.e., the general (T, n, ε)-game), highlighting the structural equilibrium differences between a period with zero, one, or multiple players who have reputation. Section 6 discusses extensions and robustness of our results. Section 7 provides matching stylized facts and concludes.

2 Related Literature

To the best of our knowledge, we are the first to equip a standard repeated all-pay auction with commitment types and fully characterize the resulting reputation dynamics. Four features of our simple framework relate to four strands of the literature: repeated contests with private information, war-of-attrition games, reputation building with committed types, and overdissipation in contests.

First, in our setup, actions signal types and players update their beliefs about types over time.[5] This feature is shared by models of contests with private information about valuations, abilities, or effort costs rather than about fighting postures (rationality/all-in), which is instead our focus.[6] One-shot two-player contests where, before the contest, a player can send a costly signal to her rival are studied by Katsenos (2010), Fu et al. (2013), and Denter et al. (2022) with one-sided and Heijnen and Schoonbeek (2017) with two-sided asymmetric information.[7] Signaling in twice-repeated contests is studied by Catepillán et al. (2022) with one-sided and Münster (2009) and Kubitz (2022) with two-sided asymmetric information over ability (or prize valuation). Signaling in those models is often two-directional: weak types may want to appear strong and strong types may want to appear weak. Two-directional signaling complicates the analysis: none of those models go beyond two periods or two

[5] For experimental evidence that subjects do understand and react to the fact that players' actions signal privately known types (strength) in contests, see, for example, Konrad and Morath (2018). Beccuti and Möller (2022) consider a common-value best-of-three contest where players observing battle outcomes update their beliefs about the prize value. In a related strand, contestants share verifiable information prior to the contest (e.g., Kovenock et al., 2015; Wu and Zheng, 2017; Ewerhart and Lareida, 2021).

[6] Abreu and Gul (2000; p. 86) provide a well-aimed description of this structural difference between the two families of models: models with canonical private information are "concerned with uncertainty about 'fundamentals'." In contrast, models adopting the committed-type approach are "rather different in that [they] seek to model uncertainty about the strategic intent or strategic posture of the opponent rather than uncertainty about such concrete factors as seller's costs of production or buyer's valuations." We believe the committed-type approach is appropriate to capture the dynamics of building reputation for fighting all-in in the applications in the Introduction.

[7] See Fu (2006) for a two-player contest where the informed contestant moves earlier than her uninformed rival, and her effort signals her private type.

players.[8] Our approach with commitment types matches applications where reputation for toughness, rather than for weakness, plays a key role, and also gives us enough tractability to fully characterize the equilibrium for any number of periods and players. Our T-period n-player results show that going beyond the two-period, two-player model is informative.[9] For instance, having more than two players gives rise to partial participation and parallel competition between players with and without reputation. Having more than two periods gives rise to war-of-attrition dynamics with reputations that outlast those of one's rivals and are then further maintained; in fact, the very first period is special because players have not yet had the chance of building reputation over their rivals, and the very last period is special because players do not wants to further invest in their reputation.

Second, an equilibrium path of our setup shares common features with war-of-attrition games. In particular, when multiple players have reputation in a period (a reputational oligopoly), we find that players with reputation will keep on fighting all-in with strictly positive probability in the hope of outlasting all rivals' reputations, even if fighting all-in yields a strictly negative current-period payoff. These equilibrium dynamics resemble war-of-attrition games, where players typically choose a time to stop and trade off the gains from outlasting other players (i.e., stopping later) and the costs incurred as time goes by.[10] In our setup, a war of attrition endogenously arises in equilibrium, due to the interaction between the specific stage game (a standard all-pay auction) and our approach to reputation with committed "all-in" types. To the contrary of typical war-of-attrition games, rather than assuming that outlasting the others yields a benefit, we find that in equilibrium players try to outlast all rivals and obtain the payoff of a reputational monopolist, which is endogenous and reputation-dependent, as in all other contingencies payoffs are zero.

Third, our paper is related to the literature on reputation building with committed types.[11] Within this literature, an important strand is that on reputational bargaining.

[8] Krähmer (2007) models t-period two-player repeated contests with binary efforts and learning about contestants' relative abilities. However, players neither have private nor asymmetric information.

[9] Going beyond the two-period, two-player model is also relevant for applications: turf wars often occur among more than two gangs and seldom consist of two fights only. For instance, Brown (2004) reports that "five shootings over the past two months were the result of a conflict between members of the Bloods, Crips and GPAN street gangs."

[10] War of attrition games are pioneered by Maynard Smith (1974) and applied to a variety of situations: patent races (Fudenberg et al., 1983), bargaining (Ordover and Rubinstein 1986), public good provision (Bliss and Nalebuff, 1984), and price wars and exit in oligopolistic markets (Fudenberg and Tirole 1986). For a general analysis of wars of attrition, see Bulow and Klemperer (1999).

[11] Kwiek (2011) considers a repeated second-price auction with committed types, hence it lacks the all-pay feature which is the inherent feature of contests. We are aware of only one paper adopting the committed-type approach to repeated contests—Hovenkamp (2013)—which analyzes a t-period model of repeated litigation; however, stage game and thus reputation dynamics are structurally different from our setup. In particular, he analyzes litigations by modeling PAEs as long-term players proposing a settlement to short-term players (firms) who can accept or reject. Following a rejection, the long-term player litigates the claim or gives

Consider the seminal work of Abreu and Gul (2000, henceforth AG) as an example to illustrate the key differences with our setup. The stage game is a two-player dividing-a-dollar game with sequential endogenous offers, rather than a contest with simultaneous endogenous efforts; in both setups, players fight for a fixed prize, but the costs arise only from discounting in AG, whereas efforts themselves are costly in our setup. This highlights the key difference: our stage game (a contest) allows us to analyze the key variable of interest of our paper, the intensity of conflict, and its dynamics.[12] A contest as a stage game fits our applications, where costly fighting efforts are a key variable of interest. Within the literature on reputation with committed types, two closely related papers to our framework and applications are Silverman (2004) and Ghosh et al. (2019). Ghosh et al. (2019) study negotiation in the shadow of a repeated conflict between a defender who holds resources and a challenger who wants the defender's resources. Players can be of types for whom fighting is costless. In contrast to our setup, the focus is on how offers in the pre-conflict negotiation phase shape beliefs about each other's types.[13] Silverman (2004) studies a reputation model of crime with endogenous and reputation-building acts of violence, similarly to our setup. Across-period differences are crucial in our setup (e.g., reputational monopoly vs. oligopoly), while Silverman analyzes the stationary steady-state of an overlapping generation model with random matching and a continuum of players. Finally, in the stage game of Silverman (2004) and the conflict phase of Ghosh et al. (2019), two players choose binary actions—to be violent or not, to concede or not. Acharya and Grillo (2015) investigate the emergence of war in a two-player three-period game where players alternatively choose their peaceful/belligerent action and

up. Short-term players are either rational or an "impressionable type" that is intimidated whenever the long-term player engages in litigation.

[12]Similarly to our setup, also in AG, 1) payoffs are strictly positive for a reputational monopolist and zero when both players still have reputation or both have been unmasked as rational, and 2) a war of attrition emerges because no player is willing to reveal herself first as rational. However, several differences also emerge. First, in AG, rational players with reputation mix between conceding and mimicking the irrational demand for a certain maximum number of periods, while in our model they mix between a continuum of non-all-in efforts and pretending to be all-in automatons, and such mimicking could last until the end of the game. Furthermore, in AG, as the irrational demand is fixed, players with reputation essentially choose only the acceptance probability in every period; in our setup, they choose not only their probability of mimicking, but also the continuous *distribution* of non-all-in effort, which varies in every period. Second, a monopoly necessarily ends the game in AG's setup (because it is reached exactly when a player concedes to the rival's inflexible demand), while in our setup it yields an interesting continuation game with its own law of motion of beliefs, duration of reputation, and efforts exerted; see the full characterization of Proposition 2 and Section 5.2. Third, in AG, as the stage game has sequential moves, the equilibrium path goes from an oligopoly to the end of the game in a fixed number of periods, whereas in our setup, as the stage game has simultaneous moves, an oligopoly lasts for a fixed number of periods, a monopoly may last until the end of the game, and oligopolies *may or may not* devolve into a monopoly. Fourth, in our n-player setup, in contrast with AG's two-player setup, the equilibrium dynamics are affected by the *number* of players with or without reputation in a way we fully characterize. All the above differences are due to the structural differences in the stage game of our and AG's setup.

[13]For a survey of models of bargaining in the shadow of conflicts, see Baliga and Sjöström (2013).

can be of a type commited to always choosing the belligerent action. In contrast, our stage game has any number of players and a continuum of actions—i.e., fighting efforts—which allows us to discuss the role of competition among several players and, most importantly, a nuanced analysis of the intensity of conflict.[14]

Fourth, as mentioned above, we find per-period overdissipation (that is, per-period aggregate effort larger than 1) in early periods, because players with reputation fight not only for the per-period prize of 1, but also for becoming the reputational monopolist. This result provides a novel rationale for the experimentally observed overdissipation in repeated contests; see, e.g., Dechenaux, Kovenock, and Sheremeta (2015) (henceforth, DKS). As DKS document, the literature proposes several explanations for overdissipation: subjects may derive a non-monetary utility from winning, care about their relative payoffs, be prone to mistakes, and be affected by other judgmental biases, such as non-linear probability weighting and hot-hand fallacy. As our results show, reputational motives may also generate per-period overdissipation in the early periods of the game. This pattern is consistent with another empirical regularity that DKS highlight: overdissipation appears to decrease over time.[15] Hence, our framework provides an account of the fierce initial fight among contestants without relying on behavioral biases: overdissipation in initial periods is the rational response of contestants who benefit from building a reputation for toughness.

3 The Model

In each period $t \in \{1, ..., T\}$ with $2 \leq T < \infty$, a fixed set of $n \geq 2$ players simultaneously exert non-negative efforts.[16] In each period, the player exerting the highest effort wins a per-period prize of value 1, while the losers obtain 0; ties are broken evenly. Effort costs are identical to efforts and paid by all players. Players are risk neutral (the per-period payoff equals the prize won, if any, minus effort) and do not discount future payoffs.[17] Recall that, in contrast to the per-period payoff, in what follows, we refer to a player's cumulative payoff in period t as the sum of all per-period payoffs from t to T. After each period, players observe all efforts.

We assume that each player is an all-in automaton with ex-ante probability $\varepsilon \in [0, 1)$, and rational with probability $1 - \varepsilon$. Players' types are realized once and for all at the beginning

[14] The violence intensity is a variable of applied interest; for instance, an incidence of mass casualties is qualitatively different from a targeted elimination, and the evolution of violence in turf wars has attracted a lot of attention—e.g., Livingston (2011).

[15] See, e.g., Davis and Reilly (1998) and Lugovskyy et al. (2010).

[16] We assume $T < \infty$ so as to abstract from collusive agreements.

[17] Discounting would reduce the benefits of building reputation in early periods, similarly to a front-loaded prize sequence, but having no discounting is not essential for our results.

of the game and are privately known. A rational player is a standard forward-looking payoff maximizer. An all-in automaton is locked into always exerting the all-in effort, equal to 1.[18] As the behavior of all-in automatons is fixed, we focus on the analysis of rational players in what follows and omit the qualifier "rational" except when needed to avoid confusion.

In each period t, players form beliefs about each other's probability of being all-in automatons. Beliefs depend on the full history of observed (all-in or non-all-in) efforts. In particular, in any Perfect Bayesian Equilibrium, if a player always fought all-in until period t, then she is believed by others to be an all-in automaton with a strictly positive probability given by Bayes' rule; if so, we say that she "has reputation" in period t. Conversely, if a player ever exerted at least once a non-all-in effort, she is unmasked as rational, and we say that she "has no reputation" in period t (and onwards). We focus on type-symmetric Perfect Bayesian Equilibrium (TSPBE): in any period, all rational players with the same level of reputation (i.e., others' beliefs of them being automatons) use the same strategy. Note that, if multiple players have reputation in period t, then they have the same reputation level; we call such reputation level $\varepsilon_t > 0$.[19] Hence, when studying a generic period t, we can focus only on a strategy for a player without reputation and one for a player with reputation. The number of players with reputation in period t is denoted by $\nu_t \in \{0, ..., n\}$; if $\nu_t = 1$, period t is what we call a period with "reputational monopoly" and if $\nu_t \geq 2$ a period with "reputational oligopoly." As all players have reputation ε in the first period, $\nu_1 = n$ (if $\varepsilon > 0$) and $\varepsilon_1 = \varepsilon$. Finally, note that, in every period, the pair (ε_t, ν_t) is a sufficient statistic for past play: it contains all the information players need to choose their actions.

Throughout the paper, a benchmark useful to single out the effects of reputation is the version of the above model without reputation (i.e., $\varepsilon = 0$). In such a benchmark, the equilibrium expected aggregate effort equals 1 and payoffs 0 in every period—see Baye, Kovenock, and De Vries (1996).

[18] Section 6 discusses the robustness of our results to alternative choices of all-in efforts. Assuming that all-in automatons are locked into exerting only one effort simplifies the derivation of equilibrium beliefs: any non-all-in effort can only be exerted by a rational player. Hence, given that automatons exert effort 1, exerting an effort greater than 1 for a rational player is strictly dominated by 0 and thus easily ruled out in equilibrium. Alternatively, one could model all-in automatons as rational but with payoffs different from those of rational players in a way that leads them to always choose the all-in effort (though they may choose non-all-in efforts too). With such alternative specification, off-the-equilibrium path beliefs over automatons deviating to non-all-in efforts would have to be carefully analyzed.

[19] Section 6 discusses asymmetric reputation levels among players and non-type-symmetric equilibria.

4 Illustrative example: $\{T, n, \varepsilon\} = \{3, 2, 1/8\}$

To gradually build intuition on equilibrium behavior, consider the example of $\{T, n, \varepsilon\} = \{3, 2, 1/8\}$. This simple case contains many of the key forces underlying the equilibrium of the general model. Panels $(a) - (i)$ of Figure 1 depict the equilibrium effort in the unique TSPBE, distinguishing between periods when zero ($\nu_t = 0$), one ($\nu_t = 1$), and both ($\nu_t = 2$) players have reputation. In the following paragraph, we describe the equilibrium strategies, which one can verify as special cases of Propositions 1-3. We then discuss the properties of the equilibrium, which one can verify as special cases of Propositions 4-9. In doing so, we focus on the insights that apply to the general (T, n, ε)-game. Finally, we conclude this section with an intuitive description of the main changes that arise when $T > 3$, $n > 2$, and $\varepsilon \neq 1/8$.

