Beyond Incomplete Spanning: Convenience Yields and Exchange Rate Disconnect*

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Abstract

We introduce safe asset demand for dollar-denominated bonds into a tractable incomplete-market model of exchange rates. The convenience yield on dollar bonds enters as a stochastic wedge in the Euler equations for exchange rate determination. This wedge reduces the pass-through from marginal utility shocks to exchange rate movements, resolving the exchange rate volatility puzzle. The wedge also exposes the dollar's exchange rate to convenience yield shocks, giving rise to exchange rate disconnect from macro fundamentals and a quantitatively important driver of currency risk premium. This endogenous exposure identifies a novel safe-asset-demand channel by which the Fed's QE impacts the dollar.

Key Words: Exchange Rate Puzzles, Dollar, Convenience Yields, Incomplete Markets

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1 Introduction

We introduce safe asset demand for dollar bonds into an otherwise standard two-country, incomplete-market economy. The model makes progress on outstanding exchange rate puzzles that complete-market models cannot address. The analysis also uncovers a novel convenience yield mechanism through which quantitative easing affects the dollar exchange rate.

We focus on safe asset demand for dollar-denominated assets for three reasons. First, there is strong empirical evidence connecting movements in the dollar exchange rate and measures of the convenience (safety/liquidity) services that foreign investors attach to U.S. dollar safe bonds (see recent work by Avdjiev, Du, Koch, and Shin, 2019; Jiang, Krishnamurthy, and Lustig, 2021; Engel and Wu, 2021). Second, recent work on market segmentation and intermediation has found that Euler equation wedges can help to address exchange rate puzzles. In particular, in the models developed by Gabaix and Maggiori (2015) and Itskhoki and Mukhin (2021), the exchange rate is determined by the Euler equation of a specialized FX intermediary. Intermediation frictions give rise to wedges in the Euler equations of standard investors who do not operate in foreign exchange markets and/or foreign bond markets, and this research shows that these wedges can resolve exchange rate puzzles. The convenience yields we study are a form of the stochastic wedges, and they can be measured from the Covered Interest Rate Parity (C.I.P.) violations in government bond markets. In doing so, we bring data to discipline the Euler equation wedges. Third, there is a well-documented channel by which quantitative easing (QE) affects the exchange rate (see Neely, 2015; Krishnamurthy and Lustig, 2019). Since QE affects convenience yields on Treasury bonds, as outlined in Krishnamurthy and Vissing-Jorgensen (2011), the safe-asset demand ingredient may shed light on the QE-exchange rate connection.

We report four key findings in a calibrated version of the model. First, the wedges introduced by convenience yields mitigate the pass-through of shocks from the stochastic discount factors (SDF) to exchange rates. As a result, the model-implied exchange rates are not as volatile as in the complete-market model. Second, the covariance between shocks to the bond convenience yield and the SDFs substantially reduces the counter-cyclicality of exchange rates. Third, the bond convenience yield also affects the dollar's currency risk premium, and as a result generates an unconditional currency expected return that is in line with the data. Fourth, we uncover a connection between quantitative easing and exchange rates via convenience yields, which is distinct from the portfolio channel studied in Gourinchas, Ray, and Vayanos (2021); Greenwood, Hanson, Stein, and Sunderam (2020). We also present empirical evidence supporting the model's QE channel.

We derive these results in a two-country, incomplete-market economy. Investors in both countries derive convenience utility on their holdings of U.S. Treasurys. Markets are not segmented. All investors can buy U.S. Treasurys and foreign risk-free bonds, while the financial markets for other contingent claims may or may not open, which allows us to model a flexible degree of market incompleteness. We characterize the exchange rate processes that satisfy the four Euler equations for the home and foreign investors (2 investors $\times$ 2 bonds). These Euler equations encapsulate investors' attitudes towards exchange rate risk and their special desire for owning dollar-denominated U.S. Treasury bonds. In particular, we posit a pair of U.S. (dollar) and foreign log SDFs and a stochastic convenience yield process. This approach allows us to derive a tractable expression for the exchange rate as a function of the histories of U.S. and foreign SDF shocks and USD convenience yield shocks.[1]

By focusing only on the four Euler equations, our approach characterizes a family of solutions for the exchange rate process. We study the properties of the exchange rate process in this family and assess its performance vis-à-vis exchange rate puzzles. In this sense, our approach is in the tradition of Hansen and Jagannathan (1991) and the preference-free approach to FX markets (Backus, Foresi, and Telmer, 2001; Lustig, Roussanov, and Verdelhan, 2011; Chernov and Creal, 2018; Lustig and Verdelhan, 2019) rather than the international macro-finance models such as Gabaix and Maggiori (2015) and Itskhoki and Mukhin (2021). While the four Euler equations hold in most international macro models, these models also imply additional restrictions on the joint dynamics of the exchange rate and the trade balance which we do not impose. Furthermore, we allow for an arbitrary correlation structure of U.S. and foreign SDF shocks and USD convenience yield shocks when solving for the exchange rate. Including other macro considerations will restrict the correlation structure. In Appendix A, we present a two-period international macro model to explain these points further. Our minimalist approach allows us to characterize how far convenience yields in an incomplete market setting can go towards resolving exchange rate puzzles in a large class of models.

Next, we discuss the four key findings of our paper in detail. First, we make progress on the exchange rate volatility puzzle. Our model's equilibrium exchange rates in logs are less volatile than the difference of U.S. and foreign log SDFs. This result, a convenience-yield variant of the result derived in Lustig and Verdelhan (2019), helps to resolve the volatility of exchange rates vis-à-vis stock prices (Brandt, Cochrane, and Santa-Clara, 2006). In our closed-form characterization, the covariance between the SDF shocks and the exchange rate is tightly connected to the covariance between the exchange rate and the convenience yield. In the complete-markets benchmark model without convenience yields, exchange rates have to fully close the gap between the pricing kernels, absorbing all of the residual shocks. In our model with convenience yields, the convenience yields can partially act as shock absorbers too. To calibrate the model, we match the comovement of convenience yields and exchange rates reported by Jiang, Krishnamurthy, and Lustig (2021). Our calibrated model manages to match the volatility of exchange rates in the data, because the convenience yields drive a wedge between the volatility of the exchange rate and that of the SDFs.

Second, we make progress on the exchange rate disconnect puzzle (Backus and Smith, 1993; Kollmann, 1995). Convenience yield shocks impact exchange rates in our model. The equilibrium exchange rate reflects expected future interest rate spreads, currency risk premia, and USD convenience yields. In particular, the dollar appreciates when dollar bonds carry a higher convenience yield. In the case of a foreign flight to the safety of U.S. Treasurys, the foreign convenience yield

[1]The long-run expected exchange rate level is well-defined, which allows a Froot and Ramadorai (2005)-type representation. We also derive the risk premium implied by the model.

on U.S. bonds increases and causes the dollar to appreciate. This convenience yield channel counteracts the standard complete markets channel through which the foreign currency appreciates as foreign investors experience higher than average marginal utility growth in this episode. As a result, the dollar can appreciate against the foreign currency in foreign recessions. We explore these countervailing forces in a calibrated version of our model. The baseline model generates a roughly acyclical exchange rate. However, we also prove that in our model it is not possible to change the sign and deliver a pro-cyclical exchange rate. This result holds in a larger class of incomplete market models.

Third, our model generates sizable deviations from U.I.P. Foreign investors earn a negative excess return on dollars because the dollar has a positive convenience yield and because the dollar endogenously appreciates when the foreign SDF is high, thereby providing a hedge. Our model delivers plausible currency returns while matching the volatility of exchange rates. In stark contrast, Lustig and Verdelhan (2019) show that these moments cannot be matched jointly in a generic incomplete-market model without any Euler equation wedges or convenience yields. In their setting, market incompleteness reduces the exchange rate volatility and cyclicality, but it also shrinks the currency risk premium towards zero. As a result, resolving the risk premium puzzle immediately deepens the other exchange rate puzzles.

Fourth, our model sheds light on the connection between QE and exchange rates. In a large class of models with stationary exchange rates, the current exchange rate in deviation from its mean depends only on the home-foreign spread in the long rates:

$$s_t - \bar{s} \quad = \quad \lim_{T\to\infty}(T-t)(r_t^{T-t} - r_t^{*,T-t}),$$

because stationary exchange rates imply no long-run exchange rate risk. As a result, U.S. and foreign bonds are perfect substitutes over long holding periods for a long-horizon investor.

Recently, Gourinchas et al. (2021); Greenwood et al. (2020) develop equilibrium models of the joint pricing of bonds and currencies to elucidate the workings of QE. In their models, the exchange rate is stationary, so that a version of (1) applies. QE in their model affects the risk premium on long-term bonds and thus long yields and the dollar exchange rate. In our model, the exchange rate expression includes a novel convenience yield term:

$$s_t - \bar{s} \quad = \quad \lim_{T\to\infty}(T-t)(r_t^{T-t} - r_t^{*,T-t}) + \lim_{T\to\infty}\mathbb{E}_t\int_t^T \tilde{\lambda}_u^f du.$$

Holding fixed the differences in the long rates, the dollar appreciates when the future convenience yields $\tilde{\lambda}^f$ foreign investors earn on U.S. Treasurys increase.

Our paper is the first to examine this distinct convenience yield channel of QE in the foreign exchange markets, separate from the bond risk premium channel. When the Fed buys U.S. Treasurys in QE, this transaction is financed by reserve issuance. If central bank reserves are more desirable as safe assets, then the QE increases the aggregate quantity of safe assets. The convenience yield on dollar-denominated safe assets declines, and the dollar depreciates. If reserves are poor substitutes, then the QE reduces the aggregate quantity of safe assets, and we expect that the convenience yields increase and the dollar appreciates. There is both empirical and theoretical support for the proposition that shifts in the supply of safe assets induced by QE change the convenience yield on safe bonds (Krishnamurthy and Vissing-Jorgensen, 2011). Our mechanism links these convenience yield changes to movements in the dollar exchange rate.

Our paper is most closely related to Lustig and Verdelhan (2019) who examine an incomplete market setting where investors in home and foreign can invest in bonds in both countries and study the extent to which incompleteness can help resolve exchange rate puzzles. We study the same four Euler equations as in Lustig and Verdelhan (2019) but with bond convenience yields, thereby demonstrating the importance of the convenience yield ingredient.

Alvarez, Atkeson, and Kehoe (2002); Gabaix and Maggiori (2015); Dou and Verdelhan (2015); Itskhoki and Mukhin (2021); Chien, Lustig, and Naknoi (2020); Sandulescu, Trojani, and Vedolin (2020) develop international macro models with segmented markets to resolve the exchange rate disconnect puzzle. Their models sever the equilibrium exchange rate from its macro-fundamentals by introducing market segmentation and deliver a pro-cyclical exchange rate based on the model's assumed patterns in the arbitrageur's portfolio. For example, Chien et al. (2020) consider a model in which only a small pool of investors arbitrage between domestic and foreign securities. As a result, the real exchange rate is disconnected from the differences in aggregate consumption growth between U.S. and foreign. Amador, Bianchi, Bocola, and Perri (2020) also study segmented markets in the international context. Their model shows how capital flows can lead to violations of covered interest rate parity when markets are segmented. Our model does not rely on market segmentation and instead delivers results from the assumption of convenience yields on dollar bonds.

Our paper adds to the recent international finance literature exploring the macro and financial market implications when international investors earn convenience yields on foreign bonds (Valchev (2020); Jiang, Krishnamurthy, and Lustig (2020); Kekre and Lenel (2021)). Our model also fits into the literature on portfolio-balance models of exchange rates (Kouri (1975); Hau and Rey (2006); Pavlova and Rigobon (2008)), which studies the effect of portfolio holdings and flows on exchange rates, as well as the literature on frictional foreign exchange markets (Du, Tepper, and Verdelhan (2018b); Augustin, Chernov, Schmid, and Song (2020)). We focus on how special demand for dollar safe assets affects equilibrium exchange rates.

Investors seem to face an incomplete menu of assets in international financial markets, either because of transactions and capital controls or because of other frictions (Lewis, 1995). Recent theoretical contributions on market incompleteness and exchange rates include the work by Chari, Kehoe, and McGrattan (2002), Bacchetta and van Wincoop (2006), Corsetti, Dedola, and Leduc (2008), Alvarez, Atkeson, and Kehoe (2009), and Pavlova and Rigobon (2010). Our work shows that the convenience yield mechanism can be fruitfully incorporated into all of these incomplete market models.

The rest of the paper proceeds as follows. Section 2 presents our model of exchange rate determination with convenience yields. Section 3 calibrates the model and examines the its implications for a common set of exchange rate puzzles. Section 4 discusses how quantitative easing impacts currency markets through the lens of our model and provides empirical support. Proofs of all propositions are in the Appendix. The Appendix also presents a two-period international macro setting with convenience yields to elucidate the deeper foundations of our minimalist model.

2 Model

We develop an incomplete-market model of exchange rates in continuous time. Our economy has an infinite horizon. We fix a probability space $(\Omega, \mathcal{F}, P)$ and a given filtration $\mathbb{F} = \{\mathcal{F}_0 : t \geq 0\}$ satisfying the usual conditions. We assume that all stochastic processes are adapted to this filtration. There are two countries, the U.S. and foreign. Let s_t denote the log real exchange rate. A higher s_t means a stronger U.S. currency (dollar, USD). The key input into the model is that the USD is special in that U.S. and foreign investors earn a convenience yield on USD bonds.

2.1 Investor Preferences

The U.S. plays a unique role in the international financial system as the world's provider of dollar-denominated safe assets, as analyzed by Caballero, Farhi, and Gourinchas (2008), Caballero and Krishnamurthy (2009), Maggiori (2017), He, Krishnamurthy, and Milbradt (2018). We formalize this by assuming that U.S. and foreign investors derive utility from their holdings of the U.S. risk-free bond (U.S. Treasurys). Let c_t denote the U.S. households' consumption, and let $q_{H,t}$ denote the U.S. households' holding in the U.S. risk-free bond. The investors' utility is derived over consumption and the dollar value of U.S. bond holdings:

$$u(c_t, q_{H,t}) = w(c_t) + v(q_{H,t}; \theta_t),$$

where θ_t is a time-varying demand shifter for U.S. bonds. We assume that the utility is increasing in the consumption and the holding in the U.S. bonds, i.e. $w'(c_t) > 0$ and $v'(q_{H,t}; \theta_t) > 0$, and the marginal utility for holding U.S. bonds is decreasing in quantity, i.e., $v''(q) < 0$. In this way, the U.S. risk-free bond carries a convenience yield which captures its non-pecuniary benefits to U.S. and foreign investors, and is decreasing in the quantity held. We also assume that the exponentially discounted utility functions $w(\cdot)$ and $v(\cdot; \theta)$ are integrable.

The expected lifetime utility for U.S. investors is

$$\mathbb{E}_0 \left[\int_{t=0}^{\infty} e^{-\rho t} u(c_t, q_{H,t}) dt \right] = \mathbb{E}_0 \left[\int_{t=0}^{\infty} e^{-\rho t} (w(c_t) + v(q_{H,t}; \theta_t)) dt \right].$$

Similarly, for foreign investors, we assume their expected lifetime utility is

$$\mathbb{E}_0 \left[\int_{t=0}^{\infty} e^{-\rho t} u(c_t^*, q_{H,t}^*) dt \right] = \mathbb{E}_0 \left[\int_{t=0}^{\infty} e^{-\rho t} (w(c_t^*) + v(q_{H,t}^* \exp(s_t); \theta_t^*) dt \right],$$

where c_t^* denotes their aggregate consumption, θ_t^* is a time-varying demand shifter for U.S. bonds, and $q_{H,t}^*$ denotes the foreign investors' holdings of the U.S. risk-free bond in foreign currency terms. The product $q_{H,t}^* \exp(s_t)$ converts the holdings into dollar terms.

2.2 A Quartet of Euler Equations

Markets are not segmented. We assume that all investors can trade both U.S. and foreign risk-free bonds. The model analysis is centered around four Euler equations, 2 for the U.S. investors and 2 for the foreign ones. The asset markets for other risky claims, such as equity and long-term debt, may or may not be open to foreign investors. This approach allows us to model a general form of market incompleteness.

The U.S. (instantaneous) risk-free bond has a constant price $P_t = 1$ and an interest rate r_t, and the foreign risk-free bond has a constant price $P_t^* = 1$ and an interest rate r_t^*. These interest rates are determined in equilibrium from the Euler equations. We start by characterizing the optimality conditions for U.S. households.

Lemma 1. *The first-order conditions for the U.S. investor with respect to holdings in the U.S. and foreign risk-free bonds are given by*[2]

$$0 = \mathbb{E}_t\left[\frac{dM_t}{M_t}\right] + r_t + \frac{v'(q_{H,t};\theta_t)}{w'(c_t)} \tag{1}$$

$$0 = \mathbb{E}_t\left[\frac{d(M_t \exp(-s_t))}{M_t \exp(-s_t)}\right] + r_t^* \tag{2}$$

where $M_t = e^{-\rho t} w'(c_t)$ is the SDF of the U.S. investor.

See Appendix B.1 for the proof.

To interpret this result, we note that the discrete-time counterparts to these equations are given by the following expressions:

$$1 - \frac{v'(q_{H,t};\theta_t)}{w'(c_t)} = \mathbb{E}_t\left[\frac{M_{t+1}}{M_t}\exp(r_t)\right] \tag{3}$$

$$1 = \mathbb{E}_t\left[\frac{M_{t+1}}{M_t}\exp(r_t^* - \Delta s_{t+1})\right] \tag{4}$$

Given $v'(q_{H,t};\theta_t) > 0$ and $w'(c_t) > 0$, the left-hand side of Eq. (3) is less than 1. U.S. investors are willing to accept a lower risk-neutral expected return in exchange for holding the U.S. risk-free bond, whereas the risk-neutral expected return on the foreign risk-free bond is exactly 1. We refer to the gap $\frac{v'(q_{H,t};\theta_t)}{w'(c_t)}$ as the convenience yield that U.S. households attach to dollar bonds.

Similarly, the foreign households' asset pricing conditions for the foreign bond and for U.S.

[2] With some abuse of notation we use the notation $\mathbb{E}_t[dX_t]$ to represent the infinitesimal generator of a stochastic process X_t. The formal notation, which is adopted in the proof, is $\mathcal{A}[X_t]$.