Equilibrium strategies: $\{T, n, \varepsilon\} = \{3, 2, 1/8\}$**.** The equilibrium path begins with the reputational duopoly at $t = 1$, where players fight all-in with probability (w.p.) 3/7, and exert non-all-in effort (uniformly distributed on $[0, 1/2]$) otherwise—panel (a). If both players fight all-in at $t = 1$, then we move to panel (b) where players fight all-in w.p. 1/3, and exert non-all-in effort (uniformly distributed on $[0, 1/2]$) otherwise. If both players fight again all-in, then we move to panel (c) where players never fight all-in. If only one player fights all-in in (a) or (b), then the next period is a reputational monopoly; the player without reputation stays inactive w.p. 1/2, the reputational monopolist fights all-in w.p. 1/3 (or 0 in the last period), and both exert non-all-in effort (uniformly distributed on $[0, 1/2]$) otherwise—panels (e) and (f). Finally, if both players exerted non-all-in effort and thus lost reputation, then efforts are uniformly distributed on $[0, 1]$—panels (h) and (i).

Preliminary equilibrium observations. First, reputation is *costly to maintain*; fighting all-in in a certain period has a cost of 1 and *may* yield a prize of 1, thus necessarily resulting in a weakly negative current-period payoff. A direct consequence is that fighting all-in never occurs in the very last period as reputation building is useless (panels (c), (f), and (i)). More formally, a rational player never fights all-in in the very last period, as fighting all-in costs 1 and yields at best a current period prize of 1 but no benefits in terms of future reputation. Second, reputation *cannot be restored once given up*. Once a player has given up her reputation by exerting a non-all-in effort, she stops paying its maintenance costs and never fights all-in (panels (h) and (i), and the bottom graphs of panels (e) and (f)). Hence; ν_t decreases in t along the equilibrium path. Third, *reputation* ε_t *increases over time*—as long as it is maintained. Initial reputation is $\varepsilon_1 = 1/8$. In panels (b) and (e), players with reputation are believed to be all-in automatons w.p. $\varepsilon_2 = \varepsilon_1/(\varepsilon_1 + (1-\varepsilon_1)(3/7)) = 1/4$ and, in panels (c) and (f), w.p. $\varepsilon_3 = \varepsilon_2/(\varepsilon_2 + (1-\varepsilon_2)(1/3)) = 1/2$.[20]

Equilibrium when no player has reputation (panels (h) and (i)). In a period where all players are commonly known to be rational, the equilibrium is as in a standard complete information all-pay auction; players mix uniformly on $[0,1]$ and payoffs are zero.

Equilibrium with reputational monopoly (panels (e) and (f)). The player without reputation exerts effort 0 with strictly positive probability (and has 0 payoff) because of her fear of being up against an all-in automaton. More generally, such a fear discourages effort; this can be seen as the expected effort of a player without reputation in panel (e) is smaller than in the no-reputation panel (h) (likewise, in panel (f) her expected effort is smaller than in (i)). In the last period (f), the reputational monopolist cashes in on her reputational advantage and, in fact, has a positive payoff (equal to $1/2$, as she, for instance, wins with certainty with an effort of $1/2$). In the non-terminal period (e), the reputational monopolist mixes between fighting all-in and a continuum of non-all-in fighting efforts: in fact, (1) if she fights all-in, she has a current-period payoff of 0 but a positive payoff tomorrow (equal to $1/2$) by maintaining—and actually boosting—her reputation, while (2) if she fights non-all-in, she has a payoff of 0 tomorrow because she loses her reputation, but a current-period positive payoff (equal to $1/2$). Thus, her cumulative payoff is $1/2$ either way and she is indifferent between (1) boosting her reputation tomorrow, or (2) cashing in on her reputation today. Moreover, a reputational monopoly turns out to be the only contingency when a player (the monopolist) has a strictly positive cumulative equilibrium payoff in the general (T, n, ε)-game. Propositions 2 and 4-6 establish that the above-discussed qualitative properties hold more generally, in the (T, n, ε)-game. Finally, the cashing-in on reputation described above matches empirical evidence, as we discuss in Section 7; the reputation for toughness gained through a series of acts of heinous violence intimidates rivals and even allows a player to win with strictly positive probability with a negligible effort—that is, she no longer "needs" acts of heinous violence, as the threat of violence suffices.

Equilibrium with reputational oligopoly (panels (a), (b), and (c)). Players with reputation fight all-in with strictly positive probability (except in the last period). If the two

[20] Note that the resulting probability of fighting all-in is $1/3$ in *both* panels (b) and (e) and thus, in panel (f), we do not need to differentiate whether panel (f) is reached because one player gave up her reputation in panel (b) or the reputational monopolist kept her reputation in panel (e). This is why ε_3 in panel (f) does not depend on the history of the game. This convenient property is by no means general, and Section 5 fully characterizes how the equilibrium depends on the history of the game. Finally, with $\{T, n\} = \{3, 3\}$, there is no ε such that the equilibrium for $\nu_3 = 1$ does not depend on the past. Hence, we consider $n = 2$ in this section for illustrative simplicity.

players fight all-in in the first period, their first-period payoff is $-1/2$, and they enter the second period with increased fear that the other is an automaton ($\varepsilon_1 = 1/8$ and $\varepsilon_2 = 1/4$). If they fight all-in also in the second period, fear further increases ($\varepsilon_3 = 1/2$) and their second-period payoff is $-1/2$. These situations are analogous to what Luce and Raiffa refer to as a "ruinous situation" in wars of attritions (Luce and Raiffa, 1989), as both players suffer. In general, as long as multiple players have reputation, the equilibrium follows a war-of-attrition path: in every non-terminal period, each player with reputation trades off the monopolist gains she obtains if her reputation outlasts that of all others vs. the negative current-period payoff due to fighting all-in. The fierce competition among players with reputation to outlast each other drags down cumulative equilibrium payoffs to 0, in sharp contrast with a reputational monopoly.[21] Proposition 3 and 7-9 establish that the above-discussed qualitative properties hold more generally, in the (T, n, ε)-game. Finally, the above-described present losses of fighting for reputation match empirical evidence, as we discuss in Section 7: e.g., during turf wars, criminal organizations' profits are often negative (Levitt and Venkatesh, 2000).

Intensity of conflict and reputation in equilibrium. In a reputational oligopoly, the per-period expected aggregate effort may exceed 1 (e.g., equals 8/7 in panel (a)) as players fight not only for the per-period prize of 1 but also for outlasting the rivals' reputations (Proposition 8). Furthermore, overdissipation is possible only in sufficiently early periods (Corollary 2); indeed, in all panels other than (a), aggregate effort never exceeds 1. In a reputational monopoly, the per-period aggregate effort is strictly below 1, because players fight for a per-period prize of value 1 and there is no competition for outlasting the rivals' reputation (Proposition 5). One can say that reputation, when monopolized, *discourages* violence as the fighting intensity (expected aggregate effort) in a reputational monopoly is lower than in the natural benchmark without reputation at all ($\varepsilon = 0$). In contrast, in a reputational oligopoly, the expected aggregate effort may be higher than in the benchmark without reputation; thus, reputation may *encourage* violence.

[21] This structure of equilibrium payoffs is reminiscent of the complete information all-pay auction with asymmetric prize valuations (Baye, Kovenock, and De Vries; 1996). There, all equilibrium payoffs are zero if at least two players have the highest prize valuation (have reputation in our setup); and only if one player has the highest valuation (has reputation in our setup), then her equilibrium payoff is strictly positive while that of all others is zero.

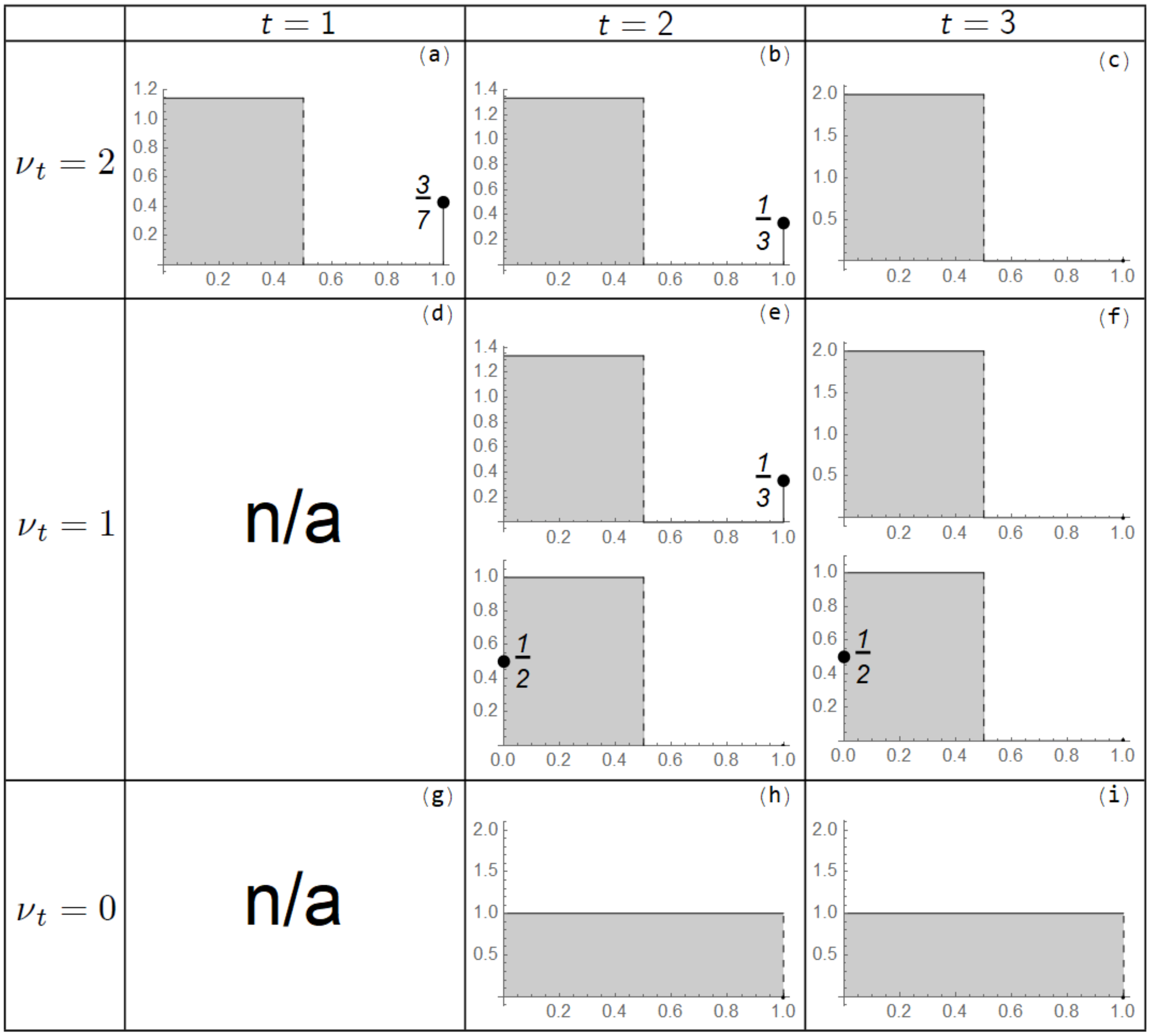

Figure 1. Unique TSPBE when $\{T, n, \varepsilon\} = \{3, 2, 1/8\}$. The figure depicts probability density functions of equilibrium efforts. In panels (e) and (f), the top (bottom) part depicts the behavior of the player with (without) reputation.

Equilibrium with $\{T, n, \varepsilon\} \neq \{3, 2, 1/8\}$. The above analysis with $\{T, n, \varepsilon\} = \{3, 2, 1/8\}$ helped grasp key intuitions, but some of its equilibrium features are not general. In fact, when $\{T, n, \varepsilon\} = \{3, 2, 1/8\}$, some forces do not show up or perfectly balance out. We illustrate some of those so-far-hidden forces by discussing the main changes to the equilibrium when, in turn, $T > 3$, $n > 2$, or $\varepsilon \neq 1/8$.

If $T > 3$, an important difference emerges between an oligopoly and a monopoly: while a reputational monopoly can be sustained indefinitely (that is, until $T < \infty$, no matter how large T is), a reputational oligopoly has a maximum duration. In fact, in an oligopoly, reciprocal beliefs of being all-in automatons increase over time, and the fear of being up against an actual automaton suffices, after seven[22] periods, to deter players from fighting all-in. In general, Proposition 7 characterizes the maximum duration of a reputational oligopoly for every initial belief ε. However, Proposition 4 shows that a similar maximum duration does not exist for a reputational monopoly; the monopolist may keep her reputation until period T because she never fears being up against an all-in automaton.

If $n > 2$, the players without reputation are inactive (exert 0 effort with certainty) in a period when at least two players have reputation: the fierce competition among the players with reputation to become the reputational monopolist deters the players without reputation from exerting effort at all (see Proposition 3). Furthermore, as discussed in Figure 1 for $n = 2$, the upper bound of the support of non-all-in efforts is $1/2$ whenever some reputation is present—that is, in all panels (a)-(f). In the general (T, n, ε)-game, such upper bound may vary with ν_t and t.[23]

If $\varepsilon < 1/8$, as ε decreases and approaches 0, the equilibrium behavior approaches a uniform on $[0, 1]$ as in panels (h) and (i). Consider now $\varepsilon > 1/8$; fighting all-in could now disappear from the equilibrium of both panels (a) and (b) because of fear of being up against an automaton.[24] As long as panel (e) can be reached (i.e., $\varepsilon < 2/3$), the reputational monopolist always fights all-in with strictly positive probability—see Proposition 2.

Many of the key insights of the general (T, n, ε)-game match stylized facts in applications ranging from turf wars to sea piracy, from mafias to litigation. The analysis of this section already delivered these main insights in a simple illustrative example, so that the reader interested in the matching stylized facts can jump to Section 7 and skip the next two more technical sections, which formally characterize the equilibrium of the general (T, n, ε)-game.

5 Equilibrium

Most of the insights from the example of Section 4 hold more generally, as this section shows formally, but some more insights emerge from the analysis of the general (T, n, ε)-game. In particular, this section fully characterizes and analyzes the unique TSPBE with any number of periods and players, and any ε, highlighting the structural equilibrium differences between periods when zero, one, or multiple players have reputation.

[22] To see this, plug $n = 2$ and $\varepsilon_t = 1/8$ into (6), *infra*.

[23] From Proposition 3, such upper bound decreases in ε_t and p_t (the probability a player with reputation exerts all-in effort), and ε_t increases while p_t decreases over time. If $n = 2$, these two effects balance out.

[24] For panel (a), $\varepsilon > 2/3$ guarantees that fighting all-in is not part of the equilibrium (so that panel (b) is never reached)—see Proposition 3. For panel (b), one can show that if $\varepsilon > 0.233$, then $\varepsilon_2 > 0.414$ and fighting all-in is not part of the equilibrium in panel (b).

5.1 Characterization

We define π_t^{ν} ($\pi_t^{-\nu}$) as the expected cumulative equilibrium payoff of a player with (without) reputation in period t, calculated as the sum of all per–period equilibrium payoffs from t to T. In what follows, we consider separately the cases when, in a general period t, zero, one, or multiple players have reputation: respectively, $\nu_t = 0$, $\nu_t = 1$, or $\nu_t \geq 2$. Recall that $\nu_1 = n$ (as long as $\varepsilon > 0$), so that at $t = 1$ we necessarily have an oligopoly.