Treasurys, respectively, are given by (see Appendix B.1 for the derivation):

$$0 = \mathbb{E}_t \left[\frac{dM_t^*}{M_t^*} \right] + r_t^* \tag{5}$$

$$0 = \mathbb{E}_t \left[\frac{d(M_t^* \exp(s_t))}{M_t^* \exp(s_t)} \right] + r_t + \frac{v'(q_{H,t}^* \exp(s_t); \theta_t^*)}{w'(c_t^*)} \tag{6}$$

Let us define $\tilde{\lambda}_t^h = \frac{v'(q_{H,t};\theta_t)}{w'(c_t)}$ as the convenience yield earned by U.S. investors on their dollar bond holdings. Likewise, define $\tilde{\lambda}_t^f = \frac{v'(q_{H,t}^* \exp(s_t);\theta_t^*)}{w'(c_t^*)}$ as the foreign investors' convenience yield on dollar bonds. Then, we can rewrite the Euler equations as:

$$0 = \mathbb{E}_t \left[\frac{dM_t}{M_t} \right] + r_t + \tilde{\lambda}_t^h, \qquad 0 = \mathbb{E}_t \left[\frac{d(M_t \exp(-s_t))}{M_t \exp(-s_t)} \right] + r_t^*,$$

$$0 = \mathbb{E}_t \left[\frac{d(M_t^* \exp(s_t))}{M_t^* \exp(s_t)} \right] + r_t + \tilde{\lambda}_t^f, \qquad 0 = \mathbb{E}_t \left[\frac{dM_t^*}{M_t^*} \right] + r_t^*.$$

These four equations arise in a large class of international macro models that may have different specifications of preferences, spanning of tradable assets, and frictions, as long as these models permit agents to trade, in an unconstrained fashion, in risk-free bonds and derive convenience yields on dollar safe assets.

Our approach is to solve for the family of equilibrium exchange rate dynamics that are consistent with these four Euler equations. In doing so, we provide a general characterization of the exchange rate dynamics in this large class of international macro models. If we were to further specify the macroeconomic environment, e.g., to derive the dynamics of the current account, the model would yield additional restrictions on the exchange rate process. Our minimal approach allows us to explore the extent to which convenience yields in an incomplete market setting can address puzzling aspects of the behavior of exchange rates.

In Appendix A we present a two-period international macro model with convenience yields on dollar bonds and trade in both countries' goods and bonds. The model reproduces the four Euler equations we study. At the same time, the model imposes additional restrictions on the relation between convenience yields and the trade balance, and it pins down a unique equilibrium from the family of solutions we characterize in the main text. We show that the two-period model qualitatively produces the forces present in our analysis, although obviously, that model cannot be explored quantitatively.

2.3 Equilibrium Forces

Before diving into the model solution, we work through a thought experiment that helps elucidate the restrictions these Euler equations impose on equilibrium exchange rates.

Suppose that at time t, there is an exogenous increase in $\tilde{\lambda}_t^h$, i.e., the foreign households' convenience yield on the dollar safe assets. For the sake of this argument, we will assume the U.S.

and foreign SDFs and the U.S. households' convenience yield remain unaffected. In the next section, we specify the joint dynamics of the SDFs and model shocks. Then, this increase in foreign households' convenience yield sets off the following chain of events.

First, consider the U.S. households' Euler equation for holding domestic bonds, reproduced in the discrete-time form below,

$$1 - \tilde{\lambda}_t^h \quad = \quad \mathbb{E}_t \left[\frac{M_{t+1}}{M_t} \exp(r_t) \right].$$

Since the U.S. households' SDF and convenience yield are assumed to be unaffected, this Euler equation implies that the dollar risk-free rate r_t does not change.

Second, from the foreign households' Euler equation for holding U.S. bonds, reproduced in discrete-time form below,

$$1 - \tilde{\lambda}_t^f \quad = \quad \mathbb{E}_t \left[\frac{M_{t+1}^*}{M_t^*} \exp(r_t + \Delta s_{t+1}) \right],$$

an increase in their convenience yield $\tilde{\lambda}_t^f$ lowers their risk-neutral expected return on holding dollar safe bonds. Since the dollar risk-free rate does not change, the exchange rate has to adjust to equilibrate this Euler equation. In particular, the dollar needs to appreciate today and create an expected depreciation to generate the lower expected return.

Lastly, if we examine the U.S. households' Euler equation for holding foreign bonds, reproduced in discrete-time form below,

$$1 \quad = \quad \mathbb{E}_t \left[\frac{M_{t+1}}{M_t} \exp(r_t^* - \Delta s_{t+1}) \right],$$

we learn that the dollar exchange rate movement also raises the expected return on purchasing foreign currency bonds from the U.S. perspective. Since the U.S. households do not derive a convenience yield on foreign bonds that can adjust, all adjustments must happen in the dollar's currency risk premium. In our equilibrium, this happens via endogenous changes in the cyclicality and volatility of the dollar exchange rate. Thus, these four Euler equations require endogenous responses in both first moments (i.e., exchange rate level and expected return) as well as second moments (i.e., currency cyclicality and volatility) in response to the shock to the convenience yield. Solving the full model involves deriving the endogenous exchange rate process that respects these forces. Section 3.5 revisits this section's experiment in the context of the solved model.

2.4 Pricing Kernel and Convenience Yield Dynamics

We posit a pair of U.S. (dollar) and foreign log SDFs. Let $m_t = \log(M_t)$ and $m_t^* = \log(M_t^*)$ denote the log SDFs. We posit that the log SDFs have the following dynamics:

$$dm_t \quad = \quad -\mu dt - \sigma dZ_t,$$

$$dm_t^* = \phi s_t dt - \sigma dZ_t^*,$$

Here, $\{Z_t, Z_t^*\}$ are standard Brownian motion processes. The Brownian increments dZ_t and dZ_t^* represent shocks to the marginal utilities of home and foreign households, which may originate from aggregate shocks to consumption/output in fully specified models.

The dynamics for the foreign SDF describe risk-free rate dynamics in the foreign country engineered to keep the real exchange rate stationary. The SDF dynamics describe an implicit monetary policy rule required for stationarity as in Engel and West (2005).

Assumption 1. *We assume that the mean-reversion parameter $\phi > 0$ is strictly positive.*

Assumption 1 implies that the foreign real interest rate is decreasing in the level of the U.S. real exchange rate. In particular, if markets are complete, the log of the real exchange rate s_t^{cm} equals the difference in the log of the SDFs:

$$\begin{aligned} s_t^{cm} &= m_t - m_t^*, \\ ds_t^{cm} &= (-\mu - \phi s_t^{cm})dt + \sigma(dZ_t^* - dZ_t), \end{aligned}$$

which is a simple stationary process.

We assume that the convenience yields derived by U.S. and foreign investors can be different. As we show later, the difference in these convenience yields is all that matters for exchange rate dynamics. We denote the difference between the two convenience yields as $\tilde{\lambda}_t = \tilde{\lambda}_t^f - \tilde{\lambda}_t^h$, and parameterize the difference as follows:

$$\tilde{\lambda}_t = \ell \frac{\exp(\lambda_t)}{\exp(\lambda_t) + 1},$$

which is bounded between 0 and ℓ. The auxiliary state variable λ_t satisfies

$$d\lambda_t = -\theta \lambda_t dt + \nu dX_t,$$

where dX_t is a standard Brownian motion on $(\Omega, \mathcal{F}, \mathcal{P})$ adapted to the filtration $\mathbb{F}$.

Finally, with slight abuse of notation, let $[dX_t, dY_t]$ denote the instantaneous conditional covariance between two diffusion processes X_t and Y_t. Formally, it is defined as $[dX_t, dY_t] = d[X, Y]_t/dt$ where $[X, Y]_t$ is the standard quadratic covariation between processes X_t and Y_t. We assume that the convenience yield shock and the SDF shocks can be pairwise correlated[3]:

$$[dZ_t, dX_t] = \rho, \qquad [dZ_t^*, dX_t] = \rho^*, \qquad [dZ_t, dZ_t^*] = \zeta.$$

Note that the home and foreign SDFs load negatively on the dZ or dZ^* shocks. We assume that $\rho^* < 0$, so that the foreign households' convenience yield on the dollar safe bonds tends to increase

[3] As the correlation matrix of dX_t dZ_t, dZ_t^* needs to be positive semidefinite, these correlation parameters need to satisfy $\det = 1 + 2\rho\rho^*\zeta - \rho^2 - \rho^{*2} - \zeta^2 \geq 0$.

when their marginal utilities rise. This correlation captures foreigners' "flight-to-Treasuries" during their recessions. When global volatility in financial markets increases, the convenience yield on U.S. Treasurys tends to increase.[4] Similarly, we assume $\rho > 0$. That is, we posit that a rising convenience yield $\tilde{\lambda}_t^h$ from the U.S. perspective is associated with an increase in U.S. marginal utility.

These correlations and their signs are the natural ones that would emerge in an international macroeconomic model. In Appendix A we develop a two-period international macro model where both U.S. and foreign households earn convenience yields on their holdings of the U.S. Treasury bond, trade both home and foreign countries' bonds and goods. We require that the net flows of capital and goods balance at the equilibrium exchange rate. In the model, an increase in foreign demand for U.S. Treasury bonds leads to foreign households cutting back on current consumption (hence higher marginal utility or SDF) in order to purchase these bonds, with the U.S. households accommodating by selling U.S. Treasury bonds, increasing holdings of foreign bonds, and increasing current consumption (hence lower marginal utility or SDF). Thus, although we work at a high level with the dynamics of the SDF in our derivations, these dynamics can be consistent with the implications from a fully-specified macroeconomic model.

2.5 Equilibrium Exchange Rate

We posit that the real exchange rate has the following dynamics[5],

$$ds_t \quad = \quad \alpha_t dt + \beta_t \sigma (dZ_t^* - dZ_t) + \gamma_t \nu dX_t, \tag{7}$$

where α_t, β_t, and γ_t are $\mathbb{F}$-adapted stochastic processes that we must solve for. β_t governs the distance from complete markets. When $\beta_t \equiv 1$, and $\gamma_t \equiv 0$, we are back in the benchmark complete markets case: the change in the log exchange rate equals the difference in the log pricing kernels. The complete markets exchange rates act as shock absorbers.

Our objective is to characterize a solution to (7) that satisfies the four pricing conditions (1),

[4] Jiang et al. (2021) use the U.S. Treasury basis to measure the convenience yield. We define the basis as the difference between the yield on a cash position in U.S. Treasurys $y_t^{\$}$ and the synthetic dollar yield constructed from a cash position in a foreign government bond, which earns a yield y_t^* in foreign currency, that is hedged back into dollars:

$$x_t^{Treas} \equiv y_t^{\$} + (f_t^1 - s_t) - y_t^*.$$

Here s_t denotes the log of the nominal exchange rate in units of foreign currency per dollar, f_t^1 denotes the log of the forward exchange rate and x_t^{Treas} measures the violation of the CIP constructed from U.S. Treasury and foreign government bond yields. A negative U.S. Treasury basis means that U.S. Treasurys are expensive relative to their foreign counterparts. The convenience yield can be inferred from the CIP deviations x_t^{Treas} in government bond markets: $(1 - \beta^{basis})(\lambda_t^f - \lambda_t^h) = -x_t^{Treas}$, where β^{basis} measures the fraction of the convenience yield earned on a synthetic Treasury constructed from a foreign currency bond. See Jiang et al. (2021) for more details.

[5] Equation (7) covers the set of equilibria in which the exchange rate is exposed to only the SDF shocks and the convenience yield shock. There exists additional exchange rate solutions that are exposed to additional shocks: $ds_t = \alpha_t dt + \beta_t \sigma (dZ_t^* - dZ_t) + \gamma_t \nu dX_t + \omega_t dW_t$. In this sense, what we are characterizing can be thought of as the "fundamental equilibria". These alternative solutions generate higher exchange rate volatility that worsens the puzzles we explore in the next section.

(2), (5) and (6). In our incomplete-market setting, there are many candidate solutions. In principle, the loading β_t of the exchange rate on the SDF shocks may vary over time. For expositional convenience, we assume it is time-invariant. This assumption allows us to characterize the exchange rate level in a tractable way. Our characterization of the exchange rate and currency return dynamics do not hinge on this assumption. Similar results obtain if we relax this assumption by replacing β by β_t.

Assumption 2. *$\beta_t \equiv \beta$ is constant.*

Then, we obtain the following family of solutions that are consistent with the four pricing conditions.

Proposition 1. *Under Assumption 2, there is a class of solutions indexed by a constant k so that,*

$$\beta = \frac{1}{2} \pm \frac{1}{2}\sqrt{\frac{\sigma^2(1-\zeta)-2k}{\sigma^2(1-\zeta)}}, \tag{8}$$

$$\gamma_t = \frac{(\rho^*-\rho)\sigma(1-2\beta) \pm \sqrt{(\rho^*-\rho)^2\sigma^2(1-2\beta)^2+4(k-\tilde{\lambda}_t)}}{2\nu}. \tag{9}$$

The log of the real exchange rate satisfies:

$$ds_t = \left(-\frac{1}{2}\tilde{\lambda}_t - \phi s_t - \mu + \frac{1}{2}\sigma\gamma_t\nu(\rho+\rho^*)\right)dt + \gamma_t\nu dX_t + \beta\sigma(dZ_t^* - dZ_t), \tag{10}$$

which loads on both the SDF shocks dZ and dZ^* and the convenience yield shock dX.

We present the proof in the appendix. For each k, there are two solutions for β. One root is between $1/2$ and 1, and the other is between 0 and $1/2$. As for γ_t, note that $(\rho^*-\rho)\sigma(1-2\beta)$ can be either positive or negative. We pick the root of γ_t with the positive sign:

$$\gamma_t = \frac{(\rho^*-\rho)\sigma(1-2\beta) + \sqrt{(\rho^*-\rho)^2\sigma^2(1-2\beta)^2+4(k-\tilde{\lambda}_t)}}{2\nu},$$

so that for $k > \tilde{\lambda}_t$, we can guarantee $\gamma_t > 0$. We focus on solutions with $\gamma_t > 0$ to arrive at the natural result that the exchange rate appreciates when the foreign convenience yield for dollar bonds rises. Finally, note that unlike β_t, γ_t is not constant and varies with the convenience yield, $\tilde{\lambda}_t$. We will calibrate β based on regression results.

Furthermore, we can solve the stochastic differential equation (10) to find a closed-form expression for the log of the real exchange rate.

Proposition 2. *The real exchange rate s_t can be expressed as*

$$s_t = f(\lambda_t) + H_t + \beta s_t^{cm}. \tag{11}$$

The first term $f(\lambda_t)$ is a function of the current convenience yield λ_t. Let $b = (\rho^* - \rho)\sigma(1-2\beta)$, then

$$
\begin{aligned}
f(\lambda) &= \frac{1}{2\nu}\{-\sqrt{b^2+4k}\log\left(2e^{\lambda/2}\left(\cosh\left(\frac{\lambda}{2}\right)\left(\sqrt{b^2+4k}\sqrt{b^2+4k-2\ell\tanh\left(\frac{\lambda}{2}\right)-2\ell}+b^2+4k-\ell\right)-\ell\sinh\left(\frac{\lambda}{2}\right)\right)\right)\\
&+ \sqrt{b^2+4k-4\ell}\log\left(2e^{\lambda/2}\left(\cosh\left(\frac{\lambda}{2}\right)\left(\sqrt{b^2+4k-4\ell}\sqrt{b^2+4k-2\ell\tanh\left(\frac{\lambda}{2}\right)-2\ell}+b^2+4k-3\ell\right)-\ell\sinh\left(\frac{\lambda}{2}\right)\right)\right)\\
&+ \lambda\left(\sqrt{b^2+4k}+b\right)\}.
\end{aligned}
$$

The second term H_t captures the history of past convenience yields:

$$
\begin{aligned}
H_t &= e^{-\phi t}H_0 + \int_0^t e^{-\phi(t-u)}h(\lambda_u)du,\\
h(\lambda_t) &= -\frac{1}{2}\tilde{\lambda}_t - \phi f - (1-\beta)\mu + \frac{1}{2}\sigma\gamma_t\nu(\rho+\rho^*) + f'\theta\lambda_t - \frac{1}{2}f''\nu^2.
\end{aligned}
$$

The third term is the real exchange rate s_t^{cm} under complete markets scaled by β, where

$$
ds_t^{cm} = (-\mu - \phi s_t^{cm})dt + \sigma(dZ_t^* - dZ_t).
$$

The proof is in the appendix. This proposition shows that the real exchange rate level is determined by not only the relative pricing kernels, as summarized by the real exchange rate s_t^{cm} under complete markets, but also the current convenience yield and the history of the convenience yields λ_t.

We also note that the exchange rate's long-run expectation $\lim_{T\to\infty}\mathbb{E}_t[s_T]$ is well-defined.

Proposition 3. *In our incomplete markets model with convenience yields, the exchange rate's long-run expectation $\lim_{T\to\infty}\mathbb{E}_t[s_T]$ is*

$$
\bar{s} \equiv \lim_{T\to\infty}\mathbb{E}_t[s_T] = \frac{1}{\phi}\left(-\frac{1}{2}\lim_{T\to\infty}\mathbb{E}_0[\tilde{\lambda}_T] - \mu + \frac{1}{2}\sigma\lim_{T\to\infty}\mathbb{E}_0[\gamma_T]\nu(\rho+\rho^*)\right).
$$

In comparison, the complete-market and no-convenience yield counterpart is

$$
\bar{s}^{cm} = \lim_{T\to\infty}\mathbb{E}_t[s_T^{cm}] = -\frac{\mu}{\phi},
$$

which does not have the "convenience yield" term $-\frac{1}{2}\lim_{T\to\infty}\mathbb{E}_0[\tilde{\lambda}_t]$ and the "endogenous risk premium" term $\frac{1}{2}\sigma\lim_{T\to\infty}\mathbb{E}_0[\gamma_t]\nu(\rho+\rho^*)$.

2.6 Exchange Rate Accounting

With this long-run expectation, the exchange rate level s_t has a forward-looking representation:

$$
s_t = \bar{s} - \lim_{T\to\infty}\mathbb{E}_t\int_t^T ds_u,
$$

where the long-run expectation of the exchange rate $\bar{s}$ is derived in Appendix B.4. Following Froot and Ramadorai (2005); Jiang et al. (2021), we can further decompose the exchange rate level in the following way.

Corollary 1. *The exchange rate level can be decomposed into*

$$s_t = \bar{s} + \lim_{T\to\infty} \mathbb{E}_t \int_t^T (r_u - r_u^*)du + \lim_{T\to\infty} \mathbb{E}_t \int_t^T \tilde{\lambda}_u^f du - \lim_{T\to\infty} \mathbb{E}_t \int_t^T rp_u du. \quad (12)$$

On the right-hand side, $\lim_{T\to\infty} \mathbb{E}_t \int_t^T (r_u - r_u^*)du$ captures expected future short rate differences, $\lim_{T\to\infty} \mathbb{E}_t \int_t^T \tilde{\lambda}_u^f du$ captures expected future convenience yields earned by the foreign investors, and $-\lim_{T\to\infty} \mathbb{E}_t \int_t^T rp_u du$ captures expected future currency risk premia from the foreign perspective plus a Jensen's term,

$$rp_u = (\frac{1}{2}\tilde{\lambda}_u + \frac{1}{2}\sigma\gamma_u\nu(\rho+\rho^*)) = -\left([dm_t^*, ds_t] + \frac{1}{2}[ds_t, ds_t]\right).$$

This decomposition in equation (12) is the equivalent of a Campbell-Shiller decomposition for exchange rates. The exchange rate level today reflects future interest rate differences (cash flows), future convenience yields, minus future risk premia (discount rates). This expression is forward-looking, which complements the backward-looking expression for the exchange rate level in equation (11). [6] The dollar appreciates when future U.S. short rates increase and dollar currency risk premia decline. Jiang et al. (2021) derive a version of this decomposition that allows for convenience yields. When foreign investors expect to earn larger convenience yields on USD bonds, the dollar appreciates in spot markets.

Alternatively, we can define a USD bond yield without convenience yield, $r_t^{xcy} = r_t + \tilde{\lambda}_t^h$. Then, the exchange rate decomposition becomes

$$s_t = \bar{s} + \lim_{T\to\infty} \mathbb{E}_t \int_t^T (r_u^{xcy} - r_u^*)du + \lim_{T\to\infty} \mathbb{E}_t \int_t^T \tilde{\lambda}_u du - \lim_{T\to\infty} \mathbb{E}_t \int_t^T rp_u du, \quad (13)$$

in which case the first term becomes the interest rate differential of USD and foreign bonds without convenience yields, and the second term becomes the differential in the convenience yields from foreign and U.S. investors' perspectives.

3 Quantitative Implications of Convenience Yields for Exchange Rates

This section discusses (1) the comovement between dollar exchange rate and flight-to-safety as in Jiang et al. (2021), (2) the partial SDF-FX pass-through and the Brandt et al. (2006) puzzle, (3) the Backus-Smith puzzle, and (4) currency risk premium in the short and long run. We evaluate the

[6] A version of this decomposition without convenience yields was derived by Campbell and Clarida (1987); Froot and Ramadorai (2005). This expression without the convenience yields holds in a large class of models.

quantitative fit of these patterns by our convenience-yield model of exchange rates. We begin by explaining our calibration choices.

3.1 Calibration Choices

We calibrate the model at the annual frequency and report our parameter values in Table 1. This set of parameter values implies that the convenience yield $\tilde{\lambda}_t$ process has an unconditional mean of 2.5% and an unconditional standard deviation of 1.6% per annum. In Jiang et al. (2021), we show how to measure the convenience yield from the deviation from CIP for government bonds ("the Treasury basis").[7].

Denote x_t^{Treas} as the Treasury basis. Then Jiang et al. (2021) show that the convenience yield is proportional to the basis:

$$(1-\beta^{basis})(\lambda_t^f - \lambda_t^h) = -x_t^{Treas},$$

where β^{basis} measures the fraction of the convenience yield earned on a synthetic U.S. Treasury constructed from a foreign currency denominated safe government bond. Jiang et al. (2021) estimate the constant of proportionality β^{basis} to be 0.9 so that the mean Treasury basis of 0.22 gives a mean convenience yield of $0.22/(1-0.9) = 2.2\%$. The standard deviation of the Treasury basis of 0.23 implies a standard deviation of the convenience yield of $0.23/(1-0.9) = 2.3\%$. Moreover, the mean-reversion parameter $\theta = 3$ implies that the convenience yield shocks have a half-life of $\log(2)/\theta = 0.23$ years. In the data, we estimate an AR(1) model of the Treasury basis and find the estimated model to have a half-life of 0.24 years.

[7] Since the Great Financial Crisis, sizable deviations from Covered Interest Parity have opened up in LIBOR markets (Du et al., 2018b), but even before the GFC, there were large, persistent deviations from CIP in government bond markets (see Du, Im, and Schreger, 2018a; Jiang et al., 2021; Du and Schreger, 2021). U.S. Treasurys are always expensive relative to synthetic Treasurys constructed from foreign bonds. In Jiang et al. (2021), we estimate that foreign investors earn convenience yields of around 200 basis points, significantly larger than the CIP deviations. Using a demand-system-based approach, Koijen and Yogo (2020) report similar estimates of the convenience yields.

TABLE 1–PARAMETER CHOICES

Symbol	Interpretation	Value	Calibration Target
	Panel A: SDF and FX		
μ	SDF drift	0	symmetry
σ	SDF shock volatility	0.5	max Sharpe ratio
ζ	SDF shock correlation	0.32	consumption growth correlation
ϕ	exchange rate persistence	0.135	half life of exchange rate shock
ρ	U.S. SDF loading on convenience shock	0.2	see text
ρ^*	foreign SDF loading on convenience shock	-0.3	see text
	Panel B: Convenience Yield		
ℓ	convenience yield level	5%	average Treasury basis
ν	convenience yield volatility	5	Treasury basis volatility
θ	convenience yield persistence	3	half life of Treasury basis shock

Notes: This table reports the values of calibrated parameters.

The SDF volatility σ is calibrated to 50% per annum, which implies that the maximal annual Sharpe ratio permitted by either country's SDF is roughly 0.5 as well. Moreover, we set $\zeta = [dZ_t, dZ_t^*]$, the correlation between U.S. and foreign SDF shocks, to 0.32, which is the average correlation between U.S. consumption growth and other G10 countries' consumption growth, using annual data from 1970 to 2018. Alternatively, we could calibrate this parameter using the average correlation between the change in the U.S. stock log price-to-dividend ratio and other G10 countries', which yields a slightly higher value of 0.48.

We assume that the correlation between the U.S. SDF shock and the convenience yield shock is $\rho = 0.2$, and the correlation between the foreign SDF shock and the convenience yield shock is $\rho^* = -0.3$. Intuitively, each country's demand for dollar safe assets should increase when its marginal utility is higher, i.e., $[dZ_t, d\lambda_t^h] < 0$ and $[dZ_t^*, d\lambda_t^f] < 0$. Since dX_t is the shock to the convenience yield differential $(\lambda_t^f - \lambda_t^h)$, we expect $\rho = [dZ_t, dX_t] > 0$ and $\rho^* = [dZ_t^*, dX_t] < 0$. While this logic determines the signs, it does not pin down values for the correlation. We calibrated the specific values of 0.2 and -0.3 to match the empirical moments in Columns (3)—(6) of Table 2. We also report alternative choices of $(\rho = 0.4, \rho^* = -0.3)$ and $(\rho = 0, \rho^* = 0)$ to illustrate how these parameters affect results.

The adjustment in interest rate in response to the exchange rate level is governed by the parameter ϕ, which we set to 0.135. This parameter value implies that the half-life of the variation in a shock to the real exchange rate is $\log(2)/\phi = 5.13$ years. In the data, we estimate an AR(1) model of the log dollar index and find the estimated model to have a half-life of 5.18 years.

To avoid imaginary roots in equation (8) for β and equation (9) for γ_t for all values of λ_t, k have to take values between $[\frac{\ell-(\rho^*-\rho)^2\sigma^2/4}{1-(\rho^*-\rho)^2/(2(1-\zeta))}, \frac{\sigma^2(1-\zeta)}{2}]$. Equivalently, β is bounded by $[0.14, 0.86]$. Each value within this range corresponds to an equilibrium solution in this system.

3.2 Exchange Rate, Flight-to-Safety, and k

Table 2 presents regression results from data generated by simulating the calibrated model. Panel A reports the benchmark calibration with flight to safety, as explained below. Panel B reports the results for the calibration without flight to safety. Panel C reports the same moments in the data. We discretize the model by a time increment of $\Delta t = 0.001$ period and simulate $50,000$ periods. As we have noted, our model generates a family of solutions indexed by k. Thus, in the table, we reported simulation results for a range of k.

Panel A reports results for the benchmark calibration of $\rho = 0.2$ and $\rho^* = -0.3$, in which case positive SDF shocks in each country raise their demand for dollar safe assets. We first note that for all these values of k, we have that $\beta < 1$. Lustig and Verdelhan (2019) show that when markets are incomplete, shocks to the SDF will pass through less than one-for-one to the exchange rate.[8] Thus, the value of $\beta < 1$ is a manifestation of their result in our setting. Market incompleteness reduces the exchange rate's exposure to fundamentals. However, the Lustig and Verdelhan (2019)

[8] In the two-period macroeconomic model in Appendix A, we show that $\beta < 1$ in equilibrium and that the R^2 of SDF shocks on exchange rate innovations is also less than 100%.

TABLE 2–SIMULATED MOMENTS

(1) k	(2) β	(3) FX-Conv Yield Coef	(4) FX Vol (%)	(5) SDF-FX Pass-Thru	(6) Exp. Log Return (%)
Panel A: $\rho = 0.2$ and $\rho^* = -0.3$					
0.04	0.14	0.59	9.76	0.10	-1.48
0.05	**0.18**	**1.18**	**11.57**	**0.11**	**-1.56**
0.06	0.22	1.66	14.11	0.13	-1.65
0.07	0.28	2.09	17.08	0.16	-1.73
0.08	0.34	2.48	20.74	0.20	-1.82
0.09	0.50	2.84	28.73	0.32	-1.98
Panel B: Data					
(1) k	(2) β	(3) FX-Conv Yield Coef	(4) FX Vol (%)	(5) SDF-FX Pass-Thru	(6) Exp. Log Return (%)
-	-	1.02—1.49	10.00	< 0	-1.89
Panel C: Complete-Market Model					
(1) k	(2) β	(3) FX-Conv Yield Coef	(4) FX Vol (%)	(5) SDF-FX Pass-Thru	(6) Exp. Log Return (%)
0.00	1.00	-6.01	57.83	1.00	-0.01
Panel D: $\rho = 0.4$ and $\rho^* = -0.3$					
(1) k	(2) β	(3) FX-Conv Yield Coef	(4) FX Vol (%)	(5) SDF-FX Pass-Thru	(6) Exp. Log Return (%)
0.03	0.10	-0.61	7.95	0.09	-1.20
0.04	0.14	-0.03	8.06	0.09	-1.10
0.05	0.19	0.44	9.81	0.10	-1.01
0.06	0.25	0.87	12.46	0.12	-0.91
0.07	0.32	1.26	16.02	0.15	-0.81
0.09	0.50	1.61	24.23	0.25	-0.62
Panel E: $\rho = \rho^* = 0$					
(1) k	(2) β	(3) FX-Conv Yield Coef	(4) FX Vol (%)	(5) SDF-FX Pass-Thru	(6) Exp. Log Return (%)
0.05	0.18	3.60	18.73	0.18	-1.35
0.06	0.21	4.18	21.54	0.21	-1.36
0.06	0.25	4.66	24.35	0.25	-1.37
0.07	0.30	5.09	27.31	0.29	-1.38
0.08	0.36	5.48	30.74	0.35	-1.40
0.09	0.50	5.85	37.75	0.50	-1.45

Notes: Calibration of parameters in Table 1. We vary the value for the remaining parameter k and report several moments from the simulated model. Column (1) reports the value of k. (2) reports the implied β parameter in the exchange rate process $ds_t = \alpha_t dt + \beta\sigma(dZ_t^* - dZ_t) + \gamma_t \nu dX_t$. (3) reports the slope coefficient in regression of Δs_t on $\Delta\tilde{\lambda}_t$. (4) reports annual FX volatility. (5) reports the slope coefficient in regression of Δs on $\Delta m - \Delta m^*$. (6) reports the annual expected log excess return on long position in the U.S. dollar. The regressions are run at quarterly frequency. Our simulation is based on a long sample of $T = 50,000 \times 1000$ time intervals.

approach does not allow one to pin down β, while our approach does.

We pin down the value of k, and hence the values of β and γ from the regression coefficient in column (3). Jiang et al. (2021) show that the dollar's real exchange rate is increasing in the convenience yield that foreign investors assign to the dollar risk-free bond. Specifically, when the U.S. Treasury's convenience yield increases by one standard deviation (0.23% as measured by Treasury basis), the dollar appreciates by 2.35%. In the post-2008 sample, the one-standard-deviation shock leads to a dollar appreciation of 3.28%. These results indicate a regression coefficient of the exchange rate movement Δs_t on the change in the convenience yield $\Delta\tilde{\lambda}_t$ of between 1.02 and 1.49 (see Panel B). We run the same regression in our simulated data and report the results for different values of k in column (3). When k is 0.05, the regression coefficient is 1.18 and in the range of the data. Once k is chosen, the values for β as well as other regression results are pinned down and reported in the table in the rest of the columns.

To illustrate these forces, Figure 1 plots β against γ_t at $\lambda_t = 0$, for the family of solutions indexed by k, by varying k. This plot is for the principal calibration of $\rho = 0.2$ and $\rho^* = -0.3$. Over most of the range of β, the convenience yield loading γ_t of the exchange rate increases as the SDF loading β on fundamentals increases. Our calibration pins down γ by matching the regression coefficient in column (3) in Table 2 and then the logic of the model pins down β.

In Panel B, we report these moments in the data, based on the sample of the dollar exchange rates relative to other G10 countries. In the next sections, we will further discuss how our model-implied moments compare to these empirical moments.

In Panel C, we report the moments implied by a complete-market model with the same parameters. In this complete-market model, the exchange rate movement is fully determined by the SDF shocks, and the implied moments miss the empirical moments by a large margin.

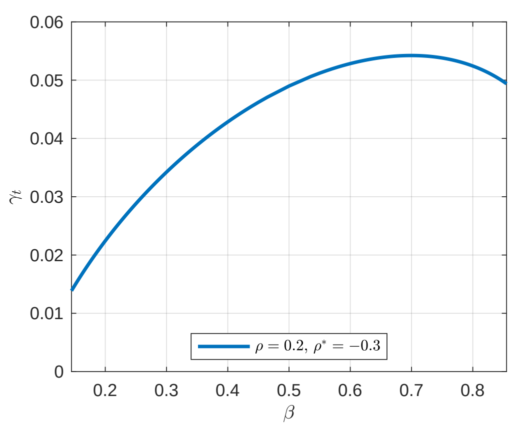


FIGURE 1. FX LOADINGS ON THE SDF AND THE CONVENIENCE YIELD SHOCKS.

Notes: We plot β and γ_t of the exchange rate process $ds_t = \alpha_t dt + \beta\sigma(dZ_t^* - dZ_t) + \gamma_t \nu dX_t$, where $\gamma_t = \frac{(\rho^*-\rho)\sigma(1-2\beta_t)\pm\sqrt{(\rho^*-\rho)^2\sigma^2(1-2\beta_t)^2+4(k-\tilde{\lambda}_t)}}{2\nu}$ by varying k and setting $\lambda_t = 0$. Calibration in Table 1.

We next consider alternative values of ρ and ρ^*, which govern the correlations between the convenience yield shock and the SDF shocks. In Panel E, we consider a greater magnitude for the correlation between the convenience yield shock and the home SDF shock by setting $\rho = 0.4$. In this parameterization, the response of the dollar appreciation to the convenience yield is dampened. To understand this result, we note that the exchange rate movement can be expressed by $ds_t = \alpha_t dt + \beta\sigma(dZ_t^* - dZ_t) + \gamma_t \nu dX_t$. Its response to the convenience yield shock is governed by both the SDF component $\beta\sigma(dZ_t^* - dZ_t)$ and the convenience yield component $\gamma_t \nu dX_t$. Specifically, a positive dX shock to the convenience yield differential always increases $\gamma_t \nu dX_t$ and appreciates the dollar. However, a positive dX shock also lowers the U.S. marginal utility and increases the foreign marginal utility, which weakens the dollar through the SDF component $\beta\sigma(dZ_t^* - dZ_t)$. When the correlations between the convenience yield shock and the SDF shocks are high enough, the SDF channel offsets the convenience yield channel, leading to a lower or even negative regression coefficient in column (2) as we see in this Panel. In comparison, our preferred choice of $\rho = 0.2$ and $\rho^* = -0.3$ manages to generate this regression coefficient that is consistent with the data.

In Panel D, we turn off the flight-to-safety channel by setting $\rho = \rho^* = 0$. Now the FX volatility is too high relative to the data; the convenience yields are not acting as a shock absorber, and the model generates too high a regression coefficient of the exchange rate movement on the convenience yield innovation.

3.3 Backus-Smith Puzzle and Exchange Rate Cyclicality

The Backus-Smith puzzle is the observation that the correlation between consumption growth and real exchange rate movement is close to zero or negative. This correlation is usually inferred from the slope coefficient in a projection of the exchange rate changes on the relative log SDF differential, which in our model can be expressed simply as:

$$\frac{[ds_t, dm_t - dm_t^*]}{[dm_t - dm_t^*, dm_t - dm_t^*]} = \beta + \frac{\gamma_t \nu(\rho^* - \rho)}{2\sigma(1-\zeta)}. \tag{14}$$

When the markets are complete and there are no convenience yields, $\beta = 1$ and $\gamma_t = 0$, and this coefficient is therefore

$$\frac{[ds_t^{cm}, dm_t - dm_t^*]}{[dm_t - dm_t^*, dm_t - dm_t^*]} = 1.$$

However, in the data, using consumption growth as a proxy for the SDF shocks, the coefficient is close to zero or negative.

This puzzle is lessened when the markets are incomplete. From Table 2, we see that this coefficient is 0.11 in our preferred calibration. This result is driven by both market incompleteness and the convenience yield. First, market incompleteness shrinks the β coefficient from 1 towards 0. That is, the first term in (14) shrinks the covariance $[ds_t, dm_t - dm_t^*]$ further towards 0. This is the channel due to market incompleteness that is highlighted by Lustig and Verdelhan (2019).