The case of $\nu_t = 0$ is equivalent to a standard complete information all-pay auction.

Proposition 1 (Baye, Kovenock, and De Vries (1996)) *In the unique TSPBE of period $t \in \{2, ..., T\}$ with $\nu_t = 0$, players exert effort on $[0, 1]$ with cumulative density function (CDF) $x^{\frac{1}{n-1}}$ and $\pi_t^{-\nu} = 0$.*

For a period t with a **reputational monopoly**, we now characterize equilibrium strategies and payoffs.[25]

Proposition 2 *Let*

$$p_t = \frac{\varepsilon_t^{\frac{1}{T-t+1}} - \varepsilon_t}{1 - \varepsilon_t} \text{ and } q_t = \varepsilon_t^{\frac{1}{(T-t+1)(n-1)}}$$

and consider the following two CDFs on $x \in \left[0, 1 - q_t^{n-1}\right]$:

$$F_t(x) = \frac{x}{(1 - \varepsilon_t)(1 - p_t)\left(x + q_t^{n-1}\right)^{\frac{n-2}{n-1}}} \text{ and } G_t(x) = \frac{\left(x + q_t^{n-1}\right)^{\frac{1}{n-1}} - q_t}{(1 - q_t)}.$$

In the unique TSPBE of period $t \in \{2, ..., T\}$ with $\nu_t = 1$, the effort of the player with reputation follows F_t w.p. $1 - p_t$, and is 1 w.p. p_t. The effort of each player without reputation is 0 w.p. q_t, and follows G_t w.p. $1 - q_t$. Also, $\pi_t^{\nu} = \varepsilon_t^{\frac{1}{T-t+1}}$ and $\pi_t^{-\nu} = 0$.

We now provide the intuition for the equilibrium quantities in Proposition 2. In a non-terminal period, the *reputational monopolist* mixes between (1) fighting all-in and boosting her reputation tomorrow, and (2) a continuum of non-all-in fighting efforts (distributed

[25] *In the last period*, one can calculate the payoff of the only player with reputation *without* deriving equilibrium strategies using the methodology in Siegel (2009). Using his terminology and considering the case of $\{T, n, \varepsilon\} = \{3, 2, 1/8\}$, the reach of the player without (with) reputation is $1 - \varepsilon_3$ (1). Thus, the payoff of the player with reputation is $1 - (1 - \varepsilon_3) = \varepsilon_3$. For instance, in panel (f) of Figure 1, $\varepsilon_3 = 1/2$, which corresponds to the mass at 0 for the player without reputation and to the payoff of the player with reputation. For non-terminal periods, the prize the player with reputation obtains by fighting all-in includes an extra component, tied to the expected level of future reputation, which depends on current equilibrium strategies via Bayes' rule, and hence the cumulative payoff cannot be computed without deriving equilibrium strategies. Therefore, using Siegel's methodology does not lead to a simplification in non-terminal periods.

according to F_t) to cash in on her reputation today.[26] Indeed, the reputational monopolist who exerts non-all-in effort gives up her reputation, and thus we can focus exclusively on the current-period payoff; for an effort $x \in \left(0, 1 - q_t^{n-1}\right]$, she obtains $\left(q_t + (1 - q_t) G_t(x)\right)^{n-1} - x$ when the $n - 1$ rivals use the strategy in Proposition 2. Also, if she exerts an arbitrarily small, but strictly positive effort, she obtains q_t^{n-1}. Hence, in equilibrium, we must have the following

$$\left(q_t + (1 - q_t) G_t(x)\right)^{n-1} - x = q_t^{n-1}. \tag{1}$$

Exerting effort $x \in \left(1 - q_t^{n-1}, 1\right)$ or $x = 0$ is not profitable. Exerting all-in effort $x = 1$ yields a zero current-period payoff (as she wins with certainty) and an endogenously-determined cumulative payoff tomorrow, which depends on others' updated beliefs about the monopolist's reputation. Calculating this payoff requires a non-trivial recursive characterization (see the Proof of Proposition 2 for details) which eventually yields $\pi_t^\nu = \varepsilon_t^{\frac{1}{T-t+1}}$. As a monopolist must be indifferent between non-all-in and all-in efforts, π_t^ν must also equal q_t^{n-1} (see (1)); this pins down the equilibrium value of q_t.

A player without reputation exerting effort $x \in \left[0, 1 - q_t^{n-1}\right]$ obtains

$$(1 - \varepsilon_t)(1 - p_t) F_t(x) \left(q_t + (1 - q_t) G_t(x)\right)^{n-2} - x = 0, \tag{2}$$

as she wins only if the reputational monopolist is rational and exerts non-all-in effort, and if she exerts effort greater than that of the monopolist and the $n-2$ players without reputation. As the support of F_t and G_t are the same, $F_t\left(1 - q_t^{n-1}\right) = 1$; this pins down the equilibrium value of p_t.

Importantly, note that a player who achieves a reputational monopoly through a series of all-in fighting efforts can cash in on her reputational monopoly (and obtain a strictly positive payoff) by exerting *non-all-in* effort. This matches stylized facts (see Section 7) and, formally, can be seen considering a reputational monopolist who exerts a strictly positive arbitrarily small effort; if all her rivals remain inactive (w.p. q_t^{n-1}), she wins 1 with a negligible effort, and thus she enjoys a strictly positive per-period payoff.

[26] In the equilibrium of a non-terminal period, a monopolist must play with positive probability *both* (1) fighting all-in, and (2) a continuum of non-all-in fighting efforts, because of the following intuitive reason. If the monopolist were to fight all-in with certainty—only (1)—, then rivals would remain inactive with certainty and the monopolist would deviate to an arbitrarily small but strictly positive effort today and obtain a payoff of 1, which is greater than the payoff $\pi_{t+1}^{\nu_t}$ that the monopolist would obtain in the following period. If the monopolist were to fight non-all-in with certainty—only (2)—, then fighting all-in would make the rival certain that she is an all-in automaton and therefore be a profitable deviation as it fully discourages rivals.

For a period t with a **reputational oligopoly**, Proposition 3 characterizes the strategies and payoffs in the TSPBE. This equilibrium is structurally different from the one in Proposition 2. The competition between players with reputation drags down their individual cumulative equilibrium payoff to 0 (and discourages players without reputation, who remain inactive). Such 0 payoff, letting p_t be the probability that a player with reputation fights all-in, implies,

$$\begin{aligned} 1 = \quad & \sum_{l=0}^{\nu_t-1} \frac{1}{l+1} \binom{\nu_t - 1}{l} \left(\varepsilon_t + (1-\varepsilon_t)\, p_t\right)^l \left(1 - \left(\varepsilon_t + (1-\varepsilon_t)\, p_t\right)\right)^{\nu_t-1-l} \\ & + (1 - (\varepsilon_t + p_t(1-\varepsilon_t)))^{\nu_t-1} \left(\frac{\varepsilon_t}{\varepsilon_t + p_t(1-\varepsilon_t)}\right)^{\frac{1}{T-t}} . \end{aligned} \tag{3}$$

The right-hand side (RHS) of (3) is the benefit of fighting all-in, and the left-hand side (LHS) is its cost. The benefit has two components. The first component is the summation that captures the expected share of the current period's prize, taking into account that there can be any number between 0 and $\nu_t - 1$ of rivals with reputation (rational or all-in automatons) who may fight all-in and tie; in fact, the probability that an individual rival with reputation exerts all-in effort is $\varepsilon_t + (1-\varepsilon_t)\, p_t$. The second component (second line of (3)) has first the probability that all the other $\nu_t - 1$ players with reputation exert non-all-in efforts today, and second the benefit of being a reputational monopolist tomorrow, which equals $\varepsilon_{t+1}^{1/(T-t)}$ from Proposition 2. In (3), ε_{t+1} is expressed as a function of ε_t and p_t using Bayes' rule.[27]

Condition (3) is key in characterizing the equilibrium for a period t with a reputational oligopoly. Whether (3) has a solution with $p_t \geq 0$ or not crucially depends on whether ε_t is small enough; intuitively, if ε_t was arbitrarily close to 1, fear of all-in automaton rivals would be high enough to guarantee that a rational player would never exert all-in effort. The upper bound that guarantees existence of an equilibrium with $p_t \geq 0$ is denoted by $\bar{\varepsilon}_t(\nu_t)$. As we show in the Proof of Proposition 3, $\bar{\varepsilon}_t(\nu_t)$ is the unique solution to

$$1 = \frac{1}{\nu_t} \sum_{l=0}^{\nu_t-1} (1-\varepsilon_t)^l + (1-\varepsilon_t)^{\nu_t-1}. \tag{4}$$

A straightfoward analysis of (4) gives the following.

Lemma 1 $\bar{\varepsilon}_t(\nu_t)$ *exists, is unique, smaller than* $2/3$, *and strictly decreasing in* ν_t.

Intuitively, the larger the number ν_t of players with reputation, the less profitable it is

[27] Bayes' rule reads $\varepsilon_{t+1} = \varepsilon_t / \left(\varepsilon_t + (1-\varepsilon_t)\, p_t\right)$.

to exert all-in effort, the smaller is the region of ε_t for which an equilibrium with all-in effort can be sustained.

We are now ready to provide the characterization of the TSPBE in a period t with a reputational oligopoly.

Proposition 3 *Consider the following CDF:*

$$F_t(x) = \frac{x^{\frac{1}{\nu_t - 1}}}{(1-\varepsilon_t)(1-p_t)} \text{ if } x \in \left[0, ((1-\varepsilon_t)(1-p_t))^{\nu_t - 1}\right].$$

In the unique TSPBE of period $t \in \{1, ..., T\}$ with $\nu_t \geq 2$, $\pi_T^{\nu} = \pi_T^{-\nu} = 0$, the effort of the $n - \nu_t$ players without reputation is 0, and

- *if $t = T$ or if $t < T$ and $\varepsilon_t \in [\bar{\varepsilon}_t(\nu_t), 1]$, the effort of each of the ν_t players with reputation follows F_t with $p_t = 0$,*
- *if $t < T$ and $\varepsilon_t \in [0, \bar{\varepsilon}_t(\nu_t)]$, the effort of each of the ν_t players with reputation follows F_t w.p. $1 - p_t$ and is 1 w.p. p_t, where p_t is the unique solution implicitly defined by* (3) *for given ε_t. Finally, $p_t \in [0, 2/3]$.*

Note that F_t in Proposition 3 has a simple interpretation, similar to that for G_t of Proposition 2. A player with reputation who exerts effort $x \in \left[0, ((1-\varepsilon_t)(1-p_t))^{\nu_t - 1}\right]$ when the rivals use the strategy described in Proposition 3 for $\varepsilon_t \in (0, \bar{\varepsilon}_t(\nu_t)]$ gives up her reputation and obtains $((1-\varepsilon_t)(1-p_t))^{\nu_t - 1} F_t(x)^{\nu_t - 1} - x$. Also, she must be indifferent between such an effort and an arbitrarily small effort that yields a 0-payoff (as she loses with certainty to the other players with reputation). Such indifference explains the equilibrium value of F_t in Proposition 3.[28]

5.2 Reputational Monopoly: Further Properties and Applications

This section describes important properties of the equilibrium under reputational monopoly other than those already characterized in Proposition 2. Proposition 4 shows that, starting in any period t with a reputational monopoly and for any reputation level $\varepsilon_t > 0$, $\nu_T = 1$ will occur with strictly positive probability even if the last period T is arbitrarily far ahead in the future: a monopoly may last indefinitely—that is, until $T < \infty$, no matter how large T is. Finally, Proposition 4 also characterizes the law of motion of reputation over time.

[28] It is not a profitable deviation for a player with reputation to exert effort $x \in \left(((1-\varepsilon_t)(1-p_t))^{\nu_t - 1}, 1\right)$.

Proposition 4 *In the unique TSPBE of period $t \in \{2, ..., T\}$ with $\nu_t = 1$,*

1. *$\forall \varepsilon_t > 0$, $\nu_T = 1$ occurs with strictly positive probability,*
2. *if $\nu_{t+1} = 1$, then $\varepsilon_{t+1} = \varepsilon_t^{\frac{T-t}{T-t+1}}$. Also, $\varepsilon_{t+1} > \varepsilon_t$.*

The intuition for the first result is that the reputational monopolist does not fear being up against all-in automatons and hence she always plays all-in effort with strictly positive probability (see Proposition 2). The second result follows by Bayes' rule and Proposition 2.

A reputational monopoly never yields per-period overdissipation, as the per-period expected aggregate effort is lower than 1 (the per-period prize).

Proposition 5 *In the unique TSPBE of period $t \in \{2, ..., T\}$ with $\nu_t = 1$, the expected aggregate effort in period t is lower than 1.*

The intuition behind Proposition 5 is as follows. When $\varepsilon_t = 0$, there is common-knowledge of rationality among players and hence the expected aggregate effort in period t is 1 (recall Proposition 1). When $\varepsilon_t > 0$, the reputational monopolist has an incentive to fight all-in to increase ε_{t+1}: pretending to be an all-in automaton becomes credible and profitable. This force tends to increase the expected aggregate effort. However, when $\varepsilon_t > 0$, the expected effort of the players without reputation decreases because of the increased fear that the reputational monopolist is an actual automaton. Proposition 5 shows that the latter effect dominates: the discouragement of players without reputation is stronger than the monopolist's incentive to fight all-in and per-period overdissipation never occurs in a reputational monopoly.

A natural benchmark is a period when no player has reputation ($\nu_t = 0$). Here, the expected aggregate effort is 1 (see Proposition 1). Proposition 5 shows that the introduction of reputational motives discourages efforts in a reputational monopoly ($\nu_t = 1$). One can say that reputation for fighting all-in *discourages* violence in repeated conflicts as the expected aggregate effort in a period of reputational monopoly is lower than in the natural benchmark without reputation ($\varepsilon_t = 0$).

We characterize the per-period win probability of the reputational monopolist.

Proposition 6 *In the unique TSPBE of period $t \in \{2, ..., T\}$ with $\nu_t = 1$, the player with reputation wins in period t w.p.*

$$\frac{1}{n} + \frac{(n-1)^2}{n} \frac{\varepsilon_t^{\frac{1}{T-t+1}} - \left(\varepsilon_t^{\frac{1}{T-t+1}}\right)^{\frac{n}{n-1}}}{1 - \varepsilon_t^{\frac{1}{T-t+1}}}. \tag{5}$$

The expression in (5) is greater than $1/n$; the reputational monopolist is more likely to win than any player without reputation. And finally, one can use (5) to show the following.

Corollary 1 *In the unique TSPBE of period $t \in \{2, ..., T\}$ with $\nu_t = 1$, the player with reputation wins with a probability that is increasing in ε_t and decreasing in n and t.*

The win probability of the reputational monopolist increases in ε_t because players without reputation are discouraged by higher levels of the monopolist's reputation. The win probability of the reputational monopolist also intuitively decreases in n (the number of rivals without reputation). It also decreases in t; for a fixed reputation level of the monopolist, the earlier she achieves such a reputation, the larger her win probability.

5.3 Reputational Oligopoly: Further Properties and Applications

This section describes important properties of the equilibrium under reputational oligopoly other than those already characterized in Proposition 3. Proposition 7 shows that, in any period t with a reputational oligopoly and for any reputation level $\varepsilon_t > 0$, the number of periods for which multiple players can maintain a reputation does not cover the entire remaining duration of the game (that is, till period T) if T is large enough. Finally, Proposition 7 also characterizes the law of motion of reputation over time.

Proposition 7 *In the unique TSPBE of period $t \in \{1, ..., T\}$ with $\nu_t \geq 2$,*

1. *$\forall \varepsilon_t > 0$, $\nu_\tau \geq 2$ occurs w.p. 0 if*

$$\tau \geq t + \frac{\log\left(\frac{1}{2}\frac{\varepsilon_t}{1-\varepsilon_t}\right)}{\log\left(\frac{2}{3}\right)} \tag{6}$$

2. *if $\varepsilon_t \in (0, \bar{\varepsilon}_t(\nu_t)]$ and $\nu_{t+1} \geq 1$, then*

$$1 - \frac{\varepsilon_{t+1}}{\nu_t \varepsilon_t}\left(1 - \left(1 - \frac{\varepsilon_t}{\varepsilon_{t+1}}\right)^{\nu_t}\right) = \left(1 - \frac{\varepsilon_t}{\varepsilon_{t+1}}\right)^{\nu_t - 1} \varepsilon_{t+1}^{\frac{1}{T-t}}, \tag{7}$$

   and $\varepsilon_{t+1} > \varepsilon_t$.

The upper bound on the number of periods with a reputational oligopoly in (6) arises because, over time, fighting all-in increases ε_t, i.e., the belief level about all-in automatons. Furthermore, since we know that $p_t \leq 2/3$ by Proposition 3, Bayes' rule (see Footnote 27) gives $\varepsilon_{t+1} > \varepsilon_t / (\varepsilon_t + 2(1-\varepsilon_t)/3)$, so that the belief follows an increasing sequence that converges to 1. And we know from Proposition 3 that fighting all-in becomes too costly for ε_t large enough ($\varepsilon_t > \bar{\varepsilon}_t(\nu_t) \in [0, 2/3]$) because of the fear of being up against actual all-in automatons. Hence, the war of attrition induced by the reputational oligopoly must end with certainty before period T, if T is sufficiently large. In fact, as mentioned in Section 4, when $\varepsilon = 1/8$ and $n = 2$, then (6) gives, approximately, that $\tau \geq t + 6.5$, and thus a reputational duopoly cannot be sustained in equilibrium beyond the seventh period—i.e., in equilibrium, $\nu_t \leq 1$ for $t \geq 8$. Note also that, for any fixed ε_1, the upper bound on the number of periods with a reputational oligopoly remains finite even if $T \to \infty$.

A reputational oligopoly may yield per-period overdissipation, as the per-period expected aggregate effort exceeds the prize at stake in that period if ε_t is small.[29]

Proposition 8 *In the unique TSPBE of period $t \in \{1, ..., T-1\}$ with $\nu_t \geq 2$, $\exists \hat{\varepsilon}_t > 0$ such that, $\forall \varepsilon_t \in (0, \hat{\varepsilon}_t)$, the expected aggregate effort in period t is strictly greater than 1.*

The intuition behind Proposition 8 is as follows. When $\varepsilon_t = 0$, the expected aggregate effort in period t is 1. When $\varepsilon_t > 0$, the players with reputation have an incentive to fight all-in in order to increase ε_{t+1}: pretending to be an all-in automaton becomes credible and profitable. The players without reputation remain inactive. Hence, for sufficiently low ε_t, per-period overdissipation occurs: players with reputation are fighting particularly hard not only for the per-period prize of 1, but also for the prospect of achieving a future reputational monopoly. To understand why per-period overdissipation requires ε_t small enough, consider the extreme case of ε_t close to 1; a player with reputation is almost certain of being up against $\nu_t - 1$ actual all-in automatons, and hence any strictly positive effort is costly and yields a negligible probability of victory.

Building on Proposition 7 and Proposition 8, we obtain that overdissipation, if any, occurs in sufficiently early battles.

Corollary 2 *In the unique TSPBE of period $t \in \{2, ..., T\}$, $\exists \bar{t} \in \{2, .., T-1\}$ s.t. overdissipation cannot occur if $t > \bar{t}$.*

As in the natural benchmark without reputation ($\nu_t = 0$) the expected aggregate effort is 1, Proposition 8 shows that the introduction of reputation may or may not encourage efforts in a reputational oligopoly ($\nu_t \geq 2$), according to the reputation level ε_t. Reputation may *encourage* violence in repeated conflicts as the expected aggregate effort may be higher than in the benchmark without reputation ($\varepsilon_t = 0$).

[29] An analytical representation of expected aggregate effort under reputational oligopoly is, in general, not tractable as (3) is a polynomial of degree up to n.

Under reputational oligopoly, each player with reputation has a per-period win probability equal to $1/\nu_t$ because players without reputation are inactive and the equilibrium we consider is type-symmetric.

Proposition 9 *In the unique TSPBE of period $t \in \{1, ..., T\}$ with $\nu_t \geq 2$, each player with reputation wins w.p. $1/\nu_t$.*

6 Extensions and Robustness

Our framework is intentionally parsimonious and thus it is natural to discuss its limitations and extensions. In particular, the tractability of our parsimonious framework that allowed us to analyze the general (T, n, ε)-game (with $2 \leq T < \infty$, $n \geq 2$ and any ε) is due to the interplay of four simplifying assumptions: the ex-ante symmetry between players, the impossibility of regaining reputation, the focus on type-symmetric equilibria, and the all-in effort equal to 1. In this section, we briefly discuss the prospects for extending our model along each of these four dimensions separately.

Asymmetries among players. From an ex-ante point of view, we considered identical players.[30] As a consequence, players enter a period either with the *same* positive reputation level $\varepsilon_t > 0$ or with no reputation at all. This property boosts the tractability of our framework and allows us to fully characterize the equilibrium for any number of players and periods. Nevertheless, having only two possible reputation levels in each period for all players may be undesirably restrictive. Introducing in the model ex-ante asymmetries that allow players to enter a certain period with different levels of (positive) reputation severely jeopardizes the tractability of our framework. However, there are alternative, tractable ways to introduce ex-ante asymmetries in the model. For instance, our analysis readily carries over if only a subset of players has the chance of building reputation from the outset, a case relevant to the applications in the Introduction. For instance, a street gang might be about to disband or relocate elsewhere (or a lawyer may be about to retire), thus only being interested in winning the current turf war (trial) rather than in building a reputation for fighting all-in (litigiousness). Such an ex-ante asymmetric setup can be captured with a simple adjustment of our framework: out of the n players, $l \in \{1, ..., n-1\}$ are long-term players who fight in every period (from 1 to T), while the other $n - l$ are short-term players,

[30] Endogenous ex-post asymmetries among players may arise in our framework because of the realization of the mixed strategies in equilibrium, which result in some players having and some others not having reputation. Such endogenous asymmetries are reminiscent of those in Bulow and Klemperer (1999)'s setup, where all but two players would give up their reputations instantaneously (a "natural oligopoly" case, as they call it) and the remaining oligopoly would continue with only two players that keep their reputation.

who are replaced by a new short-term player in every period.[31] Then, only long-term players may build a reputation, while the short-term players never fight all-in with strictly positive probability as it would result in a negative current-period payoff and no future reputational benefit. Our equilibrium characterization in Section 5 applies to such an $l-$sided model with simple adjustments.[32]

Regaining reputation. We set up our framework so that players who give up their reputation cannot regain it; e.g., a gang showing weakness once ruins its reputation forever. This assumption buys tractability, but situations where reputation can be regained may occur in some applications. For instance, media, word-of-mouth, or a new particularly violent gang member may all be windfalls capable of re-establishing the lost reputation. If one would allow players to regain reputation in a stochastic fashion over time, then it would be easy to extend our results under the assumption that players regaining reputation would automatically acquire the level of reputation equal to the current reputation of the players with reputation. However, in the perhaps more realistic setup where reputation can be regained—exogenously or endogenously—as a clean slate with belief updating starting once again from the prior ε, then we would incur in the tractability issue already highlighted in the previous paragraph, as players may enter a certain period with different levels of (positive) reputation. Importantly, our key assumption that reputation cannot be regained once given up is meant to capture, in a stylized way, the common wisdom that a beneficial reputation is much harder to gain and sustain than to lose and, conversely, a detrimental reputation can be difficult to get rid of (e.g., Levine, 2021): in the words of Gambetta (2009; p. 205), "mafiosi are so famously jealous of their professional reputation, as any loss might be fatal for them."

Type-asymmetric equilibria. Throughout the paper, we focused on equilibria where any two rational players entering a period with identical reputation behave identically. One-shot all-pay auctions with three or more players often have asymmetric equilibria; see Baye,

[31] Another equivalent extension to an ex-ante asymmetric framework is if all players are long-term, but the l players are myopic (not forward-looking).

[32] The $n-l$ short-term players play as if they were long-term players who had lost their reputation in the past, so they cannot regain reputation. Hence, long-term players who lost their reputation and short-term players jointly form the set of what we referred to as the $n-\nu_t$ players without reputation. To show in more detail how our equilibrium characterization in Section 5.1 applies to such an $l-$sided model, we modify the illustrative two-player example of Section 4 and assume one-sided private information ($l=1$); that is, only player 1 can build reputation, while player 2 is commonly known to be rational (or player 2 has a short-lived self per period). Assume also for simplicity that $T=2$ and that player 1 starts with a reputation of 1/4. Then, the first-period equilibrium is as in panel (e) of Figure 1 as the player left with reputation has a reputation of $\frac{1/8}{1/8+(1-1/8)*(3/7)}=\frac{1}{4}$ by Bayes' rule, and the second-period equilibrium is as in panel (f) or (i) of Figure 1, according to the first-period realization of the mixed strategy of player 1.

Kovenock, and De Vries, (1996, henceforth, BKD). Therefore, we consider the simplest special cases of interest; namely, $T = 2$ and $n = 3$. When $\varepsilon = 0$, reputation plays no role and payoffs in $t = 2$ are 0; in $t = 1$, there is a continuum of asymmetric equilibrium (without all-in efforts), mirroring that in BKD.[33] Similarly, when ε is sufficiently high, the fear of all-in automatons is high and thus all-in efforts are not played even in $t = 1$, so that payoffs in $t = 2$ are 0; in $t = 1$, there is a continuum of asymmetric equilibrium (without all-in efforts), once again mirroring that in BKD.[34] Interestingly, in the remaining intermediate region of ε, all-in efforts are part of the equilibrium strategies in $t = 1$ and the payoff in $t = 2$ is positive for the monopolist—if any—as in Proposition 2; in particular, in $t = 1$, there is a continuum of asymmetric equilibria, where the three players exert all-in effort w.p. q and, with the remaining probability $1 - q$, two of them further mix over $\left[0, \bar{d}\right]$, and the third over $\left[\underline{d}, \bar{d}\right]$, with $0 < \underline{d} < \bar{d} < 1$, and 0.[35] The equilibrium value of such q is the real root of a polynomial, but cannot be expressed in radicals without introducing complex numbers: i.e., a "casus irreducibilis". (A complete proof is contained at the end of the Appendix.) This suggests that a full equilibrium characterization that includes type-asymmetric equilibria in the general (T, n, ε)-game is not tractable.

All-in fighting. Throughout the paper, we assumed that the effort of players fighting all-in is set to 1. First, this is a natural choice to capture an all-in fight; 1 is the greatest effort in a one-shot all-pay auction that is not strictly dominated by 0 as any effort strictly greater than the prize value would necessarily result in a strictly negative payoff.[36] Second, consider the model generalization where the effort exerted by the automatons is $\omega \geq 0$.[37] If $\omega > 1$, fighting all-in would yield a strictly negative per-period payoff, and if $\omega < 1$ it may yield a strictly positive per-period payoff, thus "polluting" the reputational reasons to choose

[33] Namely, two players mix in $[0, (1-p)^2]$ with CDF $x/(1-p)$ and in $[(1-p)^2, 1]$ with CDF $\sqrt{x}$. The third player plays 0 w.p. $1-p$ and with CDF $(\sqrt{x} - (1-p))/p$ on $[(1-p)^2, 1]$ w.p. p, where p is a free parameter.

[34] Namely, if $\varepsilon > (9 - \sqrt{33})/8$, two players mix in $[0, (1-\varepsilon)^2(1-p)^2]$ with CDF $x/((1-p)(1-\varepsilon)^2)$ and in $[(1-\varepsilon)^2(1-p)^2, (1-\varepsilon)^2]$ with CDF $(\sqrt{x} - (1-p)(1-\varepsilon))/((1-\varepsilon)p)$, while the third player plays 0 w.p. $1-p$ and uses CDF $(1-\varepsilon)\sqrt{x}$ on $[(1-\varepsilon)^2(1-p)^2, (1-\varepsilon)^2]$ w.p. p, where p is a free parameter.

[35] Allowing players to have different mass points at all-in effort raises again the tractability issue highlighted in the two previous extensions; namely, players enter a period with different levels of (positive) reputation.

[36] Clearly, there are ways to define an "irrational" fighting posture other than fighting all-in no matter what. However, we believe our definition of all-in fighting captures fighting postures that seem to be widespread in the real-life repeated conflicts discussed in the Introduction: going all-in as a sign of resoluteness. In the realm of conflicts, an interesting though complementary action to all-in effort is the use of a nuclear weapon. For the incentives to invest in and use nuclear weapons in dynamic frameworks see, for instance, Baliga and Sjostrom (2008) and Chassang and Padró i Miquel (2010).

[37] Following the literature on "mixed committed types" (see, Mailath and Samuelson, 2006), we tried to consider the case of stochastic effort ω. However, this was not successful, as either the model lacks an equilibrium—e.g., if ω could be 1/2 or 1 with equal probability—or the belief updating process is untractable—e.g., if $\omega \sim U[\theta, 1]$ with $\theta \in (0, 1)$, because each specific realization of effort in the $[\theta, 1]$-interval could potentially entail a different updating of the belief of being automatons.

to fight all-in; the choice of $\omega = 1$ avoids these two contingencies. Nevertheless, if $\omega > 1$, our equilibrium characterization in Section 5 can be easily extended: as intuition suggests, a higher ω would increase the costs of fighting all-in in early periods and the probability that players with reputation fight all-in would decrease. Furthermore, if $\omega > 1$ and arbitrarily close to 1, then all our results carry over. If instead $\omega < 1$, then an equilibrium would not exist, in general, because of profitable deviations slightly above ω.[38] Hence, our setup is tractable and informative to capture reputation for fighting all-in, which is the most relevant case for the applications we want to capture, but not reputation for fighting meekly, which we leave to future research.