Second, convenience yield shocks also impact the dollar's exchange rate and hence affect its cyclicality. Intuitively, as the dollar's convenience yield increases in foreign high marginal utility states, the dollar appreciates and partially offsets the foreign currency appreciation driven by the high realization of the foreign SDF. When $\rho^* < \rho$, i.e., the foreign country's pricing kernel is more exposed to the convenience yield shock than the U.S., the second term in equation (14) is negative, which further reduces the slope coefficient.

Figure 2 illustrates this point. We simulate the impact of a shock to the convenience yield, traced out in the top-left panel. The top-right panel plots the paths of the SDFs m_t and m_t^*, whose innovations are correlated with the convenience yield shock under our parameterization. The foreign SDF rises, reflecting that bad news for the foreign economy is correlated with an increase in the foreign demand for U.S. dollar bonds. The home SDF decreases in our calibration since $\rho > 0$. The bottom-left panel plots the exchange rate under the complete markets benchmark. We

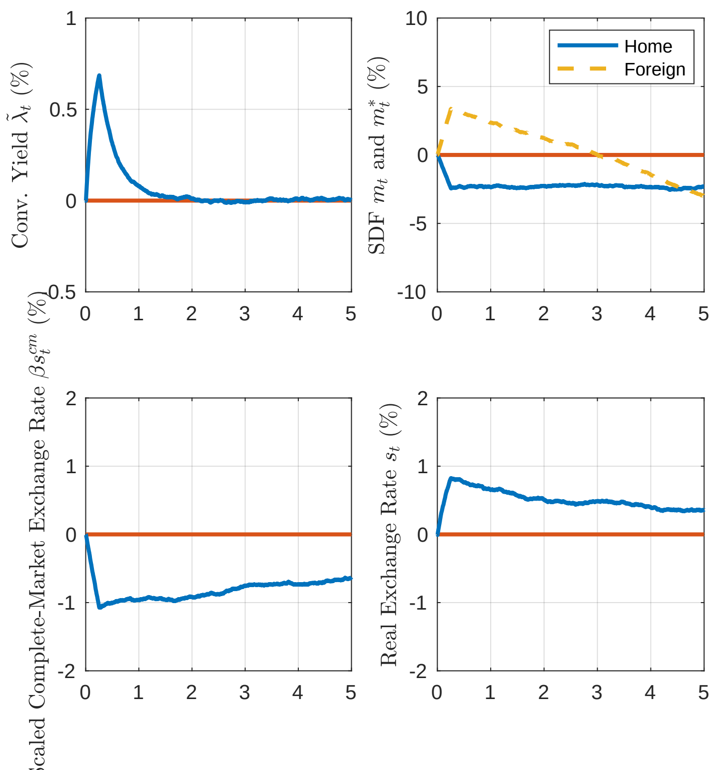


FIGURE 2. IMPULSE RESPONSE TO A CONVENIENCE YIELD SHOCK

Notes: We report the average difference between simulations in which the convenience yield λ_t jumps up by 1 standard deviation in period $(0, 0.25]$ and simulations in which all shocks have zero means. The shock to the convenience yield (top-left panel) also leads to a correlated change in the SDFs (top-right panel).

see that the home currency (dollar) depreciates, reflecting the Backus-Smith puzzle. The bottom right-panel plots the exchange rate in our incomplete markets convenience yield model. The home currency appreciates, with the convenience yield shock offsetting the change in the SDF.

Is it possible that this second effect is strong enough to reduce the Backus-Smith coefficient below zero? Our approach provides a useful, albeit negative, result. The expected excess return to a foreign investor of going long U.S. bonds relative to foreign bonds is:

$$(\mathbb{E}_t[ds_t] + r_t - r_t^*) + \frac{1}{2}[ds_t, ds_t] = -\tilde{\lambda}_t^f - [dm_t^*, ds_t],$$

where the RHS reflects the compensation the investor requires for exposure of this currency trade to the SDF as well as the convenience benefit. We can likewise write this expression in terms of the U.S. investor's return for going long foreign bonds relative to U.S. bonds:

$$(-\mathbb{E}_t[ds_t] - r_t + r_t^*) + \frac{1}{2}[ds_t, ds_t] = \tilde{\lambda}_t^h + [dm_t, ds_t].$$

The terms in parentheses in these two equations are equal but have opposing signs : they reflect the carry trade returns alternately from the foreign and the U.S. standpoints. That is, if the foreign investor is receiving an expected currency return of 2% on going long the dollar, the U.S. investor must be receiving an expected currency return of -2% on going long the foreign currency.

We can combine these equations to find:

$$-\left([dm_t, -ds_t] + \frac{1}{2}[ds_t, ds_t]\right) + \tilde{\lambda}_t^h \quad = \quad \left([dm_t^*, ds_t] + \frac{1}{2}[ds_t, ds_t]\right) + \tilde{\lambda}_t^f.$$

Rearranging terms, we obtain

Proposition 4. *The covariation between shocks to the SDF differential and exchange rates is:*

$$[dm_t - dm_t^*, ds_t] \quad = \quad [ds_t, ds_t] + \tilde{\lambda}_t.$$

The left-hand side is the numerator in our exchange rate cyclicality calculation in equation (14). Therefore, regardless of our choice in the family of exchange rate equilibria, as long as the bond convenience yield differential $\tilde{\lambda}_t$ is positive, the exchange rate cyclicality $\frac{[ds_t, dm_t - dm_t^*]}{[dm_t - dm_t^*, dm_t - dm_t^*]}$ will be positive. In particular, when we consider an incomplete-market model without convenience yields, $\tilde{\lambda}_t = 0$, then, the exchange rate cyclicality has to non-negative as $[ds_t, ds_t] > 0$. This result puts a lower bound on the exchange rate cyclicality in any possible equilibrium in the class of general diffusion models in which the investors can trade home and foreign risk-free bonds.[9]

The result is perhaps surprising. While it is the case that convenience yield shocks dampen

[9]We can generalize our setting and allow Euler equation wedges in households' valuation of the foreign risk-free bond. Specifically, suppose we rewrite the four Euler equations as

$$0 = \mathbb{E}_t\left[\frac{dM_t}{M_t}\right] + r_t + \tilde{\lambda}_t^h, \qquad 0 = \mathbb{E}_t\left[\frac{d(M_t \exp(-s_t))}{M_t \exp(-s_t)}\right] + r_t^* + \tilde{\lambda}_t^{h*},$$

the impact of the SDF shocks, as illustrated by the impulse response graph in Figure 2, it cannot be the case that the typical shock looks like a convenience yield shock. The home and foreign investors' conditional Euler equations impose a straightjacket on the comovement between the SDF and exchange rates.

On the comovement, our results offer only a partial resolution. The convenience yield mechanism reduces the cyclicality of the exchange rate substantially, but does not change the sign of the cyclicality. The argument of this section is also broader than convenience yield models. Any model operating through the SDF equations we have written down must satisfy Proposition 4. If we take the further step of associating SDF shocks with shocks to aggregates such as consumption, our result shows that it is not possible to change the sign on the consumption-exchange rate correlation.[10]

3.4 Exchange Rate Volatility Puzzle and Partial SDF-FX Pass-through

Under complete markets, in the absence of convenience yields, the real exchange rate follows the following process:

$$ds_t^{cm} - \mathbb{E}_t[ds_t^{cm}] \quad = \quad \sigma(dZ_t^* - dZ_t),$$

which does not load on the convenience yield shock dX, i.e., $\gamma_t^{cm} = 0$, and moves one-to-one with the SDF shocks, i.e., $\beta_t^{cm} = 1$. This case gives rise to an exchange rate volatility puzzle, as the SDF volatilities on the right-hand side are higher than the exchange rate volatility on the left-hand side.

Under incomplete markets with a convenience yield, the real exchange rate follows

$$ds_t - \mathbb{E}_t[ds_t] \quad = \quad \gamma_t \nu dX_t + \beta\sigma(dZ_t^* - dZ_t),$$

which loads on the convenience yield shock dX_t while having only a partial pass-through governed by $0 < \beta < 1$ from the SDF shocks to the real exchange rate movement ds_t.

Table 2 shows that in our preferred calibration, the pass-through coefficient β equals 0.18. The

$$0 = \mathbb{E}_t\left[\frac{d(M_t^* \exp(s_t))}{M_t^* \exp(s_t)}\right] + r_t + \tilde{\lambda}_t^f, \qquad 0 = \mathbb{E}_t\left[\frac{dM_t^*}{M_t^*}\right] + r_t^* + \tilde{\lambda}_t^{f*},$$

where the convenience yields $\tilde{\lambda}_t^{h*}$ and $\tilde{\lambda}_t^{f*}$ could be negative if the foreign bond has an illiquidity premium. In Appendix B.5, we show that the exchange rate cyclicality is determined by

$$[dm_t - dm_t^*, ds_t] \quad = \quad [ds_t, ds_t] + (\tilde{\lambda}_t^f - \tilde{\lambda}_t^h) - (\tilde{\lambda}_t^{f*} - \tilde{\lambda}_t^{h*}).$$

Now, the convenience yield component on the right-hand side is the difference between the foreign and home households' convenience yield differentials for the home and foreign bonds. If $\tilde{\lambda}_t^{f*} - \tilde{\lambda}_t^{h*} > \tilde{\lambda}_t^f - \tilde{\lambda}_t^h$, i.e., when the foreign households derive a higher convenience yield on the foreign bond than the home households, more so than the foreign households' convenience yield on the home bond relative to the home households, the convenience yield component is negative and the exchange rate cyclicality $[dm_t - dm_t^*, ds_t]$ can be negative.

[10] Another solution is to break the link between aggregate consumption and the SDF of the trader driving exchange rate behavior. This is the approach of market segmentation models such as Gabaix and Maggiori (2015) and Itskhoki and Mukhin (2021).

SDF volatility is 50%, but the exchange rate volatility is only 11.57%. Higher values of k lead to higher values of β and higher exchange rate volatility. This partial SDF-FX pass-through result helps resolve the volatility puzzle of Brandt et al. (2006), as the complete markets $dm - dm^*$ is more volatile than ds.

Our resolution of this volatility puzzle is similar to our resolution to the cyclicality puzzle above. Note that the conditional variance of the exchange rate movement is

$$[ds_t, ds_t] \quad = \quad \gamma_t^2 \nu^2 + 2(1-\zeta)\beta^2\sigma^2 + 2\gamma_t \nu \beta \sigma(\rho^* - \rho), \tag{15}$$

whereas under complete markets, it is

$$[ds_t^{cm}, ds_t^{cm}] \quad = \quad 2(1-\zeta)\sigma^2.$$

The reduced pass-through in our model is due to two channels. First, the pass-through coefficient β is much smaller than one. This result is related to Lustig and Verdelhan (2019), who show that incomplete markets introduce a wedge in the exchange rate movement and this wedge is always negatively correlated with the SDF differential. As a result, it offsets the exchange rate movements induced by the SDF shocks and lead to a less volatile exchange rate movement. In our model, this wedge coincides with the convenience yield, which we calibrate based on the empirical analysis in Jiang et al. (2021). This allows us to go further than Lustig and Verdelhan (2019) and nail down the extent of incomplete pass-through. Second, given $\rho^* - \rho < 0$, the last term in (15) is negative and further reduces both the volatility and the counter-cyclicality of exchange rates. This additional channel goes beyond the first channel that arise in any incomplete markets.

3.5 Currency Risk Premium Puzzles

The foreign investors' expected excess return on going long U.S. government bonds relative to foreign government bonds is given by:

$$\begin{aligned}
\Pi_t^f \quad &= \quad \frac{\mathbb{E}_t\left[d\exp(r_t + s_t - r_t^*)\right]}{\exp(r_t + s_t - r_t^*)} = \mathbb{E}_t[ds_t] + \frac{1}{2}[ds_t, ds_t] + r_t - r_t^* \\
&= \quad -\tilde{\lambda}_t^f - [dm^*, ds_t] = -\tilde{\lambda}_t^f + \beta\sigma^2(1-\zeta) + \sigma\gamma_t\nu\rho^*,
\end{aligned}$$

The first term $-\tilde{\lambda}_t^f$ captures the convenience yield: when the foreign households have a higher convenience yield on the dollar safe bonds, they are willing to accept a lower expected return. The second term $-[dm^*, ds_t]$ captures the standard risk premium: if the dollar tends to appreciate in high foreign marginal utility states, the foreign households are also willing to accept a lower expected return.

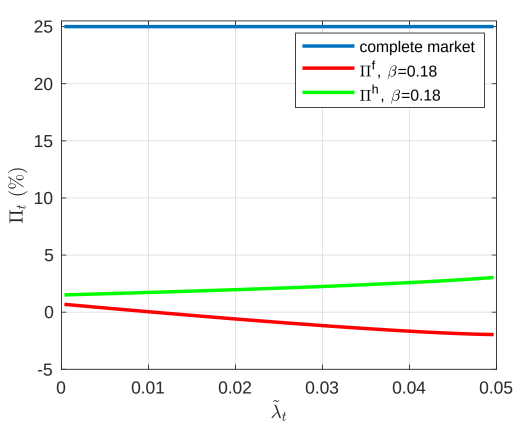
The convenience yield also drives an endogenous currency risk premium. As the foreign households' convenience yield tends to increase during foreign recessions, the dollar becomes a better hedge from the perspective of the foreign households. As a result, the convenience yield not only affects the dollar's expected return directly through the $-\tilde{\lambda}_t^f$ term but also indirectly through the currency risk premium term $\sigma\gamma_t\nu\rho^*$.

Figure 3(a) plots the dollar's expected excess returns (Π_t^f, in red) for different values of the convenience yield differential $\tilde{\lambda}_t$. On average, the dollar has a negative expected excess return from the foreign perspective. We also note that these expected excess returns vary with the state variable $\tilde{\lambda}_t$: a higher convenience yield differential on dollar safe assets leads to an even lower expected return on the dollar.

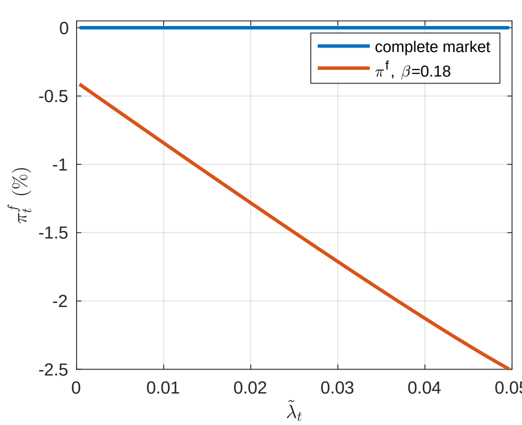
Similarly, the foreign currency's expected return Π_t^h from the perspective of the U.S. households reflects the U.S. households' convenience yield $\tilde{\lambda}_t^h$ and the covariance between the U.S. households' SDF and the exchange rate movement, $[dm_t, ds_t]$:

$$\begin{aligned} \Pi_t^h &= \frac{\mathbb{E}_t\left[d\exp(-s_t - r_t + r_t^*)\right]}{\exp(-s_t - r_t + r_t^*)} = -\mathbb{E}_t[ds_t] + \frac{1}{2}[ds_t, ds_t] - r_t + r_t^* \\ &= \tilde{\lambda}_t^h + [dm_t, ds_t] = \tilde{\lambda}_t^h + \beta\sigma^2(1-\zeta) - \sigma\gamma_t\nu\rho. \end{aligned}$$

Figure 3(a) also plots this foreign currency's expected return from the U.S. perspective, which is positive on average (Π^h, in green). As γ_t is decreasing in the convenience yield differential $\tilde{\lambda}_t$, the foreign currency's expected return is increasing in the foreign convenience yield on dollar safe assets. This result echoes the intuition in our discussion of equilibrium forces in Section 2.3: U.S. households do not derive convenience yields on foreign bonds. As a result, to enforce the U.S. households' Euler equation for holding foreign bonds with a higher expected return on the foreign currency, the foreign currency's cyclicality and volatility have to adjust to generate an endogenous currency risk premium. Specifically, when $\tilde{\lambda}_t$ is high, γ_t is low, and hence the exchange rate movement loads less on the convenience yield shock. From the U.S. households' perspective, since $\rho > 0$, a higher U.S. marginal utility is associated with a lower convenience yield



Panel (a) Expected Excess Return Levels Π_t^f and Π_t^h



Panel (b) Expected Log Excess Return π_t^f

FIGURE 3. CURRENCY EXPECTED EXCESS RETURN

differential and a weaker dollar/stronger foreign currency. As a lower γ_t weakens this hedging property, the foreign currency becomes riskier and hence has a higher risk premium.

Figure 3(b) plots the dollar's expected log excess return on going long U.S. government bonds relative to foreign government bonds, which is given by

$$\begin{aligned} \pi_t^f &= \mathbb{E}_t[d \log \exp(r_t + s_t - r_t^*)] = \mathbb{E}_t[ds_t] + r_t - r_t^* \\ &= -\frac{1}{2}(\tilde{\lambda}_t^f + \tilde{\lambda}_t^h) + \frac{1}{2}\sigma\gamma_t\nu(\rho + \rho^*), \end{aligned} \tag{16}$$

while the foreign currency's expected log excess return take the opposite value:

$$\pi_t^h = \mathbb{E}_t[d \log \exp(-r_t - s_t + r_t^*)] = -\mathbb{E}_t[ds_t] - r_t + r_t^* = -\pi_t^f.$$

As $\tilde{\lambda}_t$ increases, the log excess return on the dollar falls, while the log excess return on foreign currency rises. This behavior of the log currency risk premium is also driven by the combination of market incompleteness and the cyclicality of the convenience yield.

In Table 2 we report that the expected log return in the model is -1.56%. For comparison, in Jiang et al. (2021), we compute the returns for a foreign investor owning the entire U.S. Treasury bond index relative to their U.S. government bond index over a sample from 1980 to 2019. We report that the dollar Treasury return is 1.89% lower than the foreign bond return, which is close to the model-implied estimate of -1.56%. According to equation (16), given an average convenience yield of $\mathbb{E}[\tilde{\lambda}_t] = 1.9\%$, our model indicates that about $\frac{1}{2}\mathbb{E}[\tilde{\lambda}_t] = 0.95\%$ in the expected log return is attributable to the convenience yield, and the remaining $1.56\% - 0.95\% = 0.61\%$ is attributable to the dollar's log risk premium.

For comparison, if markets are complete, since the U.S. and the foreign SDFs have the same volatilities, the log currency risk premium on USD is zero, and the risk premium in levels equals the variance of the SDF (Bansal, 1997; Backus et al., 2001).

$$\begin{aligned} \pi_t^{cm,f} = \pi_t^{cm,h} &= 0, \\ \Pi_t^{cm,f} = \tilde{\Pi}_t^{cm,h} &= (1 - \zeta)\sigma^2. \end{aligned}$$

In this case, the log currency risk premium is too small relative to the data, whereas the level of currency risk premium is too large. These values are represented by the blue lines in Figure 3.

This result reflects a tension that constitutes the key result in Lustig and Verdelhan (2019) in an incomplete-market models without convenience yields: more market incompleteness helps rto educe exchange rate volatility and cyclicality, but it also shrinks currency risk premia towards 0. In our model, by incorporating the convenience yield, we can escape this trade-off and provide a joint resolution to all these puzzles.

Forward Premium Puzzle The SDFs have constant volatility in this model. The standard approach to introducing time variation in the conditional currency risk premium is to introduce time-varying volatility in the SDFs, which in turn can result from either changes in the quantities of risk or changes in the prices of risk. We have left out these features in order to derive a closed-form solution for the exchange rate dynamics. A more general model will be able to generate realistic variation in both the convenience yields and in the conditional currency risk premia. For example, the complete markets part of the model could be extended to generate non-zero complete markets currency risk premia by introducing asymmetries and time variation in the quantity and price of risk, as in the work of Verdelhan (2010); Colacito and Croce (2011); Farhi and Gabaix (2016).

That said, we note that while the dollar's risk premium is decreasing in the average convenience yield[11],

$$\pi_t^f \quad = \quad -\frac{1}{2}(\tilde{\lambda}_t^f + \tilde{\lambda}_t^h) + \frac{1}{2}\sigma\gamma_t\nu(\rho + \rho^*),$$

whereas the interest rate differential is decreasing in the U.S. investors' convenience yield

$$r_t - r_t^* \quad = \quad \mu + \phi s_t - \tilde{\lambda}_t^h.$$

It is natural to expect that the U.S. and foreign investors' convenience yields on the U.S. bonds are positively correlated. Then, when the demand for the safe U.S. bonds goes up, the U.S. interest rate goes down while the U.S. dollar has a lower expected return. In this way, the convenience yield also generates the forward risk premium via our incomplete-market mechanism without requiring a time-varying currency risk premium.

Long-term UIP Condition Finally, our model also has implications for the long-term UIP condition. We rewrite equation (13) as:

$$s_t - \bar{s} \quad = \quad \lim_{T\to\infty} \mathbb{E}_t \int_t^T (r_u^{xcy} - r_u^*)du + \lim_{T\to\infty} \mathbb{E}_t \int_t^T \tilde{\lambda}_u du - \lim_{T\to\infty} \mathbb{E}_t \int_t^T rp_u du.$$

[11] In fact, γ_t is also a function of $\tilde{\lambda}_t$. So the relationship between π_t^f and convenience yields is more nuanced. Under our calibration, γ_t is decreasing in $\tilde{\lambda}_t$, so the dollar exchange rate's loading on the convenience yield shock is lower when the convenience yield is higher. Since $\rho + \rho^* < 0$, the risk premium component in the dollar's expected log excess return, $\frac{1}{2}\sigma\gamma_t\nu(\rho + \rho^*)$, is increasing in $\tilde{\lambda}_t$. However, this effect is dwarfed by the convenience yield component, so the dollar's expected log excess return is still decreasing in $\tilde{\lambda}_t$ in Figure 3.

Let r_t^h denote the yield on a h-period zero coupon bond. We can combine the currency risk premium and the bond risk premium and rewrite this expression as:[12]

$$s_t - \bar{s} = \lim_{T\to\infty}(T-t)(r_t^{T-t} - r_t^{*,T-t}) + \lim_{T\to\infty}\mathbb{E}_t\int_t^T \tilde{\lambda}_u^f du. \quad (18)$$

That is, long-term UIP fails due to the additional convenience yield term, $\lim_{T\to\infty}\mathbb{E}_t\int_t^T \tilde{\lambda}_u^f du$, reflecting the sum of expected future convenience yield. A higher convenience yield leads to a stronger dollar. In the next section, we show how this mechanism can shed light on the workings of QE.

4 Quantitative Easing and the Convenience Yield Channel

Quantitative easing (QE) policies—that is, large-scale purchases of long-term bonds matched by increases in bank reserves—have been shown to affect exchange rates (Neely, 2015). In this section, we show how our model can shed light on this connection.

The long-term UIP condition (18) suggests two mechanisms by which QE may affect exchange rates. First, to the extent that QE directly impacts long-term bond risk premia and thereby the long term rate r_t^h, QE will affect exchange rates. Gourinchas et al. (2021) and Greenwood et al. (2020) bring an equilibrium model of the term structure with market segmentation along the lines of Vayanos and Vila (2021) to bear on FX markets.[13] These authors explore the impact of downward sloping demand curves for Treasurys. A decrease in the net U.S. supply of long bonds, as would occur under a QE purchase by the Fed, causes U.S. arbitrageurs to lower their required bond risk premium on long USD bonds. As a result, policymakers can control long rates and thereby impact exchange rates via a bond risk-premium channel.

Our work identifies a novel convenience yield channel through which large-scale asset purchases affect exchange rates. In particular, equation (18) shows that the dollar appreciates when future U.S. Treasury convenience yields increase, holding constant the long yields. Shifts in the supply of dollar safe assets, as happens via QE, will change convenience yields and exchange rates. The convenience yield channel is outlined in Krishnamurthy and Vissing-Jorgensen (2011). A swap of mortgage-backed securities for reserves likely increases the supply of safe assets, since reserves are a more convenient asset than mortgage-backed securities. A swap of Treasuries for reserves may increase or decrease the supply of safe assets depending on whether banks pass on

[12] To understand why the currency risk premium and bond risk premium combine to give the long rate r_t^h, Backus, Boyarchenko, and Chernov (2018); Lustig, Stathopoulos, and Verdelhan (2019) show that if the exchange rate is stationary and there is no convenience yield, the exchange rate reflects differences in long yields:

$$s_t - \bar{s} = \lim_{T\to\infty}(T-t)(r_t^{T-t} - r_t^{*,T-t}). \quad (17)$$

Because there is no difference in riskiness between holding a U.S. and a foreign bond over long holding periods, these investments have to carry the same risk premium in the limit—leading to the long-term UIP condition.

[13] In the settings of Gourinchas et al. (2021); Greenwood et al. (2020), the exchange rate is stationary. As a result, the long-term UIP condition (17) holds.

the reserve expansion by expanding deposits and the relative convenience of these deposits and Treasuries. Thus, convenience yields can either rise or fall with QE. Our theory of exchange rate connects the convenience yield with exchange rates. That is, QE that increases the convenience yield on dollar bonds should be expected to appreciate the dollar, while QE that decreases the convenience yield should be expected to depreciate the dollar. Thus our channel does not imply that QE always depreciates the dollar, as do the theories of Gourinchas et al. (2021) and Greenwood et al. (2020). Note also that the convenience yield channel assigns a special role to the U.S., to the extent that the U.S. is the world's safe asset supplier. On the other hand, the bond risk-premium channel of Gourinchas et al. (2021); Greenwood et al. (2020) is symmetric; purchases by the ECB, BoJ, or BoE also affect bond risk premia, long yields, and the exchange rate. Of course, these channels are not mutually exclusive.

4.1 Evidence for the Convenience Yield Channel

The convenience yield channel has a clear empirical counterpart. Since convenience yields can be measured using the Treasury basis, we examine the comovements between changes in the Treasury basis around QE-event dates and changes in the dollar exchange rate. The dollar exchange rate is measured as the equal-weighted G-10 cross. The basis is the 1-year U.S. Treasury against an equal-weighted currency-hedged 1-year G-10 government bond. The data is from Krishnamurthy and Lustig (2019). As we show theoretically in Jiang et al. (2021), the basis is proportional to the convenience yield on U.S. Treasury bonds relative to foreign bonds. Figure 4 presents the evidence.

We note two key patterns in this figure that are consistent with our model. First, the dollar appreciates in some of these events while it depreciates in others. Second, despite this fact, there is a clear association in both the sign and magnitude of the change in the dollar lines with changes in the basis. Table 3 presents this evidence in a regression. We regress the 2-day (Panel A) and 3-day (Panel B) change in the exchange rate against the change in the basis, controlling for the change in the relative interest rates in U.S. and foreign, which can control for shifts in the stance of monetary policy. At both horizons and measuring the basis using different maturity bonds, there is a strong relation between QE-induced changes in the basis and the dollar. Focusing on the 1-year basis in Panel A, we see that a 10 basis point change in the Treasury basis leads to a 1.66% appreciation in the dollar. From the results in Jiang et al. (2021), a 10 basis point change in the basis is equal to 1% change in the convenience yield.

Finally, we consider the impact of the Fed's dollar swap lines through the lens of our model. While our model predicts that the association between QE and exchange rate changes should not always have the same sign, its deeper prediction is between changes in the supply of dollar safe assets and exchange rates. The Fed's dollar swap lines increase the supply of dollar safe assets abroad, which lowers the convenience yield that foreign investors impute on dollar safe assets. Through our expression of dollar exchange rate determination, this action supports foreign exchange rates. There is empirical evidence for this channel. Baba and Packer (2009); Aizenman,

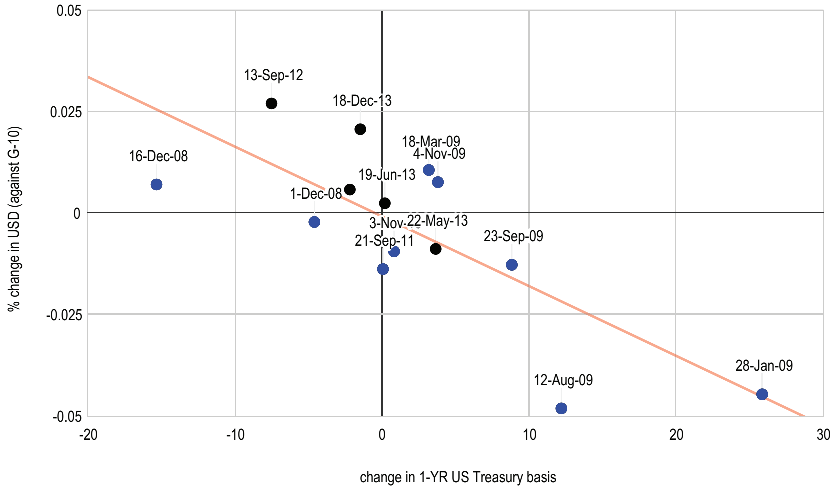


FIGURE 4. G-10 DOLLAR APPRECIATION AGAINST CHANGE IN BASIS AROUND QE EVENT DATES.

Notes: We plot the change in the Treasury basis (Δ Basis) and the change in the dollar exchange rate from the close of trading on the day prior to the event day to the close of trading 2-days later. Our sample includes 14 QE event dates.

TABLE 3–QE, BASIS, AND EXCHANGE RATE

		3M	1Y	2Y	3Y	5Y	7Y	10Y
		Panel A: 2-day window						
Δ Basis	coeff	-0.247	-0.166	-0.240	-0.225	-0.170	-0.189	-0.152
	s.e.	0.057	0.028	0.035	0.037	0.034	0.047	0.050
Δ y-diff	coeff	20.012	31.381	17.501	16.338	12.568	12.857	11.231
	s.e.	9.066	8.031	3.092	2.951	2.610	2.624	3.195
	R^2	0.637	0.828	0.837	0.800	0.751	0.697	0.563
		Panel B: 3-day window						
Δ Basis	coeff	-0.219	-0.188	-0.175	-0.183	-0.135	-0.106	-0.083
	s.e.	0.051	0.027	0.036	0.037	0.036	0.037	0.043
Δ y-diff	coeff	15.319	22.568	15.494	13.861	12.186	12.068	11.944
	s.e.	7.054	6.307	3.227	2.541	2.064	2.253	2.685
	R^2	0.624	0.811	0.745	0.779	0.778	0.724	0.643

Notes: We report regression of changes in dollar (G-10) on QE-induced changes in U.S. Treasury basis and changes in yields. We include 14 QE event dates. We include the event day and define the change in the Treasury basis (Δ Basis) and the change in the dollar from the close of trading on the day prior to the event day to the close of trading x days later. Δ y-diff is the change in the 1-year interest rate differential between the U.S. and the G-10 average.

Ito, and Pasricha (2022) present evidence that the dollar swap lines between central banks reduce the dollar-foreign currency basis (which can measure the convenience yield) and depreciate the dollar.

4.2 QE Evaluation in the Model

We next turn to our model to see how well it can capture these patterns. We do not explicitly model the relation between the convenience yield λ and the quantity of safe assets. Instead, we focus directly on inducing a shock to λ and tracing out the impact of this shock on the exchange rate. Appendix A shows how a change in the supply of safe assets affects λ in the context of fully specified macro-finance model. We discretize the model by a time increment of $\Delta t = 0.0025$ and start the model at $t = 0$. For initial values, we set $s_0 = s_0^{cm} = \bar{s}$ and $\lambda_0 = 0$, and set H_0 to satisfy

$$s_0 \quad = \quad f(\lambda_0) + H_0 + \beta s_0^{cm}.$$

We simulate dX_t, dZ_t and dZ_t^* under the normal distribution with mean zero and standard deviation $\sqrt{\Delta t}$. For the first quarter, i.e., periods $(0, 0.25]$, we introduce a positive impulse that raises all realizations of the shocks dX_t by one standard deviation. This impulse simulates a positive convenience yield shock in the first quarter. Then, we average across 100,000 simulated paths of the shocks (dX_t, dZ_t, dZ_t^*). In this way, we estimate the average response following a positive convenience yield shock at date 0. We also simulate a benchmark case in which we draw from the normal distribution with mean 0 for the entire period $t \in (0, T]$. As expected, the average responses of exchange rate and convenience yield are close to zero in this benchmark case. We report the difference between the average responses in the case of a convenience yield shock and the benchmark case.

Figure 5 reports the result. In the top-left panel, we shock the convenience yield λ_t and then let the internal dynamics of mean reversion gradually bring the convenience yield to zero over the next 10 quarters. We can think of this shock as an announcement by the central bank to purchase assets at date 0, and then slowly unwind these purchases over the next 10 quarters.

The top-left panel of the figure graphs the instantaneous convenience yield over this path. The top-right panel plots the average convenience yield between time 0 and time t $(= \frac{1}{t}\int_{k=0}^{t} \tilde{\lambda}_k dt)$. This panel gives an expectations-hypothesis-type heuristic of how different maturity bases will react to this shock. We see that the largest response is in the short maturity bases, with the effects dying out for longer maturity bases. At the one-year point, the convenience yield rises by about 0.35% (given a 1:10 ratio between Treasury basis and convenience yield, this implies a widening in the Treasury basis of 3.5 basis points). The bottom-left panel plots the complete markets exchange rate averaged across simulation paths. The last panel plots the exchange rate from the model. On impact, the exchange jumps by 1.7%, before gradually reverting to its long-run level. Thus quantitatively, our model generates a regression coefficient on the 1-year basis of -0.5, which is of the same magnitude but greater than the empirical estimates in Table 3.

The effect of this QE experiment unwinds gradually over the next several years. We note that the behavior in term H_t representing the cumulative convenience yields is also interesting. Since

$$H_t \quad = \quad \exp(-\phi t) H_0 + \int_0^t \exp(-\phi(t-u)) h(\lambda_u) du,$$

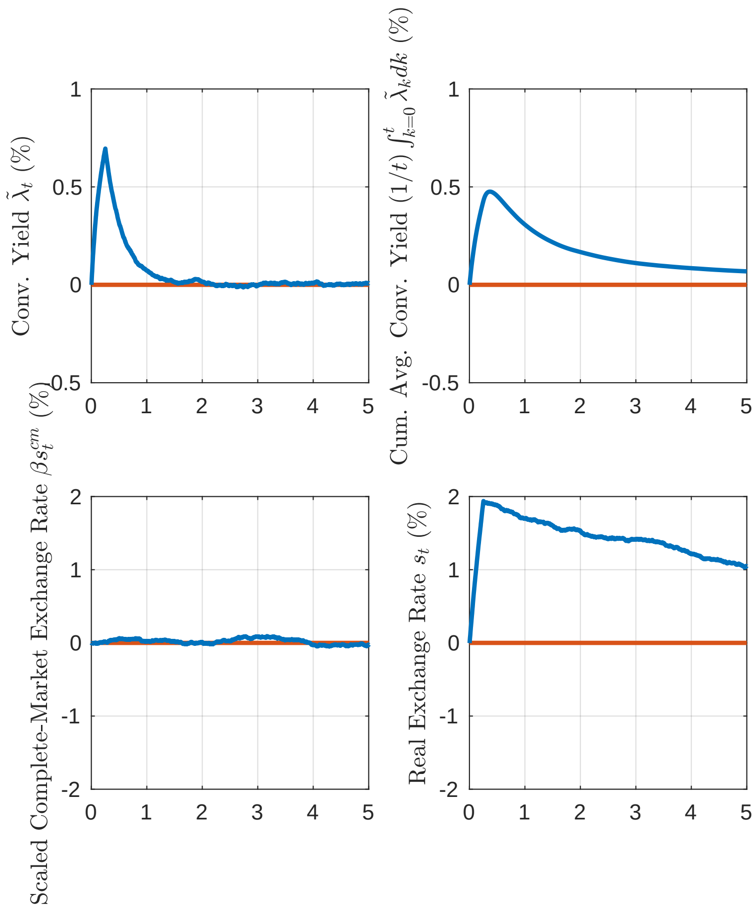


FIGURE 5. IMPULSE RESPONSE TO A PURE CONVENIENCE YIELD SHOCK.

Notes: We report the average difference between simulations in which the convenience yield λ_t jumps up by 1 standard deviation in period $(0, 0.25]$ and simulations in which all shocks have zero means. Note that we assume that there is also no change in the SDFs. The cumulative average convenience yield in the top right panel is the expected average convenience yield for the next t periods at time 0.

it aggregates influences of past convenience yields with exponential decay. As a result, the half-life of the response in the real exchange rate is longer than the half-life of the response in the spot convenience yield.