7 Matching stylized facts and conclusions

We analyze how reputational motives affect repeated conflicts. The importance of dynamic reputation effects in real life conflicts is well-established, with examples ranging from turf wars to sea piracy, from mafias to litigation. Such importance is pointed out by Donohue and Levitt (1998; p. 463), whose model of illegal markets "omits a number of potentially important considerations (e.g., private information and dynamic reputation effects)." Those considerations are our main focus. Formally, we analyze the effects of a reputation for fighting "all-in" by equipping with commitment types à la Kreps-Milgrom-Roberts-Wilson a repeated, standard all-pay auction. Three strengths of our approach emerge. First, using commitment types enhances tractability when compared to models of private information in repeated auctions where signaling is two-directional: weak types may want to appear strong and strong types may want to appear weak.[39] In our model with commitment types, signaling is one-directional: rational types may want to appear as "all-in" fighters. This enhances tractability, allows us to derive rich reputational dynamics, and matches applications where reputation for toughness, rather than for weakness, plays a key role. Second, our use of a standard all-pay auction with continuous fighting efforts allows a nuanced analysis of the intensity of conflict, as compared to binary-choice models. Third, pretending to be

[38] To intuitively illustrate why an equilibrium qualitatively similar to that depicted in Figure 1 cannot be sustained, consider a first-period equilibrium behavior that resembles that in panel (a) of Figure 1, except that the mass point is shifted down to ω. Then, a player who deviates to an effort arbitrarily close to ω, but strictly above it, would have a strictly positive per-period profit (and 0 cumulative payoff tomorrow as she would lose her reputation). This is not consistent with her being indifferent between such a deviation and a negligible effort, that results in 0 per-period and cumulative payoff. If instead ω was equal to 1, as in our main setup, then such a slightly upward deviation would not be profitable as it would result in a strictly negative per-period payoff.

[39] E.g., Hörner and Sahuguet (2007; p. 175) state: "players' incentives to misrepresent their valuations in the first stage are complex, since both sandbagging and bluffing strategies are used in equilibrium." See, also, Kubitz (2022).

"irrationally" committed to fighting all-in is in line with several applications. A topical such application is Putin's invasion of Ukraine in 2022: scholarly and popular media experts across the world are discussing whether Putin's military invasion of Ukraine is the outcome of some cost-benefit analysis, perhaps based on miscalculations or misinformation, or whether he is committed to a full-on invasion (all-in fighting) regardless of obstacles, costs, sanctions, and losses that happen along the way of the conflict (Rachman, 2022); "Putin the Rational may be pretending to be Vlad the Mad".[40]

We fully characterize the unique type-symmetric equilibrium of our model and study its properties in terms of reputation dynamics and conflict intensity. The equilibrium properties fit many of the observed features of the applications we discussed (Sections 1 and 2). In particular, we found that:

1) Players actively **build reputation for toughness** by fighting hard in a publicly observed manner. As already discussed, this is true in several real-life repeated conflicts, ranging from turf wars (e.g., Livingston, 2011) to sea piracy (e.g., Leeson, 2010), from mafias (e.g., Shvarts, 2002) to litigation (e.g., Hovenkamp, 2013). In our model, rather than imposing an exogenous benefit of building reputation, we find an endogenous benefit that arises in the unique equilibrium; this contributes to the understanding of why we often observe costly conflicts for building reputation. The importance of building a reputation for fighting all-in in real-life repeated conflicts is further stressed by the fact that, in those applications where actions are not necessarily publicly observed, players themselves tend to promote their actions' *observability*. "To spread and strengthen the reputations they build, organized criminals rely on word of mouth and cleverly capitalize on mass media. Pirates, for example, demonstrated an ability to control or 'spin' popular perception of themselves by strategically engaging in certain behaviors in front of others and releasing captives who promoted the image pirates sought," so much so that pirates have been described as "public-relations-savvy" (Leeson, 2010; p. 509). Similarly, disclosure of past litigation in trials is essential; e.g., "PAEs commonly attempt to highlight their willingness to litigate aggressively ... by referencing previous situations in which they have litigated" (Hovenkamp, 2013; p. 3).

2) Fighting to build reputation is **particularly costly**. In his extensive work on sea piracy, Leeson (2007, 2009, 2010) brings forward the insight that pirates achieved their "reputation as men who would unleash unspeakable savagery on those who crossed them"

[40] Arguments both in favor and against Putin's rationality have been put forth. Examples of the former are his worries about NATO's expansion, and his desire to occupy all of the strategic Black Sea coast, which is the Russian Navy's gateway to the Eastern Mediterranean. Examples of the latter are the narratives such as Putin seeing "the fight itself, no matter how bloody and destructive, as itself glorious and a sign that he is willing to put Russia back on top no matter what the cost – to his nation and even to himself personally", or that "going down in a blaze of glory is preferable to retreat" (Copeland, 2022).

by investing in a few battles at the outset. In fact, pirates "needed to establish a reputation for going berserk when this occurred [...] in the most brutal way they could: with heinous torture", and such "torture-for-reputation always generated immediate net costs" (Leeson, 2010; p. 504-506). For instance, "a prisoner who realized torture was forthcoming might fight back, injuring, or even killing, his torturer. Perhaps more important, torture took time ... Time pirates spent torturing captives was time they couldn't spend searching for their next prize" (Leeson, 2010; p. 505). Similarly, acts of violence in turf wars involve steep fighting costs because they are typically wasteful, risky, and disruptive of commerce; Levitt and Venkatesh (2000) detail costs and benefits of a drug-selling street gang. The cost of building reputation also solves the credibility issue related to cheap talk.[41]

3) Such costs of fighting to build reputation may outweigh the present expected benefits and yield **present losses.** First, this rationalizes the widespread observation of *overbidding* in experiments of repeated contests. In fact, we found that overbidding may emerge in early periods as players fight not only for today's prize, but for future reputation too; this is consistent with the common observations in experiments that overbidding exists but decreases over time.[42] Second, this rationalizes the common present losses often observed in applications where agents fight for some prize and reputation. Organized criminal groups are enterprises engaging in acts of violence to accumulate power for profits. Acts of heinous violence typically result in current-period losses (e.g., during turf wars, criminal organizations' profits are often negative; see Levitt and Venkatesh, 2000). "Continuous turf wars are both extremely expensive for OCGs [organized criminal groups] and disruptive of commerce" (Livingston, 2011, p. 11), so much so that warring criminal organizations' profits are often negative (Levitt and Venkatesh, 2000). Present losses may also occur in litigation: "[p]atent assertion entities (PAEs) ... frequently initiate infringement lawsuits on which they ostensibly have no chance of turning a profit .... a PAE follows through on its seemingly irrational litigation threats in order to develop a litigious reputation" (Hovenkamp, 2013; p. 2). Also, PAEs may "take cases to judgment rather than settle them even though they are

[41] It is common for criminal organizations to claim that they are serious about going all-in in turf wars. These statements are often and understandably regarded with skepticism by other criminal organizations due to the incentive to build a reputation. However, if such statements materialize in a sufficiently long series of acts of particularly heinous (hence costly) violence, then the skepticism in the eyes of other criminal organizations is replaced by fear. By the same logic, prisoners' trash talks may be regarded with skepticism by other prisoners, while, in the presence of scars from knife stabs or bullet wounds, the skepticism may turn into fear (e.g., Gambetta, 2009).

[42] See, e.g., Dechenaux et al. (2015), who also list other channels that may lead to overbidding, as we discussed in the Introduction. The fact that the desire to build a beneficial reputation leads to high efforts in the early periods is reminiscent of the "top dog effect" in the industrial organization literature (e.g., Fudenberg and Tirole, 1984), or the "frontloading effect" (e.g., Fudenberg et al.; 1983): as early laggards drop out of the competition, players fight particularly fiercely and spend many resources at the beginning of the game to gain strategic momentum and discourage rivals.

very unlikely to win those cases" (Allison et al., 2010; p. 694).

4) Such present losses of fighting for reputation are offset by (the prospect of) future reputational benefits; in fact, a player who successfully invested in reputation can **cash in on her reputation without further exerting (costly) all-in efforts**. A reputational monopolist can cash in on her reputation (and obtain a strictly positive payoff) by exerting a non-all-in effort: indeed, a monopolist wins with strictly positive probability by exerting a negligible strictly positive effort. In other words, a reputation for fighting all-in, once built to high levels, can be used to intimidate rivals and mitigate the need for further acts of heinous costly violence. This finding is in line with evidence from the literature on criminal organizations. For instance, after a sequence of acts of violence, Cosa Nostra's members leverage such a violent reputation without actually needing further acts of heinous violence "because organized crime figures have a reputation for being able to execute threats of violence" (Reuter 1985, p. 56). Also, "many mafiosi have begun their careers with violent acts (Hess 1973; Arlacchi 1983), but have subsequently relied on the reputation with which such acts provided them: 'basta la fama'" (Gambetta, 2000).[43] While, in the applications we discussed, the benefit from having built a reputation for toughness may last long, in our framework such a benefit is cashed-in in a single period (i.e., when the monopolist's effort is non-all-in). This equilibrium property is the result of our stylized setup where a single period of non-all-in effort immediately and perfectly unmasks a player as rational. One could take inspiration from the reputational literature and extend our model, for instance, to either non-persistent types, or interior probability of fighting all-in for the automatons, or imperfectly observable efforts, or endogenous selection of the all-in fighting strategy; these extensions would add realism to the model as the reputation would not necessarily be consumed in a single period and cycles of violence may emerge. This is an interesting avenue for future research.

[43] Similarly, "a reputation for violence is an effective way of lowering the future expenses of running and maintaining a criminal enterprise" (Livingston, 2011; p. 7), and "the formation of reputation capital allows criminal organizations to expand within a violently competitive environment without constantly using actual warfare to settle conflicts. Continuous turf wars are both extremely expensive for OCGs [organized criminal groups] and disruptive of commerce so the mere threat of violence presents an attractive and lower-cost form of deterrence" (Livingston, 2011, p. 11).

Appendix

Throughout the Appendix, for brevity, we use the auction terminology "bid" rather than the contest terminology "exert effort." We use the notation p_t for the probability that a player with reputation bids 1 at period t, q_t for the probability that a player without reputation bids 0, F_t (G_t) for the CDF employed by a player with (without) reputation conditional on bidding less than 1 (more than 0), and d_t for the upper bound of the support of F_t and/or G_t. These objects take different values in different equilibrium configurations. We refer to Bayes' rule as $\varepsilon_{t+1} = \varepsilon_t / \left(\varepsilon_t + (1-\varepsilon_t) p_t\right)$.

Proof of Proposition 1. See Baye, Kovenock, and De Vries (1996). ■

Proof of Proposition 2. For $t = T$, we will see that $\pi_T^\nu > 0 = \pi_T^{-\nu}$. Consider the following strategies: the player with reputation bids on $[0, d_T]$ with CDF F_T (and obtains a nonnegative payoff), and the other players bid 0 w.p. $q_T \in [0,1]$ and with CDF G_T on $[0, d_T]$ w.p. $1 - q_T$ (and obtain a zero payoff).

A reputational monopolist bidding $x \in [0, d_T]$ obtains $\left(q_T + (1-q_T) G_T(x)\right)^{n-1} - x$, but also she obtains q_T^{n-1} by bidding an arbitrarily small, but strictly positive. This gives,

$$G_T(x) = \frac{\left(q_T^{n-1} + x\right)^{\frac{1}{n-1}} - q_T}{1 - q_T}, \tag{8}$$

so $G_T(0) = 0$ and $G_T(d_T) = 1 \iff d_T = 1 - q_T^{n-1}$. Players with no reputation bidding $x \in [0, d_T]$ obtain $(1-\varepsilon_T) F_T(x) \left(q_T + (1-q_T) G_T(x)\right)^{n-2} - x = 0$, which, by (8), gives

$$F_T(x) = \frac{x}{(1-\varepsilon_T)\left(q_T^{n-1} + x\right)^{\frac{n-2}{n-1}}}, \tag{9}$$

so $F_T(0) = 0$ and $F_T\left(1 - q_T^{n-1}\right) = 1 \iff q_T^{n-1} = \varepsilon_T$. Hence, $q_T^{n-1} = \varepsilon_T = 1 - d_T$, and thus the monopolist bids on $[0, 1-\varepsilon_T]$ with CDF F_T from (9) and the other players bid 0 w.p. $\varepsilon_T^{\frac{1}{n-1}}$ and on $[0, 1-\varepsilon_T]$ with CDF G_T from (8) w.p. $1 - \varepsilon_T^{\frac{1}{n-1}}$. The monopolist obtains $\pi_T^\nu = \varepsilon_T$. This concludes the characterization of the TSPBE in the statement of the proposition for $t = T$. Throughout, we omit the proofs of uniqueness which are standard (similarly to the method of proof in Baye, Kovenock, and De Vries, 1996), but lengthy.

In the remaining of this proof, we focus on $t \in \{2, ..., T-1\}$. To calculate the equilibrium in period $t \in \{2, .., T-1\}$, we need cumulative payoffs in $t+1$. From Proposition 1, if $\nu_{t+1} = 0$, such payoffs are 0 for all players in $t+1$. If $\nu_{t+1} > 0$, we assume that the cumulative payoff of the monopolist in $t+1$ is strictly positive and that of the others is 0; $\pi_{t+1}^{\nu} > 0 = \pi_{t+1}^{-\nu}$. We verify this assumption at the end of the proof.

As the cumulative payoff of the monopolist π_t^{ν} depends on the belief level ε_t, we adopt the notation $\pi_t^{\nu}(\varepsilon_t)$. To determine equilibrium in t, one needs the cumulative payoff $\pi_{t+1}^{\nu}(\varepsilon_{t+1})$, and ε_{t+1} depends on ε_t and p_t (the probability that the monopolist bids 1 in period t). In t, we denote the resulting ε_{t+1} as function of ε_t and p_t by $\varepsilon_{t+1}(\varepsilon_t, p_t)$. To lighten notation, we indicate these dependencies only when clarity requires it.