5 Conclusion

We summarize our work by revisiting the equation describing exchange rate dynamics:

$$ds_t \quad = \quad \alpha_t dt + \beta\sigma(dZ_t^* - dZ_t) + \gamma_t \nu dX_t.$$

These exchange rate dynamics must be consistent with the four asset pricing conditions, for each of home and foreign investor in each of a home and foreign risk-free bond, where the home (dollar) bond offers convenience services to investors. The introduction of convenience yields leads to $\gamma_t > 0$ (dollar appreciates when foreign investor convenience for dollar bonds rises), and $\beta < 1$ (limited pass-through of marginal utility shocks to exchange rates). Our calibration demonstrates that convenience yields plus incomplete markets can help address exchange rate puzzles. Our joint modeling of bond and currency markets also shows how QE, as well as dollar swap lines, can affect exchange rates.

These results highlight the significance of the worldwide demand for dollar safe assets in determining the international financial markets equilibrium. We conclude by noting that our analysis, by design, only models the asset pricing determination of exchange rates. We have solved for exchange rate dynamics that are consistent with four asset-pricing Euler equations. In doing so, we have sidestepped other salient aspects of equilibrium concerning quantities, especially the bond positions of foreign/home investors and the dynamics of the current account. Next steps in research may move in this direction. For these next steps, our research points out that including convenience yields and incomplete markets are important ingredients in a richer macroeconomic model. That is, as any macroeconomic model will include the four Euler equations we work with, the solution to the macroeconomic model will be among the class of solutions we have presented.

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Appendix

A Two-period International Macroeconomic Model

In this appendix, we develop a simple, fully specified, international macroeconomics model that nests the general characterization of the exchange rate dynamics in our main text. We use it as an example of a large class of international macroeconomic models that can be characterized by our approach.

There are two periods, indexed by $t = 0, 1$. We assume that home (U.S.) and foreign households have preferences over consumption and obtain convenience services from their holdings of the home country's bonds.

A.1 Home Households

There is a unit mass of identical price-taking households in each country. Let $c_{H,t}$ denote home households' consumption of a home composite good; let $c_{F,t}$ denote home households' consumption of a foreign composite good. We define $c_t = (c_{H,t})^{\alpha}(c_{F,t})^{1-\alpha}$ as the aggregated consumption bundle. Home households' lifetime utility is

$$u = \frac{1}{1-\gamma}c_0^{1-\gamma} + v(q_{H,0}\exp(-r_0)) + \delta\frac{1}{1-\gamma}c_1^{1-\gamma},$$

where $q_{H,0}$ is the notional amount of holdings of home bonds, and r_0 is the equilibrium interest rate of the home bonds.

The only tradable assets are the home and foreign risk-free bonds. Let $q_{F,0}$ denote home households' holding in foreign bonds. The home and foreign risk-free bonds are denominated in the unit of their composite bundles. The real exchange rate s_t is also between the home composite bundle and the foreign composite bundle. Using this numeraire, the state-by-state budget constraint is

$$\begin{aligned}
\exp(-r_0)\tau_1 + y_0 p_0 &= c_0 + \exp(-r_0)q_{H,0} + \exp(-r_0^* - s_0)q_{F,0}, \\
y_1 p_1 + q_{H,0} + \exp(-s_1)q_{F,0} &= c_1 + \tau_1,
\end{aligned}$$

where y_t denotes an exogenous endowment in home goods; p_t is the price of the home good in the numeraire of the home consumption bundle. We assume that the home bonds are issued by the home country's government. τ_1 is the total par value of the issuance. The proceeds from the issuance in period 0 (i.e., $\exp(-r_0)\tau_1$) are transferred to the home households, and the bonds are paid off in period 1 using taxes collected from the home households.

The Lagrangian is

$$\begin{aligned}
& \mathbb{E}_0[\frac{1}{1-\gamma}c_0^{1-\gamma} + v(q_{H,0}\exp(-r_0)) + \delta\frac{1}{1-\gamma}(y_1 p_1 + q_{H,0} + \exp(-s_1)q_{F,0} - \tau_1)^{1-\gamma}] \\
+ \quad & \zeta_0\left[\exp(-r_0)\tau_1 + y_0 p_0 - (c_0 + \exp(-r_0)q_{H,0} + \exp(-r_0^* - s_0)q_{F,0})\right].
\end{aligned}$$

Inter-period solution The first-order conditions for investment in home and foreign bonds give two Euler equations:

$$\begin{aligned} 1 - c_0 v'(q_{H,0}\exp(-r_0)) &= \mathbb{E}_0\left[\delta\frac{c_0}{c_1}\exp(r_0)\right], \\ 1 &= \mathbb{E}_0\left[\delta\frac{c_0}{c_1}\exp(-\Delta s_1 + r_0^*)\right]. \end{aligned}$$

Intra-period solution Let p_t denote the price of the home good in units of the home consumption bundle, and let p_t^* denote the price of foreign good in units of the foreign consumption bundle. Since we assume that the foreign consumption bundle is the foreign good, we have that $p_t^* = 1$. Then, the price of the consumption bundle is $c_t = p_t c_{H,t} + c_{F,t}\exp(-s_t)$. We substitute this expression into the Lagrangian:

$$\begin{aligned} 1 - c_0^{\gamma} v'(q_{H,0}\exp(-r_0)) &= \mathbb{E}_0\left[\delta(\frac{c_0}{c_1})^{\gamma}\exp(r_0)\right], \\ 1 &= \mathbb{E}_0\left[\delta(\frac{c_0}{c_1})^{\gamma}\exp(-\Delta s_1 + r_0^*)\right]. \end{aligned}$$

The first-order conditions for home and foreign consumption imply:

$$\begin{aligned} p_0\exp(s_0) &= \frac{\alpha}{1-\alpha}\frac{c_{F,0}}{c_{H,0}}, \\ p_1\exp(s_1) &= \frac{\alpha}{1-\alpha}\frac{c_{F,1}}{c_{H,1}}. \end{aligned}$$

A.2 Foreign Households

For tractability, we assume that foreign households only consume foreign goods. Their total consumption is $c^* = c_F^*$. Then foreign utility is,

$$u^* = \frac{1}{1-\gamma}(c_0^*)^{1-\gamma} + v(q_{H,0}^*\exp(-r_0 + s_0)) + \delta\frac{1}{1-\gamma}(c_1^*)^{1-\gamma}.$$

Using foreign goods as numeraire, the foreign budget constraint at each date are:

$$\begin{aligned} \exp(-r_0^*)\tau_1^* + y_0^* p_0^* &= c_0^* + \exp(-r_0 + s_0)q_{H,0}^* + \exp(-r_0^*)q_{F,0}^*, \\ y_1^* p_1^* + q_{H,0}^*\exp(s_1) + q_{F,0}^* &= c_1^* + \tau_1^*, \end{aligned}$$

where y_t^* denotes an exogenous endowment in foreign goods. Recall that p_t^* is the price of the foreign good in the numeraire of the foreign composite bundle and that $p_t^* = 1$.

Then, the Lagrangian is

$$\mathbb{E}_0[\frac{1}{1-\gamma}(c_0^*)^{1-\gamma} + v(q_{H,0}^*\exp(-r_0 + s_0)) + \delta\frac{1}{1-\gamma}(y_1^* + q_{H,0}^*\exp(s_1) + q_{F,0}^* - \tau_1^*)^{1-\gamma}]$$

$$+ \quad \zeta_0^*[\exp(-r_0^*)\tau_1^* + y_0^* - (c_0^* + \exp(-r_0 + s_0)q_{H,0}^* + \exp(-r_0^*)q_{F,0}^*)].$$

Inter-period solution The first-order conditions imply the Euler equations for foreign households

$$\begin{aligned}
1 - (c_0^*)^\gamma v'(q_{H,0}^* \exp(-r_0 + s_0)) &= \mathbb{E}_0 \left[\delta (\frac{c_0^*}{c_1^*})^\gamma \exp(r_0 + \Delta s_1) \right], \\
1 &= \mathbb{E}_0 \left[\delta (\frac{c_0^*}{c_1^*})^\gamma \exp(r_0^*) \right].
\end{aligned}$$

A.3 Market Clearing

In the goods market:

$$\begin{aligned}
y_t &= c_{H,t}, \\
y_t^* &= c_{F,t} + c_{F,t}^*.
\end{aligned}$$

In the bond market:

$$\begin{aligned}
\tau_1 &= q_{H,0} + q_{H,0}^* \\
\tau_1^* &= q_{F,0} + q_{F,0}^*.
\end{aligned}$$

A.4 Macro Synthesis

The set of known exogenous variables is

$$(y_0, y_0^*, \theta_0, \theta_0^*, \tau_1, \tau_1^*).$$

The set of stochastic exogenous variables is

$$(y_1, y_1^*).$$

The set of endogenous variables for home and foreign households is

$$(c_{H,0}, c_{F,0}, c_{H,1}, c_{F,1}, q_{H,0}, q_{F,0}, c_{F,0}^*, c_{F,1}^*, q_{H,0}^*, q_{F,0}^*, r_0, r_0^*, p_0, p_1, s_0, s_1).$$

There are 16 endogenous variables. Once we set the known exogenous variables, the endogenous variables at period 0 are pinned-down, and the endogenous variables at period 1 will be a function of the stochastic exogenous variables (y_1, y_1^*).

The model implies the following 18 equations, two of which are redundant since the market clearing adds up to the sum of households' budget constraints. For the home households,

$$\begin{aligned}
\tau_1 \exp(-r_0) + y_0 p_0 &= (c_{H,0})^{\alpha}(c_{F,0})^{1-\alpha} + \exp(-r_0) q_{H,0} + \exp(-r_0^* - s_0) q_{F,0}, \\
(c_{H,0})^{\alpha}(c_{F,0})^{1-\alpha} &= p_0 c_{H,0} + c_{F,0} \exp(-s_0), \\
y_1 p_1 + q_{H,0} + \exp(-s_1) q_{F,0} &= (c_{H,1})^{\alpha}(c_{F,1})^{1-\alpha} + \tau_1, \\
(c_{H,1})^{\alpha}(c_{F,1})^{1-\alpha} &= p_1 c_{H,1} + c_{F,1} \exp(-s_1), \\
1 - \left((c_{H,0})^{\alpha}(c_{F,0})^{1-\alpha}\right)^{\gamma} v'(q_{H,0} \exp(-r_0)) &= \mathbb{E}_0 \left[\delta \left(\frac{(c_{H,0})^{\alpha}(c_{F,0})^{1-\alpha}}{(c_{H,1})^{\alpha}(c_{F,1})^{1-\alpha}} \right)^{\gamma} \exp(r_0) \right] \\
1 &= \mathbb{E}_0 \left[\delta \left(\frac{(c_{H,0})^{\alpha}(c_{F,0})^{1-\alpha}}{(c_{H,1})^{\alpha}(c_{F,1})^{1-\alpha}} \right)^{\gamma} \exp(-\Delta s_1 + r_0^*) \right].
\end{aligned}$$

For the foreign households,

$$\begin{aligned}
\tau_1^* \exp(-r_0^*) + y_0^* &= c_{F,0}^* + \exp(-r_0 + s_0) q_{H,0}^* + \exp(-r_0^*) q_{F,0}^*, \\
y_1^* + q_{H,0}^* \exp(s_1) + q_{F,0}^* &= c_{F,1}^* + \tau_1^*, \\
1 - (c_{F,0}^*)^{\gamma} v'(q_{H,0}^* \exp(-r_0 + s_0)) &= \mathbb{E}_0 \left[\delta \left(\frac{c_{F,0}^*}{c_{F,1}^*} \right)^{\gamma} \exp(r_0 + \Delta s_1) \right], \\
1 &= \mathbb{E}_0 \left[\delta \left(\frac{c_{F,0}^*}{c_{F,1}^*} \right)^{\gamma} \exp(r_0^*) \right].
\end{aligned}$$

Market clearing conditions are

$$\begin{aligned}
y_0 &= c_{H,0}, \\
y_1 &= c_{H,1}, \\
\tau_1 &= q_{H,0} + q_{H,0}^*, \\
y_0^* &= c_{F,0} + c_{F,0}^*, \\
y_1^* &= c_{F,1} + c_{F,1}^*, \\
\tau_1^* &= q_{F,0} + q_{F,0}^*.
\end{aligned}$$

The prices and exchange rates can be pinned down by:

$$\begin{aligned}
p_0 \exp(-s_0) &= \frac{\alpha}{1-\alpha} \frac{c_{F,0}}{c_{H,0}}, \\
p_1 \exp(-s_1) &= \frac{\alpha}{1-\alpha} \frac{c_{F,1}}{c_{H,1}}.
\end{aligned}$$

A.5 Four Euler equations

We use M_t and M_t^* to denote the two households' marginal utility in period t. We then recover, in their discrete-time forms, the four Euler equations of the main text:

$$
\begin{aligned}
1 - c_0 v'(q_{H,0} \exp(-r_0)) &= \mathbb{E}_0 \left[\frac{M_1}{M_0} \exp(r_0) \right] && \text{(A.1)} \\
1 &= \mathbb{E}_0 \left[\frac{M_1}{M_0} \exp(-\Delta s_1 + r_0^*) \right] && \text{(A.2)} \\
1 - c_0^* v^{*'}(q_{H,0}^* \exp(-r_0 + s_0)) &= \mathbb{E}_0 \left[\frac{M_1^*}{M_0^*} \exp(r_0 + \Delta s_1) \right], && \text{(A.3)} \\
1 &= \mathbb{E}_0 \left[\frac{M_1^*}{M_0^*} \exp(r_0^*) \right] && \text{(A.4)}
\end{aligned}
$$

Thus as noted, these Euler equations which we study in the main text arise in the international macro model of this appendix. It should also be apparent that they will arise in most international macro models.

The two-period model adds two elements relative to the model of the main text. First, the Ms and λs (as reflected by the v's) are endogenous objects. In the two-period model, they are driven by shocks to endowments and bond demand (θ). In particular, the macro model indicates the correlation structure that will arise endogenously. In the model of the main text, we solve the model for an arbitrary correlation structure but then take a stand on the correlations when quantitatively evaluating the model. The next sections explain further these choices of correlation parameters. Second, the macro model imposes two further equations that must be satisfied in equilibrium. These equations are that trade in goods (and bonds) needs to be balanced in both periods. With these two equations, the model pins down both s_0 and the [stochastic] s_1. In our main text, we solve for a family of exchange rate solutions that solve the four Euler equations. The macro trade balance equation further restricts the possible equilibria within this family.

A.6 Further Simplification

To solve the model further, we assume that the foreign households' utility from holding the home bonds is

$$
v^*(q_{H,0}^* \exp(-r_0 + s_0)) = \theta_0^* \frac{1}{1-\gamma} (q_{H,0}^* \exp(-r_0 + s_0))^{1-\gamma}.
$$

so that the demand for the home bonds is downward-sloping.

Also, for a positive ϵ, the home households' marginal utility from holding the home bonds is

$$
v'(q_{H,0}) = \begin{cases} \frac{1}{c_0}(1 - \frac{1}{\epsilon} q_{H,0}) & \text{if } 0 \le q_{H,0} \le \epsilon, \\ 0 & \text{if } q_{H,0} > \epsilon. \end{cases}
$$

That is,

$$v(q_{H,0}) = \begin{cases} \frac{1}{c_0}(q_{H,0} - \frac{1}{2\epsilon} q_{H,0}^2) & \text{if } 0 \leq q_{H,0} \leq \epsilon, \\ \frac{1}{2c_0}\epsilon & \text{if } q_{H,0} > \epsilon. \end{cases}$$

Then, we have a unique solution for $q_{H,0}$ that satisfies both equation (A.1) and $q_{H,0} \in (0, \epsilon)$:

$$q_{H,0} \quad = \quad \epsilon \exp(r_0) E_0 \left[\frac{M_1}{M_0} \exp(r_0) \right].$$

We take the limit of ϵ to 0 from above. Then, the model reduces to the case in which the home households hold a zero amount of home bonds in equilibrium, i.e.,

$$\begin{aligned} q_{H,0} &\rightarrow 0 \\ q_{H,0}^* &\rightarrow \tau_1. \end{aligned}$$

We further assume $y_0 = y_0^* = 1$. Note that we still have that home and foreign endowments at time 1 are uncertain. Then, we obtain the following set of equations that characterize the equilibrium:

$$\begin{aligned} 1 &= \mathbb{E}_0 \left[\delta \left(\frac{(c_{F,0})^{1-\alpha}}{(y_1)^\alpha (c_{F,1})^{1-\alpha}} \right)^\gamma \exp(r_0^*) \left(\frac{c_{F,0} y_1}{c_{F,1}} \right)^\alpha \right] \\ 1 - \theta_0^* \left(\frac{(1-\alpha)(1-c_{F,0})}{\tau_1 \exp(-r_0)(c_{F,0})^\alpha} \right)^\gamma &= \mathbb{E}_0 \left[\delta \left(\frac{1-c_{F,0}}{y_1^* - c_{F,1}} \right)^\gamma \exp(r_0) \left(\frac{c_{F,1}}{c_{F,0} y_1} \right)^\alpha \right], \\ 1 &= \mathbb{E}_0 \left[\delta \left(\frac{1-c_{F,0}}{y_1^* - c_{F,1}} \right)^\gamma \exp(r_0^*) \right] \\ c_{F,0} &= \frac{1}{1-\alpha} (c_{F,0})^\alpha \exp(-r_0) \tau_1 - \exp(-r_0^*) q_{F,0} \\ c_{F,1} &= -\frac{1}{1-\alpha} \frac{(c_{F,1})^\alpha}{(y_1)^\alpha} \tau_1 + q_{F,0}. \end{aligned}$$

The solution to these equations pin down the five unknown endogenous variables

$$(c_{F,0}, c_{F,1}, q_{F,0}, r_0, r_0^*).$$

A.7 Parameterization, Solution, and Correlations

While it is possible to algebraically analyze the equilibrium as described, we will parameterize the model and illustrate the solution. We assume that both home and foreign endowments y_1 and y_1^* follow a uniform distribution over $[0.9, 1.1]$ and we discretize this distribution at 0.01 increments. Other primitive parameters are given in Table A.1(a).

The endogenous variables are given in Table A.1(b). We note that the foreign households hold the entire outstanding amount of home bonds, i.e., $q_{H,0}^* = \tau_1 = 0.1$, whereas home households hold even more foreign bonds, i.e., $q_{F,0} = 0.3624 > q_{H,0}^*$. This saving by the home country allows the home households to purchase foreign goods for consumption in the 2nd period.

First, we compute the realized log exchange rate movement Δs_1 and the realized SDF differential $\Delta m_1 - \Delta m_1^*$. We run the regression

$$\Delta s_1 \quad = \quad \alpha + \beta(\Delta m_1 - \Delta m_1^*) + \varepsilon$$

and obtain $\beta < 1$. This partial pass-through is consistent with our characterization in the main text. We also obtain an $R^2 < 100\%$, consistent with the result in the main text regarding the presence of additional variation in exchange rates that arise from the incomplete-market wedge. For comparison, if markets are complete, β should be equal to 1, and the R^2 should be 100%.

Second, we vary the θ_0^* parameter that governs the foreign investors' utility from holding home bonds and report the equilibrium dollar exchange rate and convenience yield in period 0 in Figure A.1. Consistent with our characterization in the main text, a higher θ_0^* implies a stronger foreign demand for the U.S. bonds and hence a higher foreign convenience yield, which leads to a stronger dollar. We also report the interest rate differential between the U.S. and foreign bonds (in logs) and the expected dollar exchange rate movement in Figure A.2. Also consistent with our results in the main text, as the foreigners' demand for U.S. bonds increases, the U.S. interest rate falls relative to the foreign interest rate (Figure A.2, left panel), and the dollar's expected return falls (Figure A.2, implied from the sum of both panels). Lastly, in Figure A.3, we report the home and foreign households' marginal utilities in period 0 as functions of θ_0^*. A stronger foreign demand for U.S. bonds results in a lower U.S. marginal utility and a higher foreign marginal utility. This happens because, in equilibrium, the higher demand causes the foreign households to sell foreign goods at date 0 to the U.S. households in order to purchase U.S. bonds (as well as causing foreign households to sell some of their foreign bonds to U.S. households to finance the U.S. bond purchase). This marginal-utility association justifies our assumption of $\rho > 0$ and $\rho^* < 0$ that is made in the main text.

Finally, we also consider a shock to the supply of U.S. safe bonds, as would occur under QE or the opening of dollar swap lines, by varying the τ_1 parameter. As τ_1 increases, the total supply of the U.S. safe bond increases, and, by our simplifying assumption, is entirely absorbed by the foreign households. This supply shock lowers the equilibrium convenience yield λ_t and depreciates the dollar (Figure A.4). Moreover, consistent with our results in the main text, the U.S. interest rate rises relative to the foreign interest rate, and the dollar's expected return also rises (Figure A.5). The decline in the dollar bond's convenience yield also results in a higher U.S. marginal utility and a lower foreign marginal utility.

In sum, the model of this appendix illustrates that the forces governing exchange rate behavior that are embedded in our main text can arise in a fully-specified international macroeconomic model. The model also validates our assumptions about the correlations between marginal utilities and convenience yields. In particular, the reduced-form convenience yield shock dX captures both demand and supply shocks in the U.S. safe bond market, and produces implications for the dollar's exchange rate and expected return that are consistent with the results in the fully-specified model.

TABLE A.1–PRIMITIVE AND ENDOGENOUS VARIABLES

Panel (a) Primitive Variables		
Subjective discount factor	δ	0.95
Degree of Relative Risk Aversion	γ	4
Consumption home bias	α	0.9
Supply of home bonds	τ_1	0.1
Supply of foreign bonds	τ_1^*	0.1
Foreign investors' utility from holding home bonds	θ_0^*	0.1
Panel (b) Endogenous Variables		
Home holdings of foreign bonds	$q_{F,0}$	0.3624
Foreign holdings of home bonds	$q_{H,0}^*$	0.1
Time-0 dollar exchange rate	$\exp(s_0)$	1.9959
Avg time-1 dollar exchange rate	$\mathbb{E}\exp(s_1)$	1.9747
Time-0 home consumption of home goods	$c_{H,0}$	1
Avg time-1 home consumption of home goods	$\mathbb{E}c_{H,1}$	1
Time-0 home consumption of foreign goods	$c_{F,0}$	0.1669
Avg time-1 home consumption of foreign goods	$\mathbb{E}c_{F,1}$	0.1649
Time-0 foreign consumption of foreign goods	$c_{F,0}^*$	0.8331
Avg time-1 foreign consumption of foreign goods	$\mathbb{E}c_{F,1}^*$	0.8351
SDF-FX pass-through	β	0.1789

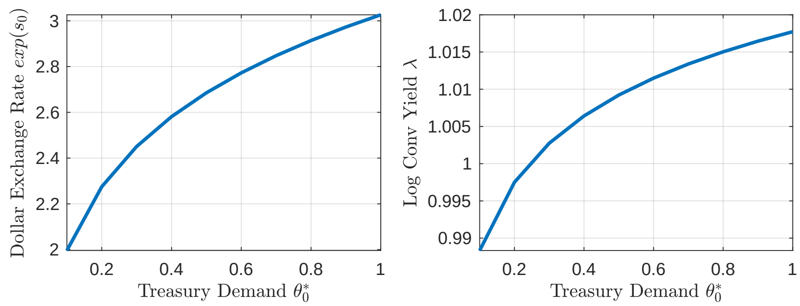

FIGURE A.1. INITIAL DOLLAR EXCHANGE RATE AS A FUNCTION OF FOREIGN TREASURY DEMAND

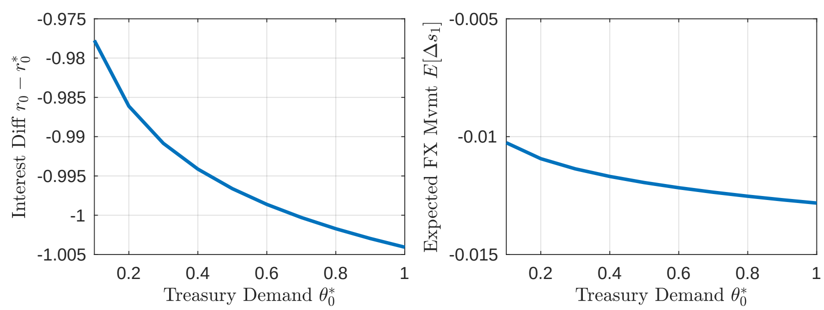

FIGURE A.2. INTEREST DIFFERENTIAL AND EXPECTED DEPRECIATION AS FUNCTIONS OF FOREIGN TREASURY DEMAND

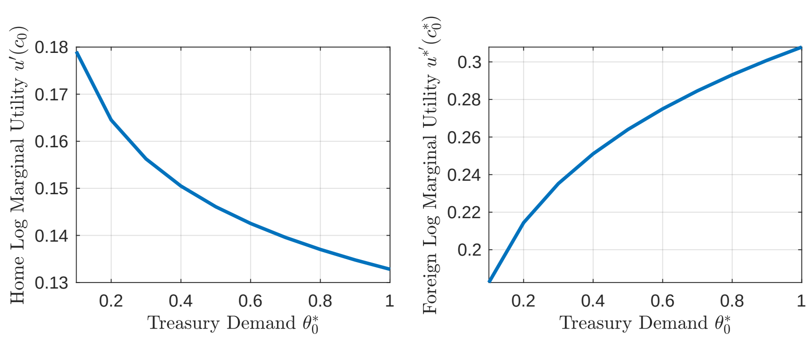

FIGURE A.3. HOME AND FOREIGN MARGINAL UTILITIES AS FUNCTIONS OF FOREIGN TREASURY DEMAND

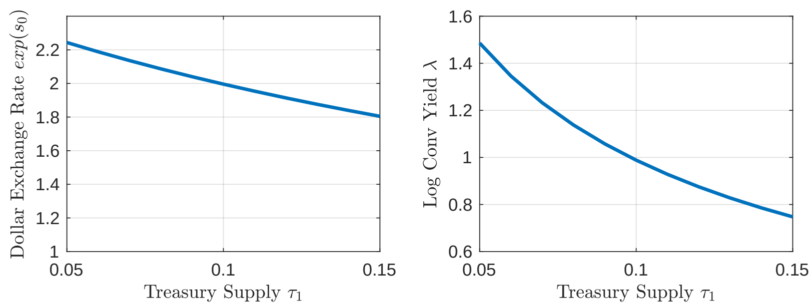

FIGURE A.4. INITIAL DOLLAR EXCHANGE RATE AS A FUNCTION OF HOME TREASURY SUPPLY

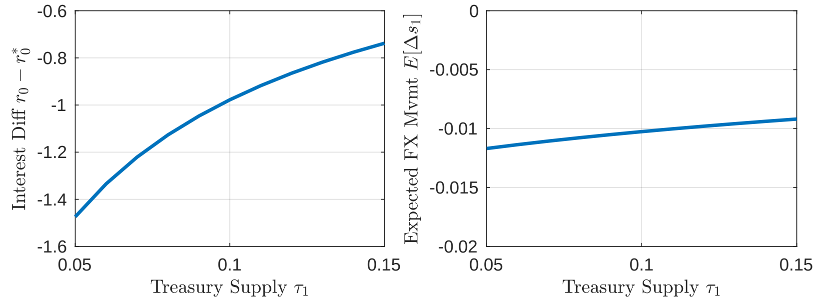

FIGURE A.5. INTEREST DIFFERENTIAL AND EXPECTED DEPRECIATION AS FUNCTIONS OF HOME TREASURY SUPPLY

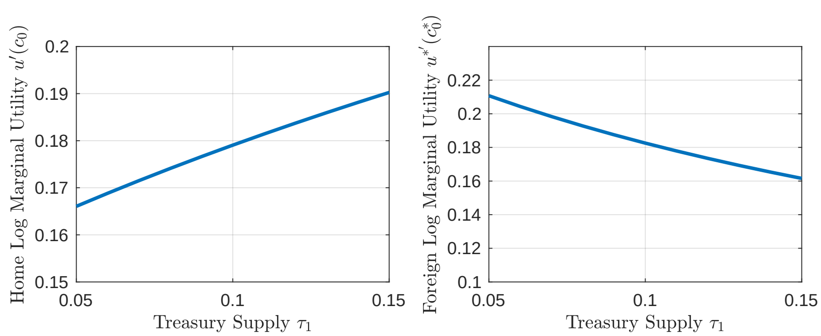

FIGURE A.6. HOME AND FOREIGN MARGINAL UTILITIES AS FUNCTIONS OF HOME TREASURY SUPPLY

B Proof

B.1 Proof of Lemma 1

The U.S. households choose consumption and bond holding processes, c, q_H, q_F, to maximize their expected lifetime utility, given initial wealth $W(0) = W$. We have the problem,

$$
\begin{aligned}
V(W) &= \sup_{c \geq 0; q_H; q_F} \mathbb{E}_0 \left[\int_{t=0}^{\infty} e^{-\rho t} u(c_t, q_{H,t}; \theta_t) dt \right], \\
W(t) &= W + \int_0^t (q_{H,s} r_s + q_{F,s} r_s^* S_s^{-1} - c_s) ds + \int_0^t q_{F,s} d(S_s^{-1}) \geq 0 \\
W(t) &= q_{H,t} + q_{F,t} S_t^{-1}
\end{aligned}
$$

In order for W to remain nonnegative, admissible policies c, q_F, q_H have the property that, for t larger than the stopping time $\inf\{s : W_s = 0\}$, $q_{H,t} = q_{F,t} = 0$ and $c_t = W(t) = 0$. That is, nonzero investment and consumption are ruled out once there is no remaining wealth.

Suppose that the exchange rate $S_t = \exp(s_t)$ follows an Ito process, then so does S_t^{-1}. Therefore, the drift and volatility terms of $W(t)$ are:

$$
\begin{aligned}
\mu(W) &= (q_H r + q_F r^* S^{-1} - c) + q_F \mu(S^{-1}) \\
\sigma(W) &= q_F \sigma(S^{-1})
\end{aligned}
$$

To save notation, we have omitted the subscript t.

The Hamilton-Jacobi-Bellman equation of the household's problem is

$$
\rho V(W) = \max_{c \geq 0; q_H; q_F} \{ u(c, q_H; \theta) + \mathcal{D}V(W) + \lambda (W - q_H - q_F S^{-1}) \} \tag{B.1}
$$

where

$$
\begin{aligned}
\mathcal{D}V(W) &= V'(W)\mu(W) + \frac{1}{2} V''(W)(\sigma(W))^2 \qquad \text{(B.2)} \\
&= V'(W)[q_H r + q_F r^* S^{-1} - c + q_F \mu(S^{-1})] + \frac{1}{2} V''(W) q_F^2 (\sigma(S^{-1}))^2
\end{aligned}
$$

We take the first-order conditions inside $\max\{\cdot\}$ of Eq.(B.1) with respect to c, q_H, q_F, λ ,

$$
\begin{aligned}
w'(c) &= V'(W) \\
v'(q_H; \theta) - V'(W) r - \lambda &= 0 \\
V'(W)[r^* S^{-1} + \mu(S^{-1})] + V''(W) q_F (\sigma(S^{-1}))^2 - \lambda S^{-1} &= 0 \\
W - q_H - q_F S^{-1} &= 0
\end{aligned}
$$

Assume we have an interior solution to the maximization problem Eq.(B.1). Then, the optimal policies $\bar{c}$, $\bar{q_H}$, $\bar{q_F}$, and $\bar{\lambda}$ satisfy:

$$
\begin{aligned}
w'(\bar{c}(W)) &= V'(W) && \text{(B.3)}\\
v'(\bar{q}_H(W);\theta) - V'(W)r &= \bar{\lambda}(W) && \text{(B.4)}\\
V'(W)[r^*S^{-1} + \mu(S^{-1})] + V''(W)\bar{q}_F(W)(\sigma(S^{-1}))^2 &= \bar{\lambda}(W)S^{-1} && \text{(B.5)}\\
W - \bar{q}_H(W) - \bar{q}_F(W)S^{-1} &= 0 && \text{(B.6)}
\end{aligned}
$$

We rewrite the HJB equation at its optimum,

$$
\begin{aligned}
\rho V(W) &= w'(\bar{c}(W)) + v(\bar{q}_H(W);\theta)\\
&+ V'(W)[\bar{q}_H(W)r + \bar{q}_F(W)r^*S^{-1} - \bar{c}(W) + \bar{q}_F(W)\mu(S^{-1})] + \frac{1}{2}V''(W)(\bar{q}_F(W))^2(\sigma(S^{-1}))^2\\
&+ \bar{\lambda}(W)(W - \bar{q}_H(W) - \bar{q}_F(W)S^{-1})\} \qquad \text{(B.7)}
\end{aligned}
$$

Assume that all elements in the HJB equation are differentiable. Under this assumption, we take the derivative of Eq.(B.7) with respect to the state variable W and then substitute in the first-order conditions, Eq.(B.3-B.5), and the budget constraint, Eq.(B.6), to find:

$$
\begin{aligned}
\rho V'(W) &= \bar{\lambda}(W) + V''(W)[\bar{q}_H(W)r + \bar{q}_F(W)r^*S^{-1} - \bar{c}(W) + \bar{q}_F(W)\mu(S^{-1})] \qquad \text{(B.8)}\\
&+ \frac{1}{2}V'''(W)(\bar{q}_F(W))^2(\sigma(S^{-1}))^2
\end{aligned}
$$

Define the optimal wealth path $W^*(t)$ as

$$
\begin{aligned}
W^*(0) &= W_0^*\\
W^*(t) &= W^*(0) + \int_0^t (\bar{q}_H(W^*(s))r_s + \bar{q}_F(W^*(s))r^*S_s^{-1} - \bar{c}(W^*(s)))ds + \int_0^t \bar{q}_F(W^*(s))d(S_s^{-1}) \geq 0
\end{aligned}
$$

Note that, at any point along the optimal wealth path, Eq.(B.8) holds.

By Ito's lemma, the drift of $V'(W^*(t))$ along the optimal path is

$$
\mathcal{A}V'(W^*(t)) = V''(W^*(t))\mu(W^*(t)) + \frac{1}{2}V'''(W^*(t))(\sigma(W^*(t)))^2 \qquad \text{(B.9)}
$$

where

$$
\begin{aligned}
\mu(W^*(t)) &= \bar{q}_H(W^*(t))r + \bar{q}_F(W^*(t))r^*S^{-1} - \bar{c}(W^*(t)) + \bar{q}_F(W^*(t))\mu(S_t^{-1})\\
\sigma(W^*(t)) &= \bar{q}_F(W^*(t))\sigma(S_t^{-1})
\end{aligned}
$$

Substitute Eq.(B.8) into Eq.(B.9),

$$
\mathcal{A}V'(W^*(t)) = \rho V'(W^*(t)) - \bar{\lambda}(W^*(t)) \qquad \text{(B.10)}
$$

Substitute Eq.(B.3) and Eq.(B.4) into Eq.(B.10),

$$\mathcal{A}[w'(\bar{c}(W^*(t)))] = (\rho - r_t)w'(\bar{c}(W^*(t))) - v'(W^*(t);\theta_t) \tag{B.11}$$

which is the Euler equation of U.S. households investing in the U.S. bond.

By Ito's lemma, the drift of $(V'(W^*(t)S_t^{-1})$ along the optimal path is

$$\begin{aligned}\mathcal{A}(V'(W^*(t)S_t^{-1}) &= [V''(W^*(t))\mu(W^*(t)) + \frac{1}{2}V'''(W^*(t))(\sigma(W^*(t)))^2]S_t^{-1} + V'(W^*(t))\mu(S_t^{-1}) \\ &+ V''(W^*(t))\sigma(W^*(t))\sigma(S^{-1}) \end{aligned} \tag{B.12}$$

Substitute Eq.(B.8) into Eq.(B.12),

$$\begin{aligned}\mathcal{A}(V'(W^*(t))S_t^{-1}) &= \rho V'(W^*(t))S_t^{-1} - \bar{\lambda}(W^*(t))S_t^{-1} + V'(W^*(t))\mu(S_t^{-1}) \\ &+ V''(W^*(t))\bar{q}_F(W^*(t))(\sigma(S_t^{-1}))^2 \end{aligned} \tag{B.13}$$

Substitute Eq.(B.4) and Eq.(B.3) into Eq.(B.13),

$$\mathcal{A}[w'(\bar{c}(W^*(t)))S_t^{-1}] = \rho w'(\bar{c}(W^*(t)))S_t^{-1} - V'(W^*(t))r_t^* S_t^{-1} \tag{B.14}$$

which is the Euler equation of U.S. households investing in the foreign bond.