Consider the following strategies; the monopolist bids on $[0, d_t]$ with CDF F_t w.p. $1-p_t$ and 1 w.p. p_t, and all other players bid 0 w.p. q_t and on $[0, d_t]$ with CDF G_t w.p. $1-q_t$. The monopolist bidding an arbitrarily small, but strictly positive, amount obtains q_t^{n-1}, and bidding $x \in (0, d_t]$ obtains $(q_t + (1-q_t) G_t(x))^{n-1} - x$. Hence,

$$G_t(x) = \frac{\left(q_t^{n-1} + x\right)^{\frac{1}{n-1}} - q_t}{1-q_t}. \tag{10}$$

Thus, $G_t(d_t) = 1$ gives

$$q_t^{n-1} = 1 - d_t. \tag{11}$$

Note that the monopolist bidding an arbitrarily small amount gives up her reputation and hence obtains zero in the next period (see Proposition 1). The monopolist bidding 1 obtains $1 + \pi_{t+1}^{\nu}(\varepsilon_{t+1}(p_t, \varepsilon_t)) - 1 = q_t^{n-1}$, which gives

$$\pi_{t+1}^{\nu}(\varepsilon_{t+1}(p_t, \varepsilon_t)) = q_t^{n-1}. \tag{12}$$

Players with no reputation obtain $(1-\varepsilon_t)(1-p_t)F_t(x)\left(q_t + (1-q_t) G_t(x)\right)^{n-2} - x = 0$ when bidding $x \in [0, d_t]$. Therefore, by (10), we have

$$F_t(x) = \frac{x}{(1-\varepsilon_t)(1-p_t)(q_t^{n-1} + x)^{\frac{n-2}{n-1}}}, \tag{13}$$

and, by (11), $F_t(d_t) = 1$ gives $d_t = (1-\varepsilon_t)(1-p_t)$, which, together with (11) and (12), implies

$$\pi_{t+1}^{\nu}(\varepsilon_{t+1}(p_t, \varepsilon_t)) = q_t^{n-1} = 1 - d_t = \varepsilon_t + (1-\varepsilon_t) p_t. \tag{14}$$

To conclude the equilibrium characterization, we compute cumulative payoffs recursively proving

$$\pi_{t+1}^{\nu}(\varepsilon_{t+1}) = \varepsilon_{t+1}^{\frac{1}{T-t}} \implies \pi_t^{\nu}(\varepsilon_t) = \varepsilon_t^{\frac{1}{T-t+1}}. \tag{15}$$

To obtain $\pi_t^{\nu}(\varepsilon_t)$ from $\pi_{t+1}^{\nu}(\varepsilon_{t+1}(p_t,\varepsilon_t))$, we use Bayes' rule

$$\varepsilon_{t+1}(p_t,\varepsilon_t)=\frac{\varepsilon_t}{\varepsilon_t+(1-\varepsilon_t)\,p_t}. \tag{16}$$

Next, $\pi_{t+1}^{\nu}(\varepsilon_{t+1}(p_t,\varepsilon_t))=(\varepsilon_{t+1}(p_t,\varepsilon_t))^{\frac{1}{T-t}}=\left(\frac{\varepsilon_t}{\varepsilon_t+(1-\varepsilon_t)p_t}\right)^{\frac{1}{T-t}}$ by the hypothesis of (15) and (16). Hence, using the extremes of (14), we have that

$$\begin{aligned}\left(\frac{\varepsilon_t}{\varepsilon_t+(1-\varepsilon_t)\,p_t}\right)^{\frac{1}{T-t}} &= \varepsilon_t+(1-\varepsilon_t)\,p_t,\\ &\iff \varepsilon_t=(\varepsilon_t+(1-\varepsilon_t)\,p_t)^{T-t+1},\\ &\iff \varepsilon_t^{\frac{1}{T-t+1}}=\varepsilon_t+(1-\varepsilon_t)\,p_t.\end{aligned} \tag{17}$$

Hence,

$$\varepsilon_t+(1-\varepsilon_t)\,p_t=\varepsilon_t^{\frac{1}{T-t+1}}=\left(\frac{\varepsilon_t}{\varepsilon_t+(1-\varepsilon_t)\,p_t}\right)^{\frac{1}{T-t}},$$

and $\varepsilon_t+(1-\varepsilon_t)\,p_t=q_t^{n-1}$ by (14). Also,

$$\pi_t^{\nu}(\varepsilon_t)=q_t^{n-1}, \tag{18}$$

because q_t^{n-1} is the payoff that the monopolist obtains in period t by bidding an arbitrarily small, but strictly positive, amount. Thus, we have

$$\pi_t^{\nu}(\varepsilon_t)=\varepsilon_t^{\frac{1}{T-t+1}}. \tag{19}$$

The above proves (15). To fully characterize payoffs, we proceed by backward induction and repeatedly use (15). From the above analysis of the last period, $\pi_T^{\nu}(\varepsilon_T)=\varepsilon_T$. This fits the hypothesis of (15). Thus, $\pi_{T-1}^{\nu}(\varepsilon_{T-1})=\varepsilon_{T-1}^{\frac{1}{T-(T-1)+1}}=\sqrt{\varepsilon_{T-1}}$. All previous periods' cumulative payoffs follow similarly. Also, from the above analysis of the last period, $\pi_T^{-\nu}(\varepsilon_T)=0$. Then, by the considered strategies, in all previous periods $\pi_t^{-\nu}(\varepsilon_t)=0$. Those facts verify our initial assumption that, if $\nu_{t+1}=1$, the cumulative payoff of the monopolist in $t+1$ is strictly positive and that of the others is 0.

Having now the full payoff characterization, we complete the strategy characterization. First, (18) and (19) give $q_t=\varepsilon_t^{\frac{1}{(T-t+1)(n-1)}}$. Second, from (11), $d_t=1-q_t^{n-1}=1-\varepsilon_t^{\frac{1}{T-t+1}}$. Third, from (14),

$$p_t=1-\frac{d_t}{1-\varepsilon_t}=\frac{\varepsilon_t^{\frac{1}{T-t+1}}-\varepsilon_t}{1-\varepsilon_t}. \tag{20}$$

Finally, F_t follows from (13) and G_t from (10), so that the strategies match those in the statement of the proposition for $t \in \{2, \ldots, T-1\}$. ■

Proof of Lemma 1. Existence and uniqueness of $\bar{\varepsilon}_t(\nu_t)$ follows immediately from (4): the RHS strictly decreases in ε_t, takes value 2 when $\varepsilon_t = 0$ and value 0 when $\varepsilon_t = 1$. Now, we prove that $\bar{\varepsilon}_t(\nu_t) \leq 2/3$. The RHS of condition (4), evaluated at $\varepsilon_t = 2/3$, is smaller than 1 if and only if

$$
\begin{aligned}
\frac{1}{\nu_t}\sum_{l=0}^{\nu_t-1}\left(\frac{1}{3}\right)^l + \left(\frac{1}{3}\right)^{\nu_t-1} &< 1 \\
&\iff \frac{1}{\nu_t}\left(\frac{1-\left(\frac{1}{3}\right)^{\nu_t}}{1-\frac{1}{3}}\right) + \left(\frac{1}{3}\right)^{\nu_t-1} < 1 \\
&\iff \frac{3}{2\nu_t}\left((2\nu_t-1)\left(\frac{1}{3}\right)^{\nu_t} + 1\right) \leq 1 \\
&\iff (2\nu_t-1)\left(\frac{1}{3}\right)^{\nu_t-1} \leq 2\nu_t - 3 \\
&\iff \left(\frac{1}{3}\right)^{\nu_t-1} \leq 1 - \frac{2}{2\nu_t-1}.
\end{aligned}
$$

The LHS of the above-displayed inequality decreases in ν_t, its RHS increases in ν_t, and the inequality holds at $\nu_t = 2$. Therefore, the RHS of (4) is smaller than 1 for any $\nu_t \geq 2$ and any $\varepsilon_t \geq 2/3$. Hence, the solution of (4) must satisfy $\bar{\varepsilon}_t(\nu_t) \leq 2/3$.

Finally, we prove that $\bar{\varepsilon}_t(\nu_t)$ decreases in ν_t. We can rewrite (4) as

$$
\begin{aligned}
1 &= \frac{1-(1-\varepsilon_t)^{\nu_t}}{\varepsilon_t\nu_t} + (1-\varepsilon_t)^{\nu_t-1} \\
\iff &\nu_t\left(1-(1-\varepsilon_t)^{\nu_t-1}\right) = \frac{1}{\varepsilon_t} - \frac{(1-\varepsilon_t)^{\nu_t}}{\varepsilon_t} \\
\iff &\nu_t\left(1-(1-\varepsilon_t)^{\nu_t-1}\right) = \frac{1-\varepsilon_t}{\varepsilon_t} + 1 - \frac{(1-\varepsilon_t)^{\nu_t}}{\varepsilon_t} \\
\iff &\nu_t\left(1-(1-\varepsilon_t)^{\nu_t-1}\right) = \frac{1-\varepsilon_t}{\varepsilon_t}\left(1-(1-\varepsilon_t)^{\nu_t-1}\right) + 1 \\
\iff &\left(\nu_t - \frac{1-\varepsilon_t}{\varepsilon_t}\right)\left(1-(1-\varepsilon_t)^{\nu_t-1}\right) = 1.
\end{aligned}
$$

As the above-displayed equation is solved by $\bar{\varepsilon}_t(\nu_t)$, its LHS must be strictly positive at $\bar{\varepsilon}_t(\nu_t)$, so that $\nu_t > (1-\bar{\varepsilon}_t(\nu_t))/\bar{\varepsilon}_t(\nu_t)$. Therefore, around the solution $\varepsilon_t = \bar{\varepsilon}_t(\nu_t)$, the LHS of the above-displayed equation strictly increases in ν_t and in ε_t, and thus $\bar{\varepsilon}_t(\nu_t)$ strictly decreases in ν_t. ■

Proof of Proposition 3. For $t = T$, consider the following strategies; the ν_T players with reputation bid on $[0, d_T]$ with CDF F_T and the other players bid 0. A player with reputation bidding $x \in [0, d_T]$ obtains $((1-\varepsilon_T) F_T(x))^{\nu_T - 1} - x = 0$, which gives

$$F_T(x) = \frac{x^{\frac{1}{\nu_T - 1}}}{1 - \varepsilon_T}, \tag{21}$$

and $F_T(d_T) = 1$ yields $d_T = (1-\varepsilon_T)^{\nu_T - 1}$. Therefore, F_T in (21) matches the statement of the proposition for $t = T$. Any one of the $n - \nu_T$ players without reputation bidding $x \in (0, d_T]$ obtains $((1-\varepsilon_T) F_T(x))^{\nu_T} - x < 0 \iff x^{\nu_T/(\nu_T - 1)} - x < 0$; thus, such deviation is not profitable. This concludes the characterization of the TSPBE in period T. Throughout, we omit the proofs of uniqueness which are standard (similarly to the method of proof in Baye, Kovenock, and De Vries, 1996), but lengthy.

In the remaining of this proof, focus on $t \in \{1, ..., T-1\}$. To calculate equilibrium in $t \in \{1, .., T-1\}$, we need cumulative payoffs in $t+1$. From Proposition 1, if $\nu_{t+1} = 0$, then cumulative payoffs in $t+1$ are 0 for all players. As shown in Proposition 2, if $\nu_t = 1$, then $\pi_t^{\nu} > 0 = \pi_t^{-\nu}$. Assume that, in any t, if $\nu_t \geq 2$, then all cumulative payoffs are 0. We verify this assumption at the end of the proof.

As the cumulative payoff of a player with reputation π_t^{ν} depends on the belief level ε_t, we adopt notation $\pi_t^{\nu}(\varepsilon_t)$. To determine equilibrium in t, one needs the cumulative payoff $\pi_{t+1}^{\nu}(\varepsilon_{t+1})$, and ε_{t+1} depends on ε_t and p_t (the probability that the monopolist bids 1 in period t). In t, we denote the resulting ε_{t+1} as function of ε_t and p_t by $\varepsilon_{t+1}(\varepsilon_t, p_t)$. To lighten notation, we indicate these dependencies only when clarity requires it.

Consider the following strategies: players with reputation bid with CDF F_t on $[0, d_t]$ w.p. $1 - p_t$ and 1 w.p. p_t, and players without reputation bid 0. A player with reputation bidding $x \in [0, d_t]$ obtains $((1-\varepsilon_t)(1-p_t) F_t(x))^{\nu_t - 1} - x = 0$, implying

$$F_t(x) = \frac{x^{\frac{1}{\nu_t - 1}}}{(1-\varepsilon_t)(1-p_t)}. \tag{22}$$

Note that the player with reputation bidding an arbitrarily small amount gives up her reputation and hence obtains zero in the next period (see Proposition 1).

A player without reputation bidding $x \in [0, d_t]$ obtains $(1-\varepsilon_t)^{\nu_t}(1-p_t)^{\nu_t} F_t(x)^{\nu_t} - x < 0$, which holds true by (22), and thus she has no incentive to deviate to bidding $x \in [0, d_t]$. Since $F_t(d_t) = 1$, we have that, using (22),

$$d_t = ((1-\varepsilon_t)(1-p_t))^{\nu_t - 1}. \tag{23}$$

(22) and (23) give the strategy in the statement of the proposition for $t \in \{1, ..., T-1\}$.

Next, we determine p_t. A player with reputation bidding 1 must obtain a cumulative payoff equal to 0; this implies that the cost of 1 must equal the future cumulative payoff in case of becoming a monopolist plus the current period's prize accounting for ties, or

$$\begin{aligned} 1 &= (1-\varepsilon_t)^{\nu_t-1}(1-p_t)^{\nu_t-1}\pi^{\nu}_{t+1}(\varepsilon_{t+1}(p_t,\varepsilon_t)) \\ &+ \sum_{l=0}^{\nu_t-1}\frac{1}{l+1}\binom{\nu_t-1}{l}(\varepsilon_t+(1-\varepsilon_t)p_t)^l(1-(\varepsilon_t+(1-\varepsilon_t)p_t))^{\nu_t-1-l} \\ \Longleftrightarrow\ 1 &= \frac{1}{\nu_t}\sum_{l=0}^{\nu_t-1}(1-\varepsilon_t)^l(1-p_t)^l+(1-\varepsilon_t)^{\nu_t-1}(1-p_t)^{\nu_t-1}\pi^{\nu}_{t+1}(\varepsilon_{t+1}(p_t,\varepsilon_t)), \end{aligned}$$

which is equivalent to (3) as $\pi^{\nu}_{t+1}(\varepsilon_{t+1}) = \varepsilon_{t+1}^{\frac{1}{T-t}} = \left(\frac{\varepsilon_t}{\varepsilon_t+(1-\varepsilon_t)p_t}\right)^{\frac{1}{T-t}}$ (by Proposition 2 and Bayes' rule). Thus, the above-displayed equation can be written as

$$\frac{1-\frac{1-(1-(\varepsilon_t+p_t(1-\varepsilon_t)))^{\nu_t}}{(\varepsilon_t+p_t(1-\varepsilon_t))\nu_t}}{(1-(\varepsilon_t+p_t(1-\varepsilon_t)))^{\nu_t-1}} = \left(\frac{\varepsilon_t}{\varepsilon_t+(1-\varepsilon_t)p_t}\right)^{\frac{1}{T-t}}. \tag{24}$$

Using notation $z \equiv 1-(\varepsilon_t+p_t(1-\varepsilon_t))$, we rewrite (24) as

$$\begin{aligned} 1-\frac{1-z^{\nu_t}}{(1-z)\nu_t} &= z^{\nu_t-1}\left(\frac{\varepsilon_t}{1-z}\right)^{\frac{1}{T-t}}, \\ 1-\frac{1+z+z^2+...+z^{\nu_t-1}}{\nu_t} &= z^{\nu_t-1}\left(\frac{\varepsilon_t}{1-z}\right)^{\frac{1}{T-t}}. \end{aligned} \tag{25}$$

We next show that a solution $z \in [0, 1-\varepsilon_t]$ to (25) exists if and only if ε_t is sufficiently small, and this solution is unique. It then follows that, when a solution $z \in [0, 1-\varepsilon_t]$ to (25) exists, then there is a unique solution $p_t \in [0,1]$ to (24).