Denote $M(t) = e^{-\rho t}w'(\bar{c}(W^*(t)))$. Then, Euler equations Eq.(B.11) and Eq.(B.14) can be rewritten as

$$\begin{aligned} 0 &= \mathbb{E}_t\left[\frac{dM_t}{M_t}\right] + r_t + \frac{v'(q_{H,t};\theta_t)}{w'(c_t)}, \\ 0 &= \mathbb{E}_t\left[\frac{d(M_t\exp(-s_t))}{M_t\exp(-s_t)}\right] + r_t^*. \end{aligned}$$

B.2 Proof of Proposition 1

Recall that the real pricing kernels are

$$\begin{aligned} dM_t &= M_t(-\mu + \frac{1}{2}\sigma^2)dt - M_t\sigma dZ_t \\ dM_t^* &= M_t^*(\phi s_t + \frac{1}{2}\sigma^2)dt - M_t^*\sigma dZ_t^* \end{aligned}$$

Rewrite the first Euler equation, we have

$$\begin{aligned} 0 &= \mathcal{A}[\int dM_t + M_t r_t dt + M_t\tilde{\lambda}_t^h dt] \\ r_t &= \mu - \frac{1}{2}\sigma^2 - \tilde{\lambda}_t^h \end{aligned}$$

The fourth Euler equation becomes

$$\begin{aligned} 0 &= \mathcal{A}\left[\int dM_t^* + M_t^* r_t^* dt\right] \\ r_t^* &= -\phi s_t - \frac{1}{2}\sigma^2 \end{aligned}$$

Notice

$$\begin{aligned} d\exp(s_t) &= \exp(s_t)ds_t + \frac{1}{2}\exp(s_t)[ds_t, ds_t]dt \\ d\exp(-s_t) &= -\exp(-s_t)ds_t + \frac{1}{2}\exp(-s_t)[ds_t, ds_t]dt \end{aligned}$$

The second Euler equation becomes

$$\begin{aligned} 0 &= \mathcal{A}\left[\int d(M_t\exp(-s_t)) + r_t^* M_t \exp(-s_t)dt\right] \\ 0 &= -\mu + \frac{1}{2}\sigma^2 - \mathcal{A}[s_t] + \frac{1}{2}[ds_t, ds_t] + [-\sigma dZ_t, -ds_t] + r_t^* \qquad \text{(B.15)} \end{aligned}$$

The third Euler equation becomes

$$\begin{aligned} 0 &= \mathcal{A}\left[\int M_t^*\exp(s_t)\tilde{\lambda}_t^f dt + M_t^*\exp(s_t)r_t dt + d(M_t^*\exp(s_t))\right] \\ 0 &= \tilde{\lambda}_t^f + \phi s_t + \frac{1}{2}\sigma^2 + r_t + \mathcal{A}[s_t] + \frac{1}{2}[ds_t, ds_t] + [-\sigma dZ_t^*, ds_t] \qquad \text{(B.16)} \end{aligned}$$

The sum of equation (B.15) and equation (B.16) gives

$$\tilde{\lambda}_t^h - \tilde{\lambda}_t^f = [ds_t, ds_t] - \sigma[dZ_t^* - dZ_t, ds_t],$$

which is

$$-\tilde{\lambda}_t = [ds_t, ds_t] - \sigma[dZ_t^* - dZ_t, ds_t]$$

Plug in the conjecture

$$ds_t = \alpha_t dt + \beta_t\sigma(dZ_t^* - dZ_t) + \gamma_t \nu dX_t,$$

then

$$-\tilde{\lambda}_t = \gamma_t^2\nu^2 + 2\beta_t^2\sigma^2(1-\zeta) + 2\gamma_t\nu\beta_t(\rho^* - \rho)\sigma - 2\beta_t\sigma^2(1-\zeta) - (\rho^* - \rho)\sigma\gamma_t\nu$$

Suppose for a certain constant k,

$$-k = 2(1-\zeta)\beta_t^2\sigma^2 - 2(1-\zeta)\beta_t\sigma^2$$

$$k - \tilde{\lambda}_t \quad = \quad \gamma_t^2 \nu^2 + 2\gamma_t \nu \beta_t (\rho^* - \rho)\sigma - (\rho^* - \rho)\sigma\gamma_t \nu$$

The solutions are

$$\begin{aligned}
\beta_t &= \frac{1}{2} \pm \frac{1}{2}\sqrt{\frac{\sigma^2(1-\zeta) - 2k}{\sigma^2(1-\zeta)}}, \\
\gamma_t &= \frac{(\rho^* - \rho)\sigma(1-2\beta_t) \pm \sqrt{(\rho^* - \rho)^2\sigma^2(1-2\beta_t)^2 + 4(k - \tilde{\lambda}_t)}}{2\nu}.
\end{aligned}$$

which have real roots for all possible values of λ_t if and only if

$$k \leq \frac{\sigma^2(1-\zeta)}{2}$$

and

$$k \geq \frac{\ell - (\rho^* - \rho)^2\sigma^2/4}{1 - (\rho^* - \rho)^2/(2(1-\zeta))}$$

When the upper bound of k is obtained, $\beta_t = 1/2$. When the lower bound of k is obtained,

$$\beta_t \quad = \quad \frac{1}{2} \pm \frac{1}{2\sigma}\sqrt{\frac{\sigma^2(1-\zeta) - 2\ell}{(1-\zeta) - (\rho^* - \rho)^2/2}}$$

which bounds the range of possible value of β_t.

Lastly, we also solve α_t from

$$\begin{aligned}
-\alpha_t &= \tilde{\lambda}_t^f + \phi s_t + \frac{1}{2}\sigma^2 + r_t + \frac{1}{2}[ds_t, ds_t] + [-\sigma dZ_t^*, ds_t] \\
&= \tilde{\lambda}_t + \phi s_t + \mu + \frac{1}{2}[ds_t, ds_t] + [-\sigma dZ_t^*, ds_t] \\
&= \frac{1}{2}\tilde{\lambda}_t + \phi s_t + \mu - \frac{1}{2}\sigma[dZ_t^*, ds_t] - \frac{1}{2}\sigma[dZ_t, ds_t] \\
\alpha_t &= -\frac{1}{2}\tilde{\lambda}_t - \phi s_t - \mu + \frac{1}{2}\sigma(\gamma_t \nu \rho^* + \beta_t \sigma - \beta_t \sigma \zeta) + \frac{1}{2}\sigma(\gamma_t \nu \rho - \beta_t \sigma + \beta_t \sigma \zeta) \\
&= -\frac{1}{2}\tilde{\lambda}_t - \phi s_t - \mu + \frac{1}{2}\sigma\gamma_t \nu(\rho + \rho^*)
\end{aligned}$$

B.3 Proof of Proposition 2

Recall the definition of the real exchange rate under complete markets, we have

$$d(s_t - \beta s_t^{cm}) \quad = \quad \left(-\frac{1}{2}\tilde{\lambda}_t - \phi(s_t - \beta s_t^{cm}) - (1-\beta)\mu + \frac{1}{2}\sigma\gamma_t \nu(\rho + \rho^*)\right) dt + \gamma_t \nu dX_t \quad \text{(B.17)}$$

We conjecture

$$\begin{aligned}
s_t - \beta s_t^{cm} &= f(\lambda_t) + H_t \\
H_t &= \exp(-\phi t) H_0 + \int_0^t \exp(-\phi(t-u)) h(\lambda_u) du
\end{aligned}$$

which implies

$$\begin{aligned}
dH_t &= \left(-\phi \exp(\phi(-t)) H_0 + h(\lambda_t) - \phi \int_0^t \exp(\phi(u-t)) h(\lambda_u) du\right) dt \\
&= \left(-\phi \exp(\phi(-t)) H_0 + h(\lambda_t) - \phi(H_t - \exp(\phi(-t)) H_0)\right) dt \\
&= (h(\lambda_t) - \phi H_t)\, dt
\end{aligned}$$

We note

$$\begin{aligned}
d(s_t - \beta s_t^{cm}) &= f' d\lambda_t + \frac{1}{2} f'' [d\lambda_t, d\lambda_t]^2 dt + dH_t \\
&= f'(-\theta \lambda_t dt + \nu dX_t) + \frac{1}{2} f'' \nu^2 dt + (h(\lambda_t) - \phi H_t)\, dt
\end{aligned}$$

and this has to match equation (B.17).

Matching dX_t term,

$$f' = \gamma_t = \frac{b + \sqrt{b^2 + 4(k - \tilde{\lambda}_t)}}{2\nu}$$

where $b = (\rho^* - \rho)\sigma(1 - 2\beta_t)$. Then,

$$\begin{aligned}
f(\lambda) &= \frac{1}{2\nu}\{-\sqrt{b^2+4k}\log\left(2e^{\lambda/2}\left(\cosh\left(\frac{\lambda}{2}\right)\left(\sqrt{b^2+4k}\sqrt{b^2+4k-2\ell\tanh\left(\frac{\lambda}{2}\right)-2\ell}+b^2+4k-\ell\right)-\ell\sinh\left(\frac{\lambda}{2}\right)\right)\right) \\
&+ \sqrt{b^2+4k-4\ell}\log\left(2e^{\lambda/2}\left(\cosh\left(\frac{\lambda}{2}\right)\left(\sqrt{b^2+4k-4\ell}\sqrt{b^2+4k-2\ell\tanh\left(\frac{\lambda}{2}\right)-2\ell}+b^2+4k-3\ell\right)-\ell\sinh\left(\frac{\lambda}{2}\right)\right)\right) \\
&+ \lambda\left(\sqrt{b^2+4k}+b\right)\}
\end{aligned}$$

and

$$f''(\lambda) = \frac{\frac{\ell e^{2\lambda}}{(e^\lambda+1)^2} - \frac{\ell e^\lambda}{e^\lambda+1}}{\nu\sqrt{b^2 + 4\left(k - \frac{\ell e^\lambda}{e^\lambda+1}\right)}}$$

Matching dt term,

$$h(\lambda_t) = -\frac{1}{2}\tilde{\lambda}_t - \phi f - (1-\beta)\mu + \frac{1}{2}\sigma\gamma_t\nu(\rho + \rho^*) + f'\theta\lambda_t - \frac{1}{2} f'' \nu^2$$

Since γ_t is also a function of λ_t, we confirm the conjecture that $h(\lambda_t)$ is a function only of λ_t.

So

$$s_t \quad = \quad f(\lambda_t) + H_t + \beta s_t^{cm}$$

B.4 Long-Term Expectation of Log Exchange Rate

Since,

$$ds_t \quad = \quad \left(-\frac{1}{2}\tilde{\lambda}_t - \phi s_t - \mu + \frac{1}{2}\sigma\gamma_t\nu(\rho+\rho^*)\right) dt + \gamma_t \nu dX_t + \beta\sigma(dZ_t^* - dZ_t),$$

then,

$$d(e^{\phi t}s_t) = e^{\phi t}\left(-\frac{1}{2}\tilde{\lambda}_t - \mu + \frac{1}{2}\sigma\gamma_t\nu(\rho+\rho^*)\right) dt + e^{\phi t}\gamma_t\nu dX_t + e^{\phi t}\beta\sigma(dZ_t^* - dZ_t)$$

The solution of the above Stochastic Differential Equation is:

$$s_T = e^{-\phi T}s_0 + \int_0^T e^{\phi(t-T)}\left(-\frac{1}{2}\tilde{\lambda}_t - \mu + \frac{1}{2}\sigma\gamma_t\nu(\rho+\rho^*)\right) dt + \int_0^T e^{\phi(t-T)}\gamma_t\nu dX_t + \int_0^T e^{\phi(t-T)}\beta\sigma(dZ_t^* - dZ_t)$$

Recall that

$$\begin{aligned}
\gamma_t \quad &= \quad \frac{(\rho^*-\rho)\sigma(1-2\beta_t) \pm \sqrt{(\rho^*-\rho)^2\sigma^2(1-2\beta_t)^2 + 4(k-\tilde{\lambda}_t)}}{2\nu}, \\
|(1-2\beta_t)| \quad &= \quad \sqrt{\frac{\sigma^2(1-\zeta)-2k}{\sigma^2(1-\zeta)}},
\end{aligned}$$

then γ_t is bounded,

$$|\gamma_t| \quad \leq \quad \frac{|\rho^*-\rho|\sqrt{\frac{\sigma^2(1-\zeta)-2k}{(1-\zeta)}} + \sqrt{\frac{\sigma^2(1-\zeta)-2k}{(1-\zeta)}(\rho^*-\rho)^2 + 4k}}{2\nu}.$$

Hence, for s_T, the integrands in the stochastic integrals are all $\mathcal{H}^2$, and the stochastic integrals are Martingales with expectation 0. Then,

$$\begin{aligned}
\lim_{T\to\infty}\mathbb{E}_0[s_T] \quad &= \quad \lim_{T\to\infty} e^{-\phi T}s_0 + \lim_{T\to\infty}\mathbb{E}_0[\int_0^T e^{\phi(t-T)}\left(-\frac{1}{2}\tilde{\lambda}_t - \mu + \frac{1}{2}\sigma\gamma_t\nu(\rho+\rho^*)\right) dt] \\
&= \quad \lim_{T\to\infty}\int_0^T e^{\phi(t-T)}\left(-\frac{1}{2}\mathbb{E}_0[\tilde{\lambda}_t] - \mu + \frac{1}{2}\sigma\mathbb{E}_0[\gamma_t]\nu(\rho+\rho^*)\right) dt \\
&= \quad \frac{1}{\phi}\left(-\frac{1}{2}\lim_{T\to\infty}\mathbb{E}_0[\tilde{\lambda}_T] - \mu + \frac{1}{2}\sigma\lim_{T\to\infty}\mathbb{E}_0[\gamma_T]\nu(\rho+\rho^*)\right).
\end{aligned}$$

B.5 Backus-Smith Puzzle

Recall that $\tilde{\lambda}_t^h$ denotes the convenience yield earned by U.S. investors on their dollar bond holdings, and $\tilde{\lambda}_t^f$ denotes the foreign investors' convenience yield on dollar bonds. Likewise, define $\tilde{\lambda}_t^{h*}$ as the convenience yield earned by U.S. investors on their foreign bond holdings, and $\tilde{\lambda}_t^{f*}$ as the foreign investors' convenience yield on their foreign bonds. Then, we can rewrite the four Euler equations as

$$0 = \mathbb{E}_t\left[\frac{dM_t}{M_t}\right] + r_t + \tilde{\lambda}_t^h, \qquad 0 = \mathbb{E}_t\left[\frac{d(M_t\exp(-s_t))}{M_t\exp(-s_t)}\right] + r_t^* + \tilde{\lambda}_t^{h*},$$
$$0 = \mathbb{E}_t\left[\frac{d(M_t^*\exp(s_t))}{M_t^*\exp(s_t)}\right] + r_t + \tilde{\lambda}_t^f, \qquad 0 = \mathbb{E}_t\left[\frac{dM_t^*}{M_t^*}\right] + r_t^* + \tilde{\lambda}_t^{f*}.$$

Recall the real pricing kernels

$$\begin{aligned} dM_t &= M_t(-\mu + \frac{1}{2}\sigma^2)dt - M_t\sigma dZ_t, \\ dM_t^* &= M_t^*(\phi s_t + \frac{1}{2}\sigma^2)dt - M_t^*\sigma dZ_t^*. \end{aligned}$$

Rewrite the first Euler equation, we have

$$\begin{aligned} 0 &= \mathcal{A}[\int dM_t + M_t r_t dt + M_t\tilde{\lambda}_t^h dt] \\ r_t &= \mu - \frac{1}{2}\sigma^2 - \tilde{\lambda}_t^h \end{aligned}$$

The fourth Euler equation becomes

$$\begin{aligned} 0 &= \mathcal{A}\left[\int dM_t^* + M_t^* r_t^* dt + M_t^*\tilde{\lambda}_t^{f*}dt\right] \\ r_t^* &= -\phi s_t - \frac{1}{2}\sigma^2 - \tilde{\lambda}_t^{f*} \end{aligned}$$

Recall that

$$\begin{aligned} d\exp(s_t) &= \exp(s_t)ds_t + \frac{1}{2}\exp(s_t)[ds_t, ds_t]dt, \\ d\exp(-s_t) &= -\exp(-s_t)ds_t + \frac{1}{2}\exp(-s_t)[ds_t, ds_t]dt. \end{aligned}$$

The second Euler equation becomes

$$\begin{aligned} 0 &= \mathcal{A}\left[\int d(M_t\exp(-s_t)) + r_t^* M_t\exp(-s_t)dt + \tilde{\lambda}_t^{h*}M_t\exp(-s_t)dt\right] \\ 0 &= -\mu + \frac{1}{2}\sigma^2 - \mathcal{A}[s_t] + \frac{1}{2}[ds_t, ds_t] + [dm_t, -ds_t] + r_t^* + \tilde{\lambda}_t^{h*} \end{aligned} \tag{B.18}$$

The third Euler equation becomes

$$
\begin{aligned}
0 &= \mathcal{A}\left[\int M_t^* \exp(s_t)\tilde{\lambda}_t^f dt + M_t^* \exp(s_t) r_t dt + d(M_t^* \exp(s_t))\right] \\
0 &= \tilde{\lambda}_t^f + \phi s_t + \frac{1}{2}\sigma^2 + r_t + \mathcal{A}[s_t] + \frac{1}{2}[ds_t, ds_t] + [dm_t^*, ds_t] \qquad \text{(B.19)}
\end{aligned}
$$

Then, the sum of equation (B.18) and equation (B.19) gives

$$
[dm_t - dm_t^*, ds_t] = [ds_t, ds_t] + (\tilde{\lambda}_t^f - \tilde{\lambda}_t^h) - (\tilde{\lambda}_t^{f*} - \tilde{\lambda}_t^{h*}).
$$

Redefine $\tilde{\lambda}_t = (\tilde{\lambda}_t^f - \tilde{\lambda}_t^h) - (\tilde{\lambda}_t^{f*} - \tilde{\lambda}_t^{h*})$, then we again have

$$
[dm_t - dm_t^*, ds_t] = [ds_t, ds_t] + \tilde{\lambda}_t
$$