The LHS of (25) strictly decreases in z and the RHS strictly increases. Thus, at most one solution exists. If $z = 0$ (or $p_t = 1$), the LHS of (25) is strictly greater than its RHS. Then, a unique solution $z \in [0, 1-\varepsilon_t]$ to (25) exists if and only if, at $z = 1-\varepsilon_t$ (or $p_t = 0$), the LHS of (25) is strictly smaller than its RHS, or equivalently,

$$\begin{aligned} & 1-\frac{1+(1-\varepsilon_t)+...+(1-\varepsilon_t)^{\nu_t-1}}{\nu_t} < (1-\varepsilon_t)^{\nu_t-1} \\ \Longleftrightarrow\ & \nu_t\left(1-(1-\varepsilon_t)^{\nu_t-1}\right) < 1+(1-\varepsilon_t)+...+(1-\varepsilon_t)^{\nu_t-1} \\ \Longleftrightarrow\ & \nu_t\left(1-(1-\varepsilon_t)^{\nu_t-1}\right) < \frac{1-(1-\varepsilon_t)^{\nu_t}}{\varepsilon_t}. \end{aligned} \tag{26}$$

We now show that (26) is satisfied if and only if ε_t is sufficiently small (later, we consider the remaining case of ε_t sufficiently large); that is, $\varepsilon_t \leq \bar{\varepsilon}_t(\nu_t)$, where $\bar{\varepsilon}_t(\nu_t)$ is the unique solution for ε_t of (3) with $p_t = 0$, or

$$\nu_t \left(1 - (1 - \varepsilon_t)^{\nu_t - 1}\right) = \frac{1 - (1 - \varepsilon_t)^{\nu_t}}{\varepsilon_t},$$

which is equivalent to (4) by $\frac{1-(1-\varepsilon_t)^{\nu_t}}{\varepsilon_t} = \sum_{l=0}^{\nu_t - 1} (1 - \varepsilon_t)^l$. The above-displayed equation is uniquely solved by $\varepsilon_t = \bar{\varepsilon}_t(\nu_t)$ by Lemma 1. Now, note that the LHS of the above-displayed equation increases in ε_t and its RHS decreases as it equals $\sum_{l=0}^{\nu_t - 1} (1 - \varepsilon_t)^l$. Therefore, (26) is satisfied for $\varepsilon_t \leq \bar{\varepsilon}_t(\nu_t)$. Hence, we proved that, $\forall \varepsilon_t \leq \bar{\varepsilon}_t(\nu_t)$, the unique equilibrium is as in the statement of the proposition. To conclude the equilibrium characterization for $\varepsilon_t \leq \bar{\varepsilon}_t(\nu_t)$, note that p_t is characterized by (24), and then p_t and (23) give d_t.

We now consider ε_t large; that is, $\varepsilon_t > \bar{\varepsilon}_t(\nu_t)$. Proceeding as for the case of $t = T$, one can show that, if all other rational players use F_t with $p_t = 0$ as described in the proposition, then any bid $x \in \left[0, (1 - \varepsilon_t)^{\nu_t - 1}\right]$ yields 0. A deviation to an all-in effort 1, yields

$$\frac{1 - (1 - \varepsilon_t)^{\nu_t}}{\varepsilon_t \nu_t} + (1 - \varepsilon_t)^{\nu_t - 1} - 1. \tag{27}$$

Note that, as discussed after (26) above, $\varepsilon_t > \bar{\varepsilon}_t(\nu_t)$ implies

$$\nu_t \left(1 - (1 - \varepsilon_t)^{\nu_t - 1}\right) > \frac{1 - (1 - \varepsilon_t)^{\nu_t}}{\varepsilon_t},$$

which in turn implies that (27) is negative. Thus, a deviation to 1 is not profitable. Therefore, when $\varepsilon_t > \bar{\varepsilon}_t(\nu_t)$, the equilibrium strategies are as in the $t = T$ case.

Next, we prove that $p_t \in [0, 2/3]$. Equation (24) is equivalent to

$$\frac{(\varepsilon_t + p_t(1-\varepsilon_t))(1 - (\varepsilon_t + p_t(1-\varepsilon_t)))^{\nu_t - 1}}{(\varepsilon_t + p_t(1-\varepsilon_t))\nu_t - 1 + (1 - (\varepsilon_t + p_t(1-\varepsilon_t)))^{\nu_t}} = \frac{\varepsilon_t \left(\frac{\varepsilon_t + (1-\varepsilon_t)p_t}{\varepsilon_t}\right)^{\frac{T-t+1}{T-t}}}{\nu_t(\varepsilon_t + (1-\varepsilon_t)p_t)}. \tag{28}$$

We now show that the LHS of (28) decreases in p_t and the RHS increases. The latter follows because $(T - t + 1) / (T - t) > 1$. To see that the LHS of (28) decreases in p_t for fixed ε_t, use the notation $z = 1 - (\varepsilon_t + p_t(1 - \varepsilon_t))$ to write the LHS of (28) as

$$\frac{(1 - z) z^{\nu_t - 1}}{(1 - z)\nu_t - 1 + z^{\nu_t}}. \tag{29}$$

The derivative of (29) with respect to z has the same sign of $\kappa(\nu_t)$ with

$$\kappa(\nu_t) \equiv -2\nu_t - z^{\nu_t} + \nu_t(\nu_t + z(-2\nu_t + (\nu_t - 1)z + 3)) + 1,$$

and we now show that $\kappa(2) > 0$ and $\kappa(\nu_t + 1) > \kappa(\nu_t)$, concluding the proof that the LHS of (28) decreases in p_t. Indeed, $\kappa(2) = (1-z)^2 > 0$ and $\kappa(\nu_t + 1) > \kappa(\nu_t) \iff (1-z)(2\nu_t(1-z) - (1 - z^{\nu_t})) > 0 \iff 2\nu_t - (1 + z + ... + z^{\nu_t - 1}) > 0$, which holds true because $1 + z + ... + z^{\nu_t-1} \leq \nu_t$ by $z < 1$. This shows that the LHS of (28) decreases in p_t and the RHS increases.

Next, we know that, if $p_t = 0$, the LHS of (28) is larger than the RHS of (28) if and only if $\varepsilon_t\nu_t\left(1 - (1-\varepsilon_t)^{\nu_t - 1}\right) < (1 - (1-\varepsilon_t)^{\nu_t})$ which holds true (see the discussion after (25)). We now show that there is no possible solution of (28) with $p_t > 2/3$ by showing that, if $p_t = 2/3$, the LHS of (28) is still smaller than the RHS. As the RHS of (28) increases in $T-t$, it is greater than its value at $T-t=1$, which is $1/\varepsilon_t\,(1/3 + 2/(3\varepsilon_t))$. This value is in turn greater than the LHS of (28) if and only if $9\varepsilon_t\nu_t(1-\varepsilon_t)^{\nu_t-1} < 3(1-\varepsilon_t)^{\nu_t} + 3^{\nu_t}((\varepsilon_t + 2)\nu_t - 3)$, which holds true because its LHS is maximized at $\varepsilon_t = 1/\nu_t$, where it takes value $9\left((\nu_t - 1)/\nu_t\right)^{\nu_t - 1} < 9$, and its RHS is larger than 9 as $3(1-\varepsilon_t)^{\nu_t} + 3^{\nu_t}((\varepsilon_t+2)\nu_t - 3) > 3^{\nu_t}((0+2)\nu_t - 3) \geq 3^2 = 9$. Hence, we proved that for any ε_t, $p_t \leq 2/3$.

Finally, we assumed that, if $\nu_t \geq 2$, $\pi_t^\nu = 0$. It remains to show that there is no equilibrium such that $\pi_t^\nu = \rho > 0$; if so, the analogue of (22) would be

$$\pi_t^\nu = \left((1-\varepsilon_t)(1-p_t)F_t(x)\right)^{\nu_t - 1} - x = \rho \iff F_t(x) = \frac{x^{\frac{1}{\nu_t - 1}} + \rho}{(1-\varepsilon_t)(1-p_t)},$$

and hence players with reputation would bid 0 with strictly positive probability, which is a contradiction because players with reputation would then find it profitable to bid an arbitrarily small, but strictly positive, amount. ■

Proof of Proposition 4. We start with the second statement. By Bayes' rule,

$$\varepsilon_{t+1} = \frac{\varepsilon_t}{\varepsilon_t + (1-\varepsilon_t)p_t}.$$

By Proposition 2,

$$p_t = \frac{\varepsilon_t^{\frac{1}{T-t+1}} - \varepsilon_t}{1 - \varepsilon_t}.$$

Hence, $\varepsilon_{t+1} = \varepsilon_t^{\frac{T-t}{T-t+1}}$. Now, since $\frac{T-t}{T-t+1} \leq 1$ and $\varepsilon_t \in (0,1)$, $\varepsilon_{t+1} > \varepsilon_t$. The first statement follows as $\varepsilon_t > 0$ implies $p_t > 0$ in every non-terminal period of reputational monopoly. ■

Proof of Proposition 5. We use the equilibrium described in Proposition 2 and notation $L \equiv T - t + 1$ to denote the number of periods remaining from t onwards. We obtain that the statement of the proposition is equivalent to

$$\begin{aligned}
& \tfrac{1-\varepsilon_t^{\frac{1}{L}}}{1-\varepsilon_t} \int_0^{1-\varepsilon_t^{\frac{1}{L}}} x dF_t(x) + \tfrac{\varepsilon_t^{\frac{1}{L}}-\varepsilon_t}{1-\varepsilon_t} + (n-1)\left(1-\varepsilon_t^{\frac{1}{L(n-1)}}\right) \int_0^{1-\varepsilon_t^{\frac{1}{L}}} x dG_t(x) \leq 1 \\
\iff\quad & \tfrac{1}{1-\varepsilon_t} \int_0^{1-\varepsilon_t^{\frac{1}{L}}} x d\left(\frac{x}{\left(\varepsilon_t^{\frac{1}{L}}+x\right)^{\frac{n-2}{n-1}}}\right) + \tfrac{\varepsilon_t^{\frac{1}{L}}-\varepsilon_t}{1-\varepsilon_t} + (n-1)\int_0^{1-\varepsilon_t^{\frac{1}{L}}} x d\left(\varepsilon_t^{\frac{1}{L}}+x\right)^{\frac{1}{n-1}} \leq 1 \\
\iff\quad & \tfrac{1}{1-\varepsilon_t}\left(-\varepsilon_t - 2\varepsilon_t^{\frac{1}{L}} + \varepsilon_t^{\frac{2}{L}} + n\varepsilon_t^{\frac{1}{L}} - \tfrac{(n-1)^2}{n}\left(\varepsilon_t^{\frac{1}{L}}\right)^{\frac{n}{n-1}} + \tfrac{1}{n}\right) \leq 1 \\
\iff\quad & (n-1)\left(n\varepsilon_t^{\frac{1}{L}} - (n-1)\left(\varepsilon_t^{\frac{1}{L}}\right)^{\frac{n}{n-1}} - 1\right) \leq n\left(\varepsilon_t^{\frac{1}{L}} - \varepsilon_t^{\frac{2}{L}}\right).
\end{aligned}$$

The RHS of the above-displayed equation is positive and its LHS is negative because it takes value 0 as $\varepsilon_t = 1$ and

$$\frac{d\left(n\varepsilon_t^{\frac{1}{L}} - (n-1)\left(\varepsilon_t^{\frac{1}{L}}\right)^{\frac{n}{n-1}} - 1\right)}{d\varepsilon_t^{\frac{1}{L}}} = n\left(1-\left(\varepsilon_t^{\frac{1}{L}}\right)^{\frac{1}{n-1}}\right) > 0.$$

This concludes the proof. ■

Proof of Proposition 6. The unique equilibrium is characterized in Proposition 2. Let $\tilde{G}_t(x) \equiv q_t + (1-q_t) G_t(x)$. We obtain

$$\tilde{G}_t(x) = \left(\varepsilon_t^{\frac{1}{T-t+1}} + x\right)^{\frac{1}{n-1}}.$$

Recalling that $L = T - t + 1$, the monopolist wins in period t with probability

$$\begin{aligned}
& \frac{\varepsilon_t^{\frac{1}{L}} - \varepsilon_t}{1-\varepsilon_t} + \frac{1-\varepsilon_t^{\frac{1}{L}}}{1-\varepsilon_t} \int_0^{1-\varepsilon_t^{\frac{1}{L}}} \tilde{G}_t(x)^{n-1} dF_t(x) \\
= \quad & \frac{1}{n} + \frac{\varepsilon^{\frac{1}{L}} - (n-1)^2 \varepsilon^{\frac{n}{L(n-1)}} + (n-1)n\varepsilon^{\frac{1}{L}} - n\varepsilon^{\frac{1}{L}}}{n\left(1-\varepsilon^{\frac{1}{L}}\right)} \\
= \quad & \frac{1}{n} + (n-1)\frac{-(n-1)\varepsilon^{\frac{n}{L(n-1)}} + n\varepsilon^{\frac{1}{L}} - \varepsilon^{\frac{1}{L}}}{n\left(1-\varepsilon^{\frac{1}{L}}\right)},
\end{aligned}$$

which is equal to the expression in (5). ■

Proof of Corollary 1. Letting $\phi \equiv \varepsilon_t^{\frac{1}{T-t+1}}$, (5) is equivalent to

$$\frac{1}{n} + \frac{(n-1)^2}{n} \frac{\phi - \phi^{\frac{n}{n-1}}}{1-\phi}, \tag{30}$$

which is increasing in ϕ as its derivative with respect to ϕ is

$$\frac{(n-1)\left((\phi - n)\phi^{\frac{1}{n-1}} + (n-1)\right)}{n(\phi-1)^2},$$

which, in turn, is positive because its numerator is decreasing in ϕ and takes value 0 at $\phi = 1$. Note that ϕ increases in ε_t and decreases in t (for fixed ε_t). Hence, the derivative of (30) with respect to n, which we label $\beta(\phi) \equiv -\phi^{\frac{n}{n-1}}(n^2 - n\log(\phi) - 1) + n^2\phi - 1$, is negative as $\lim_{\phi\to 0} \beta(\phi) = -1 \leq 0$, $\beta(1) = 0$, and $\partial^2\beta(\phi)/\partial\phi^2 = \frac{n^2\phi^{\frac{1}{n-1}-1}\log(\phi)}{(n-1)^2} < 0$. ■

Proof of Proposition 7. We begin with the proof of part 1. of the proposition. When $\varepsilon_t \in [\bar{\varepsilon}_t(\nu_t), 1]$, from Proposition 3, in $\tau = t+1$ no player has reputation and the result follows. Hence, we focus on $\varepsilon_t \in (0, \bar{\varepsilon}_t(\nu_t)]$, a range where players may bid 1 in equilibrium. Recall from Lemma 1 that $\bar{\varepsilon}_t(\nu_t) \leq 2/3$, and hence $\nu_\tau \geq 2$ is not sustainable whenever $\varepsilon_\tau > 2/3$. Thus, we derive the maximum number of periods to sustain $\nu_\tau \geq 2$ by focusing on the slowest possible path of increase in ε_τ. As $p_t \leq 2/3$ from Proposition 3, Bayes' rule implies that the slowest possible increase of ε between periods t and $t+1$ is given by

$$\varepsilon_{t+1} \geq \frac{\varepsilon_t}{\varepsilon_t + (1-\varepsilon_t)(2/3)} = \frac{3\varepsilon_t}{\varepsilon_t + 2}.$$

Therefore, $\forall \tau \geq t+1$, the lowest possible value of ε_τ equals

$$\varepsilon_\tau = \frac{3^{\tau-t}\varepsilon_t}{2^{\tau-t}(1-\varepsilon_t) + 3^{\tau-t}\varepsilon_t}, \tag{31}$$

as the above solves the difference equation $\varepsilon_{n+1} = \frac{3\varepsilon_n}{\varepsilon_n+2}$ initialized at $\varepsilon_n = \varepsilon_t$. Thus, even an increase of ε from period t to τ along the slowest possible path surpasses 2/3 when $\varepsilon_\tau \geq 2/3$ in (31), or

$$\begin{aligned} 2\left(2^{\tau-t}(1-\varepsilon_t) + 3^{\tau-t}\varepsilon_t\right) \quad &\geq \quad 3^{\tau-t+1}\varepsilon_t \iff 2^{\tau-t+1}(1-\varepsilon_t) + 2 * 3^{\tau-t}\varepsilon_t \geq 3^{\tau-t+1}\varepsilon_t \\ &\iff \quad 2^{\tau-t+1}(1-\varepsilon_t) \geq 3^{\tau-t}\varepsilon_t, \end{aligned}$$

which is equivalent to (6). This concludes the proof of part 1. of the proposition.

To prove part 2. of the proposition, note that (3) can be written as

$$1 = \frac{1-(1-(\varepsilon_t+p_t(1-\varepsilon_t)))^{\nu_t}}{(\varepsilon_t+p_t(1-\varepsilon_t))\nu_t} + (1-(\varepsilon_t+p_t(1-\varepsilon_t)))^{\nu_t-1}\left(\frac{\varepsilon_t}{\varepsilon_t+p_t(1-\varepsilon_t)}\right)^{\frac{1}{T-t}}. \quad (32)$$

We now write Bayes' rule as

$$\varepsilon_t + (1-\varepsilon_t)\,p_t = 1-(1-\varepsilon_t)(1-p_t) = \frac{\varepsilon_t}{\varepsilon_{t+1}},$$

and use this to rewrite (32) as (7). Note that existence and uniqueness of ε_{t+1} in (7) follow from existence and uniqueness of p_t demonstrated in Proposition 3 and Bayes' rule. Furthermore, Bayes' rule, together with $p_t \leq 2/3$ (see Proposition 3), implies that $\varepsilon_{t+1} > \varepsilon_t$. ■

Proof of Proposition 8. We use the equilibrium described in Proposition 3, where we focus on $\varepsilon_t \in (0, \bar{\varepsilon}_t(\nu_t)]$. The strategy of proof is to show that aggregate effort converges to 1 when $\varepsilon_t \to 0$ and increases in ε_t when $\varepsilon_t = 0$. First, we calculate the limit of p_t as $\varepsilon_t \to 0$. Second, we derive the key quantity of interest $\lim_{\varepsilon_t\to 0}(1-z)\frac{dz}{d\varepsilon_t}$, which helps us to characterize the behavior of aggregate effort near $\varepsilon_t = 0$. Third, we use these two to characterize aggregate effort.

First, using (25) in the Proof of Proposition 3 and notation $z \equiv 1-(\varepsilon_t+(1-\varepsilon_t)\,p_t)$, if $\varepsilon_t \to 0$, then the only solution is $z \to 1$, which implies $p_t \to 0$.

Second, take limits as $\varepsilon_t \to 0$ in (25) and obtain

$$\lim_{\varepsilon_t\to 0}\left(1-\frac{1+z+z^2+...+z^{\nu_t-1}}{\nu_t}\right) = 1-\frac{\nu_t}{\nu_t} = 0 = \lim_{\varepsilon_t\to 0} z^{\nu_t-1}\left(\frac{\varepsilon_t}{1-z}\right)^{\frac{1}{T-t}},$$

and therefore $\lim_{\varepsilon_t\to 0}(\varepsilon_t/(1-z))^{\frac{1}{T-t}} = 0$. Now, rewrite (25) as

$$\frac{\left(\frac{1-\frac{1+z+z^2+...+z^{\nu_t-1}}{\nu_t}}{z^{\nu_t-1}}\right)^{T-t}}{1-z} = \frac{\varepsilon_t}{(1-z)^2}. \quad (33)$$

Taking limits again and using L'Hôpital's rule, the LHS of (33), recalling that $z \to 1$ when

$\varepsilon_t \to 0$, equals

$$\lim_{z\to 1} \frac{(T-t)\left(\frac{1-\frac{1+z+z^2+...+z^{\nu_t-1}}{\nu_t}}{z^{\nu_t-1}}\right)^{T-t-1} \frac{-\frac{1}{\nu_t}\sum_{l=1}^{\nu_t-1} l z^{l-1} z^{\nu_t-1}-(\nu_t-1)\left(1-\frac{1+z+z^2+...+z^{\nu_t-1}}{\nu_t}\right) z^{\nu_t-2}}{(z^{\nu_t-1})^2}}{-1}$$

$$= \begin{cases} \frac{\nu_t-1}{2} & \text{if } T-t=1, \\ 0 & \text{if } T-t\geq 2. \end{cases}$$

Similarly, by L'Hôpital's rule, the RHS of (33) is equivalent to

$$\lim_{\varepsilon_t\to 0} \frac{\varepsilon_t}{(1-z)^2} = \lim_{\varepsilon_t\to 0} \frac{1}{-2(1-z)\frac{dz}{d\varepsilon_t}}.$$

Hence, if $T-t=1$,

$$\lim_{\varepsilon_t\to 0} (1-z)\frac{dz}{d\varepsilon_t} = \frac{1}{1-\nu_t}, \tag{34}$$

and if $T-t\geq 2$,

$$\lim_{\varepsilon_t\to 0} (1-z)\frac{dz}{d\varepsilon_t} = -\infty. \tag{35}$$

Third, recall from Proposition 3 that players without reputation bid 0. Denoting as $X_t(\varepsilon_t)$ the expected bid in period t of a player with reputation, we obtain

$$\begin{aligned} X_t(\varepsilon_t) &= (1-p_t)\int_0^{((1-\varepsilon_t)(1-p_t))^{\nu_t-1}} x * d\frac{x^{\frac{1}{\nu_t-1}}}{(1-\varepsilon_t)(1-p_t)} + p_t \\ &\iff X_t(\varepsilon_t) = \frac{(1-\varepsilon_t)^{\nu_t-1}(1-p_t)^{\nu_t}}{\nu_t} + p_t \\ &\iff \nu_t X_t(\varepsilon_t) - \nu_t = \frac{z^{\nu_t}}{1-\varepsilon_t} - \nu_t(1-p_t) \\ &\iff \nu_t(X_t(\varepsilon_t)-1)(1-\varepsilon_t) = z^{\nu_t} - \nu_t z. \end{aligned} \tag{36}$$

Implicitly differentiate (36) with respect to ε_t and, as z depends on ε_t, obtain

$$\frac{dX_t(\varepsilon_t)}{d\varepsilon_t}(1-\varepsilon_t) - (X_t(\varepsilon_t)-1) = (z^{\nu_t-1}-1)\frac{dz}{d\varepsilon_t} = (1+z+z^2+...+z^{\nu_t-2})(z-1)\frac{dz}{d\varepsilon_t}.$$

We now take the limits of the above-displayed equation and, as $\lim_{\varepsilon_t\to 0} X_t(\varepsilon_t) \to \frac{1}{\nu_t}$ from (36) and $\lim_{\varepsilon_t\to 0}(1+z+z^2+...+z^{\nu_t-2}) = \nu_t - 1$, obtain

$$\lim_{\varepsilon_t\to 0} \frac{dX_t(\varepsilon_t)}{d\varepsilon_t} = \frac{1}{\nu_t} - 1 - (\nu_t-1)\lim_{\varepsilon_t\to 0}(1-z)\frac{dz}{d\varepsilon_t}.$$

The above-displayed equation, if $T-t=1$, by (34), gives

$$\lim_{\varepsilon_t \to 0} \frac{dX_t(\varepsilon_t)}{d\varepsilon_t} = \frac{1}{\nu_t} - 1 - (\nu_t - 1)\frac{1}{1-\nu_t} = \frac{1}{\nu_t},$$

and if $T-t \geq 2$, by (35), gives

$$\lim_{\varepsilon_t \to 0} \frac{dX_t(\varepsilon_t)}{d\varepsilon_t} = +\infty.$$

The proof is concluded observing that, for $\varepsilon_t = 0$, there is full-rent dissipation (as $\nu_t X_t(0) = 1$) and for $\varepsilon_t \to 0$ the aggregate bid strictly increases in ε_t. ■

Proof of Corollary 2. Call $\bar{\tau} \equiv \left\lceil 1 + \frac{\log\left(\frac{1}{2}\frac{\varepsilon_1}{1-\varepsilon_1}\right)}{\log\left(\frac{2}{3}\right)} \right\rceil$. By Proposition 7, for any ε_1, if $\bar{\tau} \leq T-1$, then we cannot have a reputational oligopoly from period $\bar{t} = \bar{\tau}$ onwards. By Proposition 5 and Proposition 1, the expected aggregate bid is smaller than 1 in any period $t \geq \bar{t}$. If instead $\bar{\tau} > T-1$, then we set $\bar{t} = T-1$ and note that, even if a reputational oligopoly occurs in period T, Proposition 3 implies that the last-period expected aggregate effort is $(1-\varepsilon_T)^{\nu_T - 1} < 1$. ■

Proof of Proposition 9. The proposition immediately follows noting that, in the equilibrium described in Proposition 3, all players without reputation bid 0 with certainty. ■

Type-asymmetric equilibria and casus irreducibilis: Section 6.

We focus on equilibria where, whenever players bid 1, they do so with identical probability. Hence, if any two rational players have reputation in a period, their reputation lever is identical. In fact, recall from Section **6** that introducing in the model ex-ante asymmetries allows players to enter a certain period with different levels of (positive) reputation, and this severely jeopardizes the tractability of our framework.

We consider asymmetric equilibria when $T=2$, $n=3$, and $\varepsilon \in \left(0, \left(9-\sqrt{33}\right)/8\right)$. (Recall from the main body that, when $\varepsilon = 0$ or $\varepsilon \geq \frac{9-\sqrt{33}}{8}$—see Footnote 34—the payoff at $t=2$ is 0 and hence reputation building plays no role.) Consider the following strategies in $t=1$: all three players bid 1 w.p. q, players 1 and 2 play $F_1(x)$ on $x \in \left[0, \bar{d}\right]$ w.p. $1-q$, and player 3 plays $G_1(x)$ on $x \in \left[\underline{d}, \bar{d}\right]$ w.p. p and 0 w.p. $1-p-q$. Then, the equilibrium characterization in the main body implies that the payoffs in $t=2$ are 0 unless $\nu_2 = 1$; in such case, the monopolist obtains $\varepsilon/(\varepsilon + (1-\varepsilon)q)$ by Bayes' rule and Proposition 2.

Player 1 (or, equivalently, player 2) by bidding $x \in \left[0, \bar{d}\right]$ obtains a cumulative payoff equal to $(1-\varepsilon)(1-q)F_1(x)(1-\varepsilon)(1-p-q) - x = 0$, whereas if she bids $x \in \left[\underline{d}, \bar{d}\right]$, she obtains $(1-\varepsilon)(1-q)F_1(x)((1-\varepsilon)(1-p-q)+(1-\varepsilon)p*G_1(x))-x=0$. Player 3 by bidding $x\in[\underline{d},\bar{d}]$ obtains a cumulative payoff equal to $(1-\varepsilon)^2(1-q)^2F_1(x)^2-x=0$. From those conditions, we obtain

$$F_1(x)=\begin{cases}\frac{x}{(1-\varepsilon)^2(1-q)(1-p-q)} & \text{if } x\in[0,\bar{d}]\\ \frac{\sqrt{x}}{(1-\varepsilon)(1-q)} & \text{if } x\in[\underline{d},\bar{d}]\end{cases} \quad \text{and } G_1(x)=\frac{\sqrt{x}-(1-\varepsilon)(1-p-q)}{(1-\varepsilon)p} \text{ if } x\in[\underline{d},\bar{d}].$$

Using $F_1(\bar{d})=G_1(\bar{d})=1$ and $G_1(\underline{d})=0$ gives $\underline{d}=(1-\varepsilon)^2(1-p-q)^2$ and $\bar{d}=(1-\varepsilon)^2(1-q)^2$. Finally, any of the three players who bids 1 would obtain a cumulative payoff of[44]

$$(1-\varepsilon)^2(1-q)^2\left(1+\frac{\varepsilon}{\varepsilon+(1-\varepsilon)q}\right)+2(1-\varepsilon)(1-q)(\varepsilon+(1-\varepsilon)q)\cdot\frac{1}{2}+(\varepsilon+(1-\varepsilon)q)^2\cdot\frac{1}{3}-1=0,$$

which, by simple manipulations, can be written as $\tau(q,\varepsilon)\equiv Aq^3+Bq^2+Cq+D=0$ where

$$A\equiv-(1-\varepsilon)^3,\ B\equiv3(1-\varepsilon)^2(1-2\varepsilon),\ C\equiv3(1-\varepsilon)\varepsilon(4-3\varepsilon),\ D\equiv-\varepsilon\left(4\varepsilon^2-9\varepsilon+3\right).$$

The equation $\tau(q,\varepsilon)=0$ has three roots in q. One of the three roots is always negative; in fact $\tau(0,\varepsilon)=D<0$ by $\varepsilon\in\left(0,\frac{9-\sqrt{33}}{8}\right)$ and $\lim_{x\to-\infty}\tau(x,\varepsilon)=\infty>0$. Another root is larger than 1; in fact $\tau(1,\varepsilon)=2>0$ and $\lim_{x\to\infty}\tau(x,\varepsilon)=-\infty<0$. The third root has $q\in(0,1/2)$; in fact, $\tau(0,\varepsilon)=D<0$ and $\tau(1/2,\varepsilon)=\frac{5+3\varepsilon+15\varepsilon^2-7\varepsilon^3}{8}>\frac{5}{8}$ by $\varepsilon\in(0,1)$. Hence, in the only admissible solution, $q\in(0,1/2)$ and we can choose $p\in(0,1/2)$, so that $\underline{d}\in(0,1)$. However, this unique admissible real root of $\tau(q,\varepsilon)=0$ is a casus iriducibilis: it is real-valued but it cannot be expressed in radicals without using complex numbers. This can be seen as the discriminant of $\tau(q,\varepsilon)$ in q equals $27\varepsilon(1-\varepsilon)^6(12+3\varepsilon+8\varepsilon^2)>0$.

[44] One can easily see that there are no profitable deviations for any player.

